# Numerical Methods
# in Engineering and Applied Science

Lecture 13. Part 2. Numerical solution of stochastic differential equations.

For more information see *D. J. Higham. An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations. SIAM Review Vol. 43, No. 3, pp. 525–546.*

A stochastic differential equation is a generalization of the notion of differential equation taking into account a noise term. We can write an autonomous scalar stochastic differential equation in the integral form,

$$X(t) = X_0 + \int_0^t f(X(s))\mathrm{d}s + \int_0^t g(X(s))\mathrm{d}W(s), \quad 0 \leq t \leq T, \tag{1}$$

where $f$ and $g$ are scalar functions, $W$ is a *Brownian motion* (Wiener process) and the initial condition $X_0$ is a random variable.

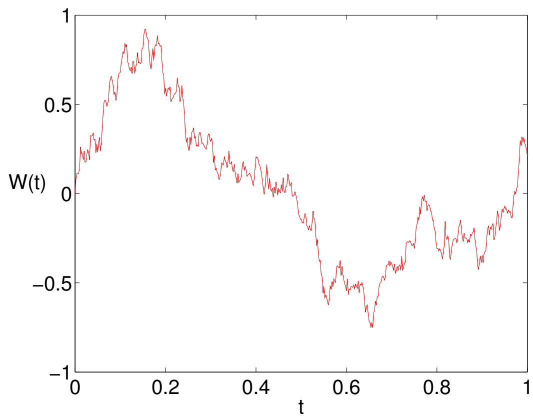
The standard Brownian motion $W(t)$ is a stochastic process depending on the time $t$ and verifying:

- $W(0) = 0$;
- (independent increments) Whatever the times $t$ and $s$ such that $t > s$, the increment $W(t) - W(s)$ is independent of the process before time $s$;
- (stationary and Gaussian increments) Whatever the times $t$ and $s$ such that $t > s$, the increment $W(t) - W(s)$ is a normal random variable zero mean and $ts$ variance; equivalently, $W(t) - W(s) \sim \sqrt{t-s}N(0,1)$;
- $W(t)$ is almost surely continuous, i.e., almost all realizations $t \longmapsto W$ are continuous.

To perform a numerical calculation we need to discretize the Brownian motion. We define $\delta t = T/N$, where $N \in \mathbb{N}$ and we write $W_j$ for $W(t_j)$ at $t_j = j\delta t$, $j = 0, ..., N$. Then, $W_0 = 0$ and

$$W_j = W_{j-1} + \mathrm{d}W_j, \quad j = 1, 2, ..., N, \tag{2}$$

where $\mathrm{d}W_j \sim \sqrt{\delta t} N(0, 1)$. $\mathrm{d}W_j$ can be simulated using a pseudo-random number generator s (e.g., randn in Matlab), or one can use experimental observations.



The stochastic differential equation (1) contains the stochastic Wiener integral,

$$\int_0^T g(t)\mathrm{d}W(t).$$

It can be defined as the limit of the sum

$$\sum_{j=0}^{N-1} g(t_j)\left(W(t_{j+1}) - W(t_j)\right), \tag{3}$$

then we obtain the *Itô integral*. If we use the sum

$$\sum_{j=0}^{N-1} g(\frac{t_j + t_{j+1}}{2})\left(W(t_{j+1}) - W(t_j)\right), \tag{4}$$

we obtain the *Stratonovich integral*. Contrary to the 'classical' integration, these two sums converge towards two different limits.

For example, if $g(t) \equiv W(t)$, we obtain for the Itô integral

$$
\begin{aligned}
&\textstyle\sum_{j=0}^{N-1} W(t_j)\left(W(t_{j+1}) - W(t_j)\right) \\
&\quad = \tfrac{1}{2}\textstyle\sum_{j=0}^{N-1}\left(W(t_{j+1})^2 - W(t_j)^2 - (W(t_{j+1}) - W(t_j))^2\right) \\
&\quad = \tfrac{1}{2}\left(W(T)^2 - W(0)^2 - \textstyle\sum_{j=0}^{N-1}(W(t_{j+1}) - W(t_j))^2\right).
\end{aligned}
$$

We notice that $W(t_{j+1}) - W(t_j) \sim \sqrt{t_{j+1} - t_j}N(0,1)$, then it can be shown that $\sum_{j=0}^{N-1}(W(t_{j+1}) - W(t_j))^2$ has the expectation $T$ and the variance $\mathcal{O}(\delta t)$, $\delta t = t_{i+1} - t_i$. In the limit of $\delta t \to 0$ we obtain

$$\int_0^T W(t)\mathrm{d}W(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T.$$

Similarly, for the Stratonovich integral we get

$$\int_0^T W(t)\mathrm{d}W(t) = \frac{1}{2}W(T)^2.$$

In what follows, we will use the Itô integral.

Examples of stochastic differential equations.

- Cox–Ingersoll–Ross (CIR) is used in financial mathematics to model the evolution of short-term interest rates

$$f(x) = a(\mu - x), \quad g(x) = b\sqrt{x}, \quad a,\ \mu \text{ et } b > 0.$$

  The solution remains strictly positive under the condition $2a\mu > b^2$. The parameter $\mu$ gives the long-term average, and $a$ gives the rate at which the process converges to equilibrium.

- The model with

$$f(x) = r(G - x), \quad g(x) = \sqrt{\varepsilon x(1 - x)}, \quad r,\ \varepsilon > 0 \text{ et } 0 < G < 1$$

  can be used to describe an individual's political evolution over time. $x(t) = 0$ corresponds to the extreme left and $x(t) = 1$ corresponds to the extreme right.

We write the stochastic differential equation (1) in the form

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t), \quad X(0) = X_0, \quad 0 \le t \le T. \tag{5}$$

Euler–Maruyama (EM) method

$$X_{j+1} = X_j + \Delta t f(X_j) + g(X_j)\left(W(t_{j+1}) - W(t_j)\right), \quad j = 0, ..., L-1, \tag{6}$$

where $\tau_j = j\Delta t$, $\Delta t = T/L$.

Example. We use the EM method to solve the linear problem

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad X(0) = X_0. \tag{7}$$

Let $\lambda = 2$, $\mu = 1$, $X_0 = 1$.We first calculate a realization of the Brownian motion $W_k$ with $\delta t = 2^{-8}$. Then, we use the EM method with the step $\Delta t = R\delta t$, $R = 4$. We use the increment

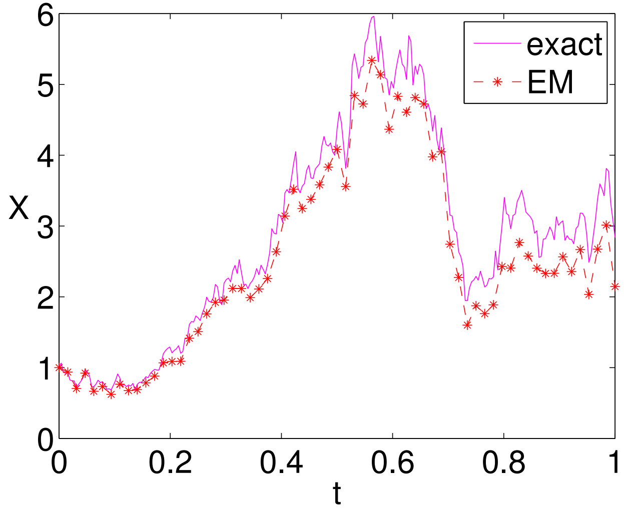
$$W(\tau_{j+1}) - W(\tau_j) = \sum_{jR+1}^{(j+1)R} \mathrm{d}W_k$$

.

To make a comparison, the exact solution

$$X(t) = X(0)\exp\left((\lambda - \frac{1}{2}\mu^2)t + \mu W(t)\right) \tag{8}$$

is evaluated at points $j\delta t$, $j = 0, ..., N$.



```
set(0,'DefaultaxesFontSize',20); set(0,'DefaulttextFontsize',20);

randn('state',100)
lambda = 2; mu = 1; Xzero = 1;
T = 1; N = 2^8; dt = 1/N;
dW = sqrt(dt)*randn(1,N);
W = cumsum(dW);

Xtrue = Xzero*exp((lambda-0.5*mu^2)*([dt:dt:T])+mu*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on

R = 4; Dt = R*dt; L = N/R;
Xem = zeros(1,L);
Xtemp = Xzero;
for j = 1:L
Winc = sum(dW(R*(j-1)+1:R*j));
Xtemp = Xtemp + Dt*lambda*Xtemp + mu*Xtemp*Winc;
Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'r--*'), hold off
xlabel('t')
ylabel('X','Rotation',0,'HorizontalAlignment','right')
legend('exact','EM');
```

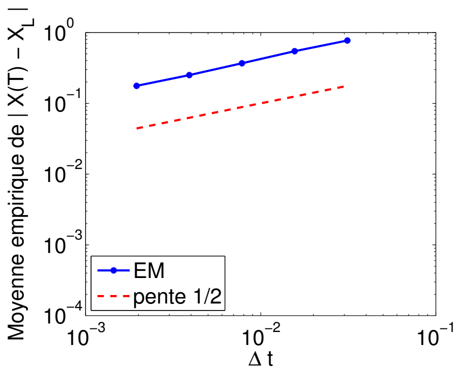
Strong convergence and weak convergence

**Definition.** A method is said to be *strongly convergent of order* $\gamma$ there exists a constant $C$ such that

$$\mathbb{E}|X_n - X(\tau)| \leq C\Delta t^{\gamma} \tag{9}$$

for any fixed $\tau = n\Delta t$ and sufficiently small $\Delta t$.

It can be shown that the Euler–Maruyama method is strongly convergent of order $\gamma = 1/2$. The figure shows the strong convergence observed in experiments with 1000 realizations of Brownian motion.



It should be noted that, in addition to the error of the numerical method, there are other sources of error, such as that related to the approximation of the expectation by the arithmetic mean, the biases of the pseudo-random number generator and rounding.

An important consequence of strong convergence can be obtained using the Markov inequality,

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}|X|}{a}, \tag{10}$$

where $a > 0$. With $a = \Delta t^{1/4}$, we obtain for the EM method

$$\mathbb{P}(|X_n - X(\tau)| \geq \Delta t^{1/4}) \leq C\Delta t^{1/4}, \tag{11}$$

where

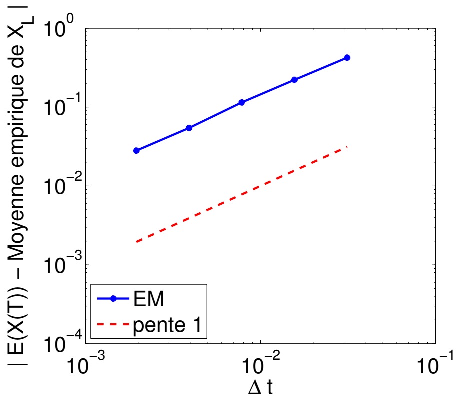
$$\mathbb{P}(|X_n - X(\tau)| < \Delta t^{1/4}) \geq 1 - C\Delta t^{1/4}. \tag{12}$$

**Definition.** A method is said to be *weakly convergent of order* $\gamma$ if there exists a constant $C$ such that, for any function $p$ of a certain class,

$$|\mathbb{E}p(X_n) - \mathbb{E}p(X(\tau))| \leq C\Delta t^\gamma \tag{13}$$

with fixed $\tau = n\Delta t$ and sufficiently small $\Delta t$.

For example, we can use the identity function $p(X) \equiv X$. The figure shows the weak convergence observed in numerical experiments with 50000 realizations of Brownian motion.



To increase the strong order of the numerical method, a correction of the stochastic term can be introduced.

Milstein method

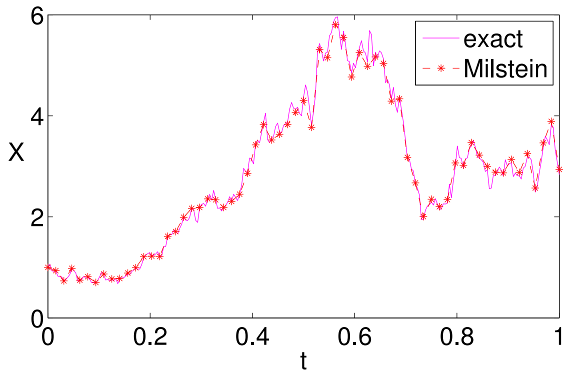
$$\begin{aligned} X_{j+1} &= X_j + \Delta t f(X_j) + g(X_j)\left(W(t_{j+1}) - W(t_j)\right) \\ &+ \frac{1}{2} g(X_j) g'(X_j) \Big( (W(\tau_{j+1}) - W(\tau_j))^2 - \Delta t \Big), \quad j = 0, ..., L-1, \end{aligned} \tag{14}$$

where $\tau_j = j\Delta t$, $\Delta t = T/L$.

Example. We use the Milstein method to solve the linear problem

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad X(0) = X_0$$

with $\lambda = 2$, $\mu = 1$, $X_0 = 1$.


exact
Milstein
X
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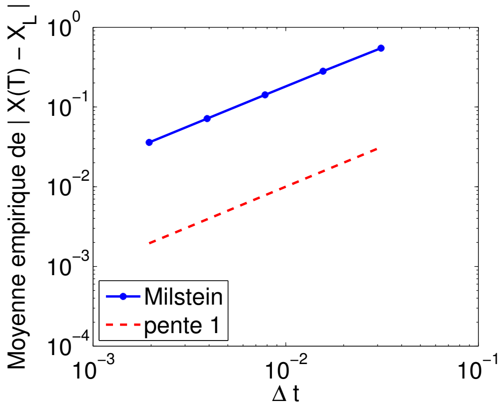
Milstein
pente 1
Moyenne empirique de | X(T) − X_L |
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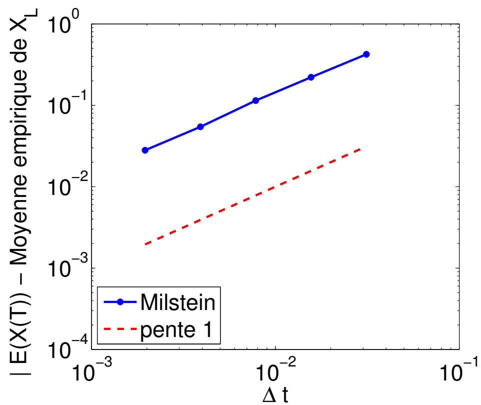
Milstein
pente 1
| E(X(T)) − Moyenne empirique de X_L |
Δ t

Absolute stability.

We consider the long-term behavior of the model problem

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad X(0) = X_0 \tag{15}$$

with $\lambda,\ \mu \in \mathbb{C}$ and $X_0 \neq 0$ with probability 1.

There are two most used notions of stability for solutions of (15).

- *Mean square stability.*

$$\lim_{t\to\infty} \mathbb{E}X(t)^2 = 0 \quad \Leftrightarrow \quad \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0. \tag{16}$$

- *Asymptotic stability.*

$$\lim_{t\to\infty} |X(t)| = 0 \text{ with probability } 1 \quad \Leftrightarrow \quad \Re(\lambda - \frac{1}{2}\mu^2) < 0. \tag{17}$$

The same two notions of stability can be used to characterize the stability of a numerical method. For example, for the Euler–Maruyama method we get

- *Mean square stability.*

$$\lim_{j\to\infty} \mathbb{E}X_j^2 = 0 \quad \Leftrightarrow \quad |1+\Delta t\lambda|^2 + \Delta t|\mu|^2 < 1. \tag{18}$$

- *Asymptotic stability.*

$$\begin{aligned} &\lim_{j\to\infty} |X_j| = 0 \text{ avec la probabilité } 1 \\ &\quad\Leftrightarrow \quad \mathbb{E}\log|1+\Delta t\lambda + \sqrt{\Delta t}\mu N(0,1)| < 0. \end{aligned} \tag{19}$$

Stability region of the EM method with $\lambda,\ \mu \in \mathbb{R}$.