A numerical experiment testing the optimality gap of nonconvex quadratic program with two quadratic constraints

Sheng Cheng Nuno C. Martins

Abstract

This document describes a numerical experiment that determines the proportion of randomly generated feasible instances of nonconvex quadratic program with two quadratic constraints (QC2QP) that has no optimality gap, in which case a solution of the QC2QP can be obtained from that of a specific semidefinite relaxation.

1 Background

We consider the following real-valued quadratic program with two quadratic constraints (QC2QP):

$$\begin{aligned} \underset{\boldsymbol{z}\in\mathbb{R}^n}{\text{minimize}} \quad & q_0(\boldsymbol{z}) = \boldsymbol{z}^T Q_0 \boldsymbol{z} + 2b_0^T \boldsymbol{z} \\ \text{subject to} \quad & q_1(\boldsymbol{z}) = \boldsymbol{z}^T Q_1 \boldsymbol{z} + 2b_1^T \boldsymbol{z} + c_1 \le 0, \\ & q_2(\boldsymbol{z}) = \boldsymbol{z}^T Q_2 \boldsymbol{z} + 2b_2^T \boldsymbol{z} + c_2 \le 0, \end{aligned} \tag{QP0}$$

where Q_0, Q_1, and Q_2 are $n \times n$-dimensional real symmetric matrices; b_0, b_1 and b_2 are n-dimensional real vectors; and c_1 and c_2 are real constants. In the sequel, we follow the notation of [3] and write the parameters in the homogeneous form:

$$M(q_0) = \begin{bmatrix} 0 & b_0^T \\ b_0 & Q_0 \end{bmatrix}, \quad M(q_i) = \begin{bmatrix} c_i & b_i^T \\ b_i & Q_i \end{bmatrix}, \quad i \in \{1,2\}. \tag{1.1}$$

The semidefinite relaxation of (QP0) that is used in [3] is

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathcal{S}^{n+1}}{\text{minimize}} \quad & M(q_0) \bullet \boldsymbol{X} \\ \text{subject to} \quad & M(q_i) \bullet \boldsymbol{X} \le 0, \qquad i \in \{1,2\}, \\ & I_{00} \bullet \boldsymbol{X} = 1, \\ & \boldsymbol{X} \succeq 0, \end{aligned} \tag{SP}$$

where $I_{00} = \left[\begin{smallmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{smallmatrix}\right]$. The Lagrange dual of (SP), which is also the dual of (QP0), is

$$\begin{aligned} \underset{\boldsymbol{Z}\in\mathcal{S}^{n+1},\boldsymbol{y}_0,\boldsymbol{y}_1,\boldsymbol{y}_2\in\mathbb{R}}{\text{maximize}} \quad & \boldsymbol{y}_0 \\ \text{subject to} \quad & \boldsymbol{y}_0 I_{00} - \boldsymbol{y}_1 M(q_1) - \boldsymbol{y}_2 M(q_2) + \boldsymbol{Z} = M(q_0), \\ & \boldsymbol{y}_i \ge 0, \qquad i \in \{1,2\}, \\ & \boldsymbol{Z} \succeq 0. \end{aligned} \tag{SD}$$

We say that there is no optimality gap when a solution of (QP0) can be computed from that of (SP).

2 Numerical experiment

We conduct a numerical experiment to determine the proportion of randomly generated feasible nonconvex QC2QP instances that has no optimality gap (and hence can be solved by the semidefinite relaxation

(SP) using the test described by [3, Theorem 3.2]). For a specific positive integer n, we generate the matrix M_i in $\mathbb{R}^{(n+1)\times(n+1)}$ whose entries are uniformly distributed in $[0, 1]$ and compute the problem data $M(q_i)$ by

$$M(q_i) = (M_i + M_i^T)/2, \qquad i \in \{0, 1, 2\}. \tag{2.1}$$

However, we only keep the problem data that satisfy the following qualifications while discarding the rest:

1. Each of the matrices Q_1 and Q_2 (corresponding to $M(q_1)$ and $M(q_2)$, respectively, in (1.1)) must have at least one negative eigenvalue.

2. The data $M(q_0)$, $M(q_1)$, and $M(q_2)$ must hold Slater's condition for (SP) and (SD).

Qualification 1 yields two consequences. First, the problem data characterize nonconvex QC2QP instances. Second, neither Q_1 or Q_2 is positive definite so that [1, Theorem 4.2] is excluded for testing the optimality gap (or, equivalently, for testing the duality gap). Qualification 2 makes sure that [3, Assumptions 2.1 and 2.3] are met.

For each dimension n, we generate 1000 instances of problem data (all satisfying the above qualifications and available online [2]), test them using [3, Theorem 3.2], and count the number of instances which do not have an optimality gap. The result is summarized in Table 1. It is interesting to note that most of the randomly generated feasible instances do not have an optimality gap and hence can be solved using a solution of the relaxation, which indicates the potential usefulness of the test.

Table 1: Result of the numerical experiment. The integer n represents the problem dimension while the integer m represents the number of instances that have no optimality gap out of a total of 1000.

n	2	3	4	5	6	7
m	855	780	775	760	744	747

References

[1] Wenbao Ai and Shuzhong Zhang. Strong duality for the CDT subproblem: a necessary and sufficient condition. *SIAM Journal on Optimization*, 19(4):1735–1756, 2009.

[2] Sheng Cheng. QC2QP-SDR-Optimality-Gap-Test. `https://github.com/Sheng-Cheng/QC2QP-SDR-Optimality-Gap-Test`, 2019.

[3] Sheng Cheng and Nuno C Martins. An optimality gap test for a semidefinite relaxation of a quadratic program with two quadratic constraints. *arXiv preprint arXiv:1907.02989*, 2019.