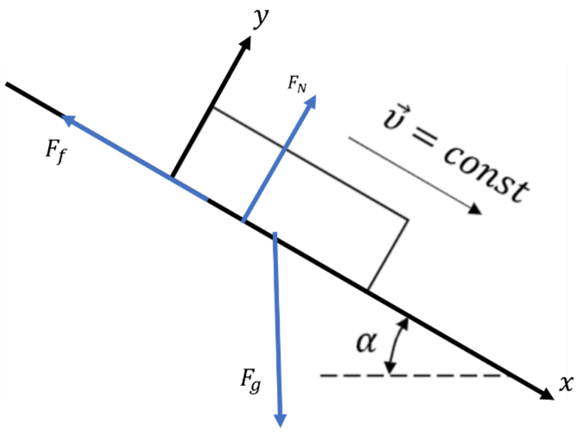
Homework 3. Continuum mechanics

(1). Problem 1

A block slides along a rough surface at an angle α with some constant velocity (Figure 1). Given that the mass of the block m, the coefficient of friction between the block and the surface f and the area of the block S, find the stress tensor describing the stresses on the bottom surface of the block.



Third Newton's Low:

$$F_f + F_{gx} = 0$$
$$F_N + F_{gy} = 0$$

Gravity:

$$F_g = mg, \qquad F_{gx} = mg\sin(\alpha), \qquad F_{gy} = -mg\cos(\alpha)$$

Consider bottom surface:

$$\sigma_{yy} = \frac{F_{gy}}{S} = -\frac{mg}{S}\cos(\alpha)$$
$$\sigma_{yx} = \frac{F_{gx}}{S} = \frac{mg}{S}\sin(\alpha)$$
$$\sigma_{yz} = 0$$

(2). Problem 2

At some point of the body in the Cartesian orthogonal coordinate system, the stress tensor is given
by its components:

$$S_{ij} = \begin{bmatrix} 150 & 45 & 180 \\ 45 & 0 & -120 \\ 180 & -120 & -60 \end{bmatrix} Pa$$

For an area with normal $n1$ =2/3,$n2$ =2/3,$n3$ =1/3, find the components of the vector $p\vec{}n$ and the magnitudes of the shear and normal stresses. Find the angle between $p\vec{}n$ and $n\vec{}\vec{}$.

$$\boldsymbol{p}_n = S\boldsymbol{n} = \begin{bmatrix} 150 & 45 & 180 \\ 45 & 0 & -120 \\ 180 & -120 & -60 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\frac{1}{3} = \begin{bmatrix} 570 \\ -30 \\ 60 \end{bmatrix}\frac{1}{3} = \begin{bmatrix} 190 \\ -10 \\ 20 \end{bmatrix} = 10\begin{bmatrix} 19 \\ -1 \\ 2 \end{bmatrix}$$

$$p_{nn} = \boldsymbol{n}^T\boldsymbol{p}_n = \frac{10}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}^T\begin{bmatrix} 19 \\ -1 \\ 2 \end{bmatrix} = \frac{380}{3}Pa$$

$$\boldsymbol{p}_{n\tau} = \sqrt{\boldsymbol{p}_n^2 - p_{nn}^2} = \frac{50}{3}\sqrt{74} = 143Pa$$

$$\cos(\widehat{\boldsymbol{p}_n\boldsymbol{n}}) = \frac{\boldsymbol{p}_n\boldsymbol{n}}{|\boldsymbol{p}_n||\boldsymbol{n}|} = \frac{\boldsymbol{p}_n\boldsymbol{n}}{|\boldsymbol{p}_n|} = \frac{\frac{380}{3}}{10\sqrt{366}} = \frac{19\sqrt{366}}{549}$$

$$\widehat{\boldsymbol{p}_n \boldsymbol{n}} = arcos\left(\frac{19\sqrt{366}}{549}\right) = 49^0$$

(3). Problem 3

Given stress tensor:

$$S_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 3 \\ 0 & 3 & 2 \end{bmatrix} MPa$$

Find σ_{11} and σ_{22}, considering that the maximum shear stress is 8.5 MPa, the two non-maximum principal stresses are −7 MPa and 3 MPa.

Let's find coordinate system where S is dioganal. For this we need to find e-vals, they will be on the main diagonal.

$$\left\| \begin{bmatrix} \sigma_{11} - \lambda & 0 & 0 \\ 0 & \sigma_{22} - \lambda & 3 \\ 0 & 3 & 2 - \lambda \end{bmatrix} \right\| = (\sigma_{11} - \lambda)\big((\sigma_{22} - \lambda)(2 - \lambda) - 9\big) = 0$$

$$\lambda_1 = \sigma_{11}$$

$$\big((\sigma_{22} - \lambda)(2 - \lambda) - 9\big) = 2\sigma_{22} - \lambda\sigma_{22} - 2\lambda + \lambda^2 - 9 = \lambda^2 - \lambda(\sigma_{22} + 2) + 2\sigma_{22} - 9 = 0$$

$$D = \sigma_{22}^2 - 4\sigma_{22} + 40$$

$$\lambda_{2,3} = \frac{(\sigma_{22} + 2) \pm \sqrt{D}}{2} = \frac{\sigma_{22}}{2} + 1 \pm \frac{\sqrt{D}}{2}$$

$$S_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 3 \\ 0 & 3 & 2 \end{bmatrix} MPa = A \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \frac{\sigma_{22}}{2} + 1 + \frac{\sqrt{D}}{2} & 0 \\ 0 & 0 & \frac{\sigma_{22}}{2} + 1 - \frac{\sqrt{D}}{2} \end{bmatrix} A^* Mpa =$$

Let $\frac{\sigma_{22}}{2} + 1 \pm \frac{\sqrt{D}}{2} = (3, -7)$

$$\frac{\sigma_{22}}{2} + 1 + \frac{\sqrt{D}}{2} = 3$$

$$\frac{\sigma_{22}}{2} + 1 - \frac{\sqrt{D}}{2} = -7$$

Then $\sigma_{22} = -6, \Rightarrow D = 6^2 - 4(-6) + 40 = 100 = 10^2$

$$\frac{\sigma_{22}}{2} + 1 + \frac{\sqrt{D}}{2} = \frac{-6}{2} + 1 + \frac{10}{2} = 3$$

$$\frac{\sigma_{22}}{2} + 1 - \frac{\sqrt{D}}{2} = \frac{-6}{2} + 1 - \frac{10}{2} = -7$$

The last $\frac{\sigma_{11} - (-7)}{2} = 8.5 \Rightarrow \sigma_{11} = 10$

Finally $\sigma_{22} = -6, \sigma_{11} = 10$

(4). Problem 4

In some plane parallel flow, the x-component of the velocity field is known:

$$v_x = -A\frac{y}{r^2}, \qquad A = const, \qquad r = \sqrt{(x^2 + y^2)}.$$

Find y-component of the motion, if it is known that the fluid is incompressible (also $vy \to 0$ for $y \to \infty$ for all x). Is the motion potential, if yes, is this statement satisfied for all points? Explain your answer.

As far as fluid is incompressible => ρ is const.

Mass conservation low: $\frac{\partial \rho}{\partial t} + div(\rho v) = 0 \Rightarrow div(\rho v) = 0 \Rightarrow div(v) = 0$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

As far as In ou case it is parallel flow, Let's chose coordinate system such that v_z parallel to that flow. So $v_z = 0$

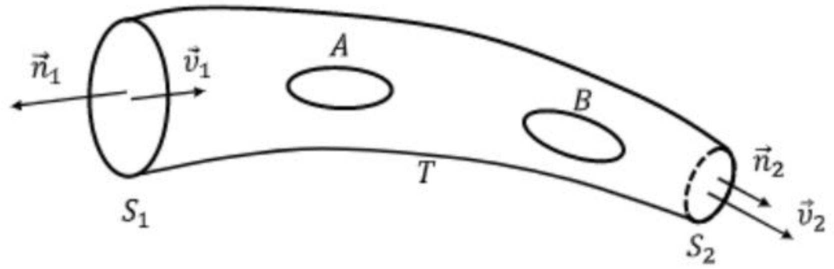
$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0$$
$$\frac{\partial v_x}{\partial x}=-\frac{\partial}{\partial x}A\frac{y}{r^2}=-\frac{\partial}{\partial x}A\frac{y}{x^2\ +y^2}=A\frac{2xy}{(x^2\ +y^2)^2}$$
$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=A\frac{2xy}{(x^2\ +y^2)^2}+\frac{\partial v_y}{\partial y}=0$$
$$\frac{\partial v_y}{\partial y}=A\frac{-2xy}{(x^2\ +y^2)^2}$$
$$v_y=\frac{Ax}{x^2+y^2}+C(x)=\frac{Ax}{r^2}+C(x)$$
For $y\to\infty, x$ is any:
$$v_y=\frac{Ax}{r^2}+C(x)=C(x)$$
From the task condition v_y must converge to 0 for any x, $y\to\infty \Rightarrow C(x)=0$
$$v=\left(-\frac{Ay}{r^2},\frac{Ax}{r^2},0\right)=\frac{A}{r^2}(-y,x,0)$$
Flow is potential if $rot(v)=0$
$$rot(v)=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}=i\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-j\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)+k\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)=$$
$$i(0-0)-j(0-0)+k\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)=k\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

$$\frac{\partial v_y}{\partial x}=\frac{\partial}{\partial x}\frac{Ax}{r^2}=\frac{A}{r^2}+x\frac{\partial}{\partial x}\frac{A}{r^2}=\frac{A}{r^2}+x\frac{\partial}{\partial x}\frac{A}{x^2+y^2}=\frac{A}{r^2}-x\frac{A2x}{(x^2+y^2)^2}=\frac{A}{r^2}-\frac{2Ax^2}{r^4}$$
$$\frac{\partial v_x}{\partial y}=-\frac{\partial}{\partial y}\frac{Ay}{r^2}=-\frac{A}{r^2}+\frac{2Ay^2}{r^4}$$
$$rot(v)=k\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)=k\left(\frac{A}{r^2}-\frac{2Ax^2}{r^4}+\frac{A}{r^2}-\frac{2Ay^2}{r^4}\right)=k\left(2\frac{A}{r^2}-\frac{2A(x^2+y^2)}{r^4}\right)=$$
$$k\left(2\frac{A}{r^2}-\frac{2A(x^2+y^2)}{r^4}\right)=k\left(2\frac{A}{r^2}-\frac{2Ar^2}{r^4}\right)=0$$
Flow is potential

(5). Stress field is described with the stress field:
$$S_{ij}=\begin{bmatrix} pgx_1 & 2\gamma x_2x_3 & -\gamma x_3^2\\ 2\gamma x_2x_3 & \frac{\beta x_2}{x_3^2} & \frac{\beta}{x_3}\\ -\gamma x_3^2 & \frac{\beta}{x_3} & 0\end{bmatrix}$$
Assume what forces act on continuum if it is in equilibrium at this moment.
$$\frac{dv}{dt}=0$$
$$\rho\frac{d\boldsymbol{v}}{dt}=\rho\boldsymbol{F}+\nabla_jS^{ij}\boldsymbol{e_i}=\boldsymbol{0}$$
$$\boldsymbol{F}=-\frac{1}{\rho}\nabla_jS^{ij}\boldsymbol{e_i}$$
$$F_1=-\frac{1}{\rho}\nabla_jS^{1j}=-\frac{1}{\rho}\frac{\partial}{\partial x_j}S^{1j}=-\frac{1}{\rho}\left(\frac{\partial}{\partial x_1}pgx_1+\frac{\partial}{\partial x_2}2\gamma x_2x_3-\frac{\partial}{\partial x_3}\gamma x_3^2\right)=$$
$$-\frac{1}{\rho}(pg+2\gamma x_3-2x_3\gamma)=-g$$

$$F_2 = -\frac{1}{\rho}\nabla_j S^{2j} = -\frac{1}{\rho}\left(0 + \frac{\beta}{x_3^2} - \frac{\beta}{x_3^2}\right) = 0$$
$$F_3 = -\frac{1}{\rho}\nabla_j S^{3j} = -\frac{1}{\rho}(0) = 0$$

(6). Problem 6 (additional: gives max +10% of your HW grade)
Two bodies A and B are stationary in a pipe through which a fluid flow. Let $S1$ and $S2$ be the cross sections of the pipe far ahead and behind bodies A and B (Figure 2), T be the surface of the pipe walls between $S1$ and $S2$.



The flow is assumed to be steady, the flow around the bodies is continuous. All the flow parameters in sections $S1$ and $S2$ are known. Find the main vector of forces acting from liquid on bodies A and B and on pipe walls T, and the flow of energy W from liquid to A, B and T:
a) neglecting the force of gravity;
b) considering the force of gravity.

$$\frac{d}{dt}\int_V \rho\boldsymbol{v}dV = \int_S \rho\boldsymbol{v}v_n dS = \int_V \rho\boldsymbol{F}dV + \int_S \boldsymbol{p}_f\boldsymbol{n}dS$$

(a) Suppose external mass forces are zero: F =0,

$$\int_S \rho\boldsymbol{v}v_n dS = \int_S \boldsymbol{p}_f\boldsymbol{n}dS$$

$S = S1 + S2 + A + B + T$, but on the A,B,T surfaces $v_n = 0$
Total force : $P = P_n - \int_S \boldsymbol{p}_f\boldsymbol{n}dS$, where P_n external forces (normal components), p_f is pressure of liquid.

$$P = P_n - \int_S \boldsymbol{p}_f\boldsymbol{n}dS = P_n - \int_{S1+S2} \rho\boldsymbol{v}v_n dS = \int_{S1+S2} p_n - \rho\boldsymbol{v}v_n dS$$

p_n is external pressure in normal on S1,S2
Low of Energy conservation in our case:

$$\frac{d}{dt}\int_V \left(\frac{v^2}{2} + u\right)\rho dV = \int_{S1+S2}\left(\frac{v^2}{2} + u\right)\rho v_n dS =$$
$$= \int_V \rho(\boldsymbol{F}\boldsymbol{v})dV + \int_{S1+S2} \boldsymbol{p}_n\boldsymbol{v}dS - \int_{S1+S2} q_n dS = \int_{S1+S2}(\boldsymbol{p}_n\boldsymbol{v} - q_n)dS =$$

Flow of energy :

$$W = \int_{S1+S2}(\boldsymbol{p}_n\boldsymbol{v} - q_n)dS - \int_S\left(\frac{v^2}{2} + u\right)\rho v_n dS = \int_{S1+S2}\left(\boldsymbol{p}_n\boldsymbol{v} - q_n - \left(\frac{v^2}{2} + u\right)\rho v_n\right)dS$$

(b) considering the force of gravity

$$\boldsymbol{F} = \boldsymbol{g}$$
$$\int_S \rho\boldsymbol{v}v_n dS = \int_V \rho\boldsymbol{F}dV + \int_S \boldsymbol{p}_f\boldsymbol{n}dS =>$$
$$\int_S \boldsymbol{p}_f\boldsymbol{n}dS = \int_S \rho\boldsymbol{v}v_n dS - \int_V \rho\boldsymbol{g}dV = \int_S \rho\boldsymbol{v}v_n dS - m\boldsymbol{g}$$
$$P = P_n - \int_S \boldsymbol{p}_f\boldsymbol{n}dS = \int_{S1+S2}(p_n - \rho\boldsymbol{v}v_n)dS - m\boldsymbol{g}$$

Low of Energy conservation in this case:

$$\int_{S1+S2}\left(\frac{v^2}{2}+u\right)\rho v_n dS=\int_V \rho(\boldsymbol{F}\boldsymbol{v})dV+\int_{S1+S2}(\boldsymbol{p}_n\boldsymbol{v}-q_n)dS$$

$$W=\int_{S1+S2}\left(\boldsymbol{p}_n\boldsymbol{v}-q_n-\left(\frac{v^2}{2}+u\right)\rho v_n\right)dS+\int_V \rho(\boldsymbol{g}\boldsymbol{v})dV$$