Contents

1 Sequences and series

1.1 Sequences

A **sequence** is any function whose domain is $\mathbb{N}$ and whose codomain is $\mathbb{R}$. A sequence is denoted as $\{x_1, x_2, ..., x_n\}$ or $\{x_n\}_{n \in \mathbb{N}}$. In general, the "$n \in \mathbb{N}$" pedix is omitted. Note that, technically, $\{x_1, x_2, ..., x_n\}$ denotes the image of the sequence, and not the function itself, but it is customary to denote sequences as such.

> **Exercise 1.1**
>
> Provide some examples of sequences.

Proof:

$$\{1, 2, 3, ..., n\} = \{n\} \qquad \{-1, 1, -1, ..., (-1)^n\} = \{(-1)^n\} \qquad \left\{1, \frac{1}{2}, \frac{1}{4}, ..., \left(\frac{1}{2}\right)^n\right\} = \left\{\left(\frac{1}{2}\right)^n\right\}$$

□

A sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be:

- **Increasing** if, for any $n < m$ with $n, m \in \mathbb{N}$, $x_n < x_m$;
- **Decreasing** if, for any $n < m$ with $n, m \in \mathbb{N}$, $x_n > x_m$.
- **Non decreasing** if, for any $n < m$ with $n, m \in \mathbb{N}$, $x_n \leq x_m$;
- **Non increasing** if, for any $n < m$ with $n, m \in \mathbb{N}$, $x_n \geq x_m$;

A sequence that possesses one of the aforementioned properties is said to be **monotone**.

> **Exercise 1.2**
>
> Do the sequences in Exercise 1.1 possess any of those properties?

Proof: $\{n\}_{n \in \mathbb{N}}$ is increasing, $\left\{\left(\frac{1}{2}\right)^n\right\}_{n \in \mathbb{N}}$ is decreasing, $\{(-1)^n\}_{n \in \mathbb{N}}$ is not monotone. □

A sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to have a **limit** $L \in \mathbb{R}$ if, given any number $\varepsilon > 0$, exists $N \in \mathbb{N}$ dependent on ε, such that for any $n > N$:

$$|x_n - L| < \varepsilon$$

In that case, $\{x_n\}_{n \in \mathbb{N}}$ is said to **converge** to L, or simply to be **convergent**. To denote that $\{x_n\}_{n \in \mathbb{N}}$ has limit L the following notation is used:

$$\lim_{n \to +\infty} x_n = L$$

> **Exercise 1.3**
>
> Prove that $\lim_{x \to +\infty} \frac{1}{n} = 0$.

Proof: Let ε be any strictly positive number, and let $N = \left[\frac{1}{\varepsilon}\right] + 1$. By the definition of N, any $n > N$ cannot lie in the interval $(0, 1]$, because it has to be at least greater than 1. Therefore, any $\frac{1}{n}$ must lie in $(0, 1]$. Since ε is strictly positive, it must be true that:

$$0 < \frac{1}{n} < \varepsilon$$

If both $\frac{1}{n}$ and ε are strictly positive, they must be greater than any negative number, and in particular greater than $-\varepsilon$. Therefore, the 0 in the inequality can be substituted with $-\varepsilon$:

$$0 < \frac{1}{n} < \varepsilon \Rightarrow -\varepsilon < \frac{1}{n} < \varepsilon \Rightarrow \left|\frac{1}{n}\right| < \varepsilon \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$$

Which is the definition of convergence. □

Note that the existence of a $L \in \mathbb{R}$ such that $\lim_{n\to+\infty} x_n = L$ is not guaranteed. That is to say, not all sequences are convergent.

Theorem 1.4 **Uniqueness of the limit**

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to+\infty} x_n = L$ with $L \in \mathbb{R}$. If such an L exists, it is unique.

Theorem 1.5

If a sequence $\{x_n\}$ is convergent then it is limited, that is to say it exists an interval $[a, b]$ such that $\{x_n\} \subset [a, b]$.

Theorem 1.6

Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences, converging to x and to y respectively. The following results hold:

- $\{x_n + y_n\}$ converges to $x + y$;
- Given $\alpha \in \mathbb{R}$, $\{\alpha x_n\}$ converges to αx;
- $\{x_n y_n\}$ converges to xy;
- $\{|x_n|\}$ converges to $|x|$;
- If $y \neq 0$ and, for any $n \in \mathbb{N}$, $y_n \neq 0$, the sequence $\left\{\frac{x_n}{y_n}\right\}$ converges to $\frac{x_n}{y_n}$;
- If $x_n < y_n$ for any $n \in \mathbb{N}$, then $x \leq y$;
- If $x_n = k$ for any $n \in \mathbb{N}$, then $\lim_{n\to+\infty} x_n = k$, $k \in \mathbb{R}$;

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to have **limit to** $+\infty$, denoted as:

$$\lim_{n\to+\infty} x_n = +\infty$$

If, for any $M > 0$, exists a $N \in \mathbb{N}$ dependent on M such that, for any $n > N$, it is true that $x_n > M$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to have **limit to** $-\infty$, denoted as:

$$\lim_{n\to+\infty} x_n = -\infty$$

If, for any $M > 0$, exists a $N \in \mathbb{N}$ dependent on M such that, for any $n > N$, it is true that $x_n < -M$.

In general, a sequence having limit to $\pm\infty$ is just called **divergent** (to $+\infty$ or to $-\infty$).

Exercise 1.7

Prove that $\lim\limits_{x\to+\infty} -2^n = -\infty$.

Proof: Let M be any strictly positive number, and let $N = [\log_2(M)] + 1$. For any $n > N$, it must be true that $2^n > M$, which in turn is equivalent to $-2^n < -M$. Therefore, $\lim\limits_{x\to+\infty} -2^n = -\infty$ □

If a sequence $\{x_n\}_{n\in\mathbb{N}}$ is neither convergent or divergent, is said to be **indeterminate**. More formally, a sequence is indeterminate if, for any $M > 0$, exists an $N \in \mathbb{N}$ dependent on M such that, for any $n > N$, it is true that $|x_n| > M$. An indeterminate sequence is denoted as:

$$\lim_{n\to+\infty} x_n = \infty$$

Notice the lack of sign on the symbol ∞, meaning that it's neither positive nor negative infinity

Note that the statements "a sequence is limited" and "a sequence has a limit" are not equivalent. Theorem 1.5 states that a sequence that has a limit (that is convergent) is limited, but a sequence that is limited might not have a limit.

Theorem 1.8

A sequence that is both limited and monotone has a limit.

Note that the opposite of Theorem 1.8 is not true. That is, if a sequence has a limit, it might not be monotone.

Theorem 1.9 **Squeeze theorem (for sequences)**

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences such that, for any $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. Suppose that:

$$\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} z_n = L$$

Then, $\lim_{n \to +\infty} y_n = L$.

Theorem 1.10

Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that, for any $n \in \mathbb{N}$, $x_n \leq y_n$. Suppose that:

$$\lim_{n \to +\infty} y_n = +\infty$$

Then, $\lim_{n \to +\infty} x_n = +\infty$.

Exercise 1.11

Prove that $\lim_{n \to +\infty} \left\{ \left(\frac{1}{2}\right)^n \right\} = 0$.

Proof: Consider the sequences $\{0^n\}$ and $\left\{\frac{1}{n}\right\}$. Both sequences are convergent:

$$\lim_{n \to +\infty} 0^n = 0 \qquad\qquad \lim_{n \to +\infty} \frac{1}{n} = 0$$

The result on the left is trivial, the result on the right was already proven in Exercise 1.3.

It is clear that, for any $n \in \mathbb{N}$, $0^n \leq \left(\frac{1}{2}\right)^n$. It is also possible to prove by induction that, for any $n \in \mathbb{N}$, $\left(\frac{1}{2}\right)^n \leq \frac{1}{n}$:

- With $n = 1$, $\frac{1}{2}^1 = \frac{1}{2}$ and $\frac{1}{1} = 1$, and clearly $\frac{1}{2} \leq 1$;
- Assuming $\left(\frac{1}{2}\right)^n \leq \frac{1}{n}$ to be true:

$$\left(\frac{1}{2}\right)^{n+1} \leq \frac{1}{n+1} \Rightarrow \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n \leq \frac{1}{n} - \frac{1}{n(n+1)} \Rightarrow \left(\frac{1}{2}\right)^n \leq \frac{2}{n} - \frac{2}{n(n+1)} \Rightarrow$$
$$\frac{1}{n} \leq \frac{2}{n} - \frac{2}{n(n+1)} \Rightarrow -\frac{1}{n} \leq -\frac{2}{n(n+1)} \Rightarrow \frac{1}{n} \geq \frac{2}{n(n+1)} \Rightarrow 1 \geq \frac{2}{n+1} \Rightarrow n+1 \geq 2 \Rightarrow n \geq 1$$

It is then possible to apply Theorem 1.9 to prove that $\lim_{n \to +\infty} \left\{ \left(\frac{1}{2}\right)^n \right\} = 0$. □

Let $\{x_n\}_{n \in \mathbb{N}}$ be an increasing sequence of natural numbers. $\{k_n\}_{n \in \mathbb{N}}$. The sequence $\left\{x_{k_n}\right\}_{n \in \mathbb{N}}$ is then said to be a **subsequence** of the sequence $\{x_n\}_{n \in \mathbb{N}}$.

Theorem 1.12

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence. If $\{x_n\}_{n \in \mathbb{N}}$ converges to L, then any subsequence of $\{x_n\}_{n \in \mathbb{N}}$ also converges to L.

Theorem 1.13

Every limited sequence has (at least) a converging subsequence.

1.2 Series

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. The following sum having an infinite amount of elements is called a **series**:

$$\sum_{n=i}^{\infty} a_n$$

The a_n elements are called the **terms** of the series. In most cases, i is equal to either 0 or 1.

A summation of this kind, even though having infinite terms, may still give a finite value. In particular, a series is said to be **convergent** if the sequence:

$$\{s_n\} = \left\{\sum_{k=i}^{n} a_k\right\} = \left\{\sum_{k=i}^{1} a_k, \sum_{k=i}^{2} a_k, \dots, \sum_{k=i}^{n} a_k\right\} = \{a_i, a_i + a_{i+1}, \dots, a_i + a_{i+1} + \dots + a_{i+n}\}$$

called the **partial sums sequence**, is itself convergent. If L is the limit of $\{s_n\}$, the series $\sum_{n=i}^{\infty} a_n$ is equal to L, and L is called the **sum** of the series:

$$\lim_{n\to+\infty} s_n = L = \sum_{n=i}^{\infty} a_n$$

Similarly, a series is said to be **divergent** (to $+\infty$ or to $-\infty$) if its partial sums sequence is divergent (to $+\infty$ or to $-\infty$, respectively). Finally, a series is said to be **indeterminate** if its partial sums sequence is indeterminate.

> **Exercise 1.14**
>
> Let r and a be two real numbers. Study the behaviour of the series
>
> $$\sum_{n=0}^{\infty} ar^n$$
>
> with respect to how r and a vary. Any series defined as such is called a **geometric series of common ratio** r.

Proof: If either a or r are equal to 0, the series is clearly equal to 0. This is why such cases are not taken into account.

The n-th partial sum of the series can be rewritten as:

$$s_n = \sum_{i=0}^{n} ar^n = ar^0 + ar^1 + ar^2 + ar^3 + \dots + ar^n = a\left(\frac{1-r^{n+1}}{1-r}\right)$$

As long as $r \neq 1$. Therefore:

$$\sum_{n=0}^{\infty} ar^n = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} a\frac{1-r^{n+1}}{1-r} = \frac{a}{1-r} - \left(\frac{a}{1-r}\right)\lim_{n\to+\infty} r^{n+1} = \frac{a}{1-r}\left(1 - \lim_{n\to+\infty} r^{n+1}\right)$$

There are three possibilities:

- If $r \geq 1$, it is easy to see that $\{r^{n+1}\}$ diverges to $+\infty$, and therefore $\sum_{n=0}^{\infty} ar^n$ diverges to $+\infty$ if a is positive or to $-\infty$ if a is negative. This is particularly evident when $r = 1$:

$$s_n = \sum_{i=0}^{n} a1^n = a1^0 + a1^1 + a1^2 + a1^3 + \dots + a1^n = a + a + a + a + \dots + a = a(n+1)$$

- If $-1 < r < 1$, (almost) as shown in Exercise 1.3, $\lim_{n\to+\infty} r^{n+1} = 0$. Therefore, $s_n = \dfrac{a}{1-r}$.
- If $r \leq -1$, it is easy to see that $\{r^{n+1}\}$ is indeterminate, and therefore $\sum_{n=0}^{\infty} ar^n$ is in turn indeterminate. This is particularly evident when $r = -1$:

$$s_n = \sum_{i=0}^{n} a(-1)^n = a(-1)^0 + a(-1)^1 + a(-1)^2 + a(-1)^3 + \dots + a(-1)^n = \cancel{a} - \cancel{a} + \cancel{a} - \cancel{a} + \dots$$

This means that s_n is either equal to a if n is even and equal to 0 if n is odd. Therefore, the limit $\lim_{n\to+\infty} s_n$ does not exist.

□

Exercise 1.15

Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges to 1. This series is called the **Mengoli series**.

Proof: Notice that, for any $k \in \mathbb{N}$ with $k > 0$:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

It is then possible to expand the series as:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{1} - \frac{1}{1+1} + \frac{1}{2} - \frac{1}{2+1} + \frac{1}{3} - \frac{1}{3+1} + \frac{1}{4} - \frac{1}{4+1} + \dots =$$
$$1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{1}{5} + \dots = 1 - \lim_{n\to+\infty} \frac{1}{n+1} = 1 - 0 = 1$$

□

Any series that can be rewritten as $\sum_{n=1}^{+\infty} b_n - b_{n+1}$ with $b \in \mathbb{R}$ is called a **telescopic series**. The series in Exercise 1.15 is an example of a telescopic series.

Exercise 1.16

Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges to $+\infty$. This series is called the **harmonic series**.

Proof: Let $\{s_n\}$ be the sequence of partial sums of the harmonic series. Notice that, for any $m \in \mathbb{N}$:

$$s_{2^m} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \dots + \left(\frac{1}{2^{m-1}+1} + \dots + \frac{1}{2^m}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = 1 + \frac{m}{2}$$

Since the first brackets contain 2 terms, the second brackets 4 terms, ecc... until the last brackets, that contain 2^{m-1} terms.

Therefore, $s_{2^m} > 1 + \frac{m}{2}$, from which stems $\lim_{n\to+\infty} s_{2^m} = +\infty$. Also notice that $\{s_{2^m}\}$ is a subsequence of $\{s_n\}$.

The sequence $\{s_n\}$ is (strictly) increasing and monotone, therefore it is either convergent or divergent to $+\infty$. If it were convergent, by Theorem 1.12 any of its subsequences would also be convergent, but since $\{s_{2^m}\}$ is not, it cannot be convergent. Then, $\{s_n\}$ ought to diverge to $+\infty$, and therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges to $+\infty$. □

Theorem 1.17 **Cauchy necessary condition for the convergence of series**

If the series $\sum_{n=i}^{\infty} a_n$ is convergent, then $\lim_{n\to+\infty} a_n = 0$.

Note that the opposite of Theorem 1.17 is not true, since it could be that $\lim\limits_{n\to+\infty} a_n = 0$ and yet the series $\sum_{n=i}^{\infty} a_n$ might not converge. The harmonic series, as shown in Exercise 1.16, is one such example.

Given a series, there is no formula that can be applied to determine whether the limit of its partial sums is a finite value, and whether it exists at all. A different approach to determine the convergence or non convergence of a series is to apply what is called a **convergence test**.

Theorem 1.18

- If the series $\sum a_n$ and $\sum b_n$ are convergent, the series $\sum(a_n + b_n)$ is also convergent, and it converges to $\sum a_n + \sum b_n$;
- If the series $\sum a_n$ is convergent then, given $\lambda \in \mathbb{R}$, the series $\sum \lambda a_n$ is also convergent, and it converges to $\lambda \sum a_n$;
- If there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, $a_n = b_n$, then the series $\sum a_n$ and $\sum b_n$ have the same behaviour.

Exercise 1.19

Study the behaviour of the series $\displaystyle\sum_{n=0}^{+\infty} \frac{17}{6^n}$.

Proof:

$$\sum_{n=0}^{+\infty} \frac{17}{6^n} = \sum_{n=0}^{+\infty} 17\frac{1}{6^n} = \sum_{n=0}^{+\infty} 17\left(\frac{1}{6}\right)^n = 17\sum_{n=0}^{+\infty}\left(\frac{1}{6}\right)^n = 17\frac{1}{1-\frac{1}{6}} = 17 \cdot \frac{6}{5} = \frac{102}{5}$$

The application of Theorem 1.18 is justified because $\sum_{n=0}^{+\infty}\left(\frac{1}{6}\right)^n$ is a geometric series of common ratio $\frac{1}{6}$, that was proven to be convergent in Exercise 1.14. □

Lemma 1.20

If the terms of a series are all strictly positive, such series is either convergent or divergent to $+\infty$.

Theorem 1.21 — Comparison test

Let $\sum_{n=i}^{+\infty} a_n$ and $\sum_{n=i}^{+\infty} b_n$ be two series. Suppose that, for any $n \in \mathbb{N}$, $0 \leq a_n \leq b_n$. Then:

1. If $\sum_{n=i}^{+\infty} b_n$ is convergent, $\sum_{n=i}^{+\infty} a_n$ is also convergent;
2. If $\sum_{n=i}^{+\infty} a_n$ is divergent, $\sum_{n=i}^{+\infty} b_n$ is also divergent.

Exercise 1.22

Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{+\infty} \frac{n+3}{n^3+25}$ are convergent.

Proof:

- The ratio $\frac{1}{n^2}$ is always strictly positive for any $n \in \mathbb{N}$ with $n > 0$. Also, for any strictly positive $n \in \mathbb{N}$, $\frac{1}{n^2} \leq \frac{2}{n^2+n}$. But $\frac{2}{n^2+n} = 2\left(\frac{1}{n(n+1)}\right)$, and the series $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ is known to converge, as shown in Exercise 1.15. Applying Theorem 1.18, the series $\sum_{n=1}^{+\infty} \frac{2}{n^2+n}$ converges and in turn, since $0 \leq \frac{1}{n^2} \leq \frac{2}{n^2+n}$ for any strictly positive integer n, applying Theorem 1.21 gives that $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges.
- The ratio $\frac{n+3}{n^3+25}$ is always strictly positive for any strictly positive interger n. It also less than $\frac{4}{n^2}$ for any strictly positive integer n. In the previous point it was shown that $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges, and therefore applying Theorem 1.18 it can be shown that $\sum_{n=1}^{+\infty} \frac{4}{n^2}$ also converges. Therefore, since $0 \leq \frac{n+3}{n^3+25} \leq \frac{4}{n^2}$ for any $n \in \mathbb{N}$ strictly positive, applying Theorem 1.21 gives that $\sum_{n=1}^{+\infty} \frac{n+3}{n^3+25}$ converges.

□

Theorem 1.23 **Limit test**

Let $\sum_{n=i}^{+\infty} a_n$ and $\sum_{n=i}^{+\infty} b_n$ be two series. Suppose that $a_n \geq 0$ and $b_n > 0$ for any $n \in \mathbb{N}$, and also suppose that:

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = L$$

- If $L \neq 0$ and $L \neq +\infty$, the two series have the same behaviour;
- If $L = 0$ and $\sum_{n=i}^{+\infty} b_n$ is convergent, then $\sum_{n=i}^{+\infty} a_n$ is also convergent;
- If $L = +\infty$ and $\sum_{n=i}^{+\infty} b_n$ is divergent, then $\sum_{n=i}^{+\infty} a_n$ is also divergent.

Exercise 1.24

Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{n+7}{n^3-8n}$ is convergent.

Proof: Consider the series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$. Since both this series and $\sum_{n=1}^{+\infty} \frac{n+7}{n^3-8n}$ have only positive terms, it is possible to apply Theorem 1.23:

$$\lim_{n\to+\infty} \frac{\frac{n+7}{n^3-8n}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n+7}{\cancel{n}(n^2-8)} \cdot n^{\cancel{2}} = \lim_{n\to+\infty} \frac{n^2+7n}{n^2-8} = \lim_{n\to+\infty} \frac{\cancel{n^2}\left(1+\frac{7}{n}\right)}{\cancel{n^2}\left(1-\frac{8}{n^2}\right)} = \frac{1+0}{1-0} = 1$$

Since $1 \neq 0$ and $1 \neq +\infty$ and since $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is known to be convergent (see Exercise 1.22), $\sum_{n=1}^{+\infty} \frac{n+7}{n^3-8n}$ is also convergent. □

Theorem 1.25 **Cauchy condensation test**

Let $\{a_n\}$ be a decreasing sequence of positive numbers. The series $\sum_{n=1}^{+\infty} a_n$ is convergent if and only if the series $\sum_{n=1}^{+\infty} 2^n a_{2^n}$ is convergent.

A series is said to be **absolutely convergent** if the sum of the absolute values of the summands is finite. More precisely, a series $\sum_{n=0}^{+\infty} a_n$ is said to be absolutely convergent if $\sum_{n=0}^{+\infty} |a_n|$ is convergent.

Theorem 1.26 **Absolute convergence test**

If a series is absolutely convergent, it is also convergent.

The opposite of Theorem 1.26 is not necessarely true, since a series can be convergent but not absolutely convergent.

Theorem 1.27 **Ratio test**

Let $\sum_{n=1}^{+\infty} a_n$ be a series. Suppose that:

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = L, \;\; a_n \neq 0$$

Then:

- If $L < 1$, the series is (absolutely) convergent;
- If $L > 1$, the series is divergent;
- If $L = 1$, the test is inconclusive.

Exercise 1.28

Prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{5^n} = \frac{1}{5}$

Proof: Since $\frac{n}{5^n}$ is always strictly positive, Theorem 1.27 can be applied:

$$\lim_{n\to+\infty} \frac{\left|\frac{n+1}{5^{n+1}}\right|}{\left|\frac{n}{5^n}\right|} = \lim_{n\to+\infty} \frac{n+1}{5\cdot\cancel{5^n}}\cdot\frac{\cancel{5^n}}{n} = \lim_{n\to+\infty} \frac{1}{5}\left(\frac{n+1}{n}\right) = \frac{1}{5}\lim_{n\to+\infty} 1+\frac{1}{n} = \frac{1}{5}(1+0) = \frac{1}{5}$$

□

Theorem 1.29 **Root test**

Let $\sum_{n=1}^{+\infty} a_n$ be a series. Suppose that:

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L$$

Then:

- If $L < 1$, the series is (absolutely) convergent;
- If $L > 1$, the series is divergent;
- If $L = 1$, the test is inconclusive.

Exercise 1.30

Prove that $\displaystyle\sum_{n=1}^{+\infty} \left(\frac{1}{n!}\right)^n = 0$

Proof: Applying Theorem 1.29:

$$\lim_{n\to+\infty} \sqrt[n]{\left|\left(\frac{1}{n!}\right)^n\right|} = \lim_{n\to+\infty} \left(\left(\frac{1}{n!}\right)^{\cancel{n}}\right)^{\cancel{\frac{1}{n}}} = \lim_{n\to+\infty} \frac{1}{n!} = 0$$

□

Theorem 1.31 **Leibniz's test**

Consider a series of the form $\displaystyle\sum_{n=0}^{+\infty} (-1)^n a_n$. Suppose that:

1. a_n is always non negative;
2. $a_{n+1} \leq a_n$ for any $n \in \mathbb{N}$;
3. $\displaystyle\lim_{n\to+\infty} a_n = 0$.

Then, the series is convergent.

Exercise 1.32

Prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is convergent.

Proof: Rewriting the series as $\displaystyle\sum_{n=1}^{+\infty} (-1)^n \left(\frac{1}{n}\right)$ and applying Theorem 1.31:

1. $\frac{1}{n}$ is always non negative;
2. $\frac{1}{n+1} \leq \frac{1}{n}$ for any $n \in \mathbb{N}$;
3. As shown in Exercise 1.3, $\displaystyle\lim_{n\to+\infty} \frac{1}{n} = 0$.

Therefore, the series is convergent. □

Theorem 1.33 **Integral test**

Let f be a function that is continuous, positive and decreasing in the interval $[N, +\infty)$. Then, the series on the left and the integral on the right have the same behaviour:

$$\sum_{n=N}^{+\infty} f(n) \qquad\qquad \int_{N}^{+\infty} f(x)dx$$

2 Linear Algebra

2.1 Matrices

2.1.1 Definition

A **matrix** is a bidimensional mathematical object, represented as follows:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \qquad A = \left(a_{i,j}\right) \; i \in \{1, ..., m\}, j \in \{1, \; n\}$$

The real numbers $a_{i,j}$ are called the **entries** or the **elements** of the matrix, while the integers i and j are the **indexes**. The numbers $a_{i,1}, a_{i,2}, ..., a_{i,n}$ with $i \in \{1, ..., m\}$ costitute a **row** of the matrix, while the numbers $a_{1,j}, a_{2,j}, ..., a_{m,j}$ with $j \in \{1, ..., n\}$ constitute a **column**. The number of rows and columns of a matrix is called its **order** or **dimension**, and is denoted as $m \times n$. Two matrices are **equal** if they have the same order and if $\left(a_{i,j}\right) = \left(b_{i,j}\right) \forall i, j$.

A matrix that has $m = n$ is a **square matrix of order** n (or order m). The elements $a_{1,1}, a_{2,2}, ..., a_{n,n}$ are called the **diagonal elements** and constitute the **diagonal** of the matrix.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \qquad A = \left(a_{i,j}\right) \; i \in \{1, ..., n\}, j \in \{1, \; n\}$$

A matrix is called a **diagonal matrix** if $\left(a_{i,j}\right) = 0$ with $i \neq j$. In other words, a matrix is diagonal if all of its non-diagonal elements are 0.

$$A = \begin{pmatrix} a_{1,1} & 0 & \dots & 0 \\ 0 & a_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n} \end{pmatrix}$$

A peculiar diagonal matrix is the **identity matrix**, denoted as I_n (where n is its order), whose diagonal elements are all 1. Another peculiar diagonal matrix is the **null matrix**, whose diagonal elements are all 0.

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \qquad O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

A matrix constituted by a single row and n columns (that is, a $1 \times n$ matrix) is also called a **row matrix**. On the other hand, a matrix constituted by a single column and n rows (that is, a $n \times 1$ matrix) is also called a **column matrix**.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \end{pmatrix} \qquad A = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix}$$

2.1.2 Operations

The **sum** between two matrices A and B having the same dimension $m \times n$ is defined as the sum, entry by entry, of A and B. Two matrices that have the same dimension are said to be *sum-conformant*. The sum between two matrices that are not sum-conformant is undefined.

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} \qquad B = \begin{pmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{m,1} & \dots & b_{m,n} \end{pmatrix} \qquad A + B = \begin{pmatrix} a_{1,1} + b_{1,1} & \dots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & \dots & a_{2,n} + b_{2,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & \dots & a_{m,n} + b_{m,n} \end{pmatrix}$$

Given a matrix A, the **opposite matrix** of A is the matrix $-A$ such that $A + (-A) = (0_{i,j})$.

The sum between matrices possesses the following properties:

- It is commutative;
- Obeys the cancellation law, meaning that $A + B = B + C$ can be simplified as $A = C$;
- It is associative;
- The null matrix is the identity element for matrix sum, since $A + O = O + A = A$ for any matrix A.

Given a $m \times p$ matrix A and a $p \times n$ matrix B, the **product** between two matrices (also called **row-by-column product**) is the $m \times n$ matrix $C = AB$ given by:

$$C = AB = (c_{i,j}) = \begin{pmatrix} \sum_{i=1}^{p} a_{1,p} \cdot b_{p,1} & \sum_{i=1}^{p} a_{1,p} \cdot b_{p,2} & \dots & \sum_{i=1}^{p} a_{1,p} \cdot b_{p,n} \\ \sum_{i=1}^{p} a_{2,p} \cdot b_{p,1} & \sum_{i=1}^{p} a_{2,p} \cdot b_{p,2} & \dots & \sum_{i=1}^{p} a_{2,p} \cdot b_{p,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{p} a_{m,p} \cdot b_{p,1} & \sum_{i=1}^{p} a_{m,p} \cdot b_{p,2} & \dots & \sum_{i=1}^{p} a_{m,p} \cdot b_{p,n} \end{pmatrix}$$

That is, the i, j-th entry of AB is given by the sum between the products corresponding entries of the i-th row of A and the j-th column of B. Two matrices that possess this property are called *product-conformant*. The product between two matrices that are not product-conformant is undefined.

Exercise 2.1

Compute the product of the following matrices:

$$A = \begin{pmatrix} -1 & 4 \\ 6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 & -4 \\ 5 & 0 & 2 \end{pmatrix}$$

Proof:

$$AB = \begin{pmatrix} -1 & 4 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & -4 \\ 5 & 0 & 2 \end{pmatrix} = \begin{pmatrix} (-1) \cdot 3 + 4 \cdot 5 & (-1) \cdot 2 + 4 \cdot 0 & (-1) \cdot (-4) + 4 \cdot 2 \\ 6 \cdot 3 + 1 \cdot 5 & 6 \cdot 2 + 1 \cdot 0 & 6 \cdot (-4) + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 17 & -2 & 12 \\ 23 & 12 & -22 \end{pmatrix}$$

□

The product between matrices possesses the following properties:

- It is not commutative, therefore $AB \neq BA$;
- Cancellation law does not hold. If $AB = BC$ holds, it does not necessarely holds that $A = C$;
- The product of two non null matrices can result in the null matrix. In other words, if $AB = 0$, neither A or B have to be the null matrix;
- The null matrix is the absorbing element for matrix multiplication. Multiplying a non null matrix by the null matrix results in the null matrix;
- It is associative;
- It is distributive with respect to the sum;
- The identity matrix is the identity element for matrix multiplication, since $AI = IA = A$ for any matrix A.

Given a matrix A, the matrix A^T that has the rows of A as its columns and the columns of A as its rows is called the **transposed** of A. If a matrix is equal to its transposed, it is said to be **symmetric**.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} \\ a_{1,2} & a_{2,2} & \dots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \dots & a_{n,n} \end{pmatrix}$$

Proposition 2.2

Let A and B be two product-conformant matrices. Then $(AB)^T = B^T A^T$.

The **product between a matrix and a scalar** is an operation that has as input a matrix A and a scalar k and has as output a matrix kA that has each entry multiplied by k.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \qquad kA = \begin{pmatrix} ka_{1,1} & ka_{1,2} & \dots & ka_{1,n} \\ ka_{2,1} & ka_{2,2} & \dots & ka_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m,1} & ka_{m,2} & \dots & ka_{m,n} \end{pmatrix}$$

The product between a matrix and a scalar possesses the following properties:

- It is associative;
- It is commutative with respect to the matrix multiplication;
- It is distributive with respect to the matrix sum;
- It is distributive with respect to the matrix transposition;

2.2 Vector Spaces

2.2.1 Definition of vector space

Let V be a set, whose elements are called **vectors**. A vector $\underline{v}$ is denoted as $\underline{v} = (v_1, v_2, ..., v_n)$, where each v_i with $1 \leq i \leq n$ is called the i-th **component** of $\underline{v}$.

Let $+$ be an operation on such set, a *sum* of vectors, that has two vectors as arguments and returns another vector. That is, foreach $\left(\underline{x}, \underline{y}\right) \in V \times V$ there exists a vector $\underline{v} \in V$ such that $\underline{x} + \underline{y} = \underline{v}$.

Let $\cdot$ be another operation, a *product* between a vector and a real number, that has a real number and a vector as argument and returns another vector. That is, foreach $\lambda \in \mathbb{R}$ and $\underline{v} \in V$ there exists a vector $\underline{w} \in V$ such that $\lambda \cdot \underline{v} = \underline{w}$.

Suppose those operations possess the following properties:

- $(V, +)$ is an Abelian group;
- The product has the distributive property, such that for every $\lambda \in \mathbb{R}$ and for every $\underline{x}, \underline{y} \in V$ it is true that $\lambda \cdot \left(\underline{x} + \underline{y}\right) = \lambda \cdot \underline{x} + \lambda \cdot \underline{y}$;
- The product has the associative property, such that for every $\lambda, \mu \in \mathbb{R}$ and for every $\underline{x} \in V$ it is true that $(\lambda + \mu) \cdot \underline{x} = \lambda \cdot \underline{x} + \mu \cdot \underline{x}$;
- For every vector $\underline{v} \in V$, it is true that $1 \cdot \underline{v} = \underline{v}$.

If that is the case, the set V is called **vector space**. It should be noted that it does not matter what the elements of a vector space actually are (be they numbers, functions, polynomals, etcetera); as long as the aforementioned properties hold for the two operations, such set shares all of the properties that a vector space possesses.

Given a vector space V, a set W is said to be a **subspace** of V if it's a subset of V and it's itself a vector space with respect to the same operations defined for V.

For the sake of readability, the product between a real number and a vector is often represented without the dot. That is to say, the expressions $\lambda \cdot \underline{x}$ and $\lambda \underline{x}$ have the same meaning.

Exercise 2.3

Denote as $\mathbb{R}^n$ the set containing all vectors of real components[1] in the n-dimensional plane. Prove that $\mathbb{R}^n$ constitutes a vector space.

Proof: It is possible to define both a sum between two vectors in the n-dimensional plane and a product between a vector in the n-dimensional space and a real number. To sum two vectors in the n-dimensional space, it suffices

[1]This is a misnomer.

to sum each component with each component. To multiply a vector in the n-dimensional space with a real number it suffices to multiply each component by that number:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} \qquad \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

Both operations obey the properties stated:

- $(\mathbb{R}^n, +)$ constitutes an Abelian group. Infact:
  - The sum has the associative property:

$$\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 + u_1 \\ v_2 + w_2 + u_2 \\ \vdots \\ v_n + w_n + u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 + u_1 \\ w_2 + u_2 \\ \vdots \\ w_n + u_n \end{pmatrix}$$

  - There exists an identity element, in the form of the vector whose components are all zero:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 + 0 \\ v_2 + 0 \\ \vdots \\ v_n + 0 \end{pmatrix} = \begin{pmatrix} 0 + v_1 \\ 0 + v_2 \\ \vdots \\ 0 + v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

  - Each vector in the n-dimensional space has an inverse element with respect to the sum, that is the same vector multiplied by -1:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + (-1) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix} = \begin{pmatrix} v_1 + (-v_1) \\ v_2 + (-v_2) \\ \vdots \\ v_n + (-v_n) \end{pmatrix} = \begin{pmatrix} -v_1 + v_1 \\ -v_2 + v_2 \\ \vdots \\ -v_n + v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

  - The sum has the commutative property:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \\ \vdots \\ w_n + v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- The product has the associative property:

$$(\lambda + \mu) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\lambda + \mu) v_1 \\ (\lambda + \mu) v_2 \\ \vdots \\ (\lambda + \mu) v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 + \mu v_1 \\ \lambda v_2 + \mu v_2 \\ \vdots \\ \lambda v_n + \mu v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix} + \begin{pmatrix} \mu v_1 \\ \mu v_2 \\ \vdots \\ \mu v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- The product has the distributive property:

$$\lambda \left(\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \right) = \lambda \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} \lambda (v_1 + w_1) \\ \lambda (v_2 + w_2) \\ \vdots \\ \lambda (v_n + w_n) \end{pmatrix} = \begin{pmatrix} \lambda v_1 + \lambda w_1 \\ \lambda v_2 + \lambda w_2 \\ \vdots \\ \lambda v_n + \lambda w_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \lambda \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

- Multiplying a vector by the number 1 leaves the vector unchanged:

$$1 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 v_1 \\ 1 v_2 \\ \vdots \\ 1 v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

□

Exercise 2.4

Denote as $\mathbb{P}_n$ the set containing all polynomials with real coefficients and degree less than or equal to n. Prove that $\mathbb{P}_n$ constitutes a vector space.

Proof: It is possible to define both a sum between two polynomials with real coefficients and degree $\leq n$ and a product between a polynomial with real coefficients and degree $\leq n$ and a real number. To sum two such polynomials it suffices to sum the coefficients of their monomials having the same degree:

$$(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) + (b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0) =$$
$$a_n x^n + a_{n-1} x^{n-1} + \ldots + (a_m + b_m) x^m + (a_{m-1} + b_{m-1}) x^{m-1} + \ldots + (a_1 + b_1) x + (a_0 + b_0)$$

To multiply a polynomial with real coefficients and degree $\leq n$ with a real number it suffices to multiply each coefficient of its monomials by such number:

$$\lambda (a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) = (\lambda a_n) x^n + (\lambda a_{n-1}) x^{n-1} + \ldots + (\lambda a_1) x + (\lambda a_0)$$

Both operations satisfy the properties required.

□

Proposition 2.5

Let V be a vector space. To prove that a set W is a subspace of V it suffices to prove that it is a subset of V and is algebraically closed with respect to the same operations defined for V.

Exercise 2.6

Consider the vector space $\mathbb{R}^3$. Prove that the set W_1 is a subspace of $\mathbb{R}^3$ while W_2 is not.

$$W_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3, x_1 + x_2 = 0 \right\} \qquad W_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3, x_2 = 2x_3 + 1 \right\}$$

Proof: The first set is a subspace of $\mathbb{R}^3$ because it is a subset of V and is algebraically closed:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ -y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ -x_1 - y_1 \\ x_3 + y_3 \end{pmatrix} \Rightarrow x_2 + y_2 = -x_1 - y_1 \Rightarrow x_2 + y_2 + (x_1 + y_1) = 0$$

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ -x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ -\lambda x_1 \\ \lambda x_3 \end{pmatrix} \Rightarrow \lambda x_2 = -\lambda x_1 \Rightarrow \lambda (x_1 + x_2) = 0$$

The second one, on the other hand, is not:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_3 + 1 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ 2y_3 + 1 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 2x_3 + 2y_3 + 2 \\ x_3 + y_3 \end{pmatrix} \Rightarrow 2x_3 + 2y_3 + 2 \neq 2(x_3 + y_3) + 1$$

□

Lemma 2.7

Let V be a vector space. The sets $\{\underline{0}\}$ and V are always subspaces of V.

2.3 Linear dependence and independence

2.3.1 Linear combinations, linear dependence and independence

Let $\{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ be a set of n vectors of a vector space V, and let $\lambda_1, \lambda_2, ..., \lambda_n$ be n real numbers (not necessarely distinct). Every summation defined as such:

$$\sum_{i=1}^{n} \lambda_i \underline{v_i} = \lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + ... + \lambda_i \underline{v_i} + ... + \lambda_n \underline{v_n}$$

Is called a **linear combination** of the vectors $\{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$, with $\lambda_1, \lambda_2, ..., \lambda_n$ as **coefficients**.

A set of vectors $\{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ is said to be **linearly independent** if the only linear combination of such vectors that equals the null vector is the one that has 0 as every coefficient. If there exists a linear combination of such vectors that is equal to the null vector and that has at least a non-zero coefficient, those vectors are said to be **linearly independent**.

> **Lemma 2.8**
>
> Let $\{\underline{v_1}, \underline{v_2}, ..., \underline{v_n}\}$ be a set of n vectors of a vector space V. If those vectors are linearly dependent, there exists at least one vector of such set that can be expressed as a linear combination of the remaining vectors, and vice versa.

Proof: If such set of vectors is linearly dependent, there must exist a linear combination of the set that equals the null vector, be it:

$$\sum_{i=1}^{n} \lambda_i \underline{v_i} = \lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + ... + \lambda_j \underline{v_j} + ... + \lambda_n \underline{v_n} = \underline{0}$$

Where there's at least a non-zero coefficient. Let's assume, without loss of generality, that the j-th coefficient is non-zero (whether such coefficient is the only one to be non-zero is irrelevant). The product of such coefficient with the corresponding vector can be moved to the right side of the equation like so:

$$\lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + ... + \lambda_n \underline{v_n} = -\lambda_j \underline{v_j}$$

Dividing both sides by $-\lambda_j$ gives:

$$-\frac{\lambda_1}{\lambda_j} \underline{v_1} - \frac{\lambda_2}{\lambda_j} \underline{v_2} - ... - \frac{\lambda_n}{\lambda_j} \underline{v_n} = \underline{v_j}$$

Each $-\frac{\lambda_i}{\lambda_j}$ is itself a real number, and therefore the expression above is a linear combination that is equal to a vector of the set.

On the other hand, assume that the j-th vector of the set is equal to a linear combination of the remaining vectors like so:

$$\lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + ... + \lambda_n \underline{v_n} = \underline{v_j}$$

Moving $\underline{v_j}$ to the left gives:

$$\lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + ... + (-1) \underline{v_j} + ... + \lambda_n \underline{v_n} = \underline{0}$$

Since -1 is a real number, the expression on the left side of the equation is indeed a linear combination of the whole set, that is equal to the null vector. □

> **Exercise 2.9**
>
> Consider the vector space $\mathbb{R}^2$. Check if the vectors $\underline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent or linearly dependent.

Proof: Consider such linear combination:

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To test whether $\underline{v_1}$ and $\underline{v_2}$ are linearly dependent or independent, it suffices to solve this system of equations:

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2\lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2\lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

Since the only solution to such system is $\lambda_1 = 0, \lambda_2 = 0$, $\underline{v_1}$ and $\underline{v_2}$ are linearly independent. □

Exercise 2.10

Consider the vector space $\mathbb{R}^2$. Check if the vectors $\underline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v_3} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ are linearly independent or linearly dependent.

Proof: Consider such linear combination:

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To test whether $\underline{v_1}, \underline{v_2}$ and $\underline{v_3}$ are linearly dependent or independent, it suffices to solve this system of equations:

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 + \lambda_2 + 2\lambda_3 = 0 \\ \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -2\lambda_3 \\ \lambda_2 = 0 \end{cases}$$

Since there are infinite solutions to such system, including non-zero solutions, $\underline{v_1}$, $\underline{v_2}$ and $\underline{v_3}$ are linearly dependent. For example, setting $\lambda_1 = 1$ results in:

$$1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which is, indeed, a correct identity. □

2.3.2 Span, bases, dimensions

A set of vectors $S = \left\{\underline{s_1}, ..., \underline{s_n}\right\}$ of a vector space V is said to **generate** V if every vector of V can be written as a linear combination of the vectors in S. That is to say, S generates V if for every $\underline{v} \in V$ there exist a set of coefficients $\lambda_1, ..., \lambda_n$ such that:

$$\underline{v} = \sum_{i=1}^{n} \lambda_i \underline{s_i} = \lambda_1 \underline{s_1} + ... + \lambda_n \underline{s_n}$$

The set that contains all possible linear combinations from vectors of a set $S = \left\{\underline{s_1}, ..., \underline{s_n}\right\}$ is also referred to as the **span** of S, denoted as $\text{span}\{S\}$. Along this line of thought, if a set V is generated by S, it also said to be *spanned* by S.

$$\text{span}\{S\} = \left\{ \underline{v} \,\middle|\, \underline{v} = \sum_{i=1}^{n} \lambda_i \underline{s_i} \right\}$$

A set of vectors that can generate a vector space and is itself linearly independent is called a **basis** for such vector space.

The cardinality of a basis is called the **dimension** of the corresponding vector space. If a vector space contains just the null vector, such vector space is said to have dimension 0.

A vector space whose dimension is a finite number is said to be a **finite-dimensional** vector space. Otherwise, it is said to be an **infinite-dimensional** vector space.

Theorem 2.11 **Steinitz exchange lemma**

Let U and W be finite subsets of a vector space V. If U is a set of linearly independent vectors and V is spanned by W, then:

1. $|U| \leq |W|$;
2. It's always possible to construct a (potentially empty) set $W' \subseteq W$ with $|W'| = |W| - |U|$ such that V is spanned by $U \cup W'$.

Theorem 2.12 **Dimension theorem for vector spaces**

Given a vector space V, any two bases of V have the same cardinality. This cardinality is the smallest number of linearly independent vectors needed to generate V.

Proof:

If S can generate V and $\underline{w}$ belongs to V, there exists a linear combination of the vectors in S such that:

$$\underline{w} = \sum_{i=1}^{n} \lambda_i \underline{s_i} = \lambda_1 \underline{s_1} + \dots + \lambda_n \underline{s_n}$$

Moving $\underline{w}$ to the right side of the equation gives:

$$\underline{0} = (-1)\underline{w} + \lambda_1 \underline{s_1} + \dots + \lambda_n \underline{s_n}$$

The expression on the right side of the equation is indeed a linear combination of $S \cup \{\underline{w}\}$, that is equal to the null vector. Since at least -1 is a non zero coefficient, such set is linearly dependent. □

Exercise 2.13

Consider the vector space $\mathbb{P}_2$. Knowing that the sets $\mathcal{B}_1 = \{1, x, x^2\}$ and $\mathcal{B}_2 = \{(x+1), (x-1), x^2\}$ are both bases for $\mathbb{P}_2$, write the polynomial $p(x) = 3x^2 + 2x - 5$ as a linear combination of each.

Proof: It's trivial to see that, for the first basis, such linear combination is $p(x)$ itself:

$$\lambda_0 1 + \lambda_1 x + \lambda_2 x^2 = -5 + 2x + 3x^2 \Rightarrow \begin{cases} \lambda_0 = -5 \\ \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases}$$

Regarding the second basis, it can be rewritten as:

$$\lambda_0(x+1) + \lambda_1(x-1) + \lambda_2 x^2 \Rightarrow \lambda_0 x + \lambda_0 + \lambda_1 x - \lambda_1 + \lambda_2 x^2 \Rightarrow (\lambda_0 - \lambda_1) + (\lambda_0 + \lambda_1)x + \lambda_2 x^2$$

Equating it term by term:

$$(\lambda_0 - \lambda_1) + (\lambda_0 + \lambda_1)x + \lambda_2 x^2 = -5 + 2x + 3x^2 \Rightarrow \begin{cases} \lambda_0 - \lambda_1 = -5 \\ \lambda_0 + \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases} \Rightarrow \begin{cases} \lambda_0 = -\frac{3}{2} \\ \lambda_1 = \frac{7}{2} \\ \lambda_2 = 3 \end{cases}$$

Therefore:

$$3x^2 + 2x - 5 = -\frac{3}{2}(x+1) + \frac{7}{2}(x-1) + 3x^2$$

□

The basis of a vector space that renders calculations the most "comfortable" is called the **canonical basis** for such vector space. Such basis is different from vector space to vector space.

Exercise 2.14

Determine the dimension of $\mathbb{R}^n$

Proof: Consider any n-dimensional vector of coordinates $a_1, a_2, \dots, a_n$. It's easy to see that such vector is equal to the following linear combination:

$$\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix} + \dots + a_n \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}$$

A set containing such vectors in linearly independent. Infact:

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 + 0 + \dots + 0 = 0 \\ 0 + \lambda_2 + \dots + 0 = 0 \\ \vdots \\ 0 + 0 + \dots + \lambda_n = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \vdots \\ \lambda_n = 0 \end{cases}$$

This set of vectors is linearly independent and can generate $\mathbb{R}^n$, therefore it's a basis for $\mathbb{R}^n$. The dimension of $\mathbb{R}^n$ is then n, since such set has cardinality n. In particular, this specific basis is the canonical basis for $\mathbb{R}^n$. □

2.3.3 Vector representation

Bases can be employed to represent vectors, no matter their nature (polynomials, tuples, matrices, ecc...), in a unique and standardised form.

Proposition 2.15

Let V be a vector space of dimension n, and let $B = \{\underline{b_1}, \dots, \underline{b_n}\}$ be a basis. Given a generic vector $\underline{v}$, let $\sum_{i=1}^{n} \lambda_i \underline{b_i} = \underline{v}$ be a linear combination employing B that is equal to $\underline{v}$, with coefficients $\lambda_1, \dots, \lambda_n$. These coefficients are unique.

Proof: Suppose that this is not the case, and that exists instead a set of coefficients $\lambda'_1, \dots, \lambda'_n$ such that $\sum_{i=1}^{n} \lambda'_i \underline{b_i} = \underline{v}$. This means that:

$$\sum_{i=1}^{n} \lambda_{\prime(i)} \underline{b_i} = \sum_{i=1}^{n} \lambda_i \underline{b_i} \Rightarrow \sum_{i=1}^{n} \lambda_{\prime(i)} \underline{b_i} - \sum_{i=1}^{n} \lambda_i \underline{b_i} = 0 \Rightarrow \sum_{i=1}^{n} \left(\lambda_{\prime(i)} - \lambda_i\right) \underline{b_i} = 0$$

Since the vectors $\underline{b_i}$ are linearly independent, the only way for this linear combination to be meaningful is to have $\left(\lambda_{\prime(i)} - \lambda_i\right) = 0$ for all $i \in \{1, \dots, n\}$. However, this means that $\lambda_{\prime(i)} = \lambda_i$ for all $i \in \{1, \dots, n\}$, which is a contradiction. □

Proposition 2.15 implies that a vector in a vector space can be uniquely identified, once a certain basis is fixed, by the coordinates of the linear combination employing said basis used to construct it.

Let V be a vector space and let $\mathcal{B} = \{\underline{\mathscr{b}_1}, \dots, \underline{\mathscr{b}_n}\}$ be a basis for V. Any vector $\underline{v}$ has two, equivalent and unique, representations with respect to $\mathcal{B}$:

$$\sum_{i=1}^{n} \lambda_i \underline{\mathscr{b}_i} = \lambda_1 \underline{\mathscr{b}_1} + \lambda_2 \underline{\mathscr{b}_2} + \dots + \lambda_n \underline{\mathscr{b}_n} \qquad \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}_{\mathcal{B}}$$

In particular, the right representation is also referred to as its **column vector representation**, or just **vector representation**, with respect to $\mathcal{B}$. Technically, the suffix $\mathcal{B}$ is necessary, otherwise it would be impossible to tell apart representations with respect to different bases. However, in general, the basis employed is either deducible from context or is not relevant, therefore in practice the suffix is often omitted.

Clearly, employing different bases to represent the same vector will give different vector representations. Different representations of the same vector can be converted into another through a simple matrix multiplication.

Proposition 2.16

Let V be a vector space, and let $\mathcal{B} = \left\{\underline{b_1}, \underline{b_2}, ..., \underline{b_n}\right\}$ and $\mathcal{B}' = \left\{\underline{b_1'}, \underline{b_2'}, ..., \underline{b_n'}\right\}$ be two bases of V. Any generic vector $\underline{x} \in V$, can be represented with respect to both bases:

$$\underline{x} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}} \qquad \underline{x} \Leftrightarrow \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}_{\mathcal{B}'}$$

There exists an invertible matrix P, independent of $\underline{x}$, such that:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} \qquad \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Proof: Being both bases constituted by vectors of the same vector space, it is possible to express the elements of $\mathcal{B}'$ as linear combinations of the elements of $\mathcal{B}$:

$$\begin{cases} \underline{b_1'} = p_{1,1}\underline{b_1} + p_{1,2}\underline{b_2} + ... + p_{1,n}\underline{b_n} = \sum_{j=1}^{n} p_{1,j}\underline{b_j} \\ \underline{b_2'} = p_{2,1}\underline{b_1} + p_{2,2}\underline{b_2} + ... + p_{2,n}\underline{b_n} = \sum_{j=1}^{n} p_{2,j}\underline{b_j} \\ \vdots \\ \underline{b_n'} = p_{n,1}\underline{b_1} + p_{n,2}\underline{b_2} + ... + p_{n,n}\underline{b_n} = \sum_{j=1}^{n} p_{n,j}\underline{b_j} \end{cases}$$

Therefore:

$$\underline{x} = \sum_{j=1}^{n} x_j'\underline{b_j'} = \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} p_{j,i}\underline{b_i} = \sum_{i=1}^{n} x_i\underline{b_i}$$

By comparing the third and fourth members of the equality term by term:

$$x_i = \sum_{j=1}^{n} p_{i,j}x_j' \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = P \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$$

Of course, it is also possible to go the other way around, expressing the elements of $\mathcal{B}$ as linear combinations of the elements of $\mathcal{B}'$:

$$\begin{cases} \underline{b_1} = q_{1,1}\underline{b_1'} + q_{1,2}\underline{b_2'} + ... + q_{1,n}\underline{b_n'} = \sum_{j=1}^{n} q_{1,j}\underline{b_j'} \\ \underline{b_2} = q_{2,1}\underline{b_1'} + q_{2,2}\underline{b_2'} + ... + q_{2,n}\underline{b_n'} = \sum_{j=1}^{n} q_{2,j}\underline{b_j'} \\ \vdots \\ \underline{b_n} = q_{n,1}\underline{b_1'} + q_{n,2}\underline{b_2'} + ... + q_{n,n}\underline{b_n'} = \sum_{j=1}^{n} q_{n,j}\underline{b_j'} \end{cases}$$

Therefore:

$$\underline{x} = \sum_{j=1}^{n} x_j\underline{b_j} = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} q_{j,i}\underline{b_i'} = \sum_{i=1}^{n} x_i'\underline{b_i'}$$

By comparing the third and fourth members of the equality term by term:

$$x_i' = \sum_{j=1}^{n} q_{i,j}x_j \Rightarrow \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & \cdots & q_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = Q \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Substituting the one in the expression of the other gives:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = PQ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = Q \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = QP \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

Since the two matrices on the edges of the equalities are the same, for these equalities to hold both matrix products PQ and QP must be equal to the identity matrix. In other words, P and Q are the inverse of each other, therefore:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} \qquad \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

□

2.4 Determinant and rank

2.4.1 Determinant

The **determinant** is a function that associates a number to a square matrix. Given a $n \times n$ matrix A, its determinant, denoted as $\det(A)$ or $| \ A \ |$, is defined recursively as follows:

$$\det(A) = \begin{cases} \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(M_{i,j}) & \text{if } n > 1 \\ a_{11} & \text{otherwise} \end{cases}$$

Where j is any column of the matrix A chosen at random and $M_{i,j}$ is the matrix obtained by removing the i-th row and j-th column from A. The formula can also be applied with respect to rows instead of columns.

When the matrix has dimension $n = 2$, the formula can actually be simplified as follows:

$$\det(A) = (a_{1,1} \cdot a_{2,2}) - (a_{2,1} \cdot a_{1,2})$$

A matrix whose determinant is equal to 0 is called a **singular matrix**.

Exercise 2.17

Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$, compute its determinant.

Proof: The fastest way to compute a determinant is to pick the row/column that has the most zeros, because the number of $\det(M_{i,j})$ to compute is the smallest. In the case of A, the best choices are: the second row, the first column, the third row and the third column. Suppose the first column is chosen:

$$\det(A) = \sum_{i=1}^{3} (-1)^{i+j} a_{i,j} \det(M_{i,j}) =$$

$$(-1)^{1+1} a_{1,1} \det(M_{1,1}) + (-1)^{2+1} a_{2,1} \det(M_{2,1}) + (-1)^{3+1} a_{3,1} \det(M_{3,1}) =$$

$$1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \cancel{- 0 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}} + 1 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = (1 \cdot 0 - 1 \cdot 2) + (2 \cdot 2 - 3 \cdot 1) = 0 - 2 + 4 - 3 = -1$$

□

Lemma 2.18

The determinant of a triangular matrix is equal to the product of the elements on its diagonal.

Proof: Consider an upper triangular matrix A and pick the first column to apply the formula:

$$\det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{n,n} \end{vmatrix} = \dots = \prod_{i=1}^{n} a_{i,i}$$

The same is achieved for a lower triangular matrix by picking the first row. □

Theorem 2.19 **Binet's Theorem**

The determinant is a multiplicative function. In other words, given two matrices A and B:

$$\det(AB) = \det(A)\det(B)$$

Proposition 2.20

The determinant is invariant with respect to transposition.

Lemma 2.21

Given a $n \times n$ matrix A and a scalar k, $\det(kA) = k^n \det(A)$.

2.4.2 Rank

Lemma 2.22

The rank of a matrix is invariant with respect to transposition. In other words, $\text{rank}(\text{A}) = \text{rank}(A^T)$.

Lemma 2.23

A matrix is invertible if and only if it has full rank.

Theorem 2.24

Let A be a square matrix. The following results are equivalent, meaning if one of these is true also the others are true:

- A is non singular;
- There exists an inverse of A;
- A is full rank;
- The rows/columns of A form a linearly independent set.

Gaussian moves are special operations that can be performed on matrices. Said operations are as follows:

- Swapping two rows/columns;
- Multiplying a row/column by a scalar;
- Summing a row/column to another row/column multiplied by a scalar.

Theorem 2.25

The application of Gaussian moves to a matrix does not change its rank.

Lemma 2.26

Let A be a square matrix, and let A' be the matrix resulting from applying the first Gaussian move to A. Then $\det(A) = -\det(A')$.

Theorem 2.27

The application of the third Gaussian move to a matrix does not change its determinant.

A matrix is in **row echelon form** if all rows having only zero entries are at the bottom and the left-most nonzero entry of every nonzero row, called the **pivot**, is on the right of the leading entry of every row above.

Exercise 2.28

Provide some examples of matrices in row echelon form.

Proof:

$$\begin{pmatrix} 0 & 4 & 1 & 5 & 2 \\ 0 & 0 & 6 & 1 & 9 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 9 & 0 & 1 \\ 0 & 0 & -4 & 2 \end{pmatrix}$$

□

Theorem 2.29

The rank of a matrix in row echelon form is equal to the number of its pivots.

Theorem 2.29 suggests another way to compute the rank of a matrix.

Exercise 2.30

Compute the rank of the matrix $A = \begin{pmatrix} 1 & -1 & 3 & 2 \\ 3 & 2 & 7 & 6 \\ 1 & 4 & 1 & 2 \end{pmatrix}$.

Proof: Theorem 2.25 guarantees that applying the third Gaussian move to A renders a matrix with the same rank. A can therefore be converted into a matrix in row echelon form as follows:

1. Substituting the third row with itself summed to the first multiplied by -1;
2. Substituting the second row with itself summed to the first multiplied by -3;
3. Substituting the second row with itself summed to the third multiplied by -1.

$$\begin{pmatrix} 1 & -1 & 3 & 2 \\ 3 & 2 & 7 & 6 \\ 1 & 4 & 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 3 & 2 \\ 3 & 2 & 7 & 6 \\ 0 & 5 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 5 & -2 & 0 \\ 0 & 5 & -2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As stated in Theorem 2.29, the rank of A is 2. □

2.4.3 Inverse matrix

Given a square matrix A, the matrix A^{-1} (if it exists) such that $AA^{-1} = A^{-1}A = I$ is called the **inverse matrix** of A. Such matrix is given by:

$$\left(a_{i,j}^{-1}\right) = \frac{(-1)^{i+j}\det\left(M_{j,i}\right)}{\det(A)}$$

Where $M_{j,i}$ is the matrix A with the j-th row and the i-th column removed.

Proposition 2.31

A matrix is invertible if and only if it's not singular.

Proof: If A is a singular matrix, its determinant is 0. Therefore, the expression for $\left(a_{i,j}^{-1}\right)$ would involve a division by 0, which is not admissible.

□

Proposition 2.32

Given an invertible matrix A, $\det(A^{-1}) = (\det(A))^{-1}$.

Proof: Applying Theorem 2.19 gives:

$$\det(AA^{-1}) = \det(A) \cdot \det(A^{-1}) = 1 \Rightarrow \det(A) = \frac{1}{\det(A^{-1})} \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

□

Exercise 2.33

Compute the inverse of the following matrix:

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$$

Proof: The determinant of A is $2 \cdot 0 - 1 \cdot 2 = -2$. Therefore, an inverse exists:

$$a_{1,1} = \frac{(-1)^{1+1}\det(M_{1,1})}{\det(A)} = \frac{0}{-2} = 0 \qquad a_{1,2} = \frac{(-1)^{1+2}\det(M_{2,1})}{\det(A)} = \frac{-2}{-2} = 1$$

$$a_{2,1} = \frac{(-1)^{2+1}\det(M_{1,2})}{\det(A)} = \frac{-1}{-2} = \frac{1}{2} \qquad a_{2,2} = \frac{(-1)^{2+2}\det(M_{2,2})}{\det(A)} = \frac{2}{-2} = -1$$

□

Lemma 2.34

A matrix is invertible if and only if the set of its rows/columns forms a linearly independent set.

2.4.4 Similar matrices

Two matrices A and B are said to be **similar** if there exist an invertible matrix P such that $A = PBP^{-1}$.

Proposition 2.35

If two matrices are similar, they have the same determinant.

Proof: Suppose A and B are similar. Then:

$$\det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1}) = \cancel{\det(P)}\det(B)\frac{1}{\cancel{\det(P)}} = \det(B)$$

□

2.4.5 Trace

The **trace** of a square matrix is defined as the sum of the elements on its diagonal:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{i,i}$$

If the inverse of a matrix is equal to his transposition, said matrix is called **orthogonal**.

2.5 Systems of Linear Equations

Any equation constituted by real numbers in the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Is called a **linear equation**. Each x_i with $i = 1, 2, ..., n$ is called a **variable** or an **unknown**. If the number of unknowns is small, variables are denoted with $x, y, z, ...$. The b term and ach a_i with $i = 1, 2, ..., n$ is instead called a **coefficient**.

Any ordered n-ple of real numbers $(k_1, k_2, ..., k_n)$ is said to be a **solution** of the previous equation if the following holds:

$$a_1 k_1 + a_2 k_2 + ... + a_n k_n = b$$

A linear equation does not necessarely have a solution

Exercise 2.36

Consider the following equations:

$$3x = 5 \qquad 2x - y = 1 \qquad 0x = 1$$

Do they have any solutions?

Proof:

- The equation $3x = 5$ has a single solution, that can be found by rearranging the terms and obtaining $x = \frac{5}{3}$;
- The equation $2x - y = 1$ has infinite solutions. Indeed, any pair of real numbers $(k, 2k - 1)$ with $k \in \mathbb{R}$ is a solution for said equation;
- The equation $0x = 1$ has no solution, since any number multiplied by 0 is always 0, which is different from 1.

□

Any set of m linear equations of n unknowns $x_1, x_2, ..., x_n$ taken into account at the same time is called a **system of linear equations of m equations of n unknowns**, or simply a **linear system**. A linear system has the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$

If each b_i with $i = 1, 2, ..., m$ is zero, the system is said to be an **homogeneous linear system**.

Exercise 2.37

What are some examples of linear equations?

Proof:

$$\begin{cases} 2x + 3y - 5z = 1 \\ 4x + y - 2z = 3 \end{cases} \qquad \begin{cases} 2x + 3y - 2z = 0 \\ 4x + 3y - 2z = 0 \\ x + 2y - z = 0 \end{cases} \qquad \begin{cases} 2x + 2y = 1 \\ 4x + y = -1 \end{cases}$$

□

Any ordered n-ple of real numbers $(k_1, k_2, ..., k_n)$ is said to be a **solution** of the linear system if it is a solution of each of the equations that constitute that system at the same time.

The set of all solutions of a linear system is called the **general solution**. *Solving* a linear system means finding, if it exists, its general solution. A linear system having at least one solution (that is, whose general solution is not the empty set), is said to be **solvable**. A linear system that is not solvable (whose general solution is the empty set) is said to be **unsolvable**.

The easiest way to solve a linear system, though not necessarely the most efficient, consists in trying to isolate each unknown and substituting its value in the other. This is allowed, since each equation ought to give the same solution at the same time. This method is informally called the *substitution method*.

Exercise 2.38

Consider the following systems of equations:

$$\begin{cases} x+y=0 \\ 2x+2y=1 \end{cases} \qquad \begin{cases} 2x+y=2 \\ 4x+2y=4 \end{cases} \qquad \begin{cases} 3x+2y=4 \\ 5x+y=1 \end{cases}$$

Are they solvable? If they are, what is their general solution?

Proof: The first linear system is unsolvable:

$$\begin{cases} x+y=0 \\ 2x+2y=1 \end{cases} \Rightarrow \begin{cases} x=-y \\ 2(-y)+2y=1 \end{cases} \Rightarrow \begin{cases} x=-y \\ -2y+2y=1 \end{cases} \Rightarrow \begin{cases} x=-y \\ 0=1 \end{cases}$$

The second linear system is solvable and has infinite solutions. More specifically, its general solution is the set of all pair of numbers in the form $(k, 2-2k)$ with $k \in \mathbb{R}$:

$$\begin{cases} 2x+y=2 \\ 4x+2y=4 \end{cases} \Rightarrow \begin{cases} y=2-2x \\ 4x+2(2-2x)=4 \end{cases} \Rightarrow \begin{cases} y=2-2x \\ 4x+4-4x=4 \end{cases} \Rightarrow \begin{cases} y=2-2x \\ 0=0 \end{cases}$$

The third linear system is solvable. Its general solution is the set containing as its single member the pair $(-\frac{2}{7}, \frac{17}{7})$.

$$\begin{cases} 3x+2y=4 \\ 5x+y=1 \end{cases} \Rightarrow \begin{cases} y=1-5x \\ 3x+2(1-5x)=4 \end{cases} \Rightarrow \begin{cases} y=1-5x \\ 3x+2-10x=4 \end{cases} \Rightarrow \begin{cases} x=-\frac{2}{7} \\ y=\frac{17}{7} \end{cases}$$

□

The term "linear system" suggests that it is possible to encode a linear system of equations into a matrix. Indeed, this is the case; for each system of equations, it is possible write a matrix product that is equivalent to said system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \iff AX = B$$

Exercise 2.39

Write the following linear system of equations as a matrix product.

$$\begin{cases} 2x-y+z=4 \\ x+3y-4z=1 \\ -x+5y+6z=0 \end{cases}$$

Proof:

$$AX = B \Rightarrow \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -4 \\ -1 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

□

The A matrix is called the **coefficient matrix**. The $(A \mid B)$ matrix, costructed by justapposing the A and B matrices, is called the **augmented matrix**.

$$(A \mid B) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

Theorem 2.40 **Rouché-Capelli Theorem**

A linear system of equations is solvable if and only if its coefficient matrix and its augmented matrix have the same rank.

Exercise 2.41

Consider the following systems of equations:

$$\begin{cases} x - y + 2z = 1 \\ 3x + y + 3z = 6 \\ x + 3y - z = -1 \end{cases} \qquad \begin{cases} x + 3y - z = -2 \\ 4x + y + z = 1 \\ 2x - 5y + 3z = 5 \end{cases}$$

Are they solvable?

Proof:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 3 \\ 1 & 3 & -1 \end{pmatrix} \qquad A \mid B = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 1 & 3 & 6 \\ 1 & 3 & -1 & -1 \end{pmatrix}$$

Since $\text{rank}(A) = 2$ and $\text{rank}(A \mid B) = 3$, Theorem 2.40 states that the linear system is unsolvable.

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 4 & 1 & 1 \\ 2 & -5 & 3 \end{pmatrix} \qquad (A \mid B) = \begin{pmatrix} 1 & 3 & -1 & -2 \\ 4 & 1 & 1 & 1 \\ 2 & -5 & 3 & 5 \end{pmatrix}$$

Since $\text{rank}(A) = 2$ and $\text{rank}(A \mid B) = 2$, Theorem 2.40 states that the linear system is solvable. □

Lemma 2.42

An homogeneous linear system is always solvable.

Proof: Let A and $(A \mid B)$ be the coefficient and augmented matrices of an homogeneous linear system. Being homogeneous, the B matrix is entirely constituted by zeros, and therefore A and $(A \mid B)$ have the same rank. Applying Theorem 2.40, it is guaranteed that said linear system is solvable. It is also easy to see that a solution is the *null solution*, constituted by all zeros. □

Notice how Theorem 2.40 only deals with the solvability of a linear system of equations, and is not concerned with how many solutions exist.

A linear system of equations is said to be **determinated** if it has as many equations as unknowns. It is instead said to be **overdeterminated** if it has more equations than unknowns or **underdeterminated** if it has more unknowns than equations.

Theorem 2.43 **Cramer Theorem**

A determinated linear system of equations has one and only solution if and only if its coefficient matrix is non singular.

Lemma 2.44

A determinated or overdeterminated linear system has either one or infinitely many solutions. The first case happens when the rank of the coefficient matrix equals the number of equations of the system, whereas the second case happens when the rank of the coefficient matrix is less than the number of its equations.

Lemma 2.45

An underdeterminated linear system is either unsolvable or, if it is solvable, has always infinitely many solutions. In particular, the number of degrees of freedom of the system is given by the difference between the number of equations and the rank of the coefficient matrix.

Theorem 2.46 **Principle of Gaussian elimination**

Let A be the coefficient matrix of a linear system of equations S. Applying any Gauss move one or more times to A gives a new matrix A' associated to a new linear system of equations S' whose general solution is the same of S.

2.6 Linear Transformations

A transformation $\phi : V \mapsto W$, with both V and W being vector spaces, is called a **linear transformation** if and only if:

$$\phi\left(\underline{v_1} + \underline{v_2}\right) = \phi\left(\underline{v_1}\right) + \phi\left(\underline{v_2}\right) \ \forall \underline{v_1}, \underline{v_2} \in V \qquad \phi(\lambda \underline{v}) = \lambda \phi(\underline{v}) \ \forall \underline{v} \in V, \lambda \in \mathbb{R}$$

In particular, if $V = W$, the transformation ϕ is said to be an **endomorphism**.

Exercise 2.47

Consider the vector space $\mathbb{R}$ (that is, the set of real numbers). Check whether the transformations $\phi_1(x) = 2x$ and $\phi_2(x) = x + 1$ are linear or not.

Proof:
- The transformation $\phi_1(x) = 2x$ is linear. Infact, given two real numbers a and b, is indeed true that $2(a + b) = 2a + 2b$, since the product between real numbers has the distributive property. Similarly, given a real number a and a real number λ, it is true that $2(\lambda a) = 2\lambda a$, since the product between real numbers has the associative property;
- The transformation $\phi_2(x) = x + 1$ is not linear. Given two real numbers a and b, it results in $\phi_2(a + b) = (a + b) + 1 = a + b + 1$, while $\phi_2(a) + \phi_2(b) = a + 1 + b + 1 = a + b + 2$.

□

A remarkable property of linear transformations is that they are equivalent to matrices. That is, for any linear transformation there exist matrices that, when multiplied, have the same effect as applying T.

Theorem 2.48 **Equivalence between linear transformations and matrices**

Let $\phi : V \mapsto W$ be a linear transformation between two vector spaces V and W. Let $\mathcal{B} = \left\{\underline{b_1}, \ldots, \underline{b_n}\right\}$ be a basis for V and $\mathcal{B}' = \left\{\underline{c_1}, \ldots, \underline{c_m}\right\}$ a basis for W. Let C and C' be the column vector representation of, respectively, a vector of V and the result of applying ϕ to said vector. There exist a matrix M_ϕ, dependent both on $\mathcal{B}$ and on $\mathcal{B}'$, such that $M_\phi C = C'$.

Proof: Consider a generic vector $\underline{x} \in V$. This vector can be written as a linear combination of the vectors of $\mathcal{B}$:

$$\underline{x} = \sum_{i=1}^{n} \lambda_i \underline{b_i}$$

Applying ϕ to $\underline{x}$ gives:

$$\phi(\underline{x}) = \phi\left(\sum_{i=1}^{n} \lambda_i \underline{b_i}\right) = \sum_{i=1}^{n} \phi\left(\lambda_i \underline{b_i}\right) = \sum_{i=1}^{n} \lambda_i \phi\left(\underline{b_i}\right)$$

The two rightmost equalities stem from the fact that ϕ is linear.

Each i-th vector $\phi\left(\underline{b_i}\right)$ is a member of W, since it's the result of applying ϕ to a vector of V. Therefore, each $\phi\left(\underline{b_i}\right)$ can itself be written as a linear combination of the vectors of $\mathcal{B}'$:

$$\phi\left(\underline{b_i}\right) = \sum_{j=1}^{m} \gamma_{j,i} \underline{c_j}$$

Substituting back in the previous expression gives:

$$\phi(\underline{x}) = \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{m} \gamma_{j,i} \underline{c_j} \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \gamma_{j,i} \underline{c_j}$$

Recall that it's always possible to encode the information regarding a vector into a column vector containing the coefficients of the linear combination used to generate it. In particular, consider the matrix for $\phi(\underline{x})$:

$$\phi(\underline{x}) \Longleftrightarrow \begin{pmatrix} \sum_{j=1}^{m} \lambda_1 \gamma_{j,1} \\ \vdots \\ \sum_{j=1}^{m} \lambda_n \gamma_{j,n} \end{pmatrix}_{\mathcal{B}'} = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m,1} & \gamma_{m,2} & \cdots & \gamma_{m,n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

However, the matrix on the right is just the column vector representation for $\underline{x}$. Renaming the entries of the matrix on the left with $\mu_1, \ldots, \mu_n$, one gets:

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}_{\mathcal{B}'} = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m,1} & \gamma_{m,2} & \cdots & \gamma_{m,n} \end{pmatrix}_{\mathcal{B}'}^{\mathcal{B}} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}_{\mathcal{B}}$$

The $\gamma_{i,j}$ matrix is the desired matrix. The i-th column is composed by the vector representation of the result of applying ϕ to the i-th vector of the basis $\mathcal{B}$. ◻

Exercise 2.49

Consider the linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ defined below. Write it as a matrix multiplication with respect to the two bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

$$T(x, y, z) = (x - z, y + z) \qquad \mathcal{B}_1 = \{(1, 0, 1), (2, 1, -1), (-2, 1, 4)\} \qquad \mathcal{B}_2 = \{(1, 2), (2, 1)\}$$

Proof: Applying the transformation to the vectors in $\mathcal{B}_1$ gives:

$$\begin{cases} T(1, 0, 1) = (1 - 1, 0 + 1) = (0, 1) = \gamma_{1,1}(1, 2) + \gamma_{2,1}(2, 1) \\ T(2, 1, -1) = (2 - (-1), 1 + (-1)) = (3, 0) = \gamma_{1,2}(1, 2) + \gamma_{2,2}(2, 1) \\ T(-2, 1, 4) = (-2 - 4, 1 + 4) = (-6, 5) = \gamma_{1,3}(1, 2) + \gamma_{2,3}(2, 1) \end{cases}$$

The coefficients are:

$$\begin{cases} 0 = \gamma_{1,1} + 2\gamma_{2,1} \\ 1 = 2\gamma_{1,1} + \gamma_{2,1} \\ 3 = \gamma_{1,2} + 2\gamma_{2,2} \\ 0 = 2\gamma_{1,2} + \gamma_{2,2} \\ -6 = \gamma_{1,3} + 2\gamma_{2,3} \\ 5 = 2\gamma_{1,3} + \gamma_{2,3} \end{cases} = \begin{cases} \gamma_{1,1} = -2\gamma_{2,1} \\ 1 = 2\left(-2\gamma_{2,1}\right) + \gamma_{2,1} \\ 3 = \gamma_{1,2} + 2\left(-2\gamma_{1,2}\right) \\ \gamma_{2,2} = -2\gamma_{1,2} \\ \gamma_{1,3} = -2\gamma_{2,3} - 6 \\ 5 = 2\left(-2\gamma_{2,3} - 6\right) + \gamma_{2,3} \end{cases} = \begin{cases} \gamma_{1,1} = -2\gamma_{2,1} \\ 1 = -3\gamma_{2,1} \\ 3 = -3\gamma_{1,2} \\ \gamma_{2,2} = -2\gamma_{1,2} \\ \gamma_{1,3} = -2\gamma_{2,3} - 6 \\ 17 = -3\gamma_{2,3} \end{cases} = \begin{cases} \gamma_{1,1} = \frac{2}{3} \\ \gamma_{2,1} = -\frac{1}{3} \\ \gamma_{1,2} = -1 \\ \gamma_{2,2} = 2 \\ \gamma_{1,3} = \frac{16}{3} \\ \gamma_{2,3} = -\frac{17}{3} \end{cases}$$

Which gives the matrix

$$M_T = \begin{pmatrix} \frac{2}{3} & -1 & \frac{16}{3} \\ -\frac{1}{3} & 2 & -\frac{17}{3} \end{pmatrix}_{\mathcal{B}_2}^{\mathcal{B}_1}$$

Consider, as an example, the triple $(3, 1, 0) \in V$. Its vector representation with respect to $\mathcal{B}_1$ is:

$$(3,1,0) = \lambda_1(1,0,1) + \lambda_2(2,1,-1) + \lambda_3(-2,1,4) = 1(1,0,1) + 1(2,1,-1) + 0(-2,1,4) \Longleftrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{B}_1}$$

Applying ϕ to $(3, 1, 0)$ gives $\phi(3, 1, 0) = (3 - 0, 1 + 0) = (3, 1)$. Its vector representation with respect to $\mathcal{B}_2$ is:

$$(3,1) = \lambda'_1(1,2) + \lambda'_2(2,1) = -\frac{1}{3}(1,2) + \frac{5}{3}(2,1) \Longleftrightarrow \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}_{\mathcal{B}_2}$$

Indeed:

$$\begin{pmatrix} \frac{2}{3} & -1 & \frac{16}{3} \\ -\frac{1}{3} & 2 & -\frac{17}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \cdot 1 + (-1) \cdot 1 + \frac{16}{3} \cdot 0 \\ -\frac{1}{3} \cdot 1 + 2 \cdot 1 + \left(-\frac{17}{3}\right) \cdot 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} - 1 + 0 \\ -\frac{1}{3} + 2 + 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$$

□

Let $T : V \mapsto W$ a linear transformation between vector spaces V and W. The set of all vectors of W that have a correspondant in V through T is called the **image** of the transformation T, and is denoted as $\mathfrak{I}(T)$. It may or may not coincide with W.

$$\mathfrak{I}(T) = \{\underline{w} \in W : \exists \underline{v} \in V \quad \text{s.t.} \;\; T(\underline{v}) = \underline{w}\}$$

The notion of image is present in every transformations, not just linear ones, but images of linear transformations possess properties that images of generic transformations don't.

Theorem 2.50

Let $T : V \mapsto W$ be a linear transformation between vector spaces V and W. $\mathfrak{I}(W)$ is a subspace of W.

Proof: By Proposition 2.5, it suffices to prove that $\underline{w_1} + \underline{w_2} \in \mathfrak{I}(W)$ holds for all $\underline{w_1}, \underline{w_2} \in \mathfrak{I}(W)$ and that $\lambda \underline{w} \in \mathfrak{I}(W)$ holds for all $\underline{w} \in \mathfrak{I}(W)$ and $\lambda \in \mathbb{R}$.

By definition, if $\underline{w} \in \mathfrak{I}(W)$ then there exists $\underline{v} \in V$ such that $T(\underline{v}) = \underline{w}$. Therefore:

$$\underline{w_1} + \underline{w_2} = T\left(\underline{v_1}\right) + T\left(\underline{v_2}\right) \qquad\qquad \lambda \underline{w} = \lambda T(\underline{v})$$

By virtue of T being linear:

$$\underline{w_1} + \underline{w_2} = T\left(\underline{v_1}\right) + T\left(\underline{v_2}\right) = T\left(\underline{v_1} + \underline{v_2}\right) \qquad\qquad \lambda \underline{w} = \lambda T(\underline{v}) = T(\lambda \underline{v})$$

In both cases, there exists a vector in V such that the application of T gives such vector, therefore $\mathfrak{I}(W)$ is algebraically closed with respect to the operations defined for W. □

Let $T : V \mapsto W$ a linear transformation between vector spaces V and W. The set of all vectors of V such that the application of T to those vectors gives the null vector (of W) is called the **kernel** of T, and is denoted as $\ker(T)$.

$$\ker(T) = \{\underline{v} \in V : T(\underline{v}) = \underline{0}\}$$

Theorem 2.51

Let $T : V \mapsto W$ be a linear transformation between vector spaces V and W. $\ker(V)$ is a subspace of V.

Proof: By Proposition 2.5, it suffices to prove that $\underline{v_1} + \underline{v_2} \in \ker(V)$ holds for all $\underline{v_1}, \underline{v_2} \in \ker(V)$ and that $\lambda \underline{v} \in \ker(V)$ holds for all $\underline{v} \in \ker(V)$ and $\lambda \in \mathbb{R}$.

By definition, if $\underline{v} \in \ker(V)$ holds, then $T(\underline{v}) = 0$. By virtue of T being linear:

$$T\left(\underline{v_1} + \underline{v_2}\right) = T\left(\underline{v_1}\right) + T\left(\underline{v_2}\right) = \underline{0} + \underline{0} = \underline{0} \qquad T(\lambda\underline{v}) = \lambda T(\underline{v}) = \lambda(\underline{0}) = \underline{0}$$

□

Let $T : V \mapsto W$ be a linear transformation between vector spaces V and W. The dimension of the image of T is called the **rank** of T, and denoted as $\mathrm{rank}(T)$, while the dimension of the kernel of T is called the **nullity** of T, and denoted as $\mathrm{null}(T)$.

Theorem 2.52 — **Rank-nullity theorem**

Let $T : V \mapsto W$ be a linear transformation between vector spaces V and W. The dimension of V is given by the sum of the rank of T and the nullity of T:

$$\dim(V) = \mathrm{rank}(T) + \mathrm{null}(T) = \dim(\ker(T)) + \dim(\mathfrak{I}(T))$$

Let $T : V \mapsto W$ be a linear transformation between vector spaces V and W. The linear transformation $T^{-1} : W \mapsto V$ is said to be the **inverse** of T if:

$$T^{-1}(T(\underline{v})) = T(T^{-1}(\underline{v})) = \underline{v}, \ \forall \underline{v} \in V$$

As for any function, a linear transformation T has an inverse if and only if it is both injective and subjective. A linear transformation that has an inverse is said to be **invertible**.

Theorem 2.53

Let $T : V \mapsto W$ be a linear transformation. If T is injective, then its nullity is 0.

Proof: If T is injective then, for any distinct $\underline{v_1}, \underline{v_2} \in V$, $T\left(\underline{v_1}\right) \neq T\left(\underline{v_2}\right)$, which is to say $T\left(\underline{v_1}\right) - T\left(\underline{v_2}\right) \neq \underline{0}$. But T is linear by definition, therefore $T\left(\underline{v_1}\right) - T\left(\underline{v_2}\right) = T\left(\underline{v_1} - \underline{v_2}\right)$. Being V a vector space, it algebraically closed with respect to the sum of vectors, therefore $\left(\underline{v_1} - \underline{v_2}\right)$ is itself a member of V distinct from $\underline{0}$, be it $\underline{v}$. In other words, if T is injective, $T(\underline{v})$ has to be different from $\underline{0}$ for any $\underline{v} \in V$, that isn't the null vector, that is to say that the kernel is only composed of the null vector, which is the definition of the nullity of a linear transformation to be 0. □

Theorem 2.54

Let $T : V \mapsto W$ be a linear transformation. If T is invertible, then V and W have the same dimension.

Proof: By Theorem 2.52, $\dim(V) = \dim(\ker(T)) + \dim(\mathfrak{I}(T))$. Being T invertible, the dimension of the image equals the dimension of the codomain W. By Theorem 2.53, $\dim(\ker(T)) = 0$. Therefore, $\dim(V) = 0 + \dim(\mathfrak{I}(T)) = \dim(W)$. □

As stated before, every result concerning linear transformations can be formulated very naturally as a result concerning matrices.

Theorem 2.55

Let A be the $m \times n$ matrix associated to the invertible linear application $T : V \mapsto W$ with respect to two bases $\mathcal{B}$ and $\mathcal{C}$. Then, m and n are equal (that is, A is a square matrix).

Proof: By Theorem 2.54, if T is an invertible linear transformation, $\dim(V) = \dim(W)$. Since the dimensions of A are $\dim(V)$ and $\dim(W)$ respectively, $m = n$. □

Indeed, it is possible to define a kernel and an image for an invertible matrix. Consider the linear transformation $T : V \mapsto W$, with respect to whom a $n \times n$ matrix A of real values can be associated. Therefore, any vector $\underline{w} \in W$ can be written as $A\underline{v}$, where $\underline{v}$ is a vector in V. Writing v as a linear combination of the canonical basis of V gives:

$$\underline{w} = A\underline{v} = A\left(\lambda_1\underline{e_1} + \lambda_2\underline{e_2} + \ldots + \lambda_n\underline{e_n}\right) = A\lambda_1\underline{e_1} + A\lambda_2\underline{e_2} + \ldots + A\lambda_n\underline{e_n}$$

But multiplying the matrix A by the canonical vector $\underline{e_i}$ simply returns the i-th column of A:

$$A\underline{e_i} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ i \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1,i} \\ a_{2,i} \\ \vdots \\ a_{n,i} \end{pmatrix}$$

Which means that any image can be written as a linear combination of the columns of A, taken as vectors:

$$\underline{w} = A\underline{v} = A\lambda_1\underline{e_1} + A\lambda_2\underline{e_2} + \dots + A\lambda_n\underline{e_n} = \lambda_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix} + \lambda_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{n,n} \end{pmatrix}$$

Being $\underline{w}$ a generic vector, this must mean that the columns of A, taken as vectors, is a set that can generate W. To know the dimension of $\mathfrak{I}(V)$, that is to say, the rank of A, it suffices to find the smallest number of column-vectors of A that is linearly independent.

By Theorem 2.52, the dimension of $\mathfrak{I}(V)$, which is equal to $\text{rank}(A)$, has to be equal to the dimension of the domain, which in this case is just $\mathbb{R}^n$ having dimension n. Therefore, for a matrix to be invertible (that is, to be the matrix associated to an invertible linear transformation), its rank has to be equal to the number of its columns or, equivalently, if its columns form a linearly independent set. If this happens, such a matrix is said to have **full rank**.

Proposition 2.16 proves that it is possible to convert the representation of a vector with respect to a given basis in the representation of the same vector to a different basis by multiplying the known representation with respect to a "conversion" matrix P. The same can be achieved with respect to matrices associated to endomorphisms.

Theorem 2.56

Let $T : V \mapsto V$ be an endomorphism of dimension n. Let A be the matrix associated to T with respect to the basis $\mathcal{B}$ (for both domain and codomain), and let A' be the matrix associated to T with respect to a different basis $\mathcal{B}'$. There exists an invertible matrix P such that:

$$A = PA'P^{-1}$$

Proof: Let $\underline{x}$ be a vector of V, and let $\underline{y}$ be the result of applying T to $\underline{x}$. Being T an endomorphism, both $\underline{x}$ and $\underline{y}$ belong to the same vector space, and can therefore be represented by the bases $\mathcal{B}$ and $\mathcal{B}'$:

$$\underline{x} \Leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}} \qquad \underline{x} \Leftrightarrow \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}_{\mathcal{B}'} \qquad \underline{y} = T(\underline{x}) \Leftrightarrow \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{\mathcal{B}} \qquad \underline{y} = T(\underline{x}) \Leftrightarrow \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}_{\mathcal{B}'}$$

By definition of associated matrix, applying T to $\underline{x}$ is equivalent to multiplying A with the representation of $\underline{x}$ with respect to $\mathcal{B}$, or multiplying A' with the representation of $\underline{x}$ with respect to $\mathcal{B}'$:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{\mathcal{B}} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}} \qquad \begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}_{\mathcal{B}'} = A' \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}_{\mathcal{B}'}$$

As stated in Proposition 2.16, there exist a matrix P that permits to convert the representation of a vector with respect to a given basis in the representation to a different basis, while the inverse matrix P^{-1} does the opposite conversion. Such conversion, since they belong to the same vector space V, can be done for both $\underline{x}$ and $\underline{y}$:

$$\begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{pmatrix} = P^{-1} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Substituting in the previous expression gives:

$$\begin{pmatrix} y'_1 \\ \vdots \\ y'_n \end{pmatrix}_{\mathcal{B}'} = A' \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}_{\mathcal{B}'} \Rightarrow P^{-1} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{\mathcal{B}} = A' P^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}} \Rightarrow P^{-1} A \cancel{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}}} = A' P^{-1} \cancel{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}}} \Rightarrow$$

$$P^{-1} A = A' P^{-1} \Rightarrow \cancel{P P^{-1}} A = P A' P^{-1} \Rightarrow A = P A' P^{-1}$$

□

2.7 Eigenvalues and eigenvectors

Let A be an $n \times n$ square matrix, and let λ be a real value. The n-dimensional vector $\underline{x}$ is said to be an **eigenvector** of A if it's not null and if:

$$A\underline{x} = \lambda \underline{x}$$

Where λ is the corresponding **eigenvalue** of A.

Retrieving the eigenvectors of a matrix A by applying such definition is not possible, since the information contained in the equation is insufficient. Infact:

$$A\underline{x} = \lambda \underline{x} \Rightarrow \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow \begin{cases} a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 + \dots + x_n a_{1,n} = \lambda x_1 \\ a_{2,1} \cdot x_1 + a_{2,2} \cdot x_2 + \dots + x_n a_{2,n} = \lambda x_2 \\ \vdots \\ a_{n,1} \cdot x_1 + a_{n,2} \cdot x_2 + \dots + x_n a_{n,n} = \lambda x_n \end{cases}$$

Even assuming the A matrix to be known, this system of equation has n equations but $n+1$ unknowns (the n components of $\underline{x}$ and λ). It is still possible to retrieve the eigenvectors of a matrix by following a different approach, by first retrieving its eigenvalues and then applying such definition.

Given a square matrix A and a real value λ, the **characteristic polynomial** of A is defined as:

$$p_A(\lambda) = \det(A - \lambda I) = c_0 + c_1 \lambda + c_2 \lambda^2 + \dots + c_{n-1} \lambda^{n-1} + (-1)^n \lambda^n$$

Where:

$$c_0 = \det(A) \qquad c_{n-1} = (-1)^{n-1} \operatorname{tr}(A)$$

Theorem 2.57

A real value is an eigenvalue for a given matrix if and only if it is a root of its characteristic polynomial.

Proof: First, suppose that $\lambda \in \mathbb{R}$ is an eigenvalue for a $n \times n$ square matrix A. By definition of eigenvalue, there must exist a non-null vector $\underline{x}$ such that $A\underline{x} = \lambda \underline{x}$. Then:

$$A\underline{x} = \lambda \underline{x} \Rightarrow A\underline{x} = \lambda I \underline{x} \Rightarrow A\underline{x} - \lambda I \underline{x} = \underline{0} \Rightarrow (A - \lambda I)\underline{x} = \underline{0}$$

This means that $\underline{x}$ is a vector that belongs to the kernel of the matrix $(A - \lambda I)$. Therefore, the nullity of $(A - \lambda I)$ can't be zero.

By Theorem 2.52, $\dim(A - \lambda I) = \operatorname{rank}(A - \lambda I) + \operatorname{null}(A - \lambda I)$. But $(A - \lambda I)$ and A have the same dimension, therefore $n = \operatorname{rank}(A - \lambda I) + \operatorname{null}(A - \lambda I)$. Since $\operatorname{null}(A - \lambda I)$ is non zero, for this equality to hold the rank of $(A - \lambda I)$ has to be less than n. By Lemma 2.23, the matrix $(A - \lambda I)$ cannot be invertible, and by Proposition 2.31 this must mean that the determinant of $(A - \lambda I)$ is 0.

Suppose then that λ is a root for the characteristic polynomial of A. This means that $\det(A - \lambda I)$ is equal to 0. By Proposition 2.31, this must mean that $(A - \lambda I)$ is not invertible, which in turn by Lemma 2.23 must mean that the rank of $(A - \lambda I)$ is less than n. By Theorem 2.52, $n = \text{rank}(A - \lambda I) + \text{null}(A - \lambda I)$, and being the rank less than n in turn implies that the kernel of $(A - \lambda I)$ does not contain just the null vector. This means that it exists a vector $\underline{x}$ such that $(A - \lambda I)\underline{x} = \underline{0}$. But then:

$$(A - \lambda I)\underline{x} = \underline{0} \Rightarrow A\underline{x} - \lambda I\underline{x} = \underline{0} \Rightarrow A\underline{x} = \lambda I\underline{x} \Rightarrow A\underline{x} = \lambda\underline{x}$$

Which is the definition of eigenvalue. □

Knowing how to compute eigenvalues, it is then possible to solve the aforementioned equation and retrieve the eigenvectors.

Eigenvectors and eigenvalues can be defined with respect to linear transformations as well. Given a linear transformation $T : V \mapsto V$, a vector $\underline{v} \in V$ is an eigenvector for T if $T\underline{v} = \lambda\underline{v}$, where λ is an eigenvalue for T. Notice how it has been imposed that the transformation T is an endomorphism, since otherwise mirroring the definition of eigenvector for matrices could not have been possible.

As stated in Theorem 2.57, to compute the eigenvalues of a matrix, it suffices to compute its characteristic polynomial. But any matrix can be associated to a linear transformation and vice versa, therefore to compute the eigenvalues of a linear transformation it suffices to compute the characteristic polynomial of the associated matrix of the linear transformation.

Theorem 2.58

Let $T : V \mapsto V$ be an endomorphism, and let A and A' be two matrices associated to T with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. The characteristic polynomials of A and A' are equivalent.

Proof: The result follows from applying Theorem 2.56 to the characteristic polynomial of one of the matrices:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(PA'P^{-1} - \lambda I\right) = \det\left(PA'P^{-1} - \lambda PIP^{-1}\right) =$$
$$\det\left(P(A'P^{-1} - \lambda IP^{-1})\right) = \det\left(P(A' - \lambda I)P^{-1}\right) = \cancel{\det(P)}\det(A' - \lambda I)\cancel{\det(P^{-1})} =$$
$$\det(A' - \lambda I) = p_{A'}(\lambda)$$

□

Theorem 2.58 justifies referring to such polynomial as the characteristic polynomial of the linear transformation itself, and not to one of the possible associated matrices to such transformation, since the choice of the matrix is irrelevant. Of course, the most convenient choice for the associated matrix is the one constructed with respect to the canonical basis, which in general requires the least amount of effort.

Theorem 2.59 **Diagonalization theorem**

- **With respect to endomorphisms**. Let $T : V \mapsto V$ be an endomorphism of dimension n that has n linearly independent eigenvectors $\underline{e}_1, \dots, \underline{e}_n$. Let E be the set that contains such vectors, forming a basis for V. Let P be the matrix associated to T with respect to the vectors in E. The matrix P is a diagonal matrix whose non-zero element are the eigenvalues of T.
- **With respect to matrices**. Let A be a $n \times n$ matrix that has n linearly independent eigenvectors $\underline{e}_1, \dots, \underline{e}_n$. Then there exist a diagonal matrix D that is similar to A, meaning that there exist a matrix P such that $A = PDP^{-1}$. In particular, the matrix D has the eigenvalues of A as non-zero elements (counted with multiplicity) and P has the eigenvectors of A as columns.

Proof:

The first point is trivial

For the second point, consider the two matrices P and D:

$$P = \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_n \end{pmatrix} = \begin{pmatrix} e_{1,1} & e_{2,1} & \cdots & e_{n,1} \\ e_{1,2} & e_{2,2} & \cdots & e_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1,n} & e_{2,n} & \cdots & e_{n,n} \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

The matrix multiplication AP is by definition equivalent to multiplying A with each column vector of P. That is, the i-th column of AP is given by multiplying the matrix A with the i-th column vector of P, giving $A\underline{e_i}$. But by definition multiplying the matrix representation of an endomorphism with one of his eigenvectors is equivalent to multiplying said eigenvector by its corresponding eigenvalue. Therefore:

$$AP = \begin{pmatrix} A\underline{e_1} & A\underline{e_2} & \dots & A\underline{e_n} \end{pmatrix} = \begin{pmatrix} \lambda_1\underline{e_1} & \lambda_2\underline{e_2} & \dots & \lambda_n\underline{e_n} \end{pmatrix}$$

Consider the matrix multiplication PD. By definition, the i-th element of such matrix is given by the sum of the products of the corresponding elements of the i-th row of P and the i-th column of D. By construction, the elements in D are zero except for the ones on its diagonal, therefore the i-th column of PD is just the i-th column vector of P multiplied by the i, i-th element of D, which is λ_i. Therefore:

$$PD = \begin{pmatrix} \lambda_1\underline{e_1} & \lambda_2\underline{e_2} & \dots & \lambda_n\underline{e_n} \end{pmatrix}$$

This shows that the two matrix products AP and PD are equivalent. Since by assumption the set of the eigenvectors of A form a basis, by Lemma 2.23 P has to be invertible. But then:

$$AP = PD \Rightarrow A\cancel{P}\cancel{P^{-1}} = PDP^{-1} \Rightarrow A = PDP^{-1}$$

□

If for a given square matrix A there exist an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, matrix A is said to be **diagonalizable**. As stated in Theorem 2.59, the matrix P is an invertible matrix whose columns are the eigenvectors of A while D is a diagonal matrix whose non-zero elements are the eigenvalues of A.

By the Fundamental Theorem of Algebra, the characteristic polynomial of any matrix will always have at least n roots, albeit they might be complex numbers. Therefore, any square matrix will always have n (not necessarely distinct) eigenvalues. Despite this, the fact that the set of its eigenvectors forms a basis for the vector space associated to A isn't always true, therefore not all matrices are diagonalizable. A matrix whose set of eigenvectors does not form a basis is said to be **defective**.

Exercise 2.60

Prove that the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is defective.

Proof: Computing the characteristic polynomial of A gives:

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 0 - \lambda & 1 - 0 \\ 0 - 0 & 0 - \lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = (-\lambda) \cdot (-\lambda) - (0 \cdot 1) = \lambda^2$$

Such polynomial has only two roots, both being 0. Therefore, the eigenvalues of A are $\lambda_1 = \lambda_2 = 0$. By applying the definition:

$$A\underline{x} = \lambda\underline{x} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \cdot x + 1 \cdot y \\ 0 \cdot x + 0 \cdot y \end{pmatrix} = \begin{pmatrix} 0 \cdot x \\ 0 \cdot y \end{pmatrix} \Rightarrow \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y = 0 \\ 0 = 0 \end{cases}$$

This means that the eigenvectors of A are all the vectors in the form $\begin{pmatrix} 0 \\ k \end{pmatrix}$ with $k \in \mathbb{R}$. Of course, the set $E = \left\{ \begin{pmatrix} 0 \\ k \end{pmatrix} \right\} \subset \mathbb{R}^2$ is not linearly independent (at least two vectors are needed) and therefore A is defective. □

Determining whether a matrix is diagonalizable through such definition can quickly become cumbersome, but there are necessary and sufficient conditions that are equivalent and that can ease the process.

Let A be a matrix and λ one of its eigenvalues. The number of times λ appears as a root of the characteristic polynomial of A is the **algebraic multiplicity** of λ, and is denoted as $m_a(\lambda)$. The dimension of the vector space generated by the set of eigenvectors that have λ as their eigenvalue is the **geometric multiplicity** of λ, and is denoted as $m_g(\lambda)$.

Theorem 2.61

For any eigenvalue λ, the following inequality holds:

$$1 \leq m_g(\lambda) \leq m_a(\lambda)$$

Theorem 2.62

A matrix is diagonalizable if and only if, for each of its eigenvalues λ_i, $m_g(\lambda_i) = m_a(\lambda_i)$.

Corollary 2.63

Any $n \times n$ matrix that has n distinct eigenvalues is diagonalizable.

Proof: If a matrix has as many distinct eigenvalues as its dimension it means that the algebraic multiplicity of any of its eigenvalues is 1. By Theorem 2.61, for any eigenvalue λ_i its geometric multiplicity must also be 1, because $1 \leq m_g(\lambda_i) \leq 1$. The fact that such matrix is diagonalizable follows from applying Theorem 2.62. □

Theorem 2.64

A symmetric matrix is always diagonalizable.

2.8 Spectral Theorem

Aside from the notions of sum between two vectors and multiplication of a vector by a scalar, which are mandatory requirements for a vector space to be defined as such, for some (but not all) vector spaces it is possible to also define other operations.

One such example is the **inner product**: given a vector space V, the inner product of two vectors $\underline{v_1}$ and $\underline{v_2}$ of V, denoted as $\langle \underline{v_1}, \underline{v_2} \rangle$, is an operation that returns a scalar and that possesses such properties:

- *Symmetry*: for any vectors $\underline{v_1}, \underline{v_2}$, $\langle \underline{v_1}, \underline{v_2} \rangle = \langle \underline{v_2}, \underline{v_1} \rangle$
- *Linearity of the first term*: For any two scalars a, b and for any vectors $\underline{v_1}, \underline{v_2}, \underline{v_3}$,

$$\langle a\underline{v_1} + b\underline{v_2}, \underline{v_3} \rangle = a\langle \underline{v_1}, \underline{v_3} \rangle + b\langle \underline{v_2}, \underline{v_3} \rangle$$

- *Positive-definiteness*: for any non-null vectors $\underline{v_1}, \underline{v_2}$, $\langle \underline{v_1}, \underline{v_2} \rangle \geq 0$.

The simplest example of an inner product is the one defined for $\mathbb{R}^n$, which is simply a matrix multiplication between a $1 \times n$ matrix and a $n \times 1$ matrix:

$$\langle \underline{x}, \underline{y} \rangle = \underline{x}^T \underline{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Indeed, such product possesses all of the properties presented above.

Any inner product allows the definition of the **norm** of a vector, which is the square root of the inner product between a vector and itself:

$$\| \underline{v} \| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$$

Since the inner product of a vector with itself is always equal or greater that zero (property 3), such square root is always well-defined.

In turn, the norm of a vector allows the definition of an **angle** between vectors:

$$\cos(\theta) = \frac{\langle \underline{x}, \underline{y} \rangle}{\| \underline{x} \| \| \underline{y} \|}$$

If the cosine of the angle between two vectors is 1, said vectors are said to be **parallel**, while if it is 0 they are said to be **orthogonal**. In particular:

$$1 = \frac{\langle \underline{x}, \underline{y} \rangle}{\parallel \underline{x} \parallel \parallel \underline{y} \parallel} \Rightarrow \langle \underline{x}, \underline{y} \rangle = \parallel \underline{x} \parallel \parallel \underline{y} \parallel \qquad 0 = \frac{\langle \underline{x}, \underline{y} \rangle}{\parallel \underline{x} \parallel \parallel \underline{y} \parallel} \Rightarrow \langle \underline{x}, \underline{y} \rangle = 0$$

Exercise 2.65

Consider the vector space $\mathbb{R}^2$. Compute the norm of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Proof:

$$\sqrt{\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle} = \sqrt{(1 \;\; 2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \sqrt{1 \cdot 1 + 2 \cdot 2} = \sqrt{5}$$

□

A basis for a vector space whose vectors are pairwise orthogonal (in other words, for any distinct $\underline{v}_i, \underline{v}_j$ in said basis, $\langle \underline{v}_i, \underline{v}_j \rangle = 0$) is called an **orthogonal basis**. In other words, a set of vectors form an orthonormal basis if, for any distinct $\underline{v}_i, \underline{v}_j$, the following holds:

$$\langle \underline{v}_i, \underline{v}_j \rangle = \begin{cases} 1 \text{ if } \underline{v}_i = \underline{v}_j \\ 0 \text{ otherwise} \end{cases}$$

In particular, if said vectors all have norm equal to 1, said basis is called an **orthonormal basis**.

If the eigenvectors of symmetric matrix form an orthonormal basis, Theorem 2.59 applies in a very specific way.

Theorem 2.66 **Spectral theorem**

Let A be a symmetric matrix whose eigenvectors can form an orthonormal basis and whose eigenvalues are all real. Then there exist two matrices P and D such that $A = PDP^T$, where P is an orthogonal matrix whose columns are the orthonormal eigenvectors of A and D is a diagonal matrix whose non-zero elements are the eigenvalues of A.

Proof: The fact that P and D with these characteristics exist stems from Theorem 2.59, since the eigenvectors of A form a basis. What has to be proved is that, under such conditions, P is orthogonal.

□

Exercise 2.67

Consider the following symmetric matrix A. Find the two matrices D and P described in Theorem 2.66.

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Proof: A is diagonalizable by virtue of Theorem 2.64. The eigenvalues can be retrieved from:

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)((-1-\lambda)^2 - 1) =$$

$$(3-\lambda)(\cancel{1} + \lambda^2 + 2\lambda \cancel{-1}) = (3-\lambda)(\lambda^2 + 2\lambda) = \lambda(3-\lambda)(\lambda+2) \Rightarrow \lambda_1 = 0, \lambda_2 = 3, \lambda_3 = -2$$

An eigenvector can then be retrieved as follows:

$$A\underline{x} = \lambda\underline{x} \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \cdot x + 1 \cdot y + 0 \cdot z \\ 1 \cdot x - 1 \cdot y + 0 \cdot z \\ 0 \cdot x + 0 \cdot y + 3 \cdot z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \Rightarrow \begin{pmatrix} y - x \\ x - y \\ 3z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

Having three eigenvalues, this amounts to solve three systems of linear equations:

$$\begin{cases} y - x = \lambda_1 x \\ x - y = \lambda_1 y \\ 3z = \lambda_1 z \end{cases} \Rightarrow \begin{cases} y - x = 0x \\ x - y = 0y \\ 3z = 0z \end{cases} \Rightarrow \begin{cases} y - x = 0 \\ x - y = 0 \\ 3z = 0 \end{cases} \Rightarrow \begin{cases} y = x \\ x = y \\ z = 0 \end{cases} \Rightarrow \begin{cases} y = x \\ 0 = 0 \\ z = 0 \end{cases} \Rightarrow \begin{pmatrix} k \\ k \\ 0 \end{pmatrix} \quad \forall k \in \mathbb{R}$$

$$\begin{cases} y - x = \lambda_2 x \\ x - y = \lambda_2 y \\ 3z = \lambda_2 z \end{cases} \Rightarrow \begin{cases} y - x = 3x \\ x - y = 3y \\ 3z = 3z \end{cases} \Rightarrow \begin{cases} y = 4x \\ x = 4y \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} y = 4x \\ x = 16x \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix} \quad \forall k \in \mathbb{R}$$

$$\begin{cases} y - x = \lambda_3 x \\ x - y = \lambda_3 y \\ 3z = \lambda_3 z \end{cases} \Rightarrow \begin{cases} y - x = -2x \\ x - y = -2y \\ 3z = -2z \end{cases} \Rightarrow \begin{cases} y = -x \\ x = -y \\ 5z = 0 \end{cases} \Rightarrow \begin{cases} y = -x \\ x = -(-x) \\ z = 0 \end{cases} \Rightarrow \begin{cases} y = -x \\ 0 = 0 \\ z = 0 \end{cases} \Rightarrow \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} \quad \forall k \in \mathbb{R}$$

Which gives:

$$A = P'DP'^{-1} \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} k & 0 & k \\ k & 0 & -k \\ 0 & k & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{2k} & \frac{1}{2k} & 0 \\ 0 & 0 & \frac{1}{k} \\ \frac{1}{2k} & -\frac{1}{2k} & 0 \end{pmatrix}$$

Out of all P matrices, the one of interest is the one whose columns vectors have norm equal to 1.

$$\| e_1 \| = \sqrt{\langle \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}, \begin{pmatrix} k \\ k \\ 0 \end{pmatrix} \rangle} = \sqrt{\begin{pmatrix} k \\ k \\ 0 \end{pmatrix} (k \;\; k \;\; 0)} = \sqrt{k \cdot k + k \cdot k + 0 \cdot 0} = \sqrt{2k^2} = \sqrt{2}\, |k|$$

$$\| e_2 \| = \sqrt{\langle \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix} \rangle} = \sqrt{\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix} (0 \;\; 0 \;\; k)} = \sqrt{0 \cdot 0 + 0 \cdot 0 + k \cdot k} = \sqrt{k^2} = |k|$$

$$\| e_3 \| = \sqrt{\langle \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix}, \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} \rangle} = \sqrt{\begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} (k \;\; -k \;\; 0)} = \sqrt{k \cdot k + (-k) \cdot (-k) + 0 \cdot 0} = \sqrt{2k^2} = \sqrt{2}\, |k|$$

It then suffices to impose the norm to be equal to 1:

$$\sqrt{2}\, |k| = 1 \Rightarrow k = \pm \frac{1}{\sqrt{2}} \qquad |k| = 1 \Rightarrow k = \pm 1 \qquad \sqrt{2}\, |k| = 1 \Rightarrow k = \pm \frac{1}{\sqrt{2}}$$

Which gives:

$$A = PDP^T \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \pm\frac{1}{\sqrt{2}} & 0 & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} & 0 & \mp\frac{1}{\sqrt{2}} \\ 0 & \pm 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \pm 1 \\ \pm\frac{1}{\sqrt{2}} & \mp\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

□

2.9 Decompositions

A symmetric matrix A is said to be **definite positive** if, for any vector $\underline{x}$, $\langle \underline{x}, A\underline{x} \rangle > 0$. It is instead said to be **semidefinite positive** if, for any vector $\underline{x}$, $\langle \underline{x}, A\underline{x} \rangle \geq 0$.

> **Theorem 2.68**
>
> If a symmetric matrix is definite positive, each one of its eigenvalues is real and strictly positive.

Proof: □

Theorem 2.69

If a symmetric matrix is definite positive, each one of its eigenvalues is real and either positive or equal to 0.

Proof: The idea is the same as in Theorem 2.68 but considering $\geq$ instead of $>$. □

Theorem 2.70 **Cholesky Decomposition**

For any positive definite matrix A there exists a lower triangular matrix L such that $A = LL^T$.

Proof: The theorem can be proven in a constructive way by defining an algorithm that recursively retrieves said L matrix.

First, the three matrices at play ought to have such form:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{1,2} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \dots & a_{n,n} \end{pmatrix} \qquad L = \begin{pmatrix} l_{1,1} & 0 & \dots & 0 \\ l_{1,2} & l_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{1,n} & l_{2,n} & \dots & l_{n,n} \end{pmatrix} \qquad L^T = \begin{pmatrix} l_{1,1} & l_{1,2} & \dots & l_{1,n} \\ 0 & l_{2,2} & \dots & l_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & l_{n,n} \end{pmatrix}$$

The $(1,1)$ entry of the product between L and L^T is given by the inner product of the first row of L and the first column of L^T:

$$l_{1,1} \cdot l_{1,1} + 0 \cdot 0 + 0 \cdot 0 + \dots + 0 \cdot 0 = l_{1,1}^2$$

This means that, for the equality $A = LL^T$ to be true, $l_{1,1}$ ought to be equal to $\sqrt{a_{1,1}}$.

The generic $(1,i)$ entry of the product between L and L^T is given by the inner product of the first row of L and the i-th column of L^T:

$$l_{1,1} \cdot l_{1,i} + 0 \cdot l_{2,i} + 0 \cdot l_{3,i} + \dots + 0 \cdot 0 = l_{1,1} l_{1,i}$$

This means that, for the equality $A = LL^T$ to be true, $a_{1,i}$ ought to be equal to $l_{1,1} l_{1,i}$, which in turn means that $l_{1,i}$ ought to be equal to $a_{1,i}/l_{1,1}$.

The $(2,2)$ entry of the product between L and L^T is given by the inner product of the second row of L and the second column of L^T:

$$l_{1,2} \cdot l_{1,2} + l_{2,2} \cdot l_{2,2} + 0 \cdot 0 + \dots + 0 \cdot 0 = l_{1,2}^2 + l_{2,2}^2$$

This means that, for the equality $A = LL^T$ to be true, $l_{2,2}$ must be equal to $\sqrt{a_{2,2} - l_{1,2}^2}$.

□

Theorem 2.59 states that if a square matrix A possesses certain properties, it can be written as a product in the form PDP^{-1}. A more generic result can be achieved for non-square matrices.

Lemma 2.71

For any matrix A, the matrices $A^T A$ and AA^T are positive semidefinite.

Proof: For a matrix to be positive definite it also needs to be symmetric. Matrix $A^T A$ is indeed symmetric since $(A^T A)^T = A^T (A^T)^T = A^T A$. Let $\underline{y} = A\underline{x}$. Then $\underline{y}^T = (A\underline{x})^T = \underline{x}^T A^T$. This means that:

$$\langle \underline{x}, A^T A \underline{x} \rangle = \underline{x}^T A^T A \underline{x} = \underline{y}^T \underline{y} = \sum_{i=1}^{n} y_i^2$$

Which, by definition, is greater or equal than 0. AA^T can be proven to be positive semidefinite following a similar line of thought. □

Theorem 2.72 **Singular Value Decomposition**

Any $m \times n$ matrix A can be written as the product $A = U\Sigma V^T$, where:

- U is a $m \times m$ orthogonal matrix whose column vectors $(\underline{u_1}, ..., \underline{u_m})$ belong to $\mathbb{R}^m$ and are called **left singular vectors**;
- Σ is a $m \times n$ matrix such that the $\sigma_{i,i}$ entries, called **singular values**, are greater or equal than 0 while the $\sigma_{i,j}$ entries with $i \neq j$ are exactly 0;
- V is a $n \times n$ orthogonal matrix whose column vectors $(\underline{v_1}, ..., \underline{v_m})$ belong to $\mathbb{R}^n$ and are called **right singular vectors**.

Proof: The theorem can be proven in a constructive way by defining an algorithm that generates said matrices, which in turn can prove that the equality holds. Assume, without loss of generality, that $m > n$.

Suppose that $A = U\Sigma V^T$. This means that:

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = \left((V^T)^T \Sigma^T U^T\right) U\Sigma V^T = V\Sigma^T \cancel{U^T} \cancel{U} \Sigma V^T = V\Sigma^T \Sigma V^T$$

By Lemma 2.71, $A^T A$ is positive semidefinite. In turn, by Theorem 2.66, it can be diagonalized as PDP^T, where the eigenvalues along the diagonal of D are non negative as of Theorem 2.69.

Since the dimension of A is $m \times n$, the dimension of $A^T A$ ought to be $n \times n$. In turn, the dimension of D ought to be $n \times n$ as well. By how Σ has been defined, the product $\Sigma^T \Sigma$ ought to be:

$$\Sigma^T \Sigma = \begin{pmatrix} \sigma_{1,1} & 0 & \dots & 0 \\ 0 & \sigma_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{n,n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{1,1} & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_{2,2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \sigma_{n,n} & \dots & 0 \end{pmatrix} = \begin{pmatrix} \sigma^2_{1,1} & 0 & \dots & 0 \\ 0 & \sigma^2_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2_{n,n} \end{pmatrix}$$

Being $\Sigma^T \Sigma$ a diagonal matrix, the representations PDP^T and $V\Sigma^T \Sigma V^T$ can be equated as long as the non-zero values of $\Sigma^T \Sigma$ are the square roots of the eigenvalues of $A^T A$ and the column vectors of V are the normalized eigenvectors of $A^T A$.

Similarly:

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T \left((V^T)^T \Sigma^T U^T\right) = U\Sigma \cancel{V^T} \cancel{V} \Sigma^T U^T = U\Sigma\Sigma^T U^T$$

By Lemma 2.71, AA^T is positive semidefinite. In turn, by Theorem 2.66, it can be diagonalized as QCQ^T, where the eigenvalues along the diagonal of C are non negative as of Theorem 2.69.

Since the dimension of A is $m \times n$, the dimension of AA^T ought to be $m \times m$. In turn, the dimension of C ought to be $m \times m$ as well. By how Σ has been defined, the product $\Sigma\Sigma^T$ ought to be:

$$\Sigma\Sigma^T = \begin{pmatrix} \sigma_{1,1} & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_{2,2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \sigma_{m,m} & \dots & 0 \end{pmatrix} \begin{pmatrix} \sigma_{1,1} & 0 & \dots & 0 \\ 0 & \sigma_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{m,m} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^2_{1,1} & 0 & \dots & 0 \\ 0 & \sigma^2_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2_{m,m} \end{pmatrix}$$

Being $\Sigma\Sigma^T$ a diagonal matrix, the representations QCQ^T and $U\Sigma\Sigma^T U^T$ can be equated as long as the non-zero values of $\Sigma\Sigma^T$ are the square roots of the eigenvalues of AA^T and the column vectors of U are the normalized eigenvectors of AA^T.

The next step consists of comparing the eigenvalues of the two matrices D and C.

By definition, an eigenvector $\underline{e_i}$ of $A^T A$ satisfies the equation $A^T A\underline{e_i} = \lambda_i \underline{e_i}$. But:

$$A^T A\underline{e_i} = \lambda_i \underline{e_i} \Rightarrow AA^T A\underline{e_i} = A\lambda_i \underline{e_i} \Rightarrow (AA^T)\left(A\underline{e_i}\right) = \lambda_i \left(A\underline{e_i}\right) \Rightarrow (AA^T)\underline{u_i} = \lambda_i \underline{u_i}$$

If $\underline{u_i}$ is guaranteed to be different from the null vector, then it is an eigenvector for AA^T. If it were to be the null vector, then:

$$A^T A\underline{e_i} = \lambda_i \underline{e_i} \Rightarrow A^T \underline{0} = \lambda_i \underline{e_i} \Rightarrow \underline{0} = \lambda_i \underline{e_i}$$

Which is true exclusively if λ_i is 0, because $\underline{e_i}$ can't be the null vector by definition of eigenvector.

This means that $A^T A$ and AA^T have the same eigenvalues (even though not necessarely the same eigenvectors) as long as said eigenvalues are not 0. In other words, if $d_{i,i}$ and $c_{i,i}$ are both different from 0, it is guaranteed that they are equal, while if one of them is equal to 0 the other one is different from 0.

To complete the proof, it is necessary to show that A is indeed equal to $U\Sigma V^T$. Consider the matrices AV and $U\Sigma$. Said matrices are constituted by the following column vectors:

$$AV = \begin{pmatrix} \underline{A_1 v_1} & \dots & \underline{A_n v_n} \end{pmatrix} \qquad\qquad U\Sigma = \begin{pmatrix} \underline{u_1 \sigma_1} & \dots & \underline{u_n \sigma_n} \end{pmatrix}$$

By the previous result, $\underline{A_i v_i}$ is equal to $\underline{u_i \sigma_i}$ whenever the corresponding eigenvalues are non zero and both zero otherwise. This means that the two matrices can be equated column by column. Therefore:

$$AV = U\Sigma \Rightarrow A\cancel{V}\cancel{V^T} = U\Sigma V^T \Rightarrow A = U\Sigma V^T$$

□

Summarizing, it is possible to perform the Singular Value Decomposition (SVD) of a matrix A by applying the algorithm stated in Theorem 2.72:

- Compute $A^T A$;
- Compute the eigenvalues of $A^T A$. The diagonal entries of Σ will be the square root of said eigenvalues;
- Compute the eigenvectors of $A^T A$. The normalized choice of eigenvectors will be the column vectors of V;
- If the i-th entry of Σ is non zero, the i-th column vector of U can be computed from the i-th column vector of V as $A\underline{e_i}/\sigma_{i,i}$;
- If the i-th entry of Σ is 0, the i-th column vector of U has to be computed from the kernel of $A\underline{x}$;
- The SVD of A is given by $U\Sigma V^T$.

It is customary to place the diagonal entries of Σ in decreasing order.

Exercise 2.73

Compute the Singular Value Decomposition of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Proof: First, it is necessary to compute $A^T A$:

$$A^T A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 1 & 1 \cdot (-1) + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 & 0 \cdot (-1) + 1 \cdot 1 \\ -1 \cdot 1 + 1 \cdot 1 & -1 \cdot 0 + 1 \cdot 1 & (-1) \cdot (-1) + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Whose eigenvalues can be retrieved from its characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 1 & 0 \\ 1 & 1 - \lambda & 1 \\ 0 & 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda) \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 2 - \lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & 1 - \lambda \\ 0 & 1 \end{vmatrix} =$$

$$(2 - \lambda)((1 - \lambda)(2 - \lambda) - 1 \cdot 1) - (1(2 - \lambda) - 0 \cdot 1) = (2 - \lambda)(2 - \lambda - 2\lambda + \lambda^2 - 1) - (2 - \lambda) =$$

$$(2 - \lambda)(\cancel{1} - 3\lambda + \lambda^2 \cancel{-1}) = \lambda(2 - \lambda)(\lambda - 3) \Rightarrow \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 0$$

The diagonal elements of Σ are therefore, from top left to bottom right: $\sqrt{3}, \sqrt{2}, 0$.

The normalized eigenvectors of $A^T A$ will be the column vectors of V:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} 2x + y = 3x \\ x + y + z = 3y \\ y + 2z = 3z \end{cases} \Rightarrow \begin{cases} y = x \\ x + z = 2y \\ y = z \end{cases} \Rightarrow \begin{cases} x = y \\ y = z \\ z = x \end{cases} \Rightarrow \begin{pmatrix} k \\ k \\ k \end{pmatrix} \; \forall k \in \mathbb{R}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} 2x + y = 2x \\ x + y + z = 2y \\ y + 2z = 2z \end{cases} \Rightarrow \begin{cases} y = 0 \\ x + z = y \\ y = 0 \end{cases} \Rightarrow \begin{cases} z = -x \\ y = 0 \\ y = 0 \end{cases} \Rightarrow \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} \; \forall k \in \mathbb{R}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} 2x + y = 0x \\ x + y + z = 0y \\ y + 2z = 0z \end{cases} \Rightarrow \begin{cases} y = -2x \\ x + z = -y \\ y = -2z \end{cases} \Rightarrow \begin{cases} y = -2x \\ z = x \\ x = z \end{cases} \Rightarrow \begin{pmatrix} k \\ -2k \\ k \end{pmatrix} \; \forall k \in \mathbb{R}$$

Out of all eigenvectors, it is necessary to pick the ones whose norm is equal to 1:

$$\| e_1 \| = \sqrt{\langle \begin{pmatrix} k \\ k \\ k \end{pmatrix}, \begin{pmatrix} k \\ k \\ k \end{pmatrix} \rangle} = \sqrt{k \cdot k + k \cdot k + k \cdot k} = \sqrt{3k^2} = \sqrt{3}\ |k| = 1 \Rightarrow k = \pm\frac{1}{\sqrt{3}}$$

$$\| e_2 \| = \sqrt{\langle \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix}, \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} \rangle} = \sqrt{k \cdot k + 0 \cdot 0 + (-k) \cdot (-k)} = \sqrt{2k^2} = \sqrt{2}\ |k| = 1 \Rightarrow k = \pm\frac{1}{\sqrt{2}}$$

$$\| e_3 \| = \sqrt{\langle \begin{pmatrix} k \\ -k \\ k \end{pmatrix}, \begin{pmatrix} k \\ -k \\ k \end{pmatrix} \rangle} = \sqrt{k \cdot k + (-2k) \cdot (-2k) + k \cdot k} = \sqrt{6k^2} = \sqrt{6}\ |k| = 1 \Rightarrow k = \pm\frac{1}{\sqrt{6}}$$

Out of the two choices for the sign, the positive one is taken for the sake of simplicity.

$A^T A$ has three eigenvalues, and only one of those is 0. Since the dimension of V is 3, the dimension of U is necessarely 2 and both eigenvectors can be computed directly from the columns of V.

$$\underline{u_1} = \frac{A\underline{e_1}}{\sigma_1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \cdot \frac{1}{\sqrt{3}} + 0 \cdot \frac{1}{\sqrt{3}} - 1 \cdot \frac{1}{\sqrt{3}} \\ 1 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ \frac{3}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{u_2} = \frac{A\underline{e_2}}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 - 1 \cdot \left(-\frac{1}{\sqrt{2}}\right) \\ 1 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 + 1 \cdot \left(-\frac{1}{\sqrt{2}}\right) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

As expected, said vectors are already normalized.

The SVD of A is therefore as follows:

$$A = U\Sigma V^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

□

3 Multivariable calculus

3.1 Partial derivatives

Let $f(x_1, x_2, ..., x_n) : \mathbb{R}^n \to \mathbb{R}$ be a function of n arguments that returns a single value. For each of its i-th argument it is possible to define the **partial derivative** with respect to x_i as:

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, ..., x_i + h, ..., x_n) - f(x_1, x_2, ..., x_i, ..., x_n)}{h}$$

That is, a partial derivative of a $f : \mathbb{R}^n \to \mathbb{R}$ function is "regular" derivative that is computed with respect to a single variable, treating all other variables as they were constants. Of course, as any regular derivative, a partial derivative may or may not exist.

A partial derivative, as a "regular" derivative, describes the rate of change of the function. The difference is that a function of n variables has n distinct directions, and its i-th partial derivative describes the rate of change of the function along the i-th axis.

> **Exercise 3.1**
>
> Compute both partial derivatives of the function $f(x, y) = \sin\left(\frac{x}{1+y}\right)$.

Proof:

$$\frac{\partial f}{\partial x} \sin\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right)\left(\frac{\partial f}{\partial x}\left(\frac{x}{1+y}\right)\right) = \cos\left(\frac{x}{1+y}\right)\left(\frac{1}{1+y}\right)\left(\frac{\partial f}{\partial x}(x)\right) = \frac{\cos\left(\frac{x}{1+y}\right)}{1+y}$$

$$\frac{\partial f}{\partial y} \sin\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right)\left(\frac{\partial f}{\partial y}\left(\frac{x}{1+y}\right)\right) = \cos\left(\frac{x}{1+y}\right)x\left(\frac{\partial f}{\partial y}\left(\frac{1}{1+y}\right)\right) = \frac{-x\cos\left(\frac{x}{1+y}\right)}{(1+y)^2}$$

□

It is possible to compute a partial derivative more than one time. That is to say, a partial derivative can be computed with respect to the result of applying a partial derivative.

For example, to denote that to the function f is first applied a partial derivative with respect to the variable x_i and then, to the result, is applied a partial derivative with respect to another variable x_j, the notation $\partial^2 f / \partial x_j \partial x_i$ is used. Notice how, in accord to the way function composition works, the order of derivation is from right to left.

Said notation can be extended to the case of computing a partial derivative for k times. When a partial derivative is computed with respect to the same variable more than once, it is possible to use the shortand notation $\partial^k f / \partial x_i^k$, meaning that a partial derivative of f is taken with respect to x_i, to which a partial derivative with respect to x_i is applied, to which a partial derivative with respect to x_i is applied, ecc...

> **Exercise 3.2**
>
> Find all second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$.

Proof: The two first partial derivatives are:

$$\frac{\partial f}{\partial x}(x^3 + x^2y^3 - 2y^2) = 3x^2 + 2xy^3 \qquad \frac{\partial f}{\partial y}(x^3 + x^2y^3 - 2y^2) = 3y^2x^2 - 4y$$

From those, it is possible to compute four second partial derivatives:

$$\frac{\partial^2 f}{\partial^2 x}(3x^2 + 2xy^3) = 6x + 2y^3 \qquad \frac{\partial^2 f}{\partial x \partial y}(3y^2x^2) - 4y = 6xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x}(3x^2 + 2xy^3) = 6xy^2 \qquad \frac{\partial^2 f}{\partial^2 y}(3y^2x^2 - 4y) = 6x^2y - 4$$

□

Notice how in Exercise 3.2, taking the partial derivative of the function first with respect to x and then with respect to y is the same as taking the derivative with respect to y and then to x. This isn't always the case, and instead happens only when the function satisfies certain conditions.

Theorem 3.3 **Schwartz's theorem**

Let f be a function defined as $f(x_1, ..., x_n) : A \subseteq \mathbb{R}^n \to \mathbb{R}$, and let $\boldsymbol{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ be a point. If some neighborhood of $\boldsymbol{p}$ is contained in A and f has continuous partial derivatives in said neighborhood, then, for any $i, j \in \{1, ..., n\}$:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} f(\boldsymbol{p}) = \frac{\partial^2 f}{\partial x_j \partial x_i} f(\boldsymbol{p})$$

Exercise 3.4

Given $f = \sin(3x + yz)$, compute $\partial^4 f / \partial z \partial y \partial^2 x$.

Proof:

$$\frac{\partial f}{\partial x}(\sin(3x + yz)) = \cos(3x + yz)\frac{\partial f}{\partial x}(3x + yz) = 3\cos(3x + yz)$$

$$\frac{\partial f}{\partial x}(3\cos(3x + yz)) = -3\sin(3x + yz)\frac{\partial f}{\partial x}(3x + yz) = -9\sin(3x + yz)$$

$$\frac{\partial f}{\partial y}(-9\sin(3x + yz)) = -9\cos(3x + yz)\frac{\partial f}{\partial y}(3x + yz) = -9z\cos(3x + yz)$$

$$\frac{\partial f}{\partial z}(-9z\cos(3x + yz)) = -9\cos(3x + yz) - 9z\frac{\partial f}{\partial z}(\cos(3x + yz)) = 9yz\sin(3x + yz) - 9\cos(3x + yz)$$

Summing up:

$$\frac{\partial^4 f}{\partial z \partial y \partial^2 x}(\sin(3x + yz)) = 9yz\sin(3x + yz) - 9\cos(3x + yz)$$

□

As stated, a partial derivative describes the rate of change of the function along the i-th axis in the n-dimensional plane. Since each axis is described by a unit vector, a derivative along the i-th axis can be conceived as the rate of change of the function along the direction described by the i-th unit vector. Since any direction can be described by a vector, it is possible to compute a derivative of a function along any arbitrary direction, not just the ones described by the n unit vectors.

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function, $\boldsymbol{v} \in \mathbb{R}^n$ a non-null vector and $\boldsymbol{c}$ a point in A. f is said to have **directional derivative** along $\boldsymbol{v}$ in $\boldsymbol{c}$, denoted as $D_{\boldsymbol{v}} f(\boldsymbol{c})$ if the following limit exists:

$$D_{\boldsymbol{v}} f(\boldsymbol{c}) = \lim_{h \to 0} \frac{f(\boldsymbol{c} + h\boldsymbol{v}) - f(\boldsymbol{c})}{h} = \lim_{h \to 0} \frac{f(c_1 + hv_1, c_2 + hv_2, ..., c_n + hv_n) - f(c_1, c_2, ..., c_n)}{h}$$

If $\boldsymbol{v}$ is the i-th unit vector, the directional derivative is just a partial derivative with respect to the i-th variable.

Let $f(x_1, ..., x_n) : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and let $\boldsymbol{p}$ be a point in A. The **gradient** of f in $\boldsymbol{p}$, denoted as $\nabla_f(\boldsymbol{p})$, is a column vector whose components are the first partial derivatives of f, arranged from the first to the n-th:

$$\nabla_f(\boldsymbol{p}) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \dots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}^T = \frac{\partial f}{\partial x_1}\hat{\imath}_1 + \frac{\partial f}{\partial x_2}\hat{\imath}_n + \dots + \frac{\partial f}{\partial x_n}\hat{\imath}_n$$

The gradient of a (n-valued) scalar function is actually a special case of a more generic matrix of a (n-valued) vectorial function, called **Jacobian matrix**:

$$\boldsymbol{J}_f(\boldsymbol{p}) = \begin{bmatrix} \dfrac{\partial \boldsymbol{f}}{\partial x_1} & \dots & \dfrac{\partial \boldsymbol{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^T_{f_1}(\boldsymbol{p}) \\ \vdots \\ \nabla^T_{f_n}(\boldsymbol{p}) \end{bmatrix}$$

The gradient vector is strongly related to maxima and minima of a function.

Theorem 3.5

If a function has a local maximum or minimum in a point and all partial derivatives in said point exist, the gradient in said point is zero.

Any point whose gradient is zero is called a **critical point** or **stationary point**. Note that, while Theorem 3.5 guarantees that maxima and minima are also critical points, the converse is not necessarely true. In particular, a critical point that is neither a maximum or a minimum is called a **saddle point**.

Similarly to the gradient, the second partial derivatives of f can be arranged in a matrix, called **Hessian matrix** and denoted as H_f:

$$H_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial^2 x_1} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dots & \dfrac{\partial^2 f}{\partial x_n \partial x_1} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial^2 x_2} & \dots & \dfrac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_n} & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} & \dots & \dfrac{\partial^2 f}{\partial^2 x_n} \end{bmatrix}$$

The Hessian matrix is strongly related the natures of a critical point of a function.

Theorem 3.6 — Second derivative test for functions having more than one variable

Consider the function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, twice (or more) differentiable. Let $\boldsymbol{p}$ be a critical point of f, and let $H_f(\boldsymbol{p})$ be the Hessian matrix of f at $\boldsymbol{p}$. Then:

- If $H_f(\boldsymbol{p})$ is positive definite, $\boldsymbol{p}$ is a local minimum;
- If $H_f(\boldsymbol{p})$ is negative definite, $\boldsymbol{p}$ is a local maximum;
- If $H_f(\boldsymbol{p})$ has at least a positive eigenvalue, a negative eigenvalue and no eigenvalue is zero, $\boldsymbol{p}$ is a saddle point;
- If none of the above is true, the test is inconclusive.