Non-Linear Dynamics.

Anders. Carlsson

第壹講 2017年8月28日

Why do we study it?

Basic vocabulary for broad range of phenomena: statistical, particle, biophysics.

1st-order differential equation

$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t)$ usually ignore explict time dependence

Newton II Law

$\frac{d^2x}{dt} = \frac{F}{m} \longrightarrow$ $x_1 = x$, $x_2 = dx/dt$ $\begin{cases} \frac{dx_2}{dt} = \frac{F(x_1)}{m} \\ \frac{dx_1}{dt} = x_2 \end{cases}$

Most ambitious goal: solve initial problem: given $\vec{x}(0) = \vec{x}_0$, solve $\vec{x}(t)$

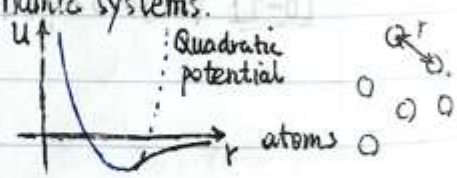
Limited goal: calculate long term behaviors

$\frac{d\vec{x}}{dt} = \hat{A}\vec{x} + \vec{v}_0$ possibilities:

1) $\vec{x}$ grows exponentially. if $\hat{A}$ has positive eigenvalues
2) $\vec{x}$ oscillates if imaginery eigenvalues
3) $\vec{x}$ decays exponentially to a constant if eigenvalues <0

Qualitative new effects caused by NL terms

1). chaotic behaviors: unpredictable
2) higher harmonic generations.
3) bistability → asympotic value depends on initial conditions
4) limit cycle: oscillation with robust amplitude - heart beat
5) strange attractors: very peculiar sets for chaotic dynamic systems.
6). phonon damping

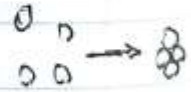


Origins of Non-Linear Terms

step functions like rectifier

anharmonic terms in potential in phonon damping

chemical reaction rates as functions of concentrations $\frac{dN_1}{dt} = kN_1^4$



partical interactions in field theory

velocity field carried along by fluids.

convective terms in fluid mechanics. $\rho\frac{d\vec{v}}{dt} = -\nabla p - \rho(v\cdot\vec{\nabla})\vec{v}$ Navier-Stokes Eq.

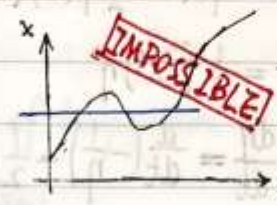
第貳講 Fixed Points and Classification 2017年8月30日

) things about Homework

1. extensions: request by the day assignment is due.
2. late penalty: 20%/day.

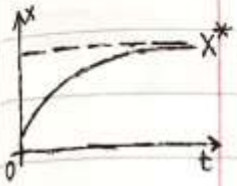
1d → 2d → 3d

$\boxed{1d}$ $\frac{dx}{dt} = f(x)$ no explicit time dependence

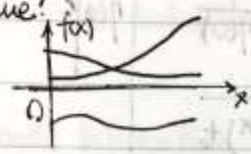


slope is determined only by x for 1st order system

1. Does x approach a limiting value at large times?
2. How fast does it approach the value?

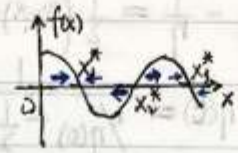


Fixed point defined by $f(x^*)=0$

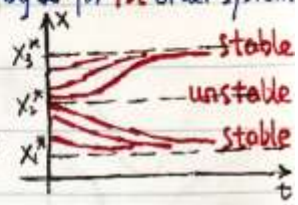


No fixed points.

$t \to 0$, $x \to \pm\infty$

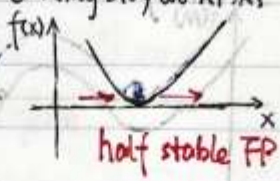


can only stay at x_1^*, x_3^*



if $f'(x^*)<0$ stable

if $f'(x^*)>0$ unstable



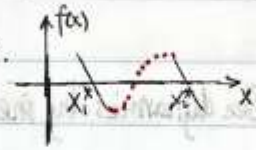
half stable FP

Long term behavior:

If there is a neighboring fixed point. $x \to x^*$ at $t \to \infty$.

If there is no neighboring fixed point. $x \to \pm\infty$ at $t \to \infty$.

There cannot be neighboring two fixed points.



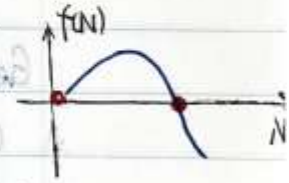
Examples. Logistic Equation

Biophysics. Logistic equation for population growth. (rabbits in backyard)

$$\frac{dN}{dt} = f(N) = rN\left(1-\frac{N}{K}\right)$$

r: birth, death rate; $\left(1-\frac{N}{K}\right)$: competition

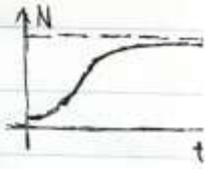
$$\frac{df}{dN} = r - \frac{2Nr}{K}$$



$N^*=0$, $\frac{df}{dN}>0$ unstable.

Stable steady-state population $N^* = K/2$, $\frac{df}{dN}<0$ stable

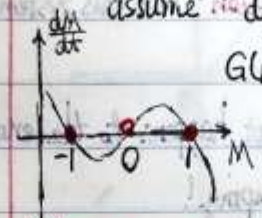
When $N \ll K$ $dN/dt = rN$. $N = N_0 e^{rt}$ exponential phase



lograthmic phase.

Condensed matter physics: Magnetization dynamic of a nanoparticle

assume $\frac{dM}{dt} = -\frac{\partial G}{\partial M}$: G drops over time until minimum.



$G(M) = -\frac{M^2}{2} + \frac{M^4}{4}$ 无奇次项 因正反磁化方向等价；M^2为负，不想在$M=0$稳定

$f(M) = \frac{dM}{dt}$.

$$\frac{df}{dM} = 1-3M^2 \begin{cases} >0, & \text{if } M=0 \\ <0, & \text{if } M=\pm 1 \end{cases}$$

第叁講 Rate of Convergence to FP

2017年9月1日

Tayler expansion near x^*

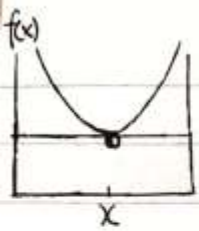
$$f(x) = f(x^*) + f'(x^*)(x-x^*)$$

let $\eta = x - x^*$

$$\frac{d\eta}{dt} \approx f'(x^*)\eta$$

$$\eta(t) = \eta(0)e^{f'(x^*)t}$$

If $f'(x^*)<0$. $\eta \to 0$ expernentially. $\tau = \frac{1}{|f'(x^*)|}$



$$\frac{d\eta}{dt} = \dot{\eta} = (\dot{\eta + x^*}) = \dot{x} = f(x)$$

$f'(x^*) = 0$

$f(x) \approx f(x^*) + \frac{1}{2} f''(x^*)(x-x^*)^2$ again $\eta = x - x^*$

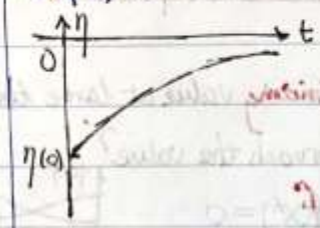
$$\frac{d\eta}{dt} = \frac{1}{2} f''(x^*)\eta^2$$

$$\frac{1}{\eta^2}\frac{d\eta}{dt} = \frac{d}{dt}\left(-\frac{1}{\eta}\right) = \frac{1}{2} f''(x^*)$$

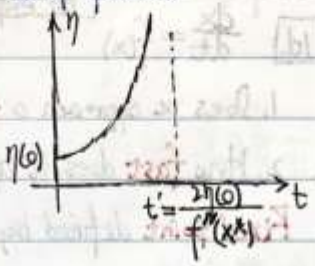
$$-\frac{1}{\eta} = \frac{1}{2} f''(x^*)t - \frac{1}{\eta(0)}$$

$$\eta(t) = \frac{1}{1/\eta(0) - \frac{1}{2} f''(x^*)t}$$

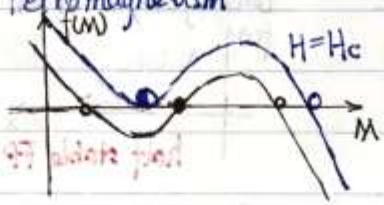
① if $\eta(0) < 0$



② If $\eta(0) > 0$



Ferromagnetism

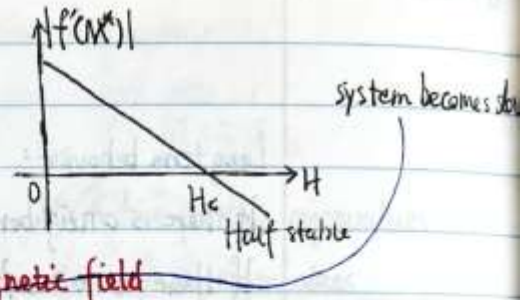


$f = M - M^3 + H$

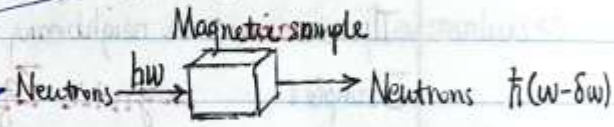
Free energy G, $f(M) = \frac{-\partial G}{\partial M}$

$G = -\frac{M^2}{2} + \frac{1}{4}M^4 - HM$

external magnetic field



See dynamics by inelastic scattering



General Facts

* ① cannot have oscillations in 1d dynamic systems

* ② always possible to describe the system with a "potential" function.

$\frac{dx}{dt} = -\mu \frac{dV}{dx}$ → systems move towards a lower energy.

$$V(x) = -\frac{1}{\mu}\int_{x_0}^{x} f(x')\,dx'$$

note: 在复数域 $\frac{dz}{dt} = f(z) = iz$ can describe oscillations question: does this system have a limit cycle?

energy analog:

Fixed Points	Zero "force"	What happens to the energy conservation?
stable FP	stable energy minimum	Damping!

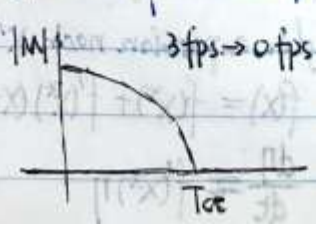
System is very non-isolated.

Bifurcation: Qualitative change in system behavior as function of control parameter.

change the # of fixed points and their stability

examples:

(1) Phase transition into magnetic state



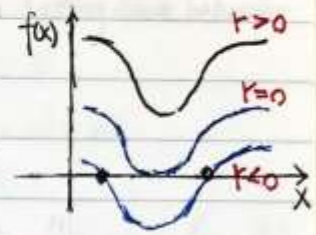
第肆講 Bifurcation: qualitative change of behavior

2017.9.6.

comes from

1) Ferromagnetic or superconductor as a function of temperature T
2) Stiff biopolymers.
3) Fluid flowing through a pipe $v<v_c$ smooth flow, $v>v_c$ turbulent

Categories of Bifurcation in 1d



"blue-sky" bifurcation : saddle-node bifurcation

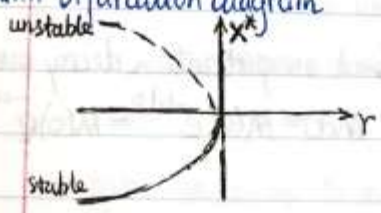
control parameter moves the curve up or down.

$f(x)$ is quadratic around the bottom

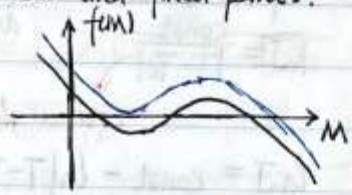
approximate around x_c^*, r_c: $f(x)=r+x^2$ "standard form"

$(x^*)^2+r=0$, $x^*=\pm\sqrt{-r}$

bifurcation diagram



the choice of parameter and fixed points.



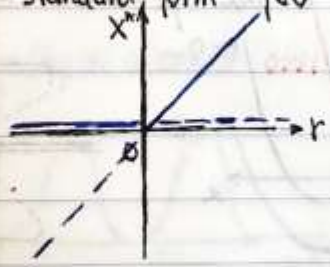
example: ferromagnetic

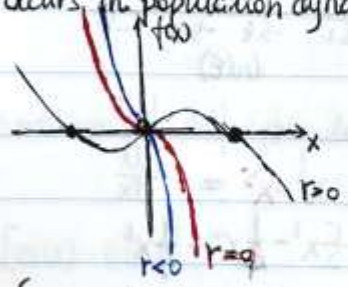
$f(M)=M-M^3+H$ H is the parameter

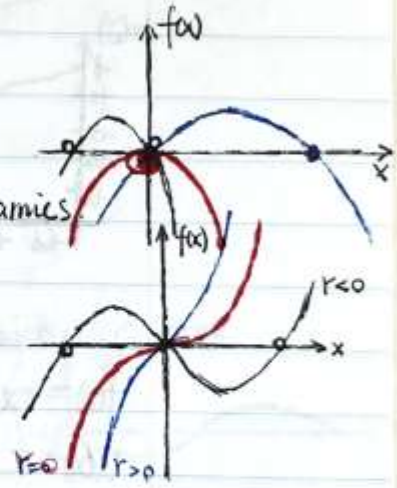
When $H<H_c$, two new fixed points develop.

Transcritical Bifurcation

standard form $f(x)=rx-x^2$ usually occurs in population dynamics







Pitchfork Bifurcation

spontaneous symmetry breaking subtypes: $\begin{cases}\text{supercritical well behaved: } f(x)=rx-x^3 \\ \text{subcritical "dangerous" } f(x)=rx+x^3\end{cases}$

time behavior: $\frac{dx}{dt}\propto x^3$. $x(t)\propto\frac{1}{\sqrt{t}}$

Apply to ferromagnetism

$$G = -\frac{r}{2}M^2 + \frac{1}{4}M^4$$

$$f(M) = -\frac{\partial G}{\partial M} = rM - M^3$$

* calculate $M(T)$

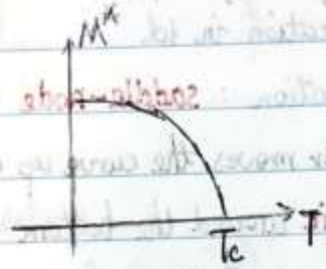
$T \Leftrightarrow r$

$M(t>\infty) \neq 0$ if $r>0$

$r \propto (T_c - T)$ if $r<0$, $T<T_c$

$x^* = \pm\sqrt{r}$

$\therefore M \propto (T_c - T)^{0.5}$



Experimental observation

$M \propto (T_c - T)^{0.325}$

第伍講 2017.5.7

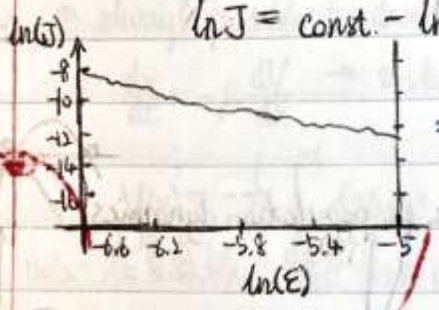
* critical slowing

If T is slightly above T_c, how quickly will an imposed magnetization decay over time?

$r<0$. If M is small $\frac{dM}{dt} = rM = -|r|M$ $\quad M(t) = M(0)e^{-|r|t} = M(0)e^{-\text{const}|T-T_c|t} = M(0)e^{-\frac{t}{\tau}}$

$\boxed{\tau} = \frac{\text{const.}}{|T-T_c|}$ critical slowing

$\ln\tau = \text{const.} - \ln|T-T_c|$



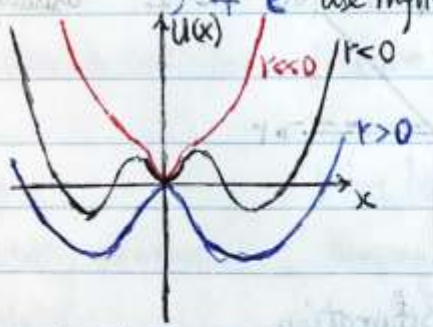
$\Rightarrow \tau \propto \frac{1}{|T-T_c|^2}$ fluctuations are important in low dimensions

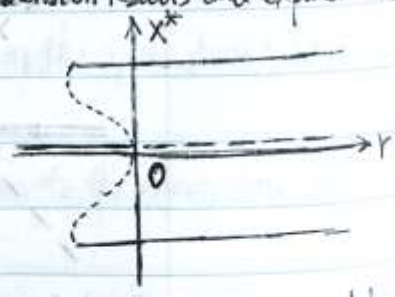
$3 = 4 - \epsilon$ use high-dimension results and expand to 3d

Subcritical pitch-fork bifurcation

$f(x) = rx + x^3 - x^5 = -\frac{dU}{dx}$

$U(x) = -\frac{r}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{6}x^6$

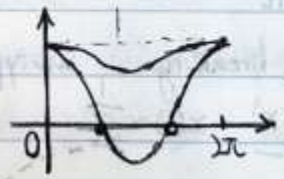




Hysteresis - history dependent behavior

Flow on a circle

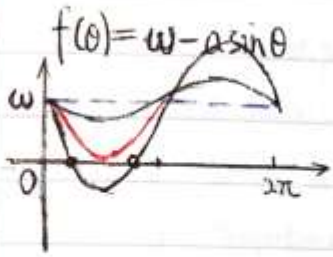
$f(\theta)$ is periodic $\quad \frac{d\theta}{dt} = f(\theta)$



$$\frac{d\theta}{dt} = \frac{1}{\eta_R}(\tau - mgl\sin\theta)$$

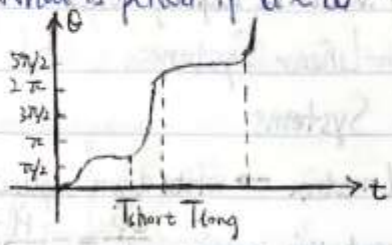
rotational drag coefficient

$f(\theta) = \omega - a\sin\theta$



$a<\omega$

What is period if $a\approx\omega$?



$T_{short}\sim\frac{1}{\omega}$

T_{long}

NOW not obvious wrong

OW obvious wrong
- wrong units
- wrong limit behavior

第四講 2017.9.11

$\boxed{T_{long}\overset{?}{=}\frac{T}{\omega-a}}$

$$T=\int_0^{2\pi}\frac{dt}{d\theta}\cdot d\theta=\int_0^{2\pi}\frac{d\theta}{\omega-a\sin\theta}$$

$\sin\theta=\cos\left(\theta-\frac{\pi}{2}\right)$

make 2 approximations.

(1) replace $f(\theta)$ with a parabola. with the same curvature at the bottom.

Expand $f(\theta)\propto\sin\theta$ around $\theta=\frac{\pi}{2}$ $\quad\sin\theta\approx 1-\frac{(\theta-\frac{\pi}{2})^2}{2!}$

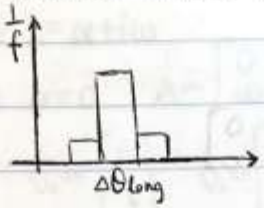
$\therefore\ f(\theta)=(\omega-a)+a(\theta-\frac{\pi}{2})^2/2$ $\quad T=\int_0^{2\pi}\frac{d\theta}{(\omega-a)+a(\theta-\frac{\pi}{2})^2/2}$

$$\approx\int_{-\infty}^{+\infty}\frac{d\theta}{(\omega-a)+a(\theta-\frac{\pi}{2})^2/2}=\pi\sqrt{\frac{2}{a(\omega-a)}}$$

(2) take integal up to infinity.

$\int_{-\infty}^{0}\frac{d\theta}{(\omega-a)+a(\theta-\frac{\pi}{2})^2/2}$ and $\int_{2\pi}^{\infty}\frac{d\theta}{(\omega-a)+a(\theta-\frac{\pi}{2})^2/2}$ are finite

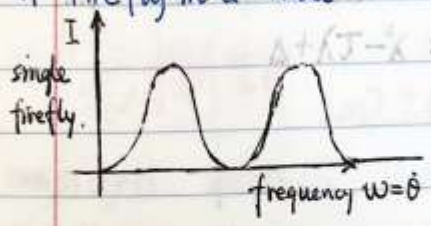
Physical picture of divergence



$\Delta\theta_{long}\propto\sqrt{\omega-a}$

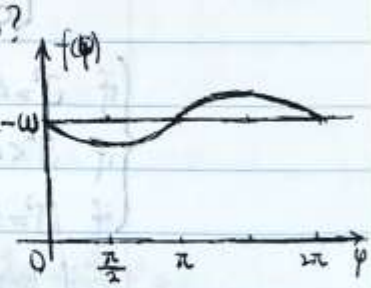
when $\omega\to a$. $\Delta\theta_{long}\to 0$.

Firefly in a crowd



What if group $\Omega=\dot\alpha$ is at another frequency?

$$\dot\theta=\omega+A\sin[\alpha(t)-\theta(t)]$$



Let $\varphi=\alpha(t)-\theta(t)$

$\dot\varphi=\Omega-\omega-A\sin[\varphi(t)]$

$A=0$. $\varphi=(\Omega-\omega)t$

$\theta=\alpha-\varphi=\omega t$, $\alpha=\Omega t$ period of $\varphi=\frac{2\pi}{\Omega-\omega}$. $T_\theta=\frac{2\pi}{\omega}$

$A\to|\Omega-\omega|$ $\quad T_\varphi\to\infty$ the system is at a fixed point

$\theta-\alpha=\text{const.}$ $\quad\theta=\Omega t+\text{const.}$ the same frequency as crowd

$A<|\Omega-\omega|$ φ has long period, small frequency.

$\theta=\alpha-\varphi$ would be slightly slower.

§ Two-Dimensional Non-Linear Systems.

Starting from Linear Systems

$\frac{d\vec{x}}{dt}=A\vec{x}$. A is a real matrix → write down explicit solution

quantum matrix: 2 component spin system $\frac{d\vec{\psi}}{dt}=-i\frac{\hat{H}}{\hbar}\vec{\psi} \Longrightarrow \vec{\psi}(t)=c_1e^{\frac{-iE_1t}{\hbar}}\vec{\psi}_1+c_2e^{\frac{-iE_2t}{\hbar}}\vec{\psi}_2$

$\vec{\psi}_1,\vec{\psi}_2$ are eigenfunctions of $\hat{H}\vec{\psi}=E\vec{\psi}$, c_1,c_2 determined by initial conditions

classical case: $\vec{x}(t)=c_1e^{\lambda_1t}\vec{v}_1+c_2e^{\lambda_2t}\vec{v}_2$

$A\vec{v}_1=\lambda_1\vec{v}_1$, $A\vec{v}_2=\lambda_2\vec{v}_2$

$\vec{x}(0)=c_1\vec{x}_1+c_2\vec{x}_2$

Comparison between Quantum & Classical

QM	CM
$\vec{\psi}$ complex	$\vec{x}$ real
$\hat{H}$ Hermittian	$\hat{A}$ real
$i\hat{H}/\hbar$ imaginary eigenvalues	$\hat{A}$'s eigenvalues real or complex.
orthogonal eigenvectors	linearly independent eigenvectors

* Chanllenges: find a matrix with only one eigenvector

第柒講 Eigenvector Analysis

2017.9.13

we find one $\begin{bmatrix}1&0\\1&1\end{bmatrix}$ questions: what is $\vec{x}(t)$ if $A=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

Eigenvalues of general $\hat{A}$

$\hat{A}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $J=Tr(A)=a+d$ $\quad Det(\hat{A}-\lambda\mathbb{I})=(a-\lambda)(d-\lambda)-bc=0$ $\quad \lambda_\pm=\frac{T\pm\sqrt{T^2-4\Delta}}{2}$

$\Delta=Det(A)=ad-bc$ $\quad =\lambda^2-T\lambda+\Delta$

$$\begin{cases}\text{if } J^2>4\Delta^2: \text{ 2 real } \lambda.\\ \text{if } J^2<4\Delta^2: \text{ 2 complex } \lambda: \text{ if } J=0: \text{ 2 imaginary } \lambda\\ \text{if } J^2=4\Delta^2. \text{ 1 real } \lambda\end{cases}$$

if $bc>0$: $T^2-4\Delta^2=(a-d)^2+4bc>0$: 2 real λ : exponential growth or decay

if $4bc<(a-d)^2$: complex λ oscillation

Fixed points

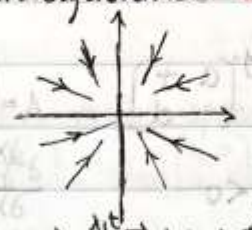
If $\Delta\neq0$ $\begin{bmatrix}0\\0\end{bmatrix}$ is the only FP

If $\Delta=0$: fixed points along eigenvector with zero eigenvalue. a line

Behaviors under Various Eigenvalues

real non-zero

(1) $\lambda_1\lambda_2<0$, $\lambda_1=\lambda_2=\lambda$. $\hat{A}=\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$ $\hat{A}\vec{x}=\lambda\vec{x}$

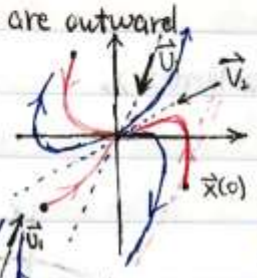


"star" trajectories

$\lambda_1,\lambda_2>0$. trajectories are outward.

(2) $0>\lambda_2>\lambda_1$ —

$\lambda_1>\lambda_2>0$ —



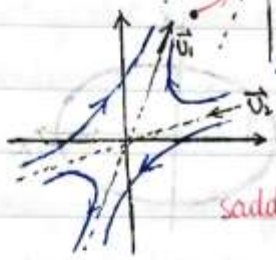
$\vec{x}(t)=(\ )e^{\lambda_1 t}\vec{v}_1+(\ )e^{\lambda_2 t}\vec{v}_2$

larger eigenvector dominates at first

stable node

unstable node

(3) $\lambda_1>0>\lambda_2$

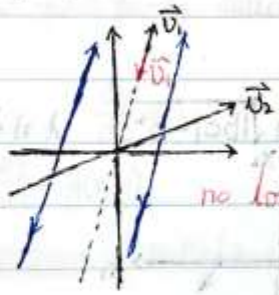


saddle point

Zero eigenvalues.

$\lambda_1>0$. $\lambda_2=0$

$\lambda\neq 0$ $\lambda_2=0$



fixed line

no long term behaviors

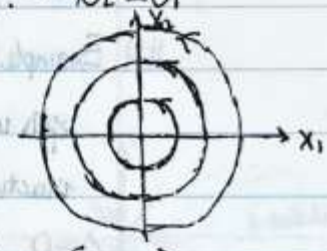
complex eigenvalue

fact: $\vec{v}_2=\vec{v}_1^*$ $\lambda_2^*=\lambda_1$ $A\vec{v}_1=\lambda_1\vec{v}_1$ $A\vec{v}_1^*=\lambda_1^*\vec{v}_1^*=\lambda_2\vec{v}_1^*$. $\Rightarrow \vec{v}_2^*=\vec{v}_1$

$\lambda=\alpha+i\omega$

(1) $\alpha=0$ $A=\begin{bmatrix}0 & -\omega\\ \omega & 0\end{bmatrix}$ $\Delta=\omega^2$. $\tau=0$ $\lambda=\pm\sqrt{-\Delta}=\pm i\omega$

$\lambda_1=i\omega$. $\lambda_2=-i\omega$



$\vec{v}_1=\begin{bmatrix}1\\ -i\end{bmatrix}$ $\vec{v}_2=\begin{bmatrix}1\\ i\end{bmatrix}$

Take $\vec{x}(0)=\begin{bmatrix}R\\ 0\end{bmatrix}=\frac{R}{2}(\vec{v}_1+\vec{v}_2)$ $\vec{x}(t)=\frac{R}{2}\left\{e^{i\omega t}\begin{bmatrix}1\\ -i\end{bmatrix}+e^{-i\omega t}\begin{bmatrix}1\\ i\end{bmatrix}\right\}=R\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix}$

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(2) $\alpha\neq 0$ spirals.

Matrix with only one eigenvalue

$\frac{d\vec{x}}{dt}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\vec{x}$ $\frac{d\vec{x}}{dt}=\hat{A}\vec{x}$ $\vec{x}(t)=e^{\hat{A}t}\vec{x}(0)$ ← $\hat{A}=I+\hat{A}t+\frac{\hat{A}^2}{2!}t^2+\cdots\cdots$

$\hat{A}^2=0$

$\Downarrow$

$\vec{x}(t)=\begin{bmatrix}1 & 0\\ t & 1\end{bmatrix}\vec{x}(0)$

$\vec{x}_1(t)=\vec{x}_1(0)$

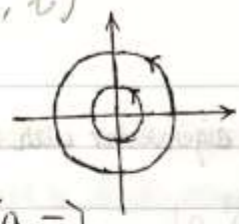
$\vec{x}_2(t)=t\vec{x}_1(0)+\vec{x}_2(0)$

$\lambda^2+\omega^2=0$ $\quad -i\omega \quad \omega$

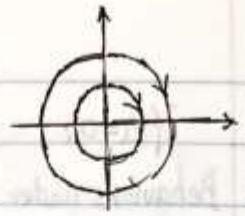
$\lambda_1 = i\omega$ $\quad (1, i)$

$\lambda_2 = -i\omega$.

$\hat{A}=\begin{bmatrix}0 & \omega\\ -\omega & 0\end{bmatrix}$ $\lambda=\pm i\omega$



$\hat{A}'=-\hat{A}$



Generally,

If $A=\begin{bmatrix}a & +\\ - & d\end{bmatrix}$ or $A=\begin{bmatrix}a & -\\ + & d\end{bmatrix}$ get rotation if $|a-d|$ is not too large.

$\frac{\partial \frac{\partial x_1}{\partial t}}{\partial x_2}<0$ $\quad \frac{\partial \frac{\partial x_2}{\partial t}}{\partial x_1}<0$ $\quad 4bc<-(a-d)^2$

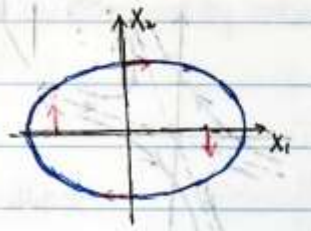
x_1 activates x_2

$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\hat{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$?

$\hat{A}=\begin{bmatrix}0 & +|b|\\ -|c| & 0\end{bmatrix}$

$\frac{\partial \frac{dx_1}{dt}}{\partial x_2}<0$.

$|b|>|c|$. draw $\frac{d\vec{x}}{dt}$ at strategic places.



at $\begin{bmatrix}1\\0\end{bmatrix}$. $\frac{d\vec{x}}{dt}=\begin{bmatrix}0\\|c|\end{bmatrix}$

at $\begin{bmatrix}0\\1\end{bmatrix}$. $\frac{d\vec{x}}{dt}=\begin{bmatrix}|b|\\0\end{bmatrix}$

$\hat{A}=\begin{bmatrix}a & |b|\\ -|c| & -a\end{bmatrix}$ $\quad J=0$ $\quad \lambda=\pm i\sqrt{|bc|-a^2}$ $\quad \lambda$ is imaginary if $|bc|>a^2$

$\pm a$ rotates the ellipse

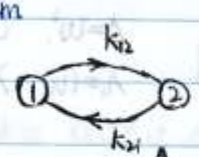
$\hat{A}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a\\-|c|\end{bmatrix}$

$\hat{A}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-|b|\\-a\end{bmatrix}$

Example in common physics: 2-D system

spin up vs. spin down in solid

inactive vs. active in biophysics



$\begin{cases}\frac{dx_1}{dt}=-k_{21}x_1+k_{12}x_2\\ \frac{dx_2}{dt}=k_{21}x_1-k_{12}x_2\end{cases}$ $\quad \frac{d\vec{x}}{dt}=\begin{bmatrix}-k_{21} & k_{12}\\ k_{21} & -k_{12}\end{bmatrix}\vec{x}$

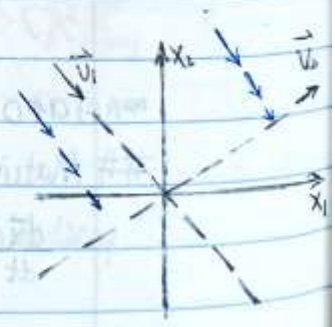
$\Delta=0$ fixed line

Do Initial Value Problems. with $\vec{x}(0)=(1,0)$

what if $k_{12}\to 0$ $\quad \frac{dx_1}{dt}=-k_{21}x_1$ $\quad t_{21}\propto\frac{1}{k_{21}}$. finite.

go on calculation $\quad \lambda_1=-(k_{12}+k_{21})$ $\quad \lambda_2=0$

$\lambda_1\neq 0$. any output of $\hat{A}\propto\begin{bmatrix}1\\-1\end{bmatrix}$. for vs. $A\begin{bmatrix}x_1\\x_2\end{bmatrix}=0$ $\quad \vec{v}_2=\begin{bmatrix}k_{12}\\k_{21}\end{bmatrix}$



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continued

$\vec{x}(0)=\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{k_{12}+k_{21}}(k_{21}\vec{v}_1+\vec{v}_2)$

$\vec{x}(t)=\frac{1}{k_{12}+k_{21}}(k_{21}\vec{v}_1e^{-(k_{12}+k_{21})t}+\vec{v}_2)$

Observations:

1). relaxation rate as expected $k_{12}+k_{21}$

2) $\vec{x}(\infty)=\left(\frac{k_{12}}{k_{12}+k_{21}}, \frac{k_{21}}{k_{12}+k_{21}}\right)$

Many possible long-term behavior → There is a conservation law.

Fixed Point Flavors.

* General stable to perturbation of problem parameters: nodes, spirals, saddle points.
* Unstable to perturbation parameter: center $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, star $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Non-Linear Systems

* Multiple stable isolated fixed points. (switch, magnetic memory in computer) spin
* Stable attracting orbit (limit cycle)

Solve time dependence

Generally we don't. :-)

Do picture analysis instead (given starting point, what is long time behavior)

* Basic approach: find fixed points, nullclines, nature of fixed points

Nullcline

line where $\frac{dx}{dt}=0$ or $\frac{dy}{dt}=0$

$\frac{dx}{dt}=f(x,y)$. $\frac{dy}{dt}=g(x,y)$

FPs are at intersections of nullclines

Stability of FPs

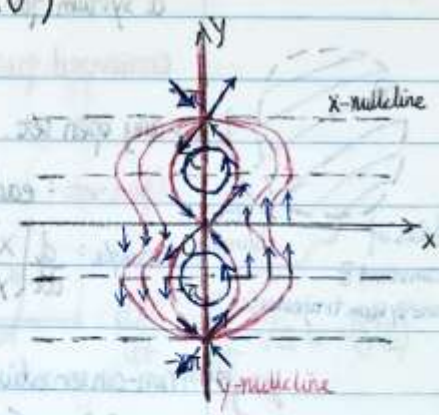
$FP=\begin{bmatrix} x^* \\ y^* \end{bmatrix}$ $u=x-x^*$, $v=y-y^*$ small distance near FP.

$$f(x,y)=f(x^*,y^*)+\frac{\partial f}{\partial x}\Big|_{x^*,y^*}(x-x^*)+\frac{\partial f}{\partial y}\Big|_{x^*,y^*}(y-y^*)+o(u^2,uv,v^2)$$

$$g(x,y)=g(x^*,y^*)+\frac{\partial g}{\partial x}\Big|_{x^*,y^*}(x-x^*)+\frac{\partial g}{\partial y}\Big|_{x^*,y^*}(y-y^*)+o(u^2,uv,v^2)$$

$$\begin{bmatrix} \dot u \\ \dot v \end{bmatrix}=\begin{bmatrix} \dot x \\ \dot y \end{bmatrix}=\begin{bmatrix} f \\ g \end{bmatrix}=\underbrace{\begin{bmatrix} \partial f/\partial x & \frac{\partial f}{\partial y} \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}_{x^*,y^*}}_{\text{Jacobian}}\begin{bmatrix} u \\ v \end{bmatrix}=\hat A\begin{bmatrix} u \\ v \end{bmatrix}$$

Same as section before



Application

$$\begin{cases} \dot x=\sin y \\ \dot y=x \end{cases}$$

two qualitatively different types of orbits. (next page)

nullclines: x-nullcline: $y=n\pi$, $n=0,\pm1,\dots$ y-nullcline: $x=0$.

Jacobian $\hat A=\begin{bmatrix} 0 & \cos y \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & (-1)^n \\ 1 & 0 \end{bmatrix}$ n is even: saddle point $\lambda_1=1, \vec v_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2=-1, \vec v_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

n is odd: center

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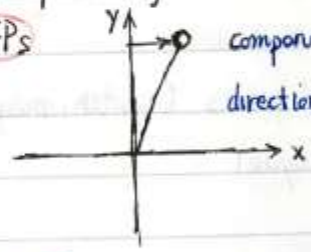
How to draw a phase portrait?

* find all the nullclines and fixed points.
* draw sketch $\begin{bmatrix} f \\ 0 \end{bmatrix}$ and $\begin{bmatrix} g \\ 0 \end{bmatrix}$ along nullclines.
* calculate types of fixed points by linearization

  sketch $[f, g]$ around FPs



components of gravity in direction of y is $\sin y$.

* Interpolate

eg: $\dot{x} = \sin y$, $\dot{y} = x$

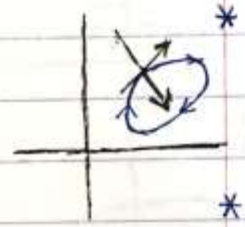
motions:

1. limited motions in phase portrait pendulum going back and forth around energy minimum
2. extended motions are clockwise or counterclockwise rotations of pendulum

Facts about Phase ~~Portraits~~ Trajectories

paths followed by x & y according to equations of motion

* Different paths never cross.



$$\frac{dx}{dt} = f(x_c, y_c)$$
$$\frac{dy}{dt} = g(x_c, y_c)$$

* A trajectory starting inside a closed orbit stays inside it.

definition: basin of attraction for an attractive fixed point: set of all points that flow into FP.

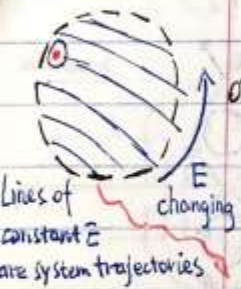
stable manifold of a saddle point: set of points that eventually flows into the saddle point.

Some Important Systems

Conservative system:

a system for which a conserved quantity exists.

conserved quantity is a quantity that is constant during all motions of the system but non-constant on any open set.



Open set: each point in an open set has a finite disk around it that is also contained in the set.

example: $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$x^2 + y^2 = \text{const}$.

non-conservative system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

All trajectories approaches origin

E on everywhere on trajectory $= E(0,0)$ $\quad E(x,y) = E(0,0)$ for $\forall x, y$

A conserved system cannot have an attractive fixed point or purely repulsive fixed points.

saddle points in conservative system? YES. and a center.

pendulum at the top is a saddle point

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$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \quad \text{who is } \omega \quad -\omega \quad \omega \quad y$$

Conservative Systems in 2D.

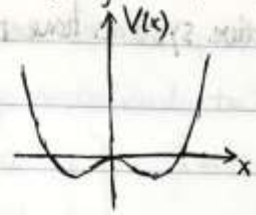
center — minimum of energy

saddle point — maximum of energy.

massive particle moving in a potential field

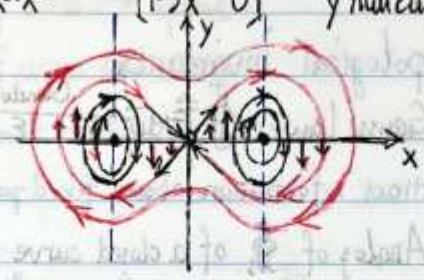
$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{1}{m}\frac{dV}{dx} \end{cases} \qquad E = \frac{1}{2}my^2 + V(x) \Rightarrow \frac{dE}{dt} = \frac{m}{2}2y\dot{y} + \dot{x}\frac{dV}{dx} = -y\frac{dV}{dx} + y\frac{dV}{dx} = 0$$

$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$



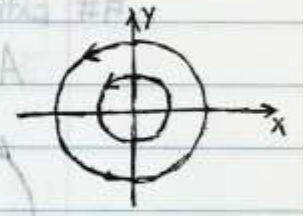
$\dot{x} = y$ $\quad A = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$ $\quad$ x nullcline: $y=0$

$\dot{y} = x - x^3$ $\quad$ y nullcline: $x=0$ or $x=\pm 1$



at $(0,0)$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\lambda = \pm 1$. saddle points

at $(0,\pm 1)$ $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $\lambda = \pm i\sqrt{2}$ eliptical center



find a NL system with $\lambda = \pm i\omega$ at $(0,0)$ that does not have closed orbits?

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\vec{r} + \alpha r^2 \vec{r}$$

doesn't contribute to Jacobian at the origin

square is smooth at 2nd derivative

* If a conservative system has a linear center, the center is a "true" center (has closed orbits).
* Center point of zero energy gradient, minimum of energy.

set of points where $E = E^* + \Delta E$

systems move along constant E paths.

Opposite Extreme: "Gradient" System

$\frac{d\vec{r}}{dt} = -\vec{\nabla}E$ $\quad$ for conservative system, $\frac{d\vec{r}}{dt} \perp \vec{\nabla}E$

for gradient system, no closed orbits.

$$\frac{dE}{dt} = \frac{d\vec{r}}{dt}\cdot\vec{\nabla}E = -(\vec{\nabla}E)^2$$

damped pendulum is not conservative

$$\ddot{\theta} = -b\dot{\theta} - \sin\theta \qquad \begin{cases} v = \dot{\theta} \\ \dot{v} = -bv - \sin\theta \end{cases} \qquad (\theta^*, v^*) = (n\pi, 0) \qquad \hat{A} = \begin{bmatrix} 0 & 1 \\ -\cos\theta & -b \end{bmatrix}$$

1). n is even, $\cos\theta = 1$.

$T = -b$. $\Delta = 1 + 1$ $\quad \lambda = \frac{-b \pm \sqrt{b^2-4}}{2}$

stable spiral if $b<2$

stable node if $b>2$

n is odd, $\lambda = \frac{-b \pm \sqrt{b^2+4}}{2}$ saddle point

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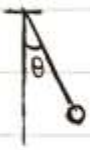
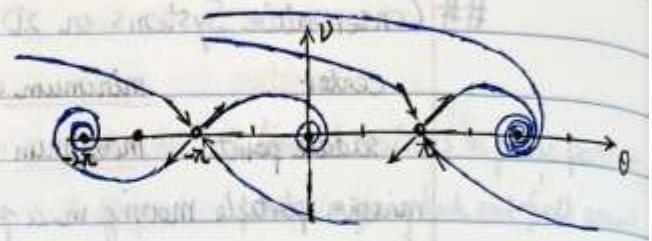


Systems in between: Damped Pendulum.

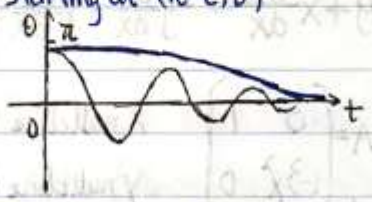
$$\begin{cases} \dot{\theta} = \nu \\ \dot{\nu} = b\nu - \sin\theta \end{cases} \qquad (\theta^*, \nu^*) = (n\pi, 0)$$

n even ...

n odd ...



Starting at $(\pi - \varepsilon, 0)$



Proof: No conservative quantities.

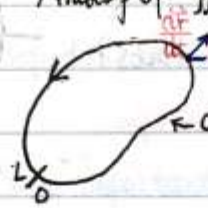
Conservative systems have no attractive fixed points.

Topological Invariants

Gauss' Law: $\oint\!\!\oint \vec{E} \cdot d\vec{\sigma} = \frac{Q_{enclosed}}{\varepsilon_0}$

Extract information about fixed points from behaviors far from FP

Analog of $\oint\!\!\oint_S$ of a closed curve. $S \to$ curve. $\vec{E} \to \frac{d\vec{r}}{dt}$.



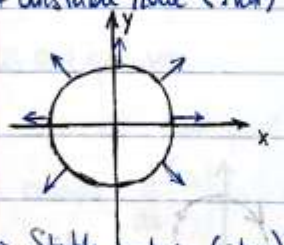
$$\text{Index} \equiv \frac{\text{Chang in } \varphi}{2\pi} = \frac{1}{2\pi}\int_0^L \frac{d\varphi}{ds}\, ds$$

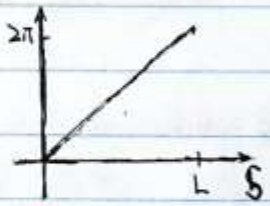
$\frac{d\vec{r}}{dt}$: hand of comoving clock.

I_c = # ccw turns of clock upon traversing curve ccw

* If a curve contains FP. I_c is determined by nature of FP.

Unstable node (star)



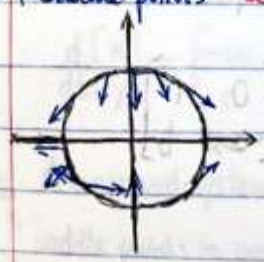


$$\Delta\varphi = \frac{2\pi \Delta s}{2\pi r} = \frac{\Delta s}{r} \qquad \frac{d\varphi}{ds} = \frac{1}{r}$$

$$I_c = \frac{1}{2\pi}\int_0^L \frac{ds}{r} = \frac{L}{2\pi r} = 1$$

Stable nodes. (star). $I_c = 1$.

Saddle points $I_c = -1$



No FPs. $I_c = 0$

Centers.

Unstable Spirals:

$$\hat{A} = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \qquad \lambda = \pm ia$$

I_c has to be a constant of a.

$\gamma = \alpha + i\omega$ $\quad e^{\alpha t}(\cos\omega t + i\sin\omega t)\vec{v}^1$

$\gamma = \alpha - i\omega$ $\quad e^{\alpha t}(\cos\omega t - i\sin\omega t)\vec{v}^2$

I_c的值	FP类型
+1	isolated stable or unstable node. center. stable and unstable spirals
−1	saddle point

第拾叁講 2017.9.27. Wed

MID TERM exam. October 11 2pm-3pm. Basically Chapter 7. "Open Brain"

Proof: spiral can go continiously to center

center: $A = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$ $\quad \lambda = a \pm i$. If $a > 0 \Rightarrow$ spiral

If $a = 0 \Rightarrow$ center

Index I_c can't change continuously, so it doesn't change at all.

Index of several FPs.

$I_c = I_c^1 + I_c^2$.

proof: I_c does not change when curve is deformed continuously.

$I_c = I_c^1 + I_c^2 + C_{12}$.

$C_{12} = \int_{top}^{bottom} \frac{d\varphi}{ds} ds + \int_{bottom}^{top} \frac{d\varphi}{ds} ds = 0$, $\frac{d\varphi}{ds}(\text{right}) = \frac{d\varphi}{ds}(\text{left})(-1)$

Start with $I_c = 0$, no FPs.

Parameters in equations changes. FP appears.

$I_c = \pm 1$ and $I_c = -1$ FPs appear in pairs. saddle node bifurcation

Closed Orbit Always has $I_c = 1$

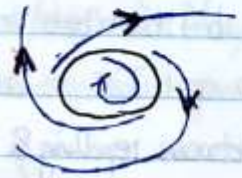
A closed orbit must contain a fixed point

§ Limit Cycles

are the isolated closed trajectories. $\Rightarrow$ Neighboring trajectories are not closed.

stable

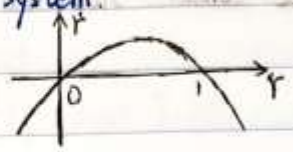
unstable

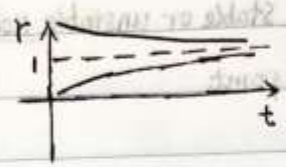


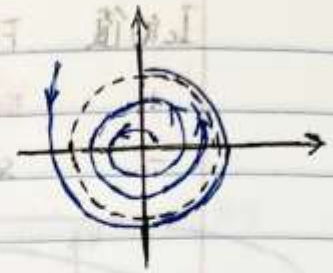
Conservative systems: no LC $\quad$ Gradient systems: no LC either

Simple limit cycle system

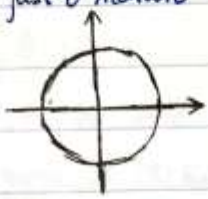
$$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{cases}$$







just θ motion



just r motion

conservative system gradient system

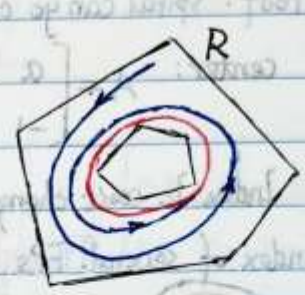
Poincaré-Bendixon Theorem: existence of limit cycles.

If (i). R is closed & bounded subset of the plane

(ii) $\frac{d\vec{r}}{dt} = f(r)$ is continuously differentiable

(iii) R contains no FPs

(iiii) $\vec{J}$ is a trajectory confined to R



Then: C is a closed orbit or approaches one at large times.

→ limit cycle.

第拾肆講 2017.9.29 Fri

Glucolutic oscillator

Fructose-6-Phorsphate ⇔ Fructose-1,6-phorsphate

+ATP +ADP

y x

$$\begin{cases} \dot{x} = -x + ay + x^2y \\ \dot{y} = b - ay - x^2y \end{cases}$$

FP: $\dot{x} + \dot{y} = b - x = 0 \Rightarrow x^* = b$

$0 = b - ay^* - b^2y^* \Rightarrow y^* = \frac{b}{a+b^2}$

Jacobian $\begin{bmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{bmatrix} = \begin{bmatrix} -1+\frac{2b^2}{a+b^2} & a+b^2 \\ \frac{-2b^2}{a+b^2} & -a-b^2 \end{bmatrix}$

positive feedback

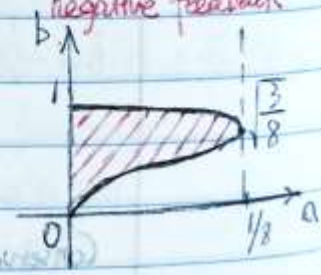
negative feedback

$\Delta = a+b^2-2b^2+2b^2 = a+b^2 > 0$

If $\tau > 0$ and $\tau < -1-a-b^2+\frac{2b^2}{a+b^2}$ FP will be repeller.

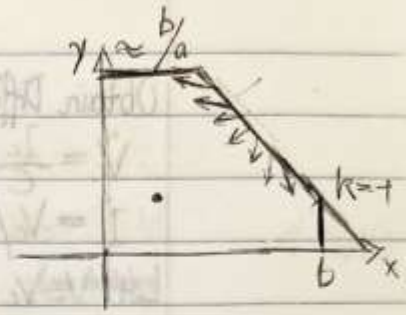
Crossover between repelling & not repelling

$J=0$ $b^2 = \frac{1}{2}(1-2a \pm \sqrt{1-8a})$ ends when $a = \frac{1}{8}$



Which side of the line is repelling?

$\frac{dI}{da} < 0$ reduces a increases I → repelling region



To prove the trapping region, look at the outside boundaries.

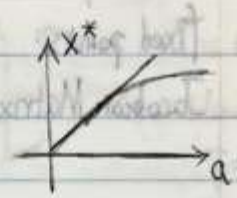
$x>0$. $y>0$ (look at $\dot{x}, \dot{y}$ at 0, they are both positive or 0).

$\dot{x}+\dot{y} = b-x<0$ if x is large $\Rightarrow \dot{x}+\dot{y} = \vec{r}\begin{pmatrix}1\\1\end{pmatrix}<0$

Reason of existence of chemical oscillator

could result from excess of negative feedback.

negative feedback could stablize ~~the~~ against purtabations.



$\dot{x} = a-x-by^2$

$\dot{y} = x-y$.

$x^* \propto \sqrt{a}$

Curl as a tool of thinking of oscillations

$C_z = [\vec{\nabla}\times(f,g,0)] = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = A_{21} - A_{12}$.

Ampere's Law. $\oint \vec{V}\cdot d\vec{s} = \int(\vec{\nabla}\times\vec{V})\cdot\hat{n}\,dA$

$\oint \frac{d\vec{r}}{dt}\cdot d\vec{s} = \int(A_{21}-A_{12})\cdot dA$ If loop is in x-y plane.

第拾伍計 2017.10.02. Mon.

MIDTERM. Weds. Oct 11. Through 7.5 (skip 7.4).

$(\nabla\times\vec{f})_z = G = A_{yx} - A_{xy}$.

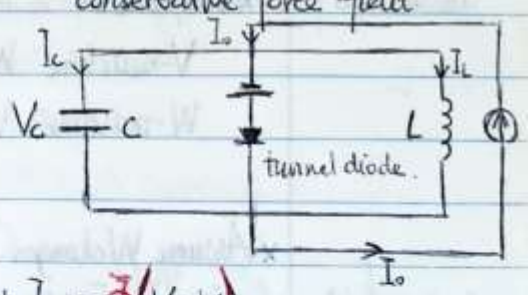
What if $G=0$ everywhere? $\frac{d\vec{r}}{dt}$ = gradient of another function ⇒ Gradient system

* What if $\nabla\times\vec{f}=0$ everywhere? conservative force field

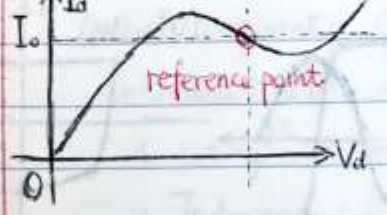
Electronic Realization of Limit Cycle.

Capacitor: $V_c = \frac{Q}{C}$ $I_c = \frac{dQ}{dt}$.

Inductor: $V_L = -L\frac{dI_L}{dt}$.



Need positive feedback for Limit Cycle and non-linearity



$I_d = \gamma(V_d-V_o)^3 - \alpha(V_d-V_o) + I_o = \Phi(V_d-V_o)$

At reference point, $dI_d/dV_d<0$, $R = -\frac{1}{\alpha} = -|R|$

LRC Circuit with normal resistence: Decaying oscillation

with negative resistence: growing oscillation

RC Circuit

$\frac{dQ}{dt} = I = \frac{-Q}{CR}$ $Q(t) = Q(0)e^{-\frac{t}{RC}} = Q(0)e^{+\frac{t}{|R|C}}$

Obtain Differentiation Equations from Kirchhoff Laws.

$$\dot{V}_C = \frac{I_C}{C} = -\frac{1}{C}\{[\Phi(V_0-V_0)+I_0]-I_0+I_L\} = \frac{-1}{C}(\Phi(V_C)+I_L)$$

$$\dot{I}_L = V_C/L$$

Let $V=V_C$, $W=I_L$.

$$\begin{cases} \dot{V} = -\frac{1}{C}(\Phi(V)+I_L) \\ \dot{W} = V/L \end{cases}$$

fixed points: $V=0$. $\Phi(V)=0$. $W=0$. Only one fixed point at $(0,0)$

Jacobian Matrix: $\hat{A} = \begin{bmatrix} \frac{\alpha}{C} & -\frac{1}{C} \\ 1/L & 0 \end{bmatrix}$ $\quad \tau = \frac{\alpha}{C}$, $\Delta = 1/LC$ $\quad \lambda_\pm = \frac{\frac{\alpha}{C} \pm \sqrt{\left(\frac{\alpha}{C}\right)^2 - \frac{4}{LC}}}{2}$

Discussion $\alpha=0$.

if α is small. $\lambda \approx \frac{\alpha}{2C} \pm i\sqrt{\frac{1}{LC}}$ unstable spiral

if α is large. $\lambda_\pm > 0$. unstable node.

Index of a spiral is ± 1. It can be surrounded by a closed trajectory

第拾陸講 2017.10.04. Wed.

At Compton 241. From 2:00 to 3:05.

Calculate $V(t)$ & $W(t)$ in Limit Cycle.

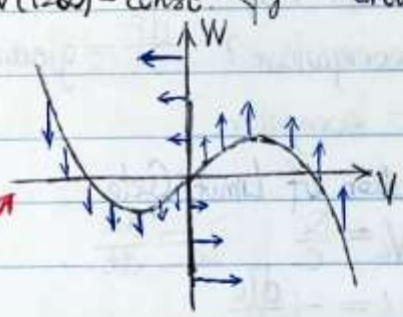
* What if $L=\infty$? $W(t)=0$ $\quad V(t=\infty) = \text{const} = \sqrt{\frac{\alpha}{\gamma}}$ around origin $\lambda = 0, \frac{\alpha}{C}$

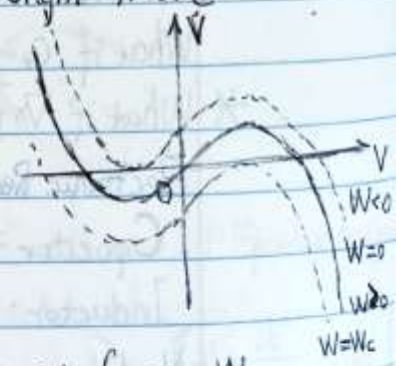
Larg L Limit * What if L large but not ∞?

× Nullclines:

V-nullcline: $W = \alpha V - \gamma V^3$

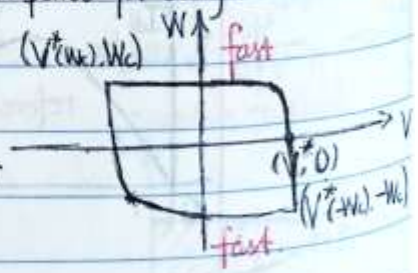
W-nullcline: $V=0$





× Assume W changes slow enough that V has time to go to its fixed point for every W.

$$\dot{V} = -\frac{1}{C}[\gamma V^3 - \alpha V + W], \quad \dot{W} = \frac{V}{L}$$



Conclusion

(1) $W_{max} = W_c$ occurs at blue-sky bifurcation

$W_{max} = \max(\alpha V - \gamma V^3)$ when $V = \sqrt{\frac{\alpha}{3\gamma}}$

$= \frac{2}{3}\sqrt{\frac{\alpha^3}{3\gamma}}$

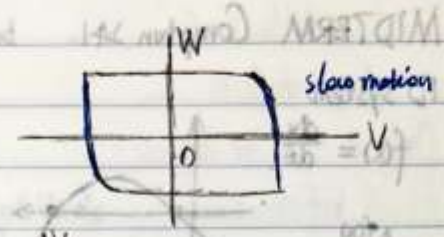
(2) $V_{max} = 2\sqrt{\frac{\alpha}{3\gamma}} = V^*(W_c)$

(3) $T = \frac{2 \cdot 2W_{max}}{|\dot{W}|}$

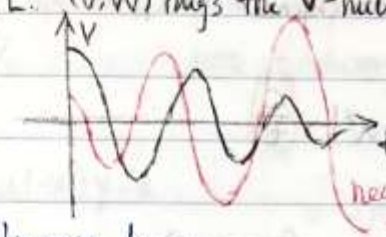
第拾柒講 2017.10.06 Fri

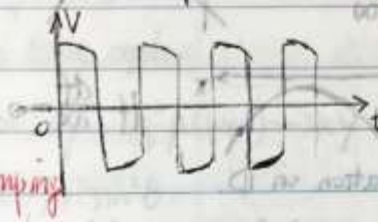
$\dot{V} = \frac{1}{C}[\alpha V - \gamma V^3 - W]$

$\dot{W} = V/L$



For larg L. (V,W) hugs the V-nullcline





T is dominated by slow motion

$T \approx \frac{8\alpha L}{3\sqrt{3}}$.

Weak Non-Linearity Limit

$\dot{V} = \frac{1}{C}(\alpha V - \gamma V^3 - W) = -\frac{1}{C}(\Phi(V) + W)$

$\dot{W} = V/L$

$\ddot{V} = -\frac{1}{C}[\dot{V}\Phi'(V) + \dot{W}]$

$\ddot{V} + \frac{3\gamma V^2 - \alpha}{C}\dot{V} + \frac{V}{LC} = 0$

Rescale in time & V

$\ddot{V} + \frac{V}{LC} = 0 \Rightarrow$ time scale $\sqrt{LC}$. $\therefore t = \sqrt{LC}\,\tau \rightarrow$ dimensionless time

$V = \sqrt{\frac{\alpha}{3\gamma}}\,x \rightarrow x^2$ term

$\ddot{x} + \alpha\sqrt{\frac{L}{C}}(x^2-1)\dot{x} + x = 0$ $\quad \mu = \alpha\sqrt{\frac{L}{C}}$

weak non-linearity : μ is small.

(1) $\mu = 0$. $\ddot{x} + x = 0$. $\omega = 1$. $x(t) = A\cos t$

(2) Determining ω & A by behaviors at fundamental frequency $\omega = 1$.

Guess $x(t) = A\cos\omega t$. determine A & ω by forcing sin & cos components of dynamical equation to vanish.

$-\omega^2 A\cos(\omega t) + \mu[A^2\cos^2(\omega t) - 1][-A\omega\sin\omega t] + A\cos(\omega t) \approx 0$

Components of LHS: $\cos\omega t$, $\sin\omega t$, $\sin 3\omega t$

$\cos(\omega t)$ component: $\cos(\omega t)[-\omega^2 A + A] = 0$. $\omega = \pm 1$.

$\sin(\omega t)$ component: $\cos^2(\omega t)\sin(\omega t) = a\sin(\omega t) + b\sin(3\omega t) \Rightarrow \left[\frac{\mu A^2}{4} - (A\omega) + \mu A\omega\right]\sin(\omega t)$

Determine a&b:

1. $\omega t = \frac{\pi}{2}$. $a - b = 0$. $a = b$.

2. Taylor expansion to ωt. $\omega t = a\omega t + 3b\omega t$. $a = b = \frac{1}{4}$.

$\left(-\frac{A^2}{4} + 1\right) = 0$ $\quad A = 2$

"Degenerate eigenvector."

第拾捌講 MIDTERM REVIEW

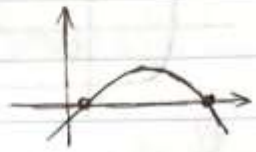
2017.10.09 Mon.

ONE. D

MIDTERM Compton 241. bring pencil/pen - 2:00 - 3:00 pm throuing 7.5 skipping 7.4.

1D system

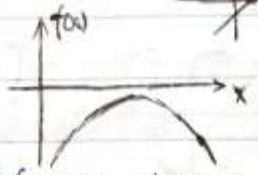
* $f(x) = \frac{dx}{dt}$.



stable FP: $\frac{df}{dx} < 0$

rate of convergence: $\left|\frac{df}{dx}\right|$.

*



If $\frac{df}{dx} = 0$. $|x(t) - x^*| \sim \left(\frac{c}{t}\right)$

Bifurcation in 1D.

* Types of bifurcation: Blue-sky/saddle node $f(x) = r + x^2$

  Transcritical $f(x) = x(r - x)$

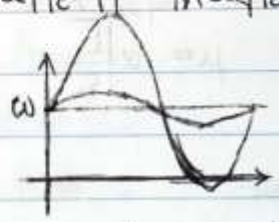
  Pitchfork supercritical: $f(x) = rx - x^3$

  subcritical: $f(x) = rx + x^3$

* Bifurcation diagram

! * $f(M) = rM - M^3$ $r \propto |T_c - T|$ $M \propto \sqrt{T_c - T}$ Above T_c, $M(T) = M(0)e^{-\text{const}(T - T_c)}$

slow fluctuations near T_c



Flows on a Circle.

* $f(\theta) = \omega - a\sin\theta$

必考! * close to bifurcation, a≈ω. the system becomes very slow. $T_{long} = \frac{1}{\sqrt{\omega - a}}$

* Firefly: simple change of variables.

$\dot{\theta} = \omega + A\sin[\alpha(t) - \theta(t)]$ $\phi = \alpha(t) - \theta(t)$ $\dot{\phi} = (\Omega - \omega) - A\sin\phi$ Fixed point: entrainment

TWO. D.

* Jacobian $A_{ij} = \partial \dot{x}_i / \partial x_j$ trace τ & determinant Δ Will be on Blackboard.

* Types of behaviors: based on discussion of τ & Δ 关键是变数时如何转向!

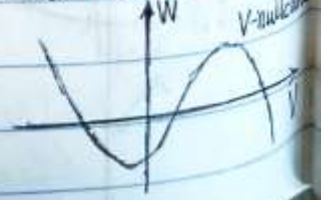
禁! * Sketch phase portrait !!!

* Conservative vs. Gradient System No attractive fixed points in Cons. No closed orbit in Grad.

? Index $I = \frac{1}{2\pi}\oint_0^L \frac{d\varphi}{ds} ds$ center node spiral → +1. saddle → -1. no FP → 0 closed orbit → 1

? Poincare Bendikson Theorem stable limit cycle.

* Von-der Pol oscillator: limit where $\dot{W}$ is small.



transition: "at top"? $T_{slow} \sim L$

第十九讲 2017.10.13. Fri

$\ddot{x}+\mu(x^2-1)\dot{x}+x=0.$

Negative damping. exponential growth.

Small μ:

1. Fourier expansion: fixed frequency & amplitude
2. $\left\langle \frac{dr^2}{dt} \right\rangle_\theta$: rate of convergence to limit cycle.

$$\begin{cases} \dot{x}=y \\ \dot{y}=\mu(1-x^2)y-x \end{cases} \qquad \frac{dr^2}{dt}=2x\dot{x}+2y\dot{y}=2xy+2y^2\mu(1-x^2)-2xy=2\mu(1-x^2)y^2$$

$$=2\mu(1-r^2\cos^2\theta)r^2\sin^2\theta.$$

$$\frac{1}{2\mu}\frac{dr^2}{dt}=r^2(\sin^2\theta-r^2\cos^2\theta\sin^2\theta) \qquad \langle\sin^2\theta\rangle=\frac{1}{2}=\langle\cos^2\theta\rangle$$

$$\left\langle\frac{1+\cos2\theta}{2}\cdot\frac{1-\cos2\theta}{2}\right\rangle=\frac{1}{4}-\frac{\langle\cos^2 2\theta\rangle}{4}=\frac{1}{8}$$

$$\frac{1}{2\mu}\left\langle\frac{dr^2}{dt}\right\rangle_\theta=r^2\left(\frac{1}{2}-\frac{r^2}{8}\right) \Rightarrow \text{Limit Cycle when } (\cdot)=0. \Rightarrow r=2$$

$$\frac{2r\,dr}{2\mu dt}=\frac{r^2}{2}\left(1-\frac{r}{2}\right)\left(1+\frac{r}{2}\right)$$

$$\frac{dr}{dt}=\mu\frac{r}{2}(2-r)(2-r)\mu.$$ Rate of convergence to LC is μ

Lienard Theorem

Physically reasonable oscillators that have negative damping near origin have LC.

$$\ddot{x}+\underset{\text{damping}}{f(x)}x+\underset{\text{restoring force}}{g(x)}=0$$

minus

assumptions.

(1) $g(-x)=-g(x)$ and $g(x)>0$ for $x>0$.

(2) $f(-x)=f(x)$

If $F(x)=\int_0^x f(x')dx'$

(3) $F(x)$ has exactly one zero at $(x=a)$

$F(x)<0$ for $0<x<a$ — negative damping around origin

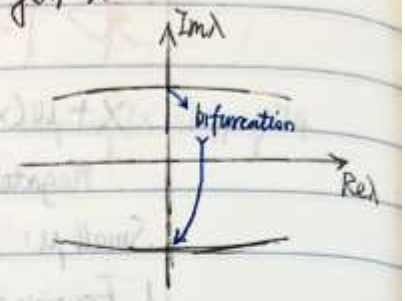
$F(x)>0$ for $x>a$.

$F(x)\to\infty$ as $x\to\infty$ — normal damping as $x\to\infty$

for von der Pol oscillators

$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$ $\quad f(x) = \mu(x^2-1)$. $\quad F(x) = \mu\left(\frac{x^3}{3} - x\right)$ $\quad g(x) = x$

§ Bifurcations in TWO-Dimensional Systems



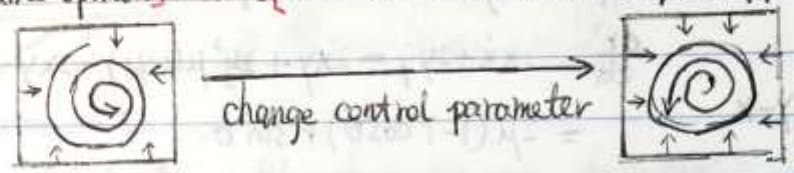
2d saddle node

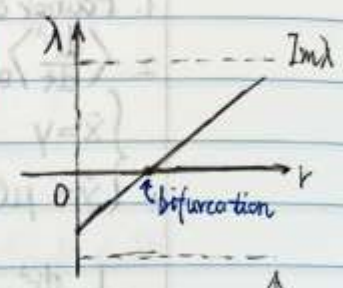
2d pitchfork.

Only in 2-D or higher:

Hopf bifurcation

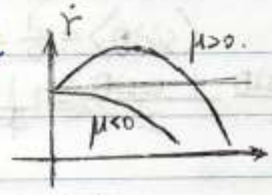
[stable spirals] become [unstable and a limit cycle appears]



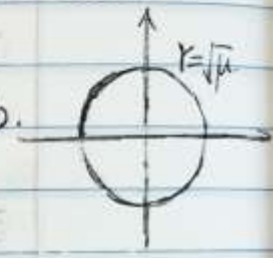


simplest hopf bif model

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega_0 + b r^2 \end{cases}$$



If $\mu > 0$. $r \to \sqrt{\mu}$ $\quad \dot{\theta} = \omega_0 + \mu b$.



Use $z = x + iy$ for normal form.

Radio motion: $\frac{dz}{dt} = \text{realnumber} \cdot z$

Rotation motion: $\frac{dz}{dt} = \text{imaginery \#} \cdot z$

Normal form: $\frac{dz}{dt} = (\mu + i) z + (\alpha + i\beta) |z|^2 \cdot z$.

第貳拾講 2017.10.18. Wed

Standard form of Hopf bifurcation: $\frac{dz}{dt} = (\mu + i) z + (\alpha + i\beta) |z|^2 z$, $z = x + iy$

Discussion

If $\frac{dz}{dt} = \mu z$, $\frac{dx}{dt} = \mu x$ $\quad \frac{dy}{dt} = \mu y$ $\quad$ radiation $\quad$ 發行講出 z

If $\frac{dz}{dt} = iz$ $\quad \frac{dx}{dt} = -y$ $\quad \frac{dy}{dx} = x$. $\quad$ circular motion

Matrix form discussion

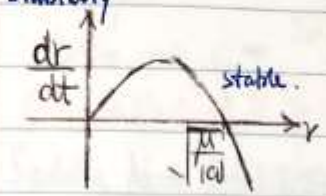
$$\frac{d\vec{r}}{dt} = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \vec{r} + \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} r^2 \vec{r} = \left((\mu + \alpha r^2) \mathbb{1} + \begin{bmatrix} 0 & -1-\beta r^2 \\ 1+\beta r^2 & 0 \end{bmatrix} \right) \vec{r}$$

radial gradient $\quad$ rotational conservative

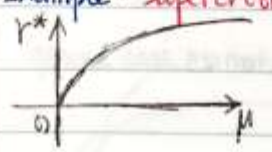
limit cycle.

If $\mu>0$ and $a<0$: unique limit cycle. radius $r=\sqrt{\frac{\mu}{|a|}}$

stability



Example: supercritical Hopf bifurcation:



radius of limit cycle varies continuously with control parameter.

Extended Van De Pol model: $\ddot{x}+\mu(x^2-1)\dot{x}+x-a=0$ Hopf bifurcation as a function of a.

Transform to 1st system

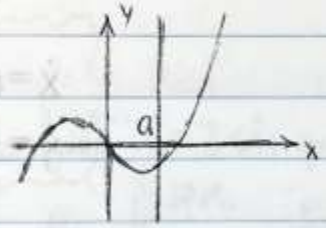
$\dot{y}=a-x \Rightarrow \ddot{x}=\mu(1-x^2)\dot{x}+\dot{y}$

$\dot{x}=-\mu\left(\frac{x^3}{3}-x\right)+y.$ $\xrightarrow[\tilde{y}=y/\mu]{\text{rescale } y}$ $\begin{cases}\dot{x}=-\mu\left(\frac{x^3}{3}-x\right)+\mu y=\mu\left[y-\left(\frac{x^3}{3}-x\right)\right]\\ \dot{y}=\frac{a-x}{\mu}\end{cases}$

Analysis

x-nullcline: $y=\frac{x^3}{3}-x$

y-nullcline: $x=a$



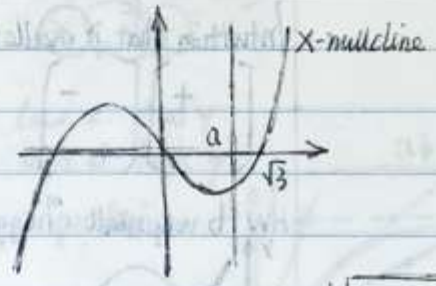
$x^*=a$

$y^*=\frac{a^3}{3}-a.$

第貳拾壹講 2017.10.20 Fri

$\ddot{x}+\mu(x^2-1)\dot{x}+x-a=0.$

$\begin{cases}\dot{x}=-\mu\left(\frac{x^3}{3}-x\right)+y\\ \dot{y}=a-x\end{cases}$ $\begin{cases}\dot{x}=\mu\left[y-\left(\frac{x^3}{3}-x\right)\right]\\ \dot{y}=\frac{a-x}{\mu}\end{cases}$



$A=\begin{bmatrix}1-a^2 & 1\\ -1 & 0\end{bmatrix}$ $\lambda=\frac{(1-a^2)\pm\sqrt{(1-a^2)^2-4}}{2}.$ if $|1-a^2|<2.$ $\lambda=\frac{(1-a^2)\pm i\sqrt{4-(1-a^2)^2}}{2}$

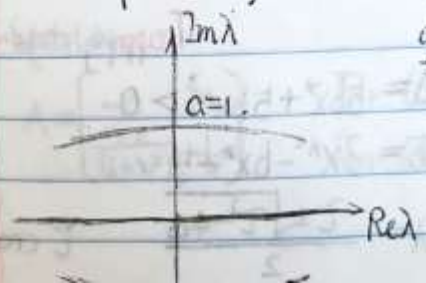
For $a>1$ $\mathrm{Re}\lambda<0.$ stable spiral

For $a<1$ $\mathrm{Re}\lambda>0.$ unstable spiral - limit cycle

Hopf Bifurcation at $a=1$.

Radius of limit cycle near bifurcation is $\propto\sqrt{|r-r_c|}$

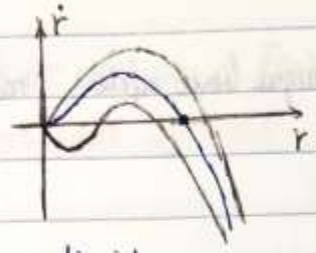
guess radius $\propto\sqrt{1-a}$ frequency almost unchanged.



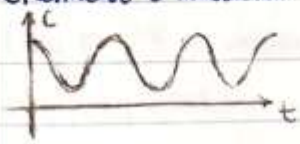
subcritical Hopf bifurcation

$\dot{r} = \mu r + r^3 - r^5$

$\dot{\theta} = \omega_0 + b r^2$



Chemical Oscillator : Chlorine dioxide



spontaneous oscillation

ClO_2 decompose drastically when separated from diluting substances.

MA malonic acid $CH_2(COOH)_2$

I_2 iodide

$MA + I_2 \rightarrow IMA + I^- + H^+$

$ClO_2 + I^- \rightarrow ClO_2^- + \frac{1}{2} I_2$

$ClO_2^- + 4I^- \rightarrow Cl^- + 2I_2 + 2H_2O$

$\frac{d[I_2]}{dt} \propto -[MA][I_2]$

MA I_2 ClO_2 ~~fast~~ slow

I^- ClO_2^- ~~slow~~ fast

$\downarrow$ $\downarrow$

x y

$$\boxed{\begin{aligned} \dot{x} &= a - x - \frac{4xy}{1+x^2} \quad (1) \\ \dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right) \quad (2) \end{aligned}}$$

第二十二講 2017.10.23. Mon.

Intuition that it oscillates

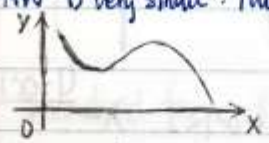
$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ $\frac{\partial \dot{x}}{\partial y} > 0, \frac{\partial \dot{y}}{\partial x} < 0.$ $\frac{\partial \dot{x}}{\partial y} < 0$ $\frac{\partial \dot{y}}{\partial x} > 0$

$b \to \infty$ long term behavior?

$y \to 1 + x^2$ very quickly. from (2)

$\dot{x} = a - x - 4x$. $x \to \frac{a}{5}$

HW: b very small : "hugging nullcline"



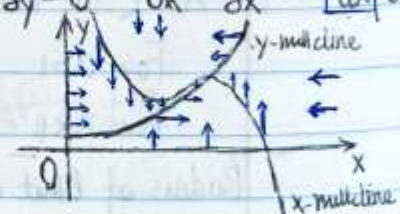
If $\frac{dy}{dx} = 0$, it follows that $\frac{d\dot{x}}{dx} = 0$.

$\dot{x} = f(x,y) = 0$. on nullcline $dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y} = 0$ $\frac{\partial \dot{x}}{\partial x} = \frac{\partial f}{\partial x} = -\boxed{\frac{dy}{dx}} \frac{\partial f}{\partial y} = 0$

Nullclines.

$\dot{x} \to y = \frac{(a-x)(1+x^2)}{4x} = \frac{a - x + ax^2 - x^3}{4x} = \frac{a}{4x} + \frac{ax}{4} - \frac{1}{4} - \frac{x^2}{4}$

$\dot{y} \to y = 1 + x^2$ or $x = 0$



Trapping region get ✓

Repeller or Not? $x^* = \frac{a}{5}$. $y^* = 1 + \frac{a^2}{25}$

$$\mathbb{J} = \begin{bmatrix} -1 - \frac{4y}{1+x^2} - \frac{8x^2y}{(1+x^2)^2}, & \\ & , \end{bmatrix}$$

$$= \begin{bmatrix} (3x^* - 5)/y^* & -4x^*/y^* \\ 2bx^*/y^* & -bx^*/y^* \end{bmatrix} = \hat{\mathbb{B}} \frac{1}{y^*}$$

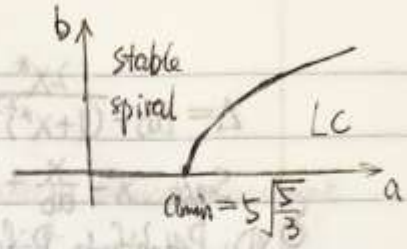
$\Delta = 5bx^* + 5bx^{*3} > 0$

$\tau = 3x^{*2} - bx^* - 5$

$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$ τ cross 0. Hopf bifurcation

几何

$\tau = 0 \rightarrow b = 3x^* - 5/x^* = \frac{3}{5}a - \frac{25}{a}$



Is $b_{out} > 0$?

$b = 0 \rightarrow a = 5\times\sqrt{\frac{5}{3}} \rightarrow$ need $a > 5\sqrt{\frac{5}{3}}$

老师画的图真单答孔连 也很清楚 甚至好于 StreamPlot

Saddle Node Bifurcation $\dot{x} = \mu - x^2$ $\dot{y} = -y$ $\mu < 0$ No FPs $\mu > 0$ stable node ×1, saddle point ×1

total $I_c = 0$ $\dot{y} = +y$ $\mu > 0$. unstable node ×1, saddle point ×1

general case:

example: Biochemical switch

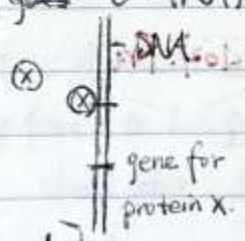
x = protein

y = gene for protein (active).

第二十三讲 2017.10.25 Wed.

For Hopf bifurcation ~~goes~~ happens $\tau = Tr(\hat{A})$ goes from <0 to >0. $\frac{\partial \dot{x}}{\partial x} > 0$ $\frac{\partial \dot{y}}{\partial y} > 0$.

#* $\begin{cases} \dot{x} = -ax + y \\ \dot{y} = \frac{x^2}{1+x^2} - by. \end{cases}$

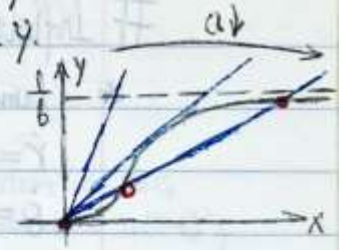


$A_{xy} = \frac{\partial \dot{x}}{\partial y} = 1 > 0$

$A_{yx} = \frac{\partial \dot{y}}{\partial x} > 0.$

$\hat{A} = \begin{bmatrix} & + \\ + & \end{bmatrix}$

2 solutions: low x low y, high x high y.



* Physical meaning of x^2 in $\dot{y}$: cooperative activation by 2 proteins.

##

x-nullcline: $y = ax$. y-nullcline: $y = \frac{x^2}{b(1+x^2)} < \frac{1}{b}$

Criterion for new fixed points: $ax = x^2/b(1+x^2) \rightarrow \frac{x^{\cancel{2}\,1}}{1+x^2} = ab$

$\frac{1}{2} > \frac{1}{1/x + x} \rightarrow ab \leq \frac{1}{2}$ bifurcation occurs.

有没有是啥样

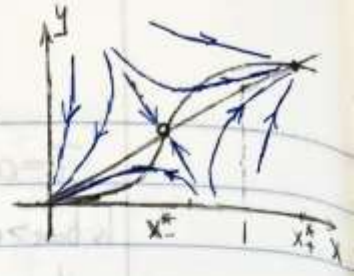
Stability

$A = \begin{bmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{bmatrix}$

For (0,0). $\lambda = -a$. $\lambda = -b$. always stable FP.

For others: $\lambda = \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4\Delta}}{2}$ $\Delta > 0$ stable node, $\Delta < 0$ saddle point

$\Delta = ab - \frac{2x^*}{(1+x^{*2})^2}$ $\quad ab\left(1-\frac{2}{1+x^{*2}}\right)$ $\quad$ If $x^*>1$ $\Delta>0$; If $x^*<1$ $\Delta<0$.

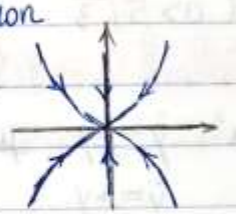


Since $x^2 - \frac{x}{ab} + 1 = 0 = (x-x^*_+)(x-x^*_-) \Longrightarrow x^*_+ x^*_- = 1$

§2D Pitchfork Bifurcation

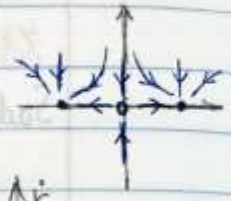
$$\begin{cases} \dot{x} = \mu x - x^3 \\ \dot{y} = -y \end{cases}$$

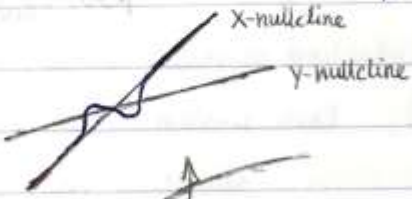
$\mu<0$



$\mu=0$

$\mu>0$

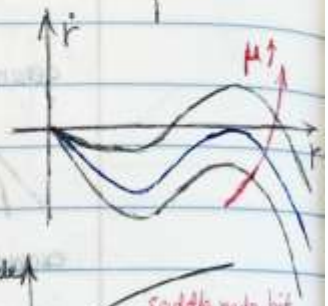


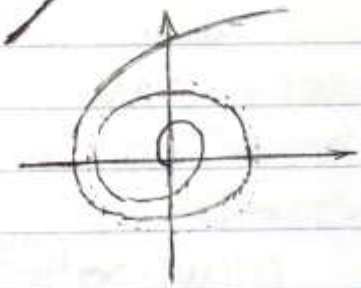


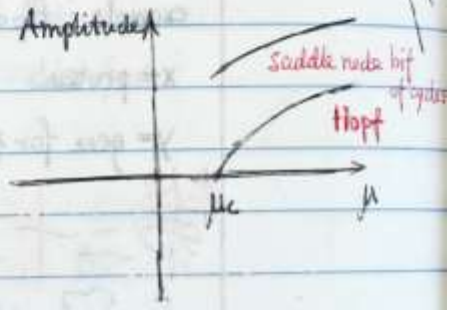
Saddle node bifurcation of cycles

$$\dot{r} = \mu r + r^3 - r^5$$

$$\dot{\theta} = \omega + br^2$$

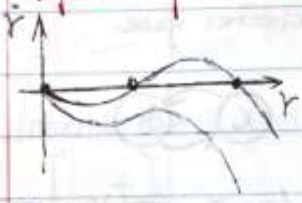






* Distinguish Hopf & Saddle node bifurcation of cycles

第二十四讲 2017.10.27. Fri

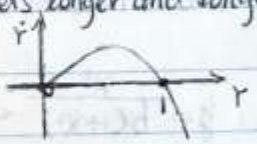


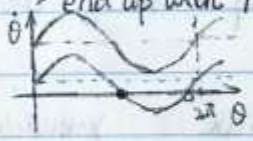
Infinite Period Bifurcation

* Limit Cycle period gets longer and longer $\rightarrow$ end up with FPs.

$\dot{r} = r(1-r^2)$

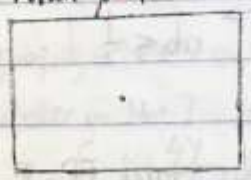
$\dot{\theta} = \mu - \sin\theta$





* When $\mu \gg 1$ LC has a fairly constant V.

When $\mu \gg 1$.

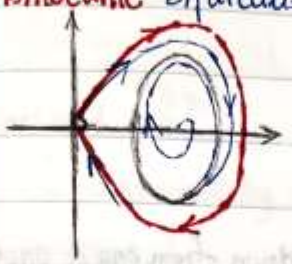


When $\mu < 1$.

What happens to **amplitude & period** going through bifurcation?

Amplitude of LC is constant. $\quad$ Period is $T \propto \sqrt{|\mu-\mu_c|}$

Homoclinic Bifurcation



$\mu<\mu_c$

$T=\infty$

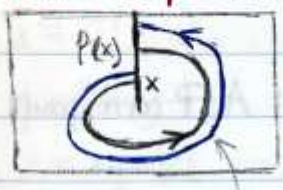
Through bifurcation amplitude is basically constant

period $T \propto \ln|\mu-\mu_c|$

	Amplitude	Period
Supercritical Hopf.	$\mu^{\frac{1}{2}}$	$O(1)$
Saddle Node.	$O(1)$	$O(1)$
Infinite Period	$O(1)$	$\mu^{-\frac{1}{2}}$
Homoclinic	$O(1)$	$\lvert\ln\mu\rvert$

Reduce 2D with rotation to 1D **Iterated Map**

"Poincare map" "surface of section" surface traverse to flow (never parallel)



Line in 2d

"surface of section" S

map from S to S: P

傳与原像重合相同. 由此得下面性质

closed orbit: fixed point of P: $P(x^*)=x^*$

$\frac{dP}{dx}>0$: trajectories can not intersect.

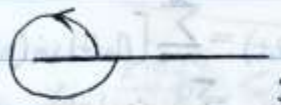
stability of limit cycle: stable $\frac{dP}{dx}<1$. unstable $\frac{dP}{dx}>1$.

for a conservative system: $\frac{dP}{dx}=1$.

concrete example.

$\frac{dr}{dt}=r(1-r^2)$ stable at $r=1$

$\frac{d\theta}{dt}=2\pi$



$S(\theta=0)$

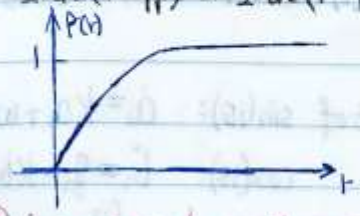
start at $r=r_0$

calculate $r_1(T=1)$

$$T=1=\int_{r_0}^{r_1}\frac{dt}{dr}dr=\int_{r_0}^{r_1}\frac{dr}{r(1-r^2)}=-\frac{1}{2}\ln\left(1-\frac{1}{r_1^2}\right)+\frac{1}{2}\ln\left(1-\frac{1}{r_0^2}\right)$$

$$r_1=\sqrt{\frac{1}{1-e^{-2}\left(1-\frac{1}{r_0^2}\right)}}$$

$$P(r)=r\cdot\sqrt{\frac{1}{1^2-e^{-2}(r^2-1)}}$$



limits $r\ll 1$ $P(r)\to er$. unstable

$r\to\infty$ $P(r)\to\frac{1}{\sqrt{1-e^2}}$

Poincare map shows there is no chaos in 2d systems

Since $\frac{dP}{dr}>0$. $\Rightarrow$ r changes monotonically.

第二十五讲 2017.10.30. Mon.

§ 3D DYNAMICAL SYSTEM.

chaotic behavior happens when trajectories jumps up & down from one & another attractors/repellers

例: Navi-Stokes Equation

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}p - \rho(\vec{v}\cdot\vec{\nabla})\vec{v} + \mu\nabla^2\vec{v}$$

density non linear convection viscosity

Dimensionless ratio of convection term to stablizing viscosity term. $\frac{\rho v L}{\mu}$ = Reynolds number

Deterministic but Unpredictable Behavior

"Deterministic Nonperiodic Flow" E. Lorenz.

convert PDE to 3 ODEs

In k-space: $\rho\dot{v}_k = \ldots\ldots - \rho\sum_q V_{k-q}(iq)V_q$ "stream function"

convolution theorem Lorenz's variables were Fourier components of

$$\begin{cases}\dot{x} = \sigma(y-x)\\ \dot{y} = rx - y - xz\\ \dot{z} = xy - bz\end{cases}$$

xy convective term.

$$J = \begin{bmatrix}-\sigma, & \sigma, & 0\\ r-z, & -1, & -x\\ y, & x, & -b\end{bmatrix}$$

A FP corresponds to steady flow.

Water wheel:

ω rotation rate.

k leakage rate parameter.

$\frac{\partial}{\partial t}$ $\frac{d}{dt}$ describes different pictures

PDE description

$$\frac{\partial m(\theta)}{\partial t} = Q(\theta) - Km(\theta) - \omega\frac{\partial m(\theta)}{\partial\theta}$$

Non linear term.

$$I\frac{d\omega(\theta)}{dt} = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta$$

Equations will not mix modes.

Fourier series: $m(\theta,t) = \sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$

$$Q(\theta) = \sum_n q_n\cos(n\theta)$$

$$* \sum_n[\dot{a}_n(t)\sin(n\theta) + \dot{b}_n(t)\cos(n\theta)] = \sum_n q_n\cos(n\theta) - K\sum_n[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$
$$- \omega\sum_n[na_n(t)\cos(nt) - nb_n(t)\sin(n\theta)]$$

Coefficient of $\sin(1\theta)$: $\dot{a}_1 = -Ka_1 + \omega b_1$ 为啥不看高阶项 没听清 ← P37.

$\cos(1\theta)$: $\dot{b}_1 = q_1 - Kb_1 + \omega a_1$ decoupled.

$$* \; I\frac{d\omega}{dt} = -\nu\omega + gra_1\int_0^{2\pi}\sin^2\theta\, d\theta \qquad \dot{\omega} = (-\nu\omega + \pi gra_1)/I$$

第二十六讲 2017.11.1 Wed.

continued... $\dot{I}$?

I determined by mass M_{total}. $\frac{dM_{tot}}{dt} = Q_{tot} - kM_{tot}$. $M_{tot} \to \frac{Q_{tot}}{k}$ exponentially with $\tau \sim \frac{1}{k}$

Fixed points!

$\omega = 0$ $a_1 = 0$. $b_1 = \frac{q_1}{k}$ symmetric distribution related to inflow q_1

k, ν will stablize while Q will distablize the system.

$\omega \neq 0$ put a_1 from $\dot{a}_1$ into b_1 equation which means $\dot{a}_1 = 0$.

$0 = q_1 - kb_1 - \frac{\omega^2}{k} b_1$ from $\dot{\omega}$: $a_1 = \frac{\nu\omega}{\pi g r}$

$b_1 = \frac{q_1 k}{\omega^2 + k^2}$ $b_1 = \frac{k}{\omega} a_1 = \frac{k\nu}{\pi g r}$

$\therefore \omega^2 = \frac{\pi g r q_1}{\nu} - k^2$ has solutions only when $\frac{\pi g r q_1}{\nu} > k^2$

§ Lorenz System

$a_1 = \frac{K\nu y}{\pi g r}$ $b_1 = -\frac{K\nu z}{\pi g r} + \frac{q_1}{K}$ $\omega = kx$. $t = \frac{J}{K}$ $b = I$.

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

$\sigma = \frac{\nu}{kI}$ → radius of water wheel

$r = \frac{\pi g r q_1}{k^2 \nu}$

Lorenz parameter

fixed points.

* $(x^*, y^*, z^*) = (0, 0, 0)$ $J = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix} \Rightarrow \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$

eigenvalues: (1) $-b$

(2). 2×2 part: $Tr() = -1 - \sigma$; $Det() = \sigma - r\sigma$. $\lambda = \frac{-(1+\sigma) \pm \sqrt{(1+\sigma)^2 - 4\sigma(1-r)}}{2}$

stable node if $\Delta > 0$. $\Leftrightarrow r < 1$.

saddle point if $\Delta < 0$ $\Leftrightarrow r > 1$

$\Rightarrow$ water wheel solution $\omega = 0$ is stable if $\frac{\pi g r_w q_1}{k^2 \nu} < 1$.

* $r > 1$. have new FPs.

$C_+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$

$C_- = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$

- Symmetry in this system.

$x \to -x$, $y \to -y$, $z \to z$ clockwise vs. counterclockwise turning of the wheel.

- stability of C^+ & C^-

$b \quad \sigma \quad r_H$

stable over a range: $1 < r < r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ hopf. Standard value: $8/3 \quad 10 \quad 24.76$

continue Lorenz System & define Concepts in Chaos

第二十七讲 2017.11.3 Fri

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

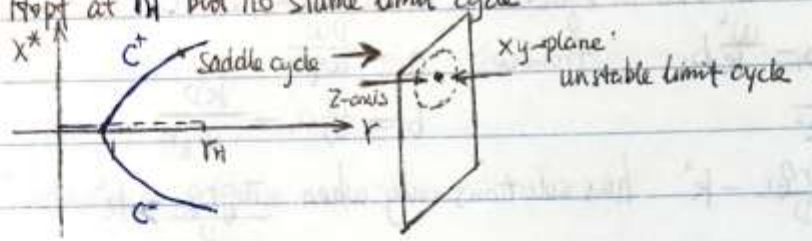
FP at origin stable until $r \geq 1$.

Pitchfork bifurcation: FP at origin becomes unstable at $r=1$. 2 new FPs.

stable for $1 < r < r_H$

new contents:

Hopf at r_H but no stable limit cycle

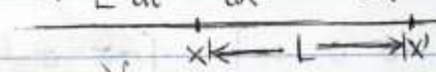


Beyond r_H. trajectories go to a strange attractor

Limiting set has finite extent

- amazing. since Lorenz system shrinks volumn in xyz-space
- fact: if $\dot{\vec{r}} = f(\vec{r})$, for volumn flowing in the system. $\boxed{\frac{1}{V}\frac{dV}{dt} = \vec{\nabla}\cdot\vec{f}}$

~~proof~~: 1d: $\frac{1}{L}\frac{dL}{dt} = \frac{df}{dx}$. $\quad \frac{df}{dx} = 0 \rightarrow f = \text{const.} \rightarrow \frac{dx}{dt} = \text{const.}$



3d: $\frac{\partial f_x}{\partial x}$: rate of system streching in x-direction

Units: $f = m/s$

$\vec{\nabla}\cdot\vec{f}$: $1/s$

$\frac{1}{V}\cdot\frac{dV}{dt}$: $1/s$

$\vec{j} = \rho\vec{v}$ with $\vec{\nabla}\cdot\vec{v} > 0$

$$\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = -\sigma - 1 - b = -(\sigma+1+b)$$

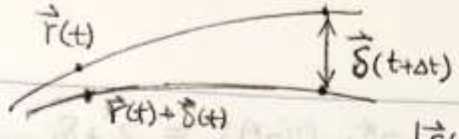
$$\frac{dV}{dt} = -(\sigma+1+b)V. \qquad V(t) = V_0 e^{-(\sigma+1+b)t}$$

3d volumn $\rightarrow 0$ as $t \rightarrow +\infty$ Fact: Lorenz attractors have dimension 2.05

Definition

Chaos: Aperiodic behavior in deterministic systems that displays sensitive dependence on initial conditions.

↳ doesn't go to Limit Cycle or Fixed Points.



Sensitive dependence: $|\vec{\delta}(t)| = |\vec{\delta}_0| e^{\lambda t}$

Limiting sets in a chaotic system are often stronge attractors

Attractor: (A)

1. ~~invariant~~ set - starting in A $\Rightarrow$ ending in A
2. ~~attracts~~ open set of initial conditions, including all those close to the attractor
3. ~~minimal~~: has no subsets satisfying (1) & (2).

Checking: stable FP. stable LC:

C time

Strange Attractor

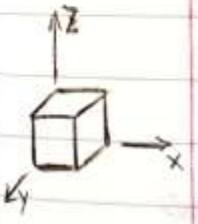
Sensitive dependence on initial ~~attractors~~ conditions. (not FPs or LCs)

$r=28$ hard to understand

larger r is easier to understand.

Sparrow in 1982. referenced in text.
numerical study of $\sigma=10$. $b=8/3$ varied r.
For $r=13.46$: unstable periodic orbits (saddle cycles)
FP stable up to $r=24.5$.
Between $22.52<r<100.795$ stable periodic orbits.

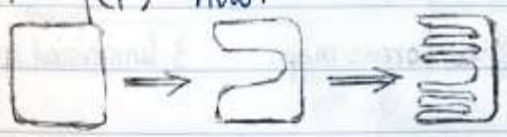
第二十八讲 2017.11.6. Mon



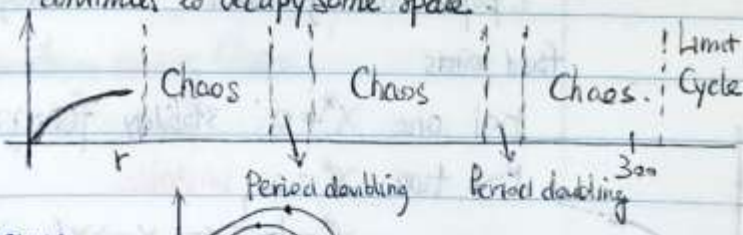
Lorenz system.

Volumn shrinks: $V = V_0 e^{-(1+b+\sigma)t}$

$\dot{\vec{r}} = f(\vec{r})$ How?



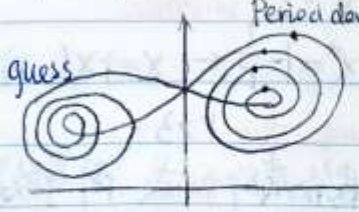
The volumn continues to shrink but the structure continues to occupy some space.



Simulations for some r.

Period doubling

Lorenz's guess



$z_1, z_2 \ldots z_n$ progressive maxima

guess $z_{n+1} = f(z_n)$

recall Poincare map.

~~Stable FP of f~~ Stable FP of f if $\left|\frac{df}{dz}\right|<1$.

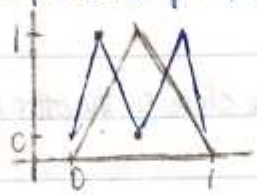
$\longrightarrow$ We observe slope is $\neq 0$.

$\left|\frac{df}{dz}\right|>1$.

$z_0 = z^* + \delta_0$. $z_1 \approx z^* + f'(z^*)\delta_0 = z^* + \delta_1$

$\delta_n = |[f'(z^*)]|^n \delta_0 = \delta_0 e^{n\ln|f'(z^*)|} = \delta_0 e^{n\lambda}$ $\lambda = \ln|f'(z^*)|$ Liapunov exponent

Simple model for $f(z)$



$f(z) = 2z$ if $z \leq \frac{1}{2}$

$= 2 - 2z$ if $z > \frac{1}{2}$

$f^2(z) = f(f(z))$

$f^2(\frac{1}{2}) = 0$

$f^2(\frac{1}{4}) = 1$

chain rule

$\left|\frac{df^2(z)}{dz}\right| = \left|\frac{df}{dz}\right|_{f^2(z)} \left|\frac{df}{dz}\right|_{f(z)} \left|\frac{df}{dz}\right|_z = 2^2$

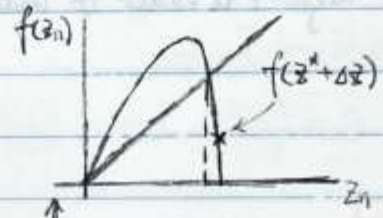
$\left|\frac{df^n(z)}{dz}\right| = 2^n$ lines get steeper & steeper

$f(z_n) = 2z_n \bmod 1$ "Lorenz map"

Lorenz prooved that if z_0 is rational, z_n forms a repeating cycle

if z_0 is irrational, z_n behavior is chaotic

第二十九讲 2017 11.08 Wed



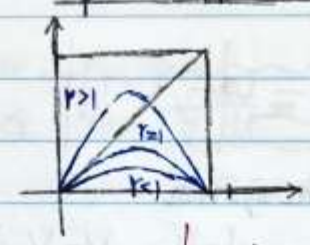
Stability criterion: $-1 < f'(z^*) < 1$.

Logistic Map

can do proofs rather than simulations only

$X_{n+1} = rX_n(1-X_n)$

map on unit interval $\rightarrow r < 4$



Logistic Map.

intuition

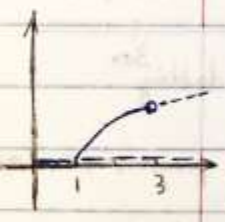
1. population dynamics 2. analogue to Lorenz map 3. universial behavior of transition to chaos

fixed points

$r<1$ one $x^* = 0$. stability $f(x) = rx(1-x)$. $f'(x) = r(1-2x)$ $f'(0) = r$ stable

$r>1$ two $x^* = 0$ unstable.

$x^* = 1 - \frac{1}{r} \leftarrow x = rx(1-x)$ stability $f'(x^*) = r(1 - \frac{2}{x^*}) = 2 - r$ stable for $1<r<3$



Period Doubling for $r>3$

某些波峰波谷有所加减，倒导致周期变化 连续情形

在 fixed point 附近波动，看 iteration map 离散情形（本例）

是 unique 的，对于两倍周期，只有两种先后顺序；对

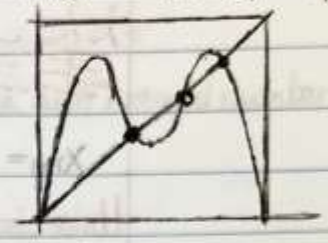
Fact: a 2-cycle is a stable fixed point of $f^{(2)}(x)$.

$f^{(2)}(x) = f(f(x)) = rrx(1-x)[1 - rx(1-x)]$ = quartic polynomial

Draw $f^{(2)}(x)$ using maxima & minima

1) Maxima: $f^{(2)}(x) = f(f(x))$ has a maximum when $f(x) = \frac{1}{2}$ $\quad f^{(2)}(x)_{max} = r\frac{1}{2}(1-\frac{1}{2}) = \frac{r}{4}$

x determined by $rx(1-x) = \frac{1}{2} \rightarrow x = \frac{1 \pm \sqrt{1-\frac{2}{r}}}{2}$



2) Local Minimum $\frac{d}{dx} f^{(2)}(x) = \frac{df}{dx}\Big|_{f(x)} \frac{df}{dx}\Big|_{x} = 0$ when $x = \frac{1}{2}$

$f^{(2)}(\frac{1}{2}) = r^2/4\,(1 - r/4) \approx 0.4$ if $r = 3.5$

第三十讲 2017.11.10. Fri

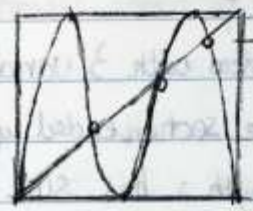
FINAL Dec 13. Wed. 2:30pm check for problems.

is easy?

当 slope < -1 时 period doubling

Fixed points of $f^{(2)}(x)$

$r = 4$.



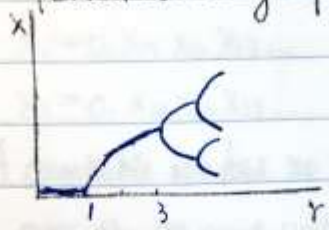
slope < -1, another period doubling

Period $= 4$

systems travels between fps of $f^{(4)}(x)$

At FPs. $\frac{d f^{(2)}(x)}{dx}\Big|_{x^*} = \left(\frac{d f(x)}{dx}\Big|_{x^*}\right)^2$ by chainrule.

period doubling of $f(x)$ is equivalent to a pitchfork bifurcation of $f^{(2)}(x)$



For logistic map specifically,

r_n = transition to having period 2^n

$r_2 = 3.449 \quad r_4 = 3.5644$.

$r_3 = 3.5441 \quad r_\infty = 3.5699$.

Periodic windows among Chaos.

Fact: several features of chaos are indep. of the system.

Order of attractors. period ≤ 6

1, 2, 2×2, 6, 5, 3, 2×3, 5, 6, 4, 6, 5, 6 for any map.

3.84

Size of windows (1970s)

$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$ for any iterated map.

$r_n = r_\infty \pm \frac{\text{const}}{\delta^n} = r_\infty \pm \text{const.}\, e^{-(\ln\delta)n}$

Belousov-Zhabotinski reaction in a stirred flow reactor

Br^- as a function of time.

Embedding theorem (Takens 1981) get a complete picture of dynamics following 1 consittuent

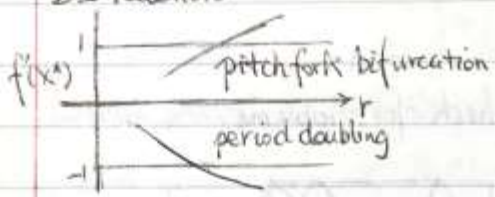
$Br(t)$ $Br(t+\Delta t)$ $Br(t+2\Delta t)$

第三十一讲 2017.11.13 Mon

$x_{n+1} = x_n(1-x_n)\cdot r$ Periods 2.4.8.16...∞ (chaos) Periodic Windows among Chaos.

Universial sequence of behaviors: U-sequence ($n \le 6$): 1.2.2×2.6.5.3.2×3.5.6.

BZ reaction



FP of $f^{(2)}$ that are not f gives period doubling

Dynamical System with 3 variables. $B(t)$. $B(t+53s)$. $B(t+106s)$

The poincare section ended up almost 1 dimensional.

The path length s. has $s(n)$. $s(n+1)$.

Lyapunov Exponent : Measure of chaotic behavior

$x = x^* + \delta \qquad x_u = x^* + \delta_u$

$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left|\frac{\delta_n}{\delta_0}\right| \longrightarrow \delta_n \propto \delta_0 e^{n\lambda}$ for large n.

If $\lambda > 0$, the behavior is chaotic

If $\lambda < 0$. the system goes to a FP.

$\delta_0 e^{n\lambda} = \prod_{i=1}^{n-1} |f'(x_i)|\, \delta_0 \qquad \lambda = \frac{1}{n}\sum_{i=1}^{n-1} \ln|f'(x_i)|$

superstable fixed point

stable FP: exponential convergence to FP.

superstable FP: faster $\leftarrow f'(x^*) = 0$. Super stable.

$x_0 = x^* + \eta_0$

$f(x^*+\eta_0) = f(x^*) + f'(x^*)\eta_0 + \underbrace{\frac{1}{2}f''(x^*)\eta_0^2}_{\eta_1} \qquad \eta_1 = \frac{1}{2}f''(x^*)\eta_0^2$

$f(x^*+\eta_1) = f(x^*) + \cancel{f'(x^*)}\, \frac{1}{2} f''(x^*)\left(\frac{1}{2}f''(x^*)\eta_0^2\right)^2 \qquad \eta_2 = \frac{1}{8} f''(x^*)^3 \eta_0^4$

$\eta_n = \text{const} \times \eta_0^{2^n} \sim e^{\ln \eta_0^{2^n}} = e^{2^n \log \eta_0}$

§ FRACTAL DIMENSION

long term behavior of chaos system

Fractal shaps that don't become smooth at small scales.

SET THEORY: diferent degrees of infinity

Countably & uncountably infinite.

A set is countable if it can be placed in one-to-one correspondance with natural numbers

Fact: positive rationals is countable.

show: list them as proper fractions as follows:

(分子+分母=2) + (分子+分母=3) +(4)+ ..... (n)

m_2 members. m_3 members. m_4 ..... m_n

$m_2 \leq 2$ $m_n \leq n$

Take number in "m" set; number of rationals ahead of it.

is $\leq m_2 + m_3 + \dots + m_n = \frac{n(n+1)}{2} \leq \infty$

Fact. $\mathbb{R}$ is uncountable.

第二十二讲 2017.11.15 Wed

Real numbers are uncountable.

$X_1 = 0.X_{11}X_{12}X_{13}\dots$ There is a real number that we'll never get to this list.

$X_2 = 0.X_{21}X_{22}X_{23}\dots$ $X_{\infty 1} =$ any digit $\neq X_{11}$

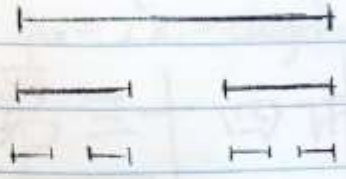
$X_3 = 0.X_{31}X_{32}X_{33}\dots$ $X_{\infty 2} =$ any digit $\neq X_{22}$

A countable set has no length

cover the 1st point with interval δ. the second $\frac{\delta}{2}$. the third $\frac{\delta}{2^2}$. total length $\delta\left[1+\frac{1}{2}+\frac{1}{2^2}+\dots\right]=2\delta$

Calculate dimension of some fractal sets.

Cantor set:



S_0 $C = \lim_{n\to\infty} S_n = S_\infty =$ collection of all points in each S_n

S_1 NOT a null set.

S_2 D. Any number that's ever an edge is in the set.

properties:

* Size as 1d object is 0. Length $= 2^n \cdot \left(\frac{1}{3}\right)^n = \left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$
* C is self-similar: $0-\frac{1}{3}$ part is 0-1 part scaled down by $\frac{1}{3}$. Structure is persistent over all length scales.
* dimension $= 0.63$.

* uncountable

Ways of measuring fractal dimension

* Similarity dimension if set is self-similiar

  Set = m copies of itself, scaled down by factor $\frac{1}{r}$

  $$d = \ln(m)/\ln(r)$$

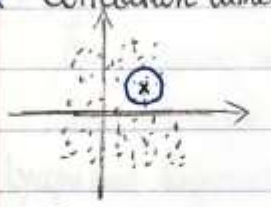
  For cantor set $d = \ln(2)/\ln(3) = 0.63$

* Box dimension : ϵ = box size $N(\epsilon)$ = # of boxes to cover the set

  $$d = \lim_{\epsilon \to 0} \frac{\ln[N(\epsilon)]}{\ln(1/\epsilon)}$$

  For cantor set $\epsilon = \frac{1}{3^n}$ $N(\epsilon) = 2^n$ $d = \lim_{n\to\infty} \frac{\log 2^n}{\log 3^n} = \log 2/\log 3$

* Correlation dimension : especially friendly to simulation



$N_x(\epsilon)$ = # of points within the radius ϵ of x

$N(\epsilon) \propto \epsilon^d$

$C(\epsilon) = \sum_x N_x(\epsilon)$ $d = \frac{\ln C}{\ln \epsilon}$

For Lorenz system $d_{corr} = 2.05$

第三十三讲 2017.11.17 Fri

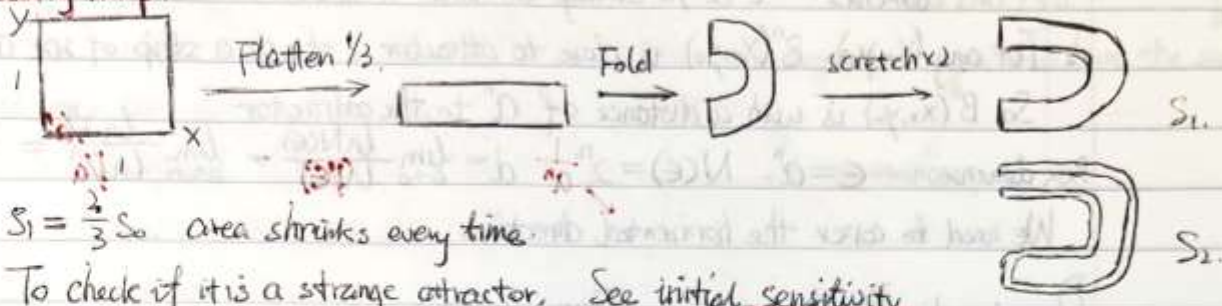
Logistic Map. Fractal Set

x at r_∞ is Cator-like with $d = 0.538$

Object	Fractal Dimension
Coastline of Norway	1.5
Great Britain	1.05
Lorenz attractor	2.05-2.06
Time course of particle in 1D 布朗运动 $x(t)$	1.5
2D random walk x vs y	2.
Diffusion limited aggregation in 2D	1.7.
in 3D	2.5
Cauliflower surface.	2.33
Broccoli	2.66

Strange Attractors.

Pastry Map.



$S_1 = \frac{2}{3} S_0$ area shrinks every time.

To check if it is a strange attractor, See initial sensitivity

$$f(x) = \begin{cases} 2x & \text{if } 0<x<\frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2}\leq x<1 \end{cases} \qquad |f'(x)|=2 \longrightarrow \text{sensitive}$$

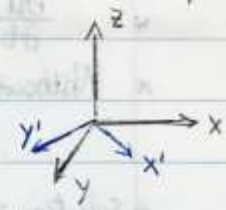
Dimension of the limiting set. Cantor $\times \mathbb{R}^1 \longrightarrow 1+\frac{\ln 2}{\ln 3} = 1.63$.

Stretching & Shrinking in different directions $\longrightarrow$ understand Lorenz sys better

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ (r-z) & -1 & x \\ y & x & -b \end{bmatrix}$$

$A_{xx} A_{yy} A_{zz} < 0$. shrinking.

If $x' = \frac{x+y}{2}$, $y' = \frac{x-y}{2}$

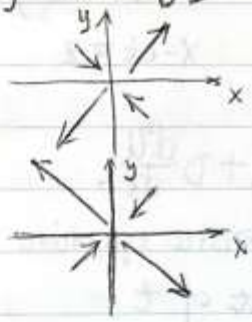


$$A_{x'x'} = \frac{A_{xx}+A_{xy}+A_{yx}+A_{yy}}{2} = \frac{r-z-1}{2} > 0 \text{ if } z<r-1$$

$$A_{y'y'} = \frac{A_{xx}+A_{yy}-A_{yx}-A_{xy}}{2} = -\sigma + \frac{z-r-1}{2} > 0 \text{ if } z>>0.$$

$z<<0$:

$z>>0$



There are positive eigenvalues of $\hat{A}$ Some directions get stretched.

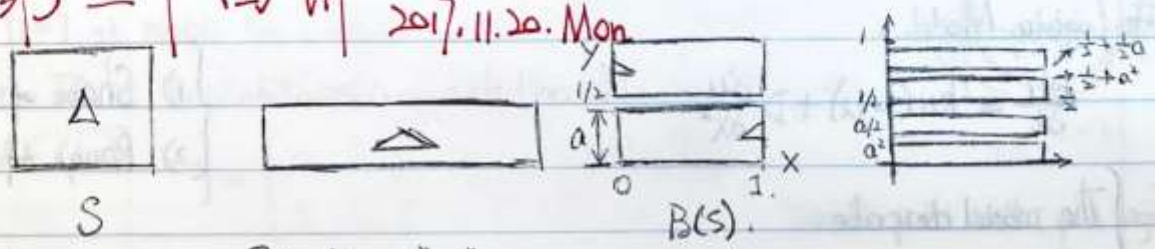
Baker's map

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{if } 0\leq x<\frac{1}{2} \\ (2x_n-1, ay_n+\frac{1}{2}) & \text{if } \frac{1}{2}\leq x<1. \end{cases}$$

Discontinuity at $x_n = \frac{1}{2}$

$x_n = 2^n x_1$ - integer. $a<\frac{1}{2}$

第三十四讲 2017.11.20. Mon



2d area $\to 0$. $S(B(S)) = 2^n \cdot a^n \to 0$ as $a<1/2$.

$A = \lim_{n\to\infty} B^n(S)$ A is an attractor

? Lyapnov exponent. related to fractal dimension

1. $\bar{A}$ has elements: 0 & 1/2 always in there. (All bottoms are in there.)
2. For any (x_0, y_0), $B^n(x_0, y_0)$ is close to attractor. ∴ its in a strip of size a^n
   So $B^n(x_0, y_0)$ is with a distance of a^n to the attractor

Box dimension $\epsilon = a^n$. $N(\epsilon) = 2^n \frac{1}{a^n}$ $d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} = \lim_{n \to \infty} \frac{\ln 2^n/a^n}{\ln 1/a^n} = \frac{\ln(2)}{\ln(1/a)} + 1.$

We need to cover the horizontal direction

Partial Differential Equations.

Dimensionality $1 \to 2 \to 3 \to \infty$

$\vec{F} \Rightarrow u(x,t)$ continuous set of arguments limiting behaviors of $u(x,t)$

Dynamics: Reaction-Diffussion Equation

* $\frac{\partial u}{\partial t} = f(u) + D\frac{\partial^2 u}{\partial x^2}$
* Without the reaction term. → Diffussion equation → spread & smoothing
  increase inhomogeneity ↳ not change total of u: $\int_{-\infty}^{+\infty} u(x,t)dx = \text{const.}$
* solutions with wave form $u(x,t) = U(x-ct)$ $x - ct \equiv z$

$$-c\frac{dU}{dz} = f(U) + D\frac{d^2U}{dz^2}$$

* Consider linear: $f(u) = ku$ $-c\frac{dU}{dz} = kU + D\frac{d^2U}{dz^2}$

solution $U(z) = e^{Kz}$ gives the characteristic equation $K = \frac{-c \pm \sqrt{c^2 - 4kD}}{2D}$

I. If $c=0$. $K = \pm i\sqrt{\frac{k}{D}}$ $u(x,t)$ is independent of t

第三十五讲 2017.11.27. Mon

Relavent to linear problems as a limit. Balance of diffusion & growth

II. If $c>0$ but not too large.

$Re(K) = -\frac{c}{2D}$

$Im(K) = \sqrt{k/D}$

Logistic Model.

$$\frac{\partial u}{\partial t} = ku(1-u) + D\frac{\partial^2 u}{\partial x^2}$$ Fisher Kolmogorov equation
1). Shape of wave solutions
2). Range of possible velocities.

The model describes
I. spread of populations of organisms
II. spread of genes in a population
III. spatial distribution of tomor cells
IV. Ca^{2+} waves inside cells.

1st step: understand behavior of corresponding uniform equation

$\frac{\partial u}{\partial t} = ku(1-u)$ $\quad u \to 1$ as $t \to \infty$

assume inhomogeneous equations have the same behavior

2nd step: Look for travelling waves

At same point $\frac{\partial^2 u}{\partial x^2} = 0$. $\frac{\partial u}{\partial t} > 0$.

Guess the wave moves to the right

Rigorous Treatment:

rescale time & space axes $t^* = kt$. $\frac{\partial u}{\partial t^*} = u(1-u)$ $\quad \boxed{\frac{\partial u}{\partial t^*} = u(1-u) + \frac{\partial^2 u}{\partial x^{*2}}}$

$x^* = \sqrt{k/D}\, x$.

$U(z) = u(x+ct)$

$\boxed{\frac{d^2U}{dz^2} + c\frac{dU}{dz} + U(1-U) = 0}$ Think of z as a "time", map onto Newton 2nd Law prob with damping force $-c\frac{dU}{dz}$ & velocity independent $-U(1-U)$

Look for solutions with $U(-\infty) = 1$. $U(\infty) = 0$.

Via assuming $V = \frac{dU}{dz}$

$$\boxed{\begin{cases} U' = V \\ V' = -U(1-U) - cV \end{cases}}$$ find FPs and use to derive possible behaviors

FPs are $(0,0)$ & $(1,0)$. $\quad J = \begin{bmatrix} 0 & 1 \\ 2U-1, & -c \end{bmatrix}$

At $(0,0)$. $J = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}$ $\quad \lambda = \frac{-c \pm \sqrt{c^2-4}}{2}$

$c \leq 0$ unstable (node?) ✓

$0 < c < 2$. stable spiral

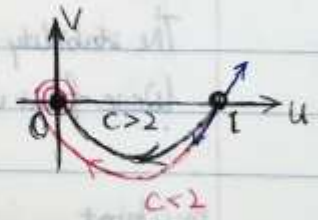
$c > 2$. stable node

第三十六讲 2017.11.29 Wed

FINAL. December 13. 2:30-4:30 pm in this room.

continue

$u = 1$ is stable for uniform system $\frac{du}{dt} = u(1-u)$



For FP at $(0,0)$ & $(1,0)$. looking for this type of path

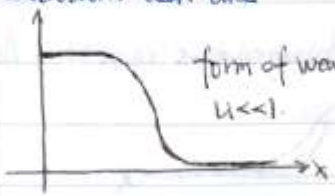
At $(1,0)$ $\quad J = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$ $\quad \lambda = \frac{-c \pm \sqrt{c^2+4}}{2}$ saddle point with pos eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Starting with $u > 0$. the system never takes $u < 0$.

$c < 2$ corresponds to unphysical solution. No solution with $c < 2$

XQuartz

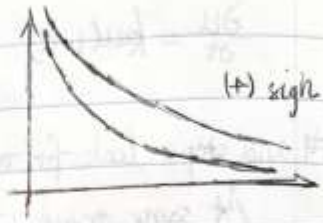
Solutions look like



$$\frac{d^2u}{dz^2} + c\frac{du}{dz} + u = 0$$

$$K^2 + cK + K = 0,$$

$$K = \frac{-c \pm \sqrt{c^2-4}}{2}$$



$$\lim_{c\to\infty} K = \lim_{c\to\infty} \frac{c}{2}\left[1+\sqrt{1-\frac{4}{c^2}}\right] = -\frac{1}{c}$$ Decay rate drops as $c\to\infty$

Zeppetella 1979

exact solution for $c = \frac{5}{\sqrt{6}}$

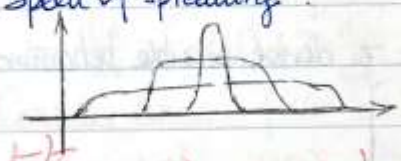
$$u(z) = \frac{1}{\left\{1+\exp\frac{z-z_0}{\sqrt{6}}\right\}^2}$$ for any z_0

linearized equation $u(1-u) \to u$ with $c = 5/\sqrt{6}$

$$u = ae^{K_+ z} + be^{K_- z}$$

Non linear term u^2 is like $e^{(K_+ + K_-)z}$

Speed of spreading.



$$c = \sqrt{2kD}$$

第二十七讲 2017.12.1 Fri

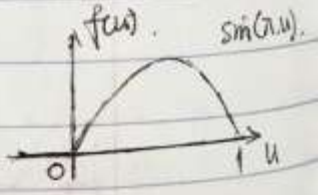
Minimum Wave Speed

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2}$$

$f(u) = ku$

$f(u) = ku(1-u)$ $D=1$. $\to c_{min} = 2$.

$f(u) = ku(1-u)$ $\to c_{min} = 2\sqrt{kD}$



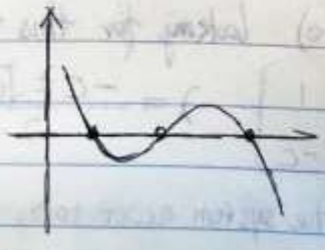
The wave front is seeing the positive feedback

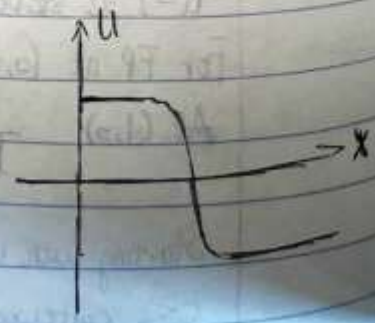
General minimum speed $c_m = 2\sqrt{f'(0)D}$

The stability of the 2D phase plane & the reaction function is opposite. Because $z = x - ct$

Wave front velocity $\to c_{min} \geq 2\sqrt{kD}$

muskrat.





Other reaction equations:

$$f(u) = u - u^3$$

For symmetry we expect $c=0$.

System Recipe

Again look for paths in U-V plane $\begin{cases} U' = V \\ V' = -U + U^3 - cV \end{cases}$

$-cU' = f(U) + U'' = U - U^3 + U''$

$$J = \begin{bmatrix} 0 & 1 \\ -f' & -c \end{bmatrix} \qquad \lambda_\pm = \frac{-c \pm \sqrt{c^2 - 4f'}}{2}$$

Fixed points $(0,0)$, $(\pm 1, 0)$

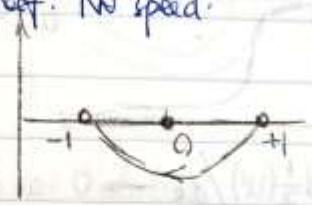
At $(0,0)$

$f' = 1 \quad \lambda_\pm = \frac{-c \pm \sqrt{c^2-4}}{2}$ stable node $c>2$, stable spiral $c<2$

At $(\pm 1, 0)$

$f' = -2 \quad \lambda_\pm = \frac{-c \pm \sqrt{c^2+8}}{2}$ saddle point always.

Proof: No speed.



A path like this exists only when $c=0$

The system is like a Newton damping.

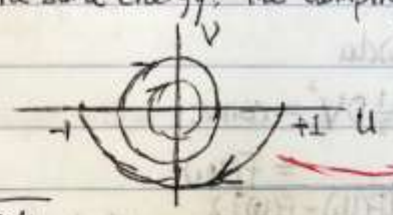
In damping the total energy is always droping. The energy of the system

$$E_{tot} = -\int \text{force}\, du + \frac{1}{2}V^2$$
$$= \int f(u)\, du + \frac{1}{2}V^2$$
$$= \frac{1}{2}U^2 - \frac{1}{4}U^4 + \frac{V^2}{2}$$

since FPs have the same energy, the damping term cannot exist.

$c=0$

$\lambda = \pm i$



Domain Wall

Width of $\overset{?}{=} \sqrt{D/k}$

第二十八讲 2017.12.04 Mon

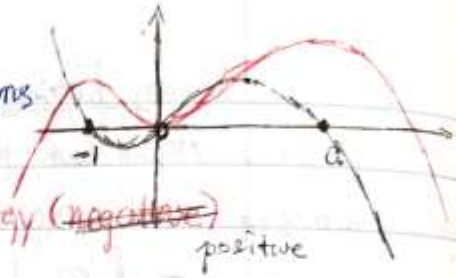
FINAL 剩一周零一天 移至24

continue

$\frac{\partial u}{\partial t} = f(u) + \nabla^2 u. \quad f(u) = u - u^3 \quad (u.v.c) = (0.9999, 0, 0)$ approx domain wall

$(0.9999\ 0, 1)$

Look for moving waves connecting FPs stable of uniform systems



$f(u) = u(1+u)(a-u) = au + (a-1)u^2 - u^3$

$F(u) = \int f(u)du = \frac{1}{2}au^2 + \frac{1}{3}(a-1)u^3 - \frac{1}{4}u^4$ potential energy (~~negative~~)

$F(a) > F(-1)$

What direction will the wave move?

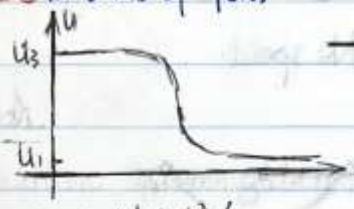
根据能量高低判断. System is losing energy from a to -1. Positive damping $c>0$?

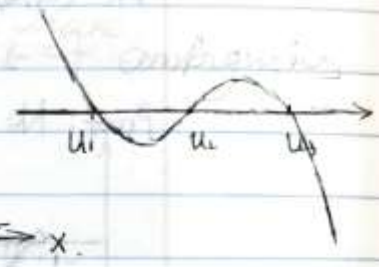
Type of FP is determined by f'

$f'<0$. saddle point

$f'>0$. stable fixed point

General Formula for approximate speed of motion c in terms of f(u)





$-cu' = f(u) + Du''$

multiply u' and integrate.

$-c\int_{-\infty}^{+\infty}[u'(z)]^2 dz = \int_{-\infty}^{+\infty} f(u)u'(z)dz$

$+\int_{-\infty}^{+\infty} Du''(z)u'(z)dz$ $\quad u''u' = d\frac{1}{2}(u')^2/dz \to 0$ at $\pm\infty$

$u'(z)dz = du$. $\int_{-\infty}^{+\infty} f(u)u'(z)dz = \int_{u_3}^{u_1} f(u)du = F(u_1) - F(u_3)$.

第二十九讲 Last Lecture on New Material 2017.12.06 Wed.

$\lim_{c\to 0}$

$\frac{\partial u}{\partial t} = f(u) + D\nabla^2 u$

回忆: 只有2个驻点时 用物理上相值为实数的方法.

$-c\frac{du}{dz} = f(u) + D\frac{d^2u}{dz^2}$

Formula for c in terms of f(u)

"Potential Energy" $F(u) = \int f(u)du$

Energy conservation: $F(u) + \frac{1}{2}Dv^2 = \text{const}$.

When $u=u_3$. $v=0$. $\quad = F(u_3)$

$$u' = \sqrt{\frac{[F(u_3) - F(u)]\cdot 2}{D}} = u'(u).$$

$\int_{-\infty}^{+\infty} f(u)u'(z)dz$

$\int_{-\infty}^{+\infty}[u'(z)]^2 dz = \left|\int_{u_1}^{u_3} u'(u)du\right|$ $\quad c = \sqrt{\frac{D}{2}}\frac{F(u_3) - F(u_1)}{\int_{u_1}^{u_3}\sqrt{F(u_3) - F(u)}\,du}$

Motion is driven by the difference in the potential energy.

Soliton

$\frac{\partial u}{\partial t} + \underbrace{6u\frac{\partial u}{\partial x}}_{\text{velocity field}} + \frac{\partial^3 u}{\partial x^3} = 0$. $\quad \omega = ck$ $\quad$ Korteweg de Veries Eqn 1895

soliton solutions

$-cu' + u''' + 6uu' = 0$

Integrate: $-cu + u'' + 3u^2 = A = 0$

$$\begin{cases} u' = v \\ v' = cu - 3u^2 \end{cases}$$

no v in RHS. no damping. Conservative System

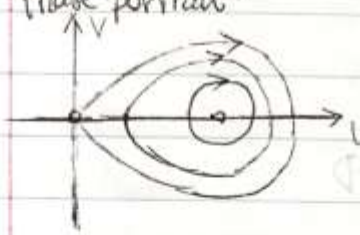
Fixed points: $(0,0)$, $(\frac{c}{3}, 0)$ $\quad J = \begin{bmatrix} 0 & 1 \\ c-6u & 0 \end{bmatrix}$

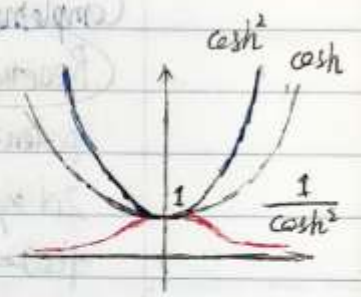
At $(0,0)$. $\lambda = \pm\sqrt{c}$. saddle point.

At $(\frac{c}{3}, 0)$ $\lambda = \pm i\sqrt{c}$. center.

$dx/dy > 0$.
$dy/dx < 0$.

Phase portrait:





Exact Solution:

$$u(\xi) = \frac{c}{2} \frac{1}{\cosh^2\left(\frac{\sqrt{c}\,\xi}{2}\right)} \qquad \cosh(\alpha) = \frac{1}{2}(e^{\alpha} + e^{-\alpha})$$

最后一课. 总复习

猜下最后一题用何公式

Chapter 8.

Hopf, 2d pitchfork, 2d saddle node bifurcation

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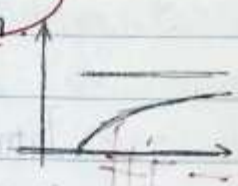
subcritical supercritical's radius of limit cycle



Poincare map: 2d flow → 1d map. Stability criterion for limit cycle (fp of P)

Chapter 9.

Lorenz system fixed point (parameters). stability of Origin $Tr(J)$? volumn contraction

Lorenz tent map $\left|\frac{dz_{n+1}}{dz_n}\right| = 2 > 1$

Chapter 10.

iterated maps. stability pitchfork bifurcation & period doubling

After 1st period doubling, fp of $f^{(2)}$ stable fp of f becomes unstable
Superstable fixed point $f'(x^*)=0$.

Chapter 11.

similar box dimension Cantor Sets.

Chapter 12.

Strange attractors Baker's map.

Complementary.

Reaction diffusion equations $\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2} D$

potential energy $F(u) = \int f(u)\, du$

2d system.

$f(u) = u(1-u)$. unphysical wave speed

$f(u) = u(u+1)(a-u)$ bistable media domain wall single wave speed

skip wave speed

skip KdV equation

Theoretical Biophysics

第一讲 Introduction

2018.1.16. Tue.

Biological Physics of Cells

How do underlying molecular interactions determine cell behavior.

1) read a present work in literature

2) computational project: repeat a paper and go further.

Papers to read:

1). take-home message

2). conceptual framework for understanding range of phenomenon.