2. Lectures 19, 20. Nov. 16,17

- ◇ Nonlinear oscillations: Mathieu equation, van der Pol equation, Duffing equation.
- ◇ Limit cycles.
- ◇ Bifurcation theory.

2.1. **Nonlinear oscillations: Mathieu equation, van der Pol equation, Duffing equation.**

2.1.1. *Some remarks about conservative and reversible systems.* Important examples of nonlinear oscillations

2.1.2. *Mathieu equation.*

$$\ddot{x} + (\alpha + \beta \cos t)\, x = 0$$

For a pendulum of length l with a vertically oscillating support at $y_0 = a\cos\omega t$, the parameters here are

$$\alpha = \left(\frac{\omega_0}{\omega}\right)^2, \quad \beta = \frac{a}{l}$$

The time is rescaled so that $\omega t = \tau$.

Note that if we take the original pendulum equation $\ddot{\phi} + \omega_0^2 \left(1 + \frac{a\omega^2}{g}\cos\omega t\right)\sin\phi = 0$ and consider small oscillations not about $\phi = 0$, but about $\phi = \pi$, which is also an equilibrium point (inverted pendulum, usually unstable), we find the following.

Let $\vartheta = \pi - \phi$, then $\sin(\phi) = \sin(\pi - \vartheta) = \sin\vartheta \approx \vartheta$, and the pendulum equation becomes

$$\ddot{\vartheta} - \omega_0^2 \left(1 + \frac{a\omega^2}{g}\cos\omega t\right)\vartheta = 0.$$

The key is that the sign of α is now negative. The sign of β does not matter, as it only depends on the phase of the parametric forcing. But the sign in front of ω_0^2 is very important. When $a = 0$, the equation $\ddot{\vartheta} - \omega_0^2\vartheta = 0$ has solutions which are exponentials, and therefore the equilibrium point $\vartheta = 0$ is unstable.

Now the question is: What if we have the parametric forcing, i.e. $a \neq 0$?

Remarkably, the oscillations can now be stable. That is, the inverted pendulum can be stabilized by periodic oscillations of its suspension point. This is the region of the narrow white area to the left of the β axis in the figure below. There exists a range of the forcing amplitudes β for which the solution is stable.

Another important point to emphasize is the following. Why does the first unstable region at $\alpha > 0$ begin at $\alpha = 1/4$?

To answer, recall that $\alpha = (\omega_0/\omega)^2$. Then the instability starts at $\omega = 2\omega_0$, that is the forcing has to occur at twice the natural frequency of the pendulum. There is a basic physical explanation of this fact – in order to pump energy into the swinging pendulum, we have to lower it when it passes through the lowest potential energy point and raise it at its points of highest potential energy. That is, during a single period of the pendulum, we lower it twice and raise it twice, which means at double the natural frequency. A child on a swing knows this well.

Example. As another example where the Mathieu equation arises, come back to Romeo and Juliet. Assume that

$$\dot{R} = -aR + bJ,$$

with a, b positive, so that Romeo has good self-control ($a > 0$, so that $R \to 0$ with time whatever the initial conditions on R in the absence of Juliet, $b = 0$). If Juliet comes into play, then she

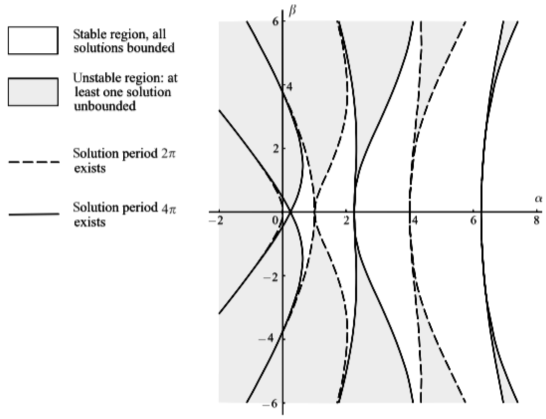

Figure 9.3 Stability diagram for Mathieu's equation $\ddot{x} + (\alpha + \beta \cos t)x = 0$.

FIGURE 2.1. Stability diagram for Mathieu equation at various parameter α, β (Jordan and Smith).

influences Romeo positively ($b > 0$). She has, unfortunately, a somewhat unstable character. Her equation is

$$\dot{J} = (c + d \sin t)\, R$$

with $c > 0$. This means that she is generally positive about Romeo ($c > 0$), however has periodic mood swings of amplitude d and that affects her response to Romeo's feelings.

If we differentiate the equation for R and substitute the equation for J, we obtain the damped Mathieu's equation

$$\ddot{R} + a\dot{R} + (-bc - bd \sin t)\, R = 0.$$

Interestingly, this corresponds to $\alpha < 0$, i.e the inverted pendulum case. Also, a could be 0, and it still would be Mathieu's equation. The situation now is unstable generally except for the narrow band of values of bd for a given bc, according to the stability diagram above. Nevertheless, the couple can oscillate stably in their relationship in some cases.

2.1.3. *Van der Pol equation and relaxation oscillations.*

$$\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0.$$

We look at what happens when μ is large to see the slow-fast dynamics of the solutions (relaxation oscillations) (Strogatz, p. 212).

Note, one can rewrite the equation as

$$\dot{x} = \mu \left(y - F(x)\right)$$
$$\dot{y} = -\frac{x}{\mu},$$

where $F(x) = x^3/3 - x$. On the $x - y$ plane, the nullcline $y = F(x)$ shows the parts of the limit cycle on which the motion switches from fast (away from $y = F$) to slow (close to $y = F$). See Fig. 2.2.

2.1.4. *Forced van der Pol equation and phase locking.* Look at the periodically forced van der Pol equation:

$$\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = a \cos(\omega t).$$

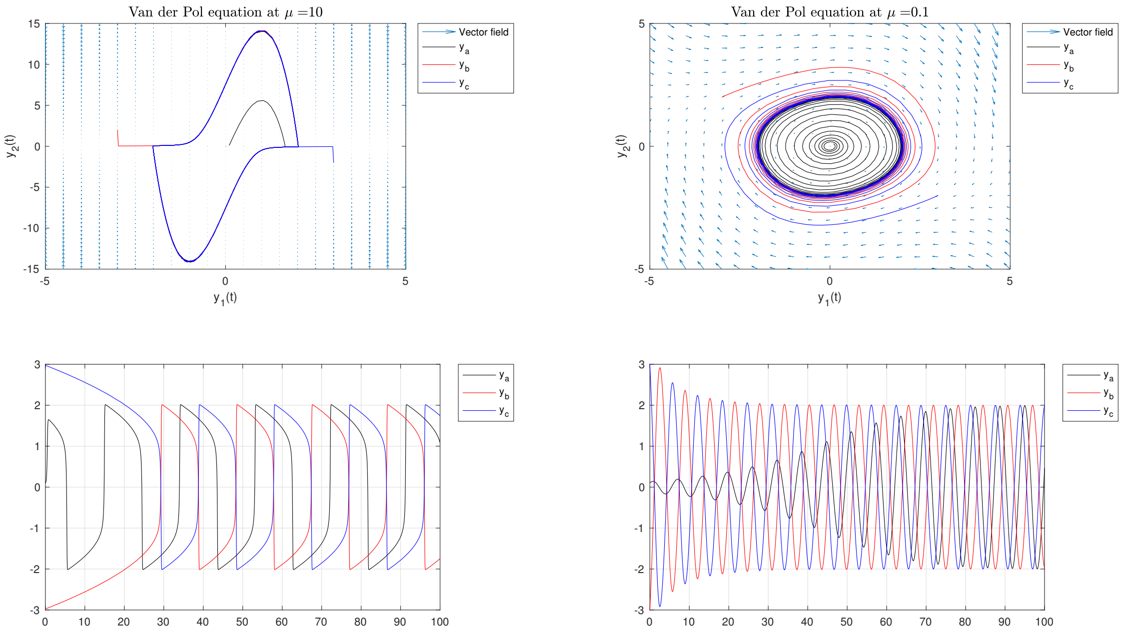

FIGURE 2.2. The phase plane and the solution of the van der Pol equation at $\mu = 10$ and $\mu = 0.1$.

We take the following parameters fixed: $\mu = 0.1$, $a = 0.5$ and vary ω in the range $0.5 \le \omega \le 1.5$. There is an interval $0.85 - 1.11$ in which there exists a phenomenon of *mode locking*, see Fig. 2.3.

2.2. **Limit cycles . . . .** A limit cycle represents a periodic solution of a nonlinear system. It is an *isolated* closed trajectory in the phase plane. Obviously, existence of limit cycles is important as it tells us if sustained *nonlinear* periodic oscillations are possible in a system.

The limit cycle will be stable or attracting if all nearby trajectories converge to it as $t \to \infty$, unstable otherwise.

The van der Pol example above showed an example of a limit cycle.

Another example can be cooked up:

$$\dot{r} = r\left(1 - r^2\right), \quad \dot{\vartheta} = 1$$

in polar coordinates will show an unstable f.p. at $r = 0$ and a stable one at $r = 1$. All trajectories will converge to $r = 1$.

In some cases it is possible to rule out the existence of limit cycles. In others one can prove they must exist.

We begin with the negative cases. A system that can be written as

$$\dot{x} = -\nabla V$$

is called a *gradient system*, and $V(x)$ is the potential.

Theorem 9. *Gradient systems cannot have closed orbits.*

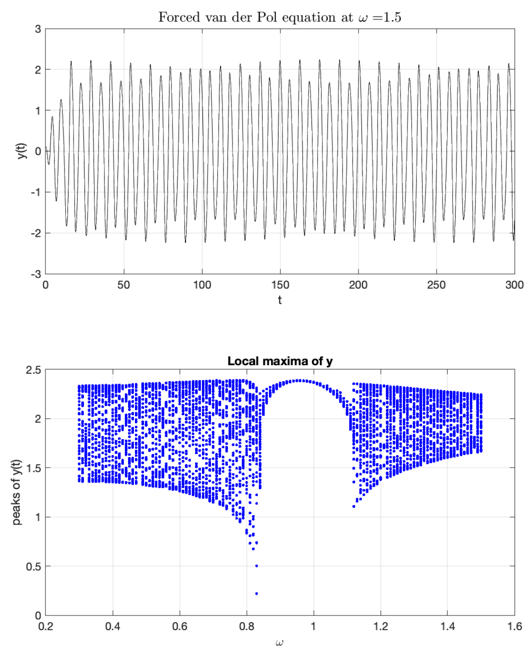

FIGURE 2.3. The maxima of solutions of the van der Pol equation as a function of the forcing frequency ω.

Proof. To prove, note that if such a closed orbit were to exist, then over one period we would have

$$\Delta V = \int_0^T \dot{V} dt = \int_0^T \nabla V \cdot \dot{x} dt = -\int_0^T |\dot{x}|^2 \, dt < 0.$$

This would be impossible since $V(x)$ must return to its initial value over a period. □

One can establish non-existence of closed orbits for non-gradient systems using a similar approach.

Example 10. Show that system

$$\ddot{x} + \dot{x}^3 + x = 0$$

has no periodic solutions.

To show this, consider

$$E = \frac{1}{2}\left(\dot{x}^2 + x^2\right).$$

The change of this function over a period is

$$\Delta E = \int_0^T \dot{E} dt = \int_0^T \dot{x}\left(\ddot{x} + x\right) dt = -\int_0^T \dot{x}^4 dt < 0.$$

This is again impossible for a closed orbit.

More generally, if for the system $\dot{x} = f(x)$, one can find a function $V(x)$ (called *Liapunov function*) such that

(1) $V > 0$ for all $x \neq x_*$ and $V(x_*) = 0$, where x_* is the fixed point, $f(x_*) = 0$,
(2) $\dot{V} < 0$ for all $x \neq x_*$,

then x_* is globally asymptotically stable, i.e. $x(t) \to x_*$ as $t \to \infty$, and so no closed orbits are possible.

Finding such a V is generally a difficult problem.

Now we look at conditions for the existence of limit cycles.

Theorem 11. *(Poincare-Bendixson theorem). Suppose R is a bounded closed subset of a plane. Let $\dot{x} = f(x)$ define a continuously differentiable vector field on an open set containing R. If R contains no fixed points and there exists a trajectory C that starts and stays in R at all times, then either C is a closed orbit or it tends to a closed orbit as $t \to \infty$.*

The main conclusion from the Poincare-Bendixson theorem is that in two dimensions one can only have simple attractors – fixed points and limit cycles. No chaos is possible in the plane. One must have at least three dimensions.

Equations of the type

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0$$

are called *Lienard's equations.* There are conditions on f and g that establish the existence of limit cycles for this equation. These are:

(1) f and g are continuously differentiable;
(2) g is an odd function;
(3) $g > 0$ at $x > 0$;
(4) f is an even function;
(5) the function $F = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative at $0 < x < a$, is positive and non-decreasing at $x > a$, and $F \to \infty$ as $x \to \infty$.

Given these conditions are satisfied, the Lienard equation has a unique stable limit cycle around the origin of the phase plane.

For example, one can apply this theorem to the van der Pol equation $\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$. Here $f = \mu\left(x^2 - 1\right)$ and $g = x$, so conditions 1-4 are satisfied. Also $F = \frac{1}{3}\mu x\left(x^2 - 3\right)$ satisfies condition 5 with $a = \sqrt{3}$.

Next we consider weakly nonlinear oscillations.

2.2.1. *Weakly nonlinear oscillations.* This time consider van der Pol oscillator at small $\mu = \epsilon \ll 1$:

$$\ddot{x} + x + \epsilon\left(x^2 - 1\right)\dot{x} = 0$$

or Duffing oscillator with weak nonlinearity

$$\ddot{x} + x + \epsilon x^3 = 0, \quad |\epsilon| \ll 1.$$

A number of interesting phenomena exist in the weakly nonlinear limit. We here discuss two:

(1) Existence of multiple-scale dynamics. Unlike the strongly nonlinear limit of relaxation oscillation, here the multiple scale dynamics occur simultaneously. That is at $\mu \gg 1$, we had slow evolution followed by rapid relaxation. Now, when $\epsilon = \mu \ll 1$, we have the evolution that involves concurrent fast and slow time scales. This will be shown below.
(2) Existence of subharmonic resonances. If we look at, for example, the forced Duffing equation $\ddot{x} + x + \epsilon x^3 = a\cos\omega t$, the presence of the cubic term has the following interesting consequence. Without it, the solution is resonant only at $\omega = 1$. With the nonlinear term, it is possible to have resonance for $\omega = 1/n$ with n some integer. This is called *subharmonic resonance.* This will be explained below.

2.2.2. *The size and period of the van der Pol limit cycle.* This time consider the van der Pol oscillator at small $\mu = \epsilon \ll 1$:

$$\ddot{x} + x + \epsilon\left(x^2 - 1\right)\dot{x} = 0.$$

tbd.

2.2.3. *Subharmonic resonance in the forced Duffing equation.* We consider

$$\ddot{x} + x + \epsilon x^3 = a\cos\omega t$$

and look to find periodic solutions $x(t)$. Suppose the period is T, $x(t+T) = x(t)$. Then, the function can be expanded in a Fourier series

$$x = a_0 + a_1\cos\nu t + b_1\sin\nu t + a_2\cos 2\nu t + b_2\sin 2\nu t + \ldots$$

with $\nu = 2\pi/T$.

We expect resonance if the left-hand side of the equation generates frequencies equal to ω. One obvious case is that of $\nu = \omega$. This is the *harmonic resonance*. The coefficients in $x(t)$ are then found by matching the left and right hand sides of the equation.

There's however, another possibility to match both sides. And this possibility arises due to the nonlinearity ϵx^3. Even with $\nu \neq \omega$, the cubic x^3 generates new frequencies that can be matched with ω because once we calculate the cube of the Fourier series, all sorts of sums and differences of the frequencies in the series will show up. For example, the cube will have terms of the type:

$$\sin^2 4\nu t \cdot \sin 9\nu t.$$

Using trigonometric identities $\sin^2(\alpha) = \frac{1}{2}(1-\cos 2\alpha)$ and $\sin\alpha\,\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$, we get:

$$\begin{aligned}\sin^2 4\nu t \cdot \sin 9\nu t &= \frac{1}{2}\left(\sin 9\nu t - \sin 9\nu t\,\cos 8\nu t\right) = \\ &= \frac{1}{2}\sin 9\nu t - \frac{1}{4}\left(\sin 17\nu t + \sin\nu t\right).\end{aligned}$$

And so terms with frequencies 4ν and 9ν in $x(t)$ have generated new frequencies ν and 17ν in addition to the original 4ν and 9ν. This is how nonlinearity gives rise to new frequencies in the solution.

So we can have resonant solutions of frequency $\nu = \omega/9$ or $\omega/17$. Such frequencies are called *subharmonic*.

2.3. Bifurcations

. The basic idea here is that a nonlinear system that depends on a parameter can go through some significant *qualitative changes* as the parameter is varied. For example, a nonlinear system that is stable can become unstable, or a system that can have multiple solutions can switch from one solution to another.

To introduce these ideas, consider the high-school physics problem of a pendulum with a rotating point of suspension. The question here is: Given the mass of the bob m and the length of the stick l, what is the angle of inclination ϕ of the pendulum for a given frequency ω of rotation of the suspension point? The surprise is that there will be two solutions if ω is bigger than $\omega_0 = \sqrt{g/l}$, the natural frequency of the pendulum.

Example 12. A pendulum of length l is rotating about its suspension point with frequency ω. The problem is to find the angle of inclination ϕ.

Writing the equations of motion in a rotating frame of reference and assuming steady state, we find that ϕ solves

$$\sin\phi\left(\cos\phi - \frac{\omega_0^2}{\omega^2}\right) = 0,$$

where $\omega_0 = \sqrt{g/l}$ is the natural frequency of oscillations of the pendulum. Note now that the above equation always has the solution $\phi = 0$, but the second factor can also be zero if $\omega > \omega_0$. Hence, at $\omega > \omega_0$ there are multiple solutions: $\phi_1 = 0$, $\phi_2 = \cos^{-1}\left(\omega_0^2/\omega^2\right)$. This value $\omega = \omega_0$ is exactly a bifurcation point.

To show stability or instability of these solutions, we need to look at the dynamics. The equations of motion are

$$m\frac{d^2}{dt^2}l\sin\phi = m\omega^2 l\sin\phi - T\sin\phi$$
$$m\frac{d^2}{dt^2}l\cos\phi = mg - T\cos\phi.$$

From here, we obtain

$$\ddot{\phi} = \omega^2\left(\cos\phi - \left(\frac{\omega_0}{\omega}\right)^2\right)\sin\phi.$$

This always has $\phi = 0$ as a fixed point. To understand its stability, linearize about $\phi = 0$:

$$\ddot{\phi} = \omega^2\left(1 - \left(\frac{\omega_0}{\omega}\right)^2\right)\phi,$$

from which we see that the fixed point is stable only if $\omega < \omega_0$ (it is a center). Otherwise, it is unstable (it is a saddle). When $\omega > \omega_0$, the other solution given above becomes stable.

Now we look at some generic types of bifurcations. First, in 1D, then in 2D.

2.3.1. *Saddle-node bifurcation.* A typical equation is

$$\dot{x} = r + x^2.$$

When $r < 0$, this has two fixed points, which disappear when r becomes positive. If we plot the fixed point as a function of r (i.e. the **bifurcation diagram**, we get that $x_c = \pm\sqrt{-r}$, when $r < 0$, of which $x_c = \sqrt{-r}$ is unstable. Otherwise, no fixed point.

Sketch the phase plane $\dot{x}$ vs x to see the dynamics near the fixed points.

2.3.2. *Transcritical bifurcation.* A typical equation is

$$\dot{x} = rx - x^2.$$

Now there are always 2 fixed points: $x_{c1} = 0$ and $x_{c2} = r$. The graph of $\dot{x}$ vs x shows that if $r < 0$, then $x_{c1} = 0$ is the stable fixed point, while the other one is unstable. However, when $r > 0$, the stability of these points switches. Now $x_{c1} = 0$ is unstable, $x_{c2} = r$ is stable. There is an *exchange of stability.*

Now the bifurcation diagram consists of two lines, $x_c = 0$ and $x_c = r$, and parts of these lines are stable, others unstable.

2.3.3. *Pitchfork bifurcation.* A typical equation is

$$\dot{x} = rx - x^3.$$

Since $rx - x^3 = x\left(r - x^2\right)$, we obtain 1 fixed point if $r < 0$ and 3 if $r > 0$. There is a trifurcation as r crosses $r = 0$ from left to right.

The phase plane $\dot{x}$ vs x shows what happens. Also, the bifurcation diagram $x_c(r)$ illustrates what happens nicely. The example of a rotating pendulum belongs to this class.

In 2D, these bifurcations are well represented by systems

$$\begin{aligned} \dot{x} &= \mu - x^2, \qquad \text{saddle-node} \\ \dot{y} &= -y \end{aligned}$$

$$\begin{aligned} \dot{x} &= \mu x - x^2, \qquad \text{transcritical} \\ \dot{y} &= -y \end{aligned}$$

$$\begin{aligned} \dot{x} &= \mu x - x^3, \qquad \text{supercritical pitchfork} \\ \dot{y} &= -y. \end{aligned}$$

Their geometry is very similar to the 1D case, except the trajectories move along y direction as well.

2.3.4. *(subcritical/supercritical) Hopf bifurcation.* This is a situation in which two complex conjugate e-values simultaneously cross the imaginary axis into the right-half plane as a parameter is varied. Then a limit cycle is born.

Consider an example

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\vartheta} &= \omega + br^2. \end{aligned}$$

At $\mu < 0$, the origin is a stable spiral, at $\mu > 0$ the origin is an unstable spiral and there is a stable limit cycle at $r = \sqrt{\mu}$.

A sketch is helpful (see the videolectures or Strogatz's book).

To see how the pair of complex conjugate e-values cross the imaginary axis, we rewrite this system in Cartesian frame. Then

$$\begin{aligned} \dot{x} &= \mu x - \omega y + h.o.t. \\ \dot{y} &= \omega x + \mu y + h.o.t. \end{aligned}$$

Thus $A = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}$ with e-values $\lambda = \mu \pm i\omega$.

Supercritical Hopf bifurcation results when a stable spiral fixed point becomes unstable and a limit cycle is born whose size grows with μ increasing above its critical value μ_c. Generally, the size will increase as $\sqrt{\mu - \mu_c}$. So, the oscillations increase in amplitude gradually with the parameter moving away from μ_c.

It is a different story with the *subcritical* Hopf bifurcation. When μ crosses the critical value, the transition to a limit cycle is sudden, its amplitude is large *immediately* after μ becomes larger than μ_c. This one is a dangerous transition that can cause damage to a system if such a Hopf bifurcation occurs. The transition to a large-amplitude limit cycle can occur either by a gradual increase of μ beyond μ_c with solution remaining close to its stable state at $\mu < \mu_c$. Or, one could give a strong kick to the solution even with $\mu < \mu_c$, and cause the system instability. In the latter case, one has linear stability, but nonlinear instability.