Homework 1. Continuum mechanics

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(1). Problem 1.

Considering that $[T]$ and $[E]$ are 3x3 matrices and $T_{ij} = 3\mu E_{ij} + 2\lambda E_{pp}\delta_{ij}$ find $T_{ij}T_{ij}$.

$$T_{ij}T_{ij} = \left(3\mu E_{ij} + 2\lambda E_{pp}\delta_{ij}\right)\left(3\mu E_{ij} + 2\lambda E_{pp}\delta_{ij}\right) =$$
$$= 9\mu^2 E_{ij}E_{ij} + 12\lambda E_{pp}\delta_{ij}\mu E_{ij} + 4\lambda^2 E_{pp}\delta_{ij}E_{pp}\delta_{ij} =$$
$$= 9\mu^2 E_{ij}E_{ij} + 12\lambda\mu E_{pp}E_{ii} + 4\lambda^2 E_{pp}E_{pp}3 =$$
$$= 9\mu^2 E_{ij}E_{ij} + 12\lambda\mu E_{pp}E_{pp} + 12\lambda^2 E_{pp}E_{pp}$$

(2). Problem 2.

$$[a_i\,] = \begin{bmatrix}2\\1\\0\end{bmatrix}, [b_i\,] = \begin{bmatrix}0\\3\\2\end{bmatrix}$$

Find $[d_i]$, if $d_k = \varepsilon_{ijk}a_ib_j$ and compare the results to $d_k = \left(\vec{a} \times \vec{b}\right) \cdot \vec{e}_k$, where $\vec{e}_k$ is the basis.

$$[d_i] = [d_k] = [\varepsilon_{ijk}a_ib_j] = \begin{bmatrix}\varepsilon_{ij1}a_ib_j\\ \varepsilon_{ij2}a_ib_j\\ \varepsilon_{ij3}a_ib_j\end{bmatrix} = \begin{bmatrix}\varepsilon_{231}a_2b_3 + \varepsilon_{321}a_3b_2\\ \varepsilon_{132}a_1b_3 + \varepsilon_{312}a_3b_1\\ \varepsilon_{123}a_1b_2 + \varepsilon_{213}a_2b_1\end{bmatrix} = \begin{bmatrix}a_2b_3 - a_3b_2\\ -a_1b_3 + a_3b_1\\ a_1b_2 - a_2b_1\end{bmatrix} = \begin{bmatrix}2\\-4\\6\end{bmatrix}$$

$$[d_i] = [d_k] = \left[\left(\vec{a} \times \vec{b}\right) \cdot \vec{e}_k\right] = \begin{bmatrix}\begin{vmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\end{bmatrix} = \begin{bmatrix}a_2b_3 - b_2a_3\\ -(a_1b_3 - b_1a_3)\\ a_1b_2 - b_1a_2\end{bmatrix} = \begin{bmatrix}2\\-4\\6\end{bmatrix}$$

As a result, $[\boldsymbol{a} \times \boldsymbol{b}]_i = \varepsilon_{ijk}a_jb_k$.

(3). Problem 3

Prove $\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ (for help refer to the lectures, make it as short as possible). Using the equation proved, find a) $\varepsilon_{ijm}\varepsilon_{jlm}$; b) $\varepsilon_{ijk}\varepsilon_{ijk}$.

$$\varepsilon_{ijm}\varepsilon_{klm} = \begin{vmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{m1} & \delta_{m2} & \delta_{m3}\end{vmatrix}\begin{vmatrix}\delta_{k1} & \delta_{k2} & \delta_{k3}\\ \delta_{l1} & \delta_{l2} & \delta_{l3}\\ \delta_{m1} & \delta_{m2} & \delta_{m3}\end{vmatrix} = |A_{ijm}|\,|B_{klm}| = |C_{ijklm}|$$

Where $C_{ijklm} = A_{ijm}B_{klm}$, transposing the second determinant does not change its value, after that:

$$C_{ijklm} = \begin{vmatrix}\delta_{ip}\delta_{kp} & \delta_{ip}\delta_{lp} & \delta_{ip}\delta_{mp}\\ \delta_{jp}\delta_{kp} & \delta_{jp}\delta_{lp} & \delta_{jp}\delta_{mp}\\ \delta_{mp}\delta_{kp} & \delta_{mp}\delta_{lp} & \delta_{mp}\delta_{mp}\end{vmatrix} = \begin{vmatrix}\delta_{ik} & \delta_{il} & \delta_{im}\\ \delta_{jk} & \delta_{jl} & \delta_{jm}\\ \delta_{mk} & \delta_{ml} & \delta_{mm}\end{vmatrix} =$$
$$= \delta_{ik}\left(\delta_{jl}\delta_{mm} - \delta_{jm}\delta_{ml}\right) - \delta_{il}\left(\delta_{jk}\delta_{mm} - \delta_{mk}\delta_{jm}\right) + \delta_{im}\left(\delta_{jk}\delta_{ml} - \delta_{jl}\delta_{mk}\right) =$$
$$= \delta_{ik}\left(3\delta_{jl} - \delta_{jl}\right) - \delta_{il}\left(3\delta_{jk} - \delta_{jk}\right) + \left(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}\right) = 2\delta_{jl}\delta_{ik} - 2\delta_{il}\delta_{jk} + \left(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}\right)$$
$$= \delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

(a) $\varepsilon_{ijm}\varepsilon_{jlm} = \delta_{ij}\delta_{jl} - \delta_{il}\delta_{jj} = \delta_{il} - 3\delta_{il} = -2\delta_{il}$

(b) $\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = 3 * 3 - \delta_{ii} = 6$

(4). Problem 4.

The components of a tensor $T = T_{ij}\vec{e}_i\vec{e}_j$ in the old coordinate system are

$$[T] = \begin{bmatrix}1 & 3 & -3\\ 3 & 0 & 0\\ -3 & 0 & 1\end{bmatrix}$$

Find the tensor components $T'_{11}, T'_{12}, T'_{31}$ in a primed coordinate system with a right-handed basis $\vec{\boldsymbol{e}}'_i$. It is known that $\boldsymbol{e}'_1$ is collinear with vector $\boldsymbol{u} = -4\boldsymbol{e}_2 + 3\boldsymbol{e}_3$ in the same direction, and that $\boldsymbol{e}_2' = \boldsymbol{e_1}$.

Let's find $\boldsymbol{e}_1'$: $\boldsymbol{e}_1' = \frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{1}{\sqrt{16+9}}(-4\boldsymbol{e}_2 + 3\boldsymbol{e}_3) = -\frac{4}{5}\boldsymbol{e}_2 + \frac{3}{5}\boldsymbol{e}_3$

$\boldsymbol{e}_2'$ is known $\boldsymbol{e}_2' = \boldsymbol{e_1}$;

$\boldsymbol{e}_3'$ may be obtained by $\boldsymbol{e}_3' = [\boldsymbol{e}_1' * \boldsymbol{e}_2'] = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \end{vmatrix} = \frac{3}{5}\boldsymbol{e_2} + \frac{4}{5}\boldsymbol{e_3}$

summing up: $[\boldsymbol{e}'] = \mathrm{A}[\boldsymbol{e}] = \begin{bmatrix} 0 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}[\boldsymbol{e}]$

For tensor transformation we have: $\mathrm{T}_{kl}' = \mathrm{T}_{ij}\alpha_{ki}\,\alpha_{lj} = \mathrm{A}T\mathrm{A}^{-1} = \begin{bmatrix} 9/25 & -21/5 & 12/25 \\ -21/5 & 1 & -3/5 \\ 12/25 & -3/5 & 16/25 \end{bmatrix}$

$T_{11}' = \frac{9}{25}; T_{12}' = -\frac{21}{5}; T_{31}' = \frac{12}{25}$

(5). Problem 5.

The components of a tensor $T = T_{ij}\vec{e}_i\vec{e}_j$ are

$$[T] = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{bmatrix}$$

Find the symmetric part and the antisymmetric part of **T**.

$T_{ij} = \dfrac{1}{2}\left(T_{ij} + T_{ji}\right) + \dfrac{1}{2}\left(T_{ij} - T_{ji}\right) = S_{ij} + A_{ij}$

$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) = \begin{bmatrix} 1 & 5 & 9 \\ 5 & 9 & 13 \\ 9 & 13 & 17 \end{bmatrix}$- symmetric part

$A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right) = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$-antisymmetric part

(6). Problem 6.

A tensor **T** is represented by the following matrix:

$$[\boldsymbol{T}] = \begin{bmatrix} 7 & 5 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Find the principal values of the tensor and their corresponding principal directions. Also find the principal scalar invariants I_1, I_2, I_3.

$$|[\boldsymbol{T} - \Lambda]| = \left|\begin{bmatrix} 7-\lambda & 5 & 0 \\ 2 & -2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{bmatrix}\right| = -\lambda^3 + 9\lambda^2 + 4\lambda - 96$$

$\lambda_{1,2,3} = 8;\ 4;\ -3$

For $\lambda_1 = 8$: $\begin{bmatrix} 7-8 & 5 & 0 \\ 2 & -2-8 & 0 \\ 0 & 0 & 4-8 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}c$, let's make it unit: $\boldsymbol{x} = \begin{bmatrix} 5/\sqrt{26} \\ 1/\sqrt{26} \\ 0 \end{bmatrix}$

Analogically for $\lambda_2 = 4$: $\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; $\lambda_3 = -3$: $\boldsymbol{x} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$;

Summing up: $S = \begin{bmatrix} 5/\sqrt{26} & 0 & -1/\sqrt{5} \\ 1/\sqrt{26} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$; - matrix of principal directions, for corresponding principal values: $\Lambda = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. Decomposition: $T = S\Lambda S^{-1}$

$$I_1 = T_{ii} = 9; I_2 = M_{ii} = -4; I_3 = det(T_{ij}) = -96$$

For both Λ and T I_i are the same.