Homework 4. Continuum mechanics

(1). Problem 1

Find the pressure distribution of a linearly viscous fluid if its flow is described by the velocity field:

$$v_1 = 0; v_2 = k(x_2^2 - x_3^2); v_3 = -2kx_2x_3.$$

Consider the body forces are represented only by the gravitation: $F = ge_1$. The pressure at origin
is $p(0,0,0) = p_0 2$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\left(\frac{\partial v_i}{\partial x_j}\right)\right) = \rho F_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j^2}$$

$F = (g, 0,0)$

$\frac{d\rho}{dt} + div(v) = 0, but\ div(v) = 0 => we\ can\ consider\ \rho = const$

$i = 1, \qquad 0 = \rho g - \frac{\partial p}{\partial x_1} => \frac{\partial p}{\partial x_1} = \rho g => p = \rho g x_1 + const(x_1)$

$i = 2, \qquad \rho\left(\frac{\partial k(x_2^2 - x_3^2)}{\partial t} + v_j\left(\frac{\partial k(x_2^2 - x_3^2)}{\partial x_j}\right)\right) = -\frac{\partial p}{\partial x_2} + \mu\frac{\partial^2 k(x_2^2 - x_3^2)}{\partial x_j^2}$

$\rho k\big(0 + v_2(2x_2) - v_3(2x_3)\big) = -\frac{\partial p}{\partial x_2} + \mu k(2-2) = -\frac{\partial p}{\partial x_2}$

$2\rho k(v_2x_2 - v_3x_3) = -\frac{\partial p}{\partial x_2}$

$\frac{\partial p}{\partial x_2} = 2\rho k(v_3x_3 - v_2x_2)$

$\frac{\partial p}{\partial x_2} = 2\rho k(-2kx_2x_3x_3 - k(x_2^2 - x_3^2)x_2)$

$p = 2\rho k\left(-kx_2^2x_3^2 - k\left(\frac{x_2^4}{4} - \frac{x_3^2x_2^2}{2}\right)\right) + const(x_2)$

$p = 2\rho k^2\left(-x_2^2x_3^2 - \frac{x_2^4}{4} + \frac{x_3^2x_2^2}{2}\right) + const(x_2)$

$p = -\rho k^2\left(x_2^2x_3^2 + \frac{x_2^4}{2}\right) + const(x_2)$

$i = 3, \qquad \rho\left(-v_j\left(\frac{\partial 2kx_2x_3}{\partial x_j}\right)\right) = -\frac{\partial p}{\partial x_3} - \mu\frac{\partial^2 2kx_2x_3}{\partial x_j^2}$

$\rho(-v_2 2kx_3 - v_3 2kx_2) = -\frac{\partial p}{\partial x_3}$

$\rho(-k(x_2^2 - x_3^2)2kx_3 + 2kx_2x_3 2kx_2) = -\frac{\partial p}{\partial x_3}$

$2\rho k^2(-(x_2^2 - x_3^2)x_3 + x_2x_3 2x_2) = -\frac{\partial p}{\partial x_3}$

$2\rho k^2\left(2x_2^2x_3 - (x_2^2x_3 - x_3^3)\right) = -\frac{\partial p}{\partial x_3}$

$\frac{\partial p}{\partial x_3} = 2\rho k^2(-x_3^3 - x_2^2x_3)$

$p = 2\rho k^2\left(-\frac{x_3^4}{4} - \frac{x_2^2x_3^2}{2}\right) + const(x_3)$

$p = -\rho k^2\left(\frac{x_3^4}{2} + x_2^2x_3^2\right) + const(x_3)$

From i =2 take all with x_2, but without x_1,from i=3, all $x_3$without x_2, x_1 and put it all in $const(x_1)$

$$const(x_1) = -\rho k^2\left(x_2^2 x_3^2 + \frac{x_2^4}{2}\right) - \rho k^2\left(\frac{x_3^4}{2}\right)$$

$$p = \rho g x_1 - \rho k^2\left(x_2^2 x_3^2 + \frac{x_2^4}{2} + \frac{x_3^4}{2}\right) + C$$

$$p(0,0,0) = p_0 => \ p = \rho g x_1 - \rho k^2\left(x_2^2 x_3^2 + \frac{x_2^4}{2} + \frac{x_3^4}{2}\right) + p_0$$

(2). Problem 2

The steady laminar flow of a viscous incompressible fluid (with viscosity coefficient μ) in a pipe is created by known pressure gradient $\left(\frac{dp}{dx_i} = \varphi\ for\ i\ = 1,2,3\right)$ and defined by the velocity field:

$$v_1 = v(x_2, x_3) = C_1\left(\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}\right) + C_2; v_2\ = 0; v_3 = 0.$$

Find coefficients $C1$ and $C2$ if the no-slip condition is satisfied on pipe walls, which an elliptical cross section described by $\left(\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2}\right) = 1$. Neglect body forces.

On the walls $v = 0 => v_1 = C_1 + C_2 => C_2 = -C_1$

$$\rho\left(v_j\left(\frac{\partial v_1}{\partial x_j}\right)\right) = -\phi + \mu\frac{\partial^2 v_1}{\partial x_j^2}$$

$$\rho\left(v_1\left(\frac{\partial v_1}{\partial x_1}\right)\right) = -\phi + \mu\left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right)$$

$$0 = -\phi + \mu\left(\frac{C_1 2}{b^2} + \frac{2C_1}{c^2}\right) = -\phi + C_1\mu\left(\frac{2}{b^2} + \frac{2}{c^2}\right)$$

$$C_1 = \frac{\phi}{\left(\frac{2}{b^2} + \frac{2}{c^2}\right)\mu} = \frac{\phi}{2\left(\frac{c^2 + b^2}{b^2 c^2}\right)\mu} = \frac{\phi b^2 c^2}{2(c^2 + b^2)\mu}$$

$$C_2 = -C_1$$

(3). Problem 3

The stress inside an elastic sphere (the material is isotropic with $E\ = 200\ GPa, v\ = 0.3$) is the following:

$$\sigma = \begin{bmatrix} 50k & 2 & 0 \\ 2 & 4 & 4 \\ 0 & 4 & -4 \end{bmatrix} GPa$$

(a) find Lame's coefficients.

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad v = \frac{\lambda}{2(\lambda + \mu)}$$

$$v = \frac{\lambda}{2(\lambda + \mu)} => 2v(\lambda + \mu) = \lambda => 2v\mu = \lambda - 2\lambda v => \mu = \frac{\lambda - 2\lambda v}{2v}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = \frac{\frac{\lambda - 2\lambda v}{2v}\left(3\lambda + 2\frac{\lambda - 2\lambda v}{2v}\right)}{\lambda + \frac{\lambda - 2\lambda v}{2v}} = \frac{\frac{\lambda - 2\lambda v}{2v}\left(\frac{6v\lambda + 2\lambda - 4\lambda v}{2v}\right)}{\frac{2v\lambda + \lambda - 2\lambda v}{2v}}$$

$$= \frac{\frac{\lambda - 2\lambda v}{2v}\left(\frac{2v\lambda + 2\lambda}{2v}\right)}{\frac{\lambda}{2v}} = \frac{\lambda - 2\lambda v}{2v}\left(\frac{2v\lambda + 2\lambda}{2v}\right) * \frac{2v}{\lambda} = \frac{\lambda - 2\lambda v}{1}\left(\frac{v + 1}{v}\right) =$$

$$\lambda(1 - 2v)\left(\frac{v + 1}{v}\right) = E$$

$$\lambda = E\left(\frac{v}{v + 1}\right)\frac{1}{(1 - 2v)} = E\left(\frac{0.3}{0.3 + 1}\right)\frac{1}{(1 - 2 * 0.3)} = E\left(\frac{0.3}{1.3}\right)\frac{1}{(0.4)} \approx 0.6 * E\ = 120GPa$$

$$\mu = \lambda \frac{1-2\nu}{2\nu} = E\left(\frac{\nu}{\nu+1}\right)\frac{1}{(1-2\nu)}\frac{1-2\nu}{2\nu} = E\left(\frac{1}{\nu+1}\right)\frac{1}{2} = E\frac{1}{2.6} = 0.4E = 80GPa$$

(b) find the maximum shear stress in the sphere, if the relative sphere volume change is k^2, where $k \in R$.

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right) = \frac{50k}{E}$$
$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right) = \frac{1}{E}\left(4 - 0.3(50k - 4)\right) = \frac{1}{E}(5.2 - 15k)$$
$$\varepsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{22} + \sigma_{11})\right) = \frac{1}{E}\left(-4 - 0.3(50k + 4)\right) = \frac{1}{E}(-5.2 - 15k)$$
$$\frac{\Delta V}{V} = \varepsilon_{ii} = \frac{1}{E}(50k + 5.2 - 15k - 5.2 - 15k) = \frac{1}{E}(20k) = k^2 => k = \frac{20}{E} = \frac{20}{200} = 0.1$$

$$\sigma = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 4 & 4 \\ 0 & 4 & -4 \end{bmatrix} GPa$$
$$\lambda_{1,2,3} = -5.7, \qquad 3.5, \qquad 7.2$$

Max shear stress: $\frac{7.2 + 5.7}{2} \approx 6.5GPa$

(4). A thin rectangular plate of dimensions 30 cm x 50 cm was formed by welding two triangular plates. The plate was elongated by 5 cm in the long direction and compressed by 3 cm in the short direction. Determine the normal stress acting perpendicular to the weld line and the shear stress acting parallel to the weld. Consider that the plate and the weld are made of steel and assume that the deformations are small.

$$\varepsilon_{11} = \frac{2\Delta x_1}{x_1} = \frac{10}{50} = 0.2$$
$$\varepsilon_{22} = \frac{-2\Delta x_2}{x_2} = \frac{-6}{30} = -0.2$$
$$\sigma_{11} = \frac{E}{1-\nu^2}(0.2 - \nu * 0.2) =$$
$$\sigma_{22} = \frac{E}{1-\nu^2}(-0.2 + \nu * 0.2) =$$

For steel: $E = 210GPa, \nu = 0.3$

$$\sigma_{11} \approx 32.3GPa$$
$$\sigma_{22} \approx -32.3GPa$$
$$\sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\boldsymbol{n} = \begin{bmatrix} -sin\phi \\ cos\phi \\ 0 \end{bmatrix}, where\ \phi = arctg\left(\frac{24}{60}\right) = 0.38$$
$$\sigma_{nn} = \boldsymbol{n\sigma n} = \begin{bmatrix} -sin\phi \\ cos\phi \\ 0 \end{bmatrix}^T \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -sin\phi \\ cos\phi \\ 0 \end{bmatrix} = \begin{bmatrix} -sin\phi \\ cos\phi \\ 0 \end{bmatrix}^T \begin{bmatrix} -\sigma_{11}sin\phi \\ \sigma_{22}cos\phi \\ 0 \end{bmatrix}$$
$$= \sigma_{11}\sin^2\phi + \sigma_{22}\cos^2\phi = \sigma_{11} * 0.1379 + \sigma_{22}0.86 = -23.3GPa$$
$$\sigma_\tau = \sqrt{(\sigma\boldsymbol{n})^2 - \sigma_{nn}} = \sqrt{(\sigma\boldsymbol{n})(\sigma\boldsymbol{n}) - \sigma_{nn}} = \sqrt{\sigma_{11}^2\sin^2\phi + \sigma_{22}^2\cos^2\phi - 23.3^2}$$
$$= \sqrt{\sigma_{11}^2 0.1379 + \sigma_{22}^2 0.86 - 23.3^2} = 22GPa$$

$|\sigma_{nn}| = 23.3GPa$

$|\sigma_\tau| = 22GPa$

(5). Problem 5

TRANSVERSELY ISOTROPIC LINEARLY ELASTIC SOLID

$$E_1 = 3GPa$$
$$E_3 = 30GPa$$
$$\nu_{32} = \nu_{31} = 0.3$$
$$\nu_{12} = \nu_{21} = 0.35$$

$$C = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix}$$

Because of symmetry:

$$-\frac{\nu_{31}}{E_3} = -\frac{\nu_{13}}{E_1} => \nu_{13} = \frac{E_1\nu_{31}}{E_3} = \frac{1}{10}0.3 = 0.03$$
$$G_{12} = \frac{E1}{2(1+\nu_{21})} = \frac{3}{2(1+0.35)} = 1.11GPa$$

We get:

$$C = \begin{bmatrix} 0.33 & -0.12 & -0.01 & 0 & 0 & 0 \\ -0.12 & 0.33 & -0.01 & 0 & 0 & 0 \\ -0.01 & -0.01 & 0.033 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} => \varepsilon = C^{-1}\sigma$$

$$\varepsilon = C^{-1}\sigma = \begin{bmatrix} 3.56 & 1.34 & 1.48 & 0 & 0 & 0 \\ 1.34 & 3.56 & 1.48 & 0 & 0 & 0 \\ 1.48 & 1.48 & 31.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.1 \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{13} \\ S_{12} \end{bmatrix}$$

$$S_{22} = 10MPa = 0.01GPa$$
$$S_{33} = 20MPa = 0.02GPa$$
$$S_{23} = 5MPa = 0.005GPa$$

$$\varepsilon = C^{-1}\sigma = \begin{bmatrix} 3.56 & 1.34 & 1.48 & 0 & 0 & 0 \\ 1.34 & 3.56 & 1.48 & 0 & 0 & 0 \\ 1.48 & 1.48 & 31.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.01 \\ 0.02 \\ 0.005 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.043 \\ 0.0652 \\ 0.6388 \\ 0.025 \\ 0 \\ 0 \end{bmatrix}$$

Elongation in x_1 is 0.043

Elongation in x_2 is0.0652

Elongation in x_3 is 0.6388

Angle between $x_2, x_3 is$ 2 $*$ 0.025

Angle between $x_1, x_3 is$ 0

Angle between $x_1, x_2 is$ 0