Mathematical Methods in Engineering and Applied Science
Problem Set 8.

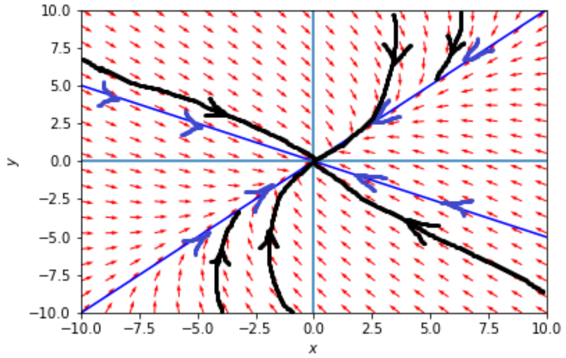
(1). lot the phase portrait and classify the fixed points for the following systems:

(a) $\dot{x} = -3x + 2y,\ \dot{y} = x - 2y;$

$u = [x, y]^T; A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix};$

$\frac{du}{dt} = Au$

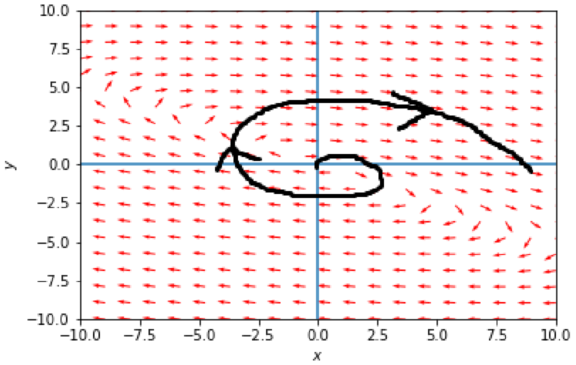
Fixed point	v_1, v_2	λ_1, λ_2	Type
(0,0)	[1,1],[-2,1]	$-1, -4$	Stable node



(b) $\dot{x} = 5x + 10y,\ \dot{y} = -x - y$

$A = \begin{bmatrix} 5 & 10 \\ -1 & -1 \end{bmatrix}$

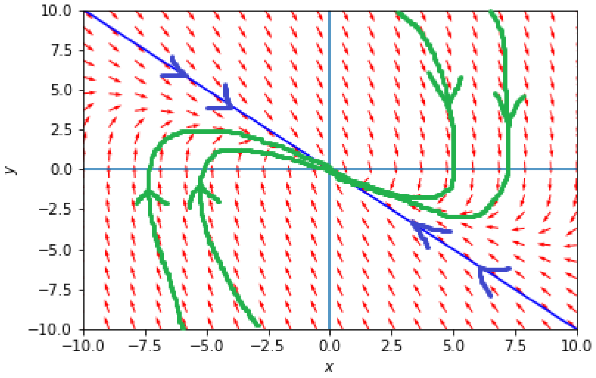
Fixed point	v_1, v_2	λ_1, λ_2	Type
(0,0)	[-3+i,1],[-3-i,1]	$2 - i, 2 + i$	Unstable spiral



(c) $\dot{x} = y,\ \dot{y} = -x - 2y$

$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Fixed point	v_1, v_2	λ_1, λ_2	Type
(0,0)	[-1,1],[-1,1]	$-1, -1$	stable node.



(2). Suppose the relationship between Romeo and Juliet is such that

$$\dot{R} = aR + bJ, \qquad \dot{J} = -bR - aJ$$

with positive a and b. Describe the type of the relationship and explain its fate depending on the initial conditions.

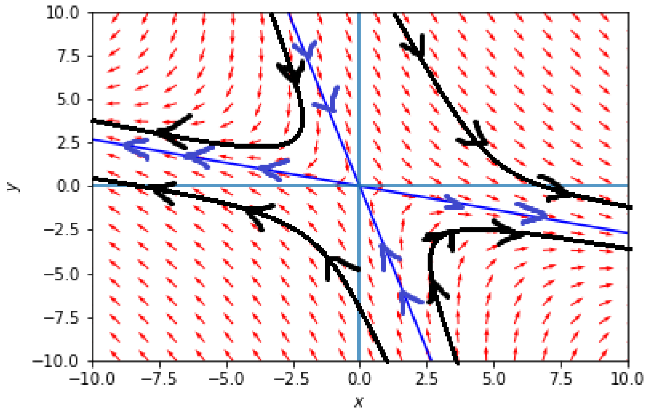
Romeo attitude towards to Juliet changes in the form of sum his attitude to J. (aR) and her attitude to him (bJ). The more R positive towards J and the more J positive towards R the more R will be positive towards J in the next moment of time.

J is opposite. The more R positive towards J and the more J positive towards R the more J will be negative towards R in the next moment of time.

Let's consider $A = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$ the Fixed point is $[R, J] = [0,0]$.

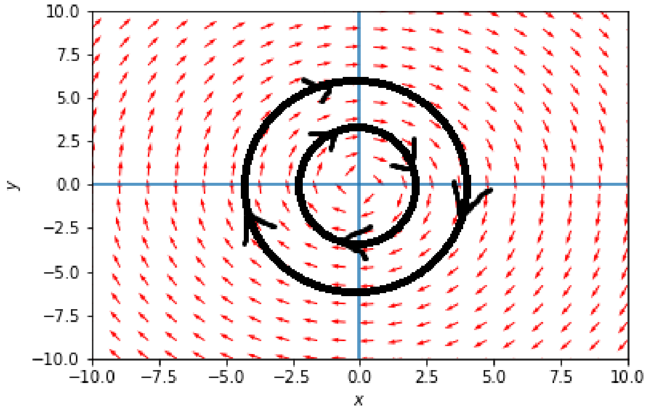
Fixed point	v_1, v_2	λ_1, λ_2	type
(0,0)	$\left[-\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b}, 1\right]$, $\left[-\frac{a}{b} - \frac{\sqrt{a^2 - b^2}}{b}, 1\right]$,	$-\sqrt{a^2 - b^2}$; $\sqrt{a^2 - b^2}$;	Saddle node

For $a > b$ e-vectors are real, and we have such phase portrait:



We can predict that in the future R will be infinitely + and J infinitely – or vice versa R will be infinitely - and J infinitely +.

For $a < b$ e-vectors becomes complex, and we have such phase portrait:



They always will change attitudes towards each other but $R^2 + J^2 = const(t)$

(3). For the system

$$\dot{x} = xy - 1, \qquad \dot{y} = x - y^3,$$

find the fixed points, classify them, sketch the neighboring trajectories and try to fill in the rest of the phase plane.

Fixed points:

$$x = \frac{1}{y} => \frac{1}{y} - y^3 = 0 => y^4 = 1$$

$$y^4 = e^{2\pi n} => y = e^{\frac{2\pi n}{4}} = e^{\frac{\pi n}{2}} = 1, -1, i, -i$$

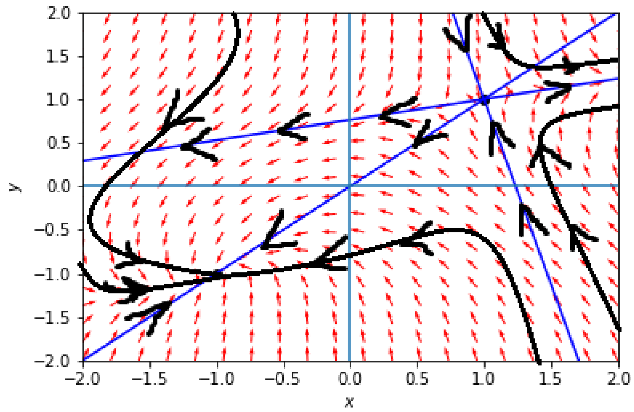
Fixed points:

x	1	-1	$-i$	i
y	1	-1	i	$-i$

$$J = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix}$$

Fixed point	v1, v2	λ_1, λ_2	Type
$(1,1)$	$[2+\sqrt{5}, 1]$ $[2-\sqrt{5}, 1]$	$-1+\sqrt{5}$ $-1-\sqrt{5}$	Saddle point
$(-1,-1)$	$[1,1]$ $[1,1]$	-2 -2	Stable node
$(-i, i)$	…		

I don't know how to represent imaginary fixed points, though there are only real.



(4). For the following model of rabbits and sheep, find the fixed points, investigate their stability and draw the phase portrait. Indicate the basins of attraction of any stable fixed point:

$$\dot{x} = x\,(3 - 2x - 2y)\,, \qquad \dot{y} = y\,(2 - x - y)\,.$$

$x\,(3 - 2x - 2y) = 0$

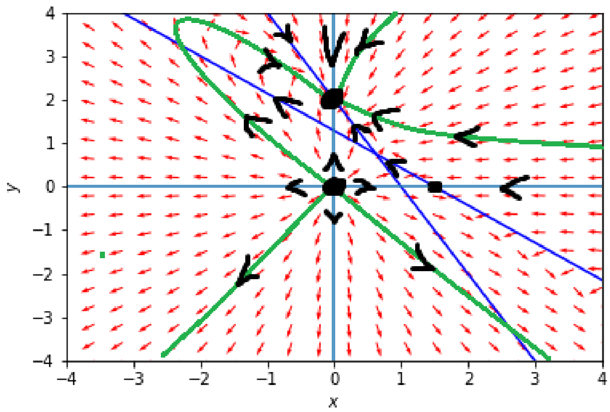
$y\,(2 - x - y) = 0$

Fixed points

x	0	0	3/2
y	0	2	0

$$J = \begin{bmatrix} -4x - 2y + 3 & -2x \\ -y & -x - 2y + 2 \end{bmatrix}$$

Fixed point	v1, v2	λ_1, λ_2	Type
(0,0)	[0,1] [1,0]	2 3	Unstable node
(0,2)	[0,1] $\left[-\frac{1}{2}, 1\right]$	-2 -1	stable node
$\left(\frac{3}{2}, 0\right)$	[1,0] $\left[-\frac{6}{7}, 1\right]$	-3 $\frac{1}{2}$	Saddle point



(5). Consider the system

$$\dot{x} = -y - x^3, \qquad \dot{y} = x$$

Show that the origin is a spiral, although the linearization predicts a center.

$$J = \begin{bmatrix} -3x^2 & -1 \\ 1 & 0 \end{bmatrix}$$

Fixed point	v1, v2	λ_1, λ_2	Type
(0,0)	$[-I, 1]$ $[I, 1]$	$-I$ I	Center

$\lambda_1, = \lambda_2^*, Re(\lambda_1) = 0 - center$

If we change coordinates to:

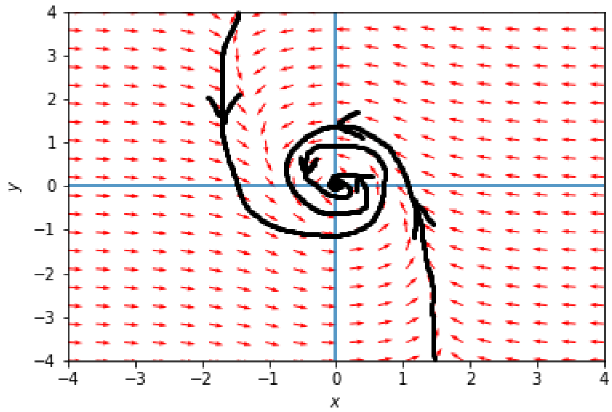
$$x = r\,cos(\phi), y = r\sin(\phi),$$

Consider derivative of $r^2 = x^2 + y^2$:

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} = 2x(-y - x^3) + 2yx = -y\,2x - x^3 2x + 2yx = -x^3 2x$$

$$\dot{r} = -\frac{x^4}{r} = -\frac{r^4\cos(\phi)}{r} = -r^3\cos(\phi)$$

Radius is always decreasing. It is spiral.



(6). he Kermack-McKendrick model of an epidemic describes the population of healthy people x (t) and sick people y (t) in terms of the equations

$$\dot{x} = -kxy$$
$$\dot{y} = kxy - ly,$$

where k, l > 0. Here, l is the death rate of the sick people, and kxy in equation for $\dot{y}$ implies that people get sick at a rate proportional to their encounters (which itself is proportional to the product of the number of sick people y and healthy people x). The parameter k measures the probability of transmission of the disease during the encounters.

(a) Find and classify the fixed points.
Fixed points: $(y = 0, x \in R)$

(b) Sketch the nullclines and the vector field.

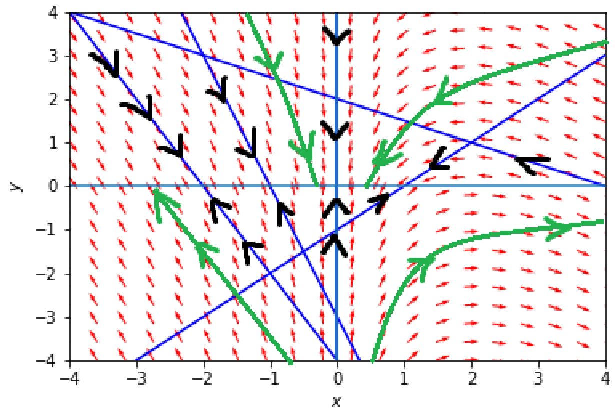
$$J = \begin{bmatrix} -\mathrm{ky} & -\mathrm{kx} \\ \mathrm{ky} & \mathrm{kx}-\mathrm{l} \end{bmatrix}$$

$$J = \begin{bmatrix} -kx + l & -\mathrm{kx} \\ 0 & 0 \end{bmatrix} v = 0$$

$$\begin{bmatrix} -kx + l & -\mathrm{kx} \\ 0 & 0 \end{bmatrix} v = (-kx + l)v1 - kxv2 = 0$$

Fixed point	v1, v2	λ_1, λ_2	Type
$(x, 0)$	$[1,0]$ $\left[-\frac{kx}{kx-l}, 1\right]$	0 $kx - l$	Saddle-nodes borderline

X:	v2	λ_2
$-\infty < x < 0$	$\left[-\frac{kx}{kx-l} < 0, 1\right]$	$kx - l < 0$
$x = 0$	$[0,1]$	$kx - l < 0$
$0 < x < \frac{l}{k}$	$\left[-\frac{kx}{kx-l} > 0, 1\right]$	$kx - l < 0$
$\frac{l}{k} < x < \infty$	$\left[-\frac{kx}{kx-l} < 0, 1\right]$	$kx - l > 0$



(c) Find a conserved quantity for the system (hint: form an ODE for dy/dx and integrate it).

$$\frac{\mathrm{dx}}{\mathrm{dt}} = -kxy, \qquad \frac{dy}{dt} = kxy - ly$$

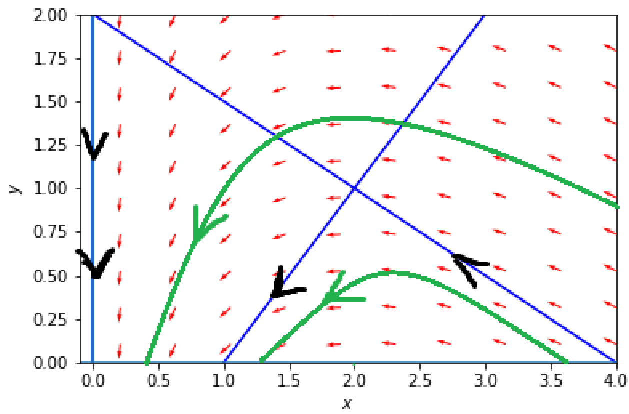
$$\frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dx} = \frac{kxy - ly}{-kxy} = \frac{-kx + l}{kx} = -1 + \frac{l}{kx}$$

$$\int dy = \int \left(-1 + \frac{l}{kx}\right) dx = -x + \frac{l}{k}\ln(x) + C = y$$

$$y + x - \frac{l}{k}\ln(x) = C$$

(d) Plot the phase portrait. What happens as t →∞?

We consider case when x > 0, y>0, because it is count of people.



Blue: nullclines

Green: trajectories.

l/k = 2

At the $t \to \infty$, $y \to 0$, $x - \frac{l}{k}\ln(x) \to C$

From initial conditions we have total number of people $S = x_0 + y_0$=>

$$y + x - \frac{l}{k}\ln(x) = S - x_0 + x_0 - \frac{l}{k}\ln(x_0) = S - \frac{l}{k}\ln(x_0) = C$$

Where S – total number of people initially, x_0 are healthy people.

$$x - \frac{l}{k}\ln(x) \to S - \frac{l}{k}\ln(x_0)$$

(e) Let (x0, y0) be the initial condition. Under what conditions on (x0, y0) will the epidemic occur? (Epidemic occurs if y (t) increases initially).

From the table (b) it is clear that v_2 lay along the x axis when $x = \frac{l}{k}$, if $x < \frac{l}{k}$, dy is negative, if $x > \frac{l}{k}$, dy is positive and epidemic occurs.

It is also clear if consider: $\dot{y} = kxy - ly = (kx - l)y$, where y > 0.

So $x_0 > \frac{l}{k}, y_0 > 0$