Seminar 2: Linear Algebra

LU, Fredholm, functions, ODEs

Andrei Goldin, andrei.goldin@skoltech.ru or via PM in Canvas

Notes on homework

– Homework is an essential element of the course that allows you to get a hands-on experience in the analysis of basic problems in mathematical methods. For this reason, you benefit the most from the course by working on the problems independently. That said, sometimes help is necessary. In those cases, you should first consult with the TAs by making an appointment or attending the TA office hours. It is allowed and in fact encouraged to have discussions with fellow students. However, after these discussions you should remain convinced that you yourself solved the problem, or at least most of it. In that case, your submitted solutions will reflect your own unique understanding. Thus, solutions and codes that you submit must be completely your own. Submissions sharing much similarity (hope this will never happen) will receive 0 points.

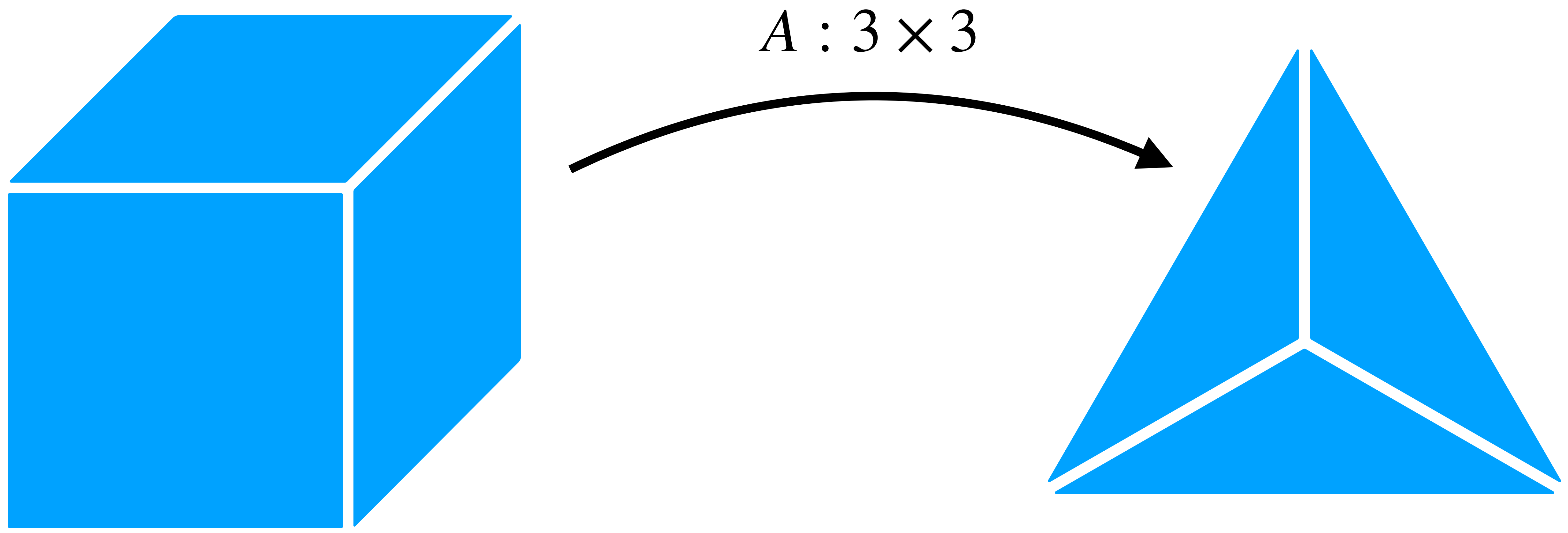
- CHEATING/COPYING FROM ANY SOURCE => ZERO POINTS
- Arithmetical mistakes, typos are forgiven for now, but…

Cube problem

(4) The columns of matrix $C = \begin{bmatrix} 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$ represent vertices of a cube. Describe transformations of the cube that result from the action on C of the following three matrices:

$$A_1 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Relate the results to the ranks of A_k and to the dimensions and bases of the four fundamental subspaces of A_k. Is there a 3×3 matrix A that can transform a cube into a tetrahedron? Explain.



1. $\dim C(A) = ?$, $\dim R(A) = ?$, $\dim N(A) = ?$

2. (LN): Theorem 6. A matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ (tall matrix) has full rank iff it maps no two distinct vectors to the same vector.

Factorisations

Square/Rectangular

$$A : m \times n, m = n, m \neq n$$

Diagonal

$$D_{ij} = 0, \ i \neq j$$

Lower/Upper triangular

$$L_{ij} = 0, \ i < j; U_{ij} = 0, \ i > j$$

Symmetric/Hermitian

$$A^T = A, A^* = A$$

Skew-symmetric/Skew-Hermitian

$$A^T = -A, A^* = -A$$

Orthogonal/Unitary

$$Q^T = Q^{-1}, U^* = U^{-1}$$

Positive (semi) definite

$$A > (\geq) 0 \iff x^T A x > (\geq) \ 0 \ \forall x \neq 0$$

Positive eigenvalues and Sylvester's criterion

LU and LDL

(1) Consider linear system

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 2 \\ x_2 + 2x_3 &= 3. \end{aligned}$$

(a) Find the LU factorization of the coefficient matrix A. Show that $U = DL^T$ with D diagonal and thus $A = LDL^T$. Find the exact solution using the LU factorization.

$A = LU$: matrix form of Gaussian elimination, actually, not any matrix admits LU, but any matrix admits LUP

APPLICATIONS OF LU: Solving systems, finding inverse or determinant

Cholesky: $A^T = A,\ A > 0 \ \Rightarrow \exists!\ L : A = LL^T,\ L_{ij} = 0,\ i < j,\ L_{ii} > 0$

$L_{ii} = 1 \Rightarrow A = LDL^T$

$A = LDL^T = LD^{1/2}D^{1/2}L^T = LD^{1/2}\left(LD^{1/2}\right)^T$

$A = CC^T = CS^{-1}S^2S^{-1}C^T = LDL^T$ **where** $S = \mathrm{diag(C)}$

Cholesky works better: more efficient (x2) and numerically more stable than LU

APPLICATIONS OF LL^T and LDL^T: Linear least squares, non-linear optimisation, Monte-Carlo simulations, Kalman filters

Proof

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ 0 & L_{22}^T \end{pmatrix} = \begin{pmatrix} L_{11}^2 & L_{11}L_{12} \\ L_{11}L_{21} & L_{22}L_{22}^T + L_{21}L_{12} \end{pmatrix}$$

$$A_{11} = L_{11}^2, A_{11} > 0 \Rightarrow L_{11} = \sqrt{A_{11}}\ , L_{12} = A_{12}/\sqrt{A_{11}}, L_{21} = A_{21}/\sqrt{A_{11}}$$

$L_{22}L_{22}^T = A_{22} - A_{12}A_{21}/A_{11} > 0$ Why?

$$\begin{pmatrix} y & -x^T \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} y \\ -x \end{pmatrix} = A_{11}y^2 - 2A_{12}xy + x^T A_{12} x > 0$$

Take $y = \dfrac{A_{12}x}{A_{11}} \quad \Rightarrow x^T A_{22} x - \|A_{12}x\|^2/A_{11} > 0 \ \forall x \neq 0$

Now find L_{22} and so on…

Example

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -1 & 7 \end{pmatrix}, A = LL^T, L = ?$$

1. $L_{11} = 2, L_{21} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 3 & -1 \\ -1 & 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} (0 \quad 1/2) = \begin{pmatrix} 3 & -1 \\ -1 & 27/4 \end{pmatrix} = L_{22} L_{22}^T$$

2. $L'_{11} = \sqrt{3}, L'_{21} = -1/\sqrt{3} \Rightarrow 27/4 - 1/3 = 77/12 = L'_{22} L'^{T}_{22}$

3. $L''_{11} = \sqrt{77/12}$

Example

$$L = \begin{pmatrix} 2 & 0 & 1 \\ 0 & \sqrt{3} & 0 \\ 1/2 & -1/\sqrt{3} & \sqrt{77/12} \end{pmatrix},$$

$$A = \hat{L}D\hat{L}^T, D = \mathrm{diag(L)} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 77/12 \end{pmatrix}, \hat{L} = LD^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & -1/3 & 1 \end{pmatrix}$$

Some numerics

(b) Solve the system using Jacobi and Gauss-Seidel iterations. How many iterations are needed to reduce the relative error of the solution to 10^{-8}?

(c) Plot in semilog scales the relative errors by both methods as a function of the number of iterations.

(d) Explain the convergence rate. Which of the methods is better and why?

$$Ax = b,\ (A_1 + A_2)x = b,\ x = A_1^{-1}\left(b - A_2 x\right)$$

$$x_{k+1} - x = -A_1^{-1}A_2(x_k - x)$$

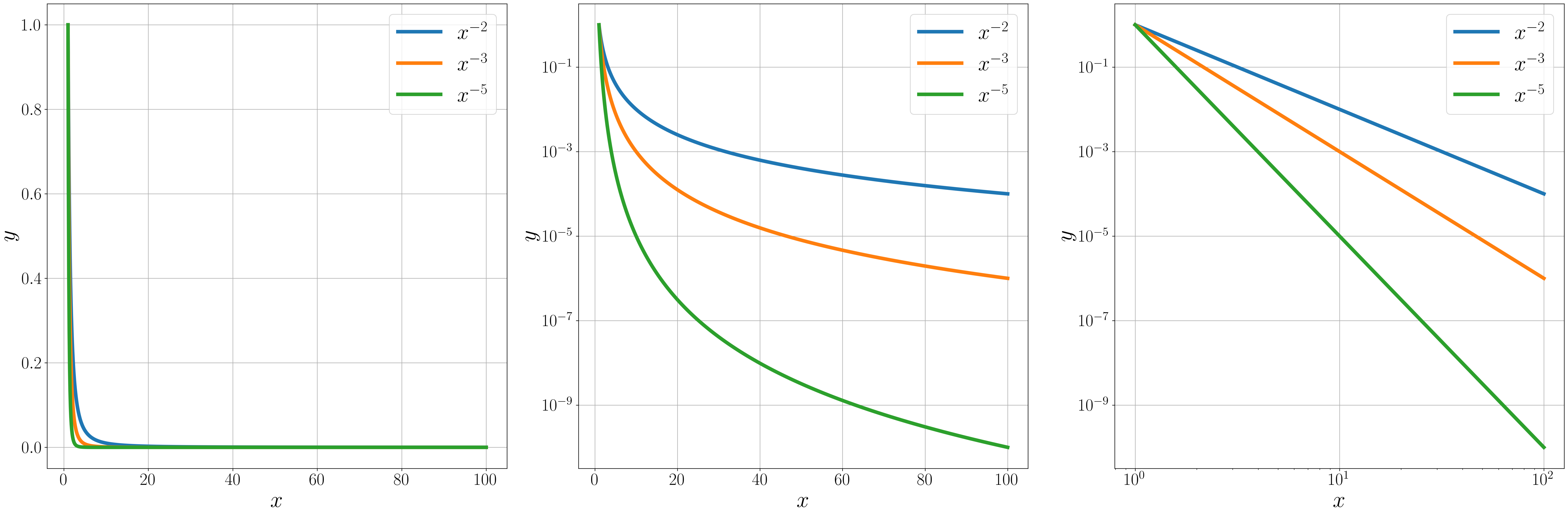
A_1 **should be easy to invert and** $A_1^{-1}A_2$ **should have** $|\lambda_i| < 1$

Jacobi: $A_1 = \mathrm{diag(A)},\ \mathrm{A_2 = A - A_1}$

Gauss−Seidel: $A_1 = \mathrm{tril(A)},\ \mathrm{A_2 = A - A_1}$

Convergence rate: $\displaystyle\lim_{k\to\infty} \frac{\|x_{k+1} - x\|}{\|x_k - x\|^q} = \mu$, **you are asked to find** q

Log scales are important!



NO LOG SCALES => NO POINTS

Matrix functions

(2) Factor these two matrices A into $S\Lambda S^{-1}$:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

Using that factorization, find for both: (a) A^3; (b) A^{-1}.

Problem: Fibonacci numbers

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

$$f_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = A f_{n-1}$$

$$f_n = A f_{n-1} = A^2 f_{n-2} = \ldots = A^{n-1} f_1$$

$$A = UDU^T \Rightarrow A^{n-1} = UD^{n-1}U^T$$

$F_n = \left(UD^{n-1}U^T\right)_{11}$ closed form expression

Solution

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -(1-\lambda)\lambda - 1 = 0, \qquad \left(\lambda - \frac{1}{2}\right)^2 = \frac{5}{4},\ \lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$$

$$v_+ = \frac{1}{\sqrt{5}\lambda_-}\begin{pmatrix} -1 \\ \lambda_- \end{pmatrix} = \begin{pmatrix} 1/\alpha_1 \\ 1/\sqrt{5} \end{pmatrix},\ v_- = \frac{1}{\sqrt{5}\lambda_+}\begin{pmatrix} 1 \\ -\lambda_+ \end{pmatrix} = \begin{pmatrix} 1/\alpha_2 \\ -1/\sqrt{5} \end{pmatrix}$$

$$\begin{pmatrix} 1/\alpha_1 & 1/\alpha_2 \\ * & * \end{pmatrix}\begin{pmatrix} \lambda_+^{n-1} & 0 \\ 0 & \lambda_-^{n-1} \end{pmatrix}\begin{pmatrix} 1/\alpha_1 & * \\ 1/\alpha_2 & * \end{pmatrix} = \begin{pmatrix} \lambda_+^{n-1}/\alpha_1^2 + \lambda_-^{n-1}/\alpha_2^2 & * \\ * & * \end{pmatrix}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Matrix functions

$$A = \begin{pmatrix} 0 & \varphi \\ -\varphi & 0 \end{pmatrix} = \varphi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \exp(A) = ?$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$A^2 = -\varphi^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A^3 = \varphi^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, A^4 = \varphi^4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\exp(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \ldots \right) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \ldots \right)$$

$$\exp(A) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$ rotation matrix

Matrix functions

$$A = \begin{pmatrix} 0 & \varphi \\ -\varphi & 0 \end{pmatrix} = \varphi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \exp(A) = ?$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0, \lambda \pm i, v_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, v_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, v_+^* v_- = 0$$

$$A = \varphi/2 \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\mathbf{exp(A) = 1 + \frac{A}{1!} + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots = S \left(1 + \frac{\Lambda}{1!} + \frac{\Lambda^2}{2!} + \ldots + \frac{\Lambda^n}{n!} + \ldots \right) S^{-1} = S \exp\left(\Lambda\right) S^{-1}}$$

$$\exp(A) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} \exp(\varphi i) & 0 \\ 0 & \exp(-\varphi i) \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

$\exp(A) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ rotation matrix, $\cos\varphi = \dfrac{\exp(i\varphi) + \exp(-i\varphi)}{2}, \ \sin\varphi = \dfrac{\exp(i\varphi) - \exp(-i\varphi)}{2i}$

Fredholm alternative

(3) Given a system $Ax = b$ with

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 2 & 3 & 4 \\ -2 & 1 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ a \\ -1 \end{bmatrix},$$

for which a there is a solution? Find the general solution of the system for that a.

Fredholm's theory (Acta Mathematica, 1903): $g(t) = \int_a^b K(s,t)f(s)\mathrm{d}s$, **find** $f(t)$ **given** $K(s,t),\ g(t)$.

APPLICATIONS: Signal processing , inverse problems, fluid mechanics, computer graphics. Operator theory, PDE theory, functional analysis: existence and uniqueness of the solutions

$$Ax = b \iff b \in C(A) \iff b \perp N\left(A^T\right) \iff y^T b = 0,\ \forall y : A^T y = 0$$

1. **If not satisfied** $\Rightarrow$ **No solution:** $\forall x\ Ax \neq b$
2. **If satisfied and** $N(A) = \{0\}$ $\Rightarrow$ **Unique solution:** $\exists x\ Ax = b$
3. **If satisfied and** $N(A) \neq \{0\}$ $\Rightarrow$ **Infinite number:** $\forall y \in N(A)\ A(x+y) = b$

Bases of $N\left(A^T\right)$ **and** $N(A)$ **are important!**

A bit of geometry

(5) For matrix $A = \begin{bmatrix} 2021 & 20 & 0 \\ 20 & 2021 & 21 \\ 0 & 21 & 2021 \end{bmatrix}$ and vector $b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, what is the most likely direction of vector $x = A^{2021}b$?

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i e_i =$$

ith coordinate

= **sum of columns** e_i = $\begin{pmatrix} | & & | & & | \\ e_1 & \dots & e_i & \dots & e_n \\ | & & | & & | \end{pmatrix} x$

Identity matrix

A bit of geometry

(5) For matrix $A = \begin{bmatrix} 2021 & 20 & 0 \\ 20 & 2021 & 21 \\ 0 & 21 & 2021 \end{bmatrix}$ and vector $b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, what is the most likely direction of vector $x = A^{2021}b$?

What if we choose another basis $v_1, \ldots, v_n$?

$$x = \sum_{i=1}^{n} \hat{x}_i v_i = \begin{pmatrix} | & & | & & | \\ v_1 & \cdots & v_i & \cdots & v_n \\ | & | & | & | & | \end{pmatrix} \hat{x} = V\hat{x} \Rightarrow \hat{x} = V^{-1}x$$

Spectral decomposition: $A = S\Lambda S^{-1}$, $Ax = S\Lambda S^{-1}x = y \;\Rightarrow \Lambda\left(S^{-1}x\right) = S^{-1}y$

We work in the basis of eigenvectors!

Directions: $\cos\left(\widehat{x, v_i}\right) = \dfrac{x^T v_i}{\|x\| \cdot \|v_i\|}$

ODE systems, normal modes

(4) Consider the system of linear differential equations

$$\frac{du}{dt} = Au, \quad \text{with } A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(a) Using the spectral factorization of A, determine the general solution of the system.

(b) Find the behavior of the solution at large t if the initial condition is $u(0) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$.

$$u = S\exp(\Lambda t)S^{-1}u_0,\ \lim_{t\to\infty} u(t) = ?$$

Change of basis? Problem 5?

What if? $A \neq S\Lambda S^{-1}$ **with diagonal** $\Lambda \iff$

There is no n **linearly independent eigenvectors.**

A **is called defective, or non-diagonalisable**

Jordan normal form

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_1 & 1 & \dots & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \lambda_2 & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix}, A = PJP^{-1} \Rightarrow AP = PJ =$$

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_1 & 1 & \dots & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \lambda_2 & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix} \Rightarrow \begin{aligned} &\left(A - \lambda_1 E\right) p_1 = 0, \\ &\left(A - \lambda_1 E\right) p_2 = p_1, \\ &\left(A - \lambda_1 E\right) p_3 = p_2, \\ &\left(A - \lambda_2 E\right) p_4 = 0, \\ &\left(A - \lambda_1 E\right) p_5 = p_4, \\ &\left(A - \lambda_3 E\right) p_6 = 0, \\ &\qquad\vdots \end{aligned}$$

Functions

$$J = \begin{pmatrix} \lambda & 1 & 0 & \dots & \dots & 0 \\ 0 & \lambda & 1 & \dots & \dots & 0 \\ 0 & 0 & \lambda & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \lambda & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$, then $f(J) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2} & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & f'(\lambda) & \cdots & \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & 0 & f(\lambda) \end{bmatrix}$

Defective system

$$\frac{\mathrm{du}}{\mathrm{dt}} = Au, A = \begin{pmatrix} 2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 2 & 0 & \dots & \dots & 0 \\ 0 & 0 & 5 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \Rightarrow \begin{aligned} &(A-2E)\, e_1 = 0, \\ &(A-2E)\, e_2 = e_1, \\ &\left(A-5E\right) e_3 = 0, \\ &\left(A-5E\right) e_4 = 0, \\ &\left(A-5E\right) e_5 = e_4, \\ &\left(A-5E\right) e_6 = e_5\,. \end{aligned}$$

Fundamental system of solutions is given as follows

$$u_1(t) = e^{2t}\mathbf{e}_1,\ u_2(t) = e^{2t}\left(\mathbf{e}_2 + t\mathbf{e}_1\right),\ u_3(t) = e^{5t}\mathbf{e}_3,$$

$$u_4(t) = e^{5t}\mathbf{e}_4,\ u_5(t) = e^{5t}\left(\mathbf{e}_5 + t\mathbf{e}_4\right),\ u_6(t) = e^{5t}\left(\mathbf{e}_6 + t\mathbf{e}_5 + \frac{1}{2}t^2\mathbf{e}_4\right).$$