Numerical Methods in Engineering and Applied Science.

Assignment 2.

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$$u'(x_i) \approx \frac{1}{h}\left(\frac{1}{6}u_{i-2} - \frac{4}{3}u_{i-1} + \frac{11}{6}u_i - \frac{5}{3}u_{i+1} + \frac{4}{3}u_{i+2} - \frac{1}{3}u_{i+3}\right) \tag{1}$$

(1). Is (1) a consistent approximation to the first derivative of u(x)?
Yes. Answer why in the next point.

(2). Use the Taylor series expansion to find the order of this approximation.

$$u(x_i + sh) = u(x_i) + sh\, u'(x_i) + \frac{(sh)^2}{2}u''(x_i) + \frac{(sh)^3}{6}u'''(x_i) + O(h^4)$$

Let's consider coefficients at $u(x_i)$.

$$u'(x_i) \approx \frac{1}{h}\left(\frac{1}{6}u_{i-2} - \frac{4}{3}u_{i-1} + \frac{11}{6}u_i - \frac{5}{3}u_{i+1} + \frac{4}{3}u_{i+2} - \frac{1}{3}u_{i+3}\right) =$$
$$= \frac{1}{h}\left(\frac{1}{6}u(x_i) - \frac{4}{3}u(x_i) + \frac{11}{6}u(x_i) - \frac{5}{3}u(x_i) + \frac{4}{3}u(x_i) - \frac{1}{3}u(x_i) + O(h)\right) = \frac{1}{h}(O(h))$$

Coefficient at $u(x_i)$ is zero.

Performing this procedure for other members in the Tailor series:

$$u'(x_i) \approx \frac{1}{h}\left(\frac{1}{6}u_{i-2} - \frac{4}{3}u_{i-1} + \frac{11}{6}u_i - \frac{5}{3}u_{i+1} + \frac{4}{3}u_{i+2} - \frac{1}{3}u_{i+3}\right)$$
$$= u'(x_i) + \frac{h^3 u''''(x_i)}{4} + O(h^4)$$

Therefore:

$$\frac{1}{h}\left(\frac{1}{6}u_{i-2} - \frac{4}{3}u_{i-1} + \frac{11}{6}u_i - \frac{5}{3}u_{i+1} + \frac{4}{3}u_{i+2} - \frac{1}{3}u_{i+3}\right) - u'(x_i)$$
$$= \frac{h^3 u''''(x_i)}{4} + O(h^4)$$

3rd order of approximation.

While h is small then it is a consistent approximation.

(3).

$$u' - \frac{\sin(x)}{3+\cos(x)}u = \frac{\cos(2x) + 3\cos(x)}{3+\cos(x)}$$

Homogenous:

$$u' - \frac{\sin(x)}{3+\cos(x)}u = 0$$
$$\int \frac{du}{u} = \int \frac{\sin(x)}{3+\cos(x)}dx$$

$$\ln(u) = \begin{cases} t = 3 + \cos(x) \\ dt = -\sin(x)\,dx \end{cases} = \int \frac{\sin(x)}{t} \frac{dt}{-\sin(x)} = -\int \frac{1}{t}\,dt$$

$$= -\ln(t) + \ln(C) = \ln\left(\frac{C}{t}\right) = \ln\left(\frac{C}{3 + cos(x)}\right)$$

$$u_h = \frac{C}{3 + cos(x)}$$

Particular solution of the equation.

$$u' - \frac{\sin(x)}{3 + \cos(x)} u = \frac{\cos(2x) + 3\cos(x)}{3 + \cos(x)}$$

Let $u = A\sin(x) + B\cos(x)$

$$u' - \frac{\sin(x)}{3 + \cos(x)} u = A\cos(x) - B\sin(x) - \frac{\sin(x)\,(A\sin(x) + B\cos(x))}{3 + \cos(x)}$$

$$= \frac{3A\cos(x) - 3B\sin(x) + A\cos^2(x) - B\sin(x)\cos(x) - \sin^2(x)\,A - \sin(x)\,B\cos(x)}{3 + \cos(x)}$$

$$= \frac{3A\cos(x) - 3B\sin(x) + A\cos(2x) - 2B\sin(2x)}{3 + \cos(x)} = \frac{\cos(2x) + 3\cos(x)}{3 + \cos(x)}$$

get:$B = 0, A = 1$

$$u_p = \sin(x)$$

Summing up:

$$u = \frac{C}{3 + cos(x)} + \sin(x)$$

Periodic boundary conditions: $x \in [0,2\pi],\ u(0) = 0$

$$u(0) = \frac{C}{3 + cos(0)} + \sin(0) = \frac{C}{3 + 1} = 0 => C = 0$$

Summing up:

$$u = \sin(x), \qquad x \in [0,2\pi]$$

(4). Use the finite-difference approximation (1) to solve this problem numerically. Plot the exact and the approximate u(x) calculated on a uniform grid with step h = π/10 over the interval from 0 to 2π.

$$u' - \frac{\sin(x)}{3 + \cos(x)} u = \frac{\cos(2x) + 3\cos(x)}{3 + \cos(x)}$$

Represent as:

$$Au + Bu = F$$

$$u^T = [u_1, u_2, \dots, u_N], \qquad u_0 = 0$$

From the task:

$$h = \frac{2\pi}{N} => N = \frac{2\pi}{h} = 20$$

Keep in mind that $u_0 = 0$ and neglect every u_0.

$$A = \frac{1}{6h}\begin{bmatrix} 11 & -10 & 8 & -2 & 0 & \dots & 0 & 1 \\ -8 & 11 & -10 & 8 & -2 & 0 & \dots & 0 \\ 1 & -8 & 11 & -10 & 8 & -2 & 0\dots & 0 \\ 0 & 1 & -8 & 11 & -10 & 8 & -2 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0\dots & 0 & 1 & -8 & 11 & -10 & 8 & -2 \\ 0 & \dots & 0 & 1 & -8 & 11 & -10 & 8 \\ -2 & 0 & \dots & 0 & 1 & -8 & 11 & -10 \\ 8 & -2 & 0 & \dots & 0 & 1 & -8 & 11 \end{bmatrix}$$

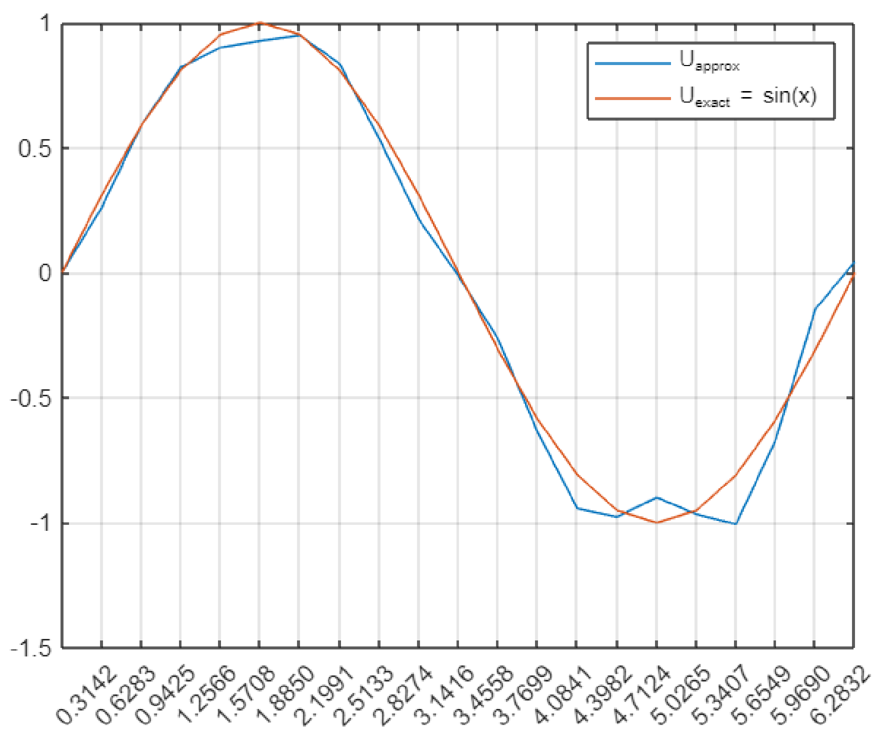
$$x_i = ih \Rightarrow \quad B = -\begin{bmatrix} \frac{\sin(h)}{3+\cos(h)} & 0 & \dots & 0 \\ 0 & \frac{\sin(2h)}{3+\cos(2h)} & 0\dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \frac{\sin(Nh)}{3+\cos(Nh)} \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{\cos(2h)+3\cos(h)}{3+\cos(h)} \\ \frac{\cos(2*2h)+3\cos(2h)}{3+\cos(2h)} \\ \dots \\ \frac{\cos(2*Nh)+3\cos(Nh)}{3+\cos(Nh)} \end{bmatrix}$$

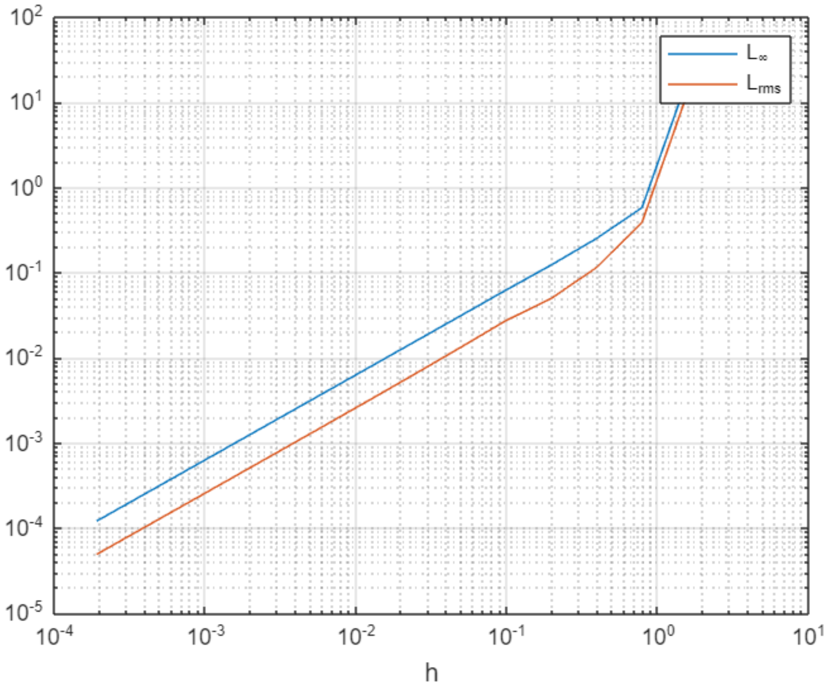
Solve $(A+B)U = F$:

$$U_{approx} = (A+B)^{-1}F$$

Plot the exact and the approximate u(x) calculated on a uniform grid with step h = π/10 over the interval from 0 to 2π.



(5). Vary h from a suitably small to a suitably large value to plot the convergence error norms (pointwise maximum and r.m.s.) as functions of h. Comment on the rate of convergence: compare your numerical observation with your theoretical result.



It is clear that $max|e_k| \approx O(h)$, $(\log(L_\infty) \approx \log(C) + p\log(\mathrm{h})$, where $\mathrm{p} = 1$) So if we divide h by 2, $max|e_k|$ becomes twice smaller.
The numerical result gives the same as theoretical.