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Subject: **Financial Engineering Project 3**
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Memo: Option Hedging

I. Introduction

Delta hedging is an option strategy that aims to reduce, or hedge, the risk associated with the price movements in the underlying asset by offsetting long and short positions. The theoretical change in premium for each basis point or $1 change in price of the underlying is delta, and the relationship between two movements is the hedge ratio. The price of a put option with a delta of -0.50 is expected to rise by 50 cents if the underlying asset falls by $1. The opposite is true as well. The delta of a call option ranges between zero and one, while the delta of a put option ranges between negative one and zero. For example, the price of a call option with a hedge ratio of 0.40 will rise 40% of the stock-price move if the price of the underlying stock increases by $1.

The future returns of each security is unknown because the stock price movement is random according to the Random Walk Theory. In order to predict future prices financial analysts use a couple of processes to predict the future price and therefore the returns. First, the Black-Scholes model, also known as the Black-Scholes-Merton model, is a model of price variation over time of financial instruments such as stocks that can, among other things, be used to determine the price of a European call option. The model assumes the price of heavily traded assets follows a geometric Brownian motion with constant drift and volatility. When applied to a stock option, the model incorporates the constant price variation of the stock, the time value of money, the option's strike price and the time to the option's expiry.

Another famous model that is used to calculate (estimate) the future prices of an entity is called the Binomial Tree Model. In finance, the binomial options pricing model provides a generalizable numerical method for the valuation of options. The binomial model was first proposed by Cox, Ross and Rubinstein in 1979. The binomial pricing model traces the evolution of the option's key underlying variables in discrete-time. This is done by means of a binomial tree, for a number of time steps between the valuation and expiration dates. Each node in the tree represents a possible price of the underlying at a given point in time. Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards through the tree towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.

For this project, stocks and calls from INTC (Intel Corp.) were bought and sold starting from shorting 5 calls whose maturity date was June 15, 2018 and strike price was $52.50. In order to make the calculation simpler, the risk-free interest rate and dividend yield rate was fixed as 2.18% and 2.25%, respectively. The hedging period was from May 1, 2018 to May 11, 2018. The real-time market results used in this project are from the Chicago Board of Options Exchange (CBOE).

Our group took advantage of our coding skills using Python 3.6 general programming language. All the functions used to generate figures and hedging results are in the attached python file named `american-call_binomial.py`.

II. Findings

Delta hedging is a great way to preserve the value of the portfolio. Black-Scholes model is used to calculate a theoretical call price using the five key determinants of an option's price: stock price, strike price, volatility, time to expiration, and risk-free interest rate. The binomial tree model however is more versatile, valuing European options and as well as American options. The option pricing using the Binomial Tree model can be improved by adding more steps to the tree. As the number of steps increases in the tree, the option pricing approximates the option value calculated by the Black-Scholes model. Lastly, the relationship between the metrics, the implied volatility reading serves as the baseline, while fluctuations in implied volatility define the relative values of the option premiums. When the two measures represent similar values, option premiums are generally considered to be fairly valued based on historical norms. Option traders seek the deviations from the state of equilibrium to take advantage of overvalued or undervalued option premiums. Historical volatility calculated in this project equaled 37.46% and implied volatility to fluctuate between 27%-29% roughly.

Dynamic hedging seemed to be better in preserving the value of the portfolio in this project. Delta hedging is useful when you believe a big move is upcoming, but you are not sure when. Since it only removes the directional risk of small moves in the underlying stock, you can still profit or lose money if that stock makes a big move. Delta hedging can be quite easy when delta coefficients are constantly provided and change in real time. In this project, we found that dynamic hedging performs better than static hedging, but the difference between historical and implied volatility was not apparent.

III. Discussion

Methods

Assumption and Historical Volatility Estimation

One of the basic assumptions worth mentioning in this project is that stock prices have lognormal property. The Black-Scholes model assumes that "percentage changes in the stock price in a short period of time are normally distributed" (Wu, 1). As seen in Equation 1, the standard deviation of the return for the short period of time Δt is $\sigma\sqrt{\Delta t}$.

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma^2\Delta t) \qquad \text{(Equation 1)}$$

where $\phi(m, v)$ denotes a normal distribution with mean m and variance v.

From this assumption, the return of stocks can be better estimated by taking natural log of the stock price ratio $ln(\frac{S_T}{S_0})$ as opposed to dividing the difference between the new and old price with the old price $\frac{S_T - S_0}{S_0}$ like we used in previous projects. Using historical data from the recent 20 days, we estimated daily volatility as follows:

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} \text{ where } u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \text{ for } i = 1, 2, \ldots, n \qquad \text{(Equation 2)}$$

Since all the inputs in every model in this project assumes annualized values, we converted this daily volatility to annual volatility by multiplying the standard deviation by $\sqrt{252}$ because there are usually 252 business days in one year. In this project, we used 20 days of recent stock price data to calculate the historical annual volatility because 20 business days is an usual length for short term hedging.

Black-Scholes Pricing Model

The Black-Scholes model assumes continuous time, which means that the evaluation follows the real market price movement more precisely. However, it can only deal with European options, which can only be exercised at their maturity dates. Potential issues with this assumption are analyzed in the *Analysis* section.

The model uses the following to evaluate option prices:

$$c = S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2) \qquad \text{(Equation 3)}$$

$$p = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \qquad \text{(Equation 4)}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Binomial Tree Pricing Model

Binomial Tree Pricing Model is versatile: even though the step is discrete, it can evaluate American options, which can be exercised at any point before their maturity dates. Since the time step is discrete, it tends to have discrepancies with the real market. However, as the number of time steps increase, the discrete model approximates continuous one, which, in this case, the Black-Scholes model but with American options. (Figure 1.)

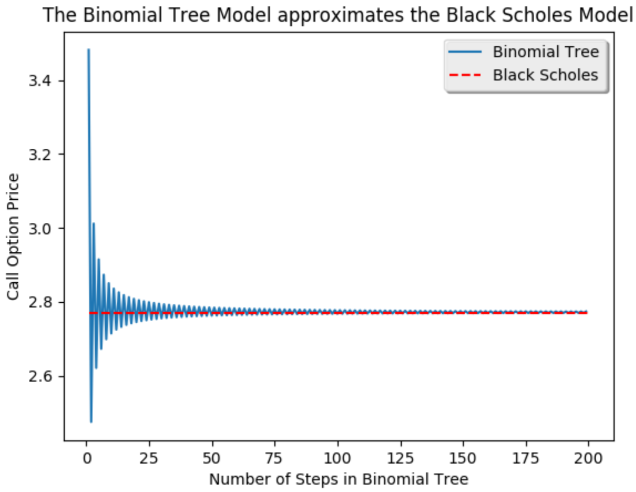

Figure1. Generated by `plotOptionPrices()` function

```
At t = 0.0
 52.39
(2.91)

At t = 0.026
 55.66   49.31
(4.62)  (1.31)

At t = 0.052
 59.14   52.39   46.41
(7.07)  (2.32)  (0.36)

At t = 0.079
 62.84   55.66   49.31   43.68
(10.34)  (4.01)  (0.74)  (0.00)

At t = 0.105
 66.77   59.14   52.39   46.41   41.11
(14.27)  (6.64)  (1.53)  (0.00)  (0.00)

At t = 0.131
 70.94   62.84   55.66   49.31   43.68   38.69
(18.44)  (10.34)  (3.16)  (0.00)  (0.00)  (0.00)

The final value of option is at t = 0: 2.91
Delta: 0.52108
```

Figure 2. Five-step Binomial Tree generated by `binomial_tree_call()` function

Option valuation using this method is, as described in the introduction, a three-step process:

1. price tree generation
2. calculation of option value at each final node
3. sequential calculation of the option value at each preceding node

We have algorithmically automated this calculation process using Python 3.6 programming language. The result of five-step binomial tree below in Figure 2. was obtained from `binomial_tree_call()` function. On the first day, the initial stock price was \$52.39 and the historical was 37.46%. By specifying the value of `N` parameter in the function, it can display arbitrary number of steps in this form. Dividend yield rate q was also taken into consideration by altering the formula for p, probability of the stock price goes up:

$$p = \frac{e^{(r-q)\Delta t} - d}{u-d} \text{ where } u = e^{\sigma\sqrt{\Delta t}} \text{and } d = e^{-\sigma\sqrt{\Delta t}}$$

Implied Volatility

One of the input variables of the Black-Scholes model, the volatility σ cannot be directly observed. Thus, historical volatility is used as an estimate. When option prices are known, the volatility can be estimated using the Black-Scholes model. Even though the equations 3 and 4 cannot be inverted to analytically calculate the value of σ, it can be estimated by numerically solving the optimization problem. Using the Microsoft Excel's Solver, all the variables except for volatility can be substituted into the equation 3 and iteratively search the volatility that fits the equation. Throughout the hedging period, the calculated implied volatilities were all very close to or slightly higher than the implied volatilities on Yahoo Finance. This difference is likely due to the difference in the available information and calculation methods.

Comparative Analysis

Black-Scholes vs. Binomial Tree

As mentioned in the *Methods* section, the Black-Scholes model cannot handle early exercises (American options). On the other hand, the Binomial Tree model can not only handle early exercises but also overcome its disadvantage of its being discrete by increasing the number of steps. Thus, it is more accurate to use binomial tree model for American call options with dividend-paying stocks. The difference is negligible, but it still exists. By running `call_option_price(N=5000)`, you can see that the value converges to \$2.7707 whereas the Black-Scholes model concludes that the call price is \$2.7692. The difference emerged from the Black-Scholes model's inability to adopt early exercises. If the hedging period and option's time to maturity had been longer, this difference would not have been negligible.

Market vs. Calculated Option Prices

The calculated option prices were about 70 cents higher than the actual market price from CBOE on the first day. On the last day, the calculation was way far off by more than a dollar. (Figure 3.)

First Day (04/30/2018)	
Calculated Option Prices	
Call (c)	**$2.77**
Put (p)	**$2.88**
Market Option Prices	
Call	$2.08
Put	$2.19

Last Day (05/11/2018)	
Calculated Option Prices	
Call (c)	**$3.73**
Put (p)	**$1.59**
Market Option Prices	
Call	$2.59
Put	$0.51

Figure 3. from *Black-Scholes Model (Q1 and Q2)* sheet

These results might be due to the inaccuracy in our historical volatility calculation. As mentioned in *Findings* section, the implied volatility ranged from 27% to 29% during the hedging period whereas our historical volatility was as high as 37.46%. This error is likely due to Intel Corp.'s Q1 earnings release on April 26, 2018. The recent 20 days of stock price data to calculate the historical volatility was until April 27, 2018. The quantitative model like this cannot take such macroeconomic events into consideration. If we hedged longer period or in a different timing when the market is a little bit more stable, we could have obtained better results.

Furthermore, the Black-Scholes model assumes that the volatility is constant through a hedging period. This assumption certainly does not hold even though it might not have affected our calculation much in this short term hedging.

Historical vs. Implied Volatility Hedging Results

However, the delta hedging was overall successful. The portfolio value fluctuation was less than plus or minus $80 out of $100,000. If we just shorted 5 calls without hedging by buying stocks, the portfolio value would have been $99,520. As seen in beta hedging from the previous project, dynamic hedging preserved the portfolio value better at the end. Figure 4. shows the portfolio value fluctuation in different types of hedging. It is hard to conclude that use of historical or implied volatility made significant differences in preserving the portfolio value, but it seems that calculating delta using implied volatility seems better. Again, if we hedged longer period, it would make a significant difference because the historical volatility is fixed whereas implied volatility changes as the input variables change.

In terms of macroeconomic events, as mentioned, the market was right after Q1 earnings release. The delta hedging could have performed even better if the market was not right after earnings release.

Figure 4. from *Implied Vol. Hedging (Q5)* sheet

IV. Limitations

Black-Scholes model uses a number of assumptions to determine the price of European options. These assumptions makes it impossible to calculate the 'true' actual value of the premiums. The model itself does not allow early exercise of the premiums making it only applicable on European options because they cannot be exercised before the maturity date. In general Black-Scholes model is not suitable for valuing warrants because they are long term options and it is quite likely that the underlying stock will pay dividends during the life of the warrant.

Binomial Tree model that is used in this project to calculate European premiums in general makes a few assumptions that in real life financial markets may not be applicable. The assumptions being as follows: There are only two possible prices for the underlying asset on the next day, the two possible prices are the up-price and the down-price, the rate of interest (r) is constant throughout the life of the option, markets are frictionless (i.e, there are no taxes and no transaction cost), and last but not least, investors are risk-neutral (i.e, investors are indifferent towards risk).

Other collective shortcomings that we decided as a group were the short time span of hedging highly influenced the option premiums. If we had more data in real time our figures of premiums would have been more accurate.

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