Seminar 1: Linear Algebra

Basics, spaces, norms

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Matrix multiplication

- Who knows how to multiply matrices?
- Who can show three different ways?

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} A_{1\star} \\ \vdots \\ A_{m\star} \end{pmatrix} = \begin{pmatrix} A_{\star 1} & \cdots & A_{\star n} \end{pmatrix}$$

Elements **Rows** **Columns**

$$A_{\star i} = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

$$A_{j\star} = (a_{j1} \quad \cdots \quad a_{jn})$$

Problems

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, AB = ?$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \dots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \dots & b_{pq} \end{pmatrix}, AB = ?$$

Solution

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Elements: $0 = 1 \cdot 1 + 2 \cdot 1 - 3 \cdot 1$

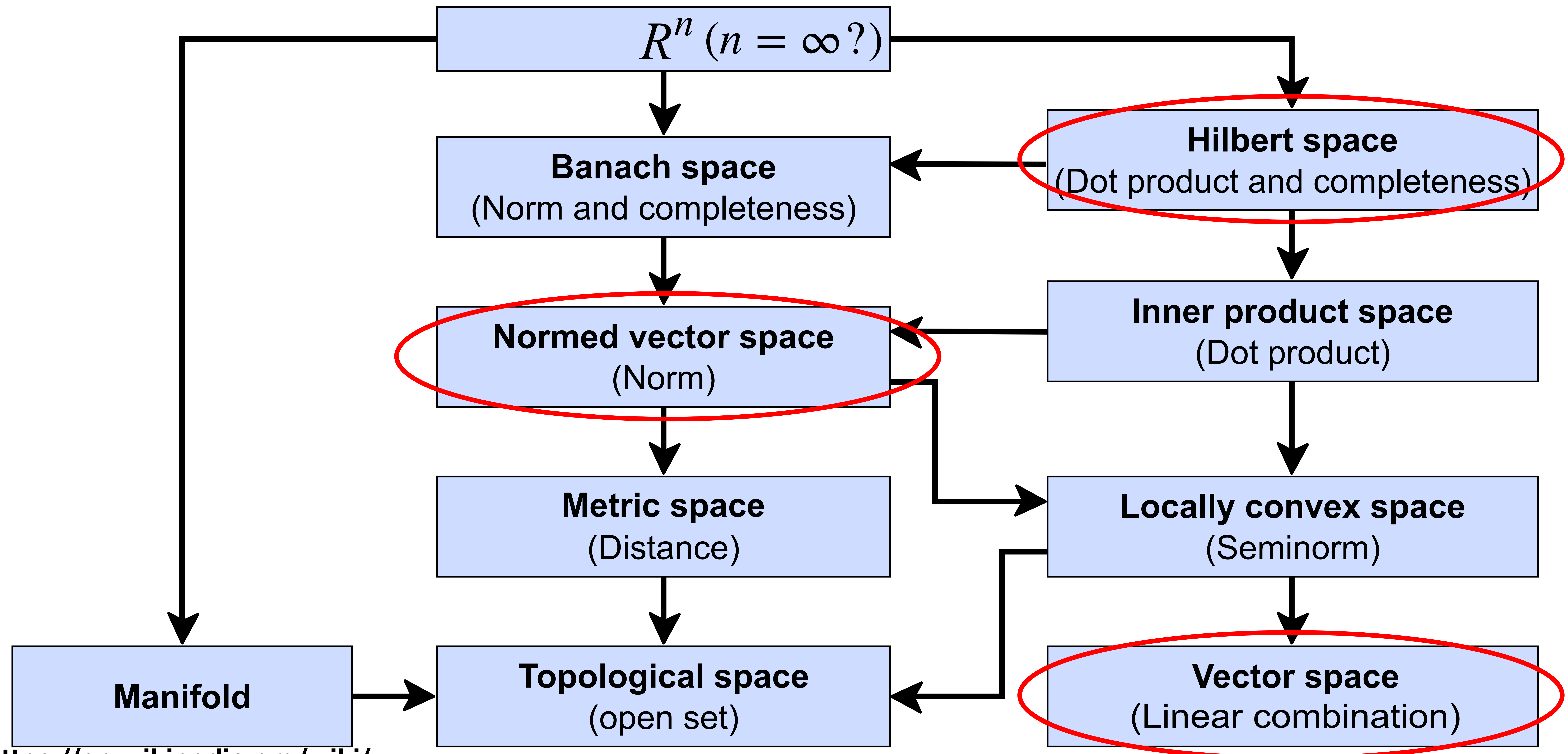
$$AB = \begin{pmatrix} 0 & 4 \\ -3 & 1 \end{pmatrix}$$

Rows: $(0 \quad 4) = 1 \cdot (1 \quad -1) + 2 \cdot (1 \quad 1) + 3 \cdot (-1 \quad 1)$

Columns: $\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot 1 + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot 1 - \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot 1$

Space

Abstract spaces



https://en.wikipedia.org/wiki/Space_(mathematics)

Space = Set + Structure

Four fundamental spaces

Vector space = Set of vectors + Linear combination

$$a, b \in V, \alpha, \beta \in \mathbb{R} \Rightarrow \alpha a + \beta b \in V$$

$\{e_1, e_2, \ldots, e_n\}$ **is basis of** V **and** $\dim V = n$ **iff**

a. $V = \operatorname{span}\{e_1, e_2, \ldots, e_n\} = \{x : x = \alpha_1 e_1 + \ldots + \alpha_n e_n,\ \alpha_i \in \mathbb{R}\}$

b. Linear independent

1. Column space $C(A)$ **or** $\operatorname{range}(A) = \left\{y : y = Az\right\}$

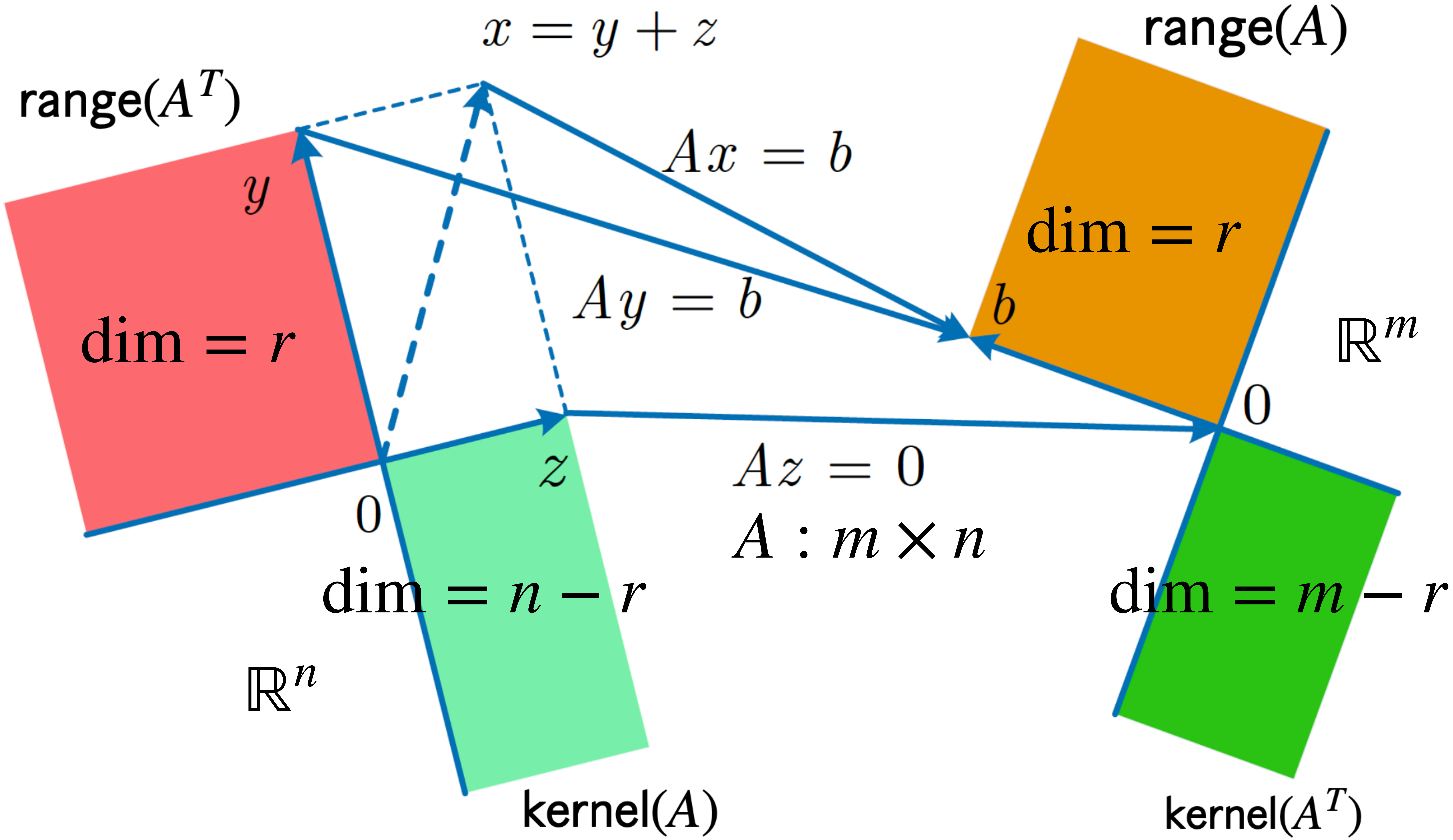
2. Nullspace $N(A)$ **or** $\operatorname{kernel}(A) = \{z : Az = 0\}$

3. Row space $C\left(A^T\right)$

4. Left nullspace $N\left(A^T\right)$

$$a \in \mathbb{R}^n \iff a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, a_i \in \mathbb{R}$$

Four fundamental spaces



$$V_1 \perp V_2 \iff \{x \in V_1, y \in V_2 \Rightarrow x \perp y\}, x \perp y \iff \langle x, y \rangle = x^T y = 0$$

Problem

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 3 & 1 \end{pmatrix}, \text{ FFS} = ?, \dim ?, \text{basis} ?$$

Solution

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 3 & 1 \end{pmatrix} \Rightarrow \begin{aligned} A_{\star 2} &= 2A_{\star 1} + 2A_{\star 3}, \\ A_{\star 4} &= A_{\star 1} + A_{\star 3}, \\ A_{\star 6} &= 4A_{\star 1} + 4A_{\star 3} - A_{\star 5} \end{aligned}$$

$$\Rightarrow x_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \\ -1 \\ -1 \end{pmatrix} \in N(A)$$

Why? $Ax_1 = 2A_{\star 1} + 2A_{\star 3} - A_{\star 2} = 0$

$$N(A) + C(A^T) = \mathbb{R}^6,\ \dim C(A^T) = r = 3, N(A) = \operatorname{span}\{x_1, x_2, x_3\}, N(A^T) = \{0\}$$

Solution 2

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 3 & 1 \end{pmatrix} \Rightarrow \tilde{A} = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \text{row echelon form}$$

$$\operatorname{rank} A = 3,\ C(A^T) = \operatorname{span}\{\tilde{A}_{1\star}, \tilde{A}_{2\star}, \tilde{A}_{3\star}\}$$

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 3 & 1 \end{pmatrix} \Rightarrow \tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \Rightarrow$$

$$\tilde{\tilde{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \Rightarrow \tilde{\tilde{\tilde{A}}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \text{column echelon form}$$

$$C(A) = \operatorname{span}\{\tilde{\tilde{A}}_{\star 1}, \tilde{\tilde{A}}_{\star 2}, \tilde{\tilde{A}}_{\star 5}\}, N(A) = \ ? \Leftarrow N(\tilde{\tilde{\tilde{A}}}) = \operatorname{span}\{e_3, e_4, e_6\}$$

Solution 2

How A and $\tilde{\tilde{\tilde{A}}}$ are related? Let's start with $\tilde{A}$…

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 3 & 1 \end{pmatrix} \Rightarrow \tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 & 2 & -2 \end{pmatrix} \Rightarrow$$

$$\tilde{A} = A \begin{pmatrix} 1 & -2 & 0 & -1 & -1 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = AO_1 \text{ and then } \tilde{\tilde{\tilde{A}}} = AO_1O_2O_3$$

$$\tilde{\tilde{\tilde{A}}}e_3 = 0 = AO_1O_2O_3e_3,\ O_1O_2O_3e_3 \in N(A)$$

Eigendecomposition

$$Ax = \lambda x, x \neq 0 \iff (A - \lambda I)\, x = 0$$

$\det(A - zI) = p(z) \Rightarrow p(\lambda) = 0$ **iff** λ **is an eigenvalue**

$$\underbrace{\dim\left(\mathrm{null}\,(A - \lambda I)\right)}_{\text{geometric multiplicity}} \leq \underbrace{p(z) = (z - \lambda)^k \ldots}_{\text{algebraic multiplicity}}$$

When $<$**, the matrix is defective: not nice**

When $=$**, the matrix is nondefective:** $A = XDX^{-1}$

Problem: Fibonacci numbers

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

$$f_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = Af_{n-1}$$

$$f_n = Af_{n-1} = A^2 f_{n-2} = \ldots = A^{n-1} f_1$$

$$A = UDU^T \Rightarrow A^{n-1} = UD^{n-1}U^T$$

$$F_n = \left(UD^{n-1}U^T\right)_{11}$$ closed form expression

Solution

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -(1-\lambda)\lambda - 1 = 0, \qquad \left(\lambda - \frac{1}{2}\right)^2 = \frac{5}{4}, \ \lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$$

$$v_+ = \frac{1}{\sqrt{5}\lambda_-}\begin{pmatrix} -1 \\ \lambda_- \end{pmatrix} = \begin{pmatrix} 1/\alpha_1 \\ 1/\sqrt{5} \end{pmatrix}, \ v_- = \frac{1}{\sqrt{5}\lambda_+}\begin{pmatrix} 1 \\ -\lambda_+ \end{pmatrix} = \begin{pmatrix} 1/\alpha_2 \\ -1/\sqrt{5} \end{pmatrix}$$

$$\begin{pmatrix} 1/\alpha_1 & 1/\alpha_2 \\ * & * \end{pmatrix}\begin{pmatrix} \lambda_+^{n-1} & 0 \\ 0 & \lambda_-^{n-1} \end{pmatrix}\begin{pmatrix} 1/\alpha_1 & * \\ 1/\alpha_2 & * \end{pmatrix} = \begin{pmatrix} \lambda_+^{n-1}/\alpha_1^2 + \lambda_-^{n-1}/\alpha_2^2 & * \\ * & * \end{pmatrix}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Norms

- Definition

$$\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$$
$$\|x\| \geq 0; \; \|x\| = 0 \iff x = 0$$
$$\|x + y\| \leq \|x\| + \|y\|$$
$$\|\alpha x\| = |\alpha| \|x\|$$

- Induced matrix norms

$$\|\,\|_a, \; \|\,\|_b \to \|A\|_{a,b} = \sup_x \frac{\|Ax\|_b}{\|x\|_a}$$

$$\|Ax\|_b \leq C\|x\|_a$$

- Examples

$$L_p : \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$
$$\|x\|_2 = \sqrt{x^T x}$$
$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

- Unit balls in $\mathbb{R}^2$?

Problem: norm of an outer product

$$A = uv^T, \ \|A\|_2 = ?$$

$$u = v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Solution

$$\|uv^T\| = \sup_{x \neq 0} \frac{\|uv^T x\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|u\|_2 |v^T x|}{\|x\|_2} = \|u\|_2 \|v\|_2$$

$$|v^T x| = \|v\|_2 \|x\|_2 |\cos\theta|$$

Lagrange multipliers

$\min\left(2x^2+3y^2\right)$ s.t. $x^2+y^2=1$

Explicit:

$$2x^2+3y^2=2(x^2+y^2)+y^2=2+y^2\to\min\Rightarrow\ \ y=0,\ x=\pm 1,\ \ \min=2$$

Multiplier: $2x^2+3y^2+\lambda(1-x^2-y^2)$

$$\begin{aligned}&\partial_x\!: 4x-2\lambda x=0\\ &\partial_y\!: 6y-2\lambda y=0 \quad\Rightarrow \begin{array}{l} x=0 \text{ or } \lambda=2\\ y=0 \text{ or } \lambda=3\end{array}\\ &\partial_\lambda\!: 1-x^2-y^2=0\end{aligned}$$

Two solutions: $(0,\pm 1),\ \ \max=3;\ \ (\pm 1,0),\ \ \min$