Appendix: Dynamical Systems

This appendix is a brief presentation of phase plane analysis in two-dimensional nonlinear dynamical systems. It is meant to be a tool that allows quick reference when reading certain parts of the book.

A two-dimensional nonlinear autonomous system has the form

$$\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y), \tag{5.7.1}$$

where p and q are given functions that are assumed to have continuous derivatives of all orders. By a *solution* of (5.7.1) we mean a pair of smooth functions $x = x(t), y = y(t)$ that satisfy the differential equations (5.7.1) for all t in some interval I. The interval I is often the whole real line. Graphically, we represent the solution as a curve in the xy plane, called the *phase plane.* A solution curve is called an *orbit, path*, or *trajectory* of (5.7.1). The independent variable t is regarded as a parameter along the curve and is interpreted as time. The orbits have a natural positive direction to them, namely, the direction in which they are traced out as the time parameter t increases; to indicate this direction an arrow is placed on a given orbit. Because the system is autonomous (t does not appear on the right sides), the solution is invariant under a time translation; therefore, the time t may be shifted along any orbit. A constant solution $x(t) = x_0, y(t) = y_0$ to (5.7.1) is called an *equilibrium solution*, and its orbit is a single point (x_0, y_0) in the phase plane. Clearly, such points satisfy the algebraic relations

$$p(x, y) = 0 \quad q(x, y) = 0. \tag{5.7.2}$$

Points that satisfy (5.7.2) are called *critical points* (also, *rest points* and *equilibrium points*), and each such point represents an equilibrium solution. It is evident that no other orbit can pass through a critical point at finite time t; otherwise, uniqueness of the initial value problem is violated. For the same reason, no orbits can cross. The totality of all the orbits of (5.7.1) and critical points, graphed in the phase plane, is called the *phase diagram*. The qualitative behavior of the phase diagram is determined to a large extent by the location of the critical points and the local behavior of orbits near those points. The *Poincaré—Bendixson theorem* in two dimensions characterizes the behavior of the possible orbits of (5.7.1):

(a) An orbit cannot approach a critical point in finite time; that is, if an orbit approaches a critical point, then, necessarily, $t \longrightarrow \pm\infty$.

(b) As $t \longrightarrow \pm\infty$, an orbit either approaches a critical point, moves on a closed path, approaches a closed path, or leaves every bounded set. A closed orbit is a *periodic solution*.

The Poincaré—Bendixson theorem does not hold in three or more dimensions.

In principle, orbits can be found by integrating the differential relationship

$$\frac{dy}{dx} = \frac{q(x,y)}{p(x,y)}, \tag{5.7.3}$$

which comes from dividing the two equations in (5.7.1). When this is done, information about how a given orbit depends on the time parameter t is lost; however, this is seldom crucial.

The right sides of (5.7.1) define a vector field $<p,q>$ in the phase plane; the orbits are the curves that have this vector field as their tangents. Thus the orbits are the integral curves of (5.7.3). The loci $p(x,y) = 0$ and $q(x,y) = 0$, where the vector field is vertical and horizontal, respectively, are called the *nullclines*. Determining the phase portrait of (5.7.1) is generally facilitated by graphing the nullclines and determining the direction of the vector field $<p,q>$ in the regions separated by the nullclines.

The following two results of Bendixson and Poincaré, respectively, are also helpful.

(a) If $p_x + q_y$ is of one sign in a region of the phase plane, the system (5.7.1) cannot have a closed orbit in that region.

(b) A closed orbit of (5.7.1) must surround at least one critical point.

Critical points are also classified as to their stability. A critical point is *stable* if each orbit sufficiently near the point at some time t_0 remains in a prescribed circle about the point for all $t > t_0$. If this is not the case, the critical point is *unstable*. A critical point is *asymptotically stable* if it is stable and every orbit sufficiently near the point at some time t_0 approaches the point as $t \longrightarrow \infty$. An asymptotically stable critical point is called an *attractor*.

Linear Systems

A linear system has the form

$$x' = ax + by, \tag{5.7.4}$$

$$y' = cx + dy. \tag{5.7.5}$$

Hereafter, we use a prime to denote the time derivative. We assume up front that $ad - bc$ is nonzero; then the only critical point of (5.7.4)–(5.7.5) is the origin $x = 0$, $y = 0$. We write the system in matrix form as

$$\mathbf{x}' = A\mathbf{x}, \tag{5.7.6}$$

where $\mathbf{x} = (x, y)^{\mathrm{T}}$ (the superscript T means transpose) is the vector of unknowns and A is the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By assumption, $\det A \neq 0$. Solutions of (5.7.6) are obtained by assuming

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}, \tag{5.7.7}$$

where $\mathbf{v}$ is a constant vector and λ is a constant, both to be determined. Substituting this form into (5.7.6) yields the algebraic eigenvalue problem

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{5.7.8}$$

Any eigenpair $(\lambda, \mathbf{v})$ of (5.7.8) gives a solution of (5.7.6) of the form (5.7.7). Therefore, if $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ are two eigenpairs with λ_1 and λ_2 distinct, all solutions of (5.7.6) are given by the linear combination

$$\mathbf{x}(t) = c_1\mathbf{v}_1 \exp(\lambda_1 t) + c_2\mathbf{v}_2 \exp(\lambda_2 t), \tag{5.7.9}$$

where c_1 and c_2 are arbitrary constants. This includes the case when λ_1 and λ_2 are complex conjugates; then real solutions may be found by taking the real and imaginary parts of (5.7.9). If $\lambda_1 = \lambda_2 = \lambda$, there may not be two linear independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$; if two independent eigenvectors exist, (5.7.9) remains valid; if not (there is a single eigenvector $\mathbf{v}$), the general solution to (5.7.6) is

$$\mathbf{x}(t) = c_1\mathbf{v} \exp(\lambda t) + c_2(\mathbf{w} + \mathbf{v}t) \exp(\lambda t), \tag{5.7.10}$$

for some constant vector $\mathbf{w}$ that satisfies $(A - \lambda I)\mathbf{w} = \mathbf{v}$.

Therefore, we may catalog the different types of solutions of the linear system (5.7.6), depending on the eigenvalues and eigenvectors of the coefficient matrix A. The results in the following summary come directly from the forms of the general solution (5.7.9) or (5.7.10).

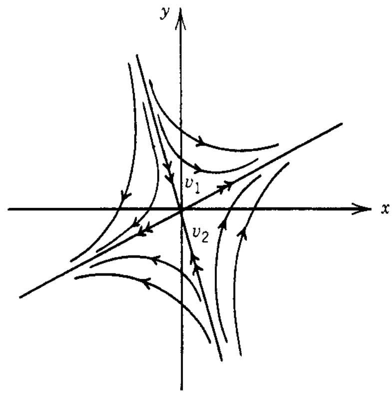

Figure 5.12 Saddle point.

Case 1. If the eigenvalues are real and have opposite signs, the critical point (0,0) is a *saddle point*, and a generic phase portrait is shown in Figure 5.12. The two orbits entering the origin are the two *stable manifolds*, and the two orbits exiting the origin are the two *unstable manifolds*; the directions of these manifolds at the critical point are determined by the two eigenvectors. The stable manifolds correspond to the negative eigenvalue and the unstable manifolds correspond to the positive eigenvalue. These special manifolds are called *separatrices*. A saddle point is unstable.

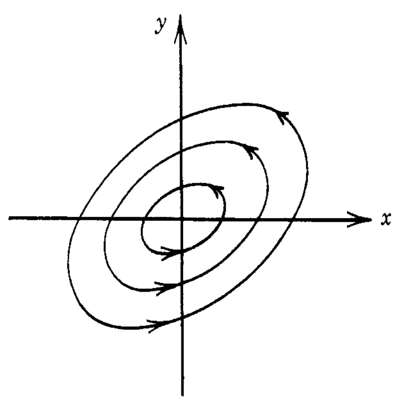

Figure 5.13 Center.

Case 2. If the eigenvalues are purely imaginary, the orbits form closed curves (ellipses) representing periodic solutions, and the origin is a *center* (see Figure 5.13). A center is stable.

Case 3. If the eigenvalues are complex (conjugates) and not purely imaginary, the orbits spiral into, or out of, the origin, depending on whether the real part of the eigenvalues is negative or positive, respectively. In this case the origin is called a *spiral point* or a focus (see Figure 5.14). If the real part of the eigenvalues is negative, the spiral point is an attractor, which is asymptotically stable; if the real part is positive, the point is unstable.

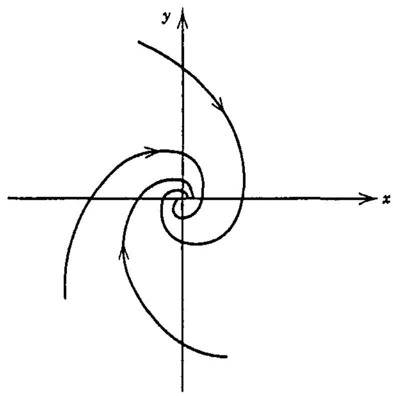

Figure 5.14 Asymptotically stable spiral.

Case 4. If the eigenvalues are real, are distinct, and have the same sign, the origin is classified as a *node*. In this case all the orbits enter the origin if the eigenvalues are negative (giving an attractor), and all the orbits exit the origin

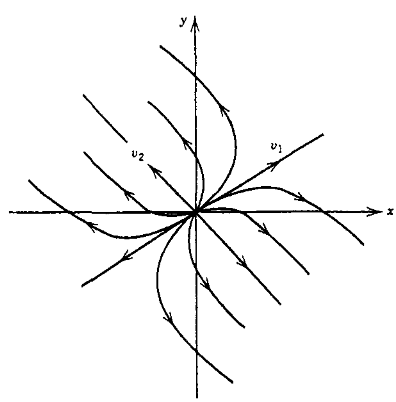

Figure 5.15 Unstable node.

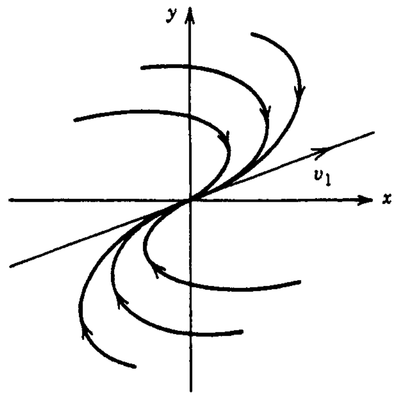

Figure 5.16 Degenerate stable node.

if the eigenvalues are positive (giving an unstable node). The detailed orbital structure near the origin is determined by the eigenvectors. One of them defines the direction in which orbits enter (or exit) the origin, and the other defines the direction approached by the orbits as t approaches minus, or plus, infinity (see Figure 5.15).

Case 5. If the eigenvalues are real and equal, the critical point at the origin is still classified as a node, but it is of different type than in case 4 because there may be only one eigenvector. In this case the node is called *degenerate* (see Figure 5.16). The node is an attractor if the eigenvalue is negative, and it is unstable if the eigenvalue is positive.

The eigenvalues of A are calculated as the roots of the characteristic equation

$$\lambda^2 - \operatorname{tr} A\lambda + \det A = 0$$

where $\operatorname{tr} A$ denotes the trace of A and $\det A$ is the determinant of A. The corresponding eigenvectors are the solution to the homogeneous equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

A useful result is that $(0, 0)$ is asymptotically stable if, and only if, $\operatorname{tr} A < 0$ and $\det A > 0$.

Nonlinear Systems

Now we return to the nonlinear system (5.7.1). The key idea is that (5.7.1) can be studied by examining a linearization near the critical points. Let $x = x_0$, $y = y_0$ be an isolated critical point of (5.7.1), meaning that there is a neighborhood of (x_0, y_0) that contains no other critical points. The local structure of the orbits near this critical point for the nonlinear system (5.7.1) can be determined, under fairly broad conditions, by examining the linearized system at that point. Let $\bar{x}(t)$ and $\bar{y}(t)$ denote small perturbations about the equilibrium (x_0, y_0), with

$$x(t) = x_0 + \bar{x}(t), \quad y(t) = y_0 + \bar{y}(t).$$

Substituting into the nonlinear system gives

$$\bar{x}' = p(x_0 + \bar{x}, y_0 + \bar{y}), \qquad \bar{y}' = q(x_0 + \bar{x}, y_0 + \bar{y}).$$

If the right sides are expanded in Taylor series, we obtain

$$\begin{aligned} \bar{x}' &= p(x_0, y_0) + p_x(x_0, y_0)\bar{x} + p_y(x_0, y_0)\bar{y} + \text{higher order terms}, \\ \bar{y}' &= q(x_0, y_0) + q_x(x_0, y_0)\bar{x} + q_y(x_0, y_0)\bar{y} + \text{higher order terms}. \end{aligned}$$

But $p(x_0, y_0) = q(x_0, y_0) = 0$, so to leading order the perturbations satisfy the linear system

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} p_x(x_0, y_0) & p_y(x_0, y_0) \\ q_x(x_0, y_0) & q_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}. \tag{5.7.11}$$

We assume that the coefficient matrix

$$J(x_0, y_0) = \begin{pmatrix} p_x(x_0, y_0) & p_y(x_0, y_0) \\ q_x(x_0, y_0) & q_y(x_0, y_0) \end{pmatrix}.$$

which is called the *Jacobian matrix* at (x_0, y_0), has a nonzero determinant. The linearized system for the small perturbations, whose behavior is determined by the eigenvalues and eigenvectors of J, dictate the local behavior of orbits for the nonlinear system near (x_0, y_0).

The main result is as follows: Let (x_0, y_0) be an isolated critical point of the nonlinear system (5.7.1). Suppose that $\det J(x_0, y_0) \neq 0$ and that $J(x_0, y_0)$ does not have purely imaginary eigenvalues. Then (5.7.1) has the same qualitative orbital structure near (x_0, y_0) as the linearized system has near $(0, 0)$.

By *qualitative structure* we mean the same stability characteristics and the same nature of critical point (saddle, node, or focus). The case that does not extend to the nonlinear system is that of a center for the linear system. In this case one must examine the higher-order terms to determine the nature of the critical point of the nonlinear system. If the Jacobian matrix for the linearized system has a zero eigenvalue, then higher-order terms play a crucial role and the nature of the critical point of the nonlinear system may differ from a node, saddle, focus, or center. For example, it may have nodal structure on one side and a saddle structure on the other. A careful analysis is required in the case when $\det J = 0$.

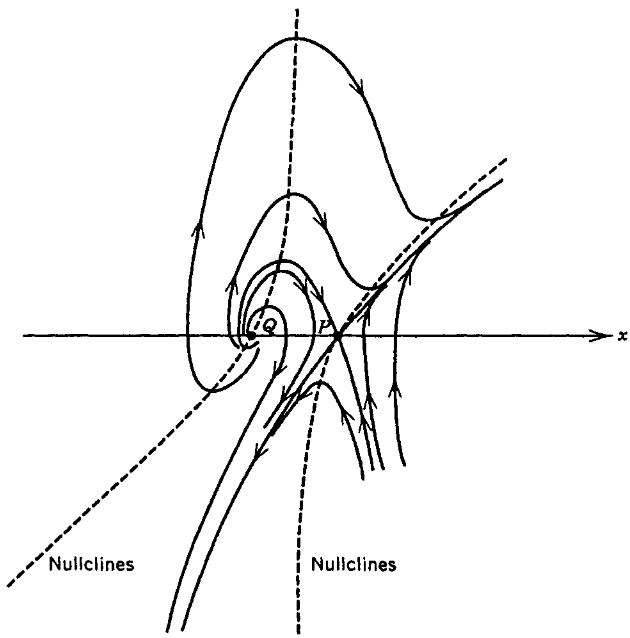

Figure 5.17 Phase portrait of the nonlinear system (5.7.12). $P : (0, 0)$ and Q : $(-1, 0)$ are the two critical points, a saddle and an unstable spiral, respectively. The nullclines $y' = 0$ are represented by the dashed lines.

The phase diagram of (5.7.1) is determined by finding all the critical points, analyzing their nature and stability, and then examining the global behavior and structure of the orbits. For example, it is an interesting problem to determine whether an orbit connects two critical points in the system; such connections are called *heteroclinic orbits.* An orbit connecting a critical point to itself is a *homoclinic orbit.*

Example. Consider the nonlinear system

$$x' = y, \qquad y' = x - y + x^2 - 2xy. \tag{5.7.12}$$

There are two critical points $P : (0, 0)$ and $Q : (-1, 0)$. The Jacobi matrix $J(0, 0)$ at P is

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

which has eigenvalues $(-1 \pm \sqrt{5})/2$. These are real with opposite signs, and therefore P is a saddle point. The Jacobi matrix $J(-1,0)$ at Q is

$$\begin{pmatrix} 0 & 1 \\ 1-2x-2y & -1-2x \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues are $(1 \pm i\sqrt{3})/2$, which are complex with positive real part. Thus Q is an unstable spiral point. The nullclines are $y = 0$ (where $x' = 0$ and the vector field is vertical) and $y = x(1+x)/(1+2x)$ (where $y' = 0$ and the vector field is horizontal). A phase diagram is shown in Figure 5.17. The nullclines $y' = 0$ are shown dotted. There is a heteroclinic orbit exiting the spiral point Q and entering the saddle point P along a separatrix, or one of its stable manifolds. □

All computer algebra systems (MATLAB, Mathematica, Maple, and others) have packages, or easily used programs, that plot phase diagrams in two dimensions. An easy to use, outstanding ODE solver and graphics package has been developed in MATLAB by John Polking (Polking 2004).

EXERCISES

1. For the following linear systems, find the type of the critical point and its stability, find the general solution, and sketch a phase diagram.

   (a) $x' = -\frac{3}{2}x + \frac{1}{2}y, \quad y' = x - y.$

   (b) $x' = 4x - 3y, \quad y' = 6x - 7y.$

   (c) $x' = -2x - 3y, \quad y' = 3x - 2y.$

   (d) $x' = -3y, \quad y' = 6x.$

2. For the following nonlinear systems, find all the equilibria, analyze their stability, draw the nullclines and sketch a phase diagram.

   (a) $x' = x - xy, \quad y' = y - xy.$

   (b) $x' = y, \quad y' = x^2 - 1 - y.$

   (c) $x' = y + (1-x)(2-x), \quad y' = y - ax^2, \quad a > 0.$

   (d) $x' = x - y, \quad y' = -y + (5x^2)/(4 + x^2).$

3. State why the system $x' = x + x^3 - 2y, \quad y' = y^5 - 3x$ has no periodic solutions (closed cycles).

Reference Notes. A good introduction to the diffusion equation and random walks is given in the first chapter of Zauderer (2006). Also see Murray (2002, 2003) and Okubo & Levin (2001). An elementary discussion of the general similarity method is presented in Logan (1987). See also Dresner (1983) and Rogers

& Ames (1989); the latter contains an extensive bibliography as well as numerous, recent applications. The similarity method as it applies to problems in the calculus of variations is discussed in Logan (1977, 1987). For a treatment of Fisher's equation, see Murray (1977, 2002, 2003). Canosa (1973) is a classic paper on TWS to Fisher's paper. A good source for Burgers' equation is Whitham (1974); Kreiss and Lorenz (1989) discuss questions of uniqueness and existence of solutions to boundary value problems associated with Burgers' equation.

For a detailed discussion of phase plane phenomena, the reader may consult one of the many excellent books on the subject; see, for example, Strogatz (1994) or Hirsch et al. (2004). Most sophomore-level differential equation texts give introductory material (see, e.g., Logan 2006b), as do texts on mathematical biology, where phase plane methods are essential (Edelstein-Keshet 2005, Brauer & Castillo-Chavez 2001). In fact, mathematical biology texts are one of the best sources for elementary approaches and interesting applications.