Kovalev Vyacheslav

Problem Set 3. . . . .

(1) The e-value problem for longitudinal waves in an elastic rod is:

$$k^2U(t) + U''(t) = 0$$

$$U(0) = 0, \qquad U_x(l) = 0$$

(a) Explain the physical meaning of the second boundary condition.

Boundary conditions mean that displacement at the start and the end of the rod is 0. Our rod is fixed at both ends.

(b) Solve the problem analytically to find k and u.

$$k^2U(x) + U''(x) = 0$$

$$k^2\gamma^0 + \gamma^2 = 0$$

$$\gamma_{1,2} = \pm ik$$

$$U(x) = Ae^{ikx} + Be^{-ikx} = C_1\cos(kx) + C_2\sin(kx)$$

$$U(0) = 0 => C1 = 0$$

$$U(l) = 0 => C_2\sin(kl) = 0$$

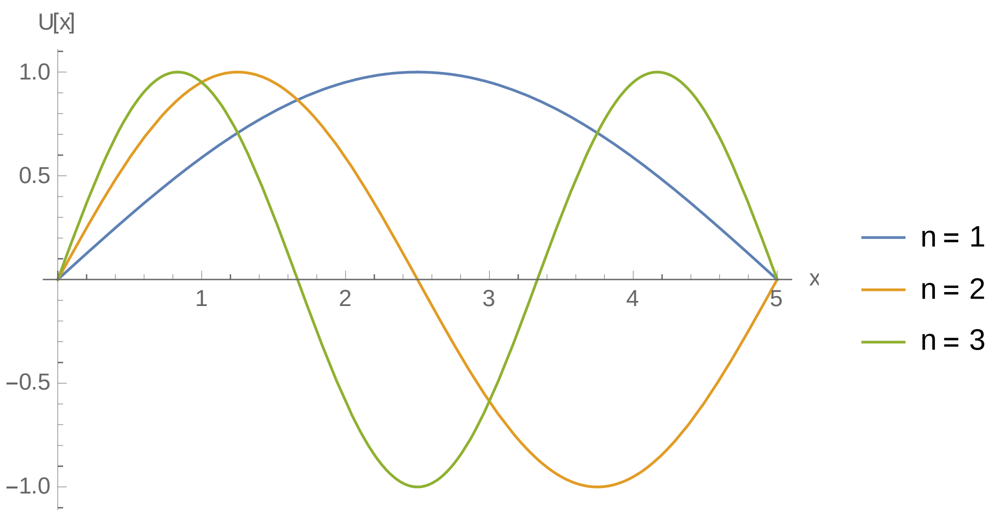
$$let\ C_2 \neq 0; then\ \ \sin(kl) = 0,$$

$$kl = \pi n; => k = \frac{\pi n}{l} : n = 1,2, \dots$$

$$U(x) = C\sin\left(\frac{\pi n}{l}x\right) : n = 1,2, \dots$$

(c) Plot the eigenfunctions that correspond to the lowest three eigenvalues

Let $C\ = 1; l = 5;$



(2) Consider elastic waves

$$\phi'' + k_0^2\left(1 - \epsilon\frac{x(l-x)}{l^2}\right)\phi = 0; \quad \phi(0) = \phi(l) = 0.$$

$$or$$

$$\phi'' + \lambda^2\big(1 - \epsilon x(1-x)\big)\phi = 0; \quad \phi(0) = \phi(1) = 0.$$

Where $\lambda^2 = \frac{\omega^2 l^2}{c_0^2}$

(a) Find the eigenvalues λ^2 numerically:

$$\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2} + \lambda^2\big(1 - \epsilon x_i(1-x_i)\big)\phi_i = 0$$

$$\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2} = -\lambda^2\big(1 - \epsilon x_i(1-x_i)\big)\phi_i$$

Where $x_i = ih$; $h = \frac{l}{n+1}$

$$\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2} = -\lambda^2\big(1 - \epsilon ih(1-ih)\big)\phi_i$$

$$\frac{1}{h^2}A\phi = -\lambda^2 B\phi;$$

Where (considering $\phi_0 = 0, \phi_{n+1} = 0$):

$$A = \begin{bmatrix} -2 & 1 & .. & 0 \\ 1 & -2 & 1 & .. \\ .. & .. & ... & .. \\ 0 & .. & 1 & -2 \end{bmatrix}; \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ ... \\ \phi_n \end{bmatrix};$$

$$B = \begin{bmatrix} 1-\epsilon h(1-h) & 0 & ... & ... & 0 \\ 0 & ... & ... & ... & ... \\ .. & ... & 1-\epsilon ih(1-ih) & ... & ... \\ ... & ... & ... & ... & 0 \\ 0 & ... & ... & 0 & 1-\epsilon nh(1-hn) \end{bmatrix}$$

Perform some transformations:

$$-\frac{1}{h^2}B^{-1}A\phi = \lambda^2\phi;$$

Of course, B^{-1} consists of $\frac{1}{1-\epsilon ih(1-ih)}$ elements on the main diagonal.

This is the problem of e-values, e-vectors:

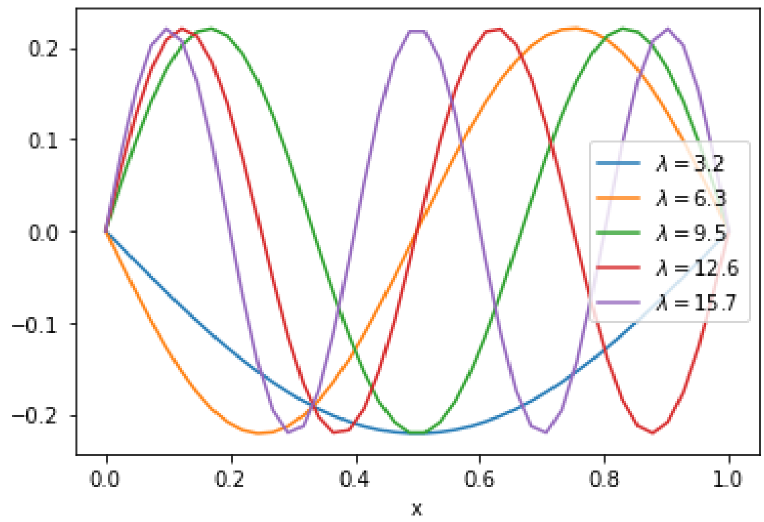
I assume $h = \frac{1}{n+1}$ (in the task it is got $h = \frac{l}{n+1}$), but if $x_i = ih$ =>$x_{n+1} = (n+1) * \frac{l}{n+1} = l$, it contradicts to condition that x_{n+1} must be 1, because we apply rescaling. So, I will use h such $h = \frac{1}{n+1}$ and $n = 20$ for several ϵ

λ		
$\epsilon = 0.1$	$\epsilon = 0$	$\pi n \ (for\ n = 1,2..5)$
3.17	3.14	3.14
6.32	6.26	6.28
9.43	9.35	9.42
12.49	12.38	12.57
15.48	15.34	15.71

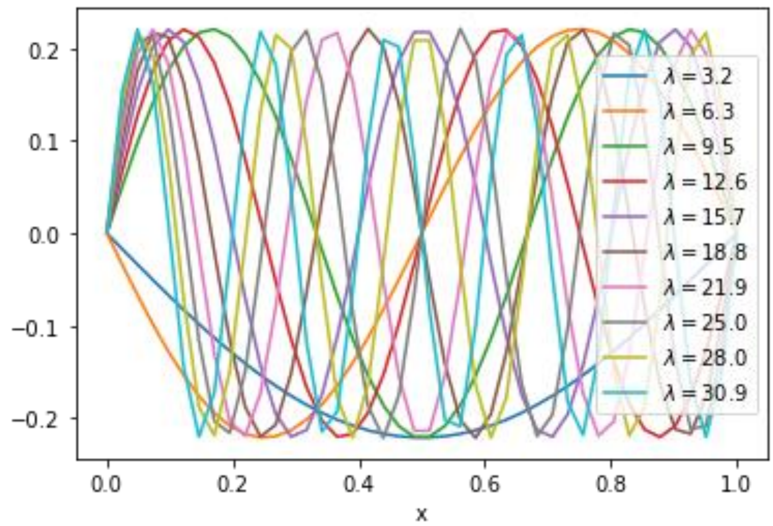
Last column corresponds to case then $\epsilon = 0$ => our equation turns to task (1) and analytical solution is $\phi_n = \sin(\pi n x)\,; => \lambda = \pi n$

n should be chosen as large as it is possible, because it makes ϕ'' more accurate. I chose n = 20, because it is enough to plot $\sin(5\pi)$ by 20 points. (When ϵ is small our solution supposed to be like $\phi_n = \sin(\pi n x)$)

(b) When $\epsilon = 0.1$ $(n = 40)$

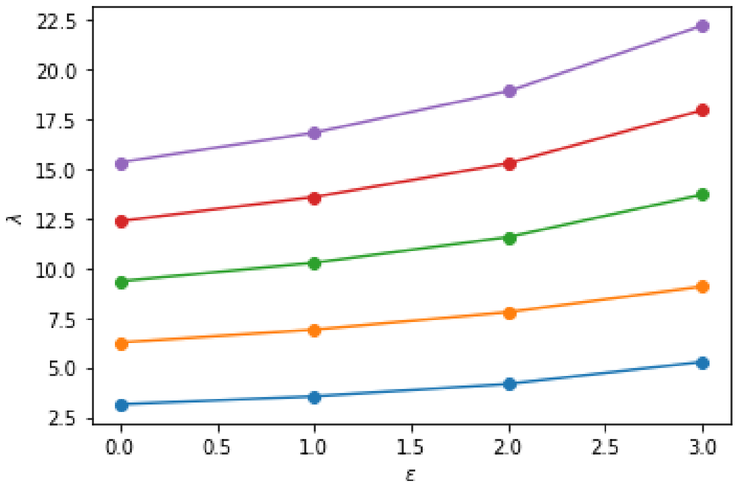


"on a separate plot show the first ten numerically found eigenvalues λ"



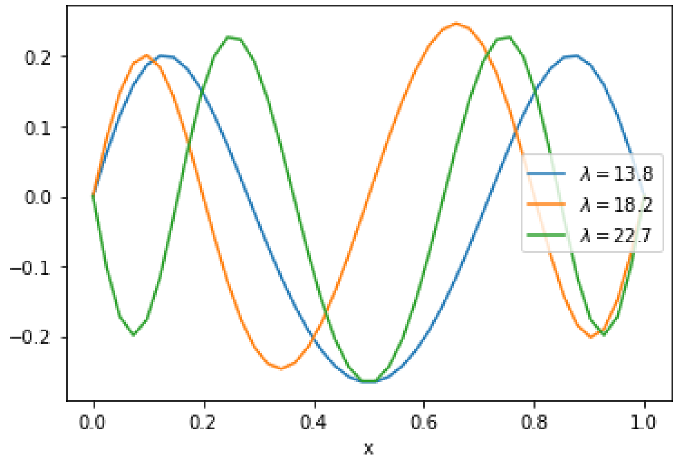
“How do they compare with those at $\epsilon = 0$?”: Almost the same λ.

(c) "Plot λ (ϵ) for the first five eigenvalues. "

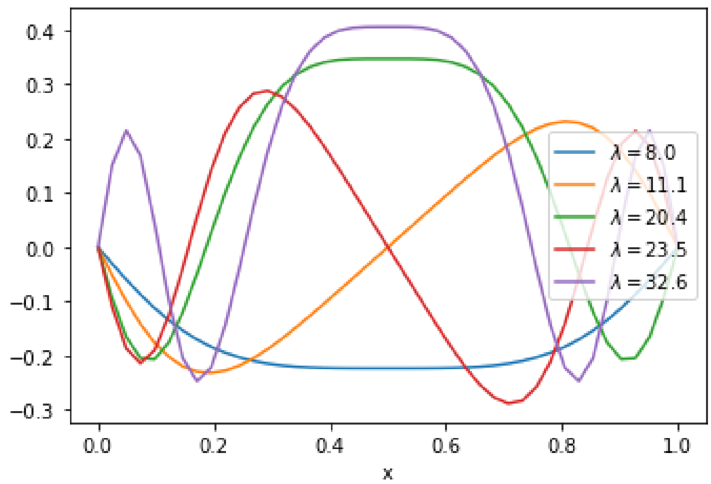


How do eigenvalues and eigenfunctions change when ϵ increases?:

For instance, $\epsilon = 3$ as you can see it increases amplitude on the middle of x. So, the more ϵ the more influence on the middle of rod.



(d) What happens when ϵ becomes close to 4?



I got such eigenvectors and eigenvalues, but further increase of ϵ results in complex λ.

(3) $A = \begin{bmatrix} 5 & -3 \\ 0 & 4 \end{bmatrix}$

(a) Find the SVD of A and illustrate on paper how $A = U\Sigma V^T$ transforms a vector x into Ax by a sequence of three transformations

$$AA^T = U\Sigma^2 U^T$$

Let's find e-values, and e-vectors for AA^T

$$\det(AA^T - \lambda I) = 0$$

$$\lambda_1 = 10;\ \lambda_2 = 40;$$

For λ_1:

$$(AA^T - \lambda I)\mathrm{x} = \begin{bmatrix} 24 & -12 \\ -12 & 6 \end{bmatrix} x = 0$$

$$x = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix};\ x' = \frac{x}{||x||} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

For λ_2:

$$(AA^T - \lambda I)\mathrm{x} = \begin{bmatrix} -6 & -12 \\ -12 & -24 \end{bmatrix} x = 0$$

$$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}; x' = \frac{x}{||x||} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

Result: $U = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}; \Sigma^2 = \begin{bmatrix} 10 & 0 \\ 0 & 40 \end{bmatrix}$

Analogically for

$$A^T A = V\Sigma^2 V^T$$

For $\lambda_1 = 10 : x_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$; For $\lambda_2 = 40 : x_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$$V^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

In conclusion $A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 10^{\frac{1}{2}} & 0 \\ 0 & 40^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

$$Ax = U\Sigma V^T x$$

$$x' = V^T x - rotates\ vector\ x, because ||x'|| = ||x||$$

$$Ax = U\Sigma x'$$

$$x'' = \Sigma x' - statches\ out\ every\ component\ of\ x'$$

$$Ax = Ux''$$

$$x''' = Ux'' - rotates\ vector\ x''$$

$$Ax = x'''$$

(b) Compare the result from (a) with how $A = S\Lambda S^{-1}$ transforms x into Ax.

$$Ax = S\Lambda S^{-1}x = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} x$$

The difference in transformation $S^{-1}x$ and S.When U and V^T only rotate vector, S and S^{-1} may change the norm of x too.

(4) Given column vectors a and b, find the SVD of

(a) $A = ab^T$

$$A = U\Sigma V^T = ab^T$$

Such as rank(A) =1

$$U\Sigma V^T = \sum_i^r \sigma_i u_i v_i^T = \sigma_1 u_1 v_1^T = ab^T$$

Let $v_1^T = \frac{b^T}{||b||}$; then $u_1 = \frac{a}{||a||}$; => $\sigma_1 = ||a|| * ||b||$

$$ab^T = \frac{a}{||a||} ||a|| * ||b|| \frac{b^T}{||b||}$$

(b) $A = ab^T + ba^T$ when $a^T b = 0$

$$Ab = ab^T b + ba^T b = ab^T b$$

$A = U\Sigma V^T => Ab = U\Sigma V^T b = ab^T b => U\Sigma V^T = ab^T$ – it is the prev. task

$$A = U\Sigma V^T = \frac{a}{||a||} ||a|| * ||b|| \frac{b^T}{||b||}$$

$$A = U\Sigma U^T$$

(5) For matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) using SVD, find the orthonormal basis for the column space, C (A), and complete it with the basis for the left nullspace, $N(A^T)$ to form a basis for R^4. Write down the 4 × 4 orthogonal matrix U.

$A = U\Sigma V^T$, where U is orthonormal basis for C (A).

$$AA^T = U\Sigma^2 U^T$$

Found nonzero e-values and corresponded orthonormal e-vectors:

$$\Sigma^2 = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}; U = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C(A) = \{c_1U_1 + c_2U_2 : \forall\ c_1, c_2\}$$

Basis for $N(A^T)$

$$A^T x = 0$$

Easy to show $x = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix};$

Full U:

$$U = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) Do the same for the row space.

$A = U\Sigma V^T$, where V^T is orthonormal basis for R(A).

$$A^T A = V\Sigma^2 V^T$$

Found nonzero e-values and corresponded orthonormal e-vectors:

$$\Sigma^2 = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}; V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$R(A) \ = \ \{c_1 V_1^T + c_2 V_2^T \ : \forall \ c_1, c_2\}$$

Basis for $N(A)$

$$Ax = 0$$

Easy to show $x = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$;

Full V:

$$V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

(c) Write the full SVD:

$$A = U\Sigma V^T = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

(d) What is the best rank-1 approximation of A?

$$A = \sqrt{2} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + \sqrt{6} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$ – the closest to A, because standard deviation is smallest.