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1. Introduction

1.1. Introduction

A **signal** is defined as "something" that carries information. In general, this "something" is a pattern of variations of a physical quantity that can be manipulated, stored, or transmitted by physical processes. Sadly, this definition is not particularly informative, since it encompasses a wide range of eterogeneous physical phenomena. A notable property of signals is that they can be represented or encoded in many equivalent ways, convertible into one another.

The most natural language to describe signals is mathematics. Signals can have one or more than one dimension, depending on how many variables are needed to describe them. Some examples of one-dimensional signals are:

- Sound, like music or human speech;
- A sensor's output, like those of a thermal sensor or of a motion sensor;
- Physiological signals, like EEGs;
- Financial data, like market trends and exchange rates.

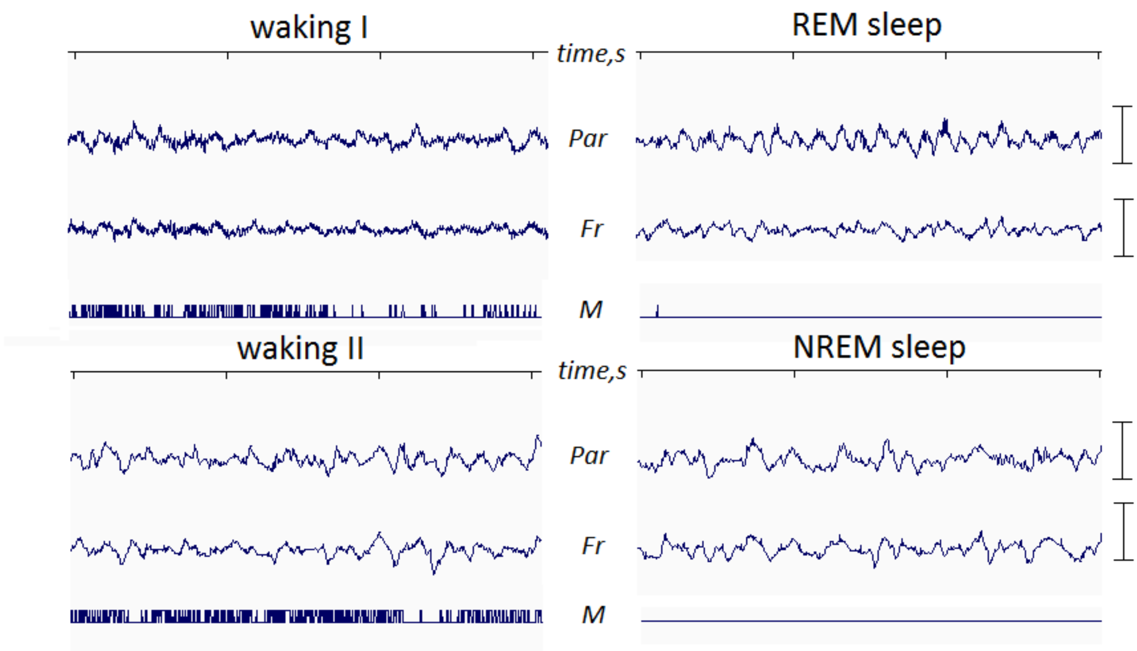

Figure 1: EEG of a mouse during different stages of sleep. [Original image by Andrii Cherninskyi, licensed under CC BY-SA 4.0.]

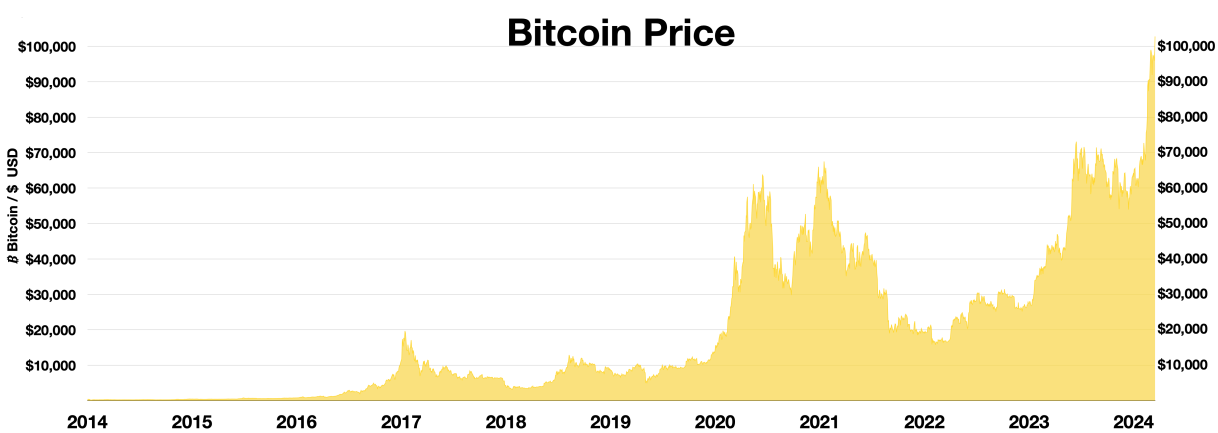

Figure 2: Bitcoin/US Dollar exchange date from 2014 to 2024. [Original image by Wikideas1, licensed under CC0. Original file in webp format.]

From one dimensional signals it is possible to generalize to multidimensional signals. Examples of two dimensional signals are images (photographs, thermal captures, radiographies, ...). Examples of three dimensional signals are 3D models (point clouds, meshes, ...).

A **system** is any process or apparatus that has a signal as input, performs some manipulation on such signal and then returns another signal as output. The output signal can be the original signal but in a different representation or a completely different signal altogether.

Many signals, such as sound, are naturally thought of as a pattern of variations in time. The evolution of a signal with respect to time is described by what's called the **time waveform**, a function $s(t)$ with a single independent variable t, representing time, and whose output is a displacement or disturbance of sort. $s(t)$ can be of arbitrary complexity, and may not be possible to write it as a closed-form expression, but it exists nonetheless. As a matter of fact, it is possible to directly refer to a signal by its waveform.

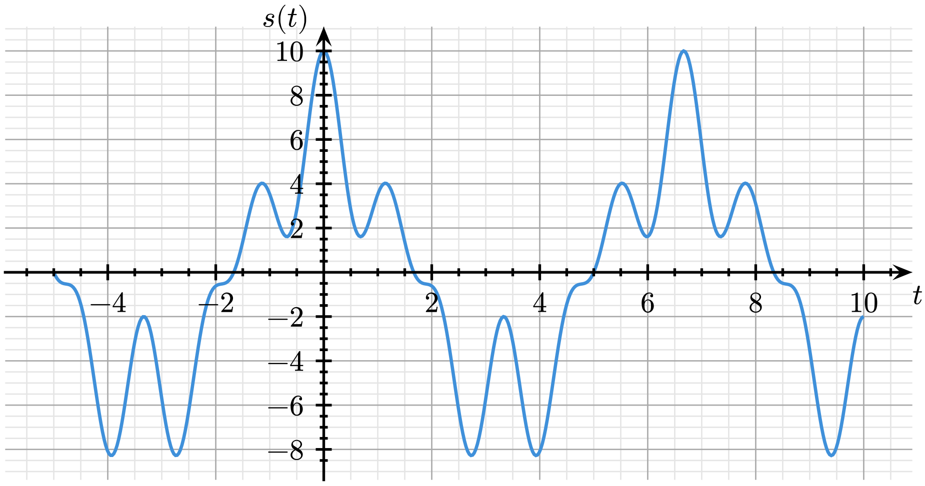

Figure 3: The plot of a continuous time signal.

Most real-world signals vary continuously, meaning that their time waveform has the entire number line as its domain. For this reason they are called **continuous time signals**. However, most systems and in particular all digital systems cannot operate with continuous quantities, only discrete quantities. For this reason, it is convenient to convert a continuous time signal into a **discrete time signal**, by quantizing or discretizing its wave form.

The most intuitive way to quantize a time signal is by sampling it at isolated, equally spaced points in time[1]. The newly obtained signal is still a function s of time, but having $\mathbb{Z}$ instead of $\mathbb{R}$ as domain. To distinguish between a continuous time waveform and a discrete time waveform, the latter uses square brackets instead of round brackets. $s[n]$ is related to $s(n)$ in the following way:

$$s[t] = \begin{cases} s(nT_s) & \text{if } n \in \mathbb{Z} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Where T_s is the **sampling period**, the time interval between one instance of sampling and the next. Without knowing what the sampling period is, $s[n]$ is a mere vector of numbers with no semantics.

[1]It's also possible to have unequally spaced samples, but the mathematical underpinning is hard to tackle.

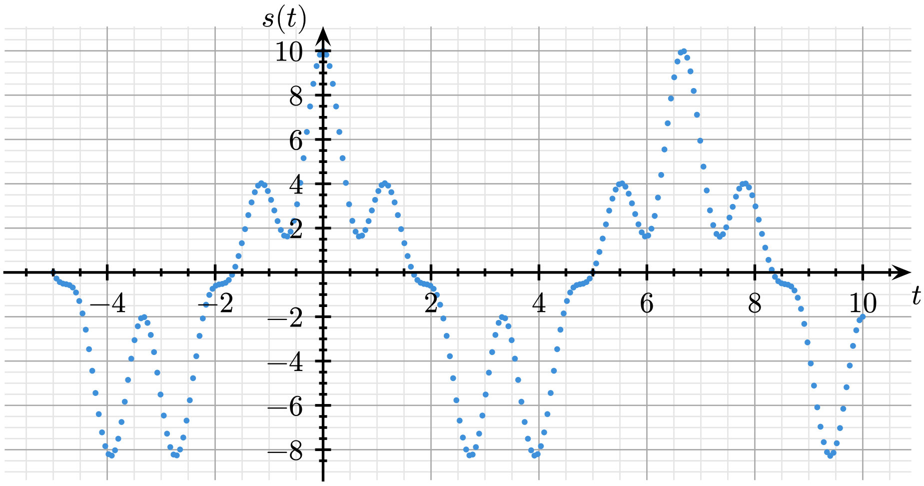

Figure 4: The signal in Figure 3, sampled with a sampling period of 0.06

Quantizing a signal necessarely entail a loss of information, because all (countably infinite) points between two sampled time instants are lost. However, if the number of sampled points is sufficient, the original signal can be reconstructed with a surprinsingly high degree of accuracy. The sampling period should not be too small, or the number of points would require too much memory to be stored. There has to be some tradeoff between sampling accuracy (quality) and space occupied (quantity). There is no silver bullet when choosing a sampling period: the choice is problem-dependent.

Not all signals can be thought of as being time dependent. For example, (still) images clearly do not depend on time. A better representation for an image would be a function of two independent variables $p(x, y)$, representing the spatial coordinates. The output of the function is the intensity of the color, having chosen an appropriate encoding. As for signals depending on time, sampling is also possible for images.

Systems, not only signals, can be represented as functions. Consider a system that has continuous signals both as argument and as return value: this is referred to as a **continuous-time system**. A one-dimensional continuous-time system can be represented as a function T that has a continuous signal $x(t)$ as input and another continuous signal $y(t)$ as output:

$$y(t) = \mathrm{T}\{x(t)\}$$

Consider instead a system that has discrete signals both as argument and as return value: this is referred to as a **discrete-time system**. A one-dimensional discrete-time system can be represented as a function T that has a discrete signal $x(t)$ as input and another discrete signal $y(t)$ as output:

$$y[n] = \mathrm{T}\{x[n]\}$$

Alongside the mathematical representation, systems are also represented using **block diagrams**, diagrams where the each rectangle (block) denotes a sub-component of a system and the arrows denote the flow of operation. To represent a continuous-time system that has a one-dimensional signal $x(t)$ as input and another one-dimensional signal $y(t)$ as output, one would do the following:

$$\xrightarrow{x(t)} \boxed{T\{\cdot\}} \xrightarrow{y(t) = \mathrm{T}\{x(t)\}}$$

Figure 5: Block diagram representation of $\mathrm{T}\{x(t)\}$.

One example of a system is a **sampler**: a sampler has a continuous signal as input and, given a certain sampling period, returns the vector of sampled points as output. A sampler is often referred to as an **ideal continuous-to-discrete converter** because no real-world sampler can possibly compute the value of the signal at each point in time with perfect accuracy (it is still a valid theoretical model, however). The block diagram representation of a sampler would look like this:

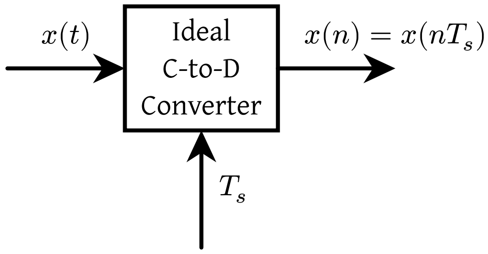

Figure 6: Block diagram representation of a sampler.

1.2. Sinusoids

Signals can have an arbitrary complicated equation, but such equations can be constructed from the ground up starting from simple building blocks[2].

The simplest one-dimensional continuous time signals are the **sinusoidal signals**, or **sinusoids** for short. The equation describing a sinusoidal signal has the following general form:

$$s(t) = A\cos(\omega t + \varphi)$$

Where $\cos$ is the trigonometric cosine function. Note that sinusoids are periodical functions (the cosine function multiplied by a constant), hence $s(t) = s(t + 2\pi)$ for any time instant t. As for the other components:

- A is the **amplitude**, the maximum value that the signal can attain (the height of any "spike"). Since $\cos$ oscillates between $+1$ and -1, a sinusoid oscillates between $-A$ and $+A$;
- ω is the **radian frequency**, the number of oscillations that the signal makes every 2π seconds;
- φ is the **phase**, the displacement from 0.

The radian frequency is the **cyclic frequency**, or just **frequency** for short, multiplied by 2π. The frequency f is the number of oscillations that the signal makes every second. The **period** T is the time the signal takes to make an entire oscillation. The frequency and the period are the reciprocal of each other.

$$f = 2\pi\omega \qquad\qquad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

Notice that choosing $f = 0$ gives a perfectly valid sinusoid: a constant function equal to its amplitude (since $\cos(0) = 1$). In the context of signals, a sinusoid having frequency 0 is often called DC (as in "direct current").

[2]Note that, again, there is no difference between a signal and the equation that models it, because the equation captures all information related to the signal. As a matter of fact, the expressions "the signal having equation $s(t)$" and "the signal $s(t)$" are equivalent

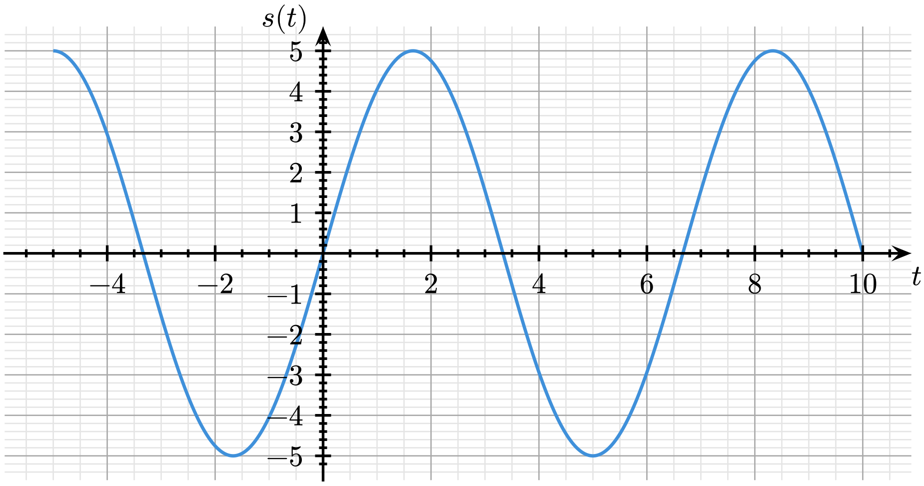

Figure 7: Plot of the sinusoidal signal $s(t) = 5\cos(0.3\pi t + 1.5\pi)$.

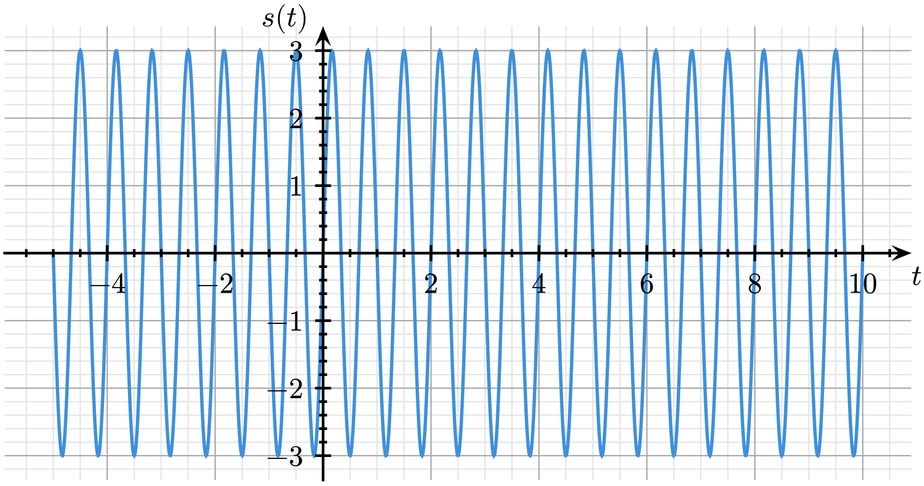

Figure 8: Plot of the sinusoidal signal $s(t) = 3\cos(3\pi t - 0.5\pi)$.

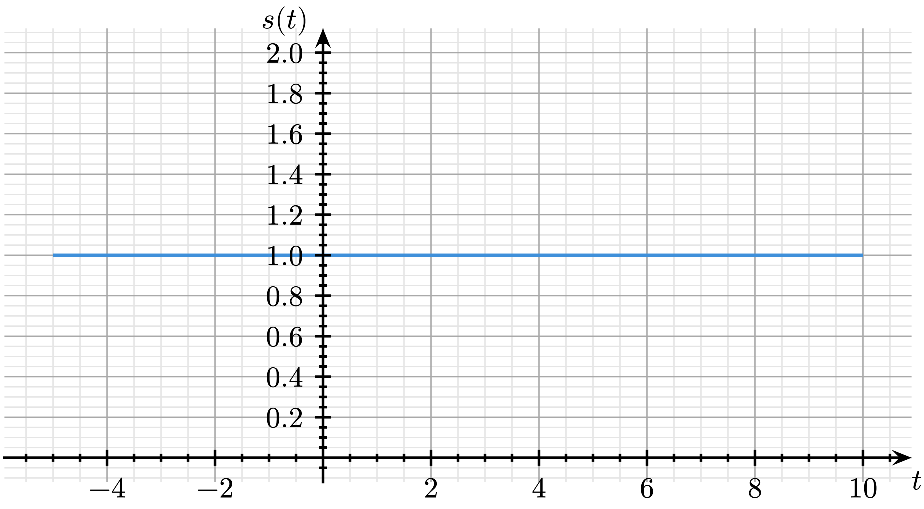

Figure 9: Plot of the sinusoidal signal $s(t) = \cos(0) = 1$ (A valid sinusoid nonetheless).

Technically speaking, it's also possible to use the sine instead of the cosine to construct sinusoidal signals, since $\sin(\omega t) = \cos(\omega t - \pi/2)$ for any ω and t. However, the cosine often takes precedence because $\cos(0) = 1$, while $\sin(0) = 0$, making it much easier in computations.

Plotting a sinusoidal function is relatively straightforward, since knowing its shape inside a single period gives the shape of the entire function. Knowing the expression of a sinusoid $A\cos(\omega t + \varphi)$, to plot it it is necessary to:

- Compute its period;

- Find any value of t that results in a peak;
- Find any value of t that results in 0;

Since ω is known, the period T is just $2\pi/\omega$. To find any t that results in a peak, let it be t_p, it suffices to observe how the peaks of a sinusoid are equal to their amplitude. This means that t_p is given by imposing $s(t_p) = A$:

$$\cancel{A} = \cancel{A}\cos(\omega t_p + \varphi) \Rightarrow \cos(\omega t_p + \varphi) = 1 \Rightarrow \omega t_p + \varphi = 2\pi k \quad \text{with } k \in \mathbb{Z}$$

Since there is no difference in choosing one peak over another, the simplest choice is $k = 0$. Solving for t_p:

$$\omega t_p + \varphi = 2\pi 0 \Rightarrow \omega t_p + \varphi = 0 \Rightarrow t_p = -\frac{\varphi}{\omega}$$

To find a t that results in 0, let it be t_c, it suffices to impose $s(t_c) = 0$ and solve for t_c:

$$0 = A\cos(\omega t_c + \varphi) \Rightarrow \cos(\omega t_c + \varphi) = 0 \Rightarrow \omega t_c + \varphi = \frac{\pi}{2} + 2\pi k \quad \text{with } k \in \mathbb{Z}$$

Choosing $k = 0$ once again:

$$\omega t_c + \varphi = \frac{\pi}{2} + 2\pi 0 \Rightarrow \omega t_c + \varphi = \frac{\pi}{2} \Rightarrow \omega t_c = \frac{\pi}{2} - \varphi \Rightarrow t_c = \frac{\pi}{2\omega} - \frac{\varphi}{\omega}$$

But $-\varphi/\omega$ is the time resulting in a peak. Moreover, since $\omega = 2\pi f$:

$$t_c = \frac{\pi}{2\omega} - \frac{\varphi}{\omega} = \frac{\cancel{\pi}}{4\cancel{\pi}f} + t_p = \frac{1}{4f} + t_p = \frac{T}{4} + t_p$$

Exercise 1.2.1: Plot the sinusoid $s(t) = 20\cos(0.6\pi t - 0.4\pi)$

Solution: The period is given by:

$$T = \frac{2\pi}{\omega} = \frac{2\cancel{\pi}}{0.6\cancel{\pi}} = \frac{2}{0.6} \approx 3.333$$

The time resulting in a peak is given by:

$$t_p = -\frac{\varphi}{\omega} = -\frac{-0.4\cancel{\pi}}{0.6\cancel{\pi}} = \frac{0.4}{0.6} \approx 0.666$$

The time resulting in a 0 crossing is given by:

$$t_c = \frac{T}{4} + t_p = \frac{3.333}{4} + 0.666 \approx 0.834 + 0.666 = 1.5$$

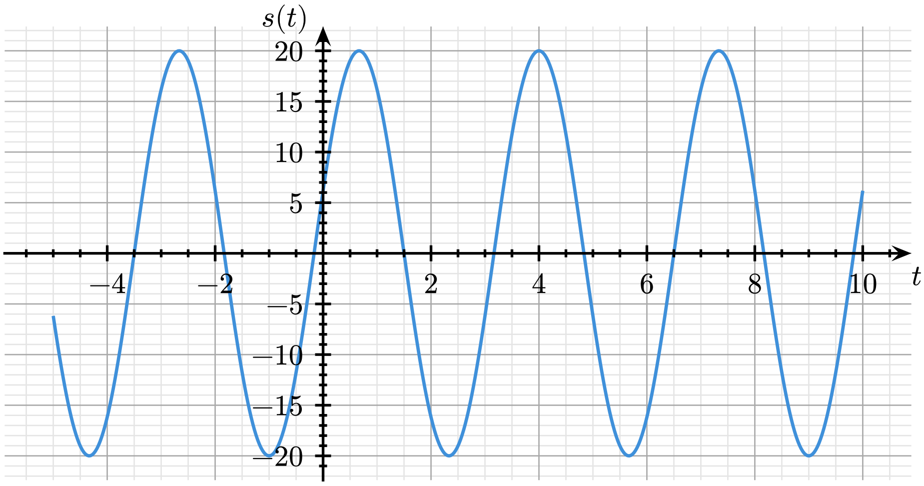

Figure 10: Plot of the sinusoidal signal $s(t) = 20\cos(0.6\pi t - 0.4\pi)$.

$\square$

Note that sketching any sinusoid by hand can be done with continuous strokes, while plotting it with a computer necessarely requires a discretization of the sinusoid. A digital plot is nothing but the quantized sinusoid with a sampling period so small that the function appears continuous.

If the equation of a sinusoid is known, there is no need to introduce a system convert from continuous to discrete: it is sufficient to evaluate the function at sufficiently many distinct and equally spaced time instants. Given a sinusoid $s(t) = A\cos(\omega t + \varphi)$ and a sampling period T_s, the vector of samples is given by solving $s(nT_s) = A\cos(\omega nT_s + \varphi)$ for a sufficient number of n.

In general, a computer plotting device performs what's called a **linear interpolation**, connecting adjacent points with a straight line: for sufficiently small segments, a straight line and a smooth curve are indistinguishable. Intuitively, having a greater number of points will result in a better approximation. In turn, a smaller sampling period will increase the number of points, because more points can "fit" into one period. This also means that a sinusoid with a smaller period (higher frequency) requires a smaller sampling period to achieve the same accuracy. As it will be clear later, there are techniques that go beyond linear interpolation and that, under the right assumptions, can reconstruct the original sinusoid with perfect accuracy.

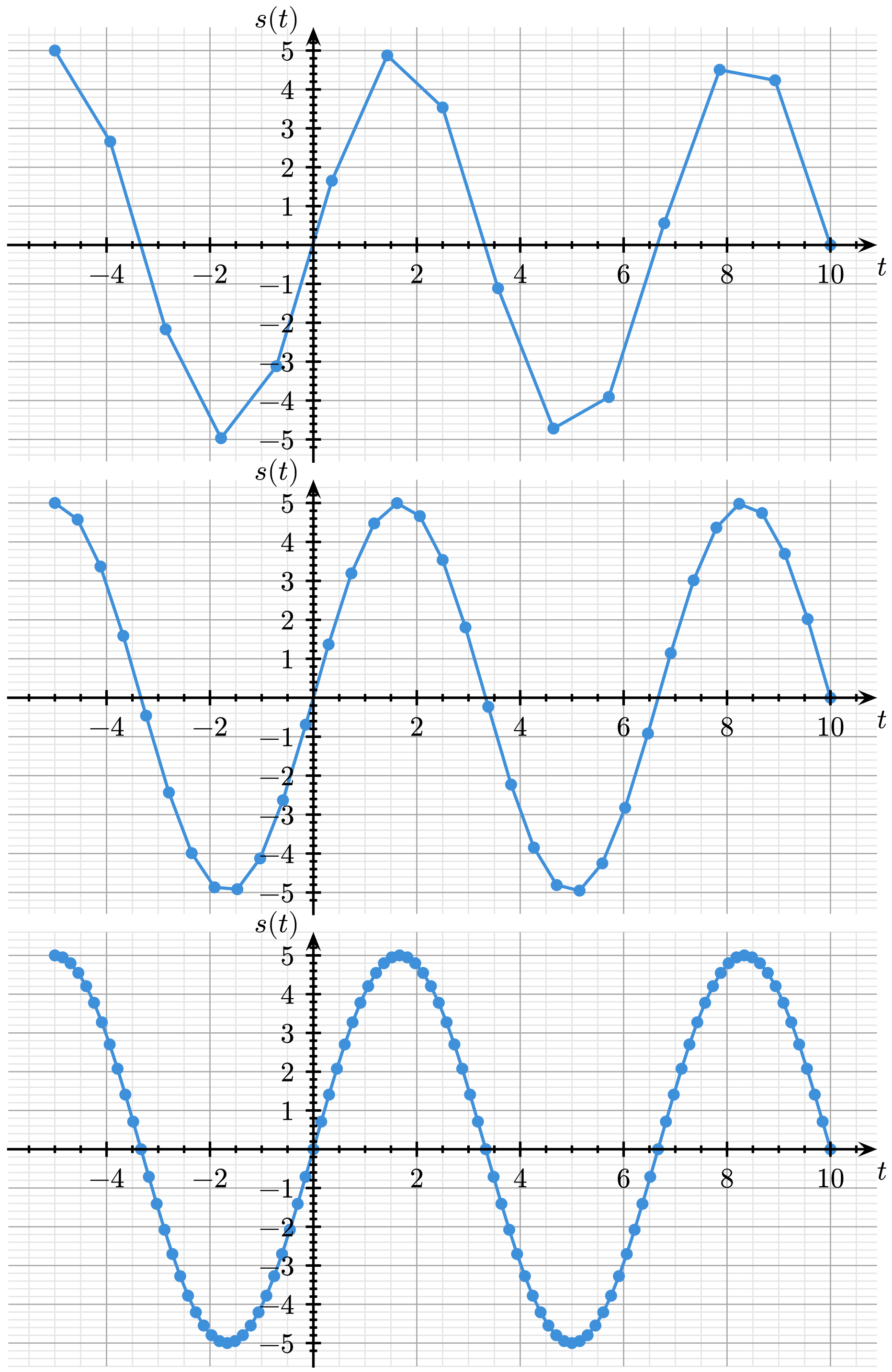

Figure 11: Plot of $s(t) = 5\cos(0.3\pi t + 1.5\pi)$ with three different choices of the sampling period.

If so desired, it's also possible to go the other way around, from the plot of a sinusoid to its equation. To do so:

- Determine the period, which is the time interval between any two points having the same value of $s(t)$. The simplest choices are either two adjacent 0 crossings or two adjacent peaks;
- Compute the radian frequency from the period as $2\pi/T$;
- Determine the amplitude, which is just the height of any peak;
- Compute the phase by using the formula $t = -\varphi/\omega$ backwards, solving for φ instead of t. The value of t is t_p, the time instant closest to 0 that results in a peak.

Exercise 1.2.2: What is the equation of the following sinusoid?

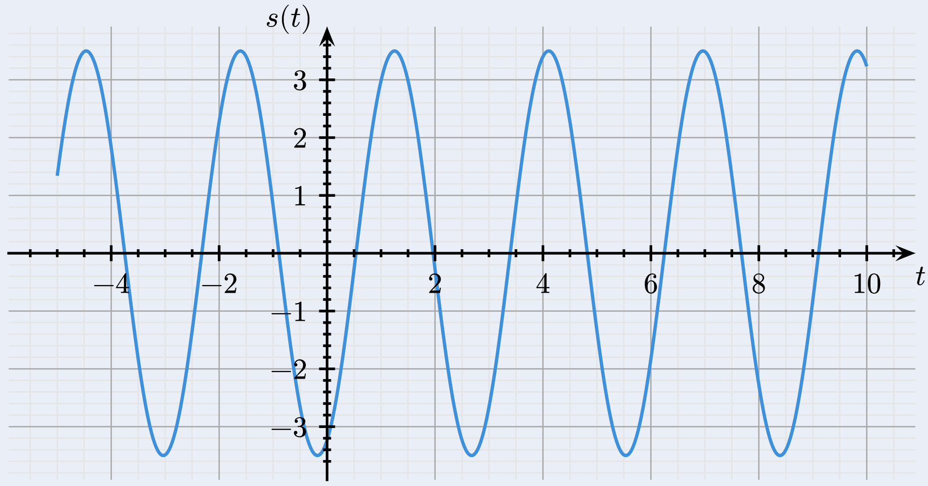

Figure 12: Plot of a sinusoid, whose equation is to be determined.

Solution:

- The sinusoid has its first peak (starting from 0) at about 1.25 and its second peak at about 4.1, hence its period is $4.1 - 1.25 = 2.85$;
- Its radian frequency is $2\pi/2.85 \approx 0.7\pi$;
- The amplitude is about 3.5;
- The phase is given by $\varphi = -\omega t_p = -0.7\pi \cdot 1.25 = -0.875$.

Which means that the sought for equation is:

$$s(t) = 3.5\cos(0.7\pi t - 0.875\pi)$$

□

Most real waves can hardly be modeled by a simple sinusoid functions, since some attenuation over time or over distance has to be taken into account. Given a real value α that represents the dampening of the strength of the signal, a more accurate waveform has the following equation:

$$s(t) = A(t)\cos(\omega t + \varphi) = Ae^{-t/\alpha}\cos(\omega t + \varphi)$$

Where the amplitude has now a time dependence, instead of being a constant. Since the negative exponential is a decreasing function, the amplitude of the signal will decrease over time.

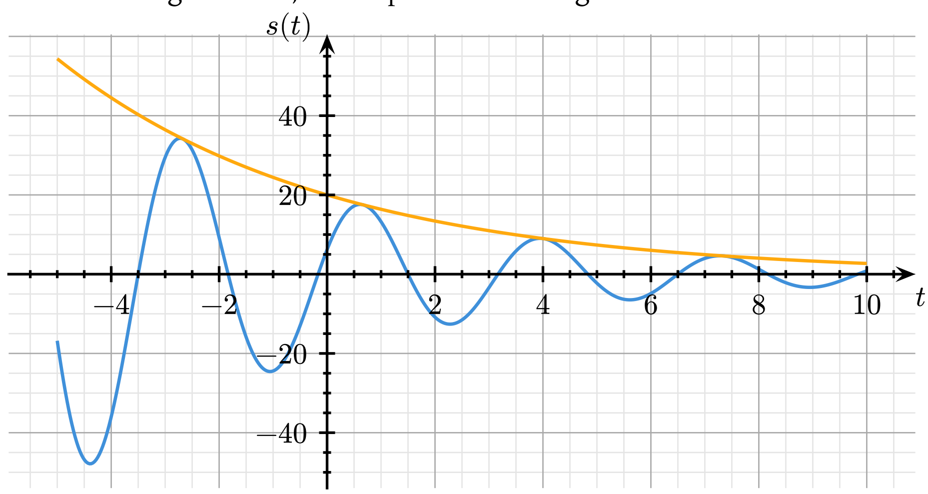

Figure 13: In blue, the plot of the sinusoidal signal $s(t) = 20e^{-t/5} \cos(0.6\pi t - 0.4\pi)$, using a sampling period of $T_s = 0.03$. In orange, the time-dependent amplitude $A(t) = 20e^{-t/5}$.

Another way to construct elaborate signals is to sum more sinusoids. The problem of summing sinusoids is that computing the sum of cosines (or sines, for that matter) is tedious. However, it's possible to rewrite the sinusoid in a slightly different form that aides in performing mathematical manipulations.

Recall how a complex number in polar form can be converted into the exponential form[3]:

$$re^{j\theta} = r\cos(\theta) + rj\sin(\theta)$$

Given an amplitude A, a radian frequency ω_0 and a phase φ, a **complex exponential signal** is defined as:

$$z(t) = Ae^{j(\omega_0 t + \varphi)} = A\cos(\omega_0 t + \varphi) + Aj\sin(\omega_0 t + \varphi)$$

Whose magnitude is the constant A and whose argument is the time-dependent expression $\omega_0 t + \varphi$. Both the real part and the imaginary part of the complex number represent a sinusoid; the two sinusoids have the same amplitude and frequency, differing only by a phase factor of $\pi/2$. This is because:

$$z(t) = A\cos(\omega_0 t + \varphi) + Aj\sin(\omega_0 t + \varphi) = A\cos(\omega_0 t + \varphi) + Aj\cos\left(\omega_0 t + \varphi - \frac{\pi}{2}\right)$$

In particular, notice how:

$$s(t) = \Re\{z(t)\} = \Re\left\{Ae^{j(\omega_0 t + \varphi)}\right\} = A\cos(\omega_0 t + \varphi)$$

Which means that the sinusoids so far considered are just the real part of complex exponential signals.

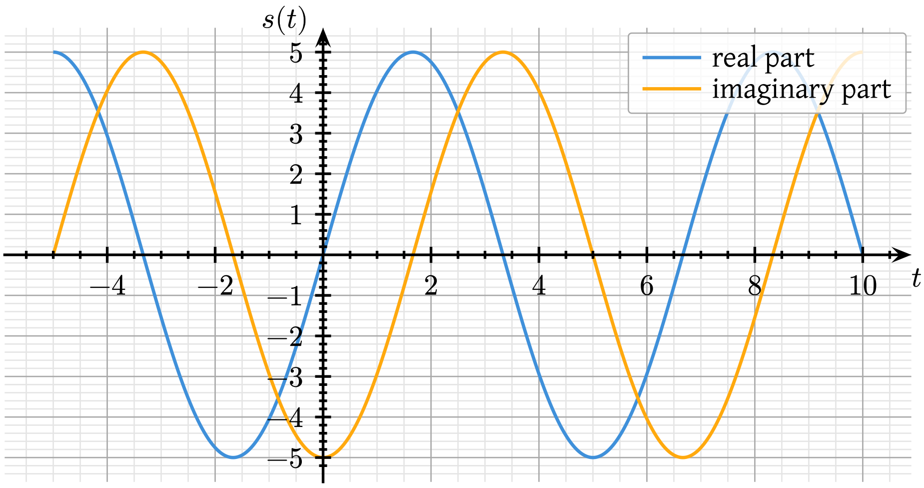

Figure 14: Plot of the complex exponential $z(t) = 5\cos(0.3\pi t + 1.5\pi) + 5j\cos(0.3\pi t + 1.5\pi - 0.5\pi)$, with both the real part and the imaginary part. The two differ by $\pi/2$.

Recall that the geometric interpretation of the multiplication of two complex numbers is a rotation in the complex plane (angles are added and magnitudes are scaled). The complex exponential signal $z(t)$ can be written as the product of two complex numbers:

$$z(t) = Ae^{j(\omega_0 t + \varphi)} = Ae^{j\omega_0 t + j\varphi} = Ae^{j\omega_0 t}e^{j\varphi} = Xe^{j\omega_0 t}$$

Where $X = Ae^{j\varphi}$ is called the **complex amplitude**, or **phasor**[4]. This means that $z(t)$ is the product of a complex constant (the phasor) and a complex-valued function dependent on time $e^{j\omega_0 t}$.

[3] It is customary to use j instead of i to denote the imaginary unit when dealing with signals, because i is often used to denote the intensity of a signal. This notation is commonplace in all fields of engineering.

[4] The term phasor is common in electrical circuit theory

By writing $\theta(t) = \omega_0 t + \varphi$, that is by expliciting the time dependence of the angle, it's also possible to write a complex exponential signal as:

$$z(t) = Xe^{j\omega_0 t} = Ae^{j\theta(t)}$$

At a given time instant t, the value of the complex exponential signal $z(t)$ is a complex number whose magnitude is A and whose angle is $\theta(t)$.

Consider the representation of $z(t) = Ae^{j\theta(t)}$ in the complex plane: as t increases, the complex number only changes in angle but not in magnitude, as the time dependency is only present in the angle. This means that the corresponding vector in the complex plane keeps rotating without ever changing in magnitude. This is why a complex exponential signal is also called **rotating phasor**.

The "speed" at which the vector rotates, meaning how much area of the plane is traversed as time increases, depends on the radian frequency ω_0: the higher ω_0, the "faster" the rotation. Moreover, the sign of the radian frequency determines the direction of the rotation: if ω_0 is positive, the rotation is counterclockwise, since the angle θ increases; if ω_0 is negative, the rotation is clockwise, since the angle θ decreases. Rotating phasors are said to have **positive frequency** if they rotate counterclockwise, and **negative frequency** if they rotate clockwise.

A rotating phasor makes one complete revolution every time the angle $\theta(t)$ changes by 2π radians. The time it takes to make one revolution is also equal to the period T_0 of the complex exponential signal, so:

$$\omega_0 T_0 = (2\pi f_0)T_0 = 2\pi \Rightarrow T_0 = \frac{1}{f_0}$$

Notice that the phase φ defines where the phasor is pointing when $t = 0$.

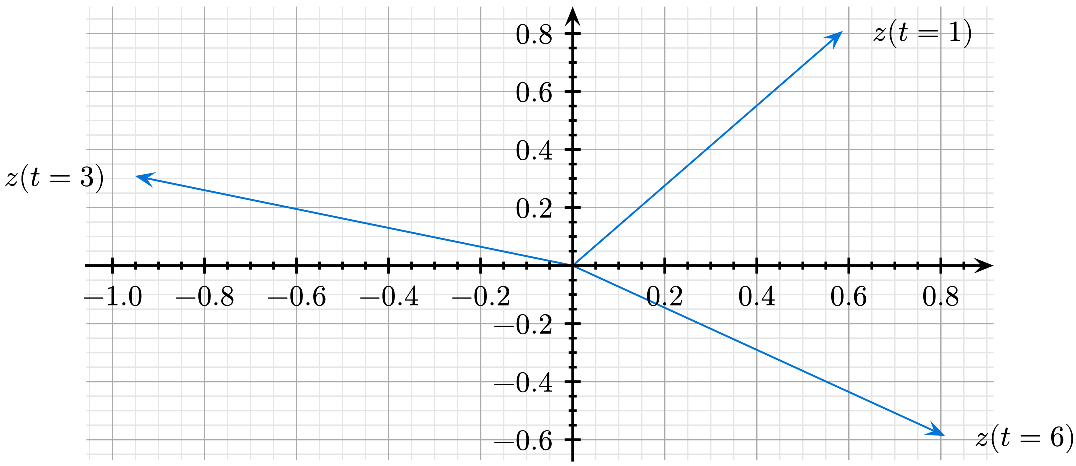

Figure 15: From the complex exponential signal $z(t) = 5e^{j(0.3\pi t + 1.5\pi)}$, discarding the phasor one gets $e^{j0.3\pi t}$.

As already hinted at, complex exponential signals allow one to compute the sum of sinusoids with ease. This is remarkably true when summing sinusoids having the same radian frequency.

Theorem 1.2.1 (Phasor addition rule): Let $A_k \cos(\omega_0 t + \varphi_k)$ with $k \in \{1, 2, ..., n\}$ be a family of n sinusoids, all having the same radian frequency. Then their sum is still a sinusoid. In particular:

$$\sum_{k=1}^{n} A_k \cos(\omega_0 t + \varphi_k) = A \cos(\omega_0 t + \varphi)$$

Where:

$$A = \left\| \sum_{k=1}^{n} A_k e^{j\varphi_k} \right\| \qquad \varphi = \arg\left(\sum_{k=1}^{n} A_k e^{j\varphi_k} \right)$$

Proof: Recall that, for any sinusoid:

$$A \cos(\omega_0 t + \varphi) = \Re\left\{ A e^{j(\omega_0 t + \varphi)} \right\}$$

Then:

$$\sum_{k=1}^{n} A_k \cos(\omega_0 t + \varphi_k) = \sum_{k=1}^{n} \Re\left\{ A_k e^{j(\omega_0 t + \varphi_k)} \right\} = \sum_{k=1}^{n} \Re\left\{ A_k e^{j\omega_0 t} e^{j\varphi_k} \right\}$$

But the sum of the real part of n complex numbers is the real part of their sum, hence:

$$\sum_{k=1}^{n} A_k \cos(\omega_0 t + \varphi_k) = \sum_{k=1}^{n} \Re\left\{ A_k e^{j\omega_0 t} e^{j\varphi_k} \right\} = \Re\left\{ \sum_{k=1}^{n} A_k e^{j\omega_0 t} e^{j\varphi_k} \right\} =$$
$$= \Re\left\{ e^{j\omega_0 t} \sum_{k=1}^{n} A_k e^{j\varphi_k} \right\}$$

The sum of complex numbers is itself a complex number. With $A e^{j\varphi} = \sum_{k=1}^{n} A_k e^{j\varphi_k}$:

$$\sum_{k=1}^{n} A_k \cos(\omega_0 t + \varphi_k) = \Re\left\{ e^{j\omega_0 t} \sum_{k=1}^{n} A_k e^{j\varphi_k} \right\} = \Re\left\{ e^{j\omega_0 t} A e^{j\varphi} \right\} =$$
$$= \Re\left\{ A e^{j(\omega_0 t + \varphi)} \right\} = A \cos(\omega_0 t + \varphi)$$

$\square$

The only caveat to using Theorem 1.2.1 is that the sum $\sum_{k=1}^{n} A_k e^{j\varphi_k}$, in order to be computed, requires the k phasors to be converted in rectangular form.

Exercise 1.2.3: Compute the sum of the two sinusoids:

$$s_1(t) = 1.7 \cos\left(20\pi t + \frac{70\pi}{180} \right) \qquad s_2(t) = 1.9 \cos\left(20\pi t + \frac{200\pi}{180} \right)$$

Solution: $s_1(t)$ and $s_2(t)$, written as the real part of a complex exponential, are:

$$s_1(t) = \Re\left\{1.7e^{j(20\pi t + 70\pi/180)}\right\} = \Re\left\{1.7e^{j20\pi t}e^{j70\pi/180}\right\}$$
$$s_2(t) = \Re\left\{1.9e^{j(20\pi t + 200\pi/180)}\right\} = \Re\left\{1.9e^{j20\pi t}e^{j200\pi/180}\right\}$$

Which gives the two phasors:

$$X_1 = 1.7e^{j\frac{70\pi}{180}} = 1.7\left(\cos\left(\frac{70\pi}{180}\right) + j\sin\left(\frac{70\pi}{180}\right)\right) \approx 1.7(0.34 + j0.94) = 0.58 + j1.60$$
$$X_2 = 1.9e^{j\frac{200\pi}{180}} = 1.9\left(\cos\left(\frac{200\pi}{180}\right) + j\sin\left(\frac{200\pi}{180}\right)\right) \approx 1.9(-0.94 - j0.34) = -1.79 - j0.65$$

And their sum is:

$$X = X_1 + X_2 = (0.58 + j1.60) + (-1.79 - j0.65) = -1.21 + j0.95$$

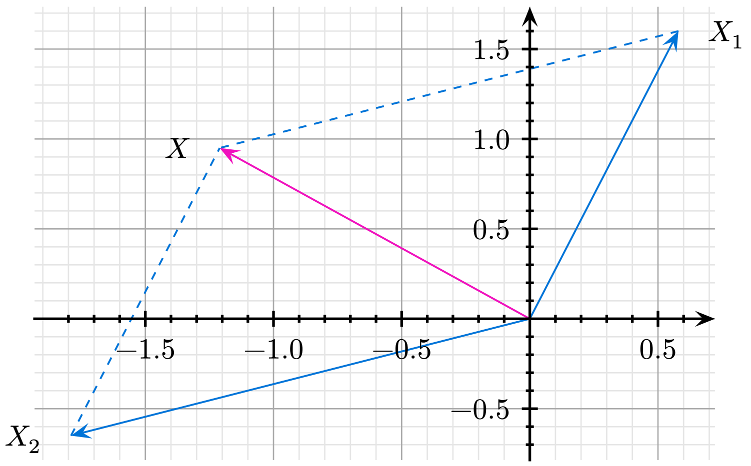

Figure 16: The two phasors X_1 and X_2 and their sum X, plotted on the complex plane.

The magnitude and the argument of X are:

$$\|X\| = \sqrt{(-1.21)^2 + (0.95)^2} \approx 1.54 \qquad \arg(X) = \tan^{-1}\left(\frac{0.95}{-1.21}\right) \approx \frac{142\pi}{180}$$

Which means that:

$$s(t) = s_1(t) + s_2(t) = 1.54\cos\left(20\pi t + \frac{142\pi}{180}\right)$$

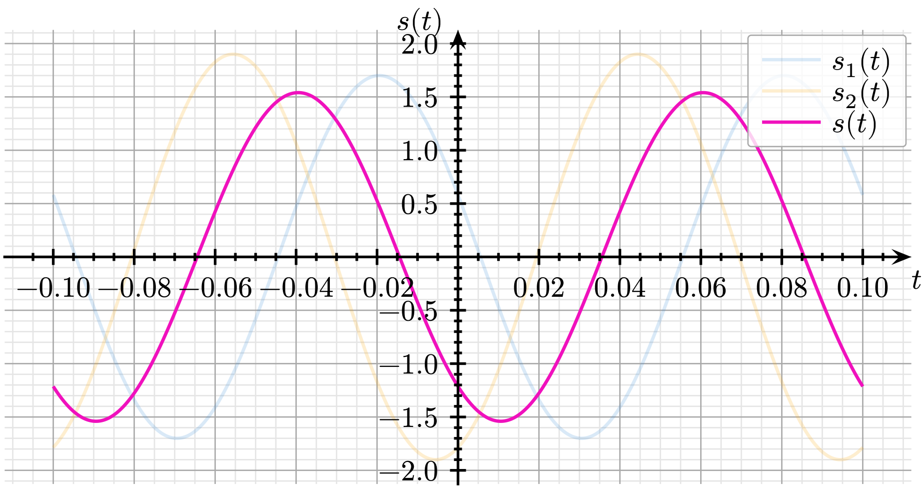

Figure 17: Plot of the sinusoids $s_1(t) = 1.7\cos(20\pi t + 70\pi/180)$ and $s_2(t) = 1.9\cos(20\pi t + 200\pi/180)$ and of their sum $s(t) = 1.54\cos(20\pi t + 142\pi/180)$.

□

1.3. Spectrum

The most general and powerful method for producing new signals from sinusoids is the *additive linear combination* (a linear combination that can include a scalar constant). A wide class of signals can be represented in the form:

$$x(t) = A_0 + \sum_{k=1}^{n} A_k \cos(2\pi f_k t + \varphi_k)$$

Where the amplitude, frequency and phase of each signal can differ. Or, expliciting the phasors and setting $A_0 = X_0$ for clarity:

$$x(t) = X_0 + \sum_{k=1}^{n} \Re\{A_k e^{j\varphi_k} e^{j2\pi f_k t}\} = X_0 + \sum_{k=1}^{n} \Re\{X_k e^{j2\pi f_k t}\}$$

As stated, this formula has the advantage of simplifying calculations when having to sum multiple sinusoids. However, it introduces the need to extract a real part out of a complex number. A different way to write a sinusoid as the sum of complex number without having to distinguish between a real and an imaginary part is hinted at by the inverse Euler formula for the cosine:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{e^{j\theta} + \left(e^{j\theta}\right)^*}{2}$$

Since a sinusoid is in the form $s(t) = A\cos(\omega_0 t + \varphi)$, it's also possible to write is as:

$$s(t) = A\cos(\omega_0 t + \varphi) = A\left(\frac{e^{j(\omega_0 t + \varphi)} + e^{-j(\omega_0 t + \varphi)}}{2}\right) = \frac{A}{2}e^{j(\omega_0 t + \varphi)} + \frac{A}{2}e^{-j(\omega_0 t + \varphi)} =$$

$$= \frac{A}{2}e^{j\omega_0 t}e^{j\varphi} + \frac{A}{2}e^{-j\omega_0 t}e^{-j\varphi} = \frac{1}{2}Xe^{j\omega_0 t} + \frac{1}{2}\left(Xe^{j\omega_0 t}\right)^* = \frac{1}{2}z(t) + \frac{1}{2}z^*(t) = \Re\{z(t)\}$$

This formula has an interesting interpretation. The sinusoid $s(t)$ is actually composed of a positive frequency complex exponential $\frac{1}{2}Xe^{j\omega_0 t}$ and a negative frequency complex exponential $\frac{1}{2}\left(Xe^{j\omega_0 t}\right)^*$. The two have the same amplitude, the same phase in modulus and the same radian frequency in modulus. In other words, any sinusoid can be represented as the sum of two complex rotating phasors that are rotating in opposite directions (the angles have opposite sign) starting from phasors that are complex conjugates of each other.

A negative radian frequency implies a negative cyclical frequency. It's possible to make sense of a negative frequency by interpreting the sign of the frequency as the direction in which the wave is moving with respect to a given frame of reference. If the frequency is positive, the wave is moving *towards* the observer, if the frequency is negative, the wave is moving *away* from the observer.

Using this formula to express a sum of sinusoids:

$$x(t) = X_0 + \sum_{k=1}^{n} \frac{X_k}{2}e^{j2\pi f_k t} + \left(\frac{X_k}{2}e^{j2\pi f_k t}\right)^*$$

Where each sinusoid is decomposed into the real part of the sum of two rotating phasors having the same frequency in magnitude but opposite in sign, rotating in opposite direction.

The **spectrum** of a signal is its representation as a sum of multiple sinusoids. In particular, the last equation is called the **two-sided representation** of a signal because it uses n positive frequencies along with the corresponding n complex amplitudes and n negative frequencies along with the corresponding n complex amplitudes, plus the constant term X_0. Specifically, the two-sided representation of a signal is the set of pairs:

$$\left\{(0, X_0), \left(f_1, \frac{1}{2}X_1\right), \left(-f_1, \frac{1}{2}X_1^*\right), \ldots, \left(f_n, \frac{1}{2}X_n\right), \left(-f_n, \frac{1}{2}X_n^*\right)\right\}$$

Where each pair $\left(f_k, \frac{1}{2}X_k\right)$ denotes one contribution to the total, with its frequency and phasor.

It is common to refer to the spectrum as the **frequency-domain representation** of the signal, which encapsulates the smallest amount of information needed to reconstruct it. In contrast, the **time-domain representation** gives the values of the time waveform itself, its explicit form. When the spectrum of a signal is small, meaning that it doesn't have that many components, it is referred to as a **sparse** spectrum.

Exercise 1.3.1: What is the spectrum of the following signal?

$$x(t) = 10 + 14\cos\left(200\pi t - \frac{\pi}{3}\right) + 8\cos\left(500\pi t + \frac{\pi}{2}\right)$$

Solution: Writing the second and third term of the sum as the real part of a rotating phasor:

$$s_1 = 14\cos\left(200\pi t - \frac{\pi}{3}\right) = \Re\left\{14e^{j(200\pi t - \pi/3)}\right\} = \Re\left\{14e^{j200\pi t}e^{-j\pi/3}\right\}$$
$$s_2 = 8\cos\left(500\pi t + \frac{\pi}{2}\right) = \Re\left\{8e^{j(500\pi t + \pi/2)}\right\} = \Re\left\{8e^{j500\pi t}e^{j\pi/2}\right\}$$

Which gives the two phasors $X_1 = 14e^{-j\pi/3}$ and $X_2 = 8e^{j\pi/2}$. The two frequencies are:

$$f_1 = \frac{\omega_1}{2\pi} = \frac{200\cancel{\pi}}{2\cancel{\pi}} = 100 \qquad\qquad f_2 = \frac{\omega_2}{2\pi} = \frac{500\cancel{\pi}}{2\cancel{\pi}} = 250$$

Giving the spectrum:

$$\left\{(0, 10), \left(100, 7e^{-j\pi/3}\right), \left(-100, 7e^{j\pi/3}\right), \left(250, 4e^{j\pi/2}\right), \left(-250, 4e^{-j\pi/2}\right)\right\}$$

Or, written as a sum:

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{j2\pi(-100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{j2\pi(-250)t}$$

□

To simplify the notation, it is useful to introduce the following convention:

$$a_k = \begin{cases} A_0 & \text{if } k = 0 \\ \frac{1}{2}A_k e^{j\varphi_k} & \text{if } k \neq 0 \end{cases}$$

Which allows one to rewrite the two half-phasors sum representation as a sum that involves a single term:

$$x(t) = X_0 + \sum_{k=1}^{n} \frac{X_k}{2}e^{j2\pi f_k t} + \left(\frac{X_k}{2}e^{j2\pi f_k t}\right)^* = \sum_{k=-n}^{n} a_k e^{j2\pi f_k t}$$

Assuming $f_0 = 0$, $X_{-k} = X_k^*$ and $f_{-k} = -f_k$.

Enumerating the n pairs (f_k, a_k) is not particularly informative. A useful representation of a sum of sinusoids is the **spectrum plot**, a stem plot where each stem is positioned in correspondence with the frequency f_k of the k-th pair and the length of the stem is proportional (or equal) to the magnitude of a_k. Each stem is also referred to as a **spectral line** and is also labeled with the value of a_k itself. Spectra of signals comprised of individual sinusoids, and not of a sum of sinusoids, are often called **line spectra**.

Exercise 1.3.2: What is the spectrum plot of the sinusoidal signal in Exercise 1.3.1?

Solution:

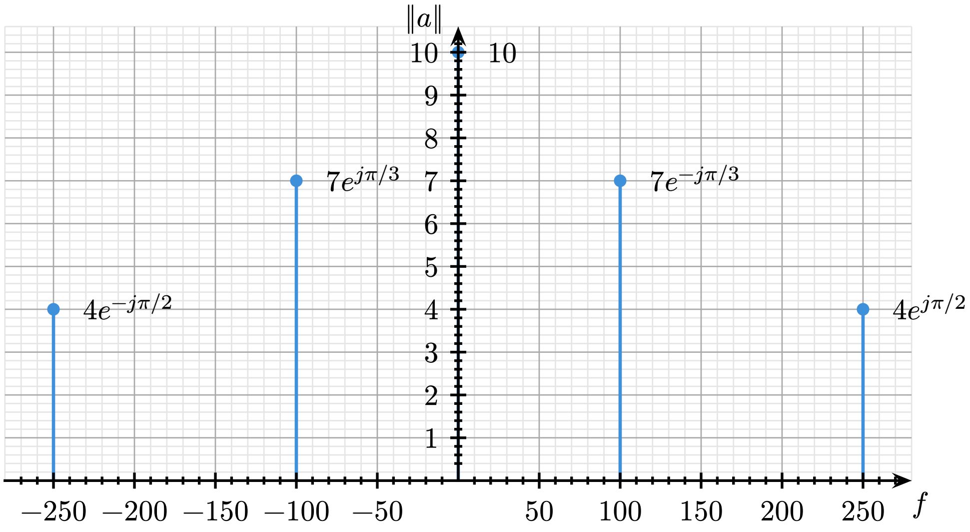

Figure 18: Spectrum plot of $x(t) = 10 + 14\cos(200\pi t - \frac{\pi}{3}) + 8\cos(500\pi t + \frac{\pi}{2})$.

□

This plot makes it easy to see the relative location of the frequencies and the relative amplitudes of the sinusoidal components. Moreover, it entirely captures the information necessary to encode a sum of sinusoids: the sum itself can be reconstructed from the information present in the plot.

In a spectrum like Exercise 1.3.2 the complex amplitude of each negative frequency component is the complex conjugate of the complex amplitude at the corresponding positive frequency component. This property is called **conjugate symmetry** and arises whenever $x(t)$ is a real signal. This is because any real number can be written as the sum of two complex numbers, one being the complex conjugate of the other, due to the inverse Euler formula.

Once a signal has been encoded as a spectrum, it becomes much easier to manipulate, in particular when applying time-domain operations. Since the spectrum consists of a set of frequency/complex amplitude pairs $S = \{(f_k, a_k)\}$, operating on a signal $x(t)$ in the time domain might influence only its frequencies, only its amplitudes or both the frequencies and the amplitudes.

The simplest example of spectrum property is scaling a signal $x(t)$ by a constant γ: all amplitudes are scaled by γ, while the frequencies remain unchanged:

$$\gamma x(t) = \gamma \sum_{k=-n}^{n} a_k e^{j2\pi f_k t} = \sum_{k=-n}^{n} (\gamma a_k) e^{j2\pi f_k t}$$

Another simple spectrum property is adding a constant c; this constant is grouped with the old DC forming a new, shifted, DC, while the other components remain unchanged:

$$c + x(t) = c + \sum_{k=-n}^{n} a_k e^{j2\pi f_k t} = \left(\sum_{k=-n}^{-1} a_k e^{j2\pi f_k t} \right) + (c + a_0) + \left(\sum_{k=1}^{n} a_k e^{j2\pi f_k t} \right)$$

Obviously, if the signal had no DC, the constant c becomes the DC.

Summing two signals (two spectra) is also trivial, even for signals that do not share the same frequencies. The resulting spectrum is the sum of the individual terms of the two spectra:

$$x_1(t) + x_2(t) = \sum_{k=-n}^{n} a_{1,k} e^{j2\pi f_{1,k} t} + \sum_{k=-n}^{n} a_{2,k} e^{j2\pi f_{2,k} t}$$

If the resulting sum happens to have duplicate frequencies, the terms can be merged together summing their phasors using Theorem 1.2.1.

Exercise 1.3.3: Compute the sum of the following signals, expressed in spectrum form.

$$x_1(t) = 10 + 7e^{-j\pi/3} e^{j2\pi(100)t} + 7e^{j\pi/3} e^{j2\pi(-100)t} + 4e^{j\pi/2} e^{j2\pi(250)t} + 4e^{-j\pi/2} e^{j2\pi(-250)t}$$
$$x_2(t) = 3e^{-j\pi/6} e^{j2\pi(100)t} + 3e^{j\pi/6} e^{j2\pi(-100)t} + 5e^{j\pi/4} e^{j2\pi(200)t} + 5e^{-j\pi/4} e^{j2\pi(-200)t}$$

Solution: The two shared frequencies can be merged using Theorem 1.2.1:

$$\begin{aligned}
7e^{-j\pi/3} + 3e^{-j\pi/6} &= 7\left(\cos\left(-\frac{\pi}{3}\right) + j\sin\left(-\frac{\pi}{3}\right)\right) + 3\left(\cos\left(-\frac{\pi}{6}\right) + j\sin\left(-\frac{\pi}{6}\right)\right) = \\
&= 3.500 - 6.062j + 2.598 - 1.500j \approx 9.7e^{-j100\pi/352} \\
7e^{j\pi/3} + 3e^{j\pi/6} &= 7\left(\cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right)\right) + 3\left(\cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right)\right) = \\
&= 3.500 + 6.062j + 2.598 + 1.500j \approx 9.7e^{j100\pi/352}
\end{aligned}$$

Which gives:

$$\begin{aligned}
x_1(t) + x_2(t) = 10 &+ 9.7e^{-j100\pi/352} e^{j2\pi(100)t} + 9.7e^{j100\pi/352} e^{j2\pi(-100)t} + 5e^{j\pi/4} e^{j2\pi(200)t} + \\
&5e^{-j\pi/4} e^{j2\pi(-200)t} + 4e^{j\pi/2} e^{j2\pi(250)t} + 4e^{-j\pi/2} e^{j2\pi(-250)t}
\end{aligned}$$

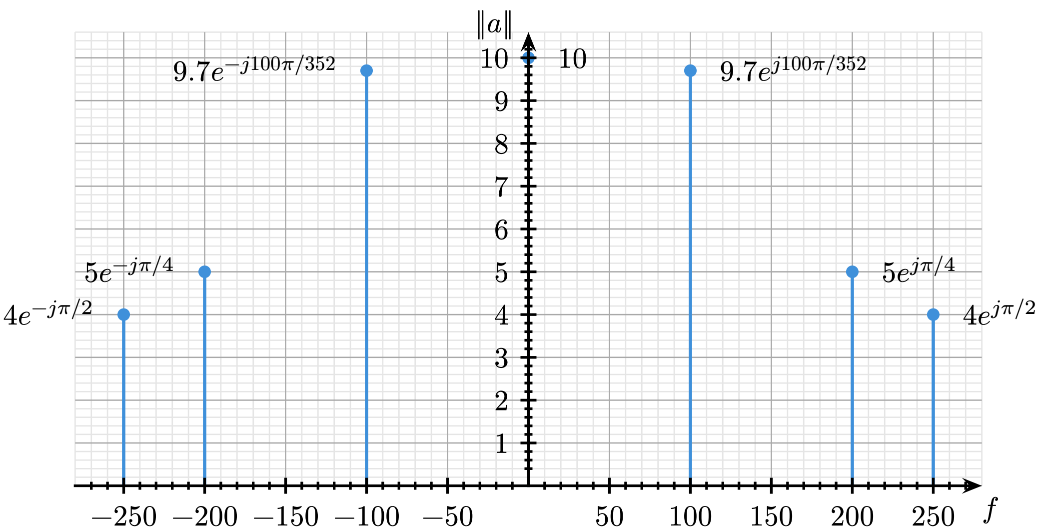

Figure 19: Spectrum plot of $x_1(t) + x_2(t)$.

□

Performing time shifting on a signal in spectrum form does not change the frequencies, but each k-th phasor is multiplied by a factor of $e^{-j2\pi f_k \tau}$:

$$x(t-\tau) = \sum_{k=-n}^{n} a_k e^{j2\pi f_k (t-\tau)} = \sum_{k=-n}^{n} a_k e^{j2\pi f_k t - j2\pi f_k \tau} = \sum_{k=-n}^{n} \left(a_k e^{-j2\pi f_k \tau}\right) e^{j2\pi f_k t}$$

Computing the derivative with respect to time on a signal in spectrum form does not change the frequencies, but each k-th phasor is multiplied by a factor of $j2\pi f_k$:

$$\begin{aligned} \frac{d}{dt} x(t) &= \frac{d}{dt} \sum_{k=-n}^{n} a_k e^{j2\pi f_k t} = \sum_{k=-n}^{n} \frac{d}{dt}\left(a_k e^{j2\pi f_k t}\right) = \sum_{k=-n}^{n} a_k \frac{d}{dt}\left(e^{j2\pi f_k t}\right) = \\ &= \sum_{k=-n}^{n} a_k e^{j2\pi f_k t} \left(\frac{d}{dt}(j2\pi f_k t)\right) = \sum_{k=-n}^{n} (a_k j2\pi f_k) e^{j2\pi f_k t} \end{aligned}$$

If a signal has a spectrum in the form $\{(a_k, f_k)\}$, multiplying it by a complex exponential $Ae^{i\varphi} e^{j2\pi f t}$ turns its spectrum into $\{(a_k A e^{i\varphi}, f_k + f)\}$:

$$Ae^{i\varphi} e^{j2\pi f t} x(t) = Ae^{i\varphi} e^{j2\pi f t} \sum_{k=-n}^{n} a_k e^{j2\pi f_k t} = \sum_{k=-n}^{n} (a_k A e^{i\varphi}) e^{j2\pi (f_k + f) t}$$

This frequency shifting happens because multiplying exponentials is done by summing their exponents. Note that the result of the multiplication might not be a real signal, meaning that conjugate symmetry might not hold.

Exercise 1.3.4: Multiply the signal $x(t)$ by $C = 2e^{j\pi/2} e^{j2\pi(9)t}$.

$$x(t) = 10 + 7e^{-j\pi/3} e^{j2\pi(10)t} + 7e^{j\pi/3} e^{j2\pi(-10)t}$$

Solution:

$$\begin{aligned} Cx(t) &= 2e^{j\pi/2} e^{j2\pi(9)t}\left(10 + 7e^{-j\pi/3} e^{j2\pi(10)t} + 7e^{j\pi/3} e^{j2\pi(-10)t}\right) = \\ &= 20e^{j\pi/2} e^{j2\pi(9)t} + 2e^{j\pi/2} e^{j2\pi(9)t} 7e^{-j\pi/3} e^{j2\pi(10)t} + 2e^{j\pi/2} e^{j2\pi(9)t} 7e^{j\pi/3} e^{j2\pi(-10)t} = \\ &= 20e^{j\pi/2} e^{j2\pi(9)t} + 14e^{j\pi/6} e^{j2\pi(19)t} + 14e^{j5\pi/6} e^{j2\pi(-1)t} \end{aligned}$$

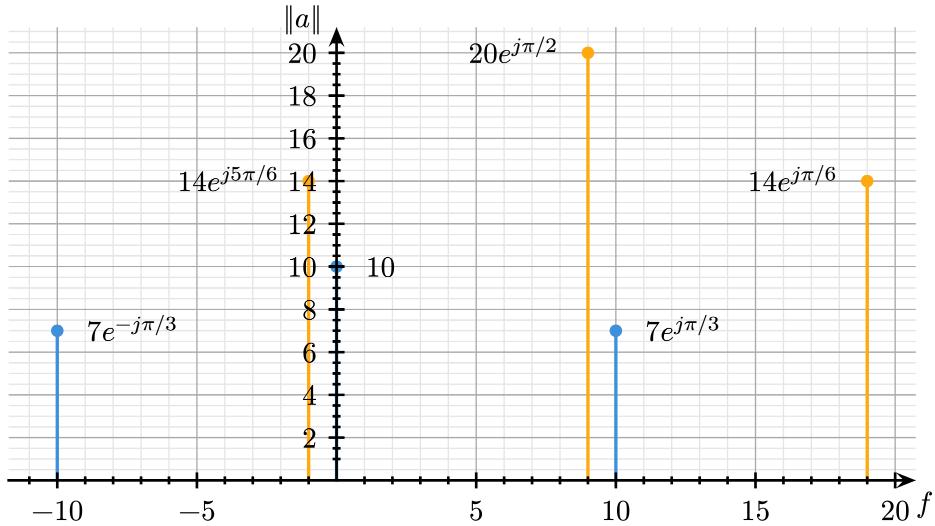

Figure 20: In blue, the spectrum plot of $x(t)$. In orange, the spectrum plot of $Cx(t)$. Notice how $Cx(t)$ does not possess the property of conjugate symmetry.

□

If a signal $x(t)$ in spectrum form is multiplied by a real sinusoid $A\cos(2\pi ft + \varphi)$, something remarkable happens. Recall that any real sinusoid (any real number, for that matter) can be broken down as the sum of a complex number and its complex conjugate:

$$x(t)A\cos(2\pi ft + \varphi) = x(t)\frac{1}{2}Ae^{j\varphi}e^{j2\pi ft} + x(t)\frac{1}{2}Ae^{-j\varphi}e^{-j2\pi ft}$$

This means that it's necessary to apply the frequency scaling twice, one for each complex exponential hereby constructed:

$$\begin{aligned} A\cos(2\pi ft + \varphi)x(t) &= A\cos(2\pi ft + \varphi)\sum_{k=-n}^{n} a_k e^{j2\pi f_k t} = \\ &= \left(\frac{1}{2}Ae^{j\varphi}e^{j2\pi ft}\sum_{k=-n}^{n} a_k e^{j2\pi f_k t}\right) + \left(\frac{1}{2}Ae^{-j\varphi}e^{-j2\pi ft}\sum_{k=-n}^{n} a_k e^{j2\pi f_k t}\right) = \\ &= \sum_{k=-n}^{n}\left(a_k\frac{1}{2}Ae^{i\varphi}\right)e^{j2\pi(f_k+f)t} + \sum_{k=-n}^{n}\left(a_k\frac{1}{2}Ae^{-i\varphi}\right)e^{j2\pi(f_k-f)t} \end{aligned}$$

Multiplying a signal by a real sinusoid effectively doubles the number of components: each component is both upscaled and shifted to the right and downscaled and shifted to the left.

Exercise 1.3.5: Multiply the signal $x(t)$ by $C = 2\cos(2\pi(12)t + \pi/4)$.

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(10)t} + 7e^{j\pi/3}e^{j2\pi(-10)t}$$

Solution: C can be written as:

$$C = 2\cos(2\pi(12)t + \pi/4) = e^{j\pi/4}e^{j2\pi(12)t} + e^{-j\pi/4}e^{-j2\pi(12)t}$$

Hence:

$$\begin{aligned}Cx(t) = &\left(e^{j\pi/4}e^{j2\pi(12)t} + e^{-j\pi/4}e^{-j2\pi(12)t}\right)\left(10 + 7e^{-j\pi/3}e^{j2\pi(10)t} + 7e^{j\pi/3}e^{j2\pi(-10)t}\right) = \\ &e^{j\pi/4}e^{j2\pi(12)t}\left(10 + 7e^{-j\pi/3}e^{j2\pi(10)t} + 7e^{j\pi/3}e^{j2\pi(-10)t}\right) + \\ &e^{-j\pi/4}e^{-j2\pi(12)t}\left(10 + 7e^{-j\pi/3}e^{j2\pi(10)t} + 7e^{j\pi/3}e^{j2\pi(-10)t}\right) = \\ &10e^{j\pi/4}e^{j2\pi(12)t} + 7e^{-j\pi/12}e^{j2\pi(22)t} + 7e^{j7\pi/12}e^{j2\pi(2)t} + \\ &10e^{-j\pi/4}e^{j2\pi(-12)t} + 7e^{-j7\pi/12}e^{j2\pi(-2)t} + 7e^{j\pi/12}e^{j2\pi(-22)t}\end{aligned}$$

In the more compact notation:

$$\left\{\left(-22, 7e^{j\pi/12}\right), \left(22, 7e^{-j\pi/12}\right), \left(12, 10e^{j\pi/4}\right), \left(-12, 10e^{-j\pi/4}\right), \left(2, 7e^{j7\pi/12}\right), \left(-2, 7e^{-j7\pi/12}\right)\right\}$$

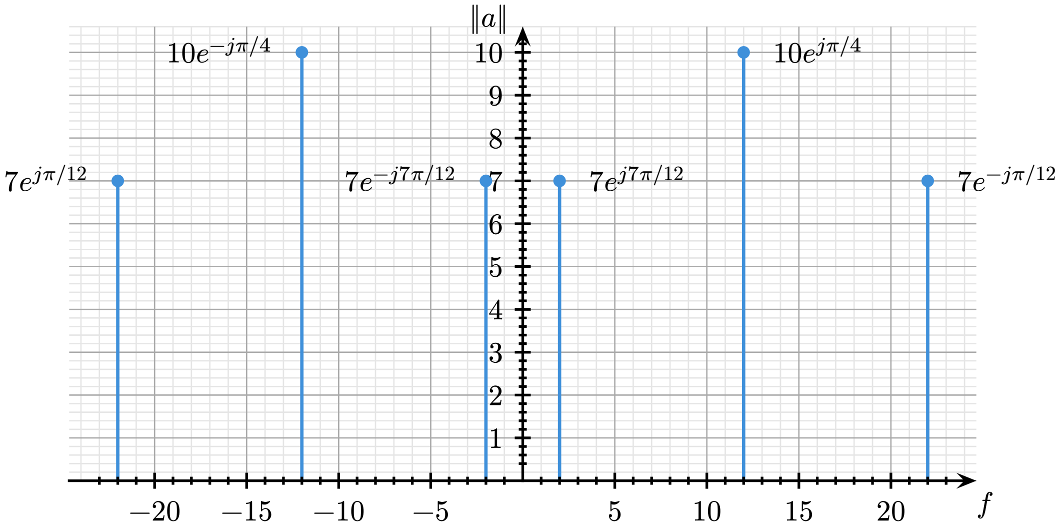

Figure 21: In blue, the spectrum plot of $x(t)$. In orange, the spectrum plot of $Cx(t)$. Notice how $Cx(t)$ does not possess the property of conjugate symmetry.

□

Note that when the frequency of the sinusoid is greater than the largest frequency in the spectrum of the signal, the upshifted and downshifted spectra do not overlap.