Contents

1. Linear Algebra

1.1. Vector Spaces

Let V be a set, whose elements are called **vectors**. A vector $\underline{v}$ is denoted as $\underline{v} = (v_1, v_2, ..., v_n)$, where each v_i with $1 \leq i \leq n$ is called the i-th **component** of $\underline{v}$.
Let $+$ be an operation on such set, a *sum* of vectors, that has two vectors as arguments and returns another vector. That is, foreach $\left(\underline{x}, \underline{y}\right) \in V \times V$ there exists a vector $\underline{v} \in V$ such that $\underline{x} + \underline{y} = \underline{v}$.
Let $\cdot$ be another operation, a *product* between a vector and a real number, that has a real number and a vector as argument and returns another vector. That is, foreach $\lambda \in \mathbb{R}$ and $\underline{v} \in V$ there exists a vector $\underline{w} \in V$ such that $\lambda \cdot \underline{v} = \underline{w}$.
Suppose those operations possess the following properties:

- $(V, +)$ is an Abelian group;
- The product has the distributive property, such that for every $\lambda \in \mathbb{R}$ and for every $\underline{x}, \underline{y} \in V$ it is true that $\lambda \cdot \left(\underline{x} + \underline{y}\right) = \lambda \cdot \underline{x} + \lambda \cdot \underline{y}$;
- The product has the associative property, such that for every $\lambda, \mu \in \mathbb{R}$ and for every $\underline{x} \in V$ it is true that $(\lambda + \mu) \cdot \underline{x} = \lambda \cdot \underline{x} + \mu \cdot \underline{x}$;
- For every vector $\underline{v} \in V$, it is true that $1 \cdot \underline{v} = \underline{v}$.

If that is the case, the set V is called **vector space**. It should be noted that it does not matter what the elements of a vector space actually are (be they numbers, functions, polynomals, etcetera); as long as the aforementioned properties hold for the two operations, such set shares all of the properties that a vector space possesses.
For the sake of readability, the product between a real number and a vector is often represented without the dot. That is to say, the expressions $\lambda \cdot \underline{x}$ and $\lambda\underline{x}$ have the same meaning.

Exercise 1.1.1: Denote as $\mathbb{R}^n$ the set containing all vectors of real components[1] in the n-dimensional plane. Prove that $\mathbb{R}^n$ constitutes a vector space.

Solution: It is possible to define both a sum between two vectors in the n-dimensional plane and a product between a vector in the n-dimensional space and a real number. To sum two vectors in the n-dimensional space, it suffices to sum each component with each component. To multiply a vector in the n-dimensional space with a real number it suffices to multiply each component by that number:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} \qquad \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

Both operations obey the properties stated:

- $(\mathbb{R}^n, +)$ constitutes an Abelian group. Infact:
  - The sum has the associative property:

$$\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 + u_1 \\ v_2 + w_2 + u_2 \\ \vdots \\ v_n + w_n + u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 + u_1 \\ w_2 + u_2 \\ \vdots \\ w_n + u_n \end{pmatrix}$$

  - There exists an identity element, in the form of the vector whose components are all zero:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 + 0 \\ v_2 + 0 \\ \vdots \\ v_n + 0 \end{pmatrix} = \begin{pmatrix} 0 + v_1 \\ 0 + v_2 \\ \vdots \\ 0 + v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

[1] This is a misnomer.

- Each vector in the n-dimensional space has an inverse element with respect to the sum, that is the same vector multiplied by -1:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + (-1)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix} = \begin{pmatrix} v_1 + (-v_1) \\ v_2 + (-v_2) \\ \vdots \\ v_n + (-v_n) \end{pmatrix} = \begin{pmatrix} -v_1 + v_1 \\ -v_2 + v_2 \\ \vdots \\ -v_n + v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- The sum has the commutative property:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \\ \vdots \\ w_n + v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- The product has the associative property:

$$(\lambda + \mu)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\lambda + \mu)v_1 \\ (\lambda + \mu)v_2 \\ \vdots \\ (\lambda + \mu)v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 + \mu v_1 \\ \lambda v_2 + \mu v_2 \\ \vdots \\ \lambda v_n + \mu v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix} + \begin{pmatrix} \mu v_1 \\ \mu v_2 \\ \vdots \\ \mu v_n \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \mu\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- The product has the distributive property:

$$\lambda\left(\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}\right) = \lambda\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} \lambda(v_1 + w_1) \\ \lambda(v_2 + w_2) \\ \vdots \\ \lambda(v_n + w_n) \end{pmatrix} = \begin{pmatrix} \lambda v_1 + \lambda w_1 \\ \lambda v_2 + \lambda w_2 \\ \vdots \\ \lambda v_n + \lambda w_n \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \lambda\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

- Multiplying a vector by the number 1 leaves the vector unchanged:

$$1\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1v_1 \\ 1v_2 \\ \vdots \\ 1v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

□

Exercise 1.1.2: Denote as $\mathbb{P}_n$ the set containing all polynomials with real coefficients and degree less than or equal to n. Prove that $\mathbb{P}_n$ constitutes a vector space.

Solution: It is possible to define both a sum between two polynomials with real coefficients and degree $\leq n$ and a product between a polynomial with real coefficients and degree $\leq n$ and a real number. To sum two such polynomials it suffices to sum the coefficients of their monomials having the same degree:

$$\begin{gathered}(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) + (b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0) = \\ a_n x^n + a_{n-1} x^{n-1} + \ldots + (a_m + b_m) x^m + (a_{m-1} + b_{m-1}) x^{m-1} + \ldots + (a_1 + b_1) x + (a_0 + b_0)\end{gathered}$$

To multiply a polynomial with real coefficients and degree $\leq n$ with a real number it suffices to multiply each coefficient of its monomials by such number:

$$\lambda(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) = (\lambda a_n) x^n + (\lambda a_{n-1}) x^{n-1} + \ldots + (\lambda a_1) x + (\lambda a_0)$$

Both operations satisfy the properties required.

□

Given a vector space V, a set W is said to be a **subspace** of V if it's a subset of V and it's itself a vector space (with respect to the same operations defined for V).

Exercise 1.1.3: Consider the vector space $\mathbb{R}^3$. Prove that the set W_1 is a subspace of $\mathbb{R}^3$ while W_2 is not.

$$W_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3, x_1 + x_2 = 0 \right\} \qquad W_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3, x_2 = 2x_3 + 1 \right\}$$

Solution: The first set is a subspace of $\mathbb{R}^3$ because it is a subset of V and is algebraically closed:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ -y_1 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ -x_1 - y_1 \\ x_3 + y_3 \end{pmatrix} \Rightarrow x_2 + y_2 = -x_1 - y_1 \Rightarrow x_2 + y_2 + (x_1 + y_1) = 0$$

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ -x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ -\lambda x_1 \\ \lambda x_3 \end{pmatrix} \Rightarrow \lambda x_2 = -\lambda x_1 \Rightarrow \lambda(x_1 + x_2) = 0$$

The second one, on the other hand, is not:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_3 + 1 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ 2y_3 + 1 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 2x_3 + 2y_3 + 2 \\ x_3 + y_3 \end{pmatrix} \Rightarrow 2x_3 + 2y_3 + 2 \neq 2(x_3 + y_3) + 1$$

$\square$

1.2. Bases and Dimension

Let $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ be a set of n vectors of a vector space V, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n real numbers (not necessarely distinct). Every summation defined as such:

$$\sum_{i=1}^{n} \lambda_i \underline{v_i} = \lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + \dots + \lambda_i \underline{v_i} + \dots + \lambda_n \underline{v_n}$$

Is called a **linear combination** of the vectors $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$, with $\lambda_1, \lambda_2, \dots, \lambda_n$ as **coefficients**.
A set of vectors $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ is said to be **linearly independent** if the only linear combination of such vectors that equals the null vector is the one that has 0 as every coefficient. If there exists a linear combination of such vectors that is equal to the null vector and that has at least a non-zero coefficient, those vectors are said to be **linearly independent**.

Theorem 1.2.1: Let $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ be a set of n vectors of a vector space V. If those vectors are linearly dependent, there exists at least one vector of such set that can be expressed as a linear combination of the remaining vectors, and vice versa.

Proof: If such set of vectors is linearly dependent, there must exist a linear combination of the set that equals the null vector, be it:

$$\sum_{i=1}^{n} \lambda_i \underline{v_i} = \lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + \dots + \lambda_j \underline{v_j} + \dots + \lambda_n \underline{v_n} = \underline{0}$$

Where there's at least a non-zero coefficient. Let's assume, without loss of generality, that the j-th coefficient is non-zero (whether such coefficient is the only one to be non-zero is irrelevant). The product of such coefficient with the corresponding vector can be moved to the right side of the equation like so:

$$\lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + \dots + \lambda_n \underline{v_n} = -\lambda_j \underline{v_j}$$

Dividing both sides by $-\lambda_j$ gives:

$$-\frac{\lambda_1}{\lambda_j}\underline{v_1} - \frac{\lambda_2}{\lambda_j}\underline{v_2} - \dots - \frac{\lambda_n}{\lambda_j}\underline{v_n} = \underline{v_j}$$

Each $-\frac{\lambda_i}{\lambda_j}$ is itself a real number, and therefore the expression above is a linear combination that is equal to a vector of the set.
On the other hand, assume that the j-th vector of the set is equal to a linear combination of the remaining vectors like so:

$$\lambda_1\underline{v_1} + \lambda_2\underline{v_2} + \dots + \lambda_n\underline{v_n} = \underline{v_j}$$

Moving $\underline{v_j}$ to the left gives:

$$\lambda_1\underline{v_1} + \lambda_2\underline{v_2} + \dots + (-1)\underline{v_j} + \dots + \lambda_n\underline{v_n} = \underline{0}$$

Since -1 is a real number, the expression on the left side of the equation is indeed a linear combination of the whole set, that is equal to the null vector. □

Exercise 1.2.1: Consider the vector space $\mathbb{R}^2$. Check if the vectors $\underline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent or linearly dependent.

Solution: Consider such linear combination:

$$\lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To test whether $\underline{v_1}$ and $\underline{v_2}$ are linearly dependent or independent, it suffices to solve this system of equations:

$$\lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2\lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2\lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

Since the only solution to such system is $\lambda_1 = 0, \lambda_2 = 0$, $\underline{v_1}$ and $\underline{v_2}$ are linearly independent. □

Exercise 1.2.2: Consider the vector space $\mathbb{R}^2$. Check if the vectors $\underline{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v_3} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ are linearly independent or linearly dependent.

Solution: Consider such linear combination:

$$\lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_3\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To test whether $\underline{v_1}$, $\underline{v_2}$ and $\underline{v_3}$ are linearly dependent or independent, it suffices to solve this system of equations:

$$\lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_3\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 + \lambda_2 + 2\lambda_3 = 0 \\ \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -2\lambda_3 \\ \lambda_2 = 0 \end{cases}$$

Since there are infinite solutions to such system, including non-zero solutions, $\underline{v_1}$, $\underline{v_2}$ and $\underline{v_3}$ are linearly dependent. For example, setting $\lambda_1 = 1$ results in:

$$1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which is, indeed, a correct identity. □

A set of vectors $S = \{\underline{s_1}, ..., \underline{s_n}\}$ of a vector space V is said to **generate** V if every vector of V can be written as a linear combination of the vectors in S. That is to say, S generates V if for every $\underline{v} \in V$ there exist a set of coefficients $\lambda_1, ..., \lambda_n$ such that:

$$\underline{v} = \sum_{i=1}^{n} \lambda_i \underline{s_i} = \lambda_1 \underline{s_1} + ... + \lambda_n \underline{s_n}$$

Theorem 1.2.2: Let S be a set of vectors of a vector space V that can generate V. Let $\underline{w} \in V$ be a random vector of V. The set of vectors $S \cup \{\underline{w}\}$ is linearly dependent.

Proof: If S can generate V and $\underline{w}$ belongs to V, there exists a linear combination of the vectors in S such that:

$$\underline{w} = \sum_{i=1}^{n} \lambda_i \underline{s_i} = \lambda_1 \underline{s_1} + ... + \lambda_n \underline{s_n}$$

Moving $\underline{w}$ to the right side of the equation gives:

$$\underline{0} = (-1)\underline{w} + \lambda_1 \underline{s_1} + ... + \lambda_n \underline{s_n}$$

The expression on the right side of the equation is indeed a linear combination of $S \cup \{\underline{w}\}$, that is equal to the null vector. Since at least -1 is a non zero coefficient, such set is linearly dependent. □

A set of vectors that can generate a vector space and is itself linearly independent is called a **basis** for such vector space. The cardinality of a basis is called the **dimension** of the corresponding vector space.

Theorem 1.2.3: A basis of a vector space has the minimum cardinality out of every set of vectors that can generate it. In other words, if a basis of a vector space has cardinality n, at least n vectors are needed to generate such space.

Exercise 1.2.3: Consider the vector space $\mathbb{P}_2$. Knowing that the sets $\mathcal{B}_1 = \{1, x, x^2\}$ and $\mathcal{B}_2 = \{(x+1), (x-1), x^2\}$ are both bases for $\mathbb{P}_2$, write the polynomial $p(x) = 3x^2 + 2x - 5$ as a linear combination of each.

Solution: It's trivial to see that, for the first basis, such linear combination is $p(x)$ itself:

$$\lambda_0 1 + \lambda_1 x + \lambda_2 x^2 = -5 + 2x + 3x^2 \Rightarrow \begin{cases} \lambda_0 = -5 \\ \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases}$$

Regarding the second basis, it can be rewritten as:

$$\lambda_0(x+1) + \lambda_1(x-1) + \lambda_2 x^2 \Rightarrow \lambda_0 x + \lambda_0 + \lambda_1 x - \lambda_1 + \lambda_2 x^2 \Rightarrow (\lambda_0 - \lambda_1) + (\lambda_0 + \lambda_1)x + \lambda_2 x^2$$

Equating it term by term:

$$(\lambda_0 - \lambda_1) + (\lambda_0 + \lambda_1)x + \lambda_2 x^2 = -5 + 2x + 3x^2 \Rightarrow \begin{cases} \lambda_0 - \lambda_1 = -5 \\ \lambda_0 + \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases} \Rightarrow \begin{cases} \lambda_0 = -\frac{3}{2} \\ \lambda_1 = \frac{7}{2} \\ \lambda_2 = 3 \end{cases}$$

Therefore:

$$3x^2 + 2x - 5 = -\frac{3}{2}(x+1) + \frac{7}{2}(x-1) + 3x^2$$

□

The basis of a vector space that renders calculations the most "comfortable" is called the **canonical basis** for such vector space. Such basis is different from vector space to vector space.

Exercise 1.2.4: Determine the dimension of $\mathbb{R}^n$

Solution: Consider any n-dimensional vector of coordinates $a_1, a_2, ..., a_n$. It's easy to see that such vector is equal to the following linear combination:

$$\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix} + \dots + a_n \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}$$

A set containing such vectors in linearly independent. Infact:

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 + 0 + \dots + 0 = 0 \\ 0 + \lambda_2 + \dots + 0 = 0 \\ \vdots \\ 0 + 0 + \dots + \lambda_n = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \vdots \\ \lambda_n = 0 \end{cases}$$

This set of vectors is linearly independent and can generate $\mathbb{R}^n$, therefore it's a basis for $\mathbb{R}^n$. The dimension of $\mathbb{R}^n$ is then n, since such set has cardinality n. In particular, this specific basis is the canonical basis for $\mathbb{R}^n$. □

Exercise 1.2.5: Determine the dimension of $\mathbb{P}_n$

Solution: Consider any polynomial of degree at most n having real coefficients:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Each monomial of such polynomial are itself polynomials of degree at most n having real coefficients. Therefore, the polynomial itself can be seen as a linear combination of the polynomials $\{x^n, x^{n-1}, ..., x^1, x^0\}$ with coefficients $a_n, a_{n-1}, ..., a_1, a_0$.
Such set of vectors is linearly independent. Infact:
This set of vectors is linearly independent and can generate $\mathbb{P}_n$, therefore it's a basis for $\mathbb{P}_n$. The dimension of $\mathbb{P}_n$ is then $n + 1$, since such set has cardinality $n + 1$. In particular, this specific basis is the canonical basis for $\mathbb{P}_n$. □

1.3. Linear Transformations

A transformation $\phi : V \mapsto W$, with both V and W being vector spaces, is called a **linear transformation** if and only if:

$$\phi\left(\underline{v_1} + \underline{v_2}\right) = \phi\left(\underline{v_1}\right) + \phi\left(\underline{v_2}\right) \ \forall \underline{v_1}, \underline{v_2} \in V \qquad \phi(\lambda \underline{v}) = \lambda \phi(\underline{v}) \ \forall \underline{v} \in V, \lambda \in \mathbb{R}$$

In particular, if $V = W$, the transformation ϕ is said to be an **endomorphism**.

Exercise 1.3.1: Consider the vector space $\mathbb{R}$ (that is, the set of real numbers). Check whether the transformations $\phi_1(x) = 2x$ and $\phi_2(x) = x + 1$ are linear or not.

Solution:

- The transformation $\phi_1(x) = 2x$ is linear. Infact, given two real numbers a and b, is indeed true that $2(a + b) = 2a + 2b$, since the product between real numbers has the distributive property. Similarly, given a real number a and a real number λ, it is true that $2(\lambda a) = 2\lambda a$, since the product between real numbers has the associative property;
- The transformation $\phi_2(x) = x + 1$ is not linear. Given two real numbers a and b, it results in $\phi_2(a + b) = (a + b) + 1 = a + b + 1$, while $\phi_2(a) + \phi_2(b) = a + 1 + b + 1 = a + b + 2$.

□

It can be shown that a linear transformation is equivalent to a manipulation of matrices.
Let $\phi : V \mapsto W$ be a linear transformation between two vector space V and W. Let $B = \left\{\underline{b_1}, ..., \underline{b_n}\right\}$ be a basis for V and $C = \left\{\underline{c_1}, ..., \underline{c_m}\right\}$ a basis for W. Each vector $\underline{x} \in V$ can be written as a linear combination of the vectors of B:

$$\underline{x} = \sum_{i=1}^{n} \lambda_i \underline{b_i}$$

Applying ϕ to $\underline{x}$ gives:

$$\phi(\underline{x}) = \phi\left(\sum_{i=1}^{n} \lambda_i \underline{b_i}\right) = \sum_{i=1}^{n} \phi\left(\lambda_i \underline{b_i}\right) = \sum_{i=1}^{n} \lambda_i \phi\left(\underline{b_i}\right)$$

The two rightmost equalities stem from the fact that ϕ is linear.
Each $\phi(b_i)$ is a vector of W, since it's the result of applying ϕ to an vector of V. This means that each $\phi(b_i)$ can itself be written as a linear combination of elements of C:

$$\phi(b_i) = \sum_{j=1}^{m} \gamma_{j,i} c_j$$

Substituting it back in the previous expression gives:

$$\phi(\underline{x}) = \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{m} \gamma_{j,i} c_j\right) = \sum_{i,j=1}^{n,m} \lambda_i \gamma_{j,i} c_j$$

This means that, fixed a given basis B, to know all the relevant information regarding a vector $\underline{x}$ of V it suffices to "store" the λ coefficients of its linear combination with respect to B in a (column) vector.
In a similar fashion, to know all the relevant information of its image $\phi(\underline{x})$ it suffices to store the $\sum_{j=1}^{m} \lambda_i \gamma_{j,i}$ coefficients of its linear combination with respect to C in a (column) vector:

$$\underline{x} \Longleftrightarrow \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \qquad \phi(\underline{x}) \Longleftrightarrow \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

Where, for clarity, each $\sum_{j=1}^{m} \lambda_i \gamma_{j,i}$ has been written simply as μ_i.
It is then possible to describe the application of the transformation ϕ as the following product of matrices:

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m,1} & \gamma_{m,2} & \cdots & \gamma_{m,n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

Exercise 1.3.2: Consider the linear application $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ defined below. Express it as a matrix multiplication with respect to the two bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ y + z \end{pmatrix} \qquad \mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \right\} \qquad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

Solution: The first step is to express the vectors of $\mathcal{B}_1$ for $\mathbb{R}^3$ evaluated in T as linear combinations of the vectors of $\mathcal{B}_2$ for $\mathbb{R}^2$:

$$T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \lambda_{1,1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_{2,1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad T\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \lambda_{1,2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_{2,2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad T\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \lambda_{1,3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_{2,3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

□