Numerical Methods
in Engineering and Applied Science

Lecture 21. Hyperbolic equations. The wave equation.

The wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \tag{1}$$

models, for example the vibrations of a string in one dimension $u = u(x,t)$, the vibrations of a thin membrane in two dimensions $u = u(x,y,t)$ or the pressure vibrations of an acoustic wave in air $u = u(x,y,z,t)$. The constant c gives the speed of propagation for the vibrations.

It can be understood as Newton's second law $(F = ma)$ and Hooke's law $(F = ku)$ combined, so that acceleration $a = u_{tt}$ is proportional to the relative displacement of u compared to its neighbours. The constant $c^2 = k/\rho$ comes from mass density and elasticity.

To derive the wave equation for a string in 1D, the key notion is that the restoring force due to tension on the string is proportional to the curvature at the point.

In the 1D wave equation, when c is a constant, it is interesting to observe that the wave operator can be factored as follows

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right). \tag{2}$$

We could then look for solutions that satisfy the individual first order equations

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{3}$$

These are *one way wave equations*, also called *advection equations*, and the general solution to the two way equation could be done by forming linear combinations of such solutions.

In 2D, similar factorization is possible, but less practical.

To solve the advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{4}$$

we notice that along the characteristic line in the x, t plane, $x - ct = \xi$, where ξ is a constant parameter, any solution $u(x, y)$ will be constant. If we take the derivative of u along the line $x = ct + \xi$, we have,

$$\frac{d}{dt}u(ct + \alpha, t) = c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0. \tag{5}$$

Therefore, u is constant on this line, and only depends on the choice of parameter ξ. If the initial data are $u(x, 0) = u_0(x)$, then we can set

$$u(x, t) = u_0(x - ct). \tag{6}$$

In the 3D wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0, \tag{7}$$

there are characteristic hyperplanes determined by constants (k_x, k_y, k_z, ω) with

$$c^2(k_x^2 + k_y^2 + k_z^2) = \omega^2. \tag{8}$$

Given any twice-differentiable function $f(x)$, the functions

$$u_1(x, y, z, t) = f(k_x x + k_y y + k_z z - \omega t) \tag{9}$$

and

$$u_1(x, y, z, t) = f(k_x x + k_y y + k_z z + \omega t) \tag{10}$$

are solutions to the wave equation. The vector $\boldsymbol{k} = (k_x, k_y, k_z)$ can be interpreted as a direction of propagation of the traveling wave. Not all solutions are of this form, since we have many characteristic hypersurfaces.

For a function $u = u(x_1, x_2, x_3, ..., x_n)$ of n independent variables, the general linear second order PDE is be of the form

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=i}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = D. \tag{11}$$

Coefficients A_{ij}, B_i, C, D are constants or functions only of the independent variables.

The PDE is *hyperbolic*, if $n-1$ of the eigenvalues of the matrix $[A_{ij}]$ are of the same sign, the other of opposite sign.

For example, for the 1D wave equation in the canonical form $u_{tt} - c^2 u_{xx} = 0$, the matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -c^2 \end{bmatrix}. \tag{12}$$

It has one positive and one negative eigenvalue, therefore, it is hyperbolic. The characteristic equation describes a parabola $t^2 - c^2 x^2 = 0$.

The Dirichlet boundary condition

$$u(0,t) = 0 \tag{13}$$

is a reasonable assumption for a vibrating string where the string is fixed at the endpoint $x = 0$.
If, on the other hand, we have a free end to the string, the physical constraint could be expressed by the Neumann boundary condition

$$u_x(0,t) = 0. \tag{14}$$

A combination of these conditions, the Robin condition,

$$au(0,t) + bu_x(0,t) = 0 \tag{15}$$

for given constants a, b leads to the classical Sturm–Liouville problems.

In higher dimensions, a boundary condition

$$u(x, y, z, t) = 0 \quad \text{on} \quad \partial\Omega \tag{16}$$

corresponds to a vibrating system that is fixed on the boundary. The condition that the normal component of the gradient vanish on the boundary,

$$\eta \cdot \nabla u(x, y, z, t) = 0 \quad \text{on} \quad \partial\Omega \tag{17}$$

where η denotes the normal to the surface.

D'Alembert's solution to the 1D wave equation.

We look for a solution $u = u(x,t)$ for the 1D wave equation with initial conditions:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad -\infty < x < \infty, \quad t > 0, \tag{18}$$

$$u(x,0) = f(x) \quad \text{for} \quad -\infty < x < \infty, \tag{19}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty. \tag{20}$$

It is a well-posed problem: it has a unique solution and it is stable. The solution is

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{21}$$

Huygens principle. In 3D (and for all odd dimensions), the solution of the wave equation at point $\boldsymbol{x}_0$, at time t_0, depends only on the initial data in an infinitesimal neighbourhood of the sphere $|\boldsymbol{x} - \boldsymbol{x}_0| = ct_0$. In particular, information from a point source travels in the form of a sphere. The wavefront is thus sharp, with a sudden onset at the start, and sudden cutoff at the end.

See *The mathematics of PDEs and the wave equation* by Michael P. Lamoureux, University of Calgary, 2006.