Numerical Methods in Engineering and Applied Science. Assignment 3.

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(1). (max. 1 point) Find a quadratic polynomial p(t) that satisfies the following conditions:

$$p(t_n) = u_n, \qquad p'(t_n) = f_n, \qquad p'(t_{n+1}) = f_{n+1} \tag{1}$$

Which time-stepping method do you obtain if you assign $u_{n+1} = p(t_{n+1})$? Write a formula and name it.

$$p(t_n) = c + at_n + bt_n^2$$
$$p'(t_n) = a + 2bt_n = f_n$$
$$p'(t_{n+1}) = a + 2bt_{n+1} = f_{n+1}$$
$$a = f_n - 2bt_n$$
$$a + 2bt_{n+1} = f_n - 2bt_n + 2bt_{n+1} = f_{n+1}$$
$$b = \frac{(f_{n+1} - f_n)}{2t_{n+1} - 2bt_n} = \frac{1}{2h}(f_{n+1} - f_n)$$
$$a = f_n - 2bt_n = f_n - \frac{1}{h}(f_{n+1} - f_n)t_n = \frac{1}{h}(hf_n - f_{n+1}t_n + f_nt_n) = \frac{1}{h}(f_n(h + t_n) - f_{n+1}t_n)$$
$$= \frac{1}{h}(f_nt_{n+1} - f_{n+1}t_n)$$
$$u_n = p(t_n) => c = u_n - at_n - bt_n^2$$
$$u_{n+1} = u_n - \frac{1}{h}(f_nt_{n+1} - f_{n+1}t_n)t_n - \frac{1}{2h}(f_{n+1} - f_n)t_n^2 + \frac{1}{h}(f_nt_{n+1} - f_{n+1}t_n)t_{n+1}$$
$$+ \frac{1}{2h}(f_{n+1} - f_n)t_{n+1}^2 =$$
$$u_n - \frac{1}{2h}(f_{n+1} - f_n)t_n^2 + (f_nt_{n+1} - f_{n+1}t_n) + \frac{1}{2h}(f_{n+1} - f_n)t_{n+1}^2 =$$
$$u_n + (f_nt_{n+1} - f_{n+1}t_n) + \frac{1}{2h}(f_{n+1} - f_n)(t_{n+1}^2 - t_n^2) =$$
$$u_n + (f_nt_{n+1} - f_{n+1}t_n) + \frac{1}{2h}(f_{n+1} - f_n)h(t_{n+1} + t_n) =$$
$$u_n + (f_nt_{n+1} - f_{n+1}t_n) + \frac{1}{2}(f_{n+1}t_{n+1} - f_nt_{n+1} + f_{n+1}t_n - f_nt_n) =$$
$$u_n + \frac{1}{2}f_nt_{n+1} - \frac{1}{2}f_{n+1}t_n + \frac{1}{2}(f_{n+1}t_{n+1} - f_nt_n) =$$
$$u_n + \frac{1}{2}f_nh + \frac{1}{2}f_{n+1}h = u_n + \frac{h}{2}(f_n + f_{n+1})$$

It is the trapezoidal rule (order 2).

(2). Consider a family of multi-step methods

$$u_{n+1} + (\theta - 2)u_n + (1 - \theta)u_{n-1} = \frac{1}{4}h\big((6 + \theta)f_{n+1} + 3(\theta - 2)f_{n-1}\big)$$

where θ is a parameter.

(a) Determine the order of consistency and the error constant of the method. Show that both do not depend on θ.

Let's use Taylor expansion for $u_{n+1}, u_{n-1}, f_{n+1}, f_{n-1}$, consider that $f_n = u'_n$.
After that subtract the right-hand side from the left. (all calculations in the program)

$$u_{n+1} + (\theta - 2)u_n + (1 - \theta)u_{n-1} - \frac{1}{4}h\big((6 + \theta)f_{n+1} + 3(\theta - 2)f_{n-1}\big) =$$
$$= -2h^2u_n'' - \frac{h^3}{3}u_n'''\theta$$

So, the order of consistency is 1. ($-2h^2u_n''$ – doesn't depend on θ)

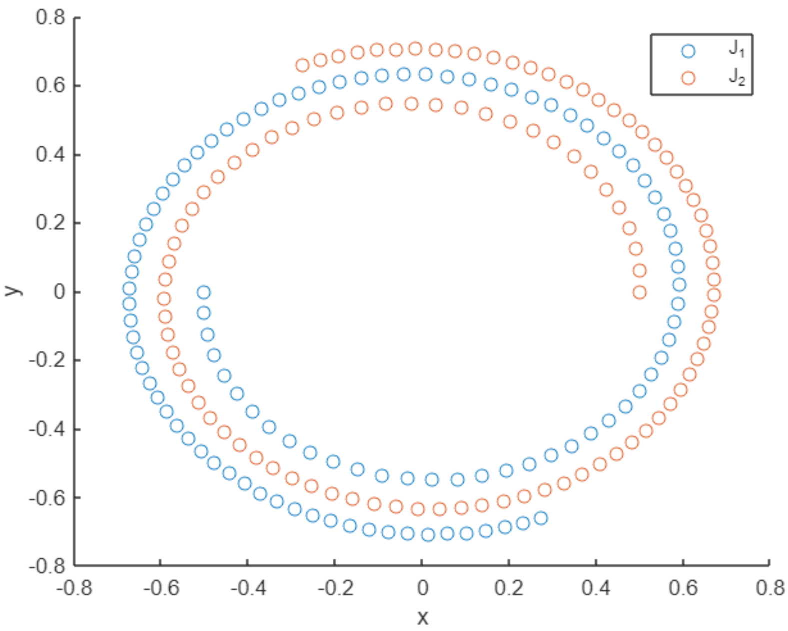
(b) For which values of θ is the method convergent?

$$\rho(z) = z^2 + (\theta - 2)z + (1 - \theta) => z_{1,2} = \frac{1}{2}(-\theta + 2 \pm \theta)$$

$$z_1 = 1; z_2 = 1 - \theta$$

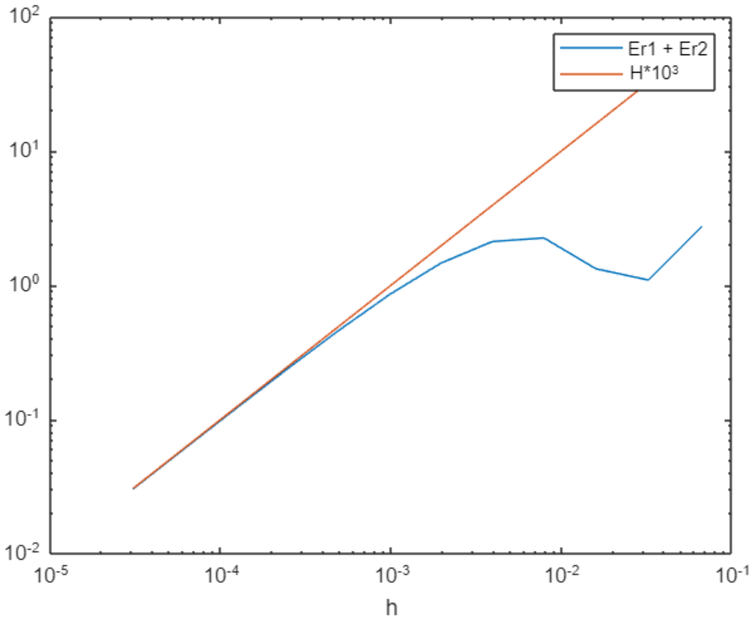
As far as $|z| \leq 1$, and if $|z| = 1$ it is simple: $\theta \in [0,2]$ => method is zero stable +consistent =>it is convergent.

(3). Using the explicit Euler method, calculate the time evolution of (xj,yj) with t ∈ [0,1] and plot y1(x1), y2(x2). On a separate plot, show the convergence of the results of the calculations at t = 1 versus the discretization step h.



J1 is the first vortex, J2 is the second.

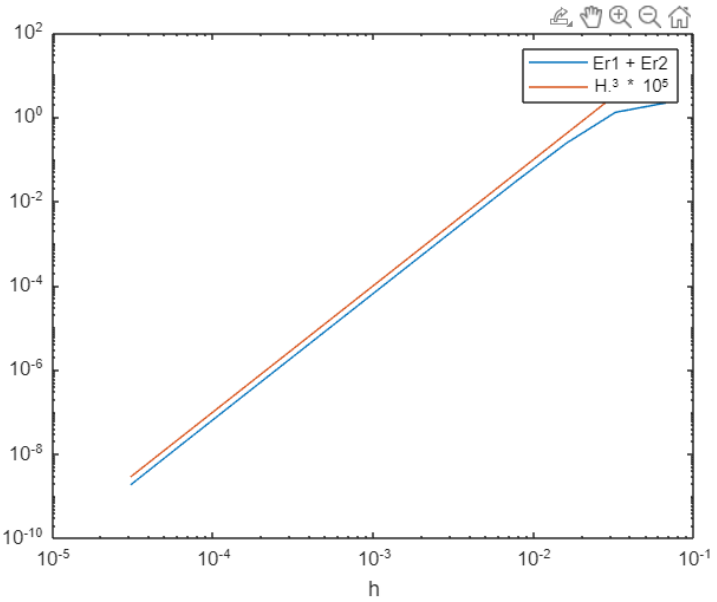
Solution must be periodic, therefore $Er_j = |\boldsymbol{J_j}(1) - \boldsymbol{J_j}(0)| = \sqrt{\Delta x^2 - \Delta y^2}$.



First order convergence.

(4). Redo the same calculations using a third-order Runge–Kutta Method

Show the convergence plot with respect to h using the logarithmic scale and comment on the slope of the line that you obtain.



You can see 3rd order convergence. So, 3rd Runge-Kutta – 3rd order convergence.