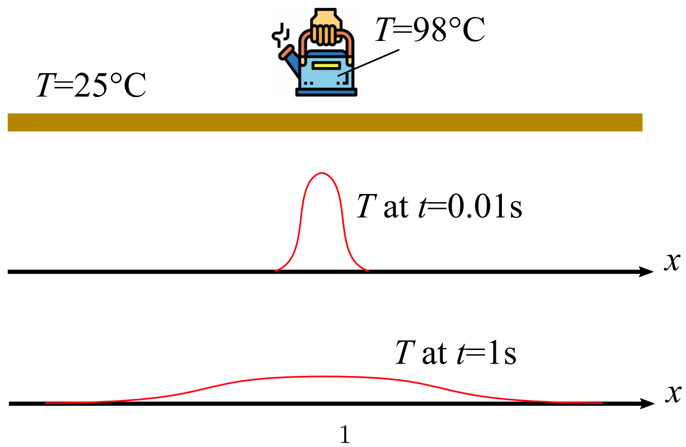
Heat equation.

Heat equation describes how the distribution of temperature evolves over time in a solid medium, as the heat flows from places where it is higher towards places where it is lower. In 1D, it is written as

$$\frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} = 0. \tag{1}$$



In 2D,

$$\frac{\partial T}{\partial t} - \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = 0. \tag{2}$$

In 3D,

$$\frac{\partial T}{\partial t} - \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = 0. \tag{3}$$

In the general vector notation,

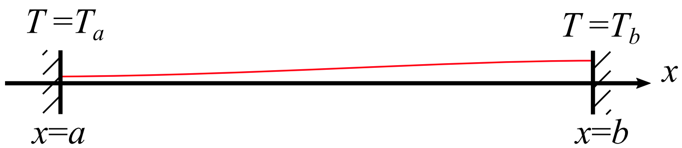
$$\frac{\partial T}{\partial t} - \kappa \Delta T = 0 \qquad \text{or} \qquad \frac{\partial T}{\partial t} - \kappa \nabla^2 T = 0. \tag{4}$$

The heat conduction problem in a finite domain requires boundary conditions. Typical boundary conditions are

- *Dirichlet* (prescribed temperature on the boundary). In 1D, this condition is written at the ends of the bar as

$$T = T_a \qquad \text{at} \qquad t = a, \tag{5}$$

$$T = T_b \qquad \text{at} \qquad t = b. \tag{6}$$



- *Neumann* (prescribed rate of heat flow). In 1D, this condition is written as

$$\frac{\partial T}{\partial x} = q_a \qquad \text{at} \qquad t = a, \tag{7}$$

$$\frac{\partial T}{\partial x} = q_b \qquad \text{at} \qquad t = b. \tag{8}$$

- If we are not concerned about boundary effects in our numerical simulation, we can use *periodic* boundary conditions

$$T\Big|_{x=a} = T\Big|_{x=b}, \tag{9}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=a} = \left.\frac{\partial T}{\partial x}\right|_{x=b}. \tag{10}$$

Let us numerically solve the 1D heat conduction problem with a prescribed initial condition and periodic boundary conditions.

$$\frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} = 0, \qquad x \in [0, \pi], \tag{11}$$

$$T\big|_{t=0} = \sin^4 x, \tag{12}$$

$$T\big|_{x=0} = T\big|_{x=\pi}, \tag{13}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = \left.\frac{\partial T}{\partial x}\right|_{x=\pi}. \tag{14}$$

Let us set the thermal diffusivity

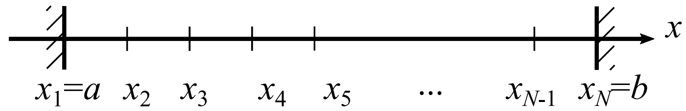
$$\kappa = 0.1 \tag{15}$$

and let us solve in the time interval

$$t \in [0, 10] \tag{16}$$

All numerical values are in some suitable physical units (e.g., x is in meters, t is in seconds etc).

To use the finite-difference method, we subdivide the domain $[a, b]$ in small intervals of length Δx, and the time interval $[0, t_{max}]$ in small intervals Δt.



Since $a = 0$ and $b = \pi$ in our example, we have $\Delta x = \pi/(N-1)$ and $x_j = (j-1)\Delta x$.

Maybe the easiest method is to use a central second-order scheme in space,

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=x_j, t=t_i} \approx \frac{T(x_j - \Delta x, t_i) - 2T(x_j, t_i) + T(x_j + \Delta x, t_i)}{\Delta x^2} \tag{17}$$

and forward first-order scheme in time,

$$\left.\frac{\partial T}{\partial t}\right|_{x=x_j, t=t_i} \approx \frac{T(x_j, t_i + \Delta t) - T(x_j, t_i)}{\Delta t}. \tag{18}$$

Let us replace

$$T(x_j, t_i) \qquad \text{by} \qquad T_j^i, \tag{19}$$

$$T(x_j, t_i + \Delta t) \qquad \text{by} \qquad T_j^{i+1}, \tag{20}$$

$$T(x_j - \Delta x, t_i) \qquad \text{by} \qquad T_{j-1}^i, \tag{21}$$

$$T(x_j + \Delta x, t_i) \qquad \text{by} \qquad T_{j+1}^i. \tag{22}$$

We obtain the following finite-difference approximation of the heat equation,

$$\frac{T_j^{i+1} - T_j^i}{\Delta t} - \kappa \frac{T_{j-1}^i - 2T_j^i + T_{j+1}^i}{\Delta x^2} = 0, \tag{23}$$

where $j = 1, 2, ..., n$ and $i = 0, 2, ..., k$.

This gives a recursion formula that will be used in the practical computation

$$T_j^{i+1} = T_j^i + \kappa \Delta t \frac{T_{j-1}^i - 2T_j^i + T_{j+1}^i}{\Delta x^2}. \tag{24}$$

For points $j = 1$ and $j = n$, we use the periodic boundary condition and obtain

$$T_1^{i+1} = T_1^i + \kappa \Delta t \frac{T_n^i - 2T_1^i + T_2^i}{\Delta x^2} \tag{25}$$

and

$$T_n^{i+1} = T_n^i + \kappa \Delta t \frac{T_{n-1}^i - 2T_n^i + T_1^i}{\Delta x^2}. \tag{26}$$

The initial condition in our example gives

$$T_j^0 = \sin^4\left((j-1)\Delta x\right). \tag{27}$$

For comparison, the exact analytical solution is

$$T(x,t) = \frac{3}{8} - \frac{1}{2} e^{-4\kappa t} \cos 2x + \frac{1}{8} e^{-16\kappa t} \cos 4x. \tag{28}$$

The following program implements this numerical scheme.

```
% heat_1d_ftcs.m
% Numerical simulation of the heat equation using
% forward-time central-space scheme.
clearvars;
close all;

% Channel length
L = pi;
% Thermal diffusivity
kappa = 0.1;
% Number of grid points
nx = 51;
% Time step size
dt = 0.002;
% Number of time steps
nt = 1000;

% Grid step
dx = L/(nx-1);
% Grid
x = dx*(0:(nx-1)).';
% Boundary conditions are periodic

```

```matlab
% Initial condition
T0 = sin(x).^4;

% Startup
t = 0;
T = T0;
T_new = T;

% Time iterations
for n = 1:nt
    % Update the values in the bulk flow
    T_new(1) = T(1) + dt * kappa *(T(nx-1)-2*T(1)+T(2))/dx^2;
    T_new(2:nx-1) = T(2:nx-1) + ...
        dt * kappa *(T(1:nx-2)-2*T(2:nx-1)+T(3:nx))/dx^2;
    % Last point holds the same value as the first point
    T_new(nx) = T_new(1);

    % Update
    t = n*dt;
    T = T_new;

    % Plot temperature profiles
    figure(1); clf;
    % Unsteady numerical solution
```

```
    plot(x,T,'.-'); hold on;

    % Plot the exact solution
    T_exact = 3/8 - 1/2*exp(-4*kappa*t)*cos(2*x) + ...
        1/8*exp(-16*kappa*t)*cos(4*x);
    plot(x,T_exact,'x'); hold on;

    % Annotations
    xlabel('x');
    ylabel('T');
    axis([0 pi 0 1]);
    pause(0.01);
end
```