Homework 4. Continuum mechanics

(1). Problem 1

Find the pressure distribution of a linearly viscous fluid if its flow is described by the velocity field:

$$v_1 = 0; v_2 = k(x_2^2 - x_3^2); v_3 = -2kx_2x_3.$$

Consider the body forces are represented only by the gravitation: $F = ge_1$. The pressure at origin
is $p(0,0,0) = p_0 2$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\left(\frac{\partial v_i}{\partial x_j}\right)\right) = \rho F_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j^2}$$

$F = (g, 0,0)$

$$\frac{d\rho}{dt} + div(v) = 0, but\ div(v) = 0 => we\ can\ consider\ \rho = const$$

$$i = 1, \qquad 0 = \rho g - \frac{\partial p}{\partial x_1} => \frac{\partial p}{\partial x_1} = \rho g => p = \rho g x_1 + const(x_1)$$

$$i = 2, \qquad \rho\left(\frac{\partial k(x_2^2 - x_3^2)}{\partial t} + v_j\left(\frac{\partial k(x_2^2 - x_3^2)}{\partial x_j}\right)\right) = -\frac{\partial p}{\partial x_2} + \mu\frac{\partial^2 k(x_2^2 - x_3^2)}{\partial x_j^2}$$

$$\rho k\big(0 + v_2(2x_2) - v_3(2x_3)\big) = -\frac{\partial p}{\partial x_2} + \mu k(2-2) = -\frac{\partial p}{\partial x_2}$$

$$2\rho k(v_2x_2 - v_3x_3) = -\frac{\partial p}{\partial x_2}$$

$$\frac{\partial p}{\partial x_2} = 2\rho k(v_3x_3 - v_2x_2)$$

$$\frac{\partial p}{\partial x_2} = 2\rho k(-2kx_2x_3x_3 - k(x_2^2 - x_3^2)x_2)$$

$$p = 2\rho k\left(-kx_2^2x_3^2 - k\left(\frac{x_2^4}{4} - \frac{x_3^2x_2^2}{2}\right)\right) + const(x_2)$$

$$p = 2\rho k^2\left(-x_2^2x_3^2 - \frac{x_2^4}{4} + \frac{x_3^2x_2^2}{2}\right) + const(x_2)$$

$$p = -\rho k^2\left(x_2^2x_3^2 + \frac{x_2^4}{2}\right) + const(x_2)$$

$$i = 3, \qquad \rho\left(-v_j\left(\frac{\partial 2kx_2x_3}{\partial x_j}\right)\right) = -\frac{\partial p}{\partial x_3} - \mu\frac{\partial^2 2kx_2x_3}{\partial x_j^2}$$

$$\rho(-v_2 2kx_3 - v_3 2kx_2) = -\frac{\partial p}{\partial x_3}$$

$$\rho(-k(x_2^2 - x_3^2)2kx_3 + 2kx_2x_3 2kx_2) = -\frac{\partial p}{\partial x_3}$$

$$2\rho k^2(-(x_2^2 - x_3^2)x_3 + x_2x_3 2x_2) = -\frac{\partial p}{\partial x_3}$$

$$2\rho k^2\left(2x_2^2x_3 - (x_2^2x_3 - x_3^3)\right) = -\frac{\partial p}{\partial x_3}$$

$$\frac{\partial p}{\partial x_3} = 2\rho k^2(-x_3^3 - x_2^2x_3)$$

$$p = 2\rho k^2\left(-\frac{x_3^4}{4} - \frac{x_2^2x_3^2}{2}\right) + const(x_3)$$

$$p = -\rho k^2\left(\frac{x_3^4}{2} + x_2^2x_3^2\right) + const(x_3)$$

From i =2 take all with x_2, but without x_1,from i=3, all $x_3$without x_2, x_1 and put it all in $const(x_1)$

$$const(x_1) = -\rho k^2\left(x_2^2x_3^2 + \frac{x_2^4}{2}\right) - \rho k^2\left(\frac{x_3^4}{2}\right)$$

$$p = \rho g x_1 - \rho k^2\left(x_2^2x_3^2 + \frac{x_2^4}{2} + \frac{x_3^4}{2}\right) + C$$

$$p(0{,}0{,}0) = p_0 => \ p = \rho g x_1 - \rho k^2\left(x_2^2x_3^2 + \frac{x_2^4}{2} + \frac{x_3^4}{2}\right) + p_0$$

(2). Problem 2

The steady laminar flow of a viscous incompressible fluid (with viscosity coefficient μ) in a pipe is created by known pressure gradient $\left(\frac{dp}{dx_i} = \varphi \ for\ i\ = 1{,}2{,}3\right)$ and defined by the velocity field:

$$v_1 = v(x_2, x_3) = C_1\left(\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}\right) + C_2; v_2\ = 0; v_3 = 0.$$

Find coefficients $C1$ and $C2$ if the no-slip condition is satisfied on pipe walls, which an elliptical cross section described by $\left(\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}\right) = 1$. Neglect body forces.

On the walls $v = 0 => v_1 = C_1 + C_2 => C_2 = -C_1$

$$\rho\left(v_j\left(\frac{\partial v_1}{\partial x_j}\right)\right) = -\phi + \mu\frac{\partial^2 v_1}{\partial x_j^2}$$

$$\rho\left(v_1\left(\frac{\partial v_1}{\partial x_1}\right)\right) = -\phi + \mu\left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right)$$

$$0 = -\phi + \mu\left(\frac{C_1 2}{b^2} + \frac{2C_1}{c^2}\right) = -\phi + C_1\mu\left(\frac{2}{b^2} + \frac{2}{c^2}\right)$$

$$C_1 = \frac{\phi}{\left(\frac{2}{b^2} + \frac{2}{c^2}\right)\mu}$$

$$C_2 = -C_1$$

(3). Problem 3

The stress inside an elastic sphere (the material is isotropic with $E\ = 200\,GPa, v\ = 0.3$) is the following:

$$\sigma = \begin{bmatrix} 50k & 2 & 0 \\ 2 & 4 & 4 \\ 0 & 4 & -4 \end{bmatrix} GPa$$

(a) find Lame's coefficients.

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad v = \frac{\lambda}{2(\lambda + \mu)}$$

$$v = \frac{\lambda}{2(\lambda + \mu)} => 2v(\lambda + \mu) = \lambda => 2v\mu = \lambda - 2\lambda v => \mu = \frac{\lambda - 2\lambda v}{2v}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = \frac{\frac{\lambda - 2\lambda v}{2v}\left(3\lambda + 2\frac{\lambda - 2\lambda v}{2v}\right)}{\lambda + \frac{\lambda - 2\lambda v}{2v}} = \frac{\frac{\lambda - 2\lambda v}{2v}\left(\frac{6v\lambda + 2\lambda - 4\lambda v}{2v}\right)}{\frac{2v\lambda + \lambda - 2\lambda v}{2v}}$$

$$= \frac{\frac{\lambda - 2\lambda v}{2v}\left(\frac{2v\lambda + 2\lambda}{2v}\right)}{\frac{\lambda}{2v}} = \frac{\lambda - 2\lambda v}{2v}\left(\frac{2v\lambda + 2\lambda}{2v}\right) * \frac{2v}{\lambda} = \frac{\lambda - 2\lambda v}{1}\left(\frac{v + 1}{v}\right) =$$

$$\lambda(1 - 2v)\left(\frac{v + 1}{v}\right) = E$$

$$\lambda = E\left(\frac{\nu}{\nu+1}\right)\frac{1}{(1-2\nu)} = 200 * 10^9\left(\frac{0.3}{0.3+1}\right)\frac{1}{(1-2*0.3)} = 200 * 10^9\left(\frac{0.3}{1.3}\right)\frac{1}{(0.4)}$$
$$= 0.6 * E$$
$$\mu = \lambda\frac{1-2\nu}{2\nu} = E\left(\frac{\nu}{\nu+1}\right)\frac{1}{(1-2\nu)}\frac{1-2\nu}{2\nu} = E\left(\frac{1}{\nu+1}\right)\frac{1}{2} = E\frac{1}{2.6} = 0.4E$$