Seminar 8

Dynamical systems

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Intro

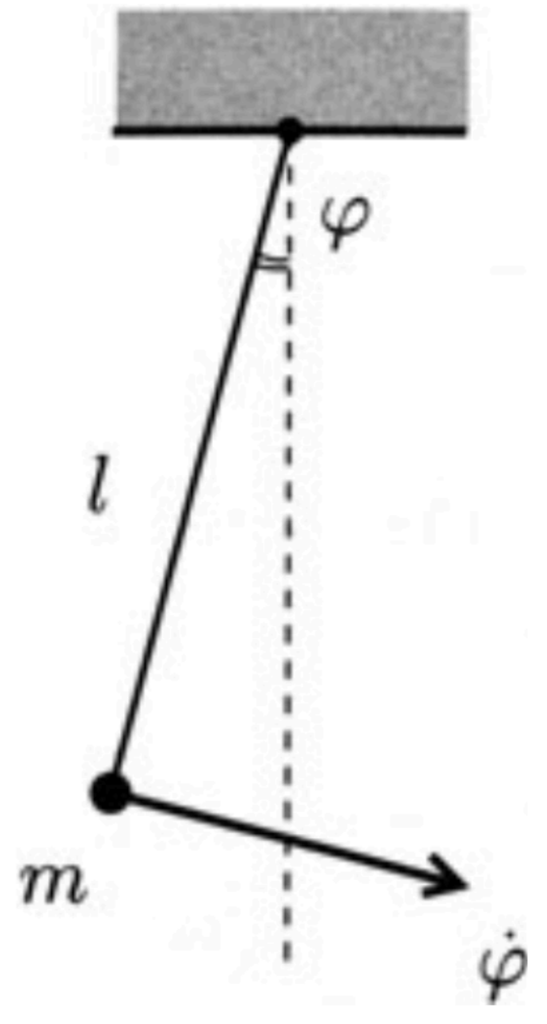
Dynamical system = Time set T + State (phase) space X + Evolution operator

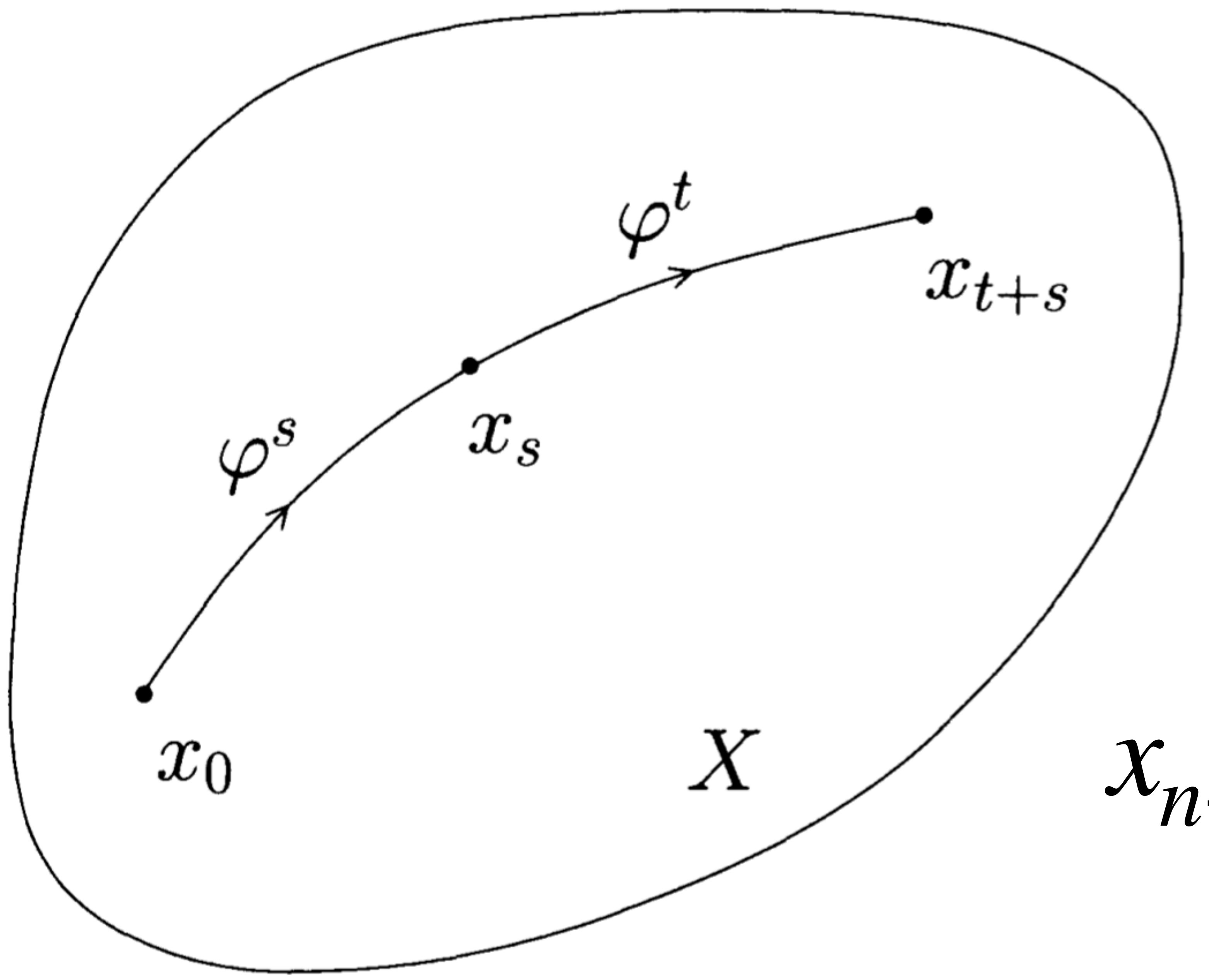
Continuous $\mathbb{R}$
Discrete $\mathbb{Z}$

All possible states of a system are characterised by the points in X

Determines the state x_t for a known the initial state x_0

Displacement and velocity
Generalised coordinates
Positions and velocities of molecules
Quantum observables
Chemical concentrations
Population of species
Iterations of calculations





$$\varphi^t : X \to X$$
$$\varphi^t x_0 = x_t$$

or

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

Autonomous systems

$$\begin{cases} \dfrac{dx_1}{dt} = f_1\left(x_1, \ldots, x_n\right), \\ \dfrac{dx_2}{dt} = f_2\left(x_1, \ldots, x_n\right), \\ \ldots \\ \dfrac{dx_n}{dt} = f_n\left(x_1, \ldots, x_n\right) \end{cases} \Rightarrow \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\frac{d}{dt}\mathbf{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} \text{ and } \frac{d}{dt}\mathbf{x}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Velocity field

Trajectory, or orbit, or path, starting in x_0 is a curve in the phase space representing a solution of the system for all possible values of $t \in T$.

Velocity vectors are tangent to the trajectory at all points.

Intersections? One point orbit? Closed path?

A phase portrait, or diagram, is a partitioning of the phase space into orbits.

i**th nullcline** $: \; f_i(x) = 0$

Fixed, or critical, or equilibrium points are the intersections of the nullclines or the solution of the system.

$$\begin{cases} f_1\left(x_1, \ldots, x_n\right) = 0, \\ f_2\left(x_1, \ldots, x_n\right) = 0, \\ \ldots \\ f_n\left(x_1, \ldots, x_n\right) = 0. \end{cases}$$

Fixed points of linear systems

$$\begin{cases} \dfrac{dx}{dt} = ax + by + e, \\ \dfrac{dy}{dt} = cx + dy + f. \end{cases}$$

$$\begin{cases} ax_0 + by_0 + e = 0, \\ cx_0 + dy_0 + f = 0. \end{cases}$$

Shift variables $\begin{aligned} u &= x - x_0, \\ v &= y - y_0 \end{aligned}$

$$\begin{cases} \dfrac{du}{dt} = au + bv, \\ \dfrac{dv}{dt} = cu + dv\,. \end{cases}$$

Fixed point is $(x_0, y_0) = (0,0)$

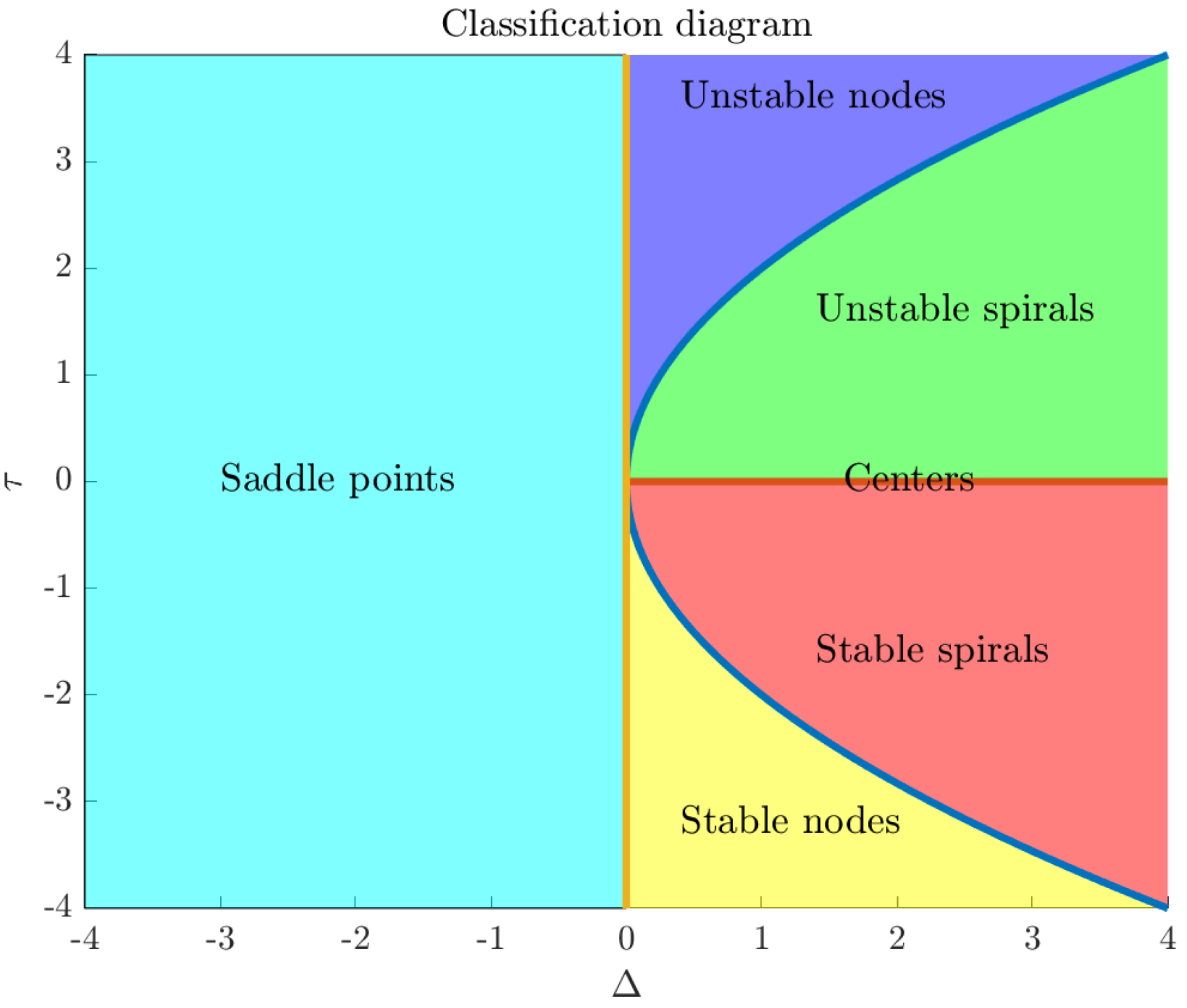
Fixed points of linear systems

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases} \quad \textbf{or } \dot{x}(t) = Ax(t)$$

$$\tau = \lambda_1 + \lambda_2 = a + d$$

$$\Delta = \lambda_1 \lambda_2 = ad - bc$$

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



Interactive version

Example

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$$

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$$

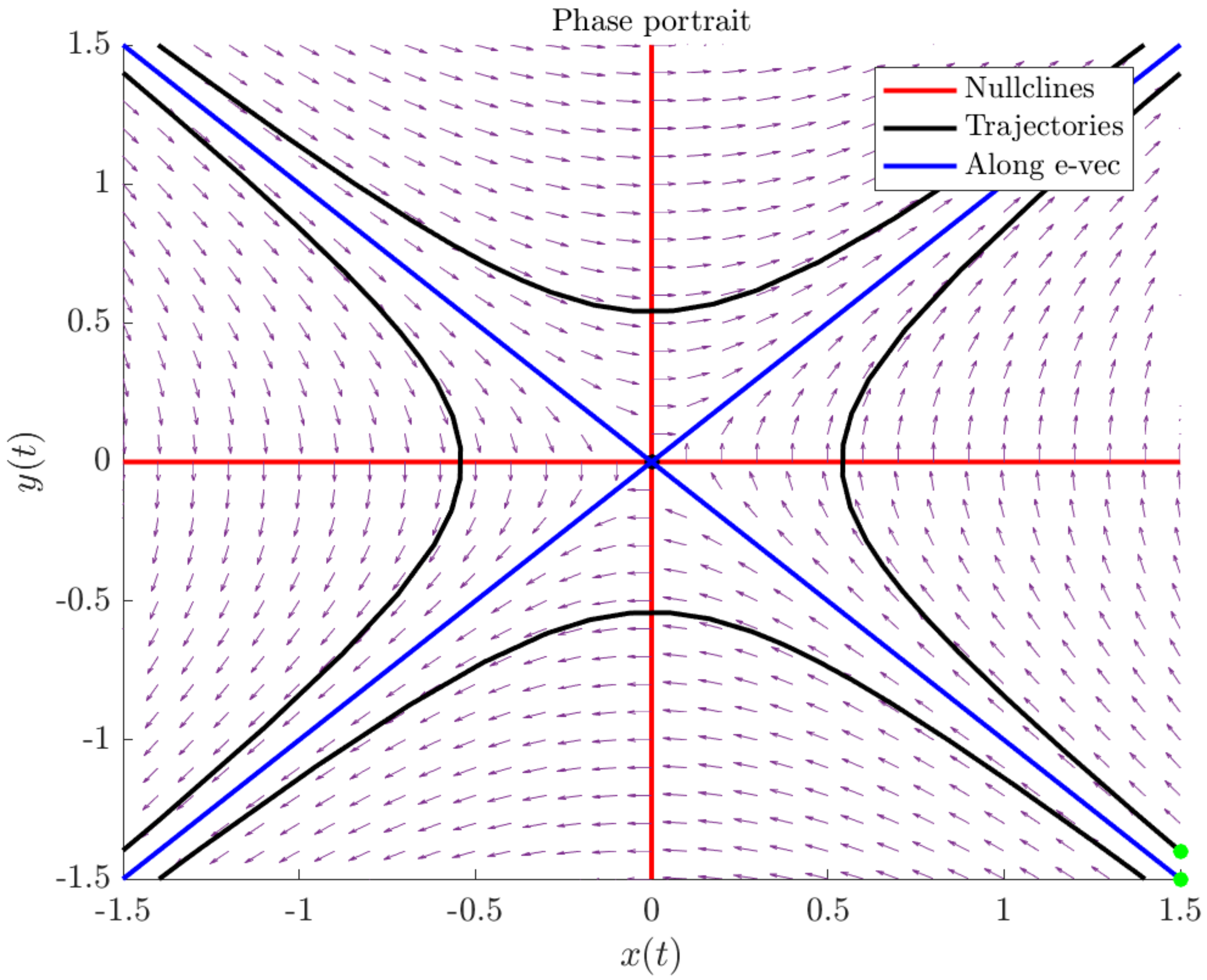
Nullclines: $y = 0$ **and** $x = 0$

E-vals: $\lambda_1 = 1,\ \lambda_2 = -1$

$$\lambda_2 < 0 < \lambda_1$$

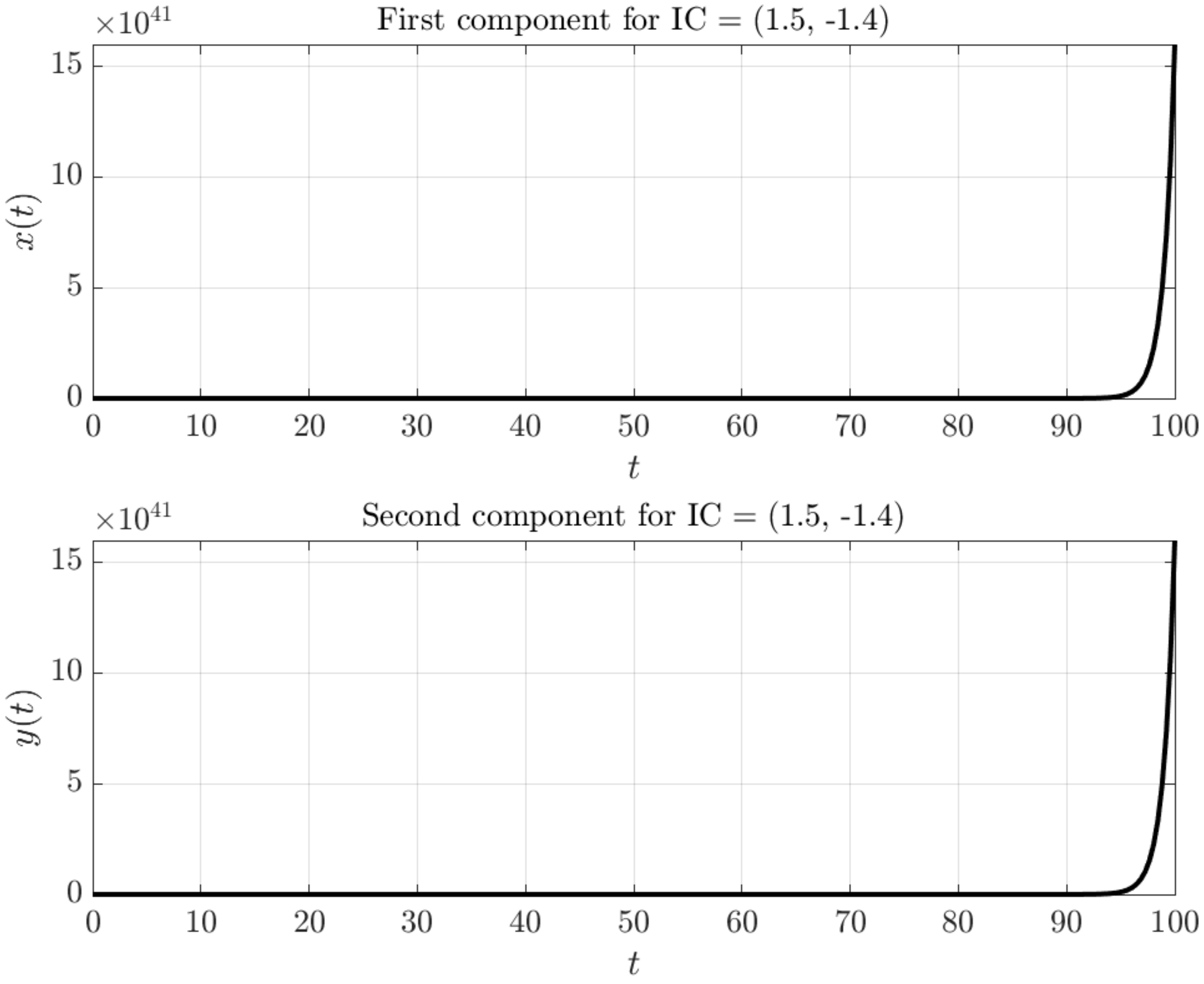
E-vecs: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution: $\begin{cases} x(t) = c_1 e^{t} + c_2 e^{-t} \\ y(t) = c_1 e^{t} - c_2 e^{-t} \end{cases}$

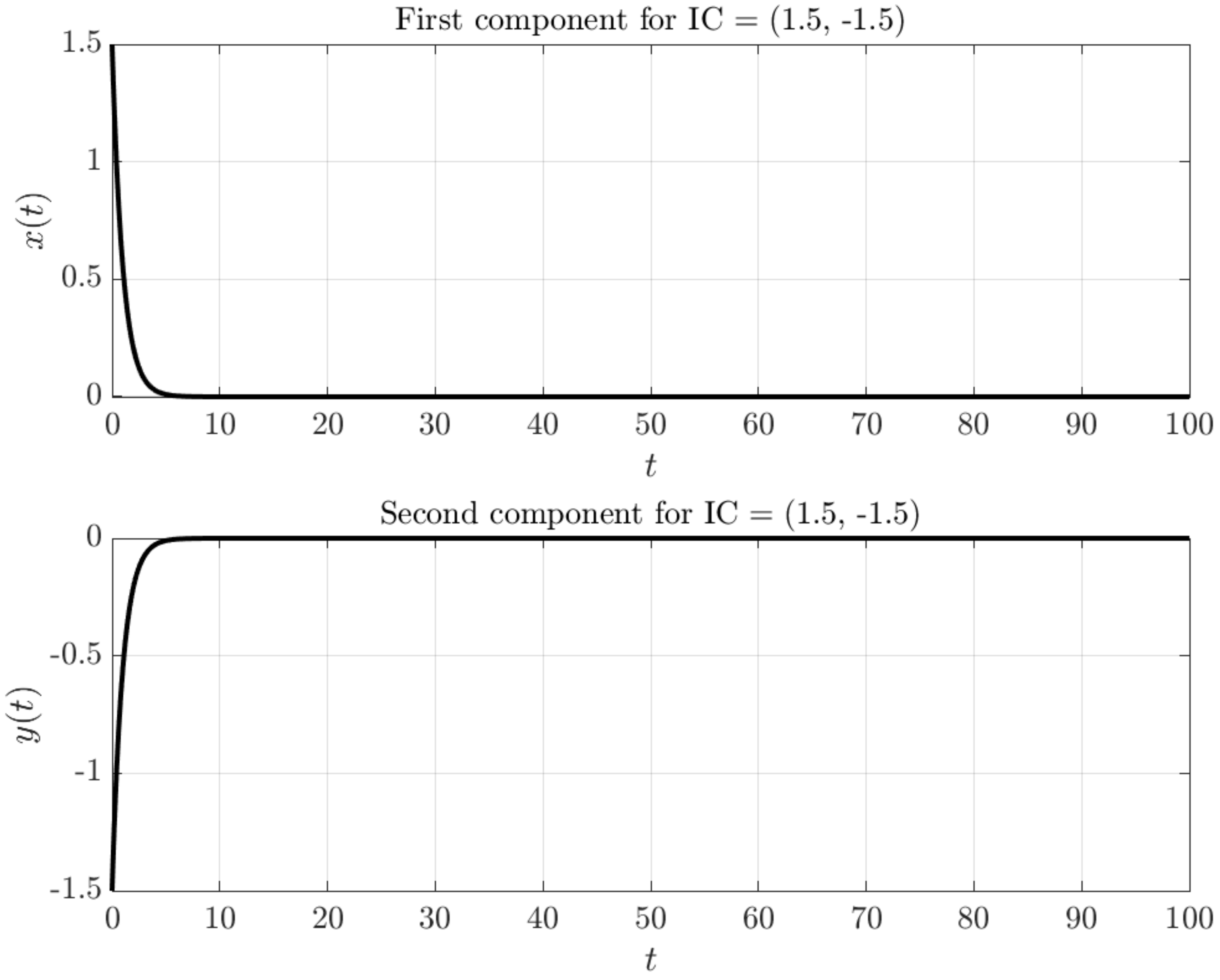


Possible solutions

Unbounded

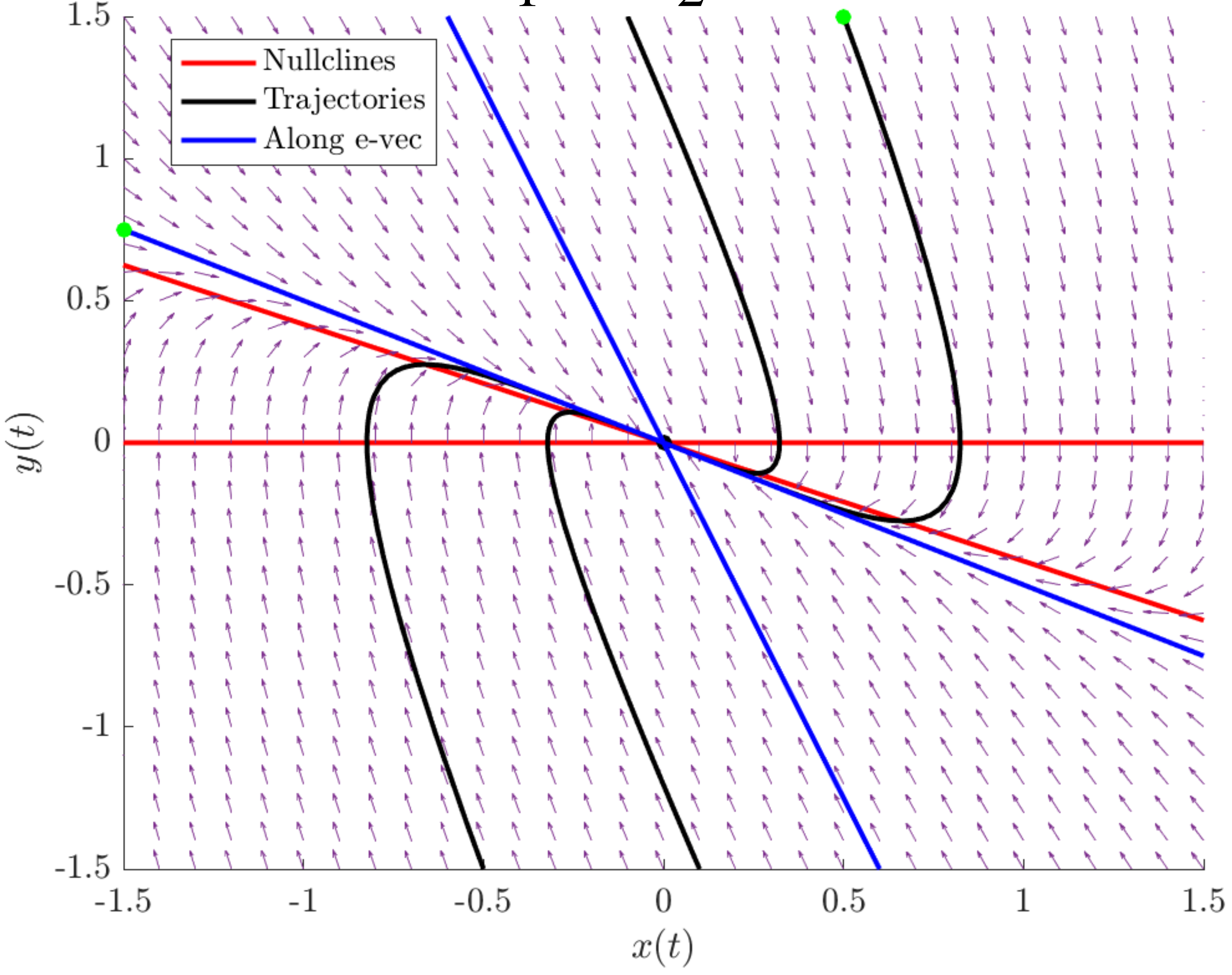


Bounded

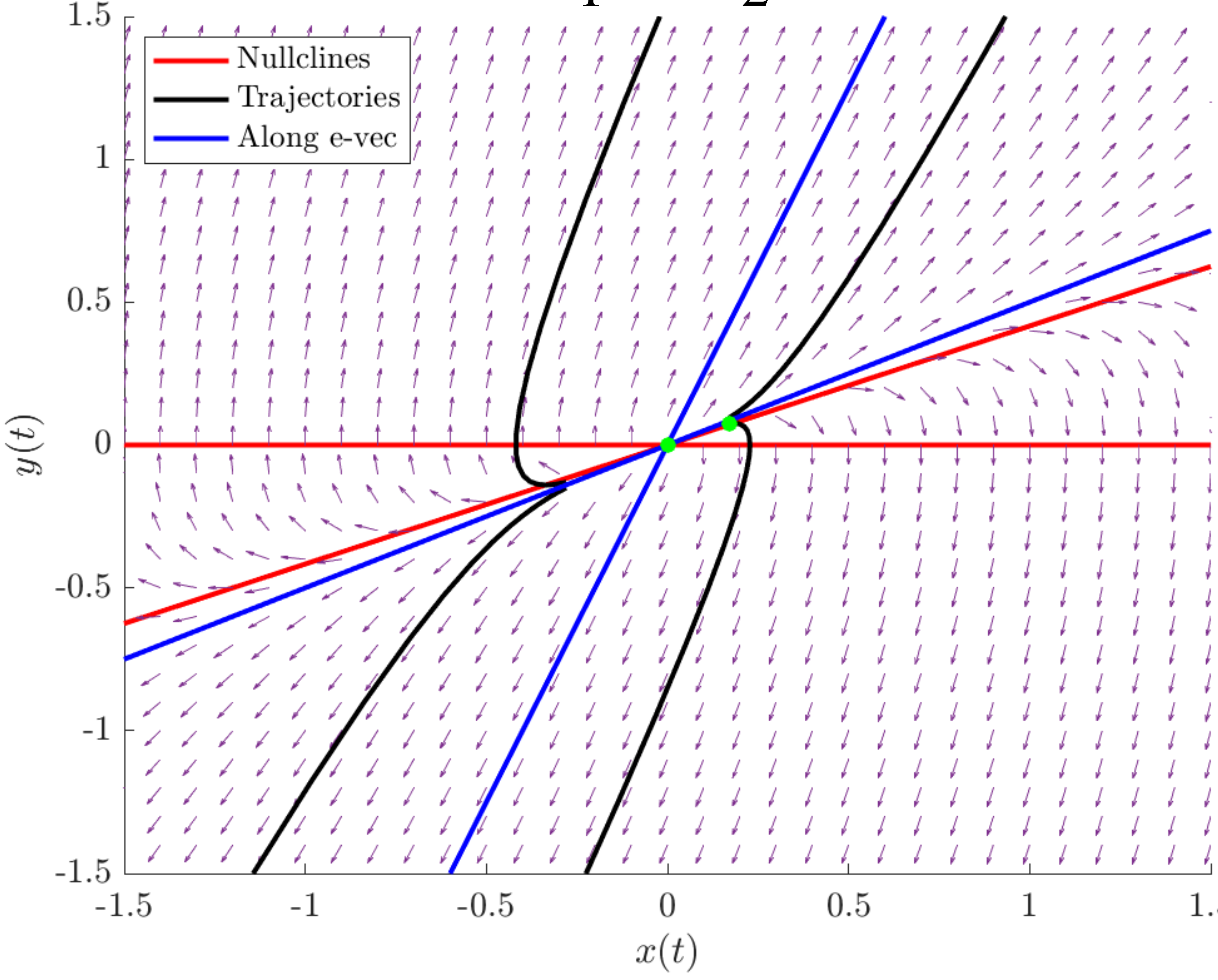


Nodes

$\lambda_1 < \lambda_2 < 0$

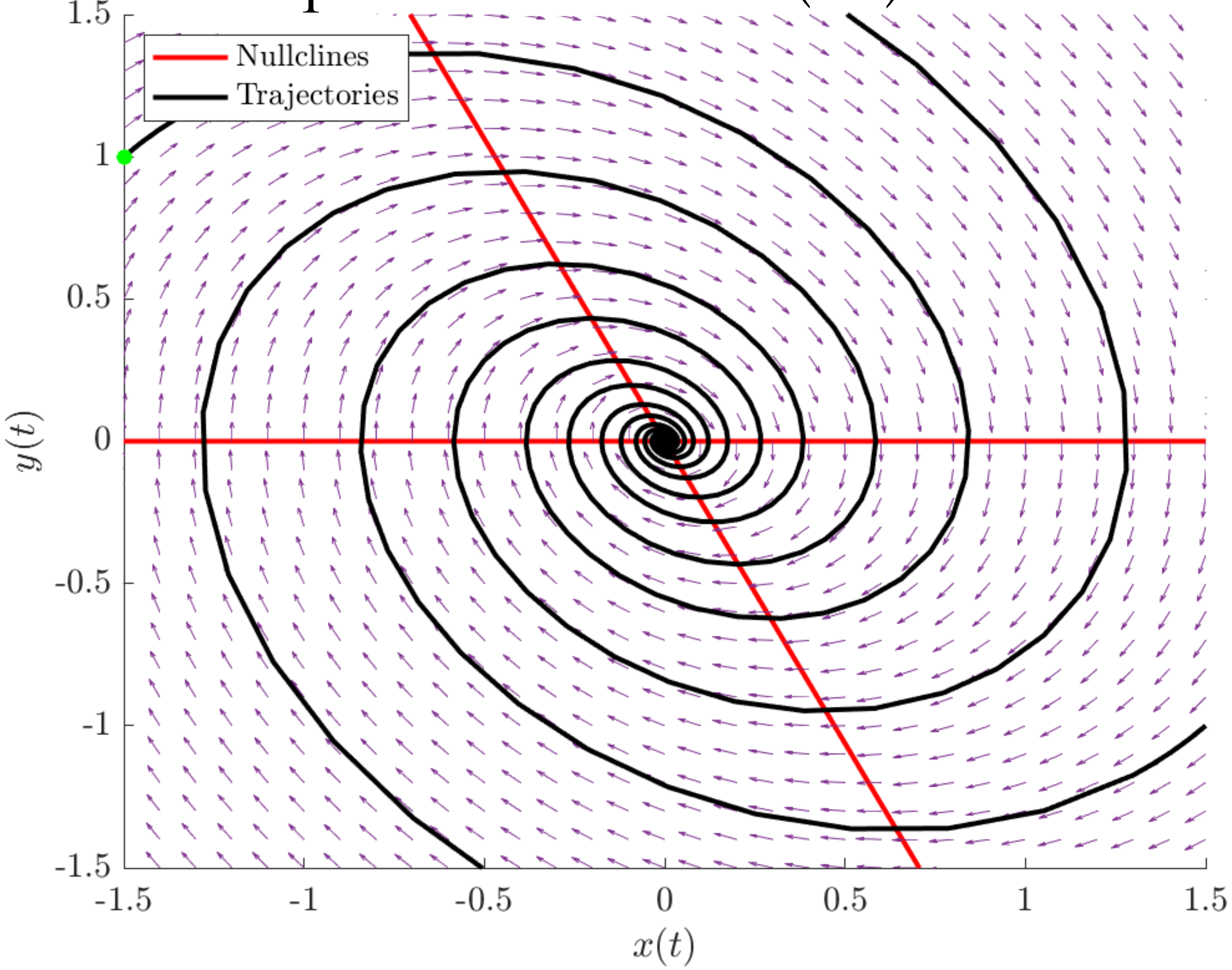


$\lambda_1 > \lambda_2 > 0$

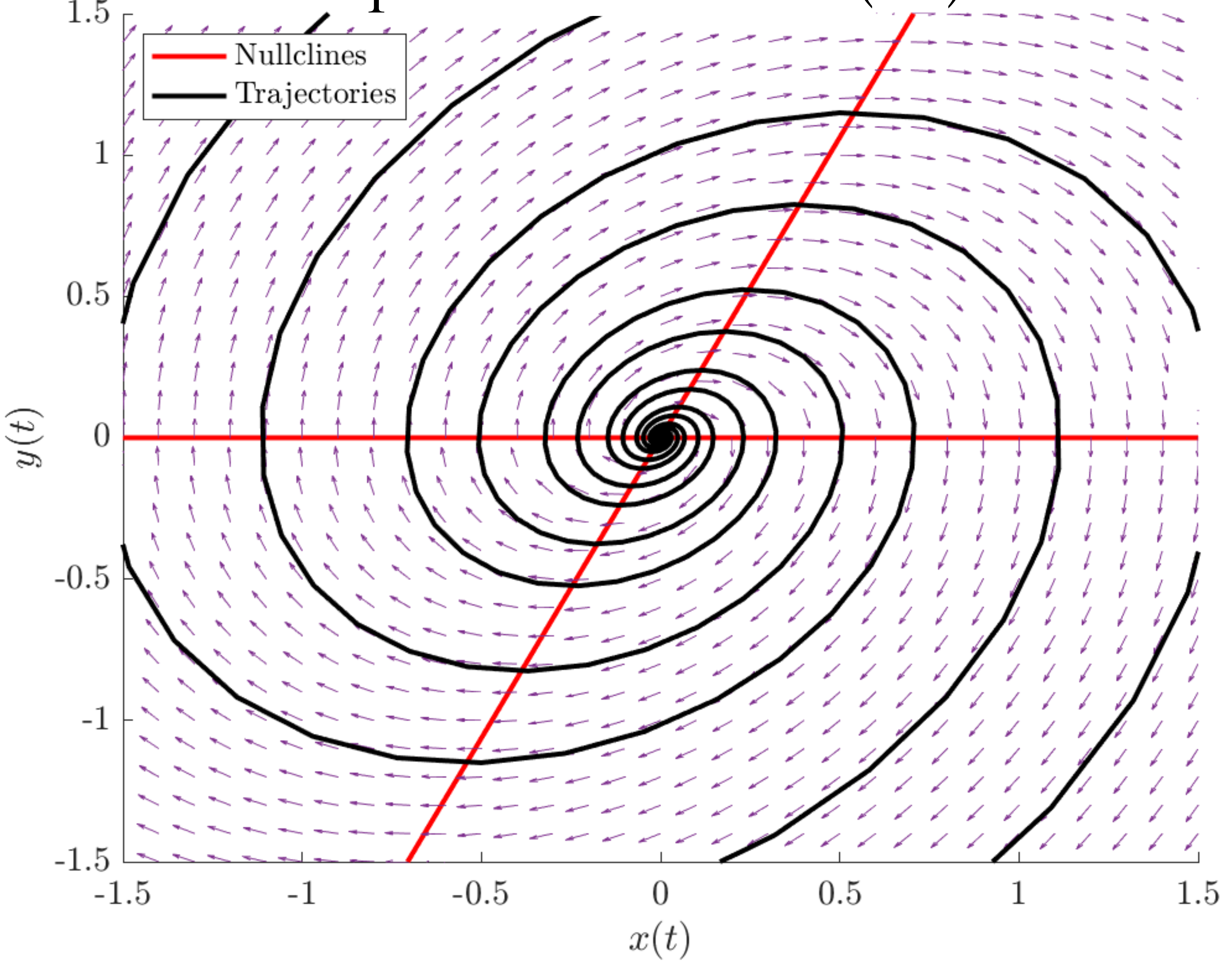


Spirals

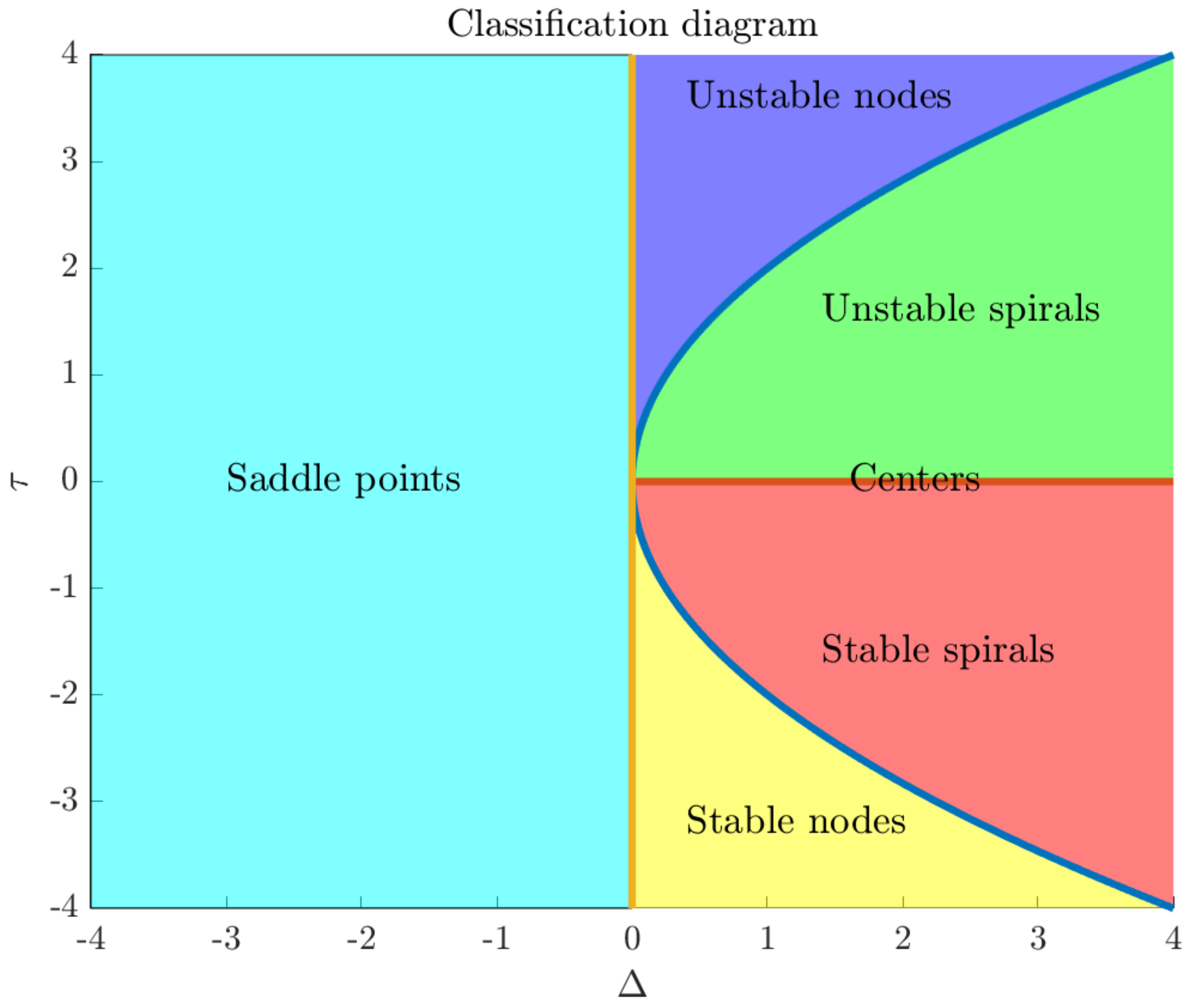
$\lambda_1^* = \lambda_2$ **with** $\mathrm{Re}\left(\lambda_1\right) < 0$



$\lambda_1^* = \lambda_2$ **with** $\mathrm{Re}\left(\lambda_1\right) > 0$



Degenerate cases



Center

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$$

Nullclines: $y = 0$ **and** $x = 0$

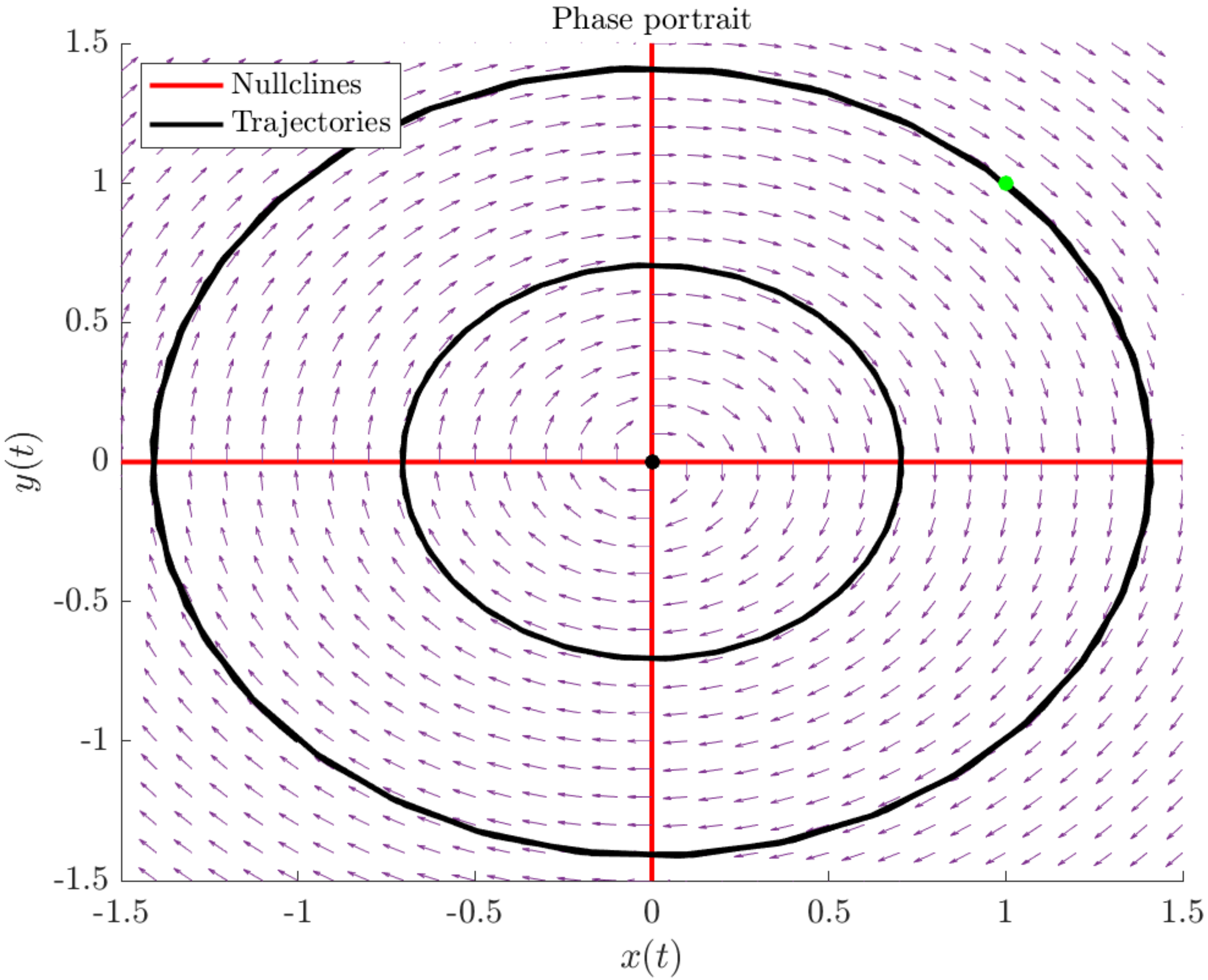
E-vals: $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_1^* = \lambda_2$

with $\mathrm{Re}\left(\lambda_1\right) = 0$

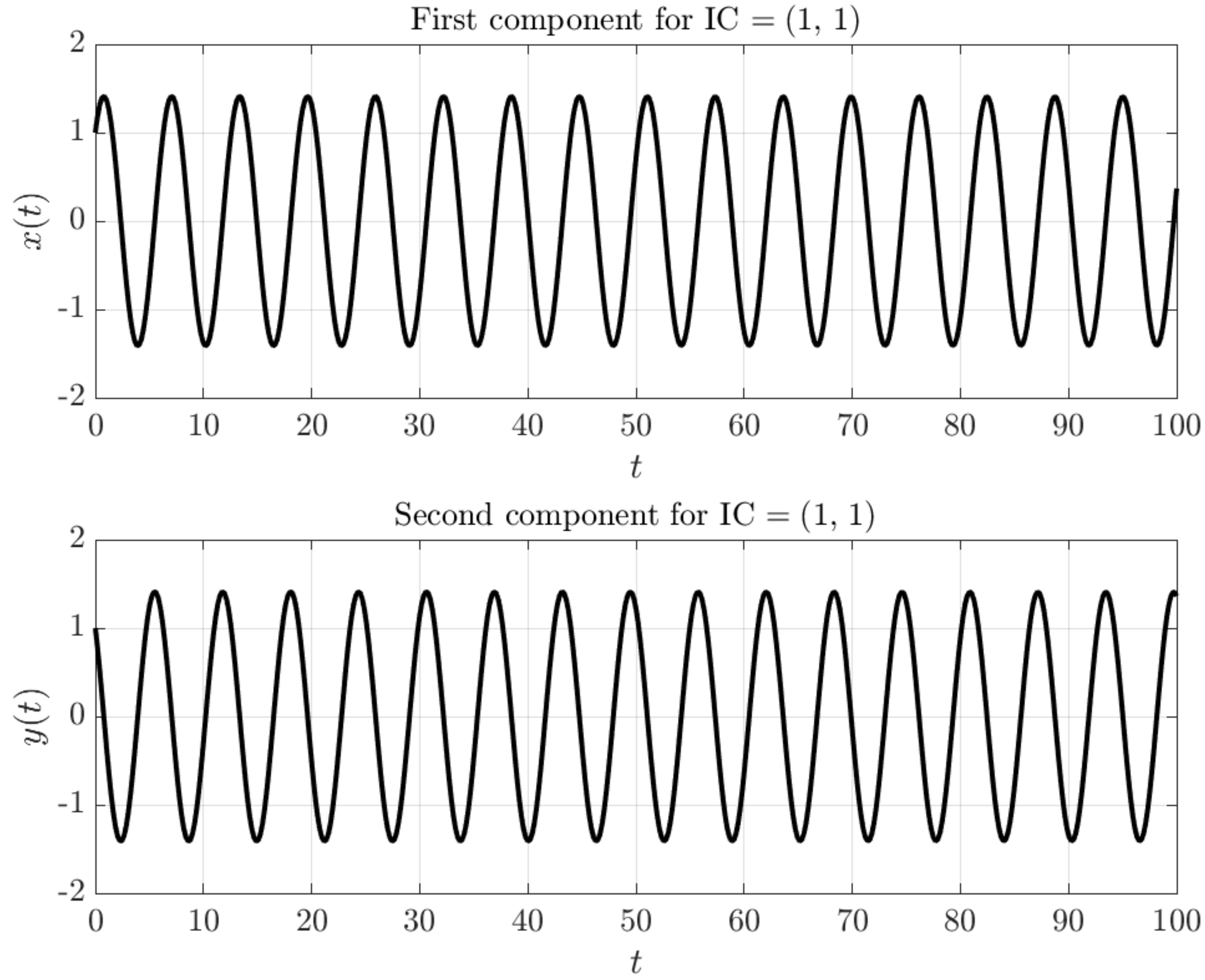
Solution:

$$\begin{cases} x(t) = c_1 \cos t + c_2 \sin t \\ y(t) = c_2 \cos t - c_1 \sin t \end{cases}$$

Neutrally stable.

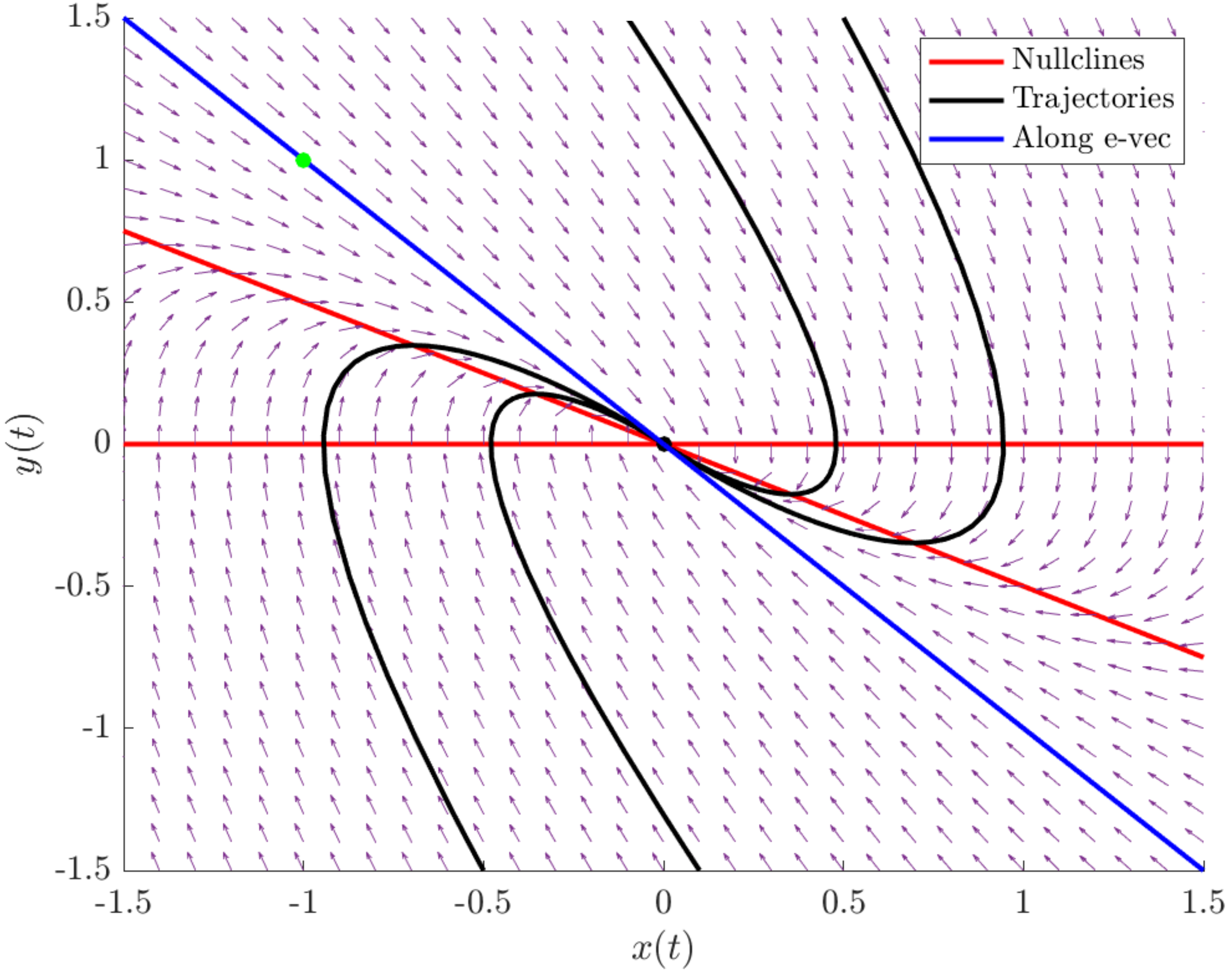


Possible solutions

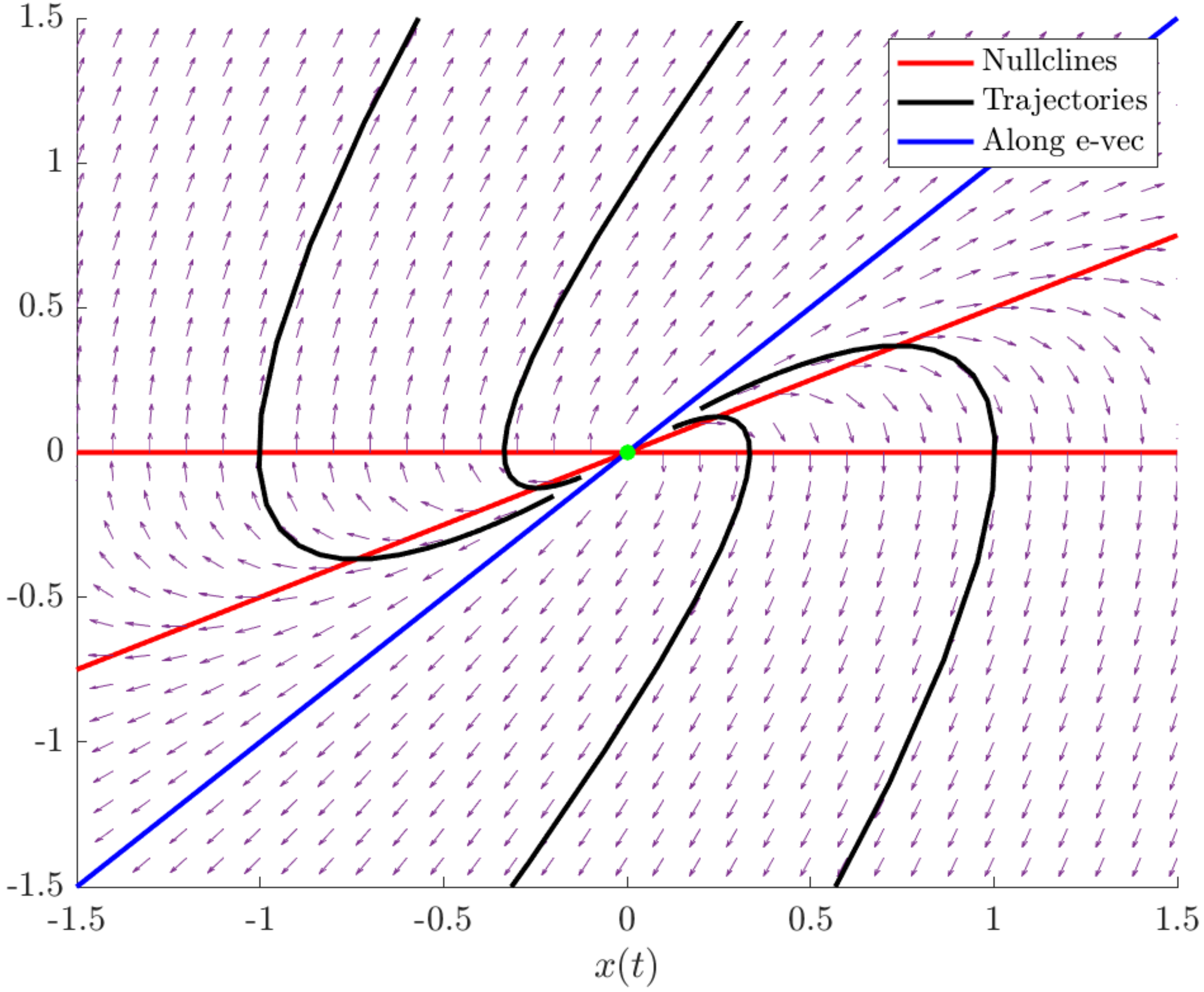


Stars

$\lambda_1 = \lambda_2 < 0$



$\lambda_1 = \lambda_2 > 0$



Nodes-spirals borderline

Saddle-nodes borderline

$$\begin{cases} \dot{x} = y \\ \dot{y} = -y \end{cases}$$

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}$$

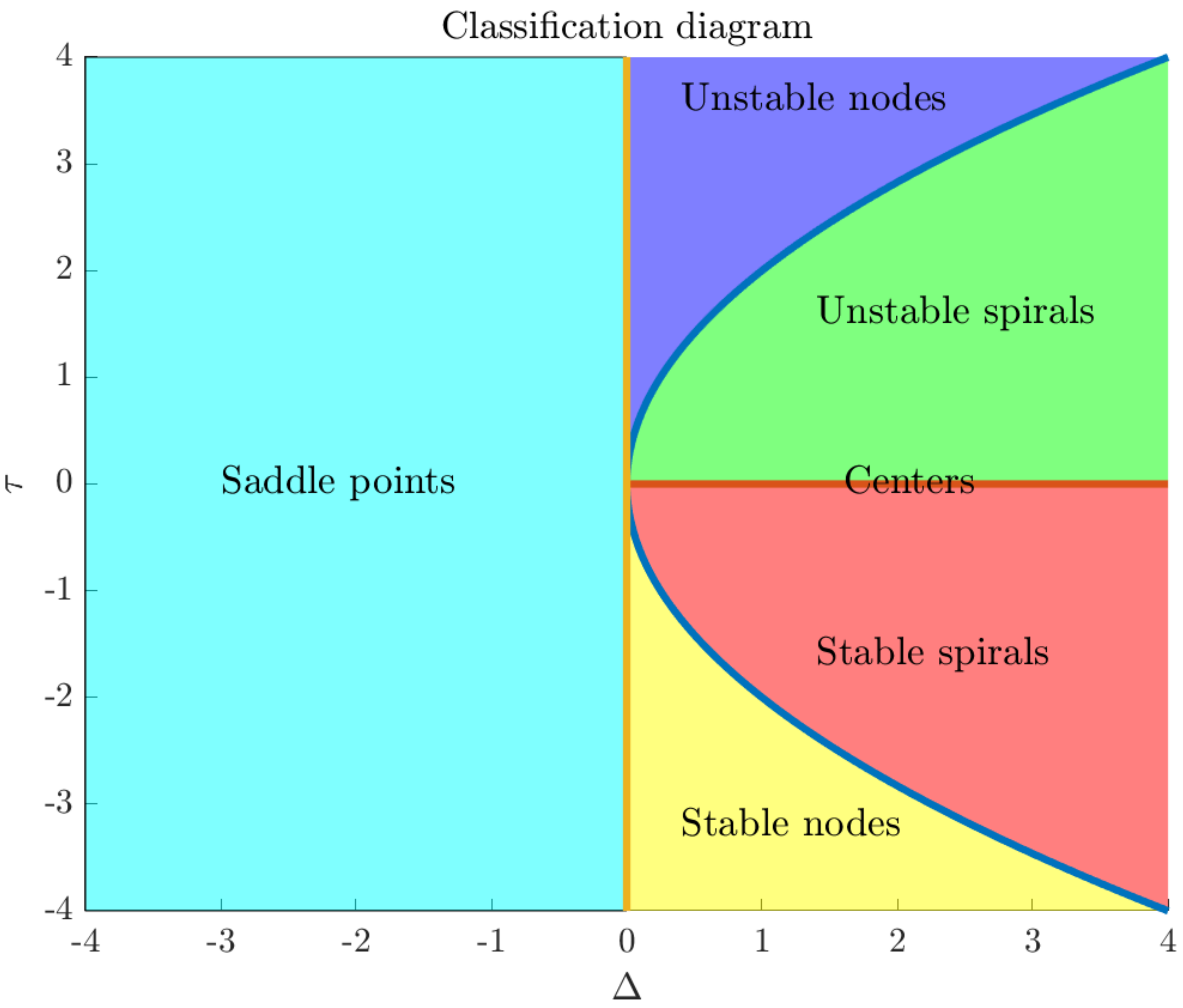
Nullclines: $y = 0$

E-vals: $\lambda_1 = -1,\ \lambda_2 = 0$**,** $\Delta = \lambda_1 \lambda_2 = 0$

E-vecs: $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution:

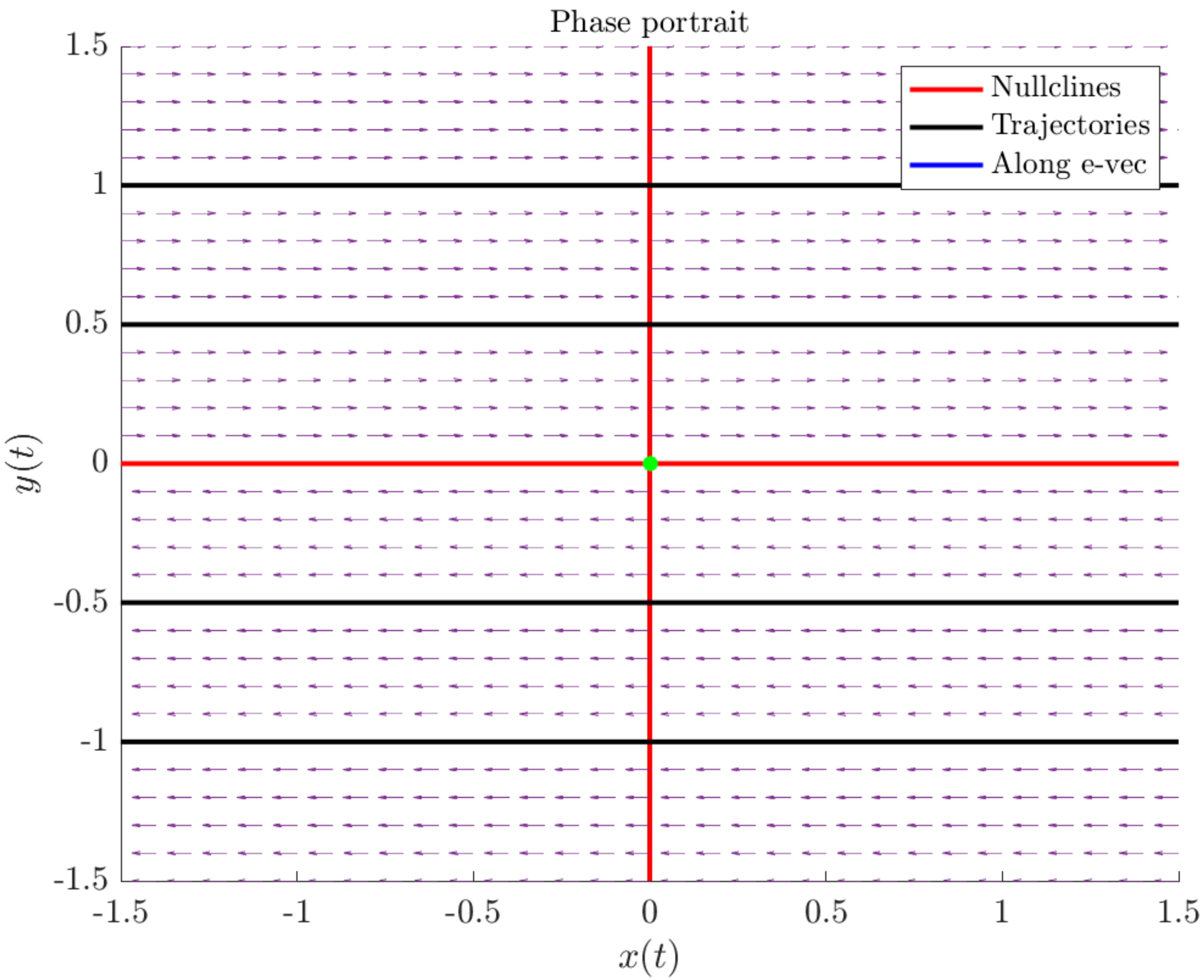
$$\begin{cases} x(t) = -c_1 e^{-t} + c_2 \\ y(t) = c_1 e^{-t} \end{cases}$$



Saddle-nodes borderline

Non-isolated fixed points

$$\lambda_1 = 0, \lambda_2 = 0$$



Non-isolated fixed points

Non-linear systems

$$\begin{cases} \dfrac{dx}{dt} = f(x,y), \\ \dfrac{dy}{dt} = g(x,y). \end{cases}$$

Shift variables

$$\bar{x}(t) = x(t) - x_0,$$
$$\bar{y}(t) = y(t) - y_0.$$

$$\frac{d\bar{x}}{dt} = f\left(x_0 + \bar{x}(t), y_0 + \bar{y}(t)\right) \approx$$

$$f\left(x_0, y_0\right) + f_x\left(x_0, y_0\right)\bar{x}(t) + f_y\left(x_0, y_0\right)\bar{y}(t)$$

$$\frac{d\bar{y}}{dt} = g\left(x_0 + \bar{x}(t), y_0 + \bar{y}(t)\right) \approx$$

$$g\left(x_0, y_0\right) + g_x\left(x_0, y_0\right)\bar{x}(t) + g_y\left(x_0, y_0\right)\bar{y}(t)$$

$$\frac{d\bar{\mathbf{x}}}{dt} = \begin{pmatrix} f_x\left(x_0, y_0\right) & f_y\left(x_0, y_0\right) \\ g_x\left(x_0, y_0\right) & g_y\left(x_0, y_0\right) \end{pmatrix} \bar{\mathbf{x}}$$

If $\det J\left(x_0, y_0\right) \neq 0$, $J\left(x_0, y_0\right)$ does not have purely imaginary eigenvalues, then non-linear system has the same qualitative orbital structure near $\left(x_0, y_0\right)$ as the linearized system has near $(0,0)$.

Be careful with

Centers: use polar coordinates or the whole phase portrait

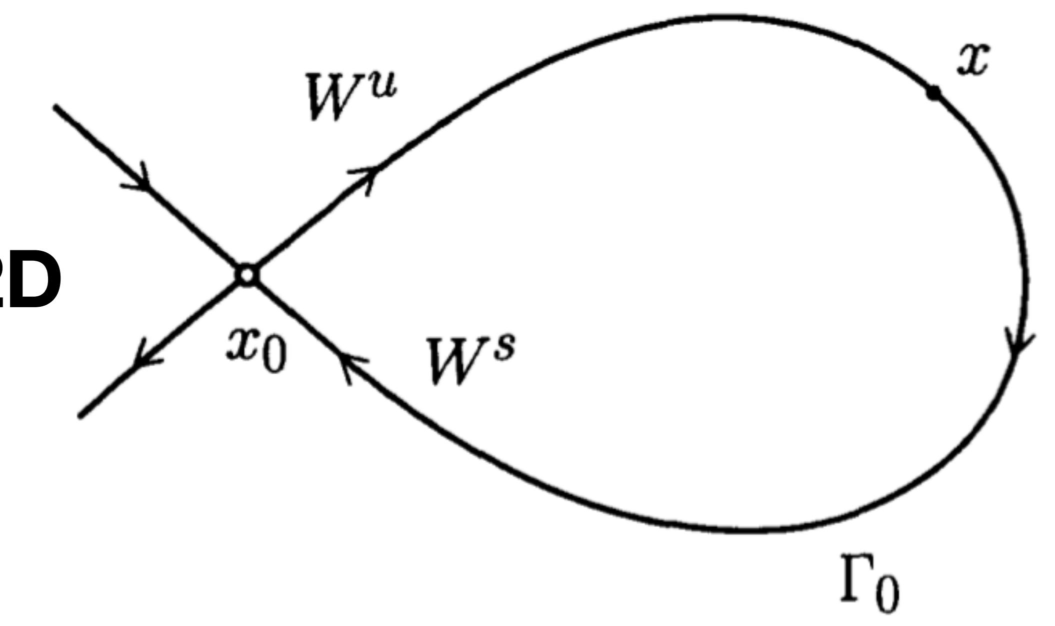
$\det J = 0$: use whole phase portrait

1. **Find all the critical points.**
2. **Analyze their nature and stability.**
3. **Sketch the phase portrait with nullclines, e-vectors, and velocity field.**
4. **Use sketching principle. Two trajectories cannot intersect: existence and uniqueness theorem for systems.**
5. **Examine the global behaviour and structure of the orbits.**

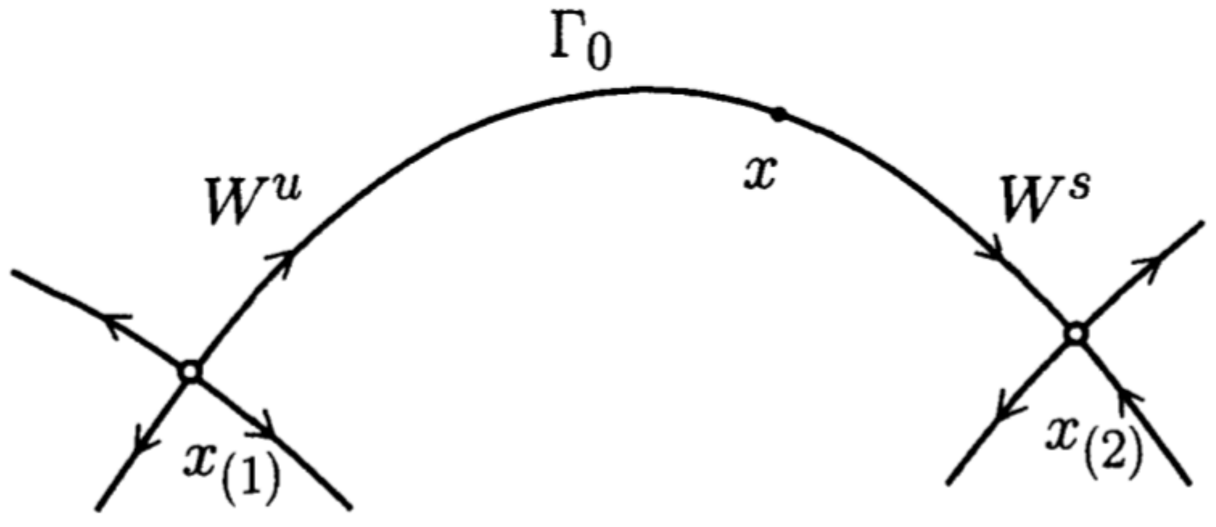
Non-linear features

Homoclinic orbits | **Heteroclinic orbits**

2D

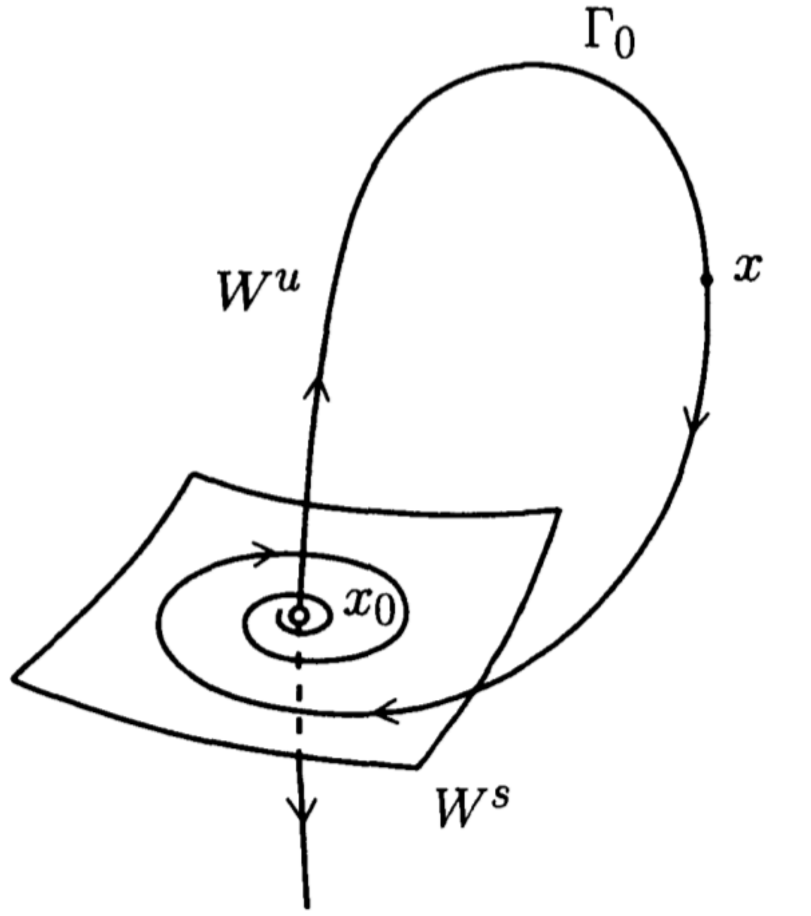


(a)

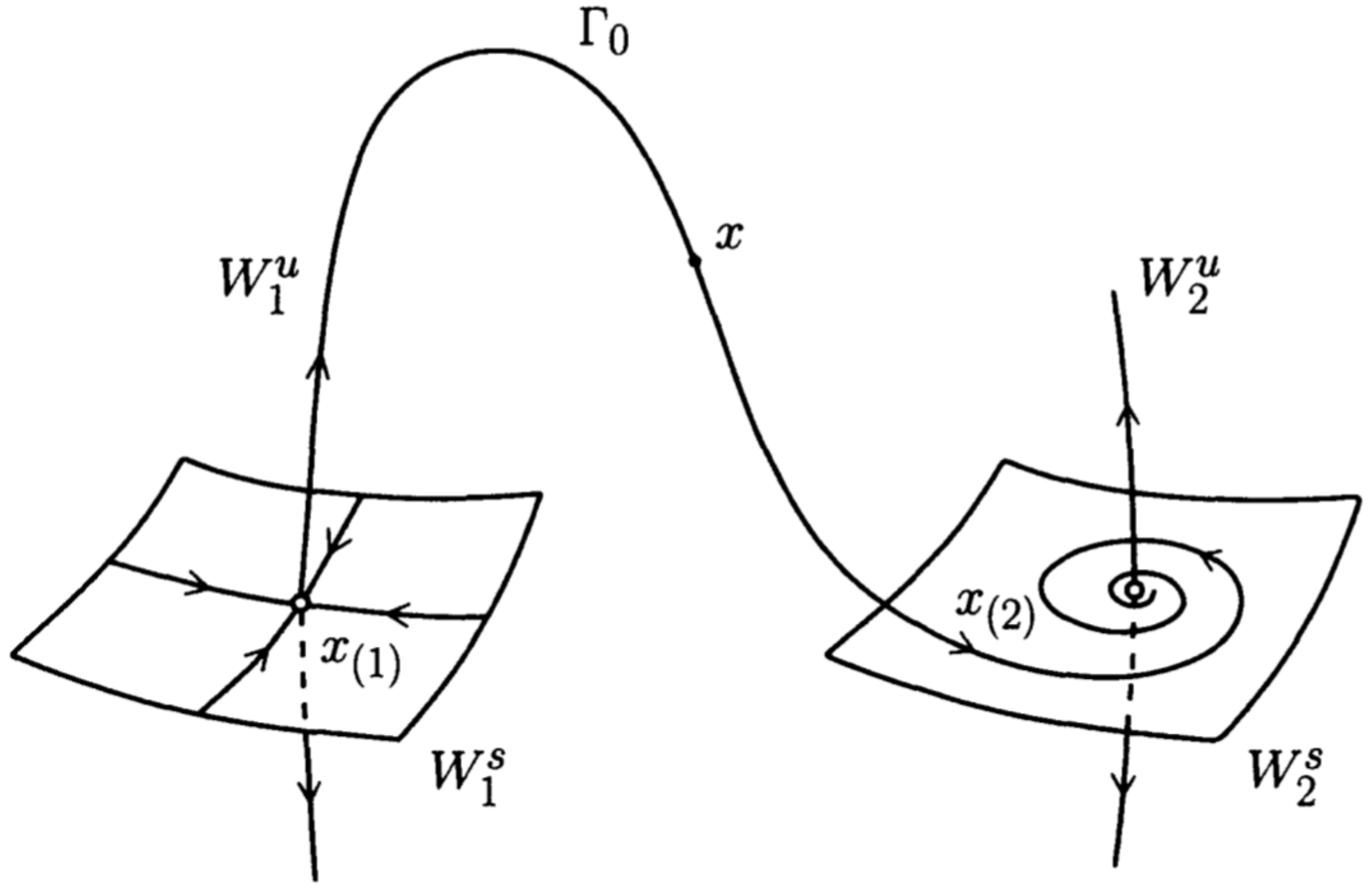


(b)

3D

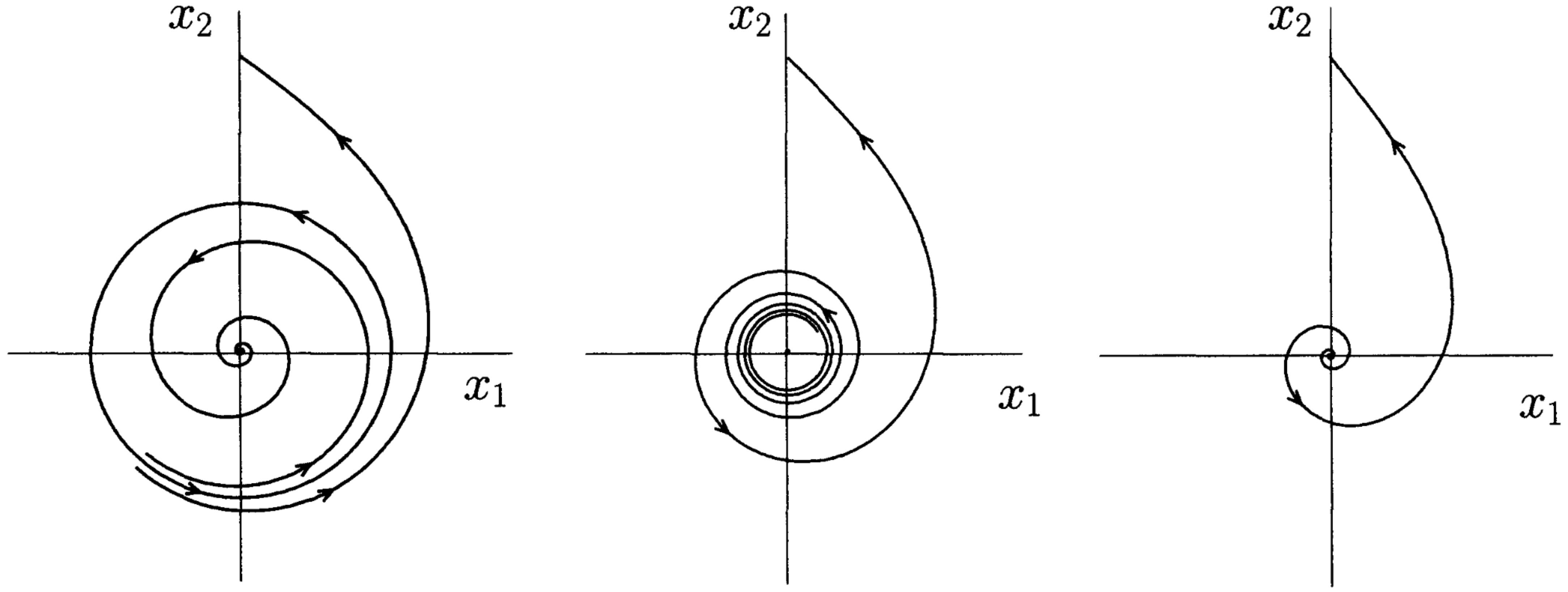


(a)



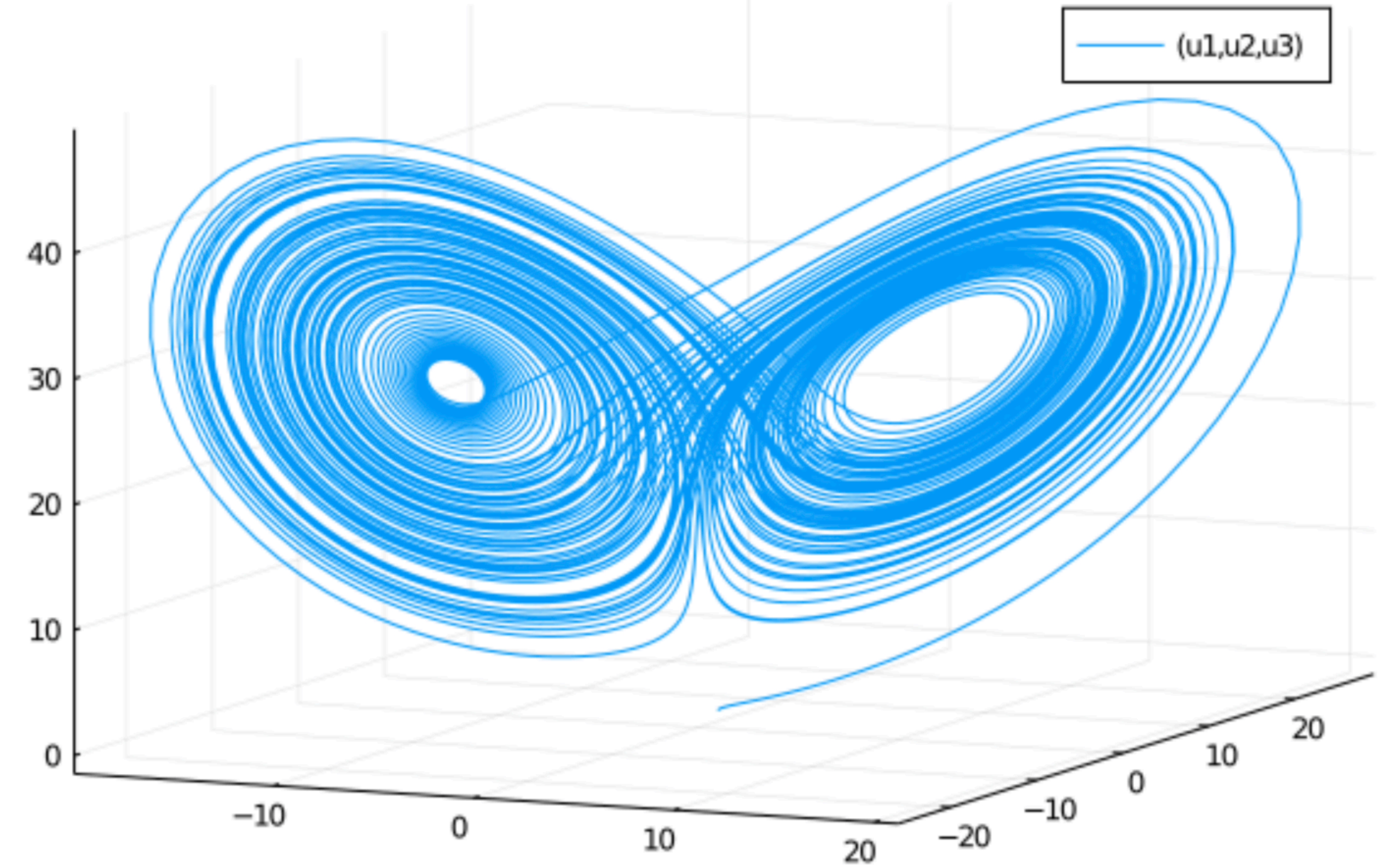
(b)

Bifurcations



$\alpha < 0$ | $\alpha = 0$ | $\alpha > 0$

Chaos



Nonlinear example 1

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x - y + x^2 - 2xy \end{cases}$$

Fixed points: $(0,0)$ **and** $(-1,0)$

Jacobian: $\begin{pmatrix} 0 & 1 \\ 1 + 2x - 2y & -1 - 2x \end{pmatrix}$

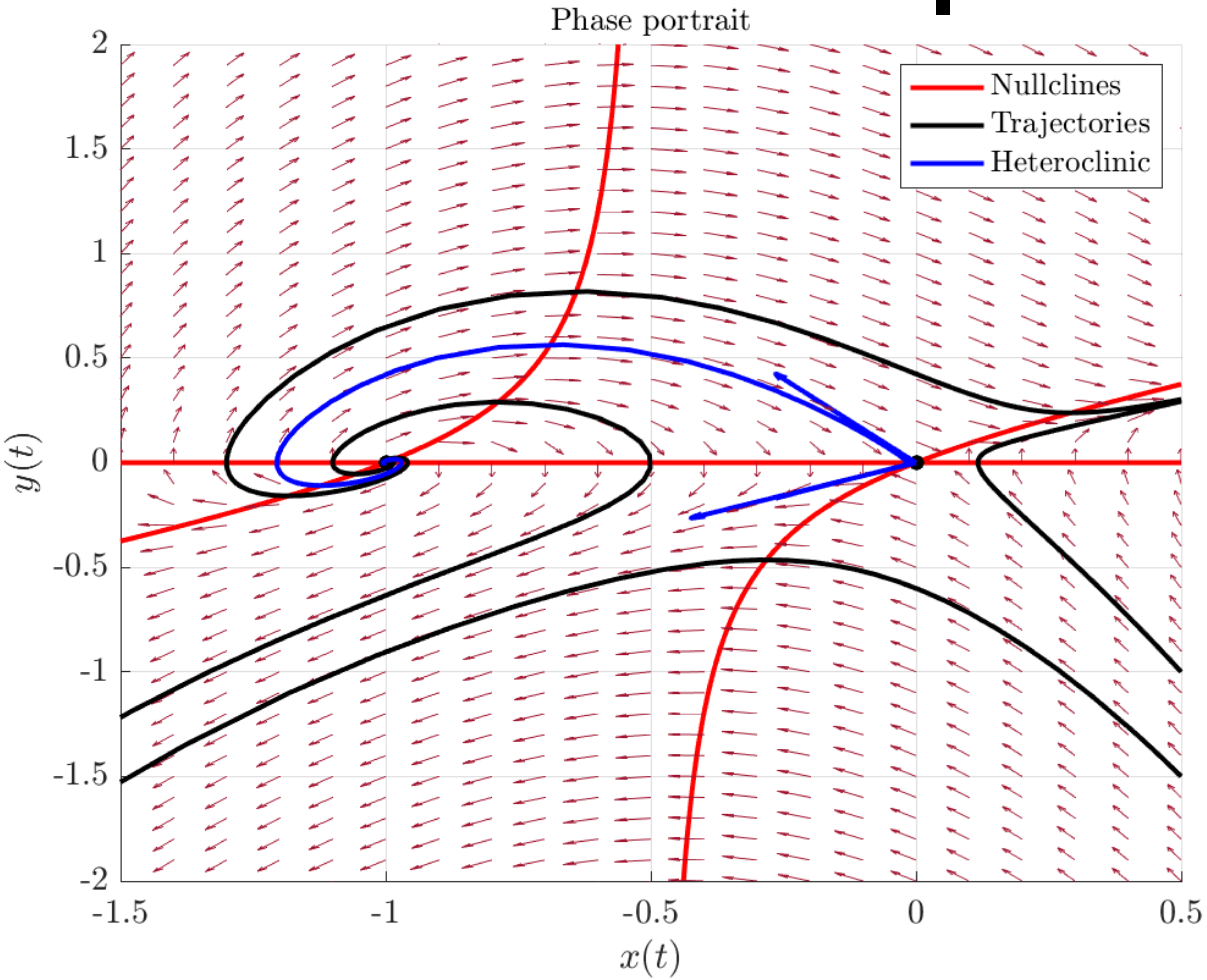
$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \det J = -1 < 0 \Rightarrow$$ **Saddle point, e-vectors:**

$v_1 = \begin{pmatrix} -0.53 \\ 0.85 \end{pmatrix}$ **for** $\lambda_1 < 0$ **and** $v_2 = \begin{pmatrix} -0.85 \\ -0.53 \end{pmatrix}$ **for** $\lambda_2 > 0$

$$J(-1,0) = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} \Rightarrow \det J = 2 > 0, \tau = 1 > 0, \tau^2 - 4\Delta = -7 < 0 \Rightarrow$$

Unstable spiral

Nonlinear example 1



Nonlinear example 2

$$\begin{cases} \dfrac{dx}{dt} = -x + x^3 \\ \dfrac{dy}{dt} = -2y \end{cases}$$

Fixed points: $(0,0)$, $(-1,0)$ **and** $(1,0)$.

Jacobian: $\begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det J = 2 > 0,\ \tau = -3 < 0,\ \tau^2 - 4\Delta = 1 > 0 \ \Rightarrow$$

Stable node, e-vectors: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **and** $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$J(-1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det J = -4 < 0 \Rightarrow$ **Saddle point, e-vectors:** $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **and** $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$J(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det J = -4 < 0 \Rightarrow$ **Saddle point, e-vectors:** $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **and** $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Nonlinear example 2