# Inverted Kalman Pendulum

## Stabilizing Inverted Pendulum with Observer-based Strategy

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**Control System Design**

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# Chapter 1

# Introduction

The inverted pendulum is a classic problem in control engineering. The rotation point of a rigid rod pendulum is connected to a horizontally moving cart. The objective of the lab sessions is to stabilize the pendulum such that its center of mass is above the rotation point.
The basic requirement of the lab is that the inverted pendulum needs to be stabilized around its point of equilibrium. The more advanced requirements are to use an observer instead of velocity measurements and to design a swing up strategy to lift the pendulum up.

## 1.1 Plant description

The plant consist of a rigid rod pendulum connected to a horizontally moving cart. The setup can be seen in Figure 1.1.

There are four sensors: two position sensors and two velocity sensors. One of the two position sensors measures the cart position and the other one measure the angle of the pendulum. The two velocity sensors are also one for the cart, and one for the pendulum.
To get the values from the sensors, we can connect the outputs to a computer.

The system is actuated by a current driven DC motor. The voltage can be tuned manually or with a computer (see Figure 1.2).

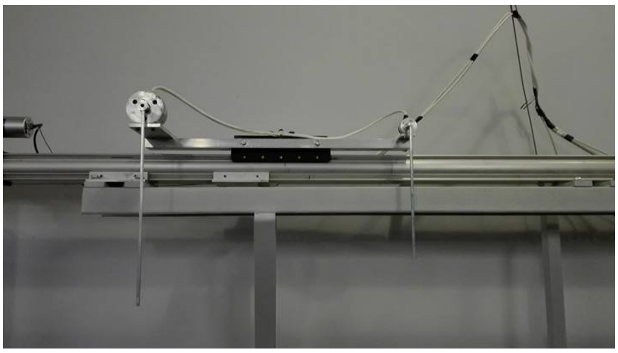


Figure 1.1: The pendulum (left side) connected to a cart

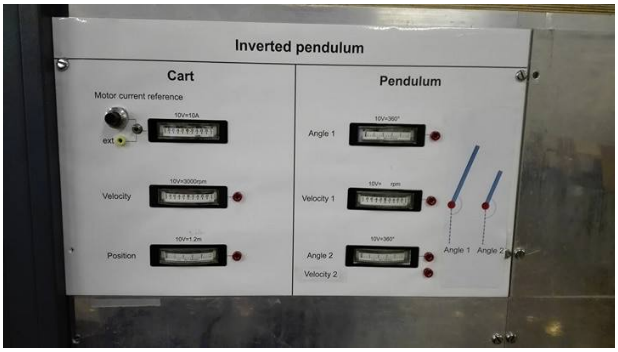



Figure 1.2: The plugs to connect the output of the sensors to the computer and to tune the input

# Chapter 2

# System Identification

## 2.1 DC motor characteristics

In order to stabilize the pendulum in the upright position, the identification of the actuator characteristics needs to be done to define its input range. Hence the saturation points and dead zone of the DC motor have been found and are shown in Table 2.1.

| | |
|---|---|
| Dead zone (+) | 0.8 A |
| Dead zone (-) | -0.7 A |
| Saturation (+) | 1.7 A |
| Saturation (-) | -1.6 A |

Table 2.1: DC motor characteristics

Once the actuator characteristics are known, different experiments can be done on the plant to identify it with the analysis of its transfer functions. But before finding the transfer functions, the first step is to modelize the system with a gray box approach.

## 2.2 Differential equations

In this section, a mechanical study of the system's motion will be done. The coordinates and forces used for the description of the system are depicted in Figure 5.1. The corresponding Lagrangian equations of motion are the following:

$$\left\{ \begin{array}{lcl} \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} & = & Gu - K_x\dot{x} \\ \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} & = & -K_\theta\dot{\theta} \end{array} \right. \tag{2.1}$$

Where:

- $Gu$ is the force applied on the cart, which is supposed to be directly proportional to the current injected in the DC motor $u$ ($G \in \mathbb{R}$)
- $K_x$ is the friction coefficient of the cart on the $\vec{x}$ direction

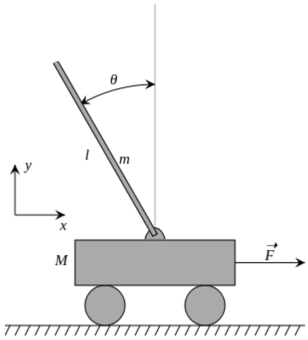



Figure 2.1: Inverted Pendulum

- $K_\theta$ is the friction coefficient of the pendulum on the $\vec{\theta}$ direction

The Lagrange variable is determined by computing the potential energy $V$ and the kinetic energy $T$ of the system:

$$V = mglcos(\theta) \tag{2.2}$$

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(ml^2+I)\dot{\theta}^2 - ml\dot{\theta}\dot{x}cos(\theta) \tag{2.3}$$

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 - ml\dot{\theta}\dot{x}cos(\theta) + \frac{1}{2}(ml^2+I)\dot{\theta}^2 - mglcos(\theta) \tag{2.4}$$

Where:

- $l$ is the half length of the pendulum
- $g$ is the gravitational acceleration
- $I$ is the moment of inertia of the pendulum
- $m$ is the mass of the pendulum
- $M$ is the mass of the cart

Injecting the expression of $L$ in equation (2.1) leads to the following differential equation of motion:

$$\begin{cases} \ddot{x} = \frac{1}{M+m}[Gu - K_x\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta] \\ \ddot{\theta} = \frac{1}{ml^2+I}[-K_\theta\dot{\theta} + mgl\sin\theta + ml\ddot{x}\cos\theta] \end{cases} \tag{2.5}$$

## 2.3 State Space Model

As the mass of the pendulum $m$ is much smaller than the mass of the cart $M$ and as the system is meant to be controlled around its upright equilibrium position ($\theta_{up} = 0$), one can suppose that the pendulum does not affect the motion of the cart around that target position. Under this assumption, the terms proportional to $\dot{\theta}^2$ and $\ddot{\theta}$ in the first line of equation (2.5) can be neglected, which leads to the following equation for the cart's motion:

$$\ddot{x} = \frac{1}{M}[Gu - K_x \dot{x}] \tag{2.6}$$

Moreover, around the upright equilibrium point $\theta_{up} = 0$, the second line of equation (2.5) can be linearized using the first order Taylor approximation:

$$\begin{cases} \sin\theta \approx \theta \\ \cos\theta \approx 1 \end{cases} \tag{2.7}$$

The linearized expression of the motion of the pendulum around the upright position is thus deduced:

$$\ddot{\tilde{\theta}} = \frac{1}{ml^2 + I}[-K_\theta \dot{\tilde{\theta}} + mgl\tilde{\theta} + ml\ddot{x}] \tag{2.8}$$

Where $\tilde{\theta}$ is defined as the gap between $\theta$ and the equilibrium point $\theta_{eq}$, with $\theta_{eq} = \theta_{up}$ in this case:

$$\tilde{\theta} = \theta - \theta_{up} = \theta \tag{2.9}$$

These combined assumptions lead to the linear system (2.10). Therefore one can consider that the motion of the inverted pendulum is described by this linear differential law around its equilibrium points:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\tilde{\theta}} \\ \ddot{\tilde{\theta}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & s(\frac{-a}{l}) & s(\frac{g}{l}) & -d \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \\ \tilde{\theta} \\ \dot{\tilde{\theta}} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ b \\ 0 \\ s(\frac{b}{l}) \end{bmatrix}}_{B} u(t) \tag{2.10}$$

Where:

- $a \approx \frac{K_x}{M}$
- $b \approx \frac{G}{M}$
- $d = \frac{K_\theta}{ml^2+I}$

Note that $s$ is a switch variable whose value changes depending on the equilibrium point around which the system is linearized (upright and down position):

$$s = \begin{cases} 1, & \text{if } \theta_{eq} = \theta_{up} = 0 \\ -1, & \text{if } \theta_{eq} = \theta_{down} = \pi \end{cases}$$

This switch will be used in the system identification as it allows to directly deduce the state space model of the system around the upright equilibrium position from experiments made around the down equilibrium position.
From an experimental point of view, the parameters of the above state space model will be

considered as uncorrelated with respect to each other. The state space model with the parameters of the system then becomes:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b & -c & -d \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ e \\ 0 \\ f \end{bmatrix}}_{B} u(t) \tag{2.11}$$

From this state space model, the transfer function of the system can be computed as the output of the system and is nothing but the 4 states denoted $X$. Mathematically, this is expressed as follows:

$$Y = I_n X \tag{2.12}$$

The system's transfer function is therefore a 4x1 matrix $G(s)$ which links each state of Y(s) with the input $U(s)$ in the Laplace domain:

$$G(s) = \frac{Y(s)}{U(s)} = I_n(sI_n - A)^{-1}B \tag{2.13}$$

Where:

$$Y(s) = \begin{bmatrix} X(s) \\ V(s) \\ \Theta(s) \\ \Omega(s) \end{bmatrix} = \begin{bmatrix} \frac{V(s)}{s} \\ V(s) \\ \Theta(s) \\ s\Theta(s) \end{bmatrix} \tag{2.14}$$

As the position of the cart $X(s)$ is found by simply integrating its velocity $V(s)$ and as the angular velocity of the pendulum $\Omega(s)$ is only the derivative of its angular position $\Theta(s)$, the system's parameters can be determined using only 2 transfer functions from $G(s)$, namely $V(s)$ and $\Theta(s)$:

$$\frac{V}{U} = \frac{e}{s+a} \qquad\qquad \frac{\Theta}{U} = \frac{f(s + \frac{b*e+f*a}{f})}{(s+a)(s^2+d*s+c)}$$

## 2.4 Identification of the parameters

In order to identify the A and B matrices of our state space model, experiments have to be done to find the good transfer functions that will allow to determine the parameters. The expected shape of these transfer functions is important for the identification process. Indeed, a Matlab script is provided to identify the system but the number of poles and zeros must be specified.

### 2.4.1 Experiments

As mention before, two transfer functions will allow to determine all the parameters of the state space representation. The transfer function of the velocity must have 1 pole and no zeros. For the transfer function of the angle, 1 zero and 3 poles are expected.

Since exciting the system around the unstable equilibrium point when the pendulum is in the upright position is not possible, all the experiments to identify the system needs to be done when

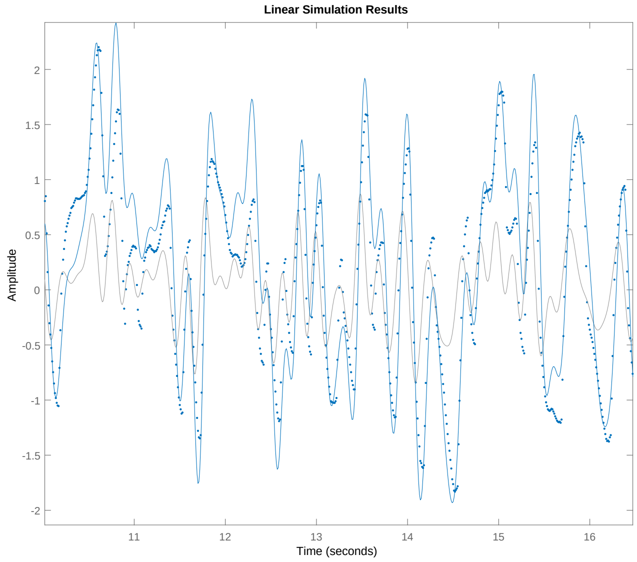



Figure 2.2: Approximated transfer function of the velocity (**gray curve**: input signal, **blue dot**: real data, **blue curve**: approximation)

the pendulum is around the stable equilibrium point.

The system has been identified by using a tuned input signal in order to obtain transfer functions closed to the situation when the cart is trying to stabilize the pendulum. The signal applied on the plant is such that the cart moves right and left at different frequencies and swings therefore the pendulum also at different frequencies. The purpose of this experiment is to excite the system in a wide frequency range in order to have a accurate model of the system around the equilibrium point.

Once the experiment done and the data acquired, the provided identification code may need to be executed several times to obtain approximate transfer functions that fit as much as possible the real data. Furthermore, the two transfer functions must have a common pole which corresponds to the parameter $a$ in the $A$ matrix of the state space model.
The input signal applied on the plant, the output measures and the output from the estimate transfer functions for the same input are shown in the Figure 2.2 and 2.3.

The transfer functions obtained are the following ones:

$$\frac{V}{U} = \frac{51.286}{s + 10.41} \qquad \frac{\Theta}{U} = \frac{21.023(s - 0.1069)}{(s + 10.94)(s^2 + 0.6249s + 36.74)}$$

Two values for the parameter $a$ are possible. As these are very close each other, the smallest one

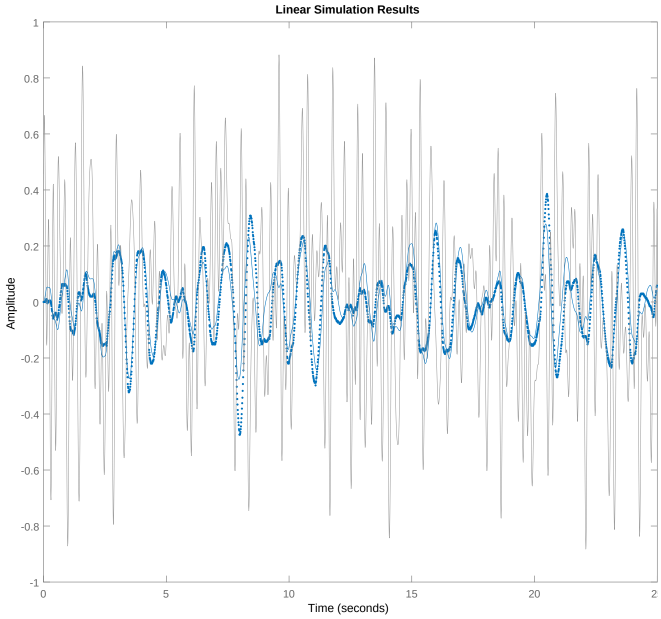



Figure 2.3: Approximated transfer function of the angle (gray curve: input signal, blue dot: real data, blue curve: approximation)

has been chosen. Hence, the state space representation can be numerically defined as follow:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -10.41 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4.3111 & -36.74 & -0.6249 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 51.2860 \\ 0 \\ 21.0230 \end{bmatrix} u(t) \tag{2.15}$$

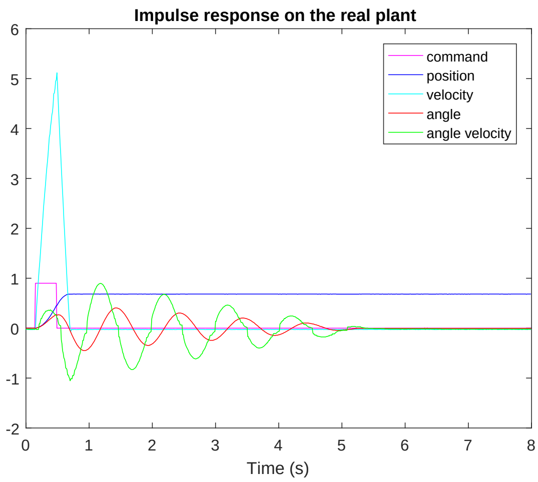
As mentioned above, these matrix $A$ and $B$ correspond to the linearized system around the stable equilibrium point. The state space model around the unstable equilibrium point is given by using the switch "s" mentioned in the system (2.10):

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -10.41 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4.3111 & 36.74 & -0.6249 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 51.2860 \\ 0 \\ -21.0230 \end{bmatrix} u(t) \tag{2.16}$$

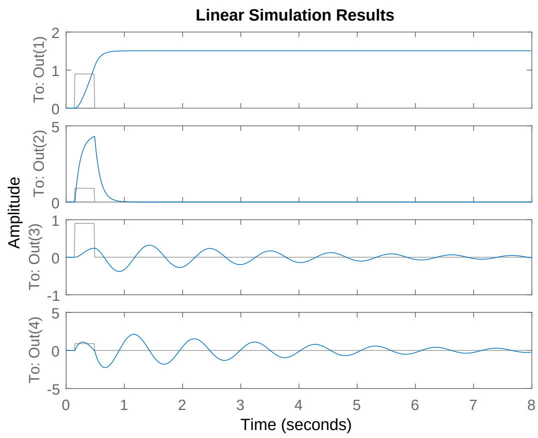
The eigenvalues of the $A$ matrix can be easily computed with *Matlab* by using the *eig* command. As expected, these are not all negative.

## 2.5 Validation of the Model

Once the identification of the system's parameters is complete, a validation of the model can be done. To do this, one can simply compare the system's impulse response with the simulated one around the pendulum's down equilibrium point (see Figure 2.4). The results of this experiment show that the evolution of the cart's position is quite different than expected. This can be explained by the fact that the identification of the system has been done in different experimental conditions (see Figure 2.2). However an inaccurate estimation of the cart's position is not very important since this state does not influence the dynamic of the pendulum.




(a) Real experiment




(b) Simulation of the experiment

Figure 2.4: Validation of the state space model

# Chapter 3

# Control Design

## 3.1 Feedback controller

The identification of the system is a necessary step to design a control law which stabilizes the pendulum around the unstable equilibrium point. The implementation of a feedback controller will allow to have a closed loop system whose all of its eigenvalues are negative. The expression of the feedback controller is:

$$u(t) = -Kx(t) \tag{3.1}$$

where $x(t)$ is the state vector of the system (position, velocity, angle, angular velocity) and $K$ is a matrix. When the control law will be implemented on the real plant, the reference will be added such that the input $u$ becomes: $u = K(reference - x)$. But in this section, the reference will be equal to 0.
Then the state space representation of the system becomes:

$$\dot{x(t)} = Ax(t) + B(-Kx(t)) \tag{3.2}$$

and can be rewritten as follow:

$$\dot{x(t)} = (A - BK)x(t) \tag{3.3}$$

The stability of the system can be reached if the value of the gain $K$ is such that all eigenvalues of the matrix $(A - BK)$ have a negative real part. The block diagram expected is shown in Figure 3.1.

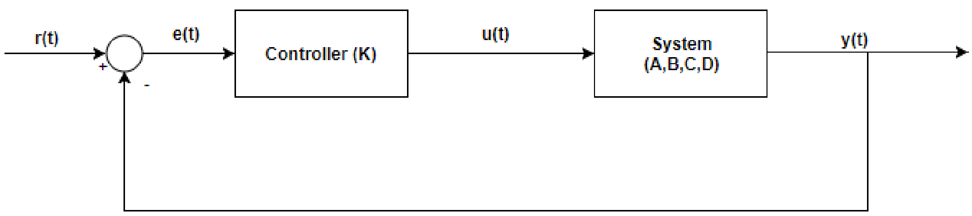



Figure 3.1: Block diagram of the closed loop

To determine the gain $K$ in order to have only negative eigenvalues, one possibility is to use the *place* command in Matlab that gives the gain $K$ by specifying the wanted eigenvalues in the close loop. An other possibility is to use a optimal regulator like a linear-quadratic regulator. The second option is the one that we chose.

## 3.2 Linear-quadratic regulator (LQR)

A easy way to find a good gain $K$ is to use a linear-quadratic regulator. Fortunately, Matlab provide a *lqr* command which gives the optimal gain $K$ for a specified state space model and constraints on the controller.

The idea behind this LQR algorithm is to calculate the gain $K$ for the control law $u = -Kx$ that minimizes the quadratic cost function $J$ subject to the system dynamics $\dot{x} = Ax + Bu$:

$$J(u) = \int_0^\infty (x^T Q x + u^T R u) dt \tag{3.4}$$

Where $Q$ and $R$ are tuning parameters and $R = R' > 0$, $Q = Q' > 0$.

The $Q$ and $R$ factors are matrices that can be seen as weighting factors. They respectively represent the energy of the error and the energy of the control action. This $Q$ matrix allow to quickly design a controller with the desired priority on certain states of the system to be stable. Since the plant has only one input, the $R$ matrix is only a scalar and has been set to 1. The $Q$ matrix has been defined by giving a high importance to the angle and angular velocity to be the closest possible to the equilibrium state. The weighting factors corresponding to these states has been respectively set to 30 and 100. The other states, the position and the velocity, have been both set to 1 meaning that these states can deviate a little bit from the equilibrium state. The $Q$ and $R$ matrix are thus the following:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix} \qquad R = 1$$

The LQR command can compute a gain $K$ by specifying the $A$, $B$, $Q$ and $R$ matrices: $K = lqr(A, B, Q, R)$. The obtained matrix $K$ for the controller is:

$$K = \begin{bmatrix} -1 & -1.8497 & -66.8128 & -14.5675 \end{bmatrix} \tag{3.5}$$

## 3.3 Simulation

Before implementing the regulator on the real plant, it is necessary to first simulate our system with *Simulink* in order to check if the control law works as expected. The Simulink model is shown in Figure 3.2.

The simulation of the regulation (Figure 3.3) has been done with the initial conditions when the pendulum is in the upright equilibrium position but with a initial angular velocity: $X_0 = [0; 0; 0; 1]$.

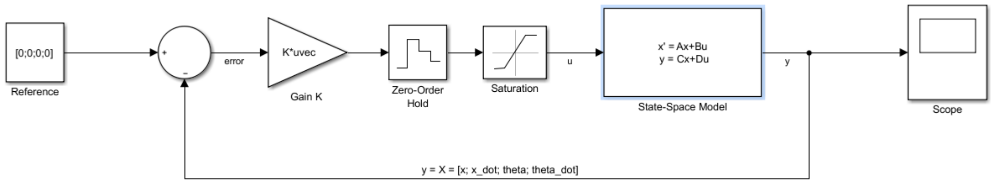



Figure 3.2: The *Simulink* model of the plant with controller

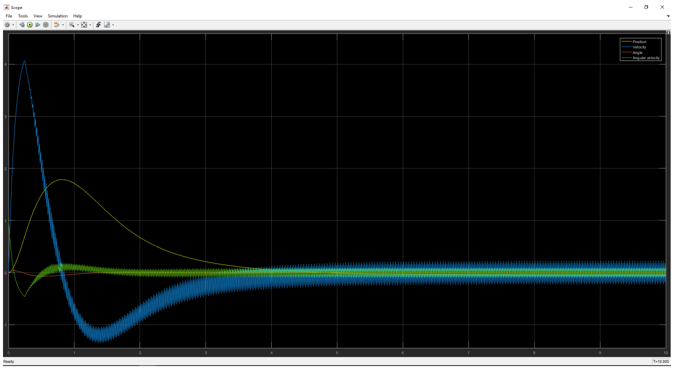
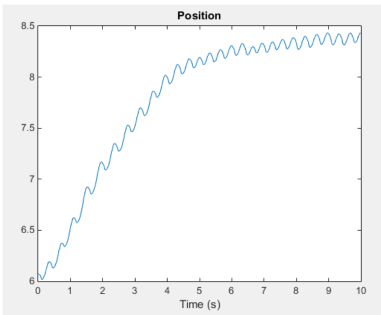


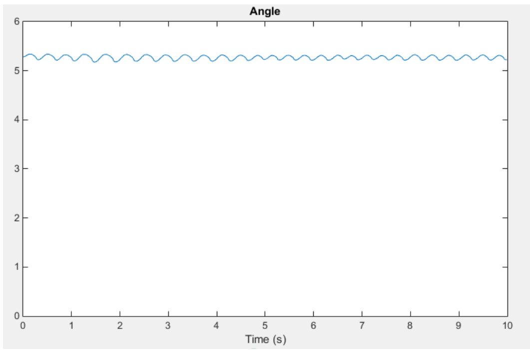
Figure 3.3: Results of the simulation with the pendulum

## 3.4 Result on the real plant

Thanks to the controller, the pendulum is stabilized in upright position. To test if the controller worked well, a change in position is asked. This change in position can be seen in Figure 3.4(a). In figure 3.4(b), we can be seen that the angle of the pendulum stays around the value $5.15V$ which corresponds to the output of the angular sensor when the pendulum is in the upright position.




(a) The position of the pendulum changes from input value 6 to $8V$




(b) The angle of the pendulum stays about constant

Figure 3.4: The measurements of the position and angle of the pendulum

# Chapter 4

# Observer

## 4.1 Feedback Estimator

In the previous section, the pendulum was controlled around its unstable equilibrium point. To do that, all the states were measured and a control feedback was applied based on the difference between the measurements of the 4 sensors and the reference state. In practice, one has not necessarily access to all the states measurement. The implementation of an observer is thus needed to fill in the lack of measurable states. To determine the minimum required measures to completely observe the system, one can compute the rank of the observability matrix $O$:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \tag{4.1}$$

Where $C$ is a matrix whose rows determine the states that are effectively observable. The Observability matrix is full rank (and thus the system is observable) if and only if the position of the cart is measurable. This is due to the fact that the motion of the cart was considered as uncorrelated with the motion of the pendulum (see equation (2.10)). It implies that a measurement related to the pendulum's motion is also needed to get a full observable system. That is why, the measurements $Y$ that were chosen to implement the estimator are:

$$Y = CX = \begin{bmatrix} x \\ \theta \end{bmatrix} \qquad whith: \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.2}$$

The estimation of the 4 states $\hat{X}$ is then introduced. This variable almost follows the same differential equation as $X$. However an extra term is added which amplifies by a factor $L$ the error between the measures $Y$ and their expected values $\hat{Y}$:

$$\begin{cases} \dot{\hat{X}} = A\hat{X} + Bu + L(Y - \hat{Y}) \\ \hat{Y} = C\hat{X} \end{cases} \tag{4.3}$$

The observer described by the system of equations (4.3) acts as a virtual plant that follows the real state evolution. Its equation can be rewritten in the state-space model form as follows:

$$\dot{\hat{X}} = (A - LC)\hat{X} + \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} u \\ Y \end{bmatrix} \tag{4.4}$$

Nevertheless the gain matrix $L$ must still be designed to ensure a fast convergence of the estimated states $\hat{X}$ towards the real states $X$. In other words, the error $E$ on the states estimation has to converge rapidly towards zero:

$$E = X - \hat{X} \tag{4.5}$$

Studying the evolution of this error with time will allow the design of such a gain $L$:

$$\dot{E} = \dot{X} - \dot{\hat{X}} \tag{4.6}$$

$$\Leftrightarrow \dot{E} = AX + Bu - (A\hat{X} + Bu + LY - LC\hat{X}) \tag{4.7}$$

$$\Leftrightarrow \dot{E} = (A - LC)E \tag{4.8}$$

The last expression of $\dot{E}$ ensures a convergence of the error towards zero if and only if the eigenvalues of $A - LC$ are strictly negative. Therefore, one can compute the gain $L$ using the *place* command of Matlab by specifying the wanted eigenvalues of the estimator. This operation can be done independently of the regulator thanks to the **separation principle** which ensures that the eigenvalues of the observer and those of the state feedback do not influence each other. As the estimator has to converge faster than the regulator, it is recommended to chose eigenvalues whose real part are more negative than those of the controller $A-BK$. In practice, the observer's eigenvalues have been arbitrarily chosen to converge faster than the controller using the *place* Matlab command:

$$L = \begin{bmatrix} 47.4204 & 5.1445 \\ -13.6144 & -0.8626 \\ 8.2296 & 36.5447 \\ 56.7638 & 176.6764 \end{bmatrix} \tag{4.9}$$

Furthermore, as only two states of the system are measured, the regulation matrix $K$ has been adapted to take this fact into account. Indeed, the weight matrix $Q$ has been changed to give more importance to the data that are effectively measured than to the data that are estimated.

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 50 \end{bmatrix} \qquad R = 1$$

Which leads to a more adapted regulation matrix $K$:

$$K = \begin{bmatrix} -1 & -1.86687 & -68.5769 & -11.9112 \end{bmatrix} \tag{4.10}$$

## 4.2 Linear Quadratic Estimator (LQE)

Another way of computing the gain $L$ is to use the linear quadratic estimator algorithm, also known as *Kalman Filter*. This algorithm minimizes the quadratic cost function $J$ just as the linear quadratic regulator (3.4) does. This can be easily computed using the *lqe* or *kalman* Matlab commands. In fact, as the notion of controllability and observability are mathematically similar, one can also compute the Kalman Filter using the *lqr* command as follows:

$$L^T = lqr(A^T, C^T, Q_N, R_N) \tag{4.11}$$

## 4.3 Computation Algorithm

As the estimator is a virtual plant implemented in the controller, it can not be considered as a continuous time system. That is why the system has to be discretized before being used in the control code. This is easily done by using the forward Euler formula:

$$\dot{\hat{X}} \approx \frac{\hat{X}[(k+1)T_s] - \hat{X}[kT_s]}{T_s} \tag{4.12}$$

As each iteration of the differential equation (4.4) are computed every $T_s$ (which is the sampling period of the controller), the latter can be rewritten using the Euler approximation 4.12:

$$\hat{X}(k+1) = T_s(A - LC + \frac{1}{Ts}I_n)\hat{X}(k) + T_s \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} u(k) \\ Y(k) \end{bmatrix} \tag{4.13}$$

As the control law has been defined in equation (3.1), the observer iterative equation becomes:

$$\hat{X}(k+1) = T_s(A - LC - BK + \frac{1}{Ts}I_n)\hat{X}(k) + T_s LY(k) \tag{4.14}$$

## 4.4 Simulation and Results

A simulation of the lab's plant is described in Figure 4.1. In order to accurately depict the plant, the offset values of the sensors have been integrated in the Simulink model. The simulation is divided in 3 main blocks:

- The System, which is described by the state space model (2.16),
- The Regulator, which is implemented using the LQR gain $K$ (4.10),
- The Observer, which state space model is (4.13).

In the simulation, the pendulum is initially placed at its upright equilibrium point and the cart is asked to move $2V$ along the x-axis while maintaining the pendulum in the same upright position. The simulated result is depicted in Figure 4.2. In the simulation, the observed states $\hat{X}$ converge rapidly to the simulated states $X$. The estimated measures and the real measures using the observer are depicted in Figures 4.3 and 4.4. The results show that:

1. Although the position of the cart $x$ is a measured value, it does not evolve exactly as expected. Unlike the simulation there is an overshoot of the position with respect to its reference value. This is probably due to the fact that the identified system is only an approximation of the reality.

2. In reality, there is a static error between the cart's target position and the position reached, which can be explained by the fact that the matrix $Q$ defined in section 4.1 gives much more importance to the regulation of the pendulum's position than to the cart's position.

3. Finally, the estimated cart's velocity roughly follows the real velocity. This is also due to the inaccurate system identification. But fortunately, these imprecisions are corrected by the regulator which maintains the pendulum's position and velocity to their reference values.

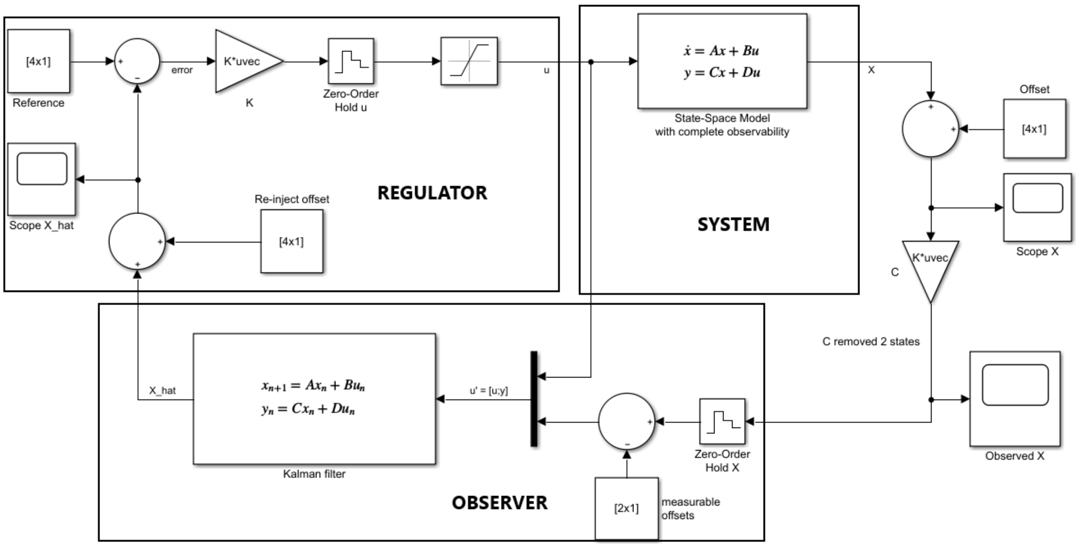



Figure 4.1: Regulation Bloc Diagram with Observer

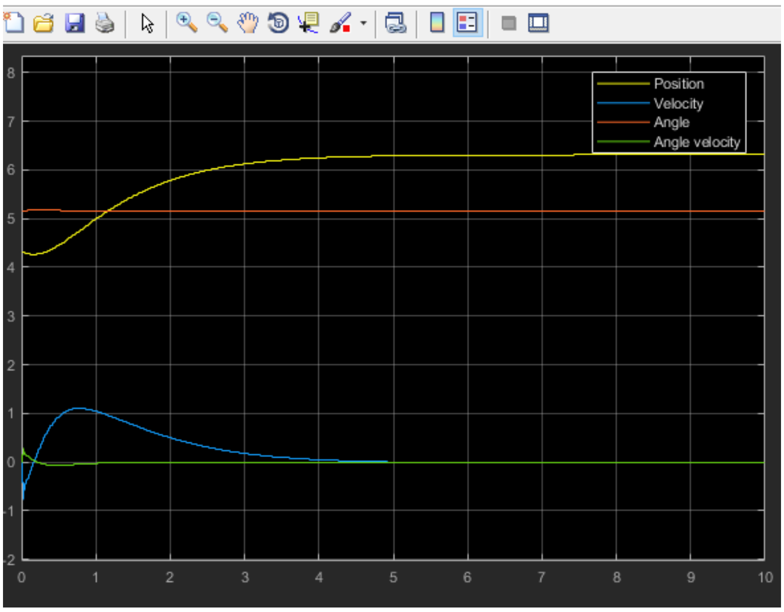



Figure 4.2: Simulation of the regulation using the Observer

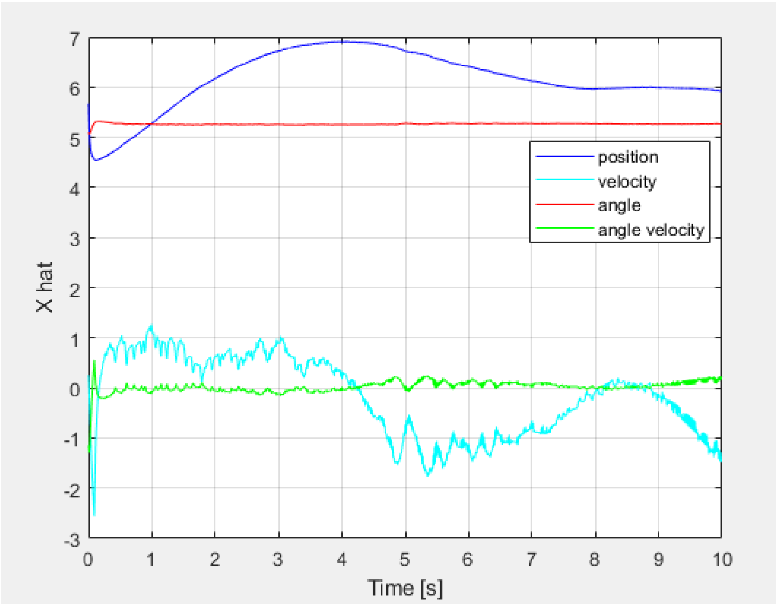



Figure 4.3: Real estimated measures

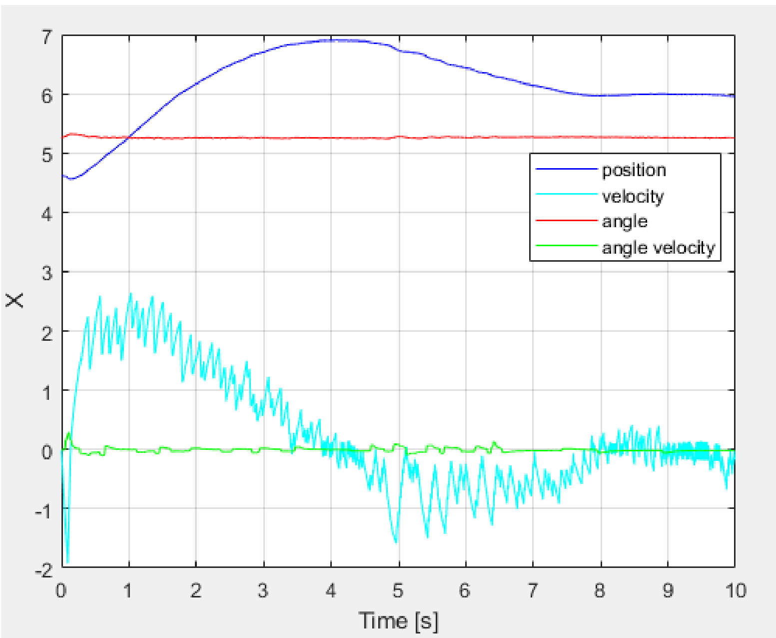



Figure 4.4: Real complete measures

# Chapter 5

# Swing up strategy

Instead of placing the pendulum in the up position and then start controlling it in its upright position, it is also possible to swing the pendulum to reach its up position.

## 5.1 First method

A first way to implement the swing up strategy is based on energy control. When the pendulum is in his point of equilibrium, the energy is minimal.
The mechanical energy in the pendulum, E, is the sum of the kinetic and potential energy. In all the formulas, the friction of the pendulum is neglected. The kinetic energy can be written as:

$$E_{kin} = \frac{1}{2}mv^2 \tag{5.1}$$

With $m$ the mass of the pendulum and $v$ the velocity.

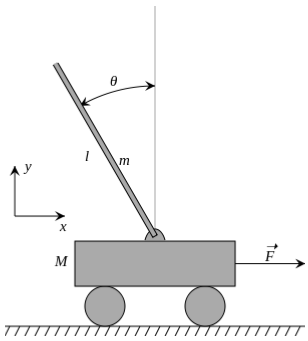



Figure 5.1: Inverted Pendulum

The position of the pendulum can be decomposed in a component in the horizontal direction and a component in the vertical direction.

$$x_p = -lsin(\theta) \tag{5.2}$$

$$y_p = lcos(\theta) \tag{5.3}$$

The velocity of the pendulum is:

$$v = (-l\dot{\theta}cos(\theta))^2 + (-l\dot{\theta}sin(\theta))^2 \tag{5.4}$$

$$v = l^2\dot{\theta}^2cos(\theta)^2 + l^2\dot{\theta}^2sin(\theta)^2 \tag{5.5}$$

$$v = l^2\dot{\theta}^2 \tag{5.6}$$

By using this equation in the equation of kinetic energy, equation (5.1) can be rewritten as:

$$E_{kin} = \frac{1}{2}ml^2\dot{\theta}^2 \tag{5.7}$$

with $\theta$ the angle between the pendulum and the upright position. The potential energy needs to be zero when the pendulum is in the upright position and can be written as:

$$E_{pot} = mgl(cos(\theta) - 1) \tag{5.8}$$

The mechanical energy of the pendulum can then be written as:

$$E = mgl(cos(\theta) - 1) + \frac{1}{2}ml^2\dot{\theta}^2 \tag{5.9}$$

The energy is zero when the pendulum is in upright position ($\theta$=0).

To control the swing up, the following control law is used:

$$\ddot{x} = -\alpha(E - \epsilon)\dot{\theta}cos(\theta) \tag{5.10}$$

with $\alpha \in \mathbb{R}^+$ a tunable parameter.
The terms of the control law can be explained as followed: when the acceleration of the cart is positive, the term $\dot{\theta}cos(\theta)$ needs to be negative. This can be seen in Figure 5.2.

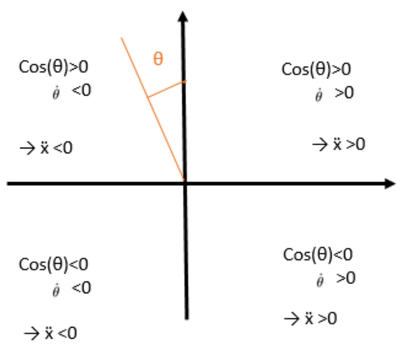



Figure 5.2: The acceleration needs to have the opposite sign of $\dot{\theta}cos(\theta)$ to swing the pendulum up

When the desired energy is reached, the acceleration of the cart needs to be zero. This is also something that can be seen in the control law.
The controllability is lost when the right hand side of equation (5.10) vanishes. This is when $\theta = \pi/2$ or when $\dot{\theta}$=0. This means that the swing up can not start from rest. It first needs to

get an impulse before it begins to work.
The control law can be rewritten as a function of the known variables ($x$, $\dot{x}, \theta, \dot{\theta}$).

$$\ddot{x} = -a_1\dot{\tilde{x}} + b_1\tilde{u} \tag{5.11}$$

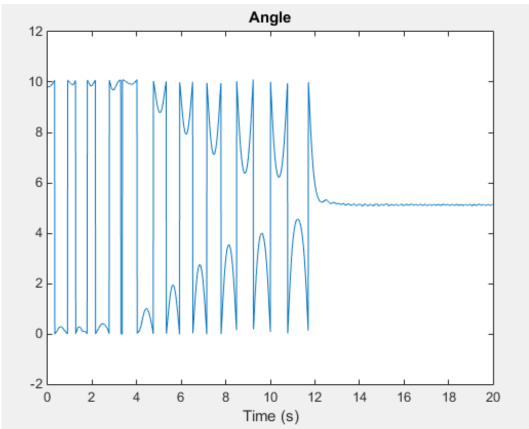
Combinig equation (5.10) and (5.11) it is possible to express all in function of $\tilde{u}$:

$$\ddot{x} = -\alpha(mgl(cos(\theta) - 1) + \frac{1}{2}ml^2\dot{\theta}^2 - \epsilon)\dot{\theta}cos(\theta) = -a_1\dot{\tilde{x}} + b_1\tilde{u} \tag{5.12}$$

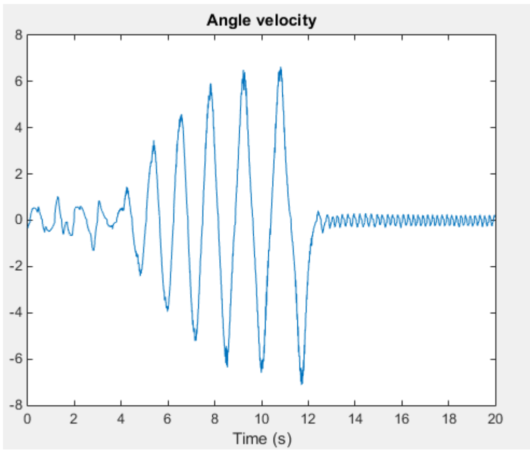
$$\tilde{u} = -\frac{alpha}{b_1}(mgl(cos(\theta) - 1) + \frac{1}{2}ml^2\dot{\theta}^2 - \epsilon)\dot{\theta}cos(\theta) + \frac{a_1}{b_1}\dot{\tilde{x}} \tag{5.13}$$

When this control law is applied on the real plant, the pendulum swings up to the upright position. After the swing up, when the angle of the pendulum is in a certain range, the control is switched from swing up control to stabilization control. In this way, the pendulum stays in upright position.

This method worked (see Figure 5.3) but was not reliable: at given moments, it did not work as expected. This is why an other swing up strategy was created.




(a) Angle




(b) Angular velocity

Figure 5.3: Stabilization of the inverted pendulum with the first swing up strategy

## 5.2 Second method

The same principle as in section 5.1 is valid: if the pendulum goes down in a certain direction, the cart moves in the other direction to give the pendulum extra velocity. An example can be seen in Figure 5.4. When the pendulum moves to the left, the cart will move to the right in order to give the pendulum more velocity

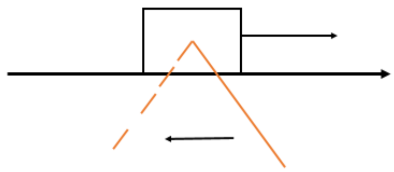
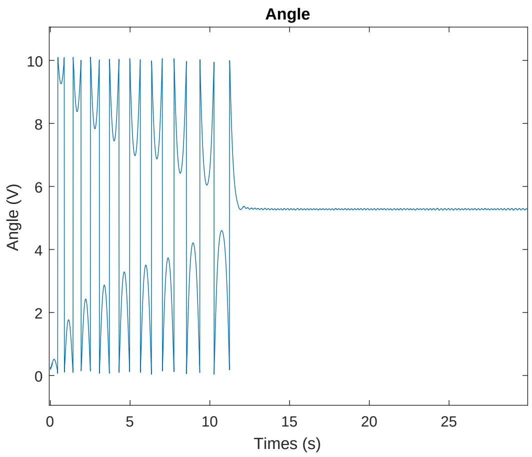


Figure 5.4: The cart and pendulum move in opposite direction

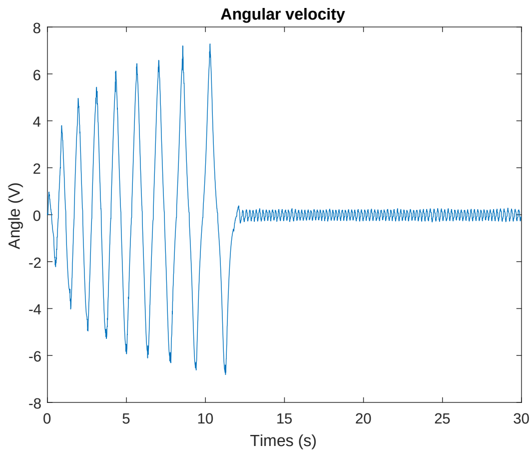
To make sure, the cart does not move all the time, an input signal is sent only when the pendulum is in a certain zone. This zone is about five degrees at each side of the down position of the pendulum. Every time the pendulum passes through this zone, the cart moves in the opposite direction of the pendulum and the velocity of the pendulum increases.

Once the pendulum is close to the upright position, the control is switched from swing up control to equilibrium control and the pendulum is stabilized around its unstable point of equilibrium.

The result with the second swing up strategy are depicted in Figure 5.5 and the pendulum takes less time to reach the upright position.




(a) Angle




(b) Angular velocity

Figure 5.5: Stabilization of the inverted pendulum with the second swing up strategy

# Chapter 6

# Conclusion

In conclusion, the project has successfully met all the specified requirements. The primary objective of stabilizing the pendulum in the upright position was achieved through the development of a state space representation of the system and the implementation of a control feedback utilizing matrix $K$, determined by a linear-quadratic regulator (LQR).

Additionally, the challenge of controlling the system with only two position sensors was addressed by simulating and testing a controller with an observer. Despite minor disparities between simulated and real-world results, the system was effectively controlled, maintaining the pendulum in the desired upright position.

Furthermore, the project aimed to implement a swing-up strategy to elevate the pendulum to its upright position. Two distinct strategies were explored, with the second strategy emerging as the most effective. This strategy involved moving the cart in the opposite direction with maximum acceleration whenever the pendulum passed through a specific zone, thereby transferring additional energy to the pendulum.

Overall, the project not only accomplished its fundamental objectives but also successfully tackled additional challenges, demonstrating innovative solutions and practical implementations in the field of pendulum control systems.