Order, fluctuations, rigidities

Simon Coste
simon.coste@inria.fr

April 11, 2021— work in progress

Abstract

This is a general review on hyperuniform point processes, also called *superhomogeneous* in the physics litterature. I describe the general theory of spectral analysis of correlations, the local-weak convergence setting for finite samples, and the fluctuations properties of hyperuniform processes. Some topics in which recent advances were made are extensively developped, such as: the links between rigidity and hyperuniformity, the maximal rigidities of stealthy hyperuniform processes, and the properties of perturbed lattices.

Contents

1 Variance formulas **5**
- 1.1 Two-point correlations for linear functionals . . . 5
- 1.2 Scaled intersection formulae . . . 7
- 1.3 Structure factor and the phase-space variance formula . . . 8
- 1.4 Number variance for simple shapes: Bessel and Sine kernels . . . 8

2 Hyperuniformity: definitions and characterizations **11**
- 2.1 On window shape dependence . . . 11
- 2.2 Integrability of the covariance measure . . . 12
- 2.3 General hyperuniformity . . . 13
- 2.4 Stealthy hyperuniform processes . . . 14
- 2.5 Stability of hyperuniformity . . . 14
- 2.6 On repulsions in hyperuniform point processes . . . 15

3 Some important examples **17**
- 3.1 Lattices . . . 17
- 3.2 Determinantal point processes . . . 17
- 3.3 Pfaffian point processes . . . 18
- 3.4 The sine-beta process . . . 20
- 3.5 A challenge: the generation of hyperuniform, amorphous patterns . . . 20

4 Density fluctuations **21**
- 4.1 Hyperuniform systems classification . . . 21
- 4.2 The minimal decay . . . 22
- 4.3 The central-limit theorem . . . 22
- 4.4 Order metrics . . . 23
- 4.5 Higher order moments than the variance . . . 23

5 Finite sample analysis **24**
- 5.1 Second-order spectral convergence and empirical structure factor . . . 24
- 5.2 The variance of the collective-coordinate . . . 25
- 5.3 How to visually see hyperuniformity . . . 25

6 Rigidity and hyperuniformity **27**
- 6.1 The variance argument . . . 27
- 6.2 Quadratic variance formula . . . 28
- 6.3 Rigidity from hyperuniformity: one and two dimensions . . . 28
- 6.4 Rigidity from hyperuniformity: three dimensions and more . . . 30
- 6.5 The link between rigidity and fluctuations: counterexamples . . . 30
- 6.6 Other examples of rigidities . . . 31

7 Extreme rigidity for stealthy hyperuniform processes **33**
- 7.1 Anticoncentration and bounded holes . . . 33
- 7.2 Maximal rigidity of SH processes . . . 34

8 The Coulomb gas **36**
- 8.1 Definition of the Hierarchical Coulomb gas . . . 36
- 8.2 Hyperuniformity and rigidity of Coulomb systems . . . 36
- 8.3 Hyperuniformity of the HCG . . . 36

9 Perturbations of point processes **37**
9.1 Fluctuations of perturbed processes . . . 37
9.2 Correlations and structure measure of perturbed point processes . . . 38

10 Reconstruction and rigidity of the independently perturbed lattice **41**
10.1 Reconstructibility: a positive result . . . 41
10.2 Rigidity and tolerance: a negative result . . . 42
10.3 Invisibility of the underlying lattice: cloaking . . . 46

11 Dependently perturbed lattices (DPL) **47**
11.1 Stable matchings . . . 47
11.2 Sliders are stealthy hyperuniform . . . 47

A Point processes and Palm technology **53**
A.1 Configurations . . . 53
A.2 Palm technology . . . 53
A.3 Intensities . . . 54
A.4 Correlations . . . 54
A.5 Cumulants of Point processes . . . 54
A.6 Ergodicity . . . 55

B Harmonic analysis **56**
B.1 Tempered distributions and the Schwartz space . . . 56
B.2 Operations on tempered distributions: convolution and multiplication . . . 56
B.3 Special Fourier transforms . . . 56
B.4 Positive, positive-definite (ppd) measures . . . 58
B.5 Harmonic analysis on lattices . . . 58

C Stationary processes and Kolmogorov's criterion **60**

D Cumulants **62**

Introduction: about this survey

Let Φ be a stationary point process on $\mathbb{R}^d$. We preliminary say that Φ is hyperuniform if it exhibits anomalously small density fluctuations:

$$\lim_{R\to\infty} \frac{\mathrm{Var}(|\Phi \cap B(0,R)|)}{|B(0,R)|} = 0. \tag{0}$$

where $|A|$ denotes the Lebesgue measure of the measurable set A and $B(0,r)$ is the euclidean ball of radius r around zero. Definition (0) is in contrast with the Poisson Point Process, where the random variable $\Phi \cap B(0,R)$ has a $\mathsf{Poi}(|B(0,R)|)$ distribution, hence the limit in the LHS is equal to its intensity $\lambda > 0$. Hyperuniform point processes can be locally disordered, but at larger scales they show better organization.

A very short history of hyperuniformity

Processes with reduced density fluctuations have been known for long: see for instance the Martin and Yalcin paper from 1980 on Coulomb gases, [57], or Lebowitz' paper [54]. Several papers in the 2000s ([35, 22]) identified hyperuniformity in several physical or statistical models.

However, the seminal paper [68] by Stillinger and Torquato (2003) seems to be the starting point of the field. Since then, Torquato and his team published quite a number of various paper exploring hyperuniformity and its properties in many directions; some of them will be quoted in this text, but we refer to the survey [67] from 2018, gathering the most significant physics papers in the field. Hyperuniform systems are now popular in the materials science community and also in the physics of particle systems.

Only recently (say, 2016) did the mathematics community become seriously interested in hyperuniformity *per se*, when the pioneering papers of Ghosh and Lebowitz ([27, 31]) and their short survey ([30]) opened the path to further investigations ([1]). A key aspect of their work is the link between hyperuniformity and rigidity, a much more studied phenomenon. Recently, Chatterjee proved hyperuniformity for hierarchical Coulomb gas ([15]), and hyperuniformity has been studied in matchings of point processes by Klatt, Last and Yogeshwaran ([48]).

Contents of this survey

The general, usual preliminaries on Palm measures, correlations, Fourier analysis, etc. are contained in the Appendices at the end of the survey. Section 1 proves the elementary formulas for the variance of linear statistics of point processes ; then in Section 2, we discuss the basic definition of hyperuniformity (especially with regards to window shape dependence). Integrable examples (notably determinantal PP) are shortly discussed in Section 3. Section 4 studies some consequences of hyperuniformity regarding number fluctuations, including the CLT. Then, Section 6 explores the links between rigidity phenomena and hyperuniformity. We prove that hyperuniformity and suitable decay of correlations imply rigidity in $d = 1, 2$ and we prove the extreme rigidity properties of stealthy hyperuniform processes (no bounded holes, anticoncentration, maximal rigidity), following Ghosh-Lebowitz. Section 9 is only on a very interesting kind of processes, perturbed lattices.

1. Variance formulas

A configuration φ is a discrete subset of $\mathbb{R}^d$. The set of all configurations, $\mathrm{Conf}(\mathbb{R}^d)$, is endowed with the σ-algebra generated by counting functions $\varphi \mapsto \#(\varphi \cap X)$, where X is a Borel subset of $\mathbb{R}^d$ and $\#(\cdot)$ is the cardinality. A point process Φ is simply a random variable on this measurable space. From now on, we will restrict our attention to translation-invariant point processes Φ, those for which Φ and $\Phi - x$ have the same distribution. The letter λ will always denote the intensity parameter: translation invariance ensures that for any measurable set A,

$$\mathbf{E}[\#(\Phi \cap A)] = \lambda |A|$$

where $|\cdot|$ is the d-dimensional Lebesgue measure.

1.1. Two-point correlations for linear functionals

The most basic object for studying the correlations and variance between two points in a point process Φ is the two-point correlation measure[1] $\rho^{(2)}$, a measure on $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y) \rho^{(2)}(\mathrm{d}x, \mathrm{d}y) := \mathbf{E}\left[\sum_{x,y \in \Phi}^{\neq} f(x,y) \right].$$

From now on, we will assume that this measure on $\mathbb{R}^d \times \mathbb{R}^d$ is locally finite. For any two measurable functions f, g, we have

$$\mathrm{Cov}\left(\sum_{x \in \Phi} f(x), \sum_{x \in \Phi} g(x) \right) = \lambda \int f(x) g(x) \mathrm{d}x + \iint f(x) g(y) \rho^{(2)}(\mathrm{d}x, \mathrm{d}y) - \lambda^2 \iint f(x) f(y) \mathrm{d}x \mathrm{d}y \qquad (1.1.1)$$

provided that each of these expressions is meaningfull; if f, g are compatcly supported functions or if they belong to the Schwartz class, it is the case. In a Poisson point process, it is known that $\rho^{(2)} = \mathrm{d}y \otimes \mathrm{d}x$, so that in this case the integral above is simply given by

$$\mathrm{Cov}\left(\sum_{x \in \Phi} f(x), \sum_{x \in \Phi} g(x) \right) = \lambda \int f(x) g(x) \mathrm{d}x.$$

For translation-invariant point processes, the two-point correlation $\rho^{(2)}$ essentially depends on the difference $x - y$ and should thus have the form '$g(x-y)\mathrm{d}y\mathrm{d}x$'. To make this rigorous, we use the Palm distribution $\mathbf{P}^\circ$, which is the unique probability distribution on $\mathrm{Conf}(\mathbb{R}^d)$ defined by the formula

$$\mathbf{P}^\circ(A) = \frac{1}{|B(0,1)|} \mathbf{E}\left[\int_{B(0,1)} \mathbf{1}_{\Phi - x \in A} \mathrm{d}x \right] \qquad (1.1.2)$$

where A is a measurable subset of $\mathrm{Conf}(\mathbb{R}^d)$. The mathematical expectation associated with the probability $\mathbf{P}^\circ$ will be noted $\mathbf{E}^\circ$. Under $\mathbf{P}^\circ$, the point process Φ almost surely has a point located at the origin, ie $\mathbf{P}^\circ(0 \in \Phi) = 1$.

The pair-correlation measure g_2 is the positive, locally finite measure on $\mathbb{R}^d$ defined by

$$g_2(X) = \frac{1}{\lambda} \mathbf{E}^\circ\left[\sum_{x \in \Phi \setminus \{0\}} \mathbf{1}_{x \in X} \right] \qquad (X \subset \mathbb{R}^d). \qquad (1.1.3)$$

Intuitively, $g_2(\mathrm{d}x)$ is the probability that Φ has a point at location $\mathrm{d}x$, given that there is a point of Φ at the origin. Thus, the probability that two points of Φ are at locations $\mathrm{d}x, \mathrm{d}y$ should be $g_2(\mathrm{d}x - \mathrm{d}y)\mathrm{d}x$, ie the

[1] See Appendix A for basic definitions on Point Processes.

probability of having a point at $\mathrm{d}x$ times the probability of having another point at $\mathrm{d}x + (\mathrm{d}x - \mathrm{d}y)$ given that there is one at $\mathrm{d}x$. The correlation measure is linked with the Ripley K-function: $K(r) = g_2(B(0,r))$. For isotropic point processes (rotation-invariant), g_2 is itself rotation-invariant: if in addition it has a density with respect to the Lebesgue measure, it has the form $\gamma(r)r^{d-1}\mathrm{d}r$ for some $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$.

Finally, we define the 'correlation measure' $\mathcal{C}$ on $\mathbb{R}^d$ as

$$\mathcal{C} = \delta_0 + \lambda[g_2 - \mathbf{1}]. \tag{1.1.4}$$

Here and in all the survey, $\mathbf{1}$ both means the constant function equal to 1, ie $\mathbf{1}(x) = 1$, and thus at the level of distributions $\mathbf{1}$ is the Lebesgue measure, because $\langle \mathbf{1}, f\rangle = \int \mathbf{1}(x)f(x)\mathrm{d}x = \int f(x)\mathrm{d}x$. For the Lebesgue measure of a set I will stick to $|X|$.

Since this is a difference between possibly infinite measures, this is not well-defined, but since all the measures at stake are locally finite, it is well-defined on bounded Borel sets. More precisely, we have

$$\mathcal{C}(X) = \delta_0(X) + \lambda g_2(X) - \lambda|X| \qquad (X \subset \mathbb{R}^d).$$

This is a measure of the repulsions in the point process. The Dirac mass δ_0 accounts for the point in itself and the $-\lambda$ times the Lebesgue measure is the mean intensity if the process was totally chaotic (Poisson). Positivity of $g_2 - |\cdot|$ indicates an attractive zone: if $g_2(\mathrm{d}x) > \lambda$, then given the presence of a point at x_0, there is a 'higher tendency' to have a point at $x_0 + \mathrm{d}x$. Similarly, negative $g_2(\mathrm{d}x) - \lambda$ indicates repulsion. In the physics litterature, the difference $g_2 - \mathbf{1}$ is often called the *total correlation function* and noted h, so that $\mathcal{C} = \delta_0 + \lambda h$.

For a Poisson point process, we saw that g_2 is nothing but the Lebesgue measure, hence $\mathcal{C} = \delta_0$: distinct points are perfectly uncorrelated, meaning that the location of one point at $\mathrm{d}x$ bears no information on the location of the other points. At the other extreme, we can consider a stationarized grid, for simplicity $\Phi = U + \mathbb{Z}^d$ where U is uniform over the elementary cell $[-1/2, 1/2]^d$. The intensity is 1 and $g_2 = \lambda^{-1}\sum_{x\in\mathbb{Z}^d, x\neq 0}\delta_x$, hence in this case

$$\mathcal{C} = -\mathbf{1} + \sum_{x\in\mathbb{Z}^d}\delta_x.$$

Theorem 1.1 (covariance measure)**.** *Let Φ be a stationary point process with intensity λ on $\mathbb{R}^d$. Then, for any functions f, g,*

$$\mathrm{Cov}\left(\sum_{x\in\Phi} f(x), \sum_{x\in\Phi} g(x)\right) = \lambda\int f \circledast g(x)\mathcal{C}(\mathrm{d}x) \tag{1.1.5}$$

where $f \circledast g(x) = \int f(y)g(x+y)\mathrm{d}y$ is the tilted convolution.

Proof. We start from (1.1.1). From the definition of the tilted convolution, the first of the three terms is $\lambda\int f(x)g(x)\mathrm{d}x = \lambda f \circledast g(0)$, and this is nothing but $\lambda\int f\circledast g(x)\delta_0(\mathrm{d}x)$. Now for the second term, we make use of the Campbell formula (we refer to the appendix for the properties of Palm distributions). This formula says that for any measurable function $F : \mathbb{R}^d \times \mathrm{Conf}(\mathbb{R}^d)$, one has

$$\mathbf{E}\left[\sum_{x\in\Phi} f(x, \Phi - x)\right] = \lambda\int_{\mathbb{R}^d}\mathbf{E}^\circ[f(x,\Phi)]\mathrm{d}x. \tag{1.1.6}$$

We apply this to the function F defined by $F(x,\varphi) = \sum_{y\in\varphi\setminus\{0\}} f(x)g(x+y)$. Clearly,

$$\begin{aligned}\sum_{x\in\Phi} F(x, \Phi - x) &= \sum_{x\in\Phi}\sum_{y\in\Phi - x, y\neq 0} f(x)g(x+y)\\ &= \sum_{x,y\in\Phi}^{\neq} f(x)g(y).\end{aligned}$$

Consequently, when we integrate, we can use Campbell's formula and then swap the integral and the expectation (positive Fubini), so that

$$\begin{aligned}\mathbf{E}\left[\sum_{x\in\Phi} F(x,\Phi-x)\right] &= \lambda\int_{\mathbb{R}^d}\mathbf{E}^\circ\left[\sum_{y\in\Phi\setminus\{0\}} f(x)g(x+y)\right]\mathrm{d}x\\ &= \lambda\mathbf{E}^\circ\left[\sum_{y\in\Phi\setminus\{0\}} f\circledast g(y)\right]\\ &= \lambda^2\int f\circledast g(y)g_2(\mathrm{d}y).\end{aligned}$$

This settles the second term in (1.1.1). The last term is easily found to be equal to $\lambda^2\int f\circledast g(x)\mathrm{d}x$. In the end, we get

$$\mathrm{Cov}\left(\sum_{x\in\Phi} f(x),\sum_{x\in\Phi} g(x)\right) = \lambda\int f\circledast g(x)\delta_0(\mathrm{d}x) + \lambda^2\int f\circledast g(x)g_2(\mathrm{d}x) - \lambda^2\int f\circledast g(x)\mathrm{d}x$$

which is our formula. □

See also [14] for a far-reaching treatment on this kind of asymptotics.

Remark 1.2. Formula (1.1.5) entails that $\int f\circledast f(x)\mathcal{C}(\mathrm{d}x)\geqslant 0$ for every measurable function f, thus the tempered measure $\mathcal{C}$ belongs to the set of positive-definite measures. This is not obvious, since in general $\mathcal{C}$ is not a positive measure itself.

1.2. Scaled intersection formulae

Apply the identity (1.1.5) to $f=g=\mathbf{1}_D$ for some measurable bounded set D (typically, D is the ball of radius R around the origin): one obtains the formula

$$\mathrm{Var}(|\Phi\cap D|) = \lambda\int_{\mathbb{R}^d}\alpha(x,D)\mathcal{C}(\mathrm{d}x) \tag{1.2.1}$$

where α is the *intersection volume*; formally, $\alpha(x,D)=\mathbf{1}_D\circledast\mathbf{1}_D(x)=|D\cap(x+D)|$ is the volume of the intersection between D and $x+D$, a quantity which can efficiently be computed when D is a simple shape such as an ellipsoid. The seminal paper [68] contains many examples of such computations.

We often like normalizing this expression with the volume of D and developping the measure $\mathcal{C}$; formula (1.2.1) will often be encountered in the litterature as

$$\frac{\mathrm{Var}(|\Phi\cap D|)}{|D|} = \lambda\left(\int_{\mathbb{R}^d}\alpha_2(x,D)\mathcal{C}(\mathrm{d}x)\right) \tag{1.2.2}$$

where $\alpha_2(x,D):=\alpha(x,D)/|D|$ denotes the *scaled* intersection volume. We will often use such formulas when $D=B(0,r)$ and we will use the letter σ to denote the scaled variance up to a constant, within a ball of radius r:

$$\sigma(r) = \frac{\mathrm{Var}(|\Phi\cap B(0,r)|)}{r^d}. \tag{1.2.3}$$

The asymptotic behaviour of this function (and consequences) is the main concern of this survey.

Example 1.3. In a Poisson point process, this formula says that $\mathrm{Var}(|\Phi\cap D|)=\lambda\alpha(0,D)=\lambda|D|$. One can also see this from the fact that the random variable $|\Phi\cap D|$ has $\mathsf{Poi}(\lambda|D|)$ distribution.

Let us now fix some function f.

1.3. Structure factor and the phase-space variance formula

Shifting from the state space to the Fourier space will prove extremely useful for studying hyperuniformity. Here, to avoir technicalities, we take the Fourier transform in the sense of tempered distributions. Since $\mathscr{F}\mathcal{C} = \mathscr{F}[\delta_0 + \lambda g_2 - \lambda \mathbf{1}]$ and since $\mathscr{F}\delta_0 = \mathbf{1}$, the structure factor can be written nicely in terms of the pair correlation.

Definition 1.4. The *structure measure* $\mathcal{S}$ of a stationary point process Φ is the Fourier transform of its correlation measure:

$$\mathcal{S} = \mathbf{1} + \lambda \mathscr{F}\left[g_2 - \mathbf{1}\right]. \tag{1.3.1}$$

In the physics litterature, one encounters this object under the name of *structure factor* or *structure function*. Remember that $\mathcal{C}$ is itself a signed positive-definite measure (see remark (1.2) Appendix B.4 at page 58). A consequence of Bochner's theorem is that $\mathcal{S}$ itself is a signed positive-definite *measure*, and not an abstract distribution, but mathematically it is not, in general, a function. Another crucial information is that, even if $\mathcal{S}$ is a priori a signed measure, it is indeed a *positive* measure as shown by the following theorem.

Theorem 1.5 (fundamental formulas)**.** *Let f, g be functions. Let Φ be a stationary point process on $\mathbb{R}^d$ with intensity λ and structure factor $\mathcal{S}$. Then,*

$$\mathrm{Cov}\left(\sum_{x\in\Phi} f(x), \sum_{x\in\Phi} g(x)\right) = \frac{\lambda}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(k)\overline{\hat{g}(k)}\mathcal{S}(\mathrm{d}k) \tag{1.3.2}$$

and

$$\mathrm{Var}\left(\sum_{x\in\Phi} f(x)\right) = \frac{\lambda}{(2\pi)^d}\int_{\mathbb{R}^d} |\hat{f}(k)|^2\mathcal{S}(\mathrm{d}k). \tag{$\hat{\star}$}$$

Proof. Combine (1.1.5) and the reciprocical Fourier relation (B.4.2). □

The functions f, g for which equations (1.3.2)-($\hat{\star}$) include all the real functions for which the direct-space formula (1.1.5) holds, and for which the reciprocical Fourier relation (B.4.2) holds. This includes a minima all compactly supported functions, all Schwartz functions, all non-smooth functions with sufficiently fast decay.

For a Poisson point process, $\mathcal{C} = \delta_0$, hence $\mathcal{S} = \mathbf{1}$. In general, since $\mathcal{S} = \mathbf{1} + \lambda\mathscr{F}[g_2 - \mathbf{1}]$, the behaviour of $\mathcal{S}$ depends on how g_2 fluctuates around $\mathbf{1}$. It can be proved (see [**?**]) that $g_2(B(0,r)) \sim |B(0,r)|$ when $r \to \infty$. If g_2 is has a smooth density with respect to the Lebesgue measure, then one can expect that $\mathscr{F}[g_2 - \mathbf{1}](k)$ vanishes at large wavevectors $|k| \to \infty$, so that the large-wavelength behavior of the structure factor is essentially $\lim_{|k|\to\infty} \mathcal{S}(k) = 1$.

Remark 1.6 (the structure kernel). Any measurable function f such that $|\hat{f}|^2 \in \ker(\mathcal{S})$ has zero variance, hence is deterministic: $\mathbf{P}$-almost surely one has

$$\sum_{x\in\Phi} f(x) = \lambda\int_{\mathbb{R}^d} f(x)\mathrm{d}x.$$

This fundamental property will be exploited many times ahead, and explains why the structure of the kernel of $\mathcal{S}$ is of paramount importance in the study of rigidities of point processes.

1.4. Number variance for simple shapes: Bessel and Sine kernels

In view of using the phase-space formula ($\hat{\star}$) applied to $f = \mathbf{1}_D$ with D some window, it will prove useful to gather here a few Fourier transforms of classical shapes such as balls and boxes.

Balls and Bessel

Let $B = B(0,r)$ be the euclidean ball of radius r. In order to express the number variance as in (0) we recall the Fourier transform of $\mathbf{1}_B$:

$$\widehat{\mathbf{1}_{B(0,r)}}(\xi) = r^{d/2}\frac{\mathrm{J}_{d/2}(r|\xi|)}{|\xi|^{d/2}}. \tag{1.4.1}$$

We also recall that the volume of $B(0,r)$ in a d-dimensional real space is r^d times the volume of the unit ball $B(0,1)$, given by

$$\kappa_d := \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}. \tag{1.4.2}$$

If Φ is a stationary point process with intensity λ, formula ($\hat{\star}$) and formula (1.4.1) shows the following fundamental representation (see for instance eq.(58) in [67]):

$$\frac{\mathrm{Var}(\Phi(B(0,r)))}{|B(0,r)|} = \frac{\lambda}{\kappa_d(2\pi)^d}\int_{\mathbb{R}^d}\frac{\mathrm{J}_{d/2}(|r\xi|)^2}{|\xi|^d}\mathcal{S}(\mathrm{d}\xi). \tag{1.4.3}$$

Equivalently, we introduce $\mathscr{J}(x) = (\kappa_d(2\pi)^d)^{-1}\mathrm{J}_{d/2}(|x|)^2/|x|^d$. This is a smooth function except at zero, and it is nonnegative and radial. They are plotted for small values of the dimension d in Figure 1.

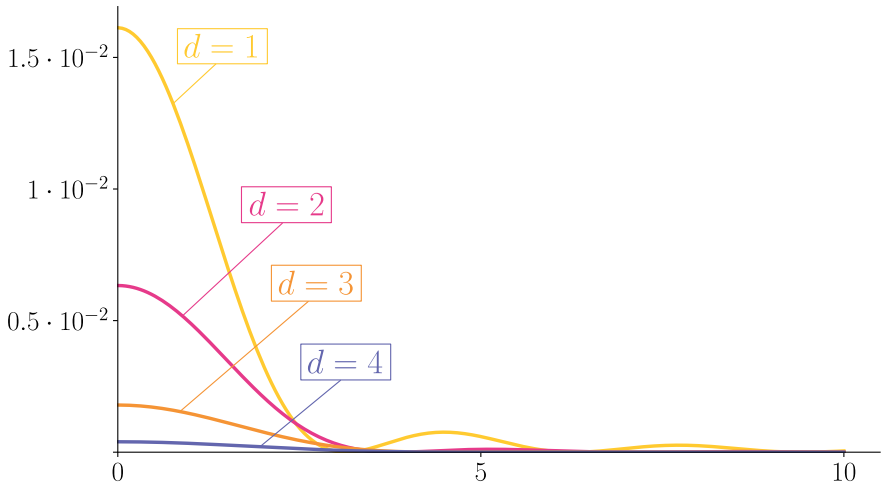

Figure 1: Plots of the (radial part of) the Bessel kernel $\mathscr{J} : t \mapsto (\kappa_d(2\pi)^d)^{-1}\mathrm{J}_{d/2}(t)^2/t^d$ for several values of the dimension d. Classical Bessel asymptotics confirm that $\mathscr{J}(t) \to 0$ as $d \to \infty$.

Parseval's identity ensures that $\int \mathscr{J}(x)\mathrm{d}x = 1$. The family of functions $\mathscr{J}_\varepsilon(x) := \varepsilon^{-d}\mathscr{J}(x/\varepsilon)$ thus forms an approximation of unity[2], which I will call the *Bessel approximations*, and we have

$$\frac{\mathrm{Var}(|\Phi \cap B(0,r)|)}{|B(0,r)|} = \lambda\int_{\mathbb{R}^d}\mathscr{J}_{1/r}(\xi)\mathcal{S}(\mathrm{d}\xi) \tag{1.4.4}$$

which goes to 'the value of $\mathcal{S}$ at zero' under some assumptions on $\mathcal{S}$ which will be described later.

Boxes and Sines

Let $C = B_\infty(0,r) = [-r,r]^d$ be the ℓ^∞-ball of radius r. We recall the Fourier transform of $\mathbf{1}_C$:

$$\widehat{\mathbf{1}_{B_\infty(0,r)}}(\xi) = 2^d\prod_{i=1}^{d}\frac{\sin(r\xi_i)}{\xi_i}. \tag{1.4.5}$$

[2]But its support is not bounded.

If Φ is a stationary point process with intensity λ, formula ($\hat{\star}$) and formula (1.4.5) show the following fundamental representation:

$$\frac{\mathrm{Var}(\Phi(B_\infty(0,r)))}{|B_\infty(0,r)|} = \frac{\lambda}{(2\pi)^d} \int_{\mathbb{R}^d} \prod_{i=1}^{d} \frac{\sin(r\xi_i)^2}{r\xi_i^2} \mathfrak{S}(\mathrm{d}\xi) \tag{1.4.6}$$

where we used $|B_\infty(0,r)| = (2r)^d$.

2. Hyperuniformity: definitions and characterizations

The initial, intuitive and natural definition for hyperuniformity happens to be (0):

$$\lim_{r\to\infty} \frac{\mathrm{Var}(\#(\Phi \cap B(0,r))}{|B(0,r)|} = 0.$$

However, there is a choice hidden in this definition: why choosing euclidean balls? Is the definition equivalent if we choose, say, balls in the ℓ^p topology, or windows with different shapes such as ovals or rectangles or other shapes, as in Figure 2?

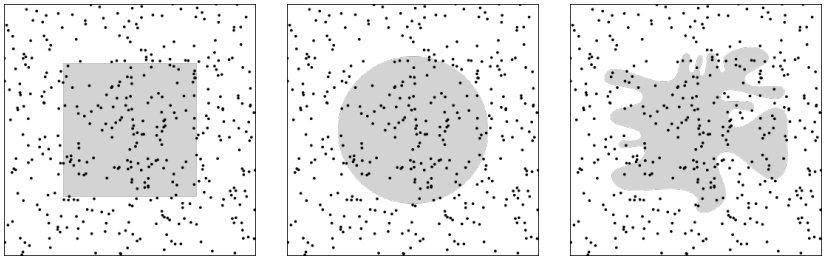

Figure 2: Does Definition (0) depend on the shape of the window in which we count the points?

This section clarifies the definition, introduces the phase-space characterization of hyperuniformity and the basic definitions linked with it: order metrics, hyperuniforms systems classification, quadratic variance formula.

2.1. On window shape dependence

The influence of window shape on number fluctuation is better seen with grids. Let $\Phi = U + \mathbb{Z}^2$ be the 2-dimensional stationarized grid. The number of points in the euclidean ball $B(0,r)$ will be noted $N(r)$, and the number of points in the ℓ^∞-ball $[-r,r]^d$ will be noted $C(r)$. We will prove that the limit in (0) is zero for $N(r)$, but not for $C(r)$.

Let us start with $N(r)$. In general, the task of counting integer points in euclidean balls is a difficult and old problem, called the *Gauss circle problem*. Indeed, a simple counting argument dating back to Gauss himself asserts that (almost surely),

$$|N(r) - \pi r^2| \leqslant 2 + 2\sqrt{2}r = O(r). \tag{2.1.1}$$

Proof. Consider the the union of every unit square, whose center is a point in $\Phi \cap B(0,r)$. This set, say V_r, has volume exactly equal to $N(r) \times 1 = N(r)$. However, it is easily seen that V_r contains the ball $B(0, r-\sqrt{2})$, and is itself contained in the ball $B(0, r+\sqrt{2})$, so a volume comparison immediately shows that $\pi(r-\sqrt{2})^2 \leqslant N(r) \leqslant \pi(r+\sqrt{2})^2$. Developping the squares yields the result. □

However, such a bound is very crude and only yields $\mathrm{Var}(N(r)) = O(\pi r^2)$. Fortunately, the bound (2.1.1) has been considerably sharpened. Finding an optimal bound is still unsolved and it is conjectured that the optimal error is $O(r^{1/2+\delta})$ for any $\delta > 0$; at the moment, there are many results showing that $|N(r) - \pi r^2| = o(r)$, which ensures that

$$\lim_{r\to\infty} \frac{\mathrm{Var}(N(r))}{|B(0,r)|} = 0.$$

This matches definition (0).

Let us now consider the square, $C(r) = \#(\mathbb{Z}^2 \cap [-r,r]^2)$. Since Φ is stationary, $C(r)$ has the same distribution as $D(r) = \#(\mathbb{Z}^2 \cap [0,2r]^2)$ which makes computations slightly easier, since if we note $U = (u,v)$ then

$$D(r) = \lceil 2r - u \rceil \lceil 2r - v \rceil.$$

This is equal to $\lceil 2r \rceil^2$ if both u and v are smaller than $p = \{2r\}$ (the fractional part), which happens with probability p^2. Similarly, it is equal to $\lceil 2r \rceil \lfloor 2r \rfloor$ with probability pq where $q = 1-p$, and it is equal to $\lfloor 2r \rfloor^2$ with probability q^2. The variance is thus given by

$$\begin{aligned}
\mathrm{Var}(C(r)) = \mathrm{Var}(D(r)) &= (p^2 \lceil 2r \rceil^4 + 2pq \lceil 2r \rceil^2 \lfloor 2r \rfloor^2 + q^2 \lfloor 2r \rfloor^4) - 16r^4 \\
&= (p \lceil 2r \rceil^2 + q \lfloor 2r \rfloor^2)^2 - 16r^4 \\
&= (p(2r+q)^2 + q(2r-p)^2)^2 - 16r^4 \\
&= (4r^2 + pq)^2 - 16r^4 \\
&= 4r^2 \left(2pq + \frac{p^2 q^2}{4r^2} \right).
\end{aligned}$$

We thus get the formula

$$\frac{\mathrm{Var}(\#(\Phi \cap [-r,r]^2))}{|[-r,r]^2|} = (1 + o(1)) 2\{2r\}(1 - \{2r\}). \tag{2.1.2}$$

The limit is oscilatory when $r \to \infty$, taking values from 0 to 1/2. For instance, taking $r = n + 1/4$ yields a limit equal to 1/2 above, which is notoriously nonzero; but this might seem counter-intuitive, since the square of side length $2n + 1/2$ is itself a totally honest convex window with self-similar growth, but the fluctuations of number density in this window is the same order as Poisson.

Actually, this example can be pushed slightly further: we can tilt the squared window by some angle θ. When $\tan(\theta)$ is a rational number, a resonnance between the Fourier transform of the indicator of the tilted square and the Fourier transform of the lattice happens, as beautifully described in [41], causing nontrivial fluctuations.

The dependence on (0) with respect to window shape hopefully vanishes under some general hypotheses discussed in the following section.

2.2. Integrability of the covariance measure

If a point process satisfies (0), then by taking $D_n = B(0,n)$ in the scaled-intersection formula (1.2.2), one has

$$0 = \lim_{n \to \infty} \lambda \int \alpha_2(x, D_n) \mathcal{C}(\mathrm{d}v).$$

The smooth functions $x \mapsto \alpha_2(x, D_n)$ converge pointwise towards the constant function 1, and we would merrily pass the limit inside the former integral and conclude that

$$\int_{\mathbb{R}^d} 1 \mathcal{C}(\mathrm{d}v) = \mathcal{C}(\mathbb{R}^d) = 0. \tag{2.2.1}$$

Equivalently, if the pair-correlation measure has a density $g_2(\mathrm{d}x) = g_2(x)\mathrm{d}x$, the former identity could be written

$$0 = 1 + \int_{\mathbb{R}^d} [g_2(x) - 1] \mathrm{d}x.$$

Such identities are essentially true and they are often seen in papers, but they carry difficulties within. When $\mathcal{C}$ is a difference between two measures, one cannot pass the limit inside the integral and this is not just a technical detail: the preceding section shows that one can find totally honest windows shapes D_n for which the limit is equal to 0, and some others for which the limit is nonzero. If one wants to rigorously use (2.2.1), one has two choices:

1. either choose a specific limiting sequence D_n growing to $\mathbb{R}^d$,

2. or make some hypothesis on $\mathcal{C}$ which will legally allow to write (2.2.1).

I will choose the second road, but most of the incoming results might also be proved in a more general way using the first one. A simple and versatile hypothesis is the following. By signed measure, I mean: a difference between two measures, $\mu = \mu_+ - \mu_-$, such that one of them is finie. This is to avoid $\infty - \infty$ when evaluating, for instance, $\mu(\mathbb{R}^d)$.

Hypothesis 2.1. The covariance measure $\mathcal{C}$ is a signed measure, or equivalently the total correlation $h = g_2 - \mathbf{1}$ is a signed measure.

By Hahn's theorem, this means that there are two measures $\mathcal{C}_+, \mathcal{C}_-$, the second one finite, such that $\mathcal{C} = \mathcal{C}_+ - \mathcal{C}_-$. This happens for instance when $\mathcal{C}$ has a bounded density $\mathscr{c}$ with respect to the Lebesgue measure, and $\mathscr{c}$ only takes negative values in a compact set. This will in general be the case for some determinantal point processes. The most blatant limitation of this hypothesis (but also, the only one I know) is that the class of lattices and perturbed lattices does not satisfy (2.1) — but they will have their own section in thus survey.

When $\mathcal{C}$ satisfies Hypothesis (2.1), one can pass to the limit inside both $\mathcal{C}_+$ and $\mathcal{C}_-$, and in this case, for any growing window shape D_n,

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathrm{Var}(|\Phi \cap D_n|)}{|D_n|} &= \lim_{n\to\infty} \int \alpha_2(x, D_n)\mathcal{C}_+(\mathrm{d}v) - \int \alpha_2(x, D_n)\mathcal{C}_-(\mathrm{d}v) \\ &= \mathcal{C}_+(\mathbb{R}^d) - \mathcal{C}_-(\mathbb{R}^d) \in (-\infty, +\infty].\end{aligned}$$

Then, (0) implies that $\mathcal{C}(\mathbb{R}^d) = 0$ whatever the shapes for D_n, provided they are nested and grow to $\mathbb{R}^d$.

Note that when Hypothesis (2.1) is fulfiled, then the Fourier transform of $\mathcal{C}$ is actually absolutely continuous with some continuous density that will be noted $\mathscr{z}$, and furthermore one has

$$|\mathscr{z}(\xi)| = \left| \int \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle} \mathcal{C}(\mathrm{d}x) \right| \leqslant |\mathcal{C}|(\mathbb{R}^d) < \infty. \tag{2.2.2}$$

Consequently, under (2.1), the structure measure assumes a particularly pleasant form: it is a bounded continuous function. We gather all the results in this section in the following.

Proposition 2.2. *Let Φ be a stationary point process on $\mathbb{R}^d$, satisfying Hypothesis (2.1). Then, the following are equivalent:*

1. $\mathcal{S}(0) = 0$.
2. $\mathcal{C}(\mathbb{R}^d) = 0$.
3. Φ *satisfies* (0).
4. Φ *satisfies* (0) *for any sequence of nested, connected, convex open sets D_n such that $D_n \nearrow \mathbb{R}^d$.*

2.3. General hyperuniformity

For any integrable function f, we define the normalized linear statistics variance of Φ at f by

$$\sigma(f) := \frac{\mathrm{Var}\left(\sum_{x\in\Phi} f(x)\right)}{\int_{\mathbb{R}^d} f(x)\mathrm{d}x}. \tag{2.3.1}$$

Definition 2.3 (hyperuniform statistics)**.** Let f be a measurable function. A stationary point process Φ is said to be hyperuniform with respect to f if there is a sequence of Schwartz functions $f_n \in \mathscr{S}(\mathbb{R}^d)$ such that

1. $f_n \to f$ pointwise,

2. $\sigma(f_n) \to 0$.

We will often say that f is a *hyperuniform statistic* for Φ, and the set of hyperuniform statistics is going to be noted $\mathrm{Hyp}(\Phi)$. The sequence (f_n) will be called an approximation sequence. Usually, we want the constant function 1 to be a hyperuniform statistic for Φ, which corresponds to the classical notion of number-hyperuniformity. Of course, my definition does not encompass the preliminary definition (0), but one can easily prove that (0) implies (1)-(2) above, by taking f_n to be smooth functions taking the value 1 on $B(0,n)$ and zero outside $B(0,n+0.1)$.

Example 2.4. An important example is when $f(x) = |x|^k$ for some $k > 0$. I will refer to this as k-hyperuniformity.

2.4. Stealthy hyperuniform processes

For most point processes, we just saw that hyperuniformity is equivalent to "$\mathcal{S}(k \to 0) = 0$". We could go further, and ask for $\mathcal{S}$ to be zero in a neighborhood of zero.

Definition 2.5. A stationary point process Φ is called *stealthy hyperuniform* (SH for short) if there is an open set U containing the origin, such that for any f which is supported inside U, then $\langle \mathcal{S}, f \rangle = 0$.

The typical example is the stationary grid $U + \mathbb{Z}^d$, or more generally any stationarize lattice $U + \mathbb{L}$ where U is uniform on the fundamental cell of $\mathbb{L}$ (see Subsection 3.1). The *spectral gap* of a SH process is the largest r such that $\mathcal{S}$ vanishes on $B(0,r)$. For instance the spectral gap of stationarized grids is 1. Such processes were notably studied, from a physicist point of view, in [73] and [69], and more recently in [31]. In Subsection 7 we prove that they have some extreme rigidity properties.

Remark 2.6 (on the lack of examples). I do not know any examples of stealthy hyperuniform processes besides grids and refinements of grids. In particular, I do not know any 'amorphous' process (locally disordered) which is stealthy. It is also interesting to see that constructing random measures which are stealthy is quite trivial: for instance, the Bochner-Khinchine theorem says that every positive measure m corresponds to a Gaussian field with structure factor m, so stealthy gaussian random fields can very easily be constructed. But to my knowledge, there is no condition for a measure to be the structure factor of a point process: for that matter, this question is asked in Gabrielli [23] and reiterated in [68]. However, it is believed that in two dimensions, there are some 3-valued processes (ie random measures Φ with values in $\{0,1,2\}$ instead of $\{0,1\}$ as for classical point processes) which are stealthy hyperuniform (Subhro Ghosh, private communication).

Remark 2.7. Nontrivially perturbed lattices cannot be hyperuniform: cf Corollary 9.4.

2.5. Stability of hyperuniformity

Let Φ be a hyperuniform point process. What are the operations on Φ that preserve hyperuniformity and those who do not? I gather here a few examples of both. Obviously, any deterministic linear transformation preserves hyperuniformity.

Example 2.8 (random rescaling preserves HU)**.** We continue Example **??**. In many examples, the function σ is bounded; this is for instance the case when Φ is a stationarized grid (one will easily check that $\sigma(r) \to 1$ when $r \to 0^+$). Consequently, if Φ is hyperuniform and σ is bounded, then the dominated convergence theorem shows that the LHS of (**??**) goes to zero as $r \to \infty$ and the rescaled process Ψ is also hyperuniform.

Example 2.9 (rotations preserve HU)**.** This is obvious, since if U is unitary then $|U\Phi \cap B(0,r)| = |\Phi \cap B(0,r)|$. However, this statement is heavily window-shape dependent.

Example 2.10 (iid perturbations preserve HU)**.** Consider a stationary point process Φ, then displace each point by a random perturbation. Formally, $\Psi = \{x + V_x : x \in \Phi\}$ were (V_x) is an iid collection of random variables with common distribution V. Then, Ψ is hyperuniform if and only if Φ is hyperuniform. This is proved in Section 9, where we will examine more thoroughly this model.

Example 2.11 (Poisson thinning is not HU)**.** Let Φ be a HU process. We thin it independentaly: to do this we simply have a collection $(\xi_x)_{x\in\Phi}$ of iid random Bernoulli variables $\mathrm{Ber}(p)$ with $p \in]0,1[$, and we set $\Phi_1 = \{x \in \Phi, \xi_x = 1\}$. We fix $r > 0$ and we note $N = |\Phi \cap B(0,r)|$ and $\bar{N} = |\Phi_1 \cap B(0,r)|$. We recall that the variance of a $\mathrm{Bin}(n,p)$ random variable is $np(1-p)$. Consequently,

$$\begin{aligned}\mathbf{E}[|\bar{N}_r - \mathbf{E}[\bar{N}_r]|^2] = \mathbf{E}\left[\,|\bar{N}_r - \mathbf{E}[\bar{N}_r]|^2 \big|\, N_r\right] &= \mathbf{E}\left[\,\mathrm{Var}(\mathrm{Bin}(N_r,p))|\, N_r\right] \\ &= \mathbf{E}[N_r p(1-p)] \\ &= \lambda|B(0,r)|p(1-p)\end{aligned}$$

which proves that Φ_1 is not hyperuniform.

Remark 2.12 (Hyperuniform thinnings). A thinning of Φ is a random subset of $\Phi_1 \subset \Phi$ and we note $\Phi_2 = \Phi \setminus \Phi_1$. If the process Φ is not hyperuniform, then Φ_1 and Φ_2 cannot be both hyperuniform. To see this we do the following simple observation (private communication from Yogeshwaran Dhandapani): if we note $\sigma(r) = \mathrm{Var}(|\Phi\cap B(0,r)|)/r^d$ and $\sigma_k(r) = \mathrm{Var}(|\Phi_k\cap B(0,r)|)/r^d$ where $k \in \{1,2\}$, then by the Cauchy-Schwarz inequality

$$|\sigma(r) - \sigma_1(r) - \sigma_2(r)| = \frac{|\mathrm{Cov}(|\Phi_1 \cap B(0,r)|, |\Phi_2 \cap B(0,r)|)|}{r^d} \leqslant \sqrt{\sigma_1(r)\sigma_2(r)}. \tag{2.5.1}$$

We suppose that σ is bounded at infinity[3]. Then, if Φ_1 is hyperuniform, we have $|\sigma(r) - \sigma_2(r)| \to 0$. Consequently, if Φ is hyperuniform, then Φ_2 is hyperuniform, but if Φ is not then Φ_2 is not. One cannot write a non-hyperuniform process as the union of two hyperuniform processes.

2.6. On repulsions in hyperuniform point processes

Based on a private communication of Günter Last.

In the paper [34], Goldman showed that for a class of determinantal point processes (later widened in [5, 58]), the Palm version of a point process can be coupled with the process itself. Let us suppose that Φ° is a point process, such that its distribution under $\mathbf{P}$ is $\mathbf{P}^\circ$. A point process is said to belong to the Palm-coupled class (PC) if there is another point process, say Ξ, such that

(i) $\Phi \cap \Xi = \varnothing$ almost surely, that is : Φ and Ξ are mutually singular.

(ii) $\Phi = (\Phi^\circ \setminus \{0\}) \cup \Xi$.

In other words, the point process Φ can be obtained by simply sampling the Palm distribution, and then adding the points of Ξ. Equivalently, to get the Palm version from the regular version, one can simply delete the points of Ξ. This is why the points in Ξ are a measure of the repulsiveness of points : they account for the points who have to be removed from the process when we condition on the appearance of a point at the origin. We recall that $\mathcal{C} = \delta_0 + \lambda[g_2 - 1]$.

Theorem 2.13. *Let Φ be a point process belonging to the Palm-coupled class. Then,*

$$\frac{\mathrm{Var}(|\Phi \cap D|)}{|D|} = \lambda\left(1 - \frac{\mathbf{E}[\#(\Xi \cap D)]}{\lambda}\right). \tag{2.6.1}$$

Consequently, it is hyperuniform if and only if $\lim_{D\nearrow\mathbb{R}^n} \mathbf{E}[\#(\Xi \cap D)] = \lambda$.

Proof. From the definition of g_2 and the Palm-coupling above, we have

$$g_2(D) = \frac{1}{\lambda}\mathbf{E}\left[\sum_{x\in\Phi} \mathbf{1}_{x\in D} - \sum_{x\in\Xi} \mathbf{1}_{x\in D}\right] = |D| - \frac{1}{\lambda}\mathbf{E}[\#(\Xi \cap D)]. \tag{2.6.2}$$

[3]Can it be unbounded?

Consequently,

$$\begin{aligned}\frac{\mathrm{Var}(|\Phi \cap D|)}{|D|} &= \lambda \int \alpha(x, D)\mathcal{C}(\mathrm{d}x) \\ &= \lambda + \lambda[|D| - \lambda^{-1}\mathbf{E}[\#(\Xi \cap D)] - |D|] \\ &= \lambda - \mathbf{E}[\#(\Xi \cap D)]\end{aligned}$$

and the theorem is proved. □

A simple consequence of this theorem is that the mean number of 'removed points', namely $\mathbf{E}[\Xi \cap \mathbb{R}^d]$, is smaller than the intensity of the point process Φ; consequently, in the PC-class, hyperuniform point processes are precisely those point processes where the mean number of points of Ξ is maximized.

3. Some important examples

In this section, I'll gather a few examples which more or less fall inside the realm of 'integrable systems'. There is a section on determinantal point processes. I will add sections on permanental point processes and pfaffian point processes. The sine-beta process clearly deserves its own subsection and we will go back at it in the rigidity section, later.

3.1. Lattices

We already took a glance at density fluctuations of grids $U + \mathbb{Z}^d$, but the same arguments apply to lattices. By *lattice*, I will always mean what physicists usually call *Bravais* lattice, ie linear deformations of $\mathbb{Z}^d$: they are the point processes defined by $G^*(\tau\mathbb{Z}^d)$, where $G \in \mathrm{SL}(\mathbb{R}^d)$ is nonsingular and[4] has determinant 1.

If $\mathbb{L}$ is any lattice, then its *dual lattice* (or reciprocical lattice) is classically defined by

$$\mathbb{L}^* = \{y \in \mathbb{R}^d : \forall x \in \mathbb{L}, \langle x, y\rangle \in \mathbb{L}\}. \tag{3.1.1}$$

but it also has the explicit expression $\mathbb{L}^* = G^{-1}\mathbb{Z}^d$. The computation of structure factor over lattices is well-known: for a summary of harmonic analysis on lattices, see Appendix B.5 at page 58. From now on, for any lattice $\mathbb{L}$, I will note $\Delta = \Delta_{\mathbb{L}}$ its measure, ie $\Delta_{\mathbb{L}} = \sum_{x\in\mathbb{L}} \delta_x$. Theorem B.4 therein says that the Fourier transform of a lattice is nothing but its dual lattice:

$$\mathscr{F}\Delta_{\mathbb{L}} = \Delta_{\mathbb{L}^*}. \tag{3.1.2}$$

This is more or less an avatar of Poisson's summation formula.

Since, for a stationarized lattice $\Phi = U + \mathbb{L}$, the pair correlation function is always the lattice with the origin removed $g_2 = \Delta_{\mathbb{L}\setminus\{0\}} = \Delta_{\mathbb{L}} - \delta_0$, we immediately see that the structure factor of Φ is given by

$$\mathcal{S} = \Delta_{\mathbb{L}^*} - \delta_0 \tag{3.1.3}$$

that is, the measure of the dual lattice with the origin removed.

3.2. Determinantal point processes

A determinantal point process (DPP) is a point process whose k-th correlation measure is absolutely continuous with respect to the Lebesgue measure[5], and has density

$$\varrho^{(k)}(x_1, \dots, x_k) = \det[(\mathbb{K}(x_i, x_j))_{i,j\in[k]}] \tag{3.2.1}$$

where $\mathbb{K} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a symmetric function called *kernel*. Such processes were introduced in physics (as Fermion models) by Odile Macchi in the 70s ([56]), and have since then attracted considerable attention. For surveys and overviews on DPPs, we refer to [62, 6] for physical and statistical point of views, and to [37, 55, 66] for complete mathematical investigations.

Proposition 3.1. *A DPP with kernel $\mathbb{K}$ is stationary if and only if*

$$\det[(\mathbb{K}(z + x_i, z + x_j))_{i,j\in[k]}] = \det[(\mathbb{K}(x_i, x_j))_{i,j\in[k]}] \tag{3.2.2}$$

for every integer k and every $z, x_1, \dots, x_k \in \mathbb{R}^d$.

[4] I restrict to $\mathrm{SL}(\mathbb{R}^d)$ for keeping the same intensity.

[5] There are more general settings, but this one will be sufficient here.

This implies that $\lambda = \mathbb{K}(x,x)$ for any x and that there is a function $\Bbbk$ such that $\mathbb{K}(x,y) = \Bbbk(x-y)$. We will from now on suppose that $\Bbbk$ is an L^2 function. A basic consequence of the definition of correlation (A.4.1) in this setting is that the two-point correlation measure $\mu^{(2)}$ is a function: for any $f, g \in \mathscr{S}(\mathbb{R}^d)$, one has

$$\mathbf{E}\left[\sum_{x\neq y} f(x)g(y)\right] = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)g(y)\varrho^{(2)}(x,y)\mathrm{d}x\mathrm{d}y$$

where $\varrho^{(2)}$ is the two-point correlation function:

$$\varrho^{(2)}(x,y) = \begin{vmatrix} \mathbb{K}(x,x) & \mathbb{K}(x,y) \\ \mathbb{K}(y,x) & \mathbb{K}(y,y) \end{vmatrix} = \lambda^2 - \Bbbk(x-y)^2. \tag{3.2.3}$$

This readily implies that $g_2 = 1 - \lambda^{-2}\Bbbk(x)^2\mathrm{d}x$ and the covariance measure is given by $\mathcal{C} = \delta_0 - \lambda^{-1}\Bbbk(x)^2\mathrm{d}x$. The structure function is then given by $\mathbf{1} - \lambda^{-1}\mathscr{F}(\Bbbk^2)$. Upon noting $\mathbb{h}(u) = \Bbbk(u)^2$ we thus see that $\mathcal{S}$ is absolutely continuous with respect to the Lebesgue measure, and its density $\mathfrak{s}$ is

$$\mathfrak{s}(\xi) = 1 - \frac{\hat{\mathbb{h}}(\xi)}{\lambda}. \tag{3.2.4}$$

The value of the density function $\mathfrak{s}$ at zero is exactly $\mathfrak{s}(0) = 1 - \hat{\mathbb{h}}(0)/\lambda = 1 - \lambda^{-1}\int \Bbbk(u)^2\mathrm{d}u$. The following proposition immediately follows.

Proposition 3.2. *A stationary determinantal point process with kernel $\mathbb{K}(x,y) = \Bbbk(x-y)$ where $\Bbbk \in L^2(\mathrm{d}x)$ is hyperuniform if and only if*

$$\mathbb{K}(0,0) = \int_{\mathbb{R}^d} \mathbb{K}(x,0)^2\mathrm{d}x. \tag{3.2.5}$$

Example 3.3. For the Ginibre process, the correlation kernel is $\mathbb{K}(x,y) = \pi^{-1}e^{xy^* - |x|^2/2 - |y|^2/2}$ (where $x, y \in \mathbb{C}$). It is obviously stationary. Some Fourier transform computations show that in this case the Fourier transform has density

$$\mathfrak{s}(\xi) = 1 - e^{-\frac{|\xi|^2}{4}}.$$

The Ginibre process is thus a class-I hyperuniform system. In dimension 2, it exactly gives the thermodynamic limit of the one-component plasma ([38]) at a specific temperature $1/\beta_0$.

Example 3.4 (Sine process)**.** The Sine Process is a DPP on the real line whose kernel is the sinc function, ie $K(x,y) = \mathbb{s}(x-y)$ where

$$\mathbb{s}(x) = \frac{\sin(\pi x)}{\pi x}. \tag{3.2.6}$$

We have $\mathbb{s}(0) = 1$. It is well known that $\mathbb{s}$ is the Fourier transform of the indicator function of $[-0.5, 0.5]$, hence its L2 norm is equal to 1 and this DPP is hyperuniform.

3.3. Pfaffian point processes

Pfaffian point processes generalize determinantal point processes. They arise when diagonalizing gaussian matrix models with real orthogonal matrices ($\beta = 1$) or symplectic matrices ($\beta = 4$). Chapter 6 in Peter Forrester's book Log gases and random matrices ([21]) contains a host of results on Pfaffian processes.

We recall the the Pfaffian of an anti-symmetric $2n \times 2n$ matrix $A = (a_{i,j})$ is defined by

$$\mathrm{Pfa}(A) = \frac{1}{2^n n!}\sum_{\sigma\in\mathfrak{S}_{2n}} \mathrm{sign}(\sigma)\prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

It is thus a polynomial in the entries of the matrix and it satisfies $\mathrm{Pfa}(A)^2 = \det(A)$. See [2, Section 5.3] for elementary results.

A point process is called Pfaffian if its k-correlation measures exist, are absolutely continuous with respect to the Lebesgue measure, and if the densities $\rho^{(k)}$ are given by

$$\rho^{(k)}(x_1,\dots,x_k) = \mathrm{Pfa}\begin{bmatrix} K(x_1,x_1)\mathrm{J} & \cdots & K(x_1,x_k)\mathrm{J} \\ \vdots & & \vdots \\ K(x_n,x_1)\mathrm{J} & \cdots & K(x_k,x_k)\mathrm{J}\end{bmatrix} \tag{3.3.1}$$

where K is a 2×2 kernel, ie $K:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}^{2\times 2}$, and must satisfy the identity

$$[K(x,y)\mathrm{J}]^t = -K(y,x)\mathrm{J} \qquad \text{with } \mathrm{J}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{3.3.2}$$

If we write $K(x,y)=(K_{i,j}(x,y)_{i,j=1,2}$, these constraints can also be written

$$\mathbb{K}_{1,2}(x,y) = -\mathbb{K}_{1,2}(y,x) \qquad \mathbb{K}_{2,2}(x,y)=\mathbb{K}_{1,1}(y,x) \qquad \mathbb{K}_{2,1}(x,y)=-\mathbb{K}_{2,1}(y,x).$$

The intensity function of a Pfaffian point process is thus simply $\varrho^{(1)}(x) = -\mathbb{K}_{1,1}(x,x)$. Just as in Proposition 3.2 for DPPs, a Pfaffian point process is stationary if and only if the RHS of (3.3.1) is invariant when $(x_1,\dots,x_k)$ is translated by $(z,\dots,z)$. This implies that $\mathbb{K}(x,y)=\mathbb{k}(x-y)$ for some $\mathbb{k}:\mathbb{R}^d\to\mathbb{C}^{2\times 2}$, and that $\mathbb{k}_{1,1}$ is a constant function equal to the intensity λ, and so is $\mathbb{k}_{2,2}$. The two-point correlation of a stationary Pfaffian point process with kernel $\mathbb{K}$ is given by

$$\begin{aligned}\varrho^{(2)}(x,y) &= \mathrm{Pfa}\begin{bmatrix} 0 & -\mathbb{K}_{1,1}(x,x) & \mathbb{K}_{1,2}(x,y) & -\mathbb{K}_{1,1}(x,y)\\ * & 0 & \mathbb{K}_{2,2}(x,y) & -\mathbb{K}_{2,1}(x,y)\\ * & * & 0 & -\mathbb{K}_{1,1}(y,y)\\ * & * & * & 0\end{bmatrix}\\ &= \mathbb{K}_{1,1}(x,x)\mathbb{K}_{1,1}(y,y)+\mathbb{K}_{1,2}(x,y)\mathbb{K}_{2,1}(x,y)-\mathbb{K}_{1,1}(x,y)\mathbb{K}_{2,2}(x,y)\\ &= \mathbb{k}_{1,1}(0)^2+\mathbb{k}_{1,2}(x-y)\mathbb{k}_{2,1}(x-y)-\mathbb{k}_{1,1}(x-y)\mathbb{k}_{2,2}(x-y)\\ &= \mathbb{k}_{1,2}(x-y)\mathbb{k}_{2,1}(x-y).\end{aligned}$$

From this, we deduce that the reduced Palm intensity is given by

$$g_2(\mathrm{d}x) = \frac{\mathbb{h}(x)}{\lambda^2}\mathrm{d}x \tag{3.3.3}$$

where $\mathbb{h}(x)=\mathbb{k}_{1,2}(x)\mathbb{k}_{2,1}(x)$. The correlation measure is in this case equal to

$$\mathcal{C}(\mathrm{d}x) = \delta_0 + \lambda\mathbb{h}(x)\mathrm{d}x - \lambda\mathbf{1} = \delta_0 - \frac{\det(\mathbb{k}(x))}{\lambda}\mathrm{d}x.$$

The structure factor is then given by

$$\mathcal{S} = \lambda^{-1}\mathscr{F}\mathcal{C} = \mathbf{1} - \frac{\mathscr{F}(\det\mathbb{k})}{\lambda}$$

Proposition 3.5. *A stationary Pfaffian point process is hyperuniform if and only if*

$$\int_{\mathbb{R}^d}\det\mathbb{k}(x)\mathrm{d}x = \mathbb{k}_{1,1}(0). \tag{3.3.4}$$

Proposition 1.4 in [13] links the hyperuniformity type of the process with its rigidity properties; it is nothing more than Theorem 6.6 thereafter, in a particular case.

Example 3.6 (Pfaffian sine process)**.** The Sine_1 process is the Pfaffian analog of the Sine process in Example 3.4. It is the Pfaffian process whose kernel is given by

$$\mathbb{K}_{\mathsf{Sine}_1}(x,y) = \begin{pmatrix}\mathbb{s}(x-y) & \int_0^{x-y}\mathbb{s}(t)\mathrm{d}t - \frac{\mathrm{sign}(x-y)}{2}\\ \mathbb{s}'(x-y) & \mathbb{s}(x-y)\end{pmatrix}$$

where $\mathbb{s}(u)=\sin(\pi u)/\pi u$ is the sincardinal function. It arises as the bulk scaling limit of the GOE, see [21, Chapter 7]. The Sine_1 process is hyperuniform: we can infer this from Proposition 3.5 and the computation of $\int\det\mathbb{k}$, but it will also be a consequence of Theorem 6.6.

3.4. The sine-beta process

The Sine_β process is a stationary point process on $\mathbb{R}$ defined in [70] and in [40] and meant to generalize Examples 3.4-3.6, the $\beta = 2$ GUE sine process and the $\beta = 1$ GOE sine process; I will not give the 'intrinsic' definition, but only the limiting definition obtained in [70, 40]. The main existence theorem is as follows: we introduce a probability distribution on $\{x_1 < x_2 < \cdots < x_n\}$ by its density

$$\mathrm{P}_{n,\beta}(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n) = \frac{1}{Z_{n,\beta}} \exp\left\{ -\beta \sum_{i=1}^{n} \frac{\lambda_i^2}{4} + \beta \sum_{j<k} \log|\lambda_j - \lambda_k| \right\} \tag{3.4.1}$$

where $Z_{n,\beta}$ is the partition function. It is known that this is the probability distribution of the ordered eigenvalues of a special kind of random matrix $H_{n,\beta}$ whose distribution is often called β-ensemble, a famous result by Dumitriu and Edelman ([19]). When $\beta = 1, 2, 4$, it is also the distribution of eigenvalues of GOE/GUE/GSE ensembles. Seen as point processes on $\mathbb{R}$, they converge, and this can be seen as a definition of Sine_β. More precisely, for every $\beta > 0$ and every fixed real number x, the random point process defined by

$$\Psi_n^x = \sum_{i=1}^{n} \delta_{n(\lambda_i - x)},$$

weakly converges as $n \to \infty$ to a stationary point process on $\mathbb{R}$ only depending on β and called Sine_β:

$$\Psi_n^x \rightsquigarrow \mathsf{Sine}_\beta \tag{3.4.2}$$

here convergence happens with respect to the vague topology for counting measures. There is an explicit representation of Sine_β using the Brownian Carousel in [70], but it is quite difficult to handle; more recently, the authors also found an infinite-dimensional Hermitian operator whose eigenvalues form a Sine_β process.

However, it is a result in [70] that Sine_β is indeed hyperuniform, and more precisely one has

$$\frac{\mathrm{Var}(|\mathsf{Sine}_\beta \cap [0, r]|)}{r} \leqslant \frac{c \log(r)}{r} \tag{3.4.3}$$

for some constant $c = c_\beta$.

Remark 3.7. The existence of a β-analog of the Ginibre process (which would called $\mathsf{Ginibre}_\beta$) is not known and the existence of higher-dimensional analogs seems far out of reach as now. Thomas Leblé knows more about this than me.

3.5. A challenge: the generation of hyperuniform, amorphous patterns

An important question in the materials science community is the efficient generation of large samples of hyperuniform point processes; by large, I mean with a few million ($n = 10^6$) points. It seems that now (April 11, 2021), the only way to do this is to start with fixed hyperuniform structures such as $\mathbb{Z}^d$, and to gently 'shuffle' the lattice by displacing every point randomly. This model, the Independently Perturbed Lattice, is of paramount importance and will occupy us through all Section 9.

However, these 'shuffled lattices' exhibit a very strong underlying structure, emerging from the original lattice $\mathbb{Z}^d$ or whatever lattice is used. Point processes with no such structure ('Bragg peaks') are often called *amorphous*. Hyperuniform DPPs, sine-β processes, Coulomb gases are amorphous, but it is quite difficult to efficiently generate very large samples of them.

At this moment, generating large samples of amorphous HU processes cannot easily be done. The question is explored in [42, 43, 45]. One strategy is to use the preceding 'shuffled lattice' model, and to artificially hide the underlying lattice [45], see Subsection 10.3. Another one is to use tesselation-based displacements, a procedure explored in [42, 43] and [47], but which seems at the moment out of reach from mathematical analysis.

4. Density fluctuations

Let Φ be a hyperuniform point process. If the variance of the points in large balls is negligible compared to the volume, how exactly does it scale? Does the variance behave like the surface area of the window, ie

$$\mathrm{Var}(|\Phi \cap D_n|) \asymp \mathrm{Area}(\partial D_n)?$$

A good description of the variance scaling is available in the physics litterature, and was globally rigorous described in [67] (Section 5.3). Recently, [1] explored these asymptotics for stationary random fields on $\mathbb{Z}^d$, and proved a CLT for hyperuniform systems as well as an insightful 'entropy degeneracy' result. We formulate these results in the setting of random point processes on $\mathbb{R}^d$.

4.1. Hyperuniform systems classification

Let us place ourselves under Hypothesis (2.1), so that the structure function $\mathcal{S}$ is actually an absolutely continuous measure with bounded density $\mathfrak{s}$ with $\mathfrak{s}(0) = 0$. Thanks to the Fourier representation of the variance derived in (1.4.4),

$$\sigma(r) := \frac{\mathrm{Var}(|\Phi \cap B(0,r)|)}{|B(0,r)|} = \lambda \int_{\mathbb{R}^d} \mathscr{J}_{1/r}(\xi)\, \mathfrak{s}(\xi) \mathrm{d}\xi,$$

we can efficiently study the behaviour of $\sigma(r)$ as dictated by the behaviour of $\mathfrak{s}(\xi)$ around zero using to classical asymptotics of Bessel functions as given by Lemma B.1 in the appendix.

Proposition 4.1. *Let Φ be a hyperuniform stationary point process on $\mathbb{R}^d$ satisfying (2.1). We suppose that there is a constant c such that $|\mathfrak{s}(\xi)| \sim c|\xi|^\alpha$ for ξ in a neighborhood of zero. Then, the following happens according to the value of α:*

1. *If $\alpha \in (0,1)$, then $\sigma(r) = O(1/r^\alpha)$.*
2. *If $\alpha = 1$, then $\sigma(r) = O(\log(r)/r)$.*
3. *If $\alpha > 1$, then $\sigma(r) = O(1/r)$.*

Proof. We suppose wlog that $\lambda = 1$. Thanks to (1.4.4), we have

$$\begin{aligned}\sigma(r) := \frac{\mathrm{Var}(|\Phi \cap B(0,r)|)}{|B(0,r)|} &= \int_{\mathbb{R}^d} \mathscr{J}_{1/r}(\xi)\, \mathfrak{s}(\xi) \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} \mathfrak{s}(x/r)\, \mathscr{J}(x) \mathrm{d}x \\ &= \int_{|x|<r} \mathfrak{s}(x/r)\, \mathscr{J}(x) \mathrm{d}x + \int_{|x|>r} \mathfrak{s}(x/r)\, \mathscr{J}(x) \mathrm{d}x =: a(r) + b(r).\end{aligned}$$

We supposed $\mathfrak{s}$ to be bounded, and we can wlog suppose it is always smaller than 1, so that $b(r) \leqslant \int_{|y|>r} \mathscr{J}(x) \mathrm{d}x$. This integral is itself $O(1/r)$ thanks to Lemma B.1 in the Appendix. Moreover, $a(r) \leqslant cr^{-\alpha} \int_{|x|<r} |x|^\alpha \mathscr{J}(x) \mathrm{d}x$, and this integral is also easy to study thanks to Lemma B.1. When $\alpha = 1$, it is equivalent to some constant times $\log(r)$, hence the case II. When $\alpha < 1$ it is bounded, and when $\alpha > 1$ it is $O(r^{-\alpha} \times r^{\alpha-1}) = O(1/r)$. □

Hyperuniform processes are frequently classified as types I, II and III, according if $\sigma(r)$ grows as $1/r$, as $\log(r)/r$ or as $r^{-\alpha}$ for some $\alpha < 1$: see the paragraph 5.3.2 in Torquato's survey [67]. We refer to the list in [67], Table 1, for many examples.

4.2. The minimal decay

One might wonder if $\sigma(r)$ might decay faster than $1/r$. It would mean that the variance of $|\Phi \cap B(0,r)|$ grows smaller than the surface of $B(0,r)$ which is cr^{d-1}. But this is actually not possible. For example, if $\mathcal{C}$ satisfies Hypothesis (2.1) and is absolutely continuous, the Riemann-Lebesgue lemma says that $\mathscr{s}(x) \to 1$ as $x \to \infty$, hence in the proof above one has $\sigma(r) \gtrsim \int_{|x|>r} \mathscr{s}(x)\mathrm{d}x \asymp 1/r$.

The same argument is also true under less restrictive assumptions, at the cost of more technicalities; in fact, the general argument was proved in a non-random setting by Józef Beck in his celebrated 1987 paper [7]. There, a general version of the following result is proved.

Theorem 4.2 (Beck's theorem ([7], Thm. 2A))**.** *Let $X = (x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of points in $\mathbb{R}^d$. Then, for every radius $r > 1$, there is a constant $c = c(d)$ and a ball $B_s = B(v,s)$ with radius $s < r$ such that*

$$|\#(X \cap B_s) - |B_s|| > cr^{d-1}. \tag{4.2.1}$$

4.3. The central-limit theorem

Hyperuniform processes satisfy a central-limit theorem. This is a remarkable phenomenon, conjectured in many physics papers but to my knowledge, properly proved very recently by Adhikari, Ghosh and Lebowitz in their paper [1] for random fields on $\mathbb{Z}^d$. We hereby reproduce the proof for general point processes on $\mathbb{R}^d$; deep inside, this is the exact same phenomenon as the CLT for determinantal point processes, see Theorem 4.6.1 in the 'GAF book', [37]. See also [14] for a more detailed and general treatment.

Theorem 4.3. *Let Φ be a hyperuniform point process satisfying (2.1), and such that $\mathrm{Var}(|\Phi \cap B_r|) \geqslant cr^{d-\alpha}$ for some $\alpha \in (0, 5d/6]$. Then, under the assumption that for every $n > 2$,*

$$\mathbf{E}[|\Phi \cap B_r|^n] = O(r^d), \tag{4.3.1}$$

the process Φ satisfies a central-limit theorem:

$$\frac{|\Phi \cap B_r| - \mathbf{E}[|\Phi \cap B_r|]}{\sqrt{\mathrm{Var}(|\Phi \cap B_r|)}} \rightsquigarrow \mathscr{N}(0,1) \tag{4.3.2}$$

when $r \to \infty$, where $\rightsquigarrow$ denotes weak convergence of probability distributions on $\mathbb{R}$.

The proof only relies on moments/cumulants convergence, and we refer to Theorem D.4 at page 62 in the Appendix. The n-th cumulant of a random variable X will be called $\mathrm{C}_n(X)$. From now on, we will note $X_r = |\Phi \cap B_r|$ and $Y_r = (X_r - \mathbf{E}[X_r])/\sqrt{\mathrm{Var}(X_r)}$.

Proof. To prove convergence towards a standard Gaussian, the strategy is to prove cumulants convergence as in Theorem D.4, ie to prove that $\mathrm{C}_n(Y_r) \to 0$ for $n \neq 2$ and that $\mathrm{C}_2(Y_r) \to 1$ as $r \to \infty$. The first cumulant is always zero. The second one is always equal to 1, because $\mathrm{C}_2(Y_r) = \mathrm{Var}(X_r)^{-1}\mathrm{C}_2(X_r - \mathbf{E}[X_r]) = \mathrm{Var}(X_r)^{-1}\mathrm{Var}(X_r) = 1$. The cumulants of ordre $n \neq 2$ are translation-invariants, ie $\mathrm{C}_n(X) = \mathrm{C}_n(X + c)$, hence by the definition of the cumulant measures κ_n in (A.5.1), we have

$$\mathrm{C}_n(Y_r) = \frac{\mathrm{C}_n(X_r)}{\mathrm{Var}(X_r)^{n/2}} = \frac{\kappa_n(B_r \times \cdots \times B_r)}{\mathrm{Var}(X_r)^{n/2}}.$$

Let us assume that n is fixed. If (4.3.1) is true, there is a constant K_n such that $\mathbf{E}[|\Phi \cap B_r|^h] \leqslant K_n r^d$ for every $k \leqslant n$. Going back to the definition of cumulants we see that there is another constant K_n' (certainly depending on n) such that $\kappa_n(B \times \cdots \times B) \leqslant K_n' r^d$. Our assumption on Φ entails that $\mathrm{Var}(X_r)^{n/2} \geqslant K_n'' r^{n(d-\alpha)/2}$ for some constant K_n'' and some $\alpha \in [0,1)$. But then, $\mathrm{C}_n(Y_r) = O(r^{d-n(d-\alpha)/2})$. This goes to zero as $r \to \infty$ if and only if $d < n(d-\alpha)/2$, ie $n \geqslant d/(2(d-\alpha))$. But when $\alpha < 5d/6$, the latter is strictly smaller than 3, hence we always have $n \geqslant 3 > d/(2(d-\alpha))$. □

The restriction of the couple (α, d) is only apparent: hyperuniform systems cannot have $\alpha > 1$, hence there is no restriction when $d > 2$.

Hypothesis (4.3.1) is trickier. It is formulated here just as in [1], in order to be sufficiently problem-independent; however, according to the proof, the 'optimal' assumption would be to check that the n-th moment/cumulant of $|\Phi \cap B_r|$ is $o(\mathrm{Var}(|\Phi \cap B_r|)^{n/2}) = o(r^{n(d-\alpha)/2})$ as $r \to \infty$. In [1], the authors say that this hypothesis "connects to the classical theory of particle number fluctuations" as in the first papers on charge fluctuations ([57]). I am personnally not aware of a simple, general method to check such hypothesis.

4.4. Order metrics

For Poisson point processes, one has $\mu_\circ = \lambda \mathbf{1}$, hence the term $\lambda^{-1}\mu_\circ - \mathbf{1}$ is exactly equal to zero. To compare the degree of order/disorder in a point process, we have different quantities which are called order metrics and widely used in physics ([68]).

Definition 4.4. Let Φ be a stationary point process on $\mathbb{R}^d$ with intensity λ and reduced Palm intensity $\mu_\circ$. We suppose that $\mu_\circ$ has a density ϱ with respect to the Lebesgue measure on $\mathbb{R}^d$. Then, the τ order metric (TOM) is defined as

$$\tau = \tau(\Phi) := \int_{\mathbb{R}^d} \left| \frac{\varrho(x)}{\lambda} - 1 \right|^2 \mathrm{d}x = |\lambda^{-1}\varrho - 1|_2^2. \tag{4.4.1}$$

The bigger the order metric, the stronger the order (at all scales) in the point process, with $\tau = 0$ being characteristic of Poisson processes. If τ is finite, then by Parseval's relation the Fourier transform of the total correlation function is also an L^2-function, hence in this case

$$\tau = \frac{1}{\lambda^2} \int_{\mathbb{R}^d} |\mathcal{S}(\xi) - 1|^2 \mathrm{d}\xi. \tag{4.4.2}$$

Another metric is sometimes used, the hyperuniform order metric (HOM). It is defined only for 'class I HU processes', ie processes with density fluctuations scaling as the window surface:

$$\sigma(r) := \frac{\mathrm{Var}(|\Phi \cap B(0,r)|)}{r^{d-1}} \asymp 1.$$

For such class-I HU processes, the HOM is defined as the Cesaro average of σ:

$$\Lambda = \Lambda(\Phi) := \lim_{r\to\infty} \frac{1}{r} \int_0^r \sigma(t)\mathrm{d}t. \tag{4.4.3}$$

If Φ is a type-I process, the function σ is bounded away from 0 and ∞, hence Λ is a positive number.

4.5. Higher order moments than the variance

[74]

5. Finite sample analysis

Hyperuniformity, as defined earlier through (0), is a property of the probability distribution of Φ, but in practice we will only have access to finite-size approximations of such processes, for instance to $\Phi \cap B(0,n)$ as in computer simulations. Estimating the structure factor of Φ by only resorting to finite-sized approximations is really important and is central to most experimental papers in physics. This section justifies this point of view.

5.1. Second-order spectral convergence and empirical structure factor

Let Φ be a stationary point process on $\mathbb{R}^d$ with intensity λ and Palm measure $\mathbf{P}^\circ$. We recall that the pair correlation is the measure defined as

$$g_2(A) = \frac{1}{\lambda}\mathbf{E}^\circ\left[\sum_{x\in\Phi\setminus\{0\}} \mathbf{1}_{x\in A}\right]. \tag{5.1.1}$$

Let us note $N_r = \#(\Phi \cap B(0,r))$ the number of points of Φ in an accessible window of observation of radius r — we'll restrict to B_r. The Fourier transform of the configuration of these N_r points is the function defined for $k \in \mathbb{R}^d$ by

$$\mathrm{n}_r(k) = \sum_{x\in\Phi\cap B_r} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle}. \tag{5.1.2}$$

It is a bounded, periodic function of k. In the physics litterature, n_r is sometimes encountered as the *collective-coordinate function*.

In the thermodynamic $r\to\infty$ limit, the function ν contains all the information relative to the structure factor $\mathcal{S}$ of Φ. More precisely, let us note $\mathcal{S}_r$ the *empirical structure factor* in a box of radius r:

$$\mathcal{S}_r(k) = \frac{|\mathrm{n}_r(k)|^2}{N_r}. \tag{5.1.3}$$

We can write

$$\mathcal{S}_r(k) = \frac{1}{N_r}\sum_{x,y} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\overline{\mathrm{e}^{-\mathrm{i}\langle k,y\rangle}} = 1 + \frac{1}{N_r}\sum_{x\neq y} \mathrm{e}^{-\mathrm{i}\langle k,x-y\rangle} = 1 + \mathscr{F}(g_{2,r}) \tag{5.1.4}$$

where $g_{2,r}$ is the empirical version of the pair correlation function, namely the random point measure defined by

$$g_{2,r} = \frac{1}{N_r}\sum_{\substack{x,y\in\Phi\cap B_r\\ x\neq y}} \delta_{x-y}.$$

The main fact about these measures is contained in the next Theorem and the subsequent proposition.

Theorem 5.1 (Empirical structure factor)**.** *If Φ is stationary and ergodic, then almost surely $g_{2,r}$ converges towards g_2 in the Schwartz space, and $\mathcal{S}_r - \delta_0$ converges towards $\mathcal{S}$ as $r\to\infty$ in the sense of distributions.*

The proof comes from the following proposition.

Proposition 5.2. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be integrable with respect to g_2. Then, almost-surely,*

$$\lim_{r\to\infty} g_{2,r}(f) = g_2(f). \tag{5.1.5}$$

Proof of Proposition 5.2. Take $f \in L^1(g_2)$. We introduce two random variables (for ease of notation, we'll systematically drop the '$\in\Phi$' parts in the sum, all the points x, y belong to Φ):

$$Y_r(f) = \frac{1}{N_r}\sum_{x\in B_r}\sum_{y\neq x} f(x-y) \qquad\qquad X_r(f) = \frac{1}{N_r}\sum_{x\in B_r}\sum_{y\in B_r, x\neq y} f(x-y).$$

By the ergodic theorem [REF], one has almost surely $\lim_{r\to\infty} Y_r(f) = \mathbf{E}^\circ[\sum_{x\neq 0} f(x)] = \int f(u)g_2(\mathrm{d}u)$. We now prove that $X_r(f) - Y_r(f) \to 0$ almost surely. The difference $X_r(f) - Y_r(f)$ is equal to

$$\frac{1}{N_r}\sum_{x\in B_r}\sum_{y\notin B_r} f(x-y).$$

We split the contributions x, y to this sum in two parts, according if $|x-y| > a$ or $|x-y| \leqslant a$, for some a to be chosen later, and we note $\mathrm{I}_r(>a)$ and $\mathrm{I}_r(\leqslant a)$ these contributions: we prove that they are both small.

We upper bound $\mathrm{I}(\geqslant a)$ by dropping the $y \in B_r$ constraint and we get

$$\mathrm{I}_r(\geqslant a) \leqslant \frac{1}{N_r}\sum_{x\in B_r}\sum_{y\neq x} |f_a(x-y)|$$

where $f_a(u) = f(u)\mathbf{1}_{|u|\geqslant a}$. By the Ergodic theorem again, this converges towards $\int |f_a(u)|g_2(\mathrm{d}u)$. But since g_2 integrates f, we see that $\int |f_a|g_2 \to 0$ when $a \to \infty$ so we can choose a big enough to ensure that $\limsup_{r\to\infty} \mathrm{I}_r(\geqslant a) \leqslant \varepsilon/2$.

As for the term $\mathrm{I}_r(<a)$, we note that the only couples of points x, y which can contribute to the sum are those for which x is within distance a from the boundary of B_r. Let us note $A(r,a)$ the annulus $\{r-a \leqslant |u| < r\}$. Since f is bounded we have

$$\begin{aligned}\mathrm{I}_r(<a) &\leqslant \frac{|f|_\infty}{N_r}\sum_{x\in A(r,a)} \#(\Phi\cap B(x,a))\\ &\leqslant \frac{|f|_\infty}{N_r}\sum_{x\in B_r} \#(\Phi\cap B(x,a)) - \frac{N_{r-a}}{N_r}\frac{|f|_\infty}{N_{r-a}}\sum_{x\in B_{r-a}} \#(\Phi\cap B(x,a)).\end{aligned}$$

At large but fixed a, we have $N_{r-a}/N_r \to 1$ almost surely, hence the two terms in the difference above both converge (by the Ergodic theorem again) towards $|f|_\infty g_2(B(0,a))$, so for fixed a, $\lim_{r\to\infty} \mathrm{I}_r(<a) = 0$, concluding the proof. □

Proof the Theorem. Once Proposition 5.2 is proved, here is how the Theorem follows. First, note that the space of Schwartz functions belongs to $L^1(g_2)$. Since $\mathscr{S}(\mathbb{R}^d)$ is separable, we can choose a dense sequence f_n and consider the event $E = \cup E_n$, where E_n is the event of the proposition. As a countable union of probability 1 events, E has itself probability one. From this, it is easy to see that on the event E, for every $f \in \mathscr{S}(\mathbb{R}^d)$, we have $g_{2,r}(f) \to g_2(f)$. Thus, on the event E, the measures $g_{2,r}$ converge (as tempered distributions) towards g_2. Since the Fourier transform is continuous, $\mathscr{F}(g_{2,r}) \to \mathscr{F}(g_2)$. We saw in (5.1.4) that $\mathcal{S}_r = 1 + \mathscr{F}(g_{2,r})$, so subtracting off $\delta_0 = \mathscr{F}(\mathbf{1})$, we get $\lim_{r\to\infty} \mathcal{S}_r - \delta_0 = 1 + \mathscr{F}(g_2 - \mathbf{1}) = \mathcal{S}$. □

5.2. The variance of the collective-coordinate

Let us note $f_{r,k}(x) = \mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\mathbf{1}_{x\in B_r}$, so that $\mathrm{n}_r(k) = \sum f_{r,k}(x)$. The variance formula says that

$$\mathrm{Var}(\mathrm{n}_r(k)) = \frac{\lambda}{(2\pi)^d}\int |\widehat{f_{r,k}}(\xi)|^2 \mathcal{S}(\mathrm{d}\xi).$$

However, $\widehat{f_{r,k}}(\xi) = \widehat{\mathbf{1}_{B_r}}(\xi + k)$, and thanks to (1.4.1),

$$\widehat{\mathbf{1}_{B_r}}(\xi+k) = r^{d/2}\frac{\mathrm{J}_{d/2}(r|k+\xi|)}{|k+\xi|^{d/2}}. \tag{5.2.1}$$

5.3. How to visually see hyperuniformity

The principal difficulty with long-range orders in point processes such as hyperuniformity, is that is is sometimes undetectable with our human eyes: this type of order is thus often marketed as *hidden order* in the

physics litterature. However, the use of Fourier transform can sometimes allow for a more direct detection of hyperuniformy. As seen earlier, a point process is HU if its structure factor vanishes at the origin. Theorem (5.1) says that, at least for ergodic processes, the empirical structure factor is almost surely a good approximation of the real one.

This justifies the pictures in Figure 3, in which I simulated stationary point processes in a finite box and then represented the corresponding structure factor.

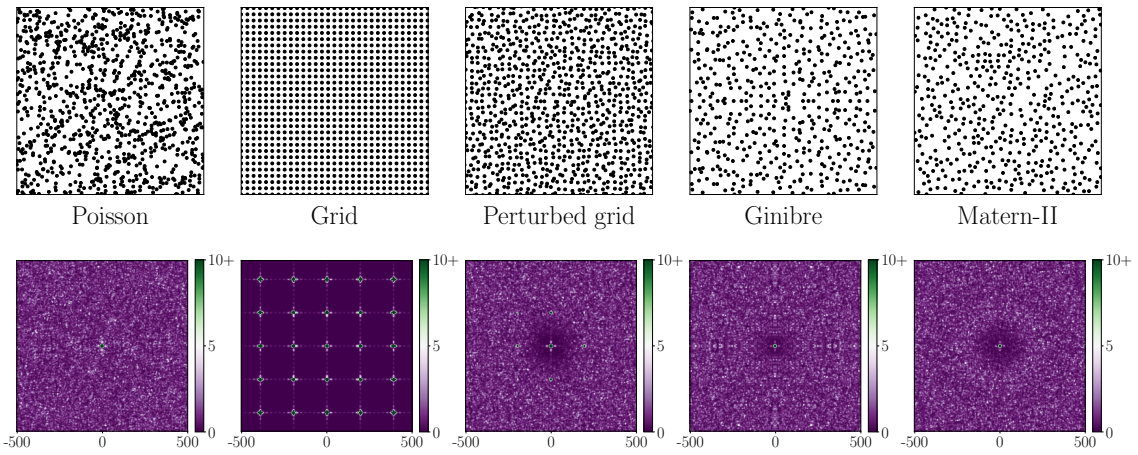

Figure 3: Several realizations of planar point processes with around 1000 points, drawn with a colorplot representation of their structure factor (below). The color represents $\mathcal{S}(\xi)$ for $\xi = (\xi_1, \xi_2)$ in a box of size 1000 around the origin. The darker, the closer to zero. The presence of a darker shape around zero indicates a hyperuniform point process ($\mathcal{S}(\xi) \to 0$).

A number of point processes we'll be studying are also isotropic, and in this case their structure factor is radial. It is thus easier to plot *angular averages* of the empirical structure factor, ie

$$\mathcal{s}_r(t) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{S}_r((t\cos(\theta), t\sin(\theta))\mathrm{d}\theta.$$

6. Rigidity and hyperuniformity

Definition 6.1. Let Φ be a random point process on $\mathbb{R}^d$. A measurable function f is a rigidity statistic for Φ if for any compact K, the random variable

$$\sum_{x\in\Phi\cap K} f(x)$$

is a measurable function of $\Phi\cap K^c$. The set of rigidity statistics of Φ will be noted $\mathrm{Rig}(\Phi)$.

When $f\equiv 1$ is a rigidity statistic, we simply say that Φ is *number-rigidity*, which is the most common rigidity studied in the litterature. Another very important kind of rigidity is when $f(x)=x^k$, which we call k-rigidity: when $k=0$ this is simply rigidity, when $k=1$ it means that $\Phi\cap K^c$ determines the center of mass of $\Phi\cap K$, etc (see [26]).

A maximal notion of rigidity is *maximal rigidity*, where we ask the whole configuration $\Phi\cap K$ to be a measurable function of $\Phi\cap K^c$. Such maximally rigid processes will be encountered in Subsection 7.

6.1. The variance argument

The basic tool for proving rigidity is an elegant Borel-Cantelli-type argument. It can be found in [29] and in many subsequent papers. We give a more general version than for number-rigidity.

Proposition 6.2. *Let Φ be a point process. Suppose that there is a sequence of measurable functions (f_n), such that $f_n\to f$ pointwise, and*

$$\sum_{n=0}^{\infty}\mathrm{Var}\left(\sum_{x\in\Phi} f_n(x)\right)<\infty. \tag{6.1.1}$$

Then f is a rigidity statistic for Φ.

Proof. Note $S_n=\sum f_n(x)$ and $e_n=\mathbf{E}[S_n]$ and $\varepsilon_n=\mathrm{Var}(S_n)$. Let s_n be a sequence of numbers going to zero and such that $\sum\varepsilon_n/s_n<\infty$ (such a sequence always exists). Then by Chebyshev's inequality,

$$\sum_{n=1}^{\infty}\mathbf{P}(|S_n-e_n|>\sqrt{s_n})\leqslant\sum_{n=1}^{\infty}\varepsilon_n/s_n<\infty.$$

By the Borel-Cantelli lemma, almost surely one has $|S_n-e_n|\leqslant\sqrt{s_n}=o(1)$ for any n large enough. Define

$$R_n=\sum_{x\in\Phi\cap K^c} f_n(x)$$

which is a measurable function of $\Phi\cap K^c$. Then, by the properties of f_n, the following statements are almost surely true:

$$\begin{aligned}\sum_{x\in\Phi\cap K} f(x)&=\lim_{n\to\infty}\sum_{x\in K\cap\Phi} f_n(x)\\&=\lim_{n\to\infty} S_n-R_n\\&=\lim_{n\to\infty} e_n-R_n\end{aligned}$$

and the latter is a limit of random variables which are $\Phi\cap K^c$-measurables, hence it is overall $\Phi\cap K^c$-measurable and Φ is f-rigid. □

Thus, proving rigidity amounts to proving serious variance bounds. Note that one can also relax further this kind of criteria: the core argument of the proof is that there is some measurable function f_n with $f_n\to f$ pointwise, and such that $\sum_{x\in\Phi} f_n(x)$ is $o(1)$-close to a $\Phi\cap K^c$-measurable random variable — in our case, its own expectation e_n.

Remark 6.3 (rigidity criterion and hyperuniformity). Note that if such functions (f_n) do exist, then the variance $\mathrm{Var}(\sum f_n(x))$ goes to zero. If the functions are chosen so that $\int f_n(x)\mathrm{d}x$ is bounded away from zero, this immediately implies $\sigma(f_n) \to 0$, hence f is a hyperuniform statistic for Φ in the sense of Definition 2.3. The assumption of the preceding proposition is thus extremely strong, showing both hyperuniformity *and* rigidity.

6.2. Quadratic variance formula

Theorem 6.4. *Let Φ be a hyperuniform point process satisfying Hypothesis (2.1). Then,*

$$\mathrm{Var}\left(\sum_{x\in\Phi} g(x)\right) = -\frac{\lambda}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |g(u)-g(u+v)|^2 \mathcal{C}(\mathrm{d}v)\mathrm{d}u \qquad (g\in\mathscr{S}(\mathbb{R}^d)). \tag{6.2.1}$$

This formula is essential in the theory of determinantal/pfaffian point processes, where it is at the heart of the paper [11], see also [13, Subsec. 2.2]; it was later used in full generality by Ghosh and Lebowitz in their paper [27] (look at equation (4) therein) to extend the results in [11]. Those results will be shown later on at Subsection 6.3 page 28.

Note that the integrated function $(u,v)\mapsto |g(u)-g(u+v)|^2$ is not a Schwartz function, it is only a bounded function. Without any hypothesis on $\mathcal{C}$, the integral above might be ill defined.

Proof of (6.2.1). Develop the right hand side. It is equal to the sum of three terms:

$$-\frac{\lambda}{2}\int |g(u)|^2\mathcal{C}(\mathrm{d}v)\mathrm{d}u - \frac{\lambda}{2}\int |g(u+v)|^2\mathcal{C}(\mathrm{d}v)\mathrm{d}u + \lambda\int g(u)g(u+v)\mathcal{C}(\mathrm{d}v)\mathrm{d}u.$$

The first two ones are zero because for hyperuniform point processes satisfying Hypothesis (2.1) we have seen in Proposition 2.2 that $\mathcal{C}(\mathbb{R}^d)=0$. The last one is equal to the variance in the LHS of (6.2.1) thanks to the covariance formula (1.1.5). □

6.3. Rigidity from hyperuniformity: one and two dimensions

A crucial result obtained by Ghosh and Lebowitz is that in dimensions 1 and 2, hyperuniformity implies rigidity, as long as the correlations are sufficiently decaying ([27]). This is no longer true — and actually pretty mysterious — for higher dimensions, as discussed in the next section. By 'sufficiently decaying', we mean that the measure has four finite moments.

Hypothesis 6.5. A signed measure m has strong decay if its moments of order $1,2,3,4$ are finite.

Theorem 6.6 ([27])**.** *Let Φ be a stationary point process on $\mathbb{R}^d$ with $d=1$ or 2. If $\mathcal{C}$ has strong decay and Φ is hyperuniform, then Φ is rigid.*

Note that by the definition of $\mathcal{C}$, the existence of moments for $\mathcal{C}$ is equivalent with the existence of moments for the measure $\mu_o - \lambda\mathbf{1}$.

Proof. We only do the proof for $d=2$, the proof for $d=1$ being essentially simpler. The goal is to prove rigidity using the variance criterion 6.1.1, and to construct a measurable function f taking the value 1 on $B(0,r)$, such that $\mathrm{Var}(\sum f(x)) < \varepsilon$ for arbitrary $\varepsilon > 0$.

We choose some $R>1$. Let g be a smooth function supported on $B(0,R)$, smaller than 1 everywhere, and taking the value 1 in $B(0,1)$. We set $g_b(x)=g(x/b)$ and $S_b=\sum_{x\in\Phi} g_b(x)$. We start with (6.2.1):

$$\mathrm{Var}(S_b) = \frac{-\lambda}{2}\iint |g_b(u)-g_b(u+v)|^2\mathcal{C}(\mathrm{d}v)\mathrm{d}u.$$

First of all, note that if $|u| > bR$ and $|u+v| > bR$, the integrand is zero, hence we only have to integrate on the union of $A_1 = \{|u| \leqslant bR\}$ and $A_2 = \{|u+v| \leqslant bR\}$; by the triangle inequality, we thus have[6]

$$\begin{aligned}\mathrm{Var}(S_b) &\leqslant \frac{\lambda}{2}\iint_{A_1\cup A_2} |g_b(u) - g_b(u+v)|^2 |\mathcal{C}|(\mathrm{d}v)\mathrm{d}u \\ &\leqslant \frac{\lambda}{2}\iint_{A_1} |g_b(u) - g_b(u+v)|^2 |\mathcal{C}|(\mathrm{d}v)\mathrm{d}u + \frac{\lambda}{2}\iint_{A_2} |g_b(u) - g_b(u+v)|^2 |\mathcal{C}|(\mathrm{d}v)\mathrm{d}u \\ &\leqslant \mathrm{I}_1 + \mathrm{I}_2\end{aligned}$$

and we separately bound those terms. To do this, we write the Taylor development of g_b as

$$g_b(u+v) = g_b(u) + b^{-1}\langle \nabla g(u/b), v\rangle + h(u,v)$$

where $|h(u,v)| \leqslant c_1|v|^2/b^2$ for some constant c_1 depending on g (indeed, $c_1 = \sup \|H^2 g(x)\|$ where $H^2 g$ is the Hessian matrix). We have

$$\begin{aligned}|g_b(u) - g_b(u+v)|^2 &\leqslant \frac{|\nabla g(u/b)|^2|v|^2}{b^2} + \frac{2|\nabla g(u/b)||v||h(u,v)|}{b} + |h(u,v)|^2 \\ &\leqslant \frac{|\nabla g(u/b)|^2|v|^2}{b^2} + \frac{2|\nabla g(u/b)|c_1|v|^3}{b^3} + \frac{c_1|v|^4}{b^4}.\end{aligned}$$

Let c be a constant greater than first four moments of $\mathcal{C}$, ie $c \geqslant \int |x|^k |\mathcal{C}|(\mathrm{d}x), k \in \{1,2,3,4\}$. We have

$$\begin{aligned}\mathrm{I}_1 &\leqslant \int_{B(0,bR)} \int \left(\frac{|\nabla g(u/b)|^2|v|^2}{b^2} + \frac{2|\nabla g(u/b)|c_1|v|^3}{b^3} + \frac{c_1|v|^4}{b^4}\right) |\mathcal{C}|(\mathrm{d}v)\mathrm{d}u \\ &\leqslant c\int |\nabla g(y)|^2 \mathrm{d}y + \frac{2cc_0\pi b^2R^2c_1}{b^3} + \frac{\pi b^2R^2c_1c}{b^4} \\ &\leqslant c\int |\nabla g(y)|^2 \mathrm{d}y + \frac{O(cc_0c_1R^2)}{b} + \frac{O(cc_1R^2)}{b^2}\end{aligned}$$

where we also introduced $c_0 = \sup|\nabla g(x)|$, and the constants in the $O(\cdot)$ is universal. A similar bound holds for I_2. Consequently,

$$\mathrm{Var}(S_b) \leqslant c\int |\nabla g(y)|^2 \mathrm{d}y + \frac{O(cc_0c_1R^2)}{b} + \frac{O(cc_1R^2)}{b^2} \tag{6.3.1}$$

The last step is thus to find, for any $\varepsilon > 0$, a function g as described in the begining of the proof, and such that this last bound is smaller than ε. This can be done, and it is the content of the lemma thereafter. We first choose $g_{\varepsilon/21c}$, so that the first term in the RHS of (6.3.1) is smaller than $\varepsilon/3$. Then, c_0, c_1 and R are fixed (they might depend on ε), and we take b so large that the two remaining terms in the RHS of (6.3.1) are smaller than $\varepsilon/3$.

□

Lemma 6.7. *For every $\varepsilon > 0$, there is a function $g_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ which as the following properties: it is nonnegative, takes the value 1 on $B(0,1)$ and 0 outside $B(0,R_\varepsilon)$ for some $R_\varepsilon > 1$, it is of class at least $\mathscr{C}^2$, it is smaller than 1 everywhere, and it satisfies*

$$\int |\nabla g_\varepsilon(x)|^2 \mathrm{d}x < 7\varepsilon. \tag{6.3.2}$$

We explain a construction which can be found in [32].

Proof. Let us start by setting $R'_\varepsilon = \mathrm{e}^{2/\varepsilon}$. We now a real-valued function $a : [0,\infty[\to [0,\infty[$ by

$$a(t) = \begin{cases} 1 \text{ if } t \in [0,1], \\ 1 - \log(t)\varepsilon/2 \text{ if } t \in [1, R'_\varepsilon] \\ 0 \text{ else.}\end{cases}$$

[6] We recall that $|\mathcal{C}|$ is the 'absolute value' of the signed measure $\mathcal{C}$.

This function is continuous and it is smooth, except at 1 and R'_ε. Moreover, for t between 1 and R'_ε, we have $a'(t) = -\varepsilon/2t$ and $a''(t) = \varepsilon/2t^2$. These properties ensure that we can find a nonnegative $\mathscr{C}^2$ function b with $|b'(t)| \leqslant \varepsilon/t, |b''(t)| \leqslant \varepsilon/t^2$, taking the value 1 on $B(0,1)$, smaller than 1 everywhere, and being zero outside $B(0, R_\varepsilon)$ with $R_\varepsilon = 1 + \mathrm{e}^{2/\varepsilon}$.

We simply define $g_\varepsilon(x) = b(|x|)$. We have $\nabla g(x) = b'(|x|)x/2|x|$, hence

$$|\nabla g|_2^2 = 2\pi \int_1^\infty |\nabla g(r)|^2 r \mathrm{d}r\mathrm{d}\theta \leqslant 2\pi \int_1^{1+R_\varepsilon} \frac{b'(r)^2 r^3}{2r^2} \mathrm{d}r$$
$$\leqslant 2\pi \int_1^{1+R_\varepsilon} \frac{\varepsilon^2}{2r} \mathrm{d}r \leqslant \pi\varepsilon^2 \log(1 + R_\varepsilon) \leqslant 7\varepsilon.$$

This settles the lemma. □

Remark 6.8 ($d \geqslant 3$). The crucial fact in the preceding proof is that we can find g_b such that (6.3.1) is true, and in particular the first term does not depend on b. In dimensions higher than 3, there would be a b factor there and the subsequent argument would fail.

Remark 6.9. One can further relax the fourth-moment hypothesis to having decaying tails like $|x|^{-4-\delta}$, as formulated in [27].

6.4. Rigidity from hyperuniformity: three dimensions and more

Consider the perturbed lattice: start with the stationarized lattice $U + \mathbb{Z}^d$, then replace each point x by $x + H_x$ where (H_x) is a collection of iid random variables, say centered Gaussian variables with variance σ^2. The corresponding process will be Φ_σ; we will come back at length on this model on Section 9.

It turns out that there is a surprising threshold for number rigidity, as proved by Sly and Peres in their 2014 paper [60].

Theorem 6.10 (Peres and Sly)**.** *If $d \geqslant 3$, there is a $\sigma_c(d) > 0$ such that Φ_σ is number-rigid if $\sigma < \sigma_c$ and not rigid if $\sigma > \sigma_c$.*

But one can also prove that such processes always satisfy (0) and have very strong decay of correlations. The strict analogue of the results in Subsection 6.3 cannot hold, for those processes with $\sigma > \sigma_c$ are hyperuniform and not rigid. Ghosh and Lebowitz conjecture that the relation should be the other way round, namely that rigidity should imply hyperuniformity. They argue that (verbatim from [31]) "*when the variance grows like $|\Lambda|$ [the window], it behaves in an additive way for two adjacent domains which seems to suggest that surface effects become negligible for large $|\Lambda|$, which is inconsistent with number rigidity*". However, there are several interesting counter-examples.

6.5. The link between rigidity and fluctuations: counterexamples

Zeros of some Gaussian process

The existence of one-dimensional, rigid and non-hyperuniform processes is investigated in [50]. The point process at stake, noted $\mathcal{Z}$, is the set of zeroes of a Gaussian stationary process $X = (X_t)_{t\in\mathbb{R}}$, in other words

$$\mathcal{Z} = \{t \in \mathbb{R} : X_t = 0\}.$$

There are a lot of tools for studying zeroes of gaussians processes, the main one being the celebrated Kac-Rice formula [39]. The variance of such zeroes can also easily be studied and linked with the covariance function, see Geman's paper [25].

Proposition 6.11. *Let X be a one-dimensional stationary gaussian field. If the correlation measure $\mathcal{C}$ of a gaussian process X is not proportional to $\cos(cx)\mathrm{d}x$ for some c, then the set of its zeroes cannot be hyperuniform.*

On the other hand, the paper shows that if the correlation measure is a sum of cosines, like

$$\mathcal{C}(\mathrm{d}x) = \sum_n a_i \cos(x)\mathrm{d}x,$$

with (a_i) a summable sequence of non-negative numbers, then $\mathcal{Z}$ is actually rigid[7]

Intersecting hyperplanes

The Poisson line intersection process is defined as follows: first, let L be a 2-dimensional Poisson line process, see for instance [4] or [16]. It is the natural Poisson process on the set of affine hyperplanes of $\mathbb{R}^d$. Its intersection process is

$$\Phi = \{\ell_1 \cap \ell_2 : \ell_1 \neq \ell_2 \in L\}.$$

This is a stationary point process with no double point (a.s.). This process and its generalizations is extensively studied in [63]; it turns out that is is hyperfluctuating, that is:

$$\lim_{r\to\infty} \frac{\mathrm{Var}(\#(\Phi \cap B_r))}{|B_r|} = +\infty.$$

However, it is very intuitive that this process is actually maximally rigid: almost surely, any line $\ell \in L$ will intersect an infinity of other lines in L, thus giving rise to an infinity of aligned points in Φ. If one erases any finite number of points in Φ, they will still be largely enough of them to first reconstruct the lines, and then reconstruct the missing points... This intuition was found and made rigorous in [46].

6.6. Other examples of rigidities

Rigidity of Pfaffian processes

Rigidity of some Pfaffian point processes is studied in the paper [13]. The models falling under their results include symplectic and orthogonal Sines and Bessels point processes.

The sine-beta process is rigid

Rigidity of the sine-beta process Sine_β has been proved independently by Chhaibi and Najnudel in [61] and by Dereudre, Hardy, Leblé and Maïda in [18]. Here, Theorem 6.6 might apply, but at the moment (April 11, 2021) there is no way to check if the correlations of the Sine_β process have strong decay, hence rigidity has to be proved directly.

Chhaibi and Najnudel use the 'classical' way described in Subsection 6.1. They exploit the definition of Sine_β as a limit of normalized eigenvalues of Circular ensembles and then use variance estimates for those ensembles.

Dereudre et al have another technique, which is up to now the only alternative to the variance arguments, and which uses the DLR formalism. I will try in a distant future (say, June 2019) to write an account of this new method, in which hyperuniformity still plays a role.

(more to come)

k-rigidity for GAFs

Kiro and Nishry proved the k-rigidity for the zero sets of some entire functions in [44]. See also [26].

Rigidity for eigenvalues of Schrodinger operators

It is proved in [51] that eigenvalues of random Schrödinger operators are rigid (under certain assumptions on the randomness). The method uses Feynman-Kac formulas to bound the variance of some linear functionals.

[7] And in fact, more than rigid: it is completely predictable, see next subsection.

Non-compact rigidity

The rigidity defined in the former definition is a 'compact' rigidity, ie the configuration outside a compact set determines some feature of the configuration inside the compact. One can ask to what extent this can be generalized for unbounded sets: what parts of $\Phi \cap C$ are determined by $\Phi \cap C^c$? This notion has been much less studied than classical rigidity. In one dimension, Bufetov, Dabrowski and Qiu ([12]) proved 'predictable rigidity' for some one-dimensional processes: it means that the point process is completely determined by the knowledge of points on a half-line, if $\Phi \cap (-\infty, t]$ determines Φ for any t; see Appendix C. The paper [50] explores this notion for point processes which are zeroes of gaussian processes. Proposition 6 therein says that if a gaussian process is $\mathscr{C}^1$ and if the correlation function C goes to 1 along some subsequence (ie $C(x_n) \to 1$ for some $x_n \to +\infty$), then its zeroes form a predictable process.

The Kolmogorov criterion

There is an old rigidity criterion for stationary sequences (not exactly the same thing as stationary point processes) which dates back to Kolmogorov ([49]). It is in Appendix C.

7. Extreme rigidity for stealthy hyperuniform processes

This section demonstrates that SH processes are extremely rigid: points cannot accumulate too much but they also cannot leave big holes, and the whole process is maximally rigid in the sense defined right after Definition 6.1 at page 27. These beautiful results entirely rely upon the Fourier approach to hyperuniformity; everything in this section comes from [31]. In the proofs I mention a few ' facts' which are actually purely analytic and independent of our problems. Their justification is postponed to the end of this section[8].

7.1. Anticoncentration and bounded holes

Theorem 7.1 (anti-concentration)**.** *Let Φ be a SH point process on $\mathbb{R}^d$ with intensity λ and with spectral gap r or more. Then, there are two constants c_0, c_1 such that for any open set V with diameter smaller than c_0/r, the number of points of Φ in V is smaller than $\lambda c_1/r^d$.*

Proof. As usual, the key is given by the phase-space variance formula ($\hat{\star}$). If f is any function such that $\hat{f}$ is supported inside $B(0,r)$, then $\langle \mathcal{S}, |\hat{f}|^2 \rangle = 0$, hence the variance of $S(f)$ is zero and this random variable is actually almost surely equal to its expectation $\lambda \int f(x)\mathrm{d}x$. Our goal is to show the following fact.

Fact 1. *There is a (deterministic) function f and a constant c_0, and such that*

1. *f is nonnegative and $f(x) \geqslant r^d$ for any $|x| \leqslant c_0/r$,*
2. *$\hat{f}$ has support inside $B(0,r)$.*

If such things exist, then almost surely one has

$$r^d |\Phi \cap B(0, c_0/r)| \leqslant \sum_{x \in \Phi} f(x) = \lambda \int_{\mathbb{R}^d} f(x)\mathrm{d}x =: c_1.$$

One can then easily use the mass transport principle to show that this is indeed true for any $B(x, c_0/r)$, and for any open set with diameter smaller than c_0/r. □

Theorem 7.2 (Bounded holes theorem)**.** *Let Φ be a SH point process on $\mathbb{R}^d$ with intensity λ and with spectral gap r or more. There is a constant c such that almost surely, any ball of radius c/r contains at least one point of Φ.*

Proof. We will suppose that the intensity is 1. Set $R > 0, \delta = c_0/r$ with c_0 the constant in the Anti-concentration Theorem, and define the square annulus $B_\ell = \{\delta(R+\ell) \leqslant |x|_\infty < \delta(R+\ell+1)\}$ for every $\ell > 0$. For $\ell = 0$ we set $B_0 = B_\infty(0, \delta R)$.

Let f be the function whose existence was proved in Fact 1, so that almost surely one has

$$\sum_{\ell=0} \sum_{x \in \Phi \cap B_\ell} f(x) = c_1 > 0. \tag{7.1.1}$$

Define X_ℓ to be the ℓ-th term in this sum. The sketch of the proof is as follows: one can show that for $\ell > 0$, the random variable X_ℓ is $O((R+\ell)^{-2})$. Hence, if R is chosen so that there are no points in B_0, then the whole sum above is going to be $\sum_{n>R} n^{-2} = O(R^{-1})$, which shows that R cannot be too big.

Formally, let $E = E(R)$ be the event $\{\Phi \text{ has no points in } B_0\}$. If this event had zero probability for every R, this would mean that Φ has an atom at zero, which is not true since Φ is stationary. We thus choose some $R > 0$ such that $\mathbf{P}(E(R)) > 0$ and from now on we will work on this event: there are no points of Φ in B_0. The sum in (7.1.1) can thus be started at $\ell = 1$.

[8] And I will probably write them later this year.

Fact 2. *Almost surely,*

$$\forall \ell > 0, \qquad X_\ell \leqslant \frac{c_2}{r} \frac{|\Phi \cap B_\ell|}{|(R+\ell)\delta|^{d+1}} \tag{7.1.2}$$

for some constant c_2.

Now, consider the following. For every ℓ, one can cover the annulus B_ℓ with a finite number of small square boxes and with diameter δ.

Fact 3. *Actually, one can take fewer than $c_3|(R+\ell)\delta|^{d-1}$ such boxes, for some constant c_3.*

By the anticoncentration theorem, each of those boxes cannot contain more than c_1/r^d points of Φ, thus the number of points of Φ inside B_ℓ is smaller than

$$\frac{c_1 c_3 |(R+\ell)\delta|^{d-1}}{r^d}.$$

Using this bound and Fact 2 to bound every term in the LHS of (7.1.1), one gets

$$c_1 \leqslant \frac{c_1 c_3}{r^{d+1}\delta^2} \sum_{\ell=1}^{\infty} \frac{1}{(R+\ell)^2} \leqslant \frac{2c_1 c_2 c_3}{c_0^2 r^{d-1} R} = \frac{c}{r^{d-1}R}.$$

As a byproduct, we also have the bound $R \leqslant 2c_2c_3/c_0^2 r^{d-1}$. □

7.2. Maximal rigidity of SH processes

We recall that a PP is maximally rigid if $\Phi \cap K$ is a measurable function of $\Phi \cap K^c$: if the points outside the compact K are revealed, then we are able to entirely reconstruct the points of Φ in K. The proof is adapted from [31], and it is valid in any dimension d.

Theorem 7.3 (maximal rigidity)**.** *Stealthy hyperuniform point processes are maximally rigid.*

Proof. It is enough to prove such a theorem when K is a ball and for simplicity we assume that $\lambda = 1$. The set of points $\Phi \cap K$ is as finite, and is entirely characterized by the Fourier transform

$$\chi(\xi) := \sum_{x \in \Phi \cap K} \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle}$$

hence we only have to prove that χ a measurable function of $\Phi \cap K^c$. Consider a small ball $B(0, 3\delta_0)$ entirely included in U, the open set on which the structure measure $\mathcal{S}$ vanishes. Let $\delta < \delta_0$. Then the function $h = \mathbf{1}_{B(0,\delta)} * \mathbf{1}_{B(0,\delta)}$ is supported inside $B(0, 2\delta) \subset U$. Using (1.4.1) we see that the function defined by

$$f_\delta(x) = \frac{\delta^d \mathrm{J}_{d/2}(\delta|x|)^2}{|x|^d}$$

satisfies $\hat{f_\delta} = h_\delta$. We now set $F_{\mu,\delta}(z) := \mathrm{e}^{\mathrm{i}\langle z, \mu\rangle} f_\delta(z)$. Thanks to the symmetries of the Fourier transform, this function satisfies $\hat{F}_{\mu,\delta}(\xi) = h_\delta(\xi - \mu)$, hence if $|\mu| < \delta_0$ it is supported inside $B(\mu, 2\delta) \subset B(0, 3\delta_0) \subset U$: it is in the kernel of $\mathcal{S}$, hence the random variable $\sum_{x\in\Phi} F_{\mu,\delta}(x)$ has zero variance (see Remark 1.6) and one has almost surely

$$\sum_{x \in \Phi \cap K} e^{\mathrm{i}\langle x, \mu\rangle} f_\delta(x) = \sum_{x \in \Phi \cap K} F_{\mu,\delta}(x) = c(\mu, \delta) - \sum_{x \in \Phi \cap K^c} F_{\mu,\delta}(x) \tag{7.2.1}$$

where $c(\mu) = \int F_{\mu,\delta}$. The RHS is a measurable function of $\Phi \cap K^c$, say $G_\delta(\Phi \cap K^c, \mu)$. Moreover, (7.2.1) is almost surely simultaneously true for all $\mu \in B(0, \delta_0)$ with rational coordinates and δ rational and small. Note that thanks to the Bessel asymptotics $\mathrm{J}_\nu(x) \sim \Gamma(\nu+1)^{-1}(x/2)^\nu$ when $x \to 0$, we see that $f_\delta(x)/\delta^{2d} \to$

$C(d) > 0$ when $\delta \to 0$, where $C(d)$ is a constant. We divide each member of (7.2.1) by $C(d)\delta^{2d}$ and take the limit along a sequence of rational numbers (δ_n) going to zero.

$$\chi(\mu) = \sum_{x \in \Phi \cap K} e^{\mathrm{i}\langle x, \mu \rangle} = \lim_{n \to \infty} \sum_{x \in \Phi \cap K} e^{\mathrm{i}\langle x, \mu \rangle} \frac{f_{\delta_n}(x)}{C(d)\delta_n^{2d}} = \lim_{n \to \infty} \frac{G_{\delta_n}(\Phi \cap K^c, \mu)}{C(d)\delta^{2n}}.$$

This is true for all $\mu \in B(0, \delta_0)$ with rational coordinates. But χ is continuous, hence it is also a measurable function of $\Phi \cap K^c$ as requested. □

Question 7.4. Let Φ be a SH point process and Ψ an infinite subset of its points. Is Φ a measurable function of Ψ? In other words, if I remove an infinite number of points of Φ, can I recover Φ?

8. The Coulomb gas

We review in this section some questions about rigidities in three-dimensional (and higher) Coulomb systems, and a major result obtained in three dimensions by Sourav Chatterjee in 2017 in his paper [15], then recently extended to all dimensions by Ganguly and Sarkar in [24]: hyperuniformity for the *hierarchical* Coulomb gas.

8.1. Definition of the Hierarchical Coulomb gas

The Coulomb gas in d dimensions is a random set of n points in $[0,1]^d$ whose density with respect to the Lebesgue measure is

$$\frac{1}{Z(\beta)} \exp\left\{ -\beta \sum_{i \neq j} w(x_i, x_j) \right\}. \tag{8.1.1}$$

Here $Z(\beta)$ is the partition function, β is the inverse temperature and w is an interaction. In the case of the hierarchical Coulomb gas, this interaction is defined as follows. We divide the unit cube $[0,1]^d$ into 2^d small cubes of side length $1/2$, then we divide each of these cubes, etc. For $x \neq y$ in the unit cube, we set

$$w(x,y) = 2^{(d-2)(k-1)} \tag{8.1.2}$$

where k is the first integer so that x and y do not belong to the same cube with side length 2^{-k}.

The point process thus obtained will be noted Φ_n.

8.2. Hyperuniformity and rigidity of Coulomb systems

In [57], Coulomb systems in 3 dimensions were non-rigorously proved to be hyperuniform.

8.3. Hyperuniformity of the HCG

The dimension d is always supposed to be 3 or more. Suppose that U is an open set in the unit cube. Then the number of points inside U should be proportional to n, hence we would like to prove something like $\mathrm{Var}(|\Phi_n \cap U|)/n = o(1)$. This will be the content of the theorem, and for technical reasons we restrict to *admissible open sets*, which are :

- nonempty, open, connected,
- whose boundary is a smooth closed orientable surface.

Note that this encompasses virtually all the window shapes, not only balls.

Theorem 8.1 (Macroscopic hyperuniformity, [15, 24])**.** *Let U be an admissible open subset of $[0,1]^d$. Then $\mathbf{E}[|\Phi \cap U|] = |U|n$. Moreover, there are two positive constants $c(U,\beta), C(U,\beta)$ such that*

$$\frac{c(U,\beta)}{n^{1/d}} \leqslant \frac{\mathrm{Var}(|\Phi_n \cap U|)}{n} \leqslant \frac{C(U,\beta)\log(n)^{13}}{n^{1/d}}. \tag{8.3.1}$$

9. Perturbations of point processes

Let Ψ be a a stationary point process on $\mathbb{R}^d$. Each node $x \in \Psi$ is then displaced by a vector V_x; The resulting point process will be noted Ψ, although it depends on Ψ and V:

$$\Phi = \Phi_{\Psi,V} = \{x + V_x : x \in \Psi\}. \tag{9.0.1}$$

We do not specify the distribution of the V_x for the moment, but we will always ask that the resulting lattice is itself stationary and has no multiple points. Such requirements are dubbed Hypothesis (S) (for Simplicity):

$$\Phi_{\Psi,V} \text{ is a stationary, simple, point process.} \tag{S}$$

This is obviously the case for the basic model we are going to study, which is when Ψ is a stationarized lattice (say, $U + \mathbb{Z}^d$) and when (V_x) is a family of independent continuous random variables with common distribution V. It will be refered to as *independently perturbed lattice* (IPL), a model studied in numerous domains with different point of views.

9.1. Fluctuations of perturbed processes

The extent to which the fluctuations in a process are modified by pointwise perturbations depends on the strength of the perturbations. The following result generalizes [71] in the context where we do not assumptions on the independence of the V_x.

Theorem 9.1. *Let Φ be a hyperuniform point process and let $\Psi = \{x + V_x : x \in \Phi\}$ be its perturbation. Let $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function with $\delta(t) = o(t)$ and let X_δ be the number of $x \in \Phi$ such that $|V_x| \leqslant \delta(|x|)$. If X_δ is almost surely finite, then*

$$\lim_{r\to\infty} \frac{|\#(\Psi \cap B_r) - \#(\Phi \cap B_r)|}{\sqrt{|B_r|}} = 0 \tag{9.1.1}$$

in probability. Moreover, if $\mathbf{E}[X_\delta^2] < \infty$, the convergence holds in L^2 and Ψ is itself hyperuniform.

Note that the theorem is valid with no assumption on the joint distribution of the variables V_x — it is only implicitly assumed that Ψ is stationary, but they might have strong dependencies. Obviously, their dependencies will influence the choice of δ.

Proof. There are two reasons why $\#(\Phi \cap B_r)$ and $\#(\Psi \cap B_r)$ might differ: either one point x of Φ in B_r has $x + V_x$ outside B_r, or one point x of Φ outside B_r has $x + V_x$ inside B_r. These points are of two kind.
(1) Either x has $r - \delta(r) \leqslant |x| < r + \delta(r)$. There are at most $\#(\Phi \cap B_{r+\delta(r)}) - \#(\Phi \cap B_{r-\delta(r)})$ such points.
(2) Or $|x| > r + \delta(r)$ or $x < r - \delta(r)$, in which case $|V_x| > \delta(|x|)$, and there are less than X_δ such points.
Gathering both bounds yields

$$\frac{|\#(\Phi \cap B_r) - \#(\Psi \cap B_r)|}{|B_r|^{1/2}} \leqslant \frac{X_\delta}{|B_r|^{1/2}} + \frac{|\#(\Phi \cap B_{r+\delta(r)}) - \#(\Phi \cap B_{r-\delta(r)})|}{|B_r|^{1/2}}.$$

If Φ is hyperuniform, the second term goes to zero in probability as long as $\delta(r) = o(r)$. This settles the first part of the theorem, since $X_\delta/|B_r|^{1/2} \to 0$. The second part follows by taking the square expectations. □

Of course, it remains to study X_δ, depending on the distribution of V and Φ. The easiest case is when the perturbations are iid and have moment conditions. Let us note $f(x) = \mathbf{1}_{|V_x|>\delta(|x|)}$. Then, $X_\delta = \sum_{x\in\Phi} f(x)$, so by Campbell's formula

$$\mathbf{E}[X_\delta] = \int_{\mathbb{R}^2} f(u)\mathrm{d}u = \int_{\mathbb{R}^2} \mathbf{P}(V > \delta(u))\mathrm{d}u = |\mathbb{S}_{d-1}| \int_0^\infty r^{d-1}\mathbf{P}(V > \delta(r))\mathrm{d}r.$$

If δ is chosen so that this expectation is finite, then $X_\delta < \infty$ almost surely. For Gaussian random variables, $\delta(r) = r^\tau$ is enough, with $\tau < 1$. In general, if $\mathbf{E}[|V_x|^{d+\varepsilon}] < \infty$, then taking $\delta(r) = r^\tau$ for some $\tau < 1$ has the same effect. Indeed in this case $\mathbf{P}(V > \delta(r)) \leqslant c/r^{(d+\varepsilon)\tau}$, so choosing $d/(d+\varepsilon) < \tau < 1$ ensures that the integral above converges.

The preceding theorem has a very interesting consequence regarding the estimation of the spectrum of a perturbed process. We recall that the collective-cooordinate function of a perturbed lattice Φ is given by

$$\mathrm{n}_r(k) = \sum_{x \in \Phi \cap B_r} \mathrm{e}^{-\mathrm{i}\langle x,k\rangle} = \sum_{\substack{x \in \Psi \\ x+V_x \in B_r}} \mathrm{e}^{-\mathrm{i}\langle x,k\rangle}.$$

However, since this sum might involve points x far away from B_r but with V_x so large that $x + V_x$ actually falls in B_r. It is much simpler to compute the function

$$\tilde{\mathrm{n}}_r(k) = \sum_{x \in \Psi \cap B_r} \mathrm{e}^{-\mathrm{i}\langle x+V_x,k\rangle}.$$

Lemma 9.2. *If Φ and Ψ are hyperuniform, $|\mathrm{n}_r - \tilde{\mathrm{n}}_r|_\infty / \sqrt{|B_r|}$ converges to zero in probability.*

Proof. In the difference $\mathrm{n}_r - \tilde{\mathrm{n}}_r$, only the points x which are in the symmetric difference between $\Phi \cap B_r$ and $\Psi \cap B_r$ contribute, and each of these contributions is bounded by 2. Since Φ and Ψ have the same intensity, we thus have $|\mathrm{n}_r(k) - \tilde{\mathrm{n}}_r(k)| \leqslant 2|\#(\Phi \cap B_r) - \#(\Psi \cap B_r)|$ and we only have to check that this quantity divided by $|B_r|^{1/2}$ goes to zero . It is the case if both processes are hyperuniform. The proof shows that the bound is uniform in k. □

Note that any normalization like $r^{-a}(\mathrm{n}_r - \tilde{\mathrm{n}}_r)$ for $a > 1$ hence converges to zero almost surely. The most interesting consequence of this result is that one can approximate the structure factor using $\tilde{\mathrm{n}}_r$ instead of n_r:

$$\tilde{\mathcal{S}}_r := \frac{|\tilde{\mathrm{n}}_r|^2}{N_r} \to \mathcal{S} + \delta_0 \tag{9.1.2}$$

in the sense of distributions.

9.2. Correlations and structure measure of perturbed point processes

We hereby state the main computation for the statistics of perturbed point processes, which is just a refinement of an argument given in the appendices of [45] using the point of view of ergodicity to rigorously pass to the thermodynamic limit.

Theorem 9.3. *Let Ψ be a stationary, ergodic point process on $\mathbb{R}^d$ with structure factor $\mathcal{S}_\Psi$ and let $(V_x)_{x\in\Psi}$ be an iid family of random variables with common distribution V and characteristic function φ. We suppose that the perturbed point process*

$$\Phi = \{x + V_x : x \in \Psi\}$$

satisfies Hypothesis (S). Then, the structure measure $\mathcal{S}_\Phi$ of $\Phi_{\Psi,V}$ is given by

$$\mathcal{S}_\Phi = 1 - |\varphi|^2 + |\varphi|^2.\mathcal{S}_\Psi \tag{9.2.1}$$

where $f.\mathcal{S}_\Psi$ denotes the usual product between a tempered distribution and a bounded function[9]

[9] See appendix B.1. Here $\mathcal{S}_\Psi$ is a measure, so the $f.\mathcal{S}_\Psi$ is simply the measure with density f with respect to $\mathcal{S}_\Psi$.

Proof. We compute the expectation of $\tilde{\mathcal{S}}_r(k) = |\tilde{\mathrm{n}}_r(k)|^2/N_r$ and condition out the perturbations:

$$\begin{aligned}
\mathbf{E}[\tilde{\mathcal{S}}_r(k)] &= \mathbf{E}\left[1 + \frac{1}{N_r}\sum_{\substack{x,y\in\Psi\cap B_r\\ x\neq y}} \mathrm{e}^{-\mathrm{i}\langle x-y,\xi\rangle}\mathrm{e}^{-\mathrm{i}\langle V_x - V_y,\xi\rangle}\right] \\
&= \mathbf{E}\left[1 + \frac{1}{N_r}\sum_{\substack{x,y\in\Psi\cap B_r\\ x\neq y}} \mathrm{e}^{-\mathrm{i}\langle x-y,k\rangle}\mathbf{E}\left[\mathrm{e}^{-\mathrm{i}\langle V_x - V_y,k\rangle}\right]\right] \\
&= \mathbf{E}\left[1 + \frac{1}{N_r}\sum_{\substack{x,y\in\Psi\cap B_r\\ x\neq y}} \mathrm{e}^{-\mathrm{i}\langle x-y,k\rangle}\varphi(k)\varphi(k)^*\right] \\
&= 1 + |\varphi(k)|^2\mathbf{E}\left[\frac{1}{N_r}\sum_{\substack{x,y\in\Psi\cap B_r\\ x\neq y}} \mathrm{e}^{-\mathrm{i}\langle x-y,k\rangle}\right] \\
&= 1 + |\varphi(k)|^2\mathbf{E}[\mathcal{S}_\Psi(k) - 1].
\end{aligned}$$

We thus have $\mathbf{E}[\tilde{\mathcal{S}}_r(k)] = 1 - |\varphi(k)|^2 + |\varphi(k)|^2\mathbf{E}[\mathcal{S}_{\Psi,r}(k)]$. The RHS converges in the sense of distributions towards $1 - |\varphi|^2 + |\varphi|^2.\mathbf{E}[\mathcal{S}_\Psi] = 1 - |\varphi|^2 + |\varphi|^2.\mathcal{S}_\Psi$ and the LHS converges to $\mathbf{E}[\mathcal{S}_\Phi] = \mathcal{S}_\Phi$. □

We get an interesting corollary which appeared in the paper [45, Appendix B]. In the statement, by 'nontrivial', I mean that the perturbation V is not a constant.

Corollary 9.4. *Nontrivially perturbed point processes cannot be stealthy hyperuniform.*

Proof. For any positive Schwartz function $f \in \mathscr{S}(\mathbb{R}^d)$, one has

$$\langle \mathcal{S}_{\Psi,V}, f\rangle \geqslant \int_{\mathbb{R}^d} f(x)(1 - |\varphi(x)|^2)\mathrm{d}x.$$

The characteristic function of a nonconstant random variable can never have modulus 1 in a neighborhood of the origin, hence one can find functions f with arbitrary small support and for which the right-hand side of the preceding inequality is nonzero, forbidding stealthiness. □

Another crucial corollary is that we only have to inverse the Fourier Transform to get back the correlation measure of perturbed point processes. The convolution between tempered distributions is defined in(B.2.1), at page 56 in the Appendix.

Corollary 9.5. *If the random variable V is absolutely continuous with probability density function p, the correlation measure $\mathcal{C}$ of Φ is given by*

$$\mathcal{C}(\mathrm{d}x) = \delta_0 + \lambda[(p * \tilde{p}) * g_2 - \mathbf{1}] \tag{9.2.2}$$

where $\tilde{p}(t) = p(-t)$, $*$ denotes the convolution operator between tempered distributions, and g_2 is the pair correlation measure of the original point process Ψ.

Proof. The Fourier transform is injective, so we only have to check that the FT of (9.2.2) matches $\mathcal{S}_\Psi$ given in (9.2.1). It is sufficient to check that $\mathscr{F}[p * \tilde{p} * g_2 - 1] = |\varphi|^2\mathscr{F}[g_2 - 1]$. Since p is a probability density, $p * \tilde{p} * 1 = 1$, hence $\mathscr{F}[p * \tilde{p} * g_2 - 1] = \mathscr{F}[p * \tilde{p} * (g_2 - 1)] = |\varphi|^2\mathscr{F}[g_2 - 1]$. □

Remark 9.6. The preceding Corollary can easily be extended: it remains valid for any V whose distribution, say ν, can be convoluted with the measure g_2. The formula is exactly the same.

Example 9.7 (correlations of perturbed lattices)**.** Let $\Psi = U + \mathbb{L}$ be a stationarized lattice with intensity λ. The correlation measure g_2 is $\lambda^{-1}\Delta_{\mathbb{L}^*} = \lambda^{-1}\sum_{x\in\mathbb{L}, x\neq 0}\delta_x$. Then we perturb it into $\Psi = \{x + V_x : x \in \Psi\}$. If the perturbations are independent and distributed as some random variable V with probability density p, then the correlation is

$$\mathcal{C}(\mathrm{d}x) = \delta_0 + \sum_{z\in\mathbb{L}\setminus\{0\}} (p * \tilde{p})(z - x) - \lambda, \tag{9.2.3}$$

see for example [45].

References to include

Gabrielli : [23]. Recovering the underlying lattice : [71]. Sodin and Tsirelson : [65]. Rigidity and tolerance : [60]. Cloaking : [45]. Matchings : [48]. DFS on perturbed lattices : [33], IPL study : [28]

10. Reconstruction and rigidity of the independently perturbed lattice

In this section we restrict to the case $\Psi = \mathbb{L}$, with $\mathbb{L}$ a Bravais lattice such as $\mathbb{Z}^d$ for instance, and we perturb it with iid perturbations V_x, thus obtaining the point process $\Phi = \Phi_{\mathbb{L},V}$. This process is hyperuniform and its number fluctuation are easily studied. But it is rigid ? There is also a closely related question regarding what I call *reconstructibility*.

Definition 10.1. Let Φ, Ψ be two point processes. We say that Ψ is Φ-reconstructible if there is a measurable function such that $\Psi = F(\Phi)$ almost surely.

Rigidity and reconstructibility are different notions, but they both describe how processes are sensitive to perturbations: deletion of points for rigidity, spatial perturbation of points for reconstructibility.

When it comes to perturbed lattices, it turns out that their rigidity and reconstructibility depend on the dimension and the strength of the perturbations. The results in this section can be summarized as follows when the perturbations are iid Gaussians with variance σ^2.

(i) In one or two dimensions, Φ is rigid and Ψ-reconstructible.

(ii) When $d \geqslant 3$, Ψ is Ψ-reconstructible, but there is a $\sigma_c(d) > 0$ such that Φ is rigid if and only if $\sigma < \sigma_c(d)$.

For the first point, rigidity is a consequence of Theorem 6.6. The proof for reconstructibility is the same in all dimensions and done in the first subsection; in the second one we focus on the rigidity statement in the second point.

10.1. Reconstructibility: a positive result

We recall that n_r, the 'collective coordinate function', was defined in (5.1.2). The original lattice is $\Phi = \mathbb{L}$ and the perturbed one is Ψ. The perturbations are iid copies of a common random variable V. We define

$$\mathscr{L}_r(\xi) := \frac{|\mathrm{n}_r(\xi)|}{\#(\Phi \cap B(0,r))} = \frac{1}{\#(\Phi \cap B(0,r))} \left| \sum_{w \in \Phi \cap B(0,r)} \mathrm{e}^{-\mathrm{i}\langle x, \xi \rangle} \right|. \tag{10.1.1}$$

Theorem 10.2. *We suppose that $\mathbf{E}[|V|^{d+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Then, with probability one, the following holds:*

$$\lim_{r \to \infty} \mathscr{L}_r(\xi) = \begin{cases} \varphi(\xi) \text{ if } 2\pi\xi \in \mathbb{L}^* \\ 0 \text{ else} \end{cases} \qquad (\forall \xi \in \mathbb{R}^d) \tag{10.1.2}$$

where $\varphi(\xi) = \mathbf{E}[\mathrm{e}^{-\mathrm{i}\langle \xi, V \rangle}]$ is the Fourier transform of V.

This allows to reconstruct the dual lattice $\mathbb{L}^*$, and then the original lattice. It is easy to prove that (10.1.2) holds almost surely for a single $\xi \in \mathbb{R}^d$, but proving that (10.1.2) holds almost surely simultaneously for every ξ is subtler (and is the content of [71]).

Proof of a.s. convergence for any fixed ξ. Let us fix $\xi \in \mathbb{R}^d$. We note $\tilde{\mathscr{L}}_r(\xi) = |B_r|^{-1} \tilde{\mathrm{n}}_r(\xi)$. By independence,

$$\mathbf{E}[\mathscr{L}_r(\xi)] = \frac{1}{|B_r|} \sum_{x \in \mathbb{L} \cap B_r} \mathrm{e}^{-\mathrm{i}\langle x, \xi \rangle} \mathbf{E}[\mathrm{e}^{-\mathrm{i}\langle V_x, \xi \rangle}] = \frac{\varphi(\xi)}{|B_r|} \sum_{x \in \mathbb{L} \cap B_r} \mathrm{e}^{-\mathrm{i}\langle x, \xi \rangle}.$$

It is a classical result that this last term converges as $r \to \infty$ towards $\varphi(\xi)\mathbf{1}_{\xi \in 2\pi\mathbb{L}^*}$, where $\mathbb{L}^*$ is the dual lattice. We now turn to the second moment:

$$\begin{aligned}\mathbf{E}[|\tilde{\mathscr{L}}_r(\xi)|^2] &= \frac{1}{|B_r|^2}\left(\#(\mathbb{L}\cap B_r) + \sum_{\substack{x,y\in\mathbb{L}\cap B_r\\ x\neq y}} \mathrm{e}^{-\mathrm{i}\langle x-y,\xi\rangle}\varphi(\xi)\overline{\varphi(\xi)}\right)\\ &= \frac{1}{|B_r|^2}\left(\#(\mathbb{L}\cap B_r) + |\varphi(\xi)|^2\left|\sum_{x\in\mathbb{L}\cap B_r}\mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle}\right|^2 - \#(\mathbb{L}\cap B_r)|\varphi(\xi)|^2\right)\\ &= \frac{\#(\mathbb{L}\cap B_r)(1-|\varphi(\xi)|^2)}{|B_r|^2} + |\mathbf{E}[\tilde{\mathscr{L}}_r(\xi)]|^2\end{aligned}$$

which shows that $\mathrm{Var}(\tilde{L}_r(\xi)) = O(1/|B_r|) = O(r^{-d})$. In dimension $d \geqslant 2$, this is summable; we only need to apply the Borel-Cantelli lemma to prove that $\tilde{\mathscr{L}}_r(\xi) \to \varphi(\xi)\mathbf{1}_{\xi\in 2\pi\mathbb{L}^*}$ almost surely, and then we need to check that $\mathscr{L}_r(\xi) - \tilde{\mathscr{L}}_r(\xi) \to 0$ almost surely — an easy consequence of Lemma 9.2 along with the discussion after Theorem 9.1. □

For the illustration in Figure 4, I took $\mathbb{L} = G^*\tau\mathbb{Z}^2$ where G is given by

$$G = \begin{pmatrix} 1.75516512 & 0.23971277\\ -0.95885108 & 0.43879128\end{pmatrix}.$$

The perturbation V is gaussian, $V \sim \mathscr{N}(0,\sigma^2)$, and it is well known that $\varphi(\xi) = \exp\{-\sigma^2|\xi|^2/2\}$. In order to have a striking illustration of Theorem 10.2, we will plot $\xi \mapsto \mathrm{e}^{\frac{\sigma^2|\xi|^2}{2}}\mathscr{L}_r(\xi)$, which converges as $r \to \infty$ towards 1 if and only if $\xi \in \mathbb{L}^*$.

10.2. Rigidity and tolerance: a negative result

We now turn to the rigidity of Ψ as in the Peres-Sly paper [60]. In one and two dimensions, the question is settled by Theorem 6.6.

Two random variables X, Y are said to be *mutually absolutely continuous* if for any event E, we have $\mathbf{P}(X \in E) = 0$ iff $\mathbf{P}(Y \in E) = 0$. They are said to be *mutually singular* when they are supported on disjoint subsets.

Definition 10.3. A point process Φ is deletion-tolerant if there is a random variable X with values in Φ, such that the distribution of $\Phi \setminus \{Z\}$ is absolutely continuous with respect to the distribution of Φ.

In other words, one can remove the point Z from Φ without being detected. It is proved in [60] that if a process is rigid, it cannot be deletion-tolerant. We do not prove this, but we are going to check a condition on the perturbations V to ensure that a process is or is not deletion-tolerant, which is already a significant result.

The statement and proof make use of the *exponential intersection tails* property, which is defined now. First, an oriented path on Ψ is a sequence of *distinct* points in Ψ, say $(\gamma_t)_{t\geqslant 0}$ with $\gamma_{t+1} \neq \gamma_t$ for each t. Let γ, γ' be two independent and identically distributed random variables on oriented paths. We say that their common distribution ν has the exponential intersection tails property with parameter $\theta \in (0,1)$ if

$$\nu \otimes \nu\{(\gamma,\gamma') : E(\gamma,\gamma') \geqslant n\} = O(\theta^n) \tag{10.2.1}$$

where $E(\gamma,\gamma')$ is the number of common *edges* of γ and γ'. The existence of random paths with the EIT property will be discussed later.

We now introduce another constraint on these paths. We say that a random variable γ on oriented paths on Ψ has the BEIT property with parameters θ, C if it has the EIT property with parameter θ, and if in addition,

$$\forall t \geqslant 0, \qquad |\gamma_{t+1} - \gamma_t| \leqslant C. \tag{10.2.2}$$

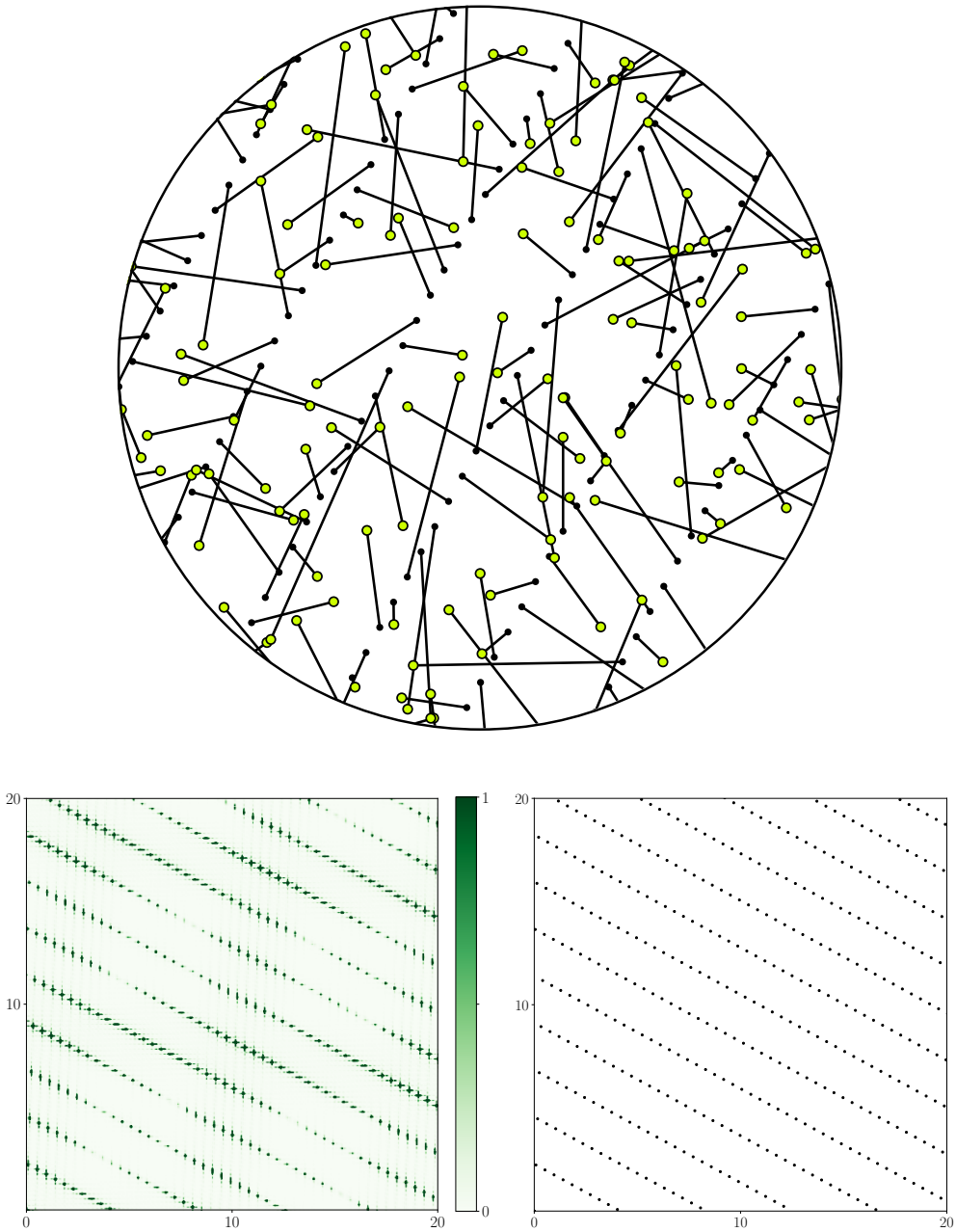

Figure 4: The above picture shows the perturbed lattice. The little black dots are the original lattice points $x \in \mathbb{L}$ while the corresponding green dots are the perturbed points $x + V_x$, where $V_x \sim \mathcal{N}(0, \sigma^2)$ where $\sigma^2 = 5$. The lower left panel is a colorplot of the function $\xi \mapsto \mathrm{e}^{\frac{\sigma^2|\xi|^2}{2}} \mathscr{L}_r(\xi)$ on the window $W = [0, 20]^2$; this function converges as $r \to \infty$ towards 1 if and only if $\xi \in \mathbb{L}^*$. The dual lattice $\mathbb{L}^*$ in the same window is displayed in the lower right panel.

In other words, the path does not do jumps with size bigger than C.

Definition 10.4. The point process Ψ is (θ, C)-escapable if, conditionnally on Ψ, there is a random variable on directed paths on Ψ with the BEIT property with parameters (θ, C). Such paths will be called 'escape paths'.

Remark 10.5. I asked these paths to have distinct points, ie no loops/self-avoiding. However, it is not a restriction. If one has a path γ which is possibly loopy but still goes go ∞, then one can erase the loops in γ with some specific procedure and get a self-avoiding path $\hat{\gamma}$ and it is clear that for any other path γ' one has $E(\gamma, \gamma') \geqslant E(\hat{\gamma}, \gamma')$.

Theorem 10.6 ([60], Proposition 2.1 slightly improved)**.** *Let Ψ be a (θ, C)-escapable point process and let V be an absolutely continuous random variable with support $\mathbb{R}^d$ and density g. Then, there is a number $\theta_c(d) > 1$ such that if*

$$\max_{|u| \leqslant C} \int_{\mathbb{R}^d} \left(\frac{g(x+u)}{g(x)} \right)^2 g(x) \mathrm{d}x < \theta_c(d) \tag{10.2.3}$$

then the independently perturbed lattice $\Phi = \Phi_{\Psi,V}$ is deletion-tolerant.

A generalization to random perturbations with smaller support (such as uniform) is in [3]. Regarding which Ψ are escapable, I refer to the discussion after the proof.

Proof. Let Ψ be the underlying point process. Everything will be conditional on Ψ, and I will emphasize this fact by using $\mathbf{E}_\Psi$ as a notation for the conditional expectation with respect to Ψ. From now, we fix a probability distribution on oriented paths on Ψ with the BEIT property conditionaly on Ψ. It will be noted by η. Let $V = (V_x)_{x \in \Psi}$ be the set of perturbations and $\gamma = (\gamma_t)_{t \in \mathbb{N}}$ be a fixed oriented path on Ψ. We note

$$V_x^\gamma = \begin{cases} V_x + \gamma_{t+1} - \gamma_t \text{ if } x \in \gamma, x = \gamma_t \\ V_x \text{ else.} \end{cases}$$

In other words, the perturbations are 'shifted' along the path γ. We note ν the distribution of $\Phi = \Phi_{\Psi,V}$, ν^γ the distribution of $\Phi^\gamma := \Phi_{\Psi,V^\gamma}$ and $\tilde{\nu}$ the distribution of $\tilde{\Phi} := \Phi_{\Psi,V^\Gamma}$ where Γ is a random path with distribution η. For any event E,

$$\tilde{\nu}(E) = \int \nu^\gamma(E) \eta(\mathrm{d}\gamma). \tag{10.2.4}$$

Lemma 10.7. *$\tilde{\nu}$ is the probability distribution of $\Phi^* := \{x + V_x : x \neq \gamma_0\} = \Phi \setminus Z$ where we noted Z the random variable $\gamma_0 + V_{\gamma_0}$.*

Proof. Each point in $\tilde{\Phi}$ is a perturbation of a point in $\Phi \setminus \gamma_0$ and these perturbations are iid. □

Let m be an integer and let ν_m be the probability distribution of

$$X_m := \{V_x : x \in \Psi \cap B(0,m)\}.$$

We similarly define ν_m^γ and $\tilde{\nu}_m$. Those are probability distributions on $\mathbb{R}^{dk}$ where $k = |\Psi \cap B(0,m)|$ is the number of points of Ψ in $B(0,m)$. Finally we set $F_m = \mathrm{d}\tilde{\nu}_m / \mathrm{d}\nu_m$ (it is a Radon-Nikodym derivative). Then,

$$\begin{aligned} \mathbf{E}_\Psi[F_m(X_m)^2] &= \int F_m(v)^2 \nu_m(\mathrm{d}v) \\ &= \int \left(\int \frac{\mathrm{d}\nu_m^\gamma}{\mathrm{d}\nu_m}(v) \eta(\mathrm{d}\gamma) \right)^2 \nu_m(\mathrm{d}v) \\ &= \int \int \int \frac{\mathrm{d}\nu_m^\gamma}{\mathrm{d}\nu_m}(v) \frac{\mathrm{d}\nu_m^{\gamma'}}{\mathrm{d}\nu_m}(v) \eta(\mathrm{d}\gamma) \eta(\mathrm{d}\gamma') \nu_m(\mathrm{d}v) \\ &= \int \int \left[\int \frac{\mathrm{d}\nu_m^\gamma}{\mathrm{d}\nu_m}(v) \frac{\mathrm{d}\nu_m^{\gamma'}}{\mathrm{d}\nu_m}(v) \nu_m(\mathrm{d}v) \right] \eta(\mathrm{d}\gamma) \eta(\mathrm{d}\gamma'). \end{aligned} \tag{10.2.5}$$

Let us focus on the term in brackets. It is clear that due to the V_{x_i} being independent, ν_m is indeed equal to the product measure

$$\bigotimes_{i=1}^{k} g(v_i)\mathrm{d}v_i.$$

Similarly, let us fix γ and note $\delta_t = \gamma_{t+1} - \gamma_t$. By independence, we have

$$\nu_m^{\gamma}(\mathrm{d}v_1, \dots, \mathrm{d}v_k) = \bigotimes_{i=1}^{k} g(v_i + \alpha_i)\mathrm{d}v_i$$

where $\alpha_i = 0$ if $v_i \notin \gamma$, and $\alpha_i = \delta_t$ if $x = \gamma_t$. A similar fact holds for γ'. Consequently, the term inside brackets in (10.2.5) above is

$$\left[\int \frac{\mathrm{d}\nu_m^{\gamma}}{\mathrm{d}\nu_m}(v)\frac{\mathrm{d}\nu_m^{\gamma'}}{\mathrm{d}\nu_m}(v)\nu_m(\mathrm{d}v)\right] = \prod_{i=1}^{k}\int \frac{g(v_i+\alpha_i)}{g(v_i)}\frac{g(v_i+\alpha_i')}{g(v_i)}g(v_i)\mathrm{d}v_i.$$

If v_i is in γ and not in γ', then clearly the integral above is equal to 1, because g is a probability density. Consequently, the non-trivial contributions in the product above correspond to the i such that $v_i \in \gamma \cap \gamma'$, say $v_i = \gamma_t = \gamma_s'$. In this case, one has

$$\begin{aligned}
\int \frac{g(v_i+\alpha_i)}{g(v_i)}\frac{g(v_i+\alpha_i')}{g(v_i)}g(v_i)\mathrm{d}v_i &= \int \frac{g(v_i+\delta_t)}{g(v_i)}\frac{g(v_i+\delta_s')}{g(v_i)}g(v_i)\mathrm{d}v_i \\
&\leqslant \sqrt{\int \left(\frac{g(v_i+\delta_t)}{g(v_i)}\right)^2 g(v_i)\mathrm{d}v_i}\sqrt{\int \left(\frac{g(v_i+\delta_s')}{g(v_i)}\right)^2 g(v_i)\mathrm{d}v_i} \\
&\leqslant \sqrt{\Sigma(\delta_t)\Sigma(\delta_s)}
\end{aligned}$$

where we noted $\Sigma(u) = \int (g(v+u)/g(v))^2 g(v)\mathrm{d}v$. We made the hypothesis that

$$\sup_{|u|\leqslant C} \Sigma(u) \leqslant \tau/\theta$$

where θ is the EIT parameter, τ is a fixed number in $(0,1)$ and C the maximum size of the steps in the path. Let us note $N_m(\gamma,\gamma')$ the number of points in $B(0,m)$ belonging to $\gamma \cap \gamma'$. Going back to (10.2.5),

$$\begin{aligned}
\mathbf{E}_\Psi[F_m(X_m))^2] &\leqslant \mathbf{E}_\Psi \iint \prod_{t\in N_m(\gamma,\gamma')} \frac{\tau}{\theta}\eta(\mathrm{d}\gamma)\eta(\mathrm{d}\gamma') \\
&= \mathbf{E}_\Psi[(\tau/\theta)^{N_m(\gamma,\gamma')}] \\
&= \sum_{n=0}^{\infty} \mathbf{P}_\Psi(N_m(\gamma,\gamma') = n)(\tau/\theta)^n \\
&\leqslant \sum_{n=0}^{\infty} \mathbf{P}_\Psi(E(\gamma,\gamma') \geqslant n)(\tau/\theta)^n \\
&= \sum_{n=0}^{\infty} O(\tau^N) \leqslant O((1-\tau)^{-1})
\end{aligned} \tag{10.2.6}$$

where we used the θ-EIT property in the last line. The random variables $F_m(X_m) = \mathrm{d}\tilde{\nu}_m/\mathrm{d}\nu_m$ are positive martingales and it is known (see [20], Section 4.3.3) that F_m converges ν-almost surely to F. Additionally, if they are L^2-bounded, then

$$\tilde{\nu}(E) = \int_E F(v)\nu(\mathrm{d}v)$$

and $\tilde{\nu} \ll \nu$. But (10.2.6) precisely says that $F_m(X_m)$ is L^2-bounded, so we proved that $\tilde{\nu} \ll \nu$ and since $\tilde{\nu}$ is the law of $\Phi^* = \Phi \setminus Z$ (see the lemma above), then the proof that Φ is deletion-tolerant is complete. □

Condition 10.2.3 simplifies when g is the density of a gaussian random variable with variance σ. A simple computation shows that it is true as soon as σ is large enough.

Let us now discuss the 'escapable' property for point processes.

1. For regular lattices, like $\mathbb{Z}^d$ or the triangular lattice, the point process is naturally embedded with a graph structure where each point x is linked with its neighbors $x+e$ where e runs through the lattice basis. Finding 'escape paths' then reduces to finding random walks on $\mathbb{Z}^d$ with the EIT property. Such a problem is discussed in [8], where it is essentially proved that paths with EIT exist when $d \geqslant 3$ and do not when $d < 3$ — actually, when $d \geqslant 4$, the simple random walk has the EIT property.

2. For general processes Φ, one can prove the existence of escape paths by using matchings (inspired by [48]): let $\tau : \mathbb{Z}^d \to \Phi$ be a matching (for instance, closest iterative neighbor, or stable matching). If γ is an escape path in $\mathbb{Z}^d$, then $\tau(\gamma)$ is an escape path in Φ and since τ is one-to-one, $\tau(\gamma)$ has the EIT iff γ has the EIT. The jump condition might not be trivial to satisfy!

10.3. Invisibility of the underlying lattice: cloaking

Let $\Phi = U + \mathbb{L}$ be a stationarized lattice and let $\Psi = \{x + V_x : x \in \Phi\}$ be its perturbed version, where the perturbations are iid versions of a common random variable V with density p. The correlation measure $\mathcal{C}$ of Ψ is given by (9.2.3):

$$\begin{aligned}\mathcal{C}(\mathrm{d}x) &= \delta_0 + \int p(u) \sum_{z \in \mathbb{L}\setminus\{0\}} p(u+z-x) - \lambda \\ &= \delta_0 + \int p(u) \sum_{z \in \mathbb{L}} p(u+z-x) - \lambda p * \tilde{p}(x) - \lambda. \end{aligned} \tag{10.3.1}$$

The point process Ψ is said to exhibit *cloaking* ([45]) if the underlying structure $\mathbb{L}$ is hidden by the perturbation, ie when there is a constant c such that

$$\sum_{z \in \mathbb{L}} p(z-x) = c \qquad \forall x \in \mathbb{R}^d. \tag{10.3.2}$$

Note that if this is true, then constant c has to be λ. If this is the case, the correlation measure does not really depend on the underlying lattice $\mathbb{L}$ and becomes $\mathcal{C} = \delta_0 + \lambda\mathbf{1} - p*\tilde{p} - \lambda\mathbf{1} = \delta_0 - \lambda p * \tilde{p}$ and the structure factor is $\mathcal{S} = 1 - |\varphi|^2$. This leads to another characterization: a perturbed lattice is cloaked if $|\varphi|^2.\mathcal{S}_{\mathbb{L}} = |\varphi|^2 \Delta_{\mathbb{L}^*}$ is identically zero, or equivalently if φ vanishes at every point in the dual lattice $\mathbb{L}^*$.

Example 10.8 (URL)**.** Consider the simple example of $\Phi = U + \tau\mathbb{Z}^d$, whose dual lattice is $\mathbb{Z}^d$. We perturb the points of Φ by independent displacements which are uniformly distributed on the box $[-a, a]^d$ for some $a > 0$. Such a model is often called *Uniformly Randomized Lattice* (URL) . The characteristic function of the generic displacement V is given by (1.4.5), ie for $\xi = (\xi_1, \dots, \xi_d)$:

$$\varphi_V(\xi) = 2^d \prod_{i=1}^{d} \frac{\sin(a\xi_i)}{\xi_i}.$$

If we choose $a \in \mathbb{N}$, this function clearly vanishes at every $\xi \in \mathbb{Z}^d$, hence we will clearly have $|\varphi|^2 \mathcal{S}_\Phi = |\varphi|^2 \Delta_{\mathbb{Z}^d} = 0$ and the perturbed lattice Ψ is cloaked.

Example 10.9 (Einstein lattice)**.** When the perturbations are standard gaussian variables, $|\varphi(k)|^2 = \mathrm{e}^{-|k|^2/2}$ which never vanishes, the gaussian perturbations can never cloak an underlying lattice. This is why in this case we can always reconstruct the underlying lattice using second-order methods ([71]), but reconstruction of cloaked lattices require higher-order methods ([45]).

11. Dependently perturbed lattices (DPL)

We now take a look at some models of perturbed lattices $\{x + V_x : x \in \mathbb{L}\}$, but where the perturbations V are no longer iid.

11.1. Stable matchings

[48]

11.2. Sliders are stealthy hyperuniform

We saw that independenly perturbed lattices cannot be stealthy. However, when the perturbations are dependent, it can be the case.

Sliders are a very simple example of dependent perturbations on lattices, notably studied in [72]. Formally, the lattice point (n, m) is going to be shifted to $(n, m) + (u_m, v_n)$, where u_i, v_j are two independent families of perturbations. The model can be described as follows: all the points in the horizontal line $L_h = \{(n, h) : n \in \mathbb{Z}\}$ of $\mathbb{Z}^2$ are to their right by the same random displacement u_h. This is called the stacked slider in [72]:

$$\Phi_1 = \{(n + u_h, h) : (n, h) \in \mathbb{Z}^2\}$$

Then, we note H_k the points in the stacked slider coming from the horizontal line with first coordinate k: $H_k = \{(k + u_n, n) : n \in \mathbb{Z}\}$, and then we shift upward each of the points in H_k by a common random displacement v_k. The resulting shifted 'line' is $\{(k+u_n, n+v_k) : n \in \mathbb{Z}\}$, and their union is the double-slider. The displacements $U = (u_i : i \in \mathbb{Z})$ and $V = (v_i, i \in \mathbb{Z})$ are supposed independent and uniformly distributed over $[0, a]$. When $a = 1$, the crystal structure is hidden. The *doubly-slided lattice* can thus be written

$$\Phi = \{(n + u_m, m + u_n) : (n, m) \in \mathbb{Z}^2\}. \tag{11.2.1}$$

The construction of sliders can easily be extended to higher-dimensional spaces, and can also be adapted to modify other processes than grids. Note that the construction in itself can easily be generalized to any point process with a stationary colouring: points with a certain colour are shifted, then points with another colour, etc.

The pair-correlation measure of the doubly-slided lattice is

$$g_2 = \sum_{n,m} \delta_{(n+u'm, m+u'_n)} \tag{11.2.2}$$

where the u'_i, v'_j are iid for $i \neq 0, j \neq 0$, and $u'_i = v'_j = 0$.

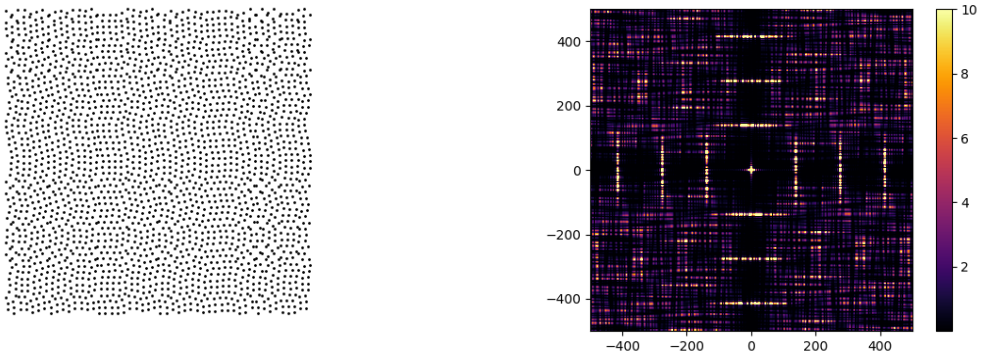

Figure 5: A doubly-slided lattice and its structure factor.

References

[1] Kartick Adhikari, Subhroshekhar Ghosh, and Joel L. Lebowitz. Fluctuation and entropy in spectrally constrained random fields, 2020.

[2] Martin Aigner. *A course in enumeration*, volume 238. Springer Science & Business Media, 2007.

[3] Yuta Arai. Rigidity and non-rigidity for uniform perturbed lattice, 2019.

[4] François Baccelli, Bartlomiej Blaszczyszyn, and Mohamed K Karray. *Random Measures, Point Processes and Stochastic Geometry*. work in progress (private communication), 2020+.

[5] François Baccelli and Eliza O'Reilly. Reach of repulsion for determinantal point processes in high dimensions. *Journal of Applied Probability*, 55(3):760–788, 2018.

[6] Rémi Bardenet, Frédéric Lavancier, Xavier MARY, and Aurélien Vasseur. On a few statistical applications of determinantal point processes. *ESAIM: Proceedings and Surveys*, 60, 2017.

[7] József Beck. Irregularities of distribution. i. *Acta Mathematica*, 159(1):1–49, 1987.

[8] Itai Benjamini, Robin Pemantle, Yuval Peres, et al. Unpredictable paths and percolation. *The Annals of Probability*, 26(3):1198–1211, 1998.

[9] C. Berg and G. Forst. *Potential Theory on Locally Compact Abelian Groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1975.

[10] Bartłomiej Błaszczyszyn. Lecture Notes on Random Geometric Models — Random Graphs, Point Processes and Stochastic Geometry, 2017. In preparation.

[11] Alexander I Bufetov. Rigidity of determinantal point processes with the airy, the bessel and the gamma kernel. *Bulletin of Mathematical Sciences*, 6(1):163–172, 2016.

[12] Alexander I Bufetov, Yoann Dabrowski, and Yanqi Qiu. Linear rigidity of stationary stochastic processes. *Ergodic Theory and Dynamical Systems*, 38(7):2493–2507, 2018.

[13] Alexander I Bufetov, Pavel P Nikitin, and Yanqi Qiu. On number rigidity for pfaffian point processes. *arXiv preprint arXiv:1810.09223*, 2018.

[14] B. Błaszczyszyn, D. Yogeshwaran, and J. E. Yukich. Limit theory for geometric statistics of point processes having fast decay of correlations. *The Annals of Probability*, 47(2):835 – 895, 2019.

[15] Sourav Chatterjee. Rigidity of the three-dimensional hierarchical coulomb gas. *Probability Theory and Related Fields*, pages 1–54, 2017.

[16] Sung Nok Chiu, Dietrich Stoyan, Wilfrid S Kendall, and Joseph Mecke. *Stochastic geometry and its applications*. John Wiley & Sons, 2013.

[17] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer Series in Statistics. Springer New York, 2013.

[18] David Dereudre, Adrien Hardy, Thomas Leblé, and Mylène Maïda. Dlr equations and rigidity for the sine-beta process, 2018.

[19] Ioana Dumitriu and Alan Edelman. Matrix models for beta ensembles. *Journal of Mathematical Physics*, 43(11):5830–5847, 2002.

[20] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[21] Peter J Forrester. *Log-gases and random matrices (LMS-34)*. Princeton University Press, 2010.

[22] A. Gabrielli, B. Jancovici, M. Joyce, J. L. Lebowitz, L. Pietronero, and F. Sylos Labini. Generation of primordial cosmological perturbations from statistical mechanical models. *Phys. Rev. D*, 67:043506, Feb 2003.

[23] Andrea Gabrielli. Point processes and stochastic displacement fields. *Physical Review E*, 70(6):066131, 2004.

[24] Shirshendu Ganguly and Sourav Sarkar. Ground States And Hyperuniformity Of The Hierarchical Coulomb Gas In All Dimensions. *arXiv e-prints*, page arXiv:1904.05321, Apr 2019.

[25] Donald Geman et al. On the variance of the number of zeros of a stationary gaussian process. *The Annals of Mathematical Statistics*, 43(3):977–982, 1972.

[26] Subhro Ghosh and Manjunath Krishnapur. Rigidity hierarchy in random point fields: random polynomials and determinantal processes, 2015.

[27] Subhro Ghosh and Joel Lebowitz. Number rigidity in superhomogeneous random point fields. *Journal of Statistical Physics*, 166(3-4):1016–1027, 2017.

[28] Subhro Ghosh, Naoto Miyoshi, and Tomoyuki Shirai. Disordered complex networks: energy optimal lattices and persistent homology, 2020.

[29] Subhroshekhar Ghosh. Determinantal processes and completeness of random exponentials: the critical case. *Probability Theory and Related Fields*, 163(3-4):643–665, 2015.

[30] Subhroshekhar Ghosh and Joel L. Lebowitz. Fluctuations, large deviations and rigidity in hyperuniform systems: a brief survey. *Indian J. Pure Appl. Math.*, 48(4):609–631, 2017.

[31] Subhroshekhar Ghosh and Joel L. Lebowitz. Generalized Stealthy Hyperuniform Processes: Maximal Rigidity and the Bounded Holes Conjecture. *Communications in Mathematical Physics*, 363(1):97–110, Oct 2018.

[32] Subhroshekhar Ghosh, Yuval Peres, et al. Rigidity and tolerance in point processes: Gaussian zeros and ginibre eigenvalues. *Duke Mathematical Journal*, 166(10):1789–1858, 2017.

[33] Subhroshekhar Ghosh and Kumarjit Saha. Transmission and navigation on disordered lattice networks, directed spanning forests and scaling limits, 2020.

[34] André Goldman et al. The palm measure and the voronoi tessellation for the ginibre process. *The Annals of Applied Probability*, 20(1):90–128, 2010.

[35] Sheldon Goldstein, JL Lebowitz, and E Speer. Large deviations for a point process of bounded variability.

[36] L. Hörmander. *The analysis of linear partial differential operators: Distribution theory and Fourier analysis*. Springer Study Edition. Springer-Verlag, 1990.

[37] John Ben Hough, Manjunath Krishnapur, Yuval Peres, et al. *Zeros of Gaussian analytic functions and determinantal point processes*, volume 51. American Mathematical Soc., 2009.

[38] B. Jancovici. Exact results for the two-dimensional one-component plasma. *Phys. Rev. Lett.*, 46:386–388, Feb 1981.

[39] Mark Kac. A correction to "on the average number of real roots of a random algebraic equation". *Bulletin of the American Mathematical Society*, 49(12):938, 1943.

[40] Rowan Killip, Mihai Stoiciu, et al. Eigenvalue statistics for cmv matrices: from poisson to clock via random matrix ensembles. *Duke Mathematical Journal*, 146(3):361–399, 2009.

[41] Jaeuk Kim and Salvatore Torquato. Effect of window shape on the detection of hyperuniformity via the local number variance. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(1):013402, 2017.

[42] Jaeuk Kim and Salvatore Torquato. Methodology to construct large realizations of perfectly hyperuniform disordered packings. *Phys. Rev. E*, 99:052141, May 2019.

[43] Jaeuk Kim and Salvatore Torquato. New tessellation-based procedure to design perfectly hyperuniform disordered dispersions for materials discovery. *Acta Materialia*, 168:143–151, 2019.

[44] Avner Kiro and Alon Nishry. Rigidity for zero sets of gaussian entire functions, 2018.

[45] Michael A. Klatt, Jaeuk Kim, and Salvatore Torquato. Cloaking the Underlying Long-Range Order of Randomly Perturbed Lattices. *arXiv e-prints*, page arXiv:2001.08161, Jan 2020.

[46] Michael A. Klatt and Günter Last. On strongly rigid hyperfluctuating random measures, 2020.

[47] Michael A Klatt, Jakov Lovrić, Duyu Chen, Sebastian C Kapfer, Fabian M Schaller, Philipp WA Schönhöfer, Bruce S Gardiner, Ana-Sunčana Smith, Gerd E Schröder-Turk, and Salvatore Torquato. Universal hidden order in amorphous cellular geometries. *Nature communications*, 10(1):811, 2019.

[48] Michael Andreas Klatt, Günter Last, and D. Yogeshwaran. Hyperuniform and rigid stable matchings. *arXiv e-prints*, page arXiv:1810.00265, Sep 2018.

[49] A Kolmogoroff. Interpolation und extrapolation von stationaren zufalligen folgen. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 5(1):3–14, 1941.

[50] Raphaël Lachièze-Rey. Variance linearity for real gaussian zeros, 2020.

[51] Pierre Yves Gaudreau Lamarre, Promit Ghosal, and Yuchen Liao. Spectral rigidity of random schrödinger operators via feynman-kac formulas, 2019.

[52] G. Last and M. Penrose. *Lectures on the Poisson Process*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2017.

[53] N.N. Lebedev and R.A. Silverman. *Special Functions and Their Applications*. Dover Books on Mathematics. Dover Publications, 1972.

[54] Joel L Lebowitz. Charge fluctuations in coulomb systems. *Physical Review A*, 27(3):1491, 1983.

[55] Russell Lyons. Determinantal probability measures. *Publications Mathématiques de l'IHÉS*, 98:167–212, 2003.

[56] Odile Macchi. The coincidence approach to stochastic point processes. *Advances in Applied Probability*, 7(1):83–122, 1975.

[57] Ph A Martin and T Yalcin. The charge fluctuations in classical coulomb systems. *Journal of Statistical Physics*, 22(4):435–463, 1980.

[58] Jesper Møller and Eliza O'Reilly. Couplings for determinantal point processes and their reduced palm distributions with a view to quantifying repulsiveness. *arXiv preprint arXiv:1806.07347*, 2018.

[59] Ferenc Móricz. Absolutely convergent fourier series and function classes. *Journal of Mathematical Analysis and Applications*, 324(2):1168 – 1177, 2006.

[60] Yuval Peres and Allan Sly. Rigidity and tolerance for perturbed lattices. *arXiv e-prints*, page arXiv:1409.4490, Sep 2014.

[61] Chhaibi Reda, Joseph Najnudel, et al. Rigidity of the sine-beta process. *Electronic Communications in Probability*, 23, 2018.

[62] Antonello Scardicchio, Chase E Zachary, and Salvatore Torquato. Statistical properties of determinantal point processes in high-dimensional euclidean spaces. *Physical Review E*, 79(4):041108, 2009.

[63] Rolf Schneider and Wolfgang Weil. *Stochastic and integral geometry*. Springer Science & Business Media, 2008.

[64] A. N. Shiryayev. *Stationary Sequences in Hilbert Space*, pages 228–271. Springer Netherlands, Dordrecht, 1992.

[65] Mikhail Sodin and Boris Tsirelson. Random complex zeroes, ii. perturbed lattice. *Israel Journal of Mathematics*, 152(1):105–124, 2006.

[66] Alexander Soshnikov. Determinantal random point fields. *Russian Mathematical Surveys*, 55(5):923, 2000.

[67] Salvatore Torquato. Hyperuniform states of matter. *Physics Reports*, 745:1–95, 2018.

[68] Salvatore Torquato and Frank H Stillinger. Local density fluctuations, hyperuniformity, and order metrics. *Physical Review E*, 68(4):041113, 2003.

[69] Salvatore Torquato, Ge Zhang, and Frank H. Stillinger. Ensemble theory for stealthy hyperuniform disordered ground states. *Phys. Rev. X*, 5:021020, May 2015.

[70] Benedek Valkó and Bálint Virág. Continuum limits of random matrices and the brownian carousel. *Inventiones mathematicae*, 177(3):463–508, 2009.

[71] Oren Yakir. Recovering the lattice from its random perturbations. *arXiv preprint arXiv:2002.01508*, 2020.

[72] G. Zhang, F. H. Stillinger, and S. Torquato. Ground states of stealthy hyperuniform potentials. ii. stacked-slider phases. *Physical Review E*, 92(2), Aug 2015.

[73] Ge Zhang, Frank H. Stillinger, and Salvatore Torquato. Can exotic disordered "stealthy" particle configurations tolerate arbitrarily large holes? *Soft Matter*, 13:6197–6207, 2017.

[74] Yuanjian Zheng and Massimo Pica Ciamarra. Spatio-temporal heterogeneity and hyperuniformity in 2d conserved lattice gas, 2020.

A. Point processes and Palm technology

This appendix summarizes the elementary formalism for point processes. I do not delve into the technical details: for excellent and exhaustive textbook-treatments of properties of stationary point processes, the reader might refer to [17, 52, 10, 4].

I will constantly make a strong abuse of notation: for any set A, the notation $|A|$ will sometimes stand for the number of elements of A, and sometimes for the Lebesgue measure of A. I will change notations in a close future.

A.1. Configurations

In this survey, we consider non-negative integer-valued Radon measures on $\mathbb{R}^d$, which are *simple* in the sense that the mass given to any singleton is either 0 or 1. Such a measure is called a *configuration* and can be identified with the discrete set of its atoms — we will do this identification whenever we want. The set of configurations on $\mathbb{R}^d$ is noted $\mathrm{Conf}(\mathbb{R}^d)$. This space is equipped with the sigma-algebra $\mathcal{H}$ generated by the counting functions $\varphi \in \mathrm{Conf}(\mathbb{R}^d) \to |\varphi \cap A|$, for any Borel set A; equivalently, $\mathcal{H}$ is generated by the sets $\{|\varphi \cap A| = n\}$, where A runs through all bounded Borel subsets of $\mathbb{R}^d$ and n is an integer.

A random point process is simply a random variable on this set. An underlying probability space $(\Omega, \mathscr{F}, \mathbf{P})$ will be fixed and will play no role in the sequel, although the bold-font letters $\mathbf{E}, \mathbf{P}$ will always refer to this underlying expectation and probability operators.

Capital greek letters such as Φ, Ψ will be my favorite notation for such random variables, the small-case letters like φ being reserved to specific elements of $\mathrm{Conf}(\mathbb{R}^d)$. If x is a point in $\mathbb{R}^d$, we will note $\varphi + x = \{y + x, y \in \varphi\}$ the set φ shifted by x.

A.2. Palm technology

We recall that a point process Φ is stationary if for any fixed $x \in \mathbb{R}^d$, the shifted process $\Phi - x$ has the same distribution as Φ. If this is the case, the intensity measure is a constant λ times the Lebesgue measure. In all this survey, λ will always denote this intensity and will always be supposed to be nonzero.

Let Φ be a random stationary PP. The Palm distribution of Φ is the probability measure $\mathbf{P}^\circ$ defined through the formula

$$\mathbf{P}^\circ(A) = \frac{1}{|B(0,1)|}\mathbf{E}\left[\int_{B(0,1)} \mathbf{1}_{\Phi - x \in A}\mathrm{d}x\right]. \tag{A.2.1}$$

Indeed, one can replace $B(0,1)$ by any other measurable set B with positive Lebesgue measure. The distribution $\mathbf{P}^\circ$ is called the Palm version of the distribution of Φ. The point process Φ under its Palm distribution almost-surely contains the origin 0. A simple extension procedure shows that for any measurable mapping f, one has

$$\mathbf{E}^\circ[f(\Phi)] = \frac{1}{\lambda|B(0,1)|}\mathbf{E}\left[\int_{B(0,1)} f(\Phi - x)\mathrm{d}x\right]. \tag{A.2.2}$$

We will not really use the properties of Palm distributions, except the elementary Campbell formula, which can be formulated in its simplest form as follows: if $f : \mathrm{Conf}(\mathbb{R}^d) \times \mathbb{R}^d$ is a nonnegative measurable function, then

$$\mathbf{E}\left[\sum_{x\in\Phi} f(x, \Phi - x)\right] = \lambda \int_{\mathbb{R}^d} \mathbf{E}^\circ[f(x,\Phi)]\mathrm{d}x. \tag{A.2.3}$$

A.3. Intensities

For every integer k, the formula

$$f \mapsto \mathbf{E}\left[\sum_{x_1,\dots,x_k\in\Phi} f(x_1,\dots,x_k)\right] \tag{A.3.1}$$

defines a unique measure μ^k on $(\mathbb{R}^d)^k$, which is called the k-point intensity measure of the process. When applied to $f(x_1,\dots,x_n) = \mathbf{1}_B(x_1)\dots\mathbf{1}_B(x_n)$ for B some Borel set, the definition immediately implies that

$$\mu^k(B^k) = \mathbf{E}[|\Phi\cap B|^k]$$

where $B^k = B\times\cdots\times B$ (k times).

A.4. Correlations

For every integer k, the formula

$$f \mapsto \mathbf{E}\left[\sum_{x_1,\dots,x_k\in\Phi}^{\neq} f(x_1,\dots,x_k)\right] \tag{A.4.1}$$

defines a unique measure $\mu^{(k)}$ on $(\mathbb{R}^d)^k$, which is called the k-point correlation measure or the factorial moment measure. In the definition, the sum is over all k-tuples of points belonging to Φ, such that all of them are distincts. It is clear that $\mu^{(1)}$ is simply the intensity measure of the process. Formula (A.4.1) immediately implies that if B is a Borel set, then

$$\mu^{(k)}(B^k) = \mathbf{E}\left[(|\Phi\cap B|)_k\right]$$

where $(a)_k := a(a-1)\cdots(a-k+1)$ is the falling factorial.

A.5. Cumulants of Point processes

The cumulants measures of a point process Φ will be useful. For properties of cumulants of random variables, see Appendix D at page 62. Let κ_n be the n-cumulant measure; it can be defined from the intensity measures μ^k through the relations

$$\int_{(\mathbb{R}^d)^n} f(x_1,\dots,x_n)\kappa_n(\mathrm{d}x_1,\dots,\mathrm{d}x_n) = \sum_{k=1}^{n}(-1)^{k-1}(k-1)!\sum_{(\pi_1,\dots,\pi_k)\in\Pi_k(n)}\prod_{\ell=1}^{k}\int_{(\mathbb{R}^d)^{|\pi_\ell|}} f(x_{i_1},\dots,x_{i_{|\pi_\ell|}})\mu^{|\pi_\ell|}(\mathrm{d}x_{i_1},\dots,\mathrm{d}x_{i_{|\pi_\ell|}}) \tag{A.5.1}$$

where $\Pi_r(n)$ denotes the set of r-partition of $[n]$, ie ordered families $(\pi_1,\dots,\pi_r)$ with $\pi_\ell\subset[n]$ nonempty and $\pi_1\cup\cdots\cup\pi_r = [n]$. The notation $|\pi_\ell|$ denotes the number of elements in the subset π_ℓ.

Let us apply (A.5.1) to $f(x_1,\dots,x_n) = \mathbf{1}_B(x_1)\dots\mathbf{1}_B(x_n)$ for some Borel set B. The integral in the right-most part of (A.5.1) is equal to

$$\mu^{|\pi_\ell|}(B\times\cdots\times B) = \mathbf{E}[|\Phi\cap B|^{|\pi_\ell|}].$$

Consequently, the real number $\kappa_n(B\times\cdots\times B)$ is nothing else than the (classical) n-th cumulant of the real-valued random variable $|\Phi\cap B|$ — see (D.0.3). With an abuse of notation, we will note this quantity $\kappa_n(B)$ for any Borel set B.

A.6. Ergodicity

A measurable subset E of $\mathrm{Conf}(\mathbb{R}^d)$ is called an invariant subset if for every $x \in \mathbb{R}^d$ we have

$$E = \{\varphi - x : \varphi \in X\}.$$

In other words, shifting every configuration φ contained in E does not change E. The collection of all invariant subsets of $\mathrm{Conf}(\mathbb{R}^d)$ forms itself a σ-algebra, called the invariant σ-algebra and noted $\mathscr{I}$.

Definition A.1 (ergodicity)**.** A stationary point process Φ on $\mathbb{R}^d$ is called *ergodic* if for every invariant subset $E \in \mathscr{I}$, one has

$$\mathbf{P}(\Phi \in E) \in \{0, 1\}. \tag{A.6.1}$$

Example A.2 (lattices are ergodic)**.** Let us take the simple example of a stationarized grid $\Phi = U + \mathbb{L}$ where $\mathbb{L}$ is a fixed lattice. For any invariant subset $E \in \mathscr{I}$, we have $\mathbf{P}(\Phi \in E) = \mathbf{P}(\mathbb{L} \in E - U) = \mathbf{P}(\mathbb{L} \in E)$ (because $E = E - U$) and this can only be zero or one for $\mathbb{L}$ is not random, hence stationarized lattices are ergodic.

It is often not so easy to prove that a point process is ergodic, but there are several criteria which can be found in the books. They notably imply that Poisson point processes are ergodic (they are indeed *mixing*, which is not the case for stationarized lattices). The second example is quite important to us, especially for Section 9.

Example A.3 (independently perturbed ergodic PPs are ergodic)**.** Let Ψ be a stationary point process, and let $\Phi = \Phi_{\Psi,V}$ be the independent perturbation of Ψ defined in (9.0.1): we take an iid family $(V_x)_{x\in\Psi}$ with common centered distribution V, and we set $\Phi = \{x + V_x : x \in \Psi\}$. I claim that if Ψ is ergodic, then Φ is also ergodic. Let us prove this.

We set

The most striking example of non-ergodic point process is obtained as follows: take for instance two independent point processes Φ_0, Φ with intensities 1 and 2, then set $\Phi = \Phi_X$ where X is an independent coin, say $X \sim \mathrm{Ber}(0.5)$. Such a process does not satisfy the ergodic theorem thereafter, hence is not ergodic itself.

The following theorem is of paramount importance since it determines a spatial law of large numbers on ergodic point processes. In the statement, 'convex averaging sequence' means a sequence (B_n) of nested compact convex sets with nonempty interior and such that $\cup B_n = \mathbb{R}^d$ — one may simply think of $B_n = B(0, n)$.

Theorem A.4 (spatial ergodic theorem)**.** *Let Φ be a stationary, ergodic point process on $\mathbb{R}^d$ with Palm distribution $\mathbf{P}^\circ$. Let (B_n) be a convex averaging sequence. Then, for any measurable function $F : \mathrm{Conf}(\mathbb{R}^d) \to \mathbb{R}$ telle que $\mathbf{E}^\circ[|F(\Phi)|] < \infty$, on a*

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{x\in\Phi\cap B_n} F(\Phi - x) = \mathbf{E}^\circ[F(\Phi)], \qquad \mathbf{P}\text{-}\textit{almost surely.} \tag{A.6.2}$$

Example A.5. Let $A = B(0, r)$ let $F(\varphi) = |\varphi \cap A| - \delta_0(A)$. The ergodic theorem entails

$$\lim_{n\to\infty} \frac{1}{|B_n|} \sum_{x\in\Phi\cap B_n} (|\Phi \cap B(x, r)| - 1) = \mathbf{E}^\circ[F(\Phi)] = \lambda^{-1}\mu_\circ(B(0, r)) = K(r) \tag{A.6.3}$$

where K is the Ripley K-function defined in Remark (**??**).

B. Harmonic analysis

B.1. Tempered distributions and the Schwartz space

We list here the few basic results we will need on tempered distributions and their Fourier transforms; they can be found in any introductory book to real analysis. We refer to Laurent Schwartz's *Cours d'analyse* or to [36].

A function $f : \mathbb{R}^d \to \mathbb{C}$ is a Schwartz function if it has derivatives at every order and if for any multi-index α, β one has

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} |x^\beta \partial_\alpha f(x)| < \infty.$$

The maps $p_{\alpha,\beta}$ are semi-norms and they generate the topology of the set $\mathscr{S}(\mathbb{R}^d)$ of every Schwartz function. A continuous linear form over $\mathscr{S}(\mathbb{R}^d)$ is called a *tempered distribution* and the set of tempered distributions is noted $\mathscr{S}'(\mathbb{R}^d)$. The Fourier transform of a Schwartz function f is the Schwartz function defined by

$$\mathscr{F}(f)(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\langle \xi, x\rangle} \mathrm{d}x$$

and the inverse Fourier transform is $\overline{\mathscr{F}} f(\xi) := \mathscr{F}(f)(-\xi)$. The Fourier transform is an isomorphism of $\mathscr{S}(\mathbb{R}^d)$ and one has the fundamental reciprocity relation

$$\mathscr{F}\overline{\mathscr{F}} = \overline{\mathscr{F}}\mathscr{F} = (2\pi)^d \mathrm{I}. \tag{B.1.1}$$

The Fourier transform of a tempered distribution $S \in \mathscr{S}'(\mathbb{R}^d)$ is then defined through duality by

$$\langle \mathscr{F} S, f\rangle = \langle S, \overline{\mathscr{F}} f\rangle.$$

This is an isomorphism on $\mathscr{S}'(\mathbb{R}^d)$ and (B.1.1) still holds.

B.2. Operations on tempered distributions: convolution and multiplication

If g is a smooth function with sub-polynomial growth, one can define the multiplication of (the tempered distribution associated with) g and a tempered distribution S through the formula

$$\langle g.S, f\rangle := \langle S, gf\rangle.$$

In general, if S is a little bit more than a tempered distribution, one can multiply S with more general functions than smooth functions. For example, if S is indeed a tempered measure, one can multiply S by any bounded measurable functions.

Convolution is more difficult.

If g is L^1, then one can define the convolution of g with a tempered distribution S through the formula

$$\langle g * S, f\rangle := \langle S, g \circledast f\rangle. \tag{B.2.1}$$

With those definitions, the group morphism property of Fourier transforms is preserved:

$$\mathscr{F}(g * S) = \hat{g}.\mathscr{F} S. \tag{B.2.2}$$

B.3. Special Fourier transforms

Let $B(0, r)$ be the euclidean ball of radius r around the origin in $\mathbb{R}^d$. Its Fourier transform is given by the following formula

$$\widehat{\mathbf{1}_{B(0,r)}}(\xi) = r^{d/2} \frac{\mathrm{J}_{d/2}(r|\xi|)}{|\xi|^{d/2}}. \tag{B.3.1}$$

where J_ν is the Bessel function of the first kind of index ν. We recall that the basic definition of the Bessel function is

$$\mathrm{J}_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$

Proof. Set $B = B(0,1)$. As $\mathbf{1}_B$ is rotation-invariant, so is its Fourier transform, hence it is enough to compute the Fourier transform at $\xi = (0,\dots,0,t)$. Let ω_k be the volume of the k-dimensional unit ball [10]. Integration by slices yields

$$\widehat{\mathbf{1}_B}(\xi) = \int_B e^{-itx_d} \mathrm{d}x_1 \dots \mathrm{d}x_d = \omega_{d-1} \int_{-1}^{1} e^{-ixt} (\sqrt{1-x^2})^{d-1} \mathrm{d}x.$$

Now the Poisson representation formula for Bessel functions, valid for any $\nu > -1/2$, is

$$\mathrm{J}_\nu(z) = \frac{(z/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)} \int_{-1}^{1} e^{-izt}(1-t^2)^{\nu-1/2} \mathrm{d}t \tag{B.3.2}$$

where J_ν is the Bessel function of the first kind of order ν. Setting $\nu = d/2$, we get

$$\widehat{\mathbf{1}_B}(\xi) = \frac{\omega_{d-1}\sqrt{\pi}\Gamma((d+1)/2)}{(t/2)^{d/2}} \mathrm{J}_{d/2}(t) = \frac{\mathrm{J}_{d/2}(|\xi|)}{|\xi|^{d/2}}.$$

By homogeneity[11], we obtain (1.4.1) □

We will need some asymptotics of Bessel functions. Such asymptotics can be easily found on any book on special functions, such as [53].

Lemma B.1. *Let ν be a positive real number. As $x \to \infty$, one has*

$$\mathrm{J}_\nu(t) = \sqrt{\frac{2}{\pi t}} \left(\cos(t - c_\nu) + O\left(\frac{1}{t}\right)\right) \tag{B.3.3}$$

where $c_\nu := \nu\pi/2 + \pi/4$. The positive real function $\mathscr{J} : x \in \mathbb{R}^d \mapsto \mathrm{J}_{d/2}(|x|)^2/|x|^d$ thus satisfies the following bound as $|x| \to \infty$:

$$\mathscr{J}(x) = O\left(\frac{1}{|x|^{d+1}}\right). \tag{B.3.4}$$

We now turn to boxes. We note $B_\infty(0,r)$ the set of $x \in \mathbb{R}^d$ with all coordinates between $-r$ and r, ie the box of side length $2r$. Its Fourier transform is given as follows.

$$\widehat{\mathbf{1}_{B_\infty(0,r)}}(\xi) = 2^d \prod_{i=1}^{d} \frac{\sin(r\xi_i)}{\xi_i}. \tag{B.3.5}$$

Proof. Set $\xi = (\xi_1,\dots,\xi_d)$. Then,

$$\begin{aligned}
\int_{B_\infty(0,1)} \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle} \mathrm{d}x &= \int_{-1}^{1} \cdots \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\langle \xi_1, x_1\rangle} \cdots \mathrm{e}^{-\mathrm{i}\langle \xi_d, x_d\rangle} \mathrm{d}x_1 \dots \mathrm{d}x_d \\
&= \prod_{i=1}^{d} \left(\int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\langle \xi_i, x\rangle} \mathrm{d}x \right) \\
&= 2^d \prod_{i=1}^{d} \frac{\sin(\xi_i)}{\xi_i}.
\end{aligned}$$

Identity (1.4.5) follows from homogeneity. □

[10] One has $\omega_k = \frac{\pi^{k/2}}{\Gamma(1+k/2)}$.

[11] If $f(x) = g(x/r)$ then $\hat{f}(\xi) = |r|^d \hat{g}(r\xi)$.

B.4. Positive, positive-definite (ppd) measures

A Borel measure m on $\mathbb{R}^d$ is ppd if it is a positive Borel measure which is also *positive definite*, which means that for any compactly supported test function f, one has

$$\int_{\mathbb{R}^d} f * \tilde{f}(x) m(\mathrm{d}x) \geqslant 0 \tag{B.4.1}$$

where $\tilde{f}(x) := f(-x)$. As we will often have to use the convolution $f * \tilde{g}$, I find it easier to use the notation $f \circledast g$ instead. We refer the reader to the first chapter of Berg and Frost' *Potential theory* book for a clear treatment of this theme ([9]).

The set of ppd measures is noted $\mathcal{P}_{\mathrm{pd}}$. A measure is *signed ppd* if it is the difference of two ppd measures. It turns out that ppd measures are the good framework for working with Fourier transform of measures which are not necessarily finite. We refer the reader to the first chapter of Berg and Frost' *Potential theory* book for a clear treatment of this theme ([9]). It is not obvious how to define Fourier transforms of infinite measures in general, even if they are locally finite; for instance, the measure $e^{x^2}\mathrm{d}x$ is locally finite but it does not define a tempered distribution, hence there is no obvious way to define its Fourier transform. However, ppd measures behave well.

Proposition B.2. $\mathcal{P}_{\mathrm{pd}} \subset \mathscr{S}'(\mathbb{R}^d)$.

Consequently, for any signed ppd m, one can define $\mathscr{F}m$, which is a tempered distribution; but Bochner's theorem comes into play and guarantees that $\mathscr{F}m$ is indeed a signed measure.

Theorem B.3 (Ultimate Bochner Theorem)**.** *The tempered Fourier transform is an isomorphism of* $\mathcal{P}_{\mathrm{pd}}$ *and the inversion formula holds.*

In other words, if m is ppd and has Fourier transform $\hat{m}$, then for any $f \in \mathscr{S}(\mathbb{R}^d)$ one has

$$\int_{\mathbb{R}^d} f(x) m(\mathrm{d}x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(x) \hat{m}(\mathrm{d}x) \tag{B.4.2}$$

which also known as the *reciprocity relation*.

B.5. Harmonic analysis on lattices

The celebrated Poisson summation formula (see [36], chapter 7) reads

$$\sum_{n \in \tau\mathbb{Z}} \hat{f}(n) = \sum_{n \in \mathbb{Z}} f(n) \tag{B.5.1}$$

for any suitable function f, here taking $f \in \mathscr{S}(\mathbb{R}^d)$ will be sufficient. In this equation, beware of the normalization: I use the 'un-normalized FT' defined as $\hat{f}(n) = \int \mathrm{e}^{-inu} f(u)\mathrm{d}u$, hence the first sum is over $\tau\mathbb{Z}$ with $\tau = 2\pi$. At the level of tempered distributions, this is exactly equivalent to

$$\mathscr{F}\Delta_{\tau\mathbb{Z}} = \Delta_{\mathbb{Z}}. \tag{B.5.2}$$

More generally, this can be extended to $\mathbb{Z}^d$ as $\mathscr{F}\Delta_{\tau\mathbb{Z}^d} = \Delta_{\mathbb{Z}^d}$. Now, suppose that $\mathbb{L} = G^*(\tau\mathbb{Z}^d)$ for some G with determinant 1. For any test function $f \in \mathscr{S}(\mathbb{R}^d)$, one has

$$\begin{aligned}
\langle \mathscr{F}\Delta_{\mathbb{L}}, f\rangle = \langle \Delta_{\mathbb{L}}, \overline{\mathscr{F}} f\rangle &= \sum_{x\in\mathbb{L}} \hat{f}(-x) \\
&= \sum_{x\in\tau\mathbb{Z}^d} \hat{f}(-G^*x) \\
&= \sum_{x\in\tau\mathbb{Z}^d} \hat{f}_*(x) = \langle \mathscr{F}\Delta_{\tau\mathbb{Z}^d}, f_*\rangle
\end{aligned}$$

where $f_*(u) = f(-G^{-1}u)$, the third line coming from symmetries of the FT. But then, this is equal to $\langle \Delta_{\mathbb{Z}}, f_* \rangle$ thanks to (B.5.2), and it is also equal to $\langle \Delta_{G^{-1}\mathbb{Z}^d}, f \rangle$. Consequently, we have proven the following classical result on dual lattices.

Theorem B.4 (lattice duality)**.** *Let $\mathbb{L} = G^*(\tau\mathbb{Z}^d)$ be a lattice with $G \in \mathrm{SL}(\mathbb{R}^d)$. Define $\mathbb{L}^* = G^{-1}\mathbb{Z}^d$. Then,*

$$\mathscr{F}\Delta_{\mathbb{L}} = \Delta_{\mathbb{L}^*}. \tag{B.5.3}$$

Example B.5. Note $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. The lattice $\mathbb{L} = \{\alpha 2\pi x\mathbf{e}_1 + \beta 2\pi y\mathbf{e}_2 : x, y \in \mathbb{Z}\}$ corresponds to the choice

$$G = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad G^{-1} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$

so that $\mathbb{L}^* = \{x\mathbf{e}_1/\alpha + y\mathbf{e}_2/\beta : x, y \in \mathbb{Z}^2\}$.

C. Stationary processes and Kolmogorov's criterion

An integer-indexed random process $X = (X_n)_{n\in\mathbb{Z}^d}$ is called stationary if the distribution of $(X_{n+k} : n \in \mathbb{Z}^d)$ does not depend on $k \in \mathbb{Z}^d$. If this is the case, let us note $c(k)$ the k-th covariance

$$c(k) := \mathrm{Cov}(X_0, X_k) = \mathrm{Cov}(X_n, X_{n+k}) \tag{C.0.1}$$

provided that these covariances exist — assuming that X_0 has a variance is sufficient, since in this case $|\mathrm{Cov}(X_0, X_n)| \leqslant \sqrt{\mathrm{Var}(X_0)\mathrm{Var}(X_n)} = \mathrm{Var}(X_0) < \infty$ by the Cauchy-Schwarz inequality and stationarity. In the sequel, we will note $\mathbb{T}^d = \mathbb{R}^d/\tau\mathbb{Z}^d$ (with $\tau = 2\pi$), and we will identify this space with $[0, 2\pi[^d$. We will naturally endow it with the Borel σ-algebra, and with the Lebesgue measure $\mathrm{d}x$. We will use the common notation for exponential functions, $\mathrm{e}_n : \xi \in \mathbb{T}^d \mapsto \mathrm{e}^{\mathrm{i}\langle\xi,n\rangle} \in \mathbb{C}$. We first recall an important result regarding spectral representations of linear combinations of stationary processes.

Proposition C.1. *Let $X = (X_n)_{n\in\mathbb{Z}^d}$ be a stationary process with covariances $(c(k) : k \in \mathbb{Z}^d)$. There is a Borel measure μ on $\mathbb{T}^d$ such that the map $\imath : \mathrm{Span}(X_n : n \in \mathbb{Z}^d) \to L^2(\mu)$ implicitly defined by $\imath(X_n) = \mathrm{e}_n$ is an $(2\pi)^d$-isometry, ie $\|Y\|^2 = (2\pi)^{-d}|\imath(Y)|^2$. Moreover, the measure μ is uniquely determined by X and satisfies $c(k) = (2\pi)^{-d}\int \mathrm{e}_k(\xi)\mu(\mathrm{d}\xi)$.*

The measure μ has total mass $\mu(\mathbb{T}^d) = \mathrm{Var}(X_0) < \infty$. It is called the spectral measure of X. An important link between X and μ is given by the following theorem, due to Kolmogorov himself in 1941.

Theorem C.2 (Kolmogorov, [49, 64])**.** *Let X be a stationary process with spectral measure μ. Let $H_0 = \mathrm{Span}(X_n : n \neq 0)$ be the linear vector space spanned by all the X_n's except X_0 and let $\mathrm{dist}(X_0, H_0)$ be the L^2-distance between X_0 and the subspace H_0, ie*

$$\mathrm{dist}(X_0, H_0) := \min\{\mathbf{E}[|X_0 - Y|^2]^{1/2} : Y \in H_0\}.$$

Note $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sg}}$ the Lebesgue decomposition of μ with respect to the Lebesgue measure, and note μ'_{ac} the density of μ_{ac} with respect to the Lebesgue measure. Then,

$$\mathrm{dist}(X_0, H_0) = \left(\int_{\mathbb{T}^d} \frac{1}{\mu'_{\mathrm{ac}}(\xi)}\mathrm{d}\xi\right)^{-1/2} \tag{C.0.2}$$

where $1/\infty$ means 0.

Proof. I follow the proof as it is given in [12]. Using the isomorphism $\imath$ in Proposition C.1, we only have to show that the distance between the constant function $1 = \imath(X_0)$ and the subspace $L_0 := \imath(H_0) \subset L^2(\mu)$ is given by $(2\pi)^d$ times the RHS of (C.0.2). For ease of notation we will also note $u(\xi) = \mu'_{\mathrm{ac}}(\xi)$. First, let us decompose $1 = p + q$ where $q \in L_0$ and $p \in L_0^\perp$, so the distance between X_0 and H_0 is equal to the norm of p.

Let us consider the measure $\nu(\mathrm{d}\xi) = p(\xi)\mu(\mathrm{d}\xi)$. As p is orthogonal to L_0, for every $n \neq 0$ we have

$$\hat{\nu}(n) = \int_{\mathbb{T}^d} \mathrm{e}_n(\xi)\nu(\mathrm{d}\xi) = \int_{\mathbb{T}^d} p(\xi)\mathrm{e}_n(\xi)\mu(\mathrm{d}\xi) = 0$$

and $\hat{\nu}(0) = \int_{\mathbb{T}^d} p(\xi)\mu(\mathrm{d}\xi)$, which is going to be called γ. These Fourier coefficients coincide with those of the measure $\gamma\mathrm{d}\xi/(2\pi)^d$, hence by injectivity $\nu = \gamma\mathrm{d}\xi/(2\pi)^d = \delta\mathrm{d}\xi$. But the Lebesgue decomposition of ν is given by $p(\xi)\mu_{\mathrm{sg}}(\mathrm{d}\xi) + p(\xi)u(\xi)\mathrm{d}\xi$, which ensures that p vanishes on the support of μ_{sg}, that almost everywhere we have $p(\xi) = \delta/u(\xi)$, and finally that

$$\text{for Lebesgue-almost every } \xi \in \mathbb{T}^d, \qquad q(\xi) = 1 - \frac{\delta}{u(\xi)}. \tag{C.0.3}$$

Together, these three facts have a crucial consequence: let us recall that $|p| = \mathrm{dist}(1, L_0)$; then,

$$|p|^2 = \int_{\mathbb{T}^d} p(\xi)^2\mu(\mathrm{d}\xi) = \int_{\mathbb{T}^d} p(\xi)^2\mu_{\mathrm{sg}}(\mathrm{d}\xi) + \delta^2\int_{\mathbb{T}^d} \frac{1}{u(\xi)^2}\mu_{\mathrm{ac}}(\mathrm{d}\xi) = \delta^2\int_{\mathbb{T}^d} \frac{1}{u(\xi)}\mathrm{d}\xi.$$

We now separately consider the two possible cases for the value of $\beta := \int u(\xi)^{-1}\mu(\mathrm{d}\xi)$.

First case: $\beta = \infty$. The preceding identities say that $|p|^2 = \delta^2\beta$. If δ was nonzero we should have $|p| = \infty$, which is absurd since $p \in L^2(\mu)$, hence $\delta = 0$ and $p = 0$ and $\mathrm{dist}(X_0, H_0) = 0$.

Second case: $\beta < \infty$. Let S be a support of μ_{ac} and define a function f by $f(\xi) = 0$ if $\xi \notin S$ and $u(\xi)^{-1}$ otherwise. We have

$$\int f(\xi)^2\mu(\mathrm{d}\xi) = \int \frac{1}{u(\xi)^2}\mu_{\mathrm{ac}}(\mathrm{d}\xi) = \beta$$

hence f is in $L^2(\mu)$. On the other hand, for each $n \neq 0$, one has

$$\langle \mathrm{e}_n, f\rangle = \int_{\mathbb{T}^d} \frac{1}{u(\xi)}\mu_{\mathrm{ac}}(\mathrm{d}\xi) = \int_{\mathbb{T}^d} \mathrm{e}_n(\xi)\mathrm{d}\xi = 0$$

which entails $f \in L_0^{\perp}$ and in particular $0 = \langle f, q\rangle = \int_{\mathbb{T}^d} q(\xi)\mathrm{d}\xi$. Thanks to (C.0.3) and the fact that the Lebesgue measure of $\mathbb{T}^d$ is $(2\pi)^d$, we finally get $\delta = \frac{(2\pi)^d}{\beta}$, and $|p|^2 = \delta^2\beta = (2\pi)^{2d}\beta^{-1}$. But as $\mathrm{dist}(X_0, H_0) = (2\pi)^{-d}\mathrm{dist}(1, L_0) = (2\pi)^{-d}|p|$, the identity (C.0.2) is proved. □

Consequently, X_0 is a linear function of $(X_n : n \neq 0)$ if and only if $\int \mu'_{\mathrm{ac}}(\xi)^{-1}\mathrm{d}\xi = \infty$: the random variable X_0 is entirely determined by all the other X_n's, and the dependence between them is linear. This is called 'linear rigidity'. Checking if a stationary process X satisfies Kolmogorov's criterion can be difficult, but Bufetov, Dabrowski and Qiu have a very nice sufficient condition in one dimension ([12], Theorem 3.1).

Theorem C.3. *Let $X = (X_n)_{n\in\mathbb{Z}}$ be a one-dimensional stationary process with covariances $(c(k) : k \in \mathbb{Z})$ and spectral measure $\mu = \mu_{\mathrm{sg}} + \mu_{\mathrm{ac}}$. If*

$$\sup_{m\geqslant 1}\left\{ m\sum_{n\geqslant m} |c(n)| \right\} < \infty \tag{C.0.4}$$

and

$$\sum_{n\in\mathbb{Z}} c(n) = 0 \tag{C.0.5}$$

then $\mathrm{dist}(X_0, H_0) = 0$*, hence X is linearly rigid.*

The second equation is obviously the same one as the $\mathcal{C}(\mathbb{R}^d) = 0$ hyperuniformity condition from stationary point processes. The proof uses tools from the theory of Fourier series and their link with the so-called Zygmund class ([59]).

D. Cumulants

Let X be a real random variable with all moments finite. Let $m_n := \mathbf{E}[X^n]$ be its n-th moment. Their exp-generating function is called the moment generating function (MGF):

$$\varphi(z) = \mathbf{E}[\mathrm{e}^{zX}] = \sum_{n=0}^{\infty} m_n \frac{z^n}{n!}.$$

The cumulants are defined as the coefficients of the analytic function $\psi(z) := \log \varphi(z)$. More precisely, the n-th cumulant $\mathrm{C}_n(X) =: c_n$ is defined as

$$\psi(z) := \log \mathbf{E}[\mathrm{e}^{zX}] = 1 + \sum_{n=1}^{\infty} c_n \frac{z^n}{n!}. \tag{D.0.1}$$

It is clear from the definition that $\mathrm{e}^{\psi(z)} = \varphi(z)$. From this identity one can extract formulas for expressing c_n in terms of m_n, as in the following theorem.

Theorem D.1 (reciprocical relations between moments and cumulants)**.** *Let X be a real-valued random variable with all moments finite. Note $m_n = \mathbf{E}[X^n]$ its moments and $c_n = \mathrm{C}_n(X)$ its cumulants. These sequences satisfy*

$$m_n = \sum_{k=1}^{n} \sum_{(\pi_1,\dots,\pi_k)\in\Pi_k(n)} \prod_{j=1}^{k} m_{|\pi_j|} \tag{D.0.2}$$

and also

$$c_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \sum_{(\pi_1,\dots,\pi_k)\in\Pi_k(n)} \prod_{j=1}^{k} m_{|\pi_j|} \tag{D.0.3}$$

where $\Pi_k(n)$ is the set of k-partitions of n and $|\pi_i|$ is the number of elements in the block $\pi_i \subset [n]$.

One can alternatively define the cumulants through (D.0.3), and then check that their exponential generating function $\psi(z) = \sum c_n z^n/n!$ is linked with the exponential generating function of m, noted φ, by the relation $\exp\psi(z) = \varphi(z)$. This result is often called the *exponential formula*. See [2], Section 3.3 for many related results.

From these relations, it is almost obvious that cumulants are homogeneous : $\mathrm{C}_n(tX) = t^n \mathrm{C}_n(X)$. On the other hand, the definition given above allows us to see that cumulants behave well with translations. Indeed, $\log \mathbf{E}[\mathrm{e}^{z(X+x)}] = zx + \log \mathbf{E}[\mathrm{e}^{zX}]$. As a consequence, only the term corresponding to $n = 1$ is modified in the RHS of (D.0.1). This shows that for every $n \geqslant 2$, one has $\mathrm{C}_n(X+x) = \mathrm{C}_n(X)$, while $\mathrm{C}_1(X+x) = \mathrm{C}_1(X)+x$.

Example D.2 (Gaussian distribution)**.** The generating function of the centered Gaussian distribution with variance σ^2 is given by $\varphi(z) = \mathrm{e}^{\sigma^2 z^2/2}$, hence $\psi(z) = \sigma^2 z^2/2$. This proves that the $\mathscr{N}(0,\sigma^2)$ distribution is the only one with all cumulants equal to zero, except the second one equal to σ^2. Marcinkiewicz's theorem states that indeed, Gaussian random variables are the only ones with only a finite number of nonzero cumulants.

Example D.3 (Poisson)**.** The MGF of a $\mathsf{Poi}(\lambda)$ random variable is given by $\varphi(z) = \mathrm{e}^{\lambda(\mathrm{e}^z-1)}$, hence the cumulant generating function is $\psi(z) = \lambda(\mathrm{e}^z - 1)$ and we immediately see that the cumulants are all equal to λ.

Theorem D.4. *Let (X_n) be a sequence of real random variables. If for every integer p,*

$$\lim_{n\to\infty} \mathrm{C}_p(X_n) = \begin{cases} \sigma^2 \text{ if } p = 2, \\ 0 \text{ else} \end{cases} \tag{D.0.4}$$

then $X_n \rightsquigarrow \mathscr{N}(0,\sigma^2)$.

Proof. If $\mathrm{C}_2(X_n) \to \sigma^2$ then (X_n) is a sequence of bounded variables in L^2, and by Chebyshev's inequality, it is tight, hence relatively compact by Prokhorov's theorem. Moreover, by the same argument, (X_n^{2k}) is uniformly integrable for every k, and by standard results, weak convergence + uniform integrability of powers implies convergence of moments. Consequently, along any subsequence converging weakly, the moments and cumulants at every order also converge. But then the limit along this subsequence has only the second cumulant which is nonzero and equal to σ^2: it is $\mathscr{N}(0, \sigma^2)$, and the whole sequence converges towards $\mathscr{N}(0, \sigma^2)$ both weakly and in moments. □

Index

Campbell formula, 53
cloaking, 46
CLT, 22
collective-coordinate function, 24
correlation measure, 54

deletion tolerance, 42
determinantal point processes, 17

empirical structure factor, 24
ergodic, 55
ergodic theorem, 55
ergodicity of point processes, 55
escapable point process, 44
extreme rigidity, 33

factorial moment measure, 54
fluctuations, 21

hyperfluctuating, 31
hyperuniformity types, 21

integrability hypothesis, 13
intensity, 54
intersections of hyperplanes, 31
invariant subset, 55

maximal rigidity, 31

order metric, 23

Palm measure (definition), 53
Palm-coupled class, 15
Pfaffian point processes, 18

quadratic variance formula, 28

reconstructibility, 41
Ripley K-function, 6, 55

scaled intersection formula, 7
scaled intersection volume, 7
sigma function, 7
sine-beta process, 20
slider models, 47
stationary process, 60
stealthy hyperuniform processes, 14
structure
    kernel, 8
    measure (definition), 8

tau order metric, 23
tolerance, 42

URL (Uniformly Randomized Lattice), 46