1 Second-order differential equations in the phase plane

Very few ordinary differential equations have explicit solutions expressible in finite terms. This is not simply because ingenuity fails, but because the repertory of standard functions (polynomials, exp, sin, and so on) in terms of which solutions may be expressed is too limited to accommodate the variety of differential equations encountered in practice. Even if a solution can be found, the 'formula' is often too complicated to display clearly the principal features of the solution; this is particularly true of implicit solutions and of solutions which are in the form of integrals or infinite series.

The qualitative study of differential equations is concerned with how to deduce important characteristics of the solutions of differential equations without actually solving them. In this chapter we introduce a geometrical device, the phase plane, which is used extensively for obtaining directly from the differential equation such properties as equilibrium, periodicity, unlimited growth, stability, and so on. The classical pendulum problem shows how the phase plane may be used to reveal all the main features of the solutions of a particular differential equation.

1.1 Phase diagram for the pendulum equation

The simple pendulum (see Fig. 1.1) consists of a particle P of mass m suspended from a fixed point O by a light string or rod of length a, which is allowed to swing in a vertical plane. If there is no friction the equation of motion is

$$\ddot{x} + \omega^2 \sin x = 0, \tag{1.1}$$

where x is the inclination of the string to the downward vertical, g is the gravitational constant, and $\omega^2 = g/a$.

We convert eqn (1.1) into an equation connecting $\dot{x}$ and x by writing

$$\begin{aligned} \ddot{x} &= \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\dot{x}^2\right). \end{aligned} \tag{1.2}$$

This representaion of $\ddot{x}$ is called the **energy transformation**. Equation (1.1) then becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\dot{x}^2\right) + \omega^2 \sin x = 0.$$

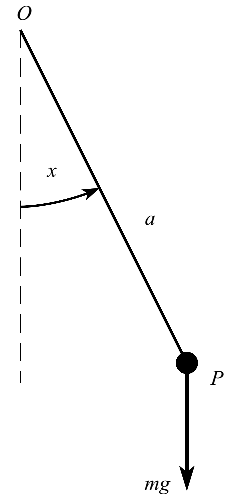

Figure 1.1 The simple pendulum, with angular displacement x.

By integrating this equation with respect to x we obtain

$$\tfrac{1}{2}\dot{x}^2 - \omega^2 \cos x = C, \tag{1.3}$$

where C is an arbitrary constant. Notice that this equation expresses **conservation of energy** during any particular motion, since if we multiply through eqn (1.3) by a constant ma^2, we obtain

$$\tfrac{1}{2}ma^2\dot{x}^2 - mga\cos x = E,$$

where E is another arbitrary constant. This equation has the form

$$E = \text{kinetic energy of } P + \text{potential energy of } P,$$

and a particular value of E corresponds to a particular free motion.

Now write $\dot{x}$ in terms of x from eqn (1.3):

$$\dot{x} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2}. \tag{1.4}$$

This is a *first-order* differential equation for $x(t)$. It cannot be solved in terms of elementary functions (see McLachlan 1956), but we shall show that it is possible to reveal the main features of the solution by working directly from eqn (1.4) without actually solving it.

Introduce a new variable, y, defined by

$$\dot{x} = y. \tag{1.5a}$$

Then eqn (1.4) becomes

$$\dot{y} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2}. \tag{1.5b}$$

Set up a frame of Cartesian axes x, y, called the **phase plane**, and plot the one-parameter family of curves obtained from (1.5b) by using different values of C. We obtain Fig. 1.2. This is called

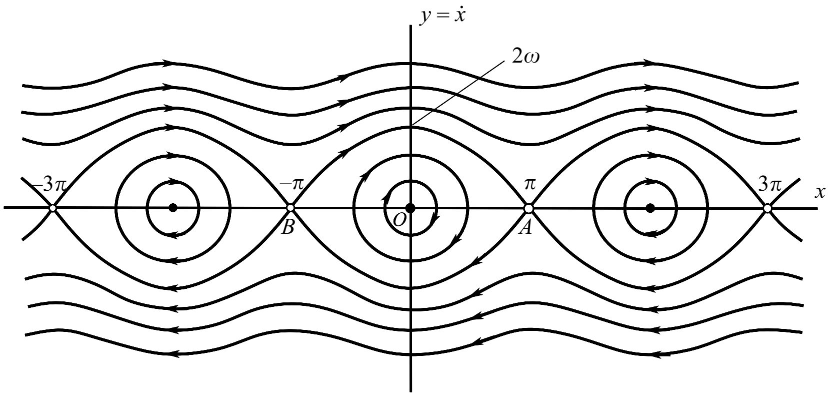

Figure 1.2 Phase diagram for the simple pendulum equation (1.1).

phase diagram for the problem, and the curves are called the **phase paths**. Various types of phase path can be identified in terms of C. On the paths joining $(-\pi, 0)$ and $(\pi, 0)$, $C = \omega^2$; for paths within these curves $\omega^2 > C > -\omega^2$; and for paths outside $C > \omega^2$. Equation (1.56) implies the 2π-periodicity in x shown in Fig. 1.2. The meaning of the arrowheads will be explained shortly.

A given pair of values (x, y), or $(x, \dot{x})$, represented by a point P on the diagram is called a **state of the system**. A state gives the angular velocity $\dot{x} = y$ at a particular inclination x, and these variables are what we sense when we look at a swinging pendulum at any particular moment. A given state $(x, \dot{x})$ serves also as a pair of **initial conditions** for the original differential equation (1.1); therefore a given state determines all subsequent states, which are obtained by following the phase path that passes through the point P: (x, y), where (x, y) is the initial state.

The **directions** in which we must proceed along the phase paths for increasing time are indicated by the arrowheads in Fig. 1.2. This is determined from (1.5a): when $y > 0$, then $\dot{x} > 0$, so that x must increase as t increases. Therefore the required direction is always *from left to right in the upper half-plane*. Similarly, the direction is always *from right to left in the lower half-plane*. The complete picture, Fig. 1.2, is the phase diagram for this problem.

Despite the non-appearance of the time variable in the phase plane display, we can deduce several physical features of the pendulum's possible motions from Fig. 1.2. Consider first the possible states of the physical equilibrium of the pendulum. The obvious one is when the pendulum hangs without swinging; then $x = 0$, $\dot{x} = 0$, which corresponds to the origin in Fig. 1.2. The corresponding *time*-function $x(t) = 0$ is a perfectly legitimate **constant solution** of eqn (1.1); the phase path degenerates to a single point.

If the suspension consists of a light rod there is a second position of equilibrium, where it is balanced vertically on end. This is the state $x = \pi$, $\dot{x} = 0$, another constant solution, represented by point A on the phase diagram. The same physical condition is described by $x = -\pi$, $x = 0$, represented by the point B, and indeed the state $x = n\pi$, $\dot{x} = 0$, where n is any integer, corresponds physically to one of these two equilibrium states. In fact we have displayed in Fig. 1.2 only part of the phase diagram, whose pattern repeats periodically; there is not in this case a one-to-one relationship between the *physical* state of the pendulum and points on its phase diagram.

Since the points O, A, B represent states of physical equilibrium, they are called **equilibrium points** on the phase diagram.

Now consider the family of closed curves immediately surrounding the origin in Fig. 1.2. These indicate **periodic motions**, in which the pendulum swings to and fro about the vertical. The **amplitude** of the swing is the maximum value of x encountered on the curve. For small enough amplitudes, the curves represent the usual 'small amplitude' solutions of the pendulum equation in which eqn (1.1) is simplified by writing $\sin x \approx x$. Then (1.1) is approximated by $\ddot{x} + \omega^2 x = 0$, having solutions $x(t) = A\cos\omega t + B\sin\omega t$, with corresponding phase paths

$$x^2 + \frac{y^2}{\omega^2} = \text{constant}$$

The phase paths are nearly ellipses in the small amplitude region.

The wavy lines at the top and bottom of Fig. 1.2, on which $\dot{x}$ is of constant sign and x continuously increases or decreases, correspond to whirling motions on the pendulum. The fluctuations in $\dot{x}$ are due to the gravitational influence, and for phase paths on which $\dot{x}$ is very large these fluctuations become imperceptible: the phase paths become nearly straight lines parallel to the x axis.

We can discuss also the **stability** of the two typical equilibrium points O and A. If the initial state is displaced slightly from O, it goes on to one of the nearby closed curves and the pendulum oscillates with small amplitude about O. We describe the equilibrium point at O as being **stable**. If the initial state is slightly displaced from A (the vertically upward equilibrium position) however, it will normally fall on the phase path which carries the state far from the equilibrium state A into a large oscillation or a whirling condition (see Fig. 1.3). This equilibrium point is therefore described as **unstable**.

An exhaustive account of the pendulum can be found in the book by Baker and Blackburn (2005).

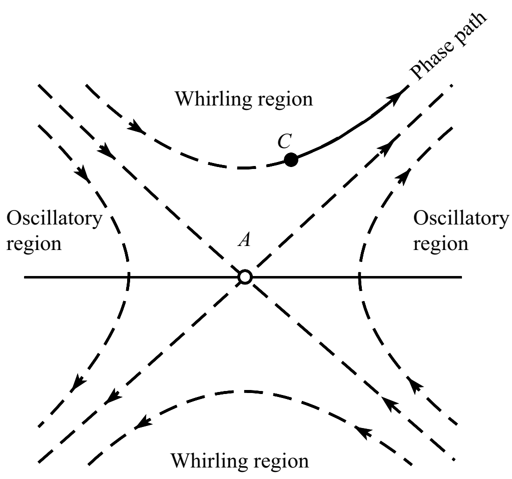

Figure 1.3 Unstable equilibrium point for the pendulum: typical displaced initial state C.

1.2 Autonomous equations in the phase plane

The second-order differential equation of general type

$$\ddot{x} = f(x, \dot{x}, t)$$

with **initial conditions**, say $x(t_0)$ and $\dot{x}(t_0)$, is an example of a **dynamical system.** The evolution or future states of the system are then given by $x(t)$ and $\dot{x}(t)$. Generally, dynamical systems are **initial-value problems** governed by ordinary or partial differential equations, or by difference equations. In this book we consider mainly those nonlinear systems which arise from ordinary differential equations.

The equation above can be interpreted as an equation of motion for a mechanical system, in which x represents displacement of a particle of unit mass, $\dot{x}$ its velocity, $\ddot{x}$ its acceleration, and f the applied force, so that this general equation expresses Newton's law of motion for the particle:

$$\text{acceleration} = \text{force per unit mass}$$

A mechanical system is in equilibrium if its state does not change with time. This implies that *an equilibrium state corresponds to a constant solution* of the differential equation, and conversely. A constant solution implies in particular that $\dot{x}$ and $\ddot{x}$ must be simultaneously zero. Note that $\dot{x} = 0$ is not alone sufficient for equilibrium: a swinging pendulum is instantaneously at rest at its maximum angular displacement, but this is obviously not a state of equilibrium. Such constant solutions are therefore the constant solutions (if any) of the equation

$$f(x, 0, t) = 0.$$

We distinguish between two types of differential equation:

(i) the **autonomous type** in which f does not depend explicitly on t;
(ii) the **non-autonomous** or **forced equation** where t appears explicitly in the function f.

A typical non-autonomous equation models the damped linear oscillator with a harmonic forcing term

$$\ddot{x} + k\dot{x} + \omega_0^2 x = F \cos \omega t,$$

in which $f(x, \dot{x}, t) = -k\dot{x} - \omega_0^2 x + F \cos \omega t$. There are no equilibrium states. Equilibrium states are not usually associated with non-autonomous equations although they can occur as, for example, in the equation (Mathieu's equation, Chapter 9)

$$\ddot{x} + (\alpha + \beta \cos t)x = 0.$$

which has an equilibrium state at $x = 0$, $\dot{x} = 0$.

In the present chapter we shall consider only *autonomous systems*, given by the differential equation

$$\ddot{x} = f(x, \dot{x}), \tag{1.6}$$

in which t is absent on the right-hand side. To obtain the representation on the phase plane, put

$$\dot{x} = y, \tag{1.7a}$$

so that

$$\dot{y} = f(x, y). \tag{1.7b}$$

This is a pair of simultaneous first-order equations, equivalent to (1.6).

The **state** of the system at a time t_0 consists of the pair of numbers $(x(t_0), \dot{x}(t_0))$, which can be regarded as a pair of initial conditions for the original differential equation (1.6). The initial state therefore determines all the subsequent (and preceding) states in a particular free motion.

In the **phase plane** with axes x and y, the state at time t_0 consists of the pair of values $(x(t_0), y(t_0))$. These values of x and y, represented by a point P in the phase plane, serve as initial conditions for the simultaneous first-order differential equations (1.7a), (1.7b), and therefore determine all the states through which the system passes in a particular motion. The succession of states given parametrically by

$$x = x(t), \qquad y = y(t), \tag{1.8}$$

traces out a curve through the initial point P: $(x(t_0), y(t_0))$, called a **phase path**, a **trajectory** or an **orbit.**

The **direction** to be assigned to a phase path is obtained from the relation $\dot{x} = y$ (eqn 1.7a). When $y > 0$, then $\dot{x} > 0$, so that x is increasing with time, and when $y < 0$, x is decreasing with time. Therefore the directions are from *left to right in the upper half-plane, and from right to left in the lower half-plane*.

To obtain a relation between x and y that defines the phase paths, eliminate the parameter t between (1.7a) and (1.7b) by using the identity

$$\frac{\dot{y}}{\dot{x}} = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Then the **differential equation for the phase paths** becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x, y)}{y}. \tag{1.9}$$

A particular phase path is singled out by requiring it to pass through a particular point P: (x, y), which corresponds to an initial state (x_0, y_0), where

$$y(x_0) = y_0. \tag{1.10}$$

The complete pattern of phase paths including the directional arrows constitutes the **phase diagram**. The time variable t does not figure on this diagram.

The **equilibrium points** in the phase diagram correspond to **constant solutions** of eqn (1.6), and likewise of the equivalent pair (1.7a) and (1.7b). These occur when *$\dot{x}$ and $\dot{y}$ are simultaneously zero*; that is to say, at points on the x axis where

$$y = 0 \quad \text{and} \quad f(x, 0) = 0. \tag{1.11}$$

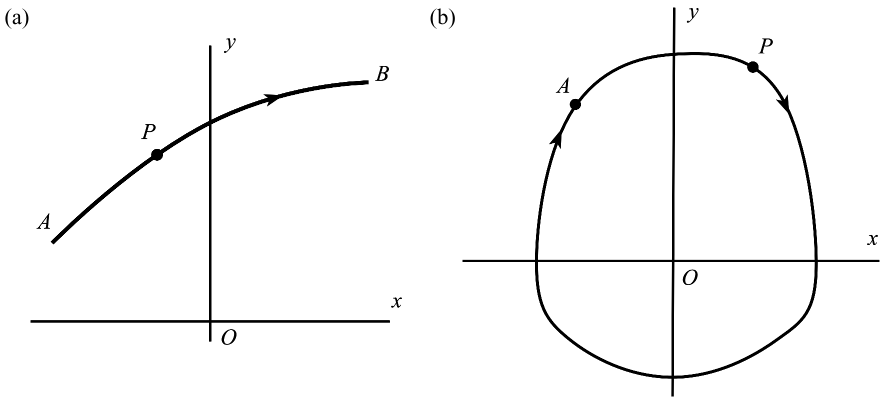

Figure 1.4 (a) The representative point P on a segment of a phase path. (b) A closed path: P leaves A and returns to A an infinite number of times.

Equilibrium points can be regarded as degenerate phase paths. At equilibrium points we obtain, from eqn (1.9),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{0}{0},$$

so they are singular points of eqn (1.9), although they are not singular points of the time-dependent equations (1.7) (see Appendix A for a description of singular points).

In the representation on the phase plane the time t is not involved quantitatively, but can be featured by the following considerations. Figure 1.4(a) shows a segment $\widehat{AB}$ of a phase path. Suppose that the system is in a state A at time $t = t_A$. The moving point P represents the states at times $t \geq t_A$; it moves steadily along $\widehat{AB}$ (from left to right in $y > 0$) as t increases, and is called a **representative point** on $\widehat{AB}$.

The velocity of P along the curve $\widehat{AB}$ is given in component form by

$$(\dot{x}(t), \dot{y}(t)) = (y, f(x, y))$$

(from (1.7)): this depends *only* on its position $P : (x, y)$, and not at all on t and t_A (this is true only for *autonomous equations*). If t_B is the time P reaches B, the time T_{AB} taken for P to move from A to B,

$$T_{AB} = t_B - t_A, \tag{1.12}$$

is *independent of the initial time* t_A. The quantity T_{AB} is called the **elapsed time** or **transit time** from A to B along the phase path.

We deduce from this observation that if $x(t)$ represents any particular solution of $\ddot{x} = f(x, \dot{x})$, then the *family of solutions* $x(t - t_1)$, where t_1 may take any value, is represented by the *same phase path and the same representative point*. The graphs of the functions $x(t)$ and $x(t - t_1)$, and therefore of $y(t) = \dot{x}(t)$ and $y(t - t_1)$, are identical in shape, but are displaced along the time axis by an interval t_1, as if the system they represent had been switched on at two different times of day.

Consider the case when a phase path is a **closed curve**, as in Fig. 1.4(b). Let A be any point on the path, and let the representative point P be at A at time t_A. After a certain interval of time T, P returns to A, having gone once round the path. Its second circuit starts at A at time $t_A + T$, but since its subsequent positions depend only on the time elapsed from its starting point, and not on its starting time, the second circuit will take the same time as the first circuit, and so on. A *closed phase path therefore represents a motion which is periodic in time*.

The converse is not true—a path that is not closed may also describe a periodic motion. For example, the time-solutions corresponding to the whirling motion of a pendulum (Fig. 1.2) are periodic.

The transit time $T_{AB} = t_B - t_A$ of the representative point P from state A to state B along the phase path can be expressed in several ways. For example,

$$\begin{aligned} T_{AB} &= \int_{t_A}^{t_B} \mathrm{d}t = \int_{t_A}^{t_B} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_{\widehat{AB}} \frac{\mathrm{d}x}{\dot{x}} = \int_{\widehat{AB}} \frac{\mathrm{d}x}{y(x)}. \end{aligned} \tag{1.13}$$

This is, in principle, calculable, given y as a function of x on the phase path. Notice that the final integral depends only on the path $\widehat{AB}$ and not on the initial time t_A, which confirms the earlier conclusion. The integral is a line integral, having the usual meaning in terms of infinitesimal contributions:

$$\int_{\widehat{AB}} \frac{\mathrm{d}x}{y} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{\delta x_i}{y(x_i)},$$

in which we follow values of x in the direction of the path by increments δx_i, appropriately signed. Therefore the δx_i are positive in the upper half-plane and negative in the lower half-plane. It may therefore be necessary to split up the integral as in the following example.

Example 1.1 *The phase paths of a system are given by the family $x + y^2 = C$, where C is an arbitrary constant. On the path with $C = 1$ the representative point moves from $A: (0, 1)$ to $B: (-1, -\sqrt{2})$. Obtain the transit time T_{AB}.*

The path specified is shown in Fig. 1.5. It crosses the x axis at the point $C: (1, 0)$, and at this point δx changes sign. On $\widehat{AC}$, $y = (1 - x)^{1/2}$, and on $\widehat{CB}$, $y = -(1 - x)^{1/2}$. Then

$$\begin{aligned} T_{AB} &= \int_{\widehat{AC}} \frac{\mathrm{d}x}{y} + \int_{\widehat{CB}} \frac{\mathrm{d}x}{y} = \int_0^1 \frac{\mathrm{d}x}{(1-x)^{1/2}} + \int_1^{-1} \frac{\mathrm{d}x}{[-(1-x)^{1/2}]} \\ &= [-2(1-x)^{1/2}]_0^1 + [2(1-x)^{1/2}]_1^{-1} = 2 + 2\sqrt{2}. \end{aligned}$$

For an expression alternative to eqn (1.13), see Problem 1.8. ●

Here we summarize the main properties of autonomous differential equations $\ddot{x} = f(x, \dot{x})$, as represented in the phase plane by the equations

$$\dot{x} = y, \qquad \dot{y} = f(x, y). \tag{1.14}$$

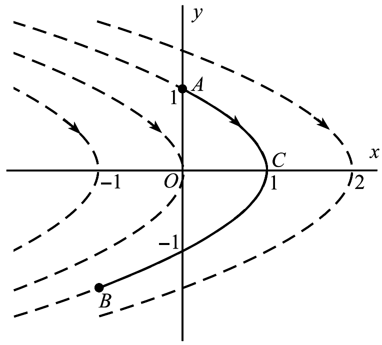

Figure 1.5 Path AB along which the transit time is calculated.

(i) *Equation for the phase paths:*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x, y)}{y}. \tag{1.15}$$

(ii) *Directions of the phase paths:* from left to right in the upper half-plane; from right to left in the lower half-plane.

(iii) *Equilibrium points:* situated at points $(x, 0)$ where $f(x, 0) = 0$; representing constant solutions.

(iv) *Intersection with the x axis:* the phase paths cut the x axis at right angles, except possibly at equilibrium points (see (ii)).

(v) *Transit times:* the transit time for the representative point from a point A to a point B along a phase path is given by the line integral

$$T_{AB} = \int_{\widehat{AB}} \frac{\mathrm{d}x}{y}. \tag{1.16}$$

(vi) *Closed paths:* closed phase paths represent periodic time-solutions $(x(t), y(t))$.

(vii) *Families of time-solutions:* let $x_1(t)$ be any particular solution of $\ddot{x} = f(x, \dot{x})$. Then the solutions $x_1(t - t_1)$, for any t_1, give the same phase path and representative point.

The examples which follow introduce further ideas.

Example 1.2 *Construct the phase diagram for the simple harmonic oscillator equation* $\ddot{x} + \omega^2 x = 0$.

This approximates to the pendulum equation for small-amplitude swings. Corresponding to equations (1.14) we have

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x.$$

There is a single equilibrium point, at $x = 0$, $y = 0$. The phase paths are the solutions of (1.15):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\omega^2 \frac{x}{y}.$$

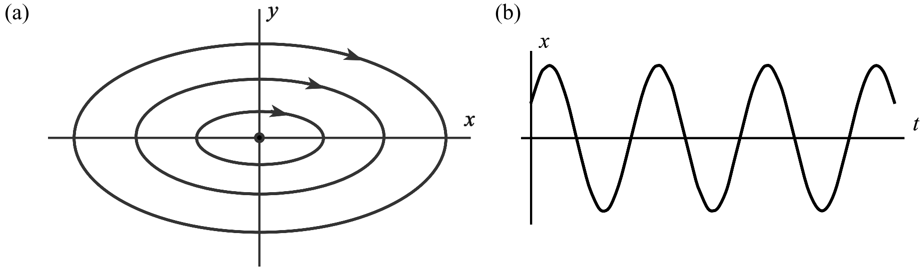

Figure 1.6 (a) centre for the simple harmonic oscillator. (b) Typical solution.

This is a separable equation, leading to

$$y^2 + \omega^2 x^2 = C,$$

where C is arbitrary, subject to $C \geq 0$ for real solutions. The phase diagram therefore consists of a family of ellipses concentric with the origin (Fig. 1.6(a)). All solutions are therefore periodic. Intuitively we expect the equilibrium point to be stable since phase paths near to the origin remain so. Figure 1.6(b) shows one of the periodic time-solutions associated with a closed path. ●

An equilibrium point surrounded in its immediate neighbourhood (not necessarily over the whole plane) by closed paths is called a **centre**. A centre is stable equilibrium point.

Example 1.3 *Construct the phase diagram for the equation* $\ddot{x} - \omega^2 x = 0$.
The equivalent first-order pair (1.14) is

$$\dot{x} = y, \qquad \dot{y} = \omega^2 x.$$

There is a single equilibrium point (0, 0). The phase paths are solutions of (1.15):

$$\frac{dy}{dx} = \omega^2 \frac{x}{y}.$$

Therefore their equations are

$$y^2 - \omega^2 x^2 = C, \tag{1.17}$$

where the parameter C is arbitrary. These paths are hyperbolas, together with their asymptotes $y = \pm\omega x$, as shown in Fig. 1.7. ●

Any equilibrium point with paths of this type in its neighbourhood is called a **saddle point**. Such a point is unstable, since a small displacement from the equilibrium state will generally take the system on to a phase path which leads it far away from the equilibrium state.

The question of stability is discussed precisely in Chapter 8. In the figures, stable equilibrium points are usually indicated by a full dot •, and unstable ones by an 'open' dot ∘.

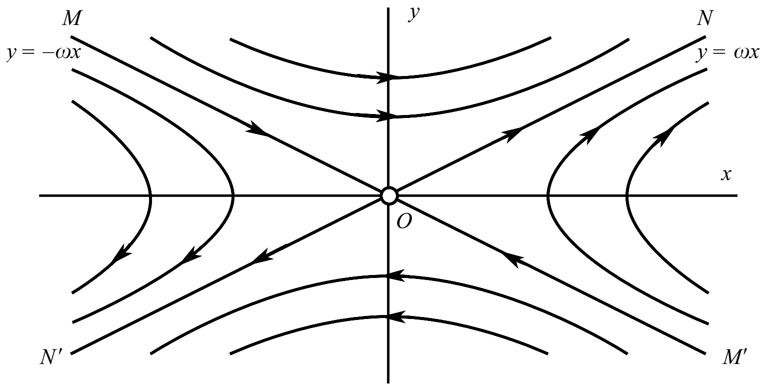

Figure 1.7 Saddle point: only the paths MO and $M'O$ approach the origin.

The differential equations in Examples 1.2 and 1.3 can be solved explicitly for x in terms of t. For Example 1.2, the general solution of $\ddot{x} + \omega^2 x = 0$ is

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{1.18}$$

where A and B are arbitrary constants. This can be written in another form by using the ordinary trigonometric identities. Put

$$\kappa = (A^2 + B^2)^{1/2}$$

and let ϕ satisfy the equations

$$\frac{A}{\kappa} = \cos\phi, \qquad \frac{B}{\kappa} = \sin\phi.$$

Then (1.18) becomes

$$x(t) = \kappa\cos(\omega t - \phi), \tag{1.19}$$

where the amplitude κ and the phase angle ϕ are arbitrary. Figure 1.6(b) shows an example of this time-solution: all values of ϕ produce the same phase path (see (vii) in the summary above). The period of every oscillation is $2\pi/\omega$, which is independent of initial conditions (known as an **isochronous oscillation**).

For Example 1.3, the time-solutions of $\ddot{x} - \omega^2 x = 0$ are given by

$$x(t) = Ae^{\omega t} + Be^{-\omega t}, \tag{1.20}$$

where A and B are arbitrary. To make a correspondence with Fig. 1.7, we require also

$$y = \dot{x}(t) = A\omega e^{\omega t} - B\omega e^{-\omega t}. \tag{1.21}$$

Assume that $\omega > 0$. Then from eqns (1.20) and (1.21), all the solutions approach infinity as $t \to \infty$, except those for which $A = 0$ in (1.20) and (1.21). The case $A = 0$ is described in the phase plane parametrically by

$$x = Be^{-\omega t}, \qquad y = -B\omega e^{-\omega t}. \tag{1.22}$$

For these paths we have

$$\frac{y}{x} = -\omega;$$

these are the paths MO and $M'O$ in Fig. 1.7, and they approach the origin as $t \to \infty$. Note that these paths represent not just one time-solution, but a whole family of time-solutions $x(t) = Be^{-\omega t}$, and this is the case for every phase path (see (vii) in the summary above: for this case put $B = \pm\, e^{-\omega t_1}$, for any value of t_1).

Similarly, if $B = 0$, then we obtain the solutions

$$x = Ae^{\omega t}, \qquad y = A\omega e^{\omega t},$$

for which $y = \omega x$: this is the line NN'. As $t \to -\infty$, $x \to 0$ and $y \to 0$. The origin is an example of a **saddle point**, characterised by a pair of incoming phase paths MO, $M'O$ and outgoing paths ON, ON', known as **separatrices**.

Example 1.4 *Find the equilibrium points and the general equation for the phase paths of* $\ddot{x} + \sin x = 0$. *Obtain the particular phase paths which satisfy the initial conditions* (*a*) $x(t_0) = 0$, $y(t_0) = \dot{x}(t_0) = 1$; (*b*) $x(t_0) = 0$, $y(t_0) = 2$.

This is a special case of the pendulum equation (see Section 1.1 and Fig. 1.2). The differential equations in the phase plane are, in terms of t,

$$\dot{x} = y, \qquad \dot{y} = -\sin x.$$

Equilibrium points lie on the x axis at points where $\sin x = 0$; that is at $x = n\pi (n = 0, \pm 1, \pm 2, \ldots)$. When n is even they are centres; when n is odd they are saddle points.

The differential equation for the phase paths is

$$\frac{dy}{dx} = -\frac{\sin x}{y}.$$

This equation is separable, leading to

$$\int y\, dy = -\int \sin x\, dx,$$

or

$$\tfrac{1}{2}y^2 = \cos x + C, \tag{i}$$

where C is the parameter of the phase paths. Therefore the equation of the phase paths is

$$y = \pm\sqrt{2}(\cos x + C)^{1/2}. \tag{ii}$$

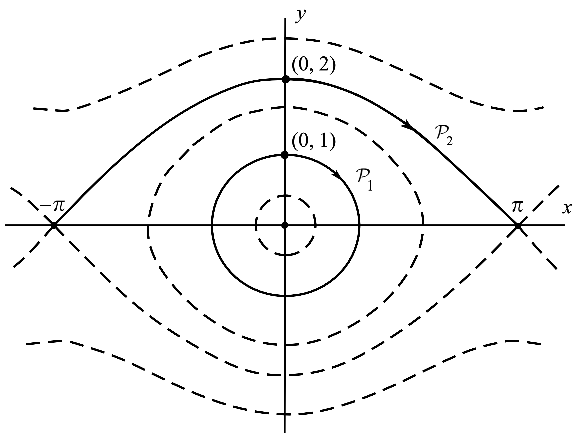

Figure 1.8 Phase paths for $\ddot{x} + \sin x = 0$.

Since y must be real, C may be chosen arbitrarily, but in the range $C \geq 1$. Referring to Fig. 1.8, or the extended version Fig. 1.2, the permitted range of C breaks up as follows:

Values of C	*Type of motion*
$C = -1$	Equilibrium points at $(n\pi, 0)$ (centres for n even; saddle points for n odd)
$-1 < C < 1$	Closed paths (periodic motions)
$C = 1$	Paths which connect equilibrium points (separatrices)
$C > 1$	Whirling motions

(a) $x(t_0) = 0, y(t_0) = 1$. From (i) we have $\frac{1}{2} = 1 + C$, so that $C = -\frac{1}{2}$. The associated phase path is (from (ii))

$$y = \sqrt{2}\left(\cos x - \tfrac{1}{2}\right)^{1/2},$$

shown as $\mathcal{P}_1$ in Fig. 1.8. The path is closed, indicating a periodic motion.

(b) $x(t_0) = 0, y(t_0) = 2$. From (i) we have $2 = 1 + C$, or $C = 1$. The corresponding phase path is

$$y = \sqrt{2}(\cos x + 1)^{1/2}.$$

On this path $y = 0$ at $x = \pm n\pi$, so that the path connects two equilibrium points (note that it does not continue beyond them). As $t \to \infty$, the path approaches $(\pi, 0)$ and emanates from $(-\pi, 0)$ at $t = -\infty$. This path, shown as $\mathcal{P}_2$ in Fig. 1.8, is called a **separatrix**, since it separates two modes of motion; oscillatory and whirling. It also connects two saddle points. ●

Exercise 1.1
Find the equilibrium points and the general equation for the phase paths of $\ddot{x} + \cos x = 0$. Obtain the equation of the phase path joining two adjacent saddles. Sketch the phase diagram.

Exercise 1.2
Find the equilibrium points of the system $\ddot{x} + x - x^2 = 0$, and the general equation of the phase paths. Find the elapsed time between the points $(-\frac{1}{2}, 0)$ and $(0, \frac{1}{\sqrt{3}})$ on a phase path.

1.3 Mechanical analogy for the conservative system $\ddot{x} = f(x)$

Consider the family of autonomous equations having the more restricted form

$$\ddot{x} = f(x). \tag{1.23}$$

Replace $\dot{x}$ by the new variable y to obtain the equivalent pair of first-order equations

$$\dot{x} = y, \qquad \dot{y} = f(x). \tag{1.24}$$

In the (x, y) phase plane, the states and paths are defined exactly as in Section 1.2, since eqn (1.23) is a special case of the system (1.6).

When $f(x)$ is nonlinear the analysis of the solutions of (1.23) is sometimes helped by considering a mechanical model whose equation of motion is the same as eqn (1.23). In Fig. 1.9, a particle P having *unit* mass is free to move along the axis Ox. It is acted on by a force $f(x)$ which depends only on the displacement coordinate x, and is counted as positive if it acts in the positive direction of the x axis. The equation of motion of P then takes the form (1.23). Note that frictional forces are excluded since they are usually functions of the velocity $\dot{x}$, and their direction of action depends on the sign of $\dot{x}$; but the force $f(x)$ depends *only* on position.

Sometimes physical intuition enables us to predict the likely behaviour of the particle for specific force functions. For example, suppose that

$$\ddot{x} = f(x) = 1 + x^2.$$

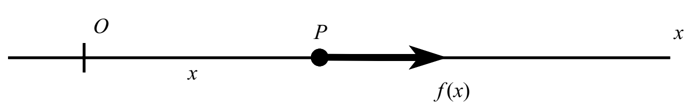

Figure 1.9 Unit particle P under the force $f(x)$.

Then $f(x) > 0$ always, so f always acts from left to right in Fig. 1.9. There are no equilibrium points of the system, so we expect that, whatever the initial conditions, P will be carried away to infinity, and there will be no oscillatory behaviour.

Next, suppose that

$$f(x) = -\lambda x, \quad \lambda > 0,$$

where λ is a constant. The equation of motion is

$$\ddot{x} = -\lambda x.$$

This is the force on the unit particle exerted by a linear spring of stiffness λ, when the origin is taken at the point where the spring has its natural length l (Fig. 1.10). We know from experience that such a spring causes oscillations, and this is confirmed by the explicit solution (1.19), in which $\lambda = \omega^2$. The cause of the oscillations is that $f(x)$ is a **restoring force**, meaning that its direction is always such as to try to drive P towards the origin.

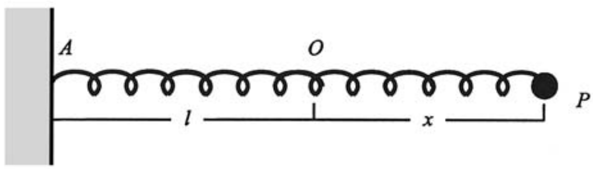

Figure 1.10 Unit particle P attached to a spring of natural length $l = AO$. The displacement of P from O is x.

Now consider a spring having a nonlinear relation between tension and extension:

$$\text{tension} = -f(x),$$

where $f(x)$ has the restoring property; that is

$$\begin{aligned} f(x) &> 0 \quad \text{for } x < 0, \\ f(0) &= 0, \\ f(x) &< 0 \quad \text{for } x > 0. \end{aligned} \tag{1.25}$$

We should expect oscillatory behaviour in this case also. The equation

$$\ddot{x} = -x^3 \tag{1.26}$$

is of this type, and the phase paths are shown in the lower diagram in Fig. 1.11 (the details are given in Example 1.5). However, the figure tells us a good deal more; that the oscillations do not consist merely of to-and-fro motions, but are strictly regular and periodic. The result is obtained from the more detailed analysis which follows.

Returning to the general case, let $x(t)$ represent a particular solution of eqn (1.23). When the particle P in Fig. 1.9 moves from a position x to a nearby position $x + \delta x$ the work $\delta\mathcal{W}$ done on P by $f(x)$ is given by

$$\delta\mathcal{W} = f(x)\delta x.$$

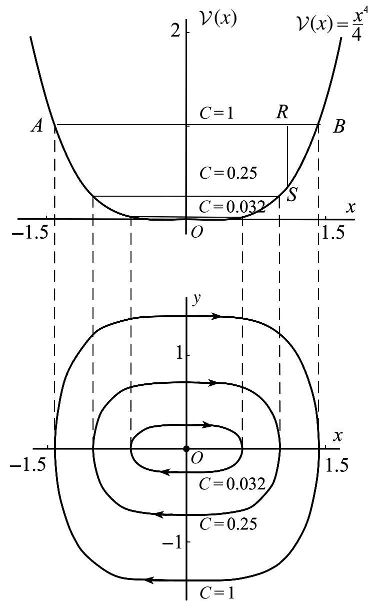

Figure 1.11

This work goes to increment the kinetic energy $\mathcal{T}$ of the (unit) particle, where $\mathcal{T} = \frac{1}{2}\dot{x}^2$:

$$\delta\mathcal{T} = \delta\mathcal{W} = f(x)\delta x.$$

Divide by δx and let $\delta x \to 0$; we obtain

$$\frac{d\mathcal{T}}{dx} = f(x). \tag{1.27}$$

Now define a function $\mathcal{V}(x)$ by the relation

$$\frac{d\mathcal{V}}{dx} = -f(x), \tag{1.28}$$

where $\mathcal{V}(x)$ is called a **potential function** for $f(x)$. Specifically,

$$\mathcal{V}(x) = -\int f(x)\, dx, \tag{1.29}$$

where $\int f(x)\, dx$ stands for any *particular* indefinite integral of $f(x)$. (Indefinite integrals involve an arbitrary constant: any constant may be chosen here for convenience, but it is necessary to stick with it throughout the problem so that $\mathcal{V}(x)$ has a single, definite, meaning.) If we specify a self-contained device that will generate the force $f(x)$, such as a spring in Fig. 1.10, or the earth's gravitation field, then $\mathcal{V}(x)$ is equal to the physical potential energy stored in the device, at any rate up to an additive constant. From (1.27) and (1.28) we obtain

$$\frac{d}{dt}(\mathcal{T} + \mathcal{V}) = 0,$$

so that during any particular motion

$$\mathcal{T} + \mathcal{V} = \text{constant}, \tag{1.30}$$

or, explicitly,

$$\frac{1}{2}\dot{x}^2 - \int f(x)\mathrm{d}x = C, \tag{1.31}$$

where C is a parameter that depends upon the particular motion and the particular potential function which has been chosen. As we range through all possible values of C consistent with real values of $\dot{x}$ we cover all possible motions. (Note that eqn (1.31) can also be obtained by using the energy transformation (1.2); or the phase-plane equation (1.9) with $\dot{x} = y$.)

In view of eqn (1.30), systems governed by the equation $\ddot{x} = f(x)$ are called **conservative systems**. From (1.31) we obtain

$$\dot{x} = \pm\sqrt{2}(C - \mathcal{V}(x))^{1/2}. \tag{1.32}$$

The equivalent equations in the phase plane are

$$\dot{x} = y, \qquad \dot{y} = f(x),$$

and (1.32) becomes

$$y = \pm\sqrt{2}(C - \mathcal{V}(x))^{1/2}, \tag{1.33}$$

which is the equation of the phase paths.

Example 1.5 *Show that all solutions of the equation*

$$\ddot{x} + x^3 = 0$$

are periodic.

Here $f(x) = -x^3$, which is a restoring force in terms of the earlier discussion, so we expect oscillations. Let

$$\mathcal{V}(x) = -\int f(x)\mathrm{d}x = \frac{1}{4}x^4,$$

in which we have set the usual arbitrary constant to zero for simplicity. From (1.33) the phase paths are given by

$$y = \pm\sqrt{2}(C - \mathcal{V}(x))^{1/2} = \pm\sqrt{2}(C - \tfrac{1}{4}x^4)^{1/2}. \tag{1.33a}$$

Figure 1.11 illustrates how the structure of the phase diagram is constructed from eqn (1.33a). In order to obtain any real values for y, we must have $C \geq 0$. In the top frame the graph of $\mathcal{V}(x) = \frac{1}{4}x^4$ is shown, together with three horizontal lines for representative values of $C > 0$. The distance RS is equal to $C - \frac{1}{4}x^4$ for $C = 1$ and a typical value of x. The relevant part of the graph, for which y in eqn (1.33a) takes real values, is the part below the line AB. Then, at the typical point on the segment, $y = \pm\sqrt{2}(RS)^{1/2}$. These two values are placed in the lower frame on Fig. 1.11.

The complete process for $C = 1$ produces a closed curve in the phase diagram, representing a periodic motion. For larger or smaller C, larger or smaller ovals are produced. When $C = 0$ there is only one point—the equilibrium point at the origin, which is a centre. ●

Equilibrium points of the system $\dot{x} = y$, $\dot{y} = f(x)$ occur at points where $y = 0$ and $f(x) = 0$, or alternatively where

$$y = 0, \qquad \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}x} = 0,$$

from (1.28). The values of x obtained are therefore those where $\mathcal{V}(x)$ has a minimum, maximum or point of inflection, and the type of equilibrium point is different in these three cases. Figure 1.12 shows how their nature can be established by the method used in Example 1.5:

$$\left.\begin{array}{l}\text{a } \textbf{minimum of } \mathcal{V}(x) \text{ generates a } \textbf{centre} \text{ (stable);}\\ \text{a } \textbf{maximum} \text{ of } \mathcal{V}(x) \text{ generates a } \textbf{saddle} \text{ (unstable);}\\ \text{a } \textbf{point of inflection} \text{ leads to a } \textbf{cusp}\text{, as shown in Fig. 1.12(c),}\end{array}\right\} \qquad (1.34)$$

Consider these results in terms of the force function $f(x)$ in the mechanical model. Suppose that $f(x_e) = 0$, so that $x = x_e, y = 0$ is an equilibrium point. If x changes sign from positive to negative as x increases through the value x_e, then it is a restoring force (eqn (1.25)). Since $\mathrm{d}\mathcal{V}/\mathrm{d}x = -f(x)$, this is also the condition for $\mathcal{V}(x_e)$ to be minimum. Therefore a restoring force always generates a centre.

If $f(x)$ changes from negative to positive through x_e the particle is repelled from the equilibrium point, so we expect an unstable equilibrium point on general grounds. Since $\mathcal{V}(x)$ has a maximum at x_e in this case, the point $(x_e, 0)$ is a saddle point, so the expectation is confirmed.

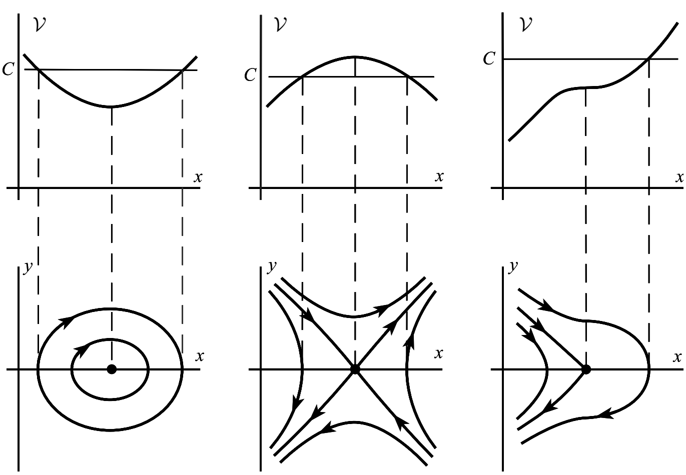

Figure 1.12 Typical phase diagrams arising from the stationary points of the potential energy.

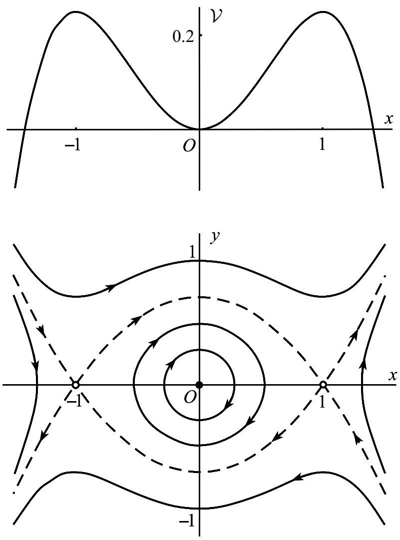

Figure 1.13 The dashed phase paths are separatrices associated with the equilibrium points at $(-1, 0)$ and $(1, 0)$.

Example 1.6 *Sketch the phase diagram for the equation* $\ddot{x} = x^3 - x$.

This represents a conservative system (the pendulum equation (1.1) reduces to a similar one after writing $\sin x \approx x - \frac{1}{6}x^3$ for moderate amplitudes). We have $f(x) = x^3 - x$, so by eqn (1.29)

$$\mathcal{V}(x) = \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4.$$

Figure 1.13 shows the construction of the phase diagram.

There are three equilibrium points: a centre at (0, 0) since $\mathcal{V}(0)$ is a minimum; and two saddle points, at $(-1, 1)$ and (1, 0) since $\mathcal{V}(-1)$ and $\mathcal{V}(1)$ are maxima. The reconciliation between the types of phase path originating around these points is achieved across special paths called separatrices, shown as broken lines (see Example 1.4 for an earlier occurrence). They correspond to values of C in the equation

$$y = \pm\surd 2(C - \mathcal{V}(x))^{1/2}$$

of $C = \frac{1}{4}$ and $C = 0$, equal to the ordinates of the maxima and minimum of $\mathcal{V}(x)$. They start or end on equilibrium points, and must not be mistaken for closed paths. ●

Example 1.7 *A unit particle P is attached to a long spring having the stress–strain property*

$$\text{tension} = x\mathrm{e}^{-x},$$

where x is the extension from its natural length. Show that the point $(0, 0)$ *on the phase diagram is a centre, but that for large disturbances P will escape to infinity.*

The equation of motion is $\ddot{x} = f(x)$, where $f(x) = x\mathrm{e}^{-x}$, so this is a restoring force (eqn (1.25)). Therefore we expect oscillations over a certain range of amplitude. However, the spring becomes very weak as x increases, so the question arises as to whether it has the strength to reverse the direction of motion if P is moving rapidly towards the right.

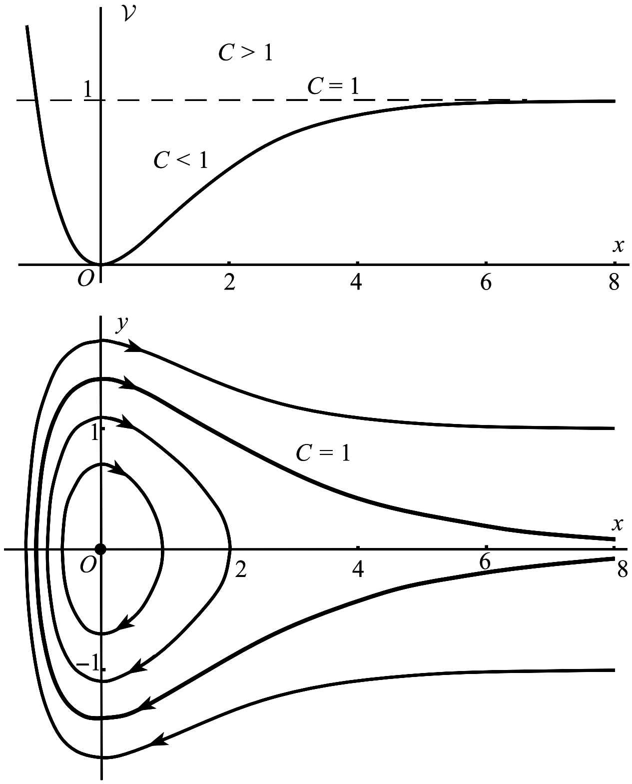

Figure 1.14

We have

$$\mathcal{V}(x) = -\int f(x)\mathrm{d}x = \int x\mathrm{e}^{-x}\mathrm{d}x = 1 - \mathrm{e}^{-x}(1+x),$$

having chosen, for convenience, a value of the arbitrary constant that causes $\mathcal{V}(0)$ to be zero. The upper frame in Fig. 1.14 shows the graph of $\mathcal{V}(x)$.

The function $\mathcal{V}(x)$ has a minimum at $x = 0$, so the origin is a centre, indicating periodic motion. As $x \to \infty$, $\mathcal{V}(x) \to 1$. The phase diagram is made up of the curves

$$y = \pm\sqrt{2}(C - \mathcal{V}(x))^{1/2}.$$

The curves are constructed as before: any phase path occupies the range in which $\mathcal{V}(x) \leq C$.

The value $C = 1$ is a critical value: it leads to the path separating the oscillatory region from the region in which P goes to infinity, so this path is a separatrix. For values of C approaching $C = 1$ from below, the ovals become increasingly extended towards the right. For $C \geq 1$ the spring stretches further and further and goes off to infinity.

The transition takes place across the path given by

$$\tfrac{1}{2}y^2 + \mathcal{V}(x) = \tfrac{1}{2}y^2 + \{1 - \mathrm{e}^{-x}(1+x)\} = C = 1.$$

The physical interpretation of $\frac{1}{2}y^2$ is the kinetic energy of P, and $\mathcal{V}(x)$ is the potential energy stored in the spring due to its displacement from equilibrium. Therefore for any motion in which the total energy is greater than 1, P will go to infinity. There is a parallel between this case and the escape velocity of a space vehicle. ●

Exercise 1.3
Find the potential function $\mathcal{V}(x)$ for the conservative system $\ddot{x} - x + x^2 = 0$. Sketch $\mathcal{V}(x)$ against x, and the main features of the phase diagram.

1.4 The damped linear oscillator

Generally speaking, equations of the form

$$\ddot{x} = f(x, \dot{x}) \tag{1.35}$$

do not arise from conservative systems, and so can be expected to show new phenomena. The simplest such system is the linear oscillator with linear damping, having the equation

$$\ddot{x} + k\dot{x} + cx = 0, \tag{1.36}$$

where $c > 0$, $k > 0$. An equation of this form describes a spring–mass system with a damper in parallel (Fig. 1.15(a)); or the charge on the capacitor in a circuit containing resistance, capacitance, and inductance (Fig. 1.15(b)). In Fig. 1.15(a), the spring force is proportional to the extension x of the spring, and the damping, or frictional force, is proportional to the velocity $\dot{x}$. Therefore

$$m\ddot{x} = -mcx - mk\dot{x}$$

by Newton's law, where c and k are certain constants relating to the stiffness of the spring and the degree of friction in the damper respectively. Since the friction generates heat, which

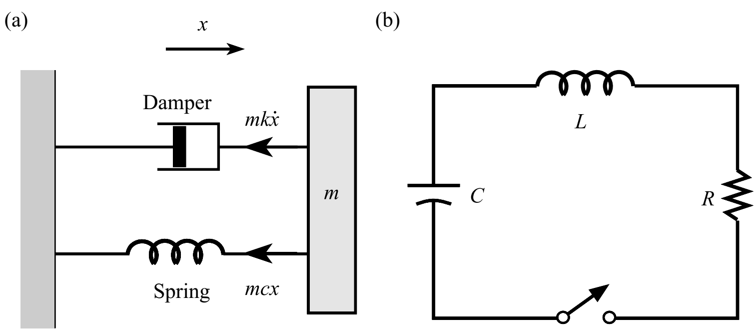

Figure 1.15 (a) Block controlled by a spring and damper. (b) Discharge of a capacitor through an (L, R, C) circuit.

is irrecoverable energy, the system is not conservative. These devices serve as models for many other oscillating systems. We shall show how the familiar features of damped oscillations show up on the phase plane.

Equation (1.36) is a standard type of differential equation, and the procedure for solving it goes as follows. Look for solutions of the form

$$x(t) = \mathrm{e}^{pt}, \tag{1.37}$$

where p is a constant, by substituting (1.37) into (1.36). We obtain the **characteristic equation**

$$p^2 + kp + c = 0. \tag{1.38}$$

This has the solutions

$$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \frac{1}{2}\{-k \pm \sqrt{(k^2 - 4c)}\}. \tag{1.39}$$

where p_1 and p_2 may be both real, or complex conjugates depending on the sign of $k^2 - 4c$.

Unless $k^2 - 4c = 0$, we have found two solutions of (1.36); $\mathrm{e}^{p_1 t}$ and $\mathrm{e}^{p_2 t}$, and the general solution is

$$x(t) = A\mathrm{e}^{p_1 t} + B\mathrm{e}^{p_2 t}, \tag{1.40}$$

where A and B are arbitrary constants which are real if $k^2 - 4c > 0$, and complex conjugates if $k^2 - 4c < 0$. If $k^2 - 4c = 0$ we have only one solution, of the form $\mathrm{e}^{-\frac{1}{2}kt}$; we need a second one, and it can be checked that this takes the form $t\mathrm{e}^{-\frac{1}{2}kt}$. Therefore, in the case of coincident solutions of the characteristic equation, the general solution is

$$x(t) = (A + Bt)\mathrm{e}^{-\frac{1}{2}kt}, \tag{1.41}$$

where A and B are arbitrary real constants.

Put

$$k^2 - 4c = \Delta, \tag{1.42}$$

where Δ is called the discriminant of the characteristic equation (1.38). The physical character of the motion depends upon the nature of the parameter Δ, as follows:

Strong damping ($\Delta > 0$)

In this case p_1 and p_2 are real, distinct and negative; and the general solution is

$$x(t) = A\mathrm{e}^{p_1 t} + B\mathrm{e}^{p_2 t}; \quad p_1 < 0, \;\; p_2 < 0. \tag{1.43}$$

Figure 1.16(a) shows two typical solutions. There is no oscillation and the t axis is cut at most once. Such a system is said to be deadbeat.

To obtain the differential equation of the phase paths, write as usual

$$\dot{x} = y, \qquad \dot{y} = -cx - ky; \tag{1.44}$$

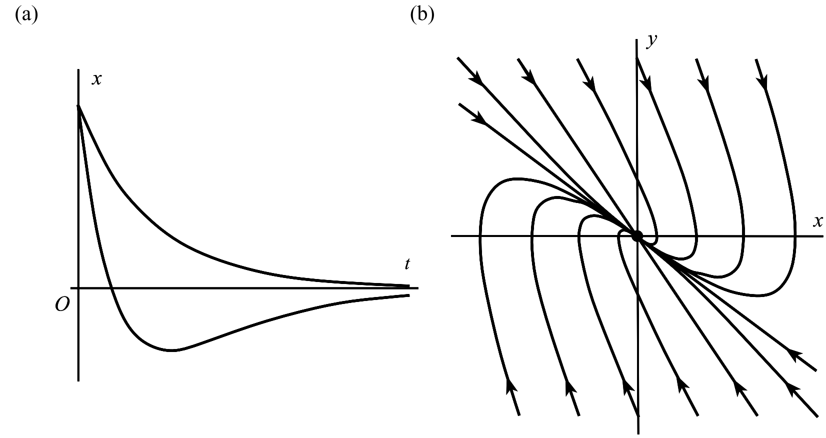

Figure 1.16 (a) Typical damped time solutions for strong damping. (b) Phase diagram for a stable node.

then

$$\frac{dy}{dx} = -\frac{cx + ky}{y}. \tag{1.45}$$

There is a single equilibrium point, at $x = 0, y = 0$. A general approach to linear systems such as (1.44) is developed in Chapter 2: for the present the solutions of (1.45) are too complicated for simple interpretation. We therefore proceed in the following way. From (1.43), putting $y = \dot{x}$,

$$x = Ae^{p_1 t} + Be^{p_2 t}, \qquad y = Ap_1 e^{p_1 t} + Bp_2 e^{p_2 t} \tag{1.46}$$

for fixed A and B, there can be treated as a parametric representation of a phase path, with parameter t. The phase paths in Fig. 1.16(b) are plotted in this way for certain values of $k > 0$ and $c > 0$. This shows a new type of equilibrium point, called a **stable node**. All paths start at infinity and terminate at the origin, as can be seen by putting $t = \pm\infty$ into (1.43). More details on the structure of nodes is given in Chapter 2.

Weak damping ($\Delta < 0$)

The exponents p_1 and p_2 are complex with negative real part, given by

$$p_1, p_2 = -\tfrac{1}{2}k \pm \tfrac{1}{2}\mathrm{i}\sqrt{(-\Delta)},$$

where $\mathrm{i} = \sqrt{-1}$. The expression (1.40) for the general solution is then, in general, complex. To extract the cases for which (1.40) delivers **real solutions**, allow A and B to be arbitrary and complex, and put

$$A = \tfrac{1}{2}Ce^{\mathrm{i}\alpha},$$

where α is real and $C = 2|A|$; and

$$B = \bar{A} = \tfrac{1}{2}Ce^{-\mathrm{i}\alpha},$$

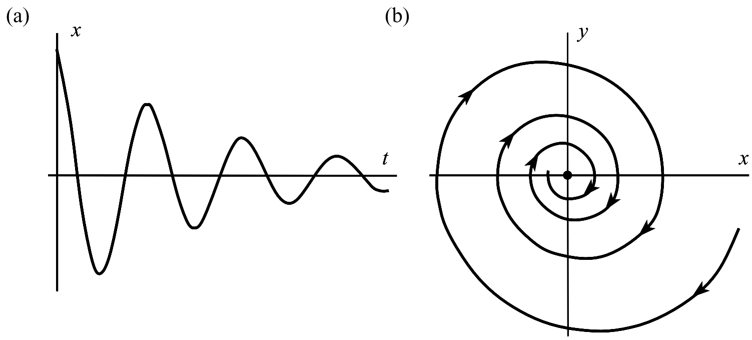

Figure 1.17 (a) Typical time solution for weak damping. (b) Phase diagram for a stable spiral showing just one phase path.

where $\bar{A}$ is the complex conjugate of A. Then (1.40) reduces to

$$x(t) = C\mathrm{e}^{-\frac{1}{2}kt} \cos\{\tfrac{1}{2}\sqrt{(-\Delta)}t + \alpha\};$$

C and α are real and arbitrary, and $C > 0$.

A typical solution is shown in Fig. 1.17(a); it represents an oscillation of frequency $(-\Delta)^{\frac{1}{2}}/(4\pi)$ and exponentially decreasing amplitude $C\mathrm{e}^{-\frac{1}{2}kt}$. Its image on the phase plane is shown in Fig. 1.17(b). The whole phase diagram would consist of a *family* of such spirals corresponding to different time solutions.

The equilibrium point at the origin is called a **stable spiral** or a **stable focus**.

Critical damping ($\Delta = 0$)

In this case $p_1 = p_2 = -\frac{1}{2}k$, and the solutions are given by (1.41). The solutions resemble those for strong damping, and the phase diagram shows a stable node.

We may also consider cases where the signs of k and c are negative:

Negative damping ($k < 0$, $c > 0$)

Instead of energy being lost to the system due to friction or resistance, energy is generated within the system. The node or spiral is then unstable, the directions being outward (see Fig. 1.18). A slight disturbance from equilibrium leads to the system being carried far from the equilibrium state (see Fig. 1.18).

Spring with negative stiffness ($c < 0$, k takes any value)

The phase diagram shows a saddle point, since p_1, p_2 are real but of opposite signs.

Exercise 1.4
For the linear system $\ddot{x} - 2\dot{x} + 2x = 0$, classify its equilibrium point and sketch the phase diagram.

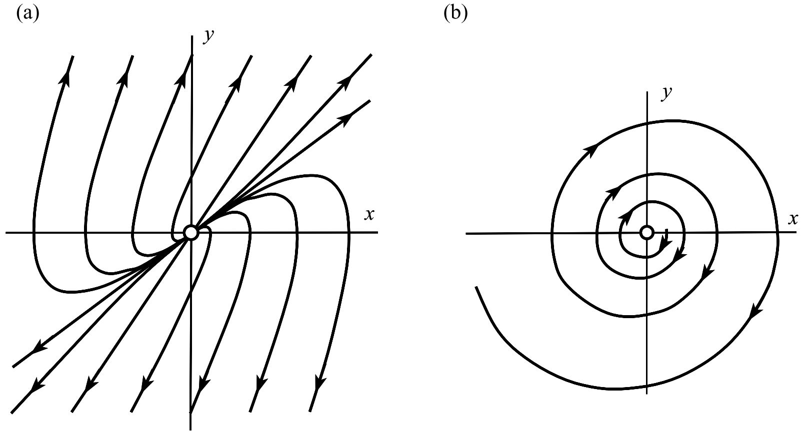

Figure 1.18 Phase diagrams for (a) the unstable node ($k < 0$, $\Delta > 0$); (b) unstable spiral ($k < 0$, $\Delta < 0$) showing just one phase path.

Exercise 1.5
Show that every phase path of

$$\ddot{x} + \varepsilon|x|\operatorname{sgn}\dot{x} + x = 0, \quad 0 < \varepsilon < 1,$$

is an isochronous spiral (that is, every circuit of the origin on every path occur in the same time).

1.5 Nonlinear damping: limit cycles

Consider the autonomous system

$$\ddot{x} = f(x, \dot{x}),$$

where f is a nonlinear function; and f takes the form

$$f(x, \dot{x}) = -h(x, \dot{x}) - g(x),$$

so that the differential equation becomes

$$\ddot{x} + h(x, \dot{x}) + g(x) = 0. \tag{1.47}$$

The equivalent pair of first-order equations for the phase paths is

$$\dot{x} = y, \qquad \dot{y} = -h(x, y) - g(x). \tag{1.48}$$

For the purposes of interpretation we shall assume that there is a single equilibrium point, and that it is at the origin (having been moved to the origin, if necessary, by a change of axes). Thus

$$h(0,0) + g(0) = 0$$

and the only solution of $h(x,0) + g(x) = 0$ is $x = 0$. We further assume that

$$g(0) = 0, \tag{1.49}$$

so that

$$h(0,0) = 0. \tag{1.50}$$

Under these circumstances, by writing eqn (1.47) in the form

$$\ddot{x} + g(x) = -h(x,\dot{x}), \tag{1.51}$$

we can regard the system as being modelled by a unit particle on a spring whose free motion is governed by the equation $\ddot{x} + g(x) = 0$ (a conservative system), but is also acted upon by an external force $-h(x,\dot{x})$ which supplies or absorbs energy. If $g(x)$ is a restoring force (eqn (1.25) with $-g(x)$ in place of $f(x)$), then we should expect a tendency to oscillate, modified by the influence of the external force $-h(x,\dot{x})$. In both the free and forced cases, equilibrium occurs when $x = \dot{x} = 0$.

Define a potential energy function for the spring system by

$$\mathcal{V}(x) = \int g(x)\,\mathrm{d}x, \tag{1.52a}$$

and the kinetic energy of the particle by

$$\mathcal{T} = \tfrac{1}{2}\dot{x}^2. \tag{1.52b}$$

The total energy $\mathcal{E}$ for the particle and spring alone is

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \tfrac{1}{2}\dot{x}^2 + \int g(x)\,\mathrm{d}x, \tag{1.53}$$

so that the rule of change of energy

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \dot{x}\ddot{x} + g(x)\dot{x}.$$

Therefore, by (1.49)

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} &= \dot{x}(-g(x) - h(x,\,\dot{x}) + g(x)) = -\dot{x}h(x,\,\dot{x}) \\ &= -yh(x,\,y)\end{aligned} \tag{1.54}$$

in the phase plane. This expression represents external the rate of supply of energy generated by the term $-h(x,\dot{x})$ representing the external force.

Suppose that, in some connected region $\mathcal{R}$ of the phase plane which contains the equilibrium point (0, 0), $\mathrm{d}\mathcal{E}/\mathrm{d}t$ is negative:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -yh(x, y) < 0 \tag{1.55}$$

(except on $y = 0$, where it is obviously zero). Consider any phase path which, after a certain point, lies in $\mathcal{R}$ for all time. Then $\mathcal{E}$ continuously decreases along the path. The effect of h resembles damping or resistance; energy is continuously withdrawn from the system, and this produces a general decrease in amplitude until the initial energy runs out. We should expect the path to approach the equilibrium point.

If

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -yh(x, y) > 0 \tag{1.56}$$

in $\mathcal{R}$ (for $y \neq 0$), the energy increases along every such path, and the amplitude of the phase paths increases so long as the paths continue to remain in $\mathcal{R}$. Here h has the effect of negative damping, injecting energy into the system for states lying in $\mathcal{R}$. ●

Example 1.8 *Examine the equation*

$$\ddot{x} + |\dot{x}|\dot{x} + x = 0$$

for damping effects.

The free oscillation is governed by $\ddot{x} + x = 0$, and the external force is given by

$$-h(x, \dot{x}) = -|\dot{x}|\dot{x}.$$

Therefore, from (1.55), the rate of change of energy

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -|\dot{x}|\dot{x}^2 = -|y|y^2 < 0$$

everywhere (except for $y = 0$). There is loss of energy along every phase path no matter where it goes in the phase plane. We therefore expect that from any initial state the corresponding phase path enters the equilibrium state as the system runs down. ●

A system may possess both characteristics; energy being injected in one region and extracted in another region of the phase plane. On any common boundary to these two regions, $\dot{x}h(x, y) = 0$ (assuming that $h(x, y)$ is continuous). The common boundary *may* constitute a phase path, and if so the energy $\mathcal{E}$ is constant along it. This is illustrated in the following example.

Example 1.9 *Examine the equation*

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$$

for energy input and damping effects.

Put $\dot{x} = y$; then

$$h(x, y) = (x^2 + \dot{x}^2 - 1)\dot{x},$$

and, from (1.55),

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -yh(x, y) = -(x^2 + y^2 - 1)y^2.$$

Therefore the energy in the particle–spring system is governed by:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} > 0 \quad \text{along the paths in the region } x^2 + y^2 < 1;$$
$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} < 0 \quad \text{along the paths in the region } x^2 + y^2 > 1;$$

The regions concerned are shown in Fig. 1.19. It can be verified that $x = \cos t$ satisfies the differential equation given above. Therefore $\dot{x} = y = -\sin t$; so that the boundary between the two regions, the circle

$$x^2 + y^2 = 1,$$

is a phase path, as shown. Along it

$$\mathcal{T} + \mathcal{V} = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 = \tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}$$

is constant, so it is a curve of constant energy $\mathcal{E}$, called an energy level.

The phase diagram consists of this circle together with paths spiralling towards it from the interior and exterior, and the (unstable) equilibrium point at the origin. All paths approach the circle. Therefore the system moves towards a state of steady oscillation, whatever the (nonzero) initial conditions. ●

The circle in Fig. 1.19 is an **isolated closed path**: 'isolated' in the sense that there is no other closed path in its immediate neighbourhood. An isolated closed path is called a **limit cycle**, and when it exists it is always one of the most important features of a physical system. Limit cycles can only occur in nonlinear systems. The limit cycle in Fig. 1.19 is a **stable limit cycle**, since if the system is disturbed from its regular oscillatory state, the resulting new path, on

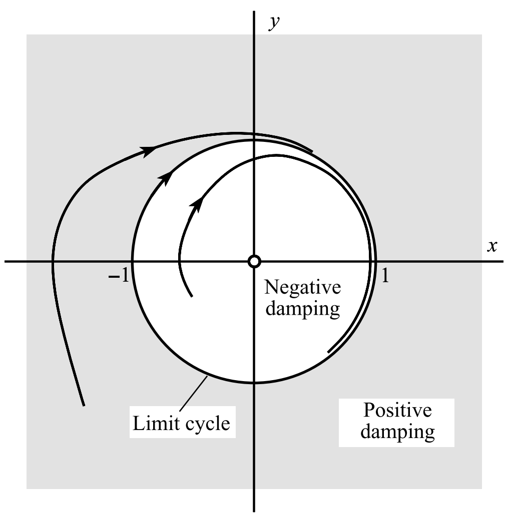

Figure 1.19 Approach of two phase paths to the stable limit cycle $x^2+y^2 = 1$ generated by $\ddot{x}+(x^2+\dot{x}^2-1)\dot{x}+x = 0$.

either side, will be attracted back to the limit cycle. There also exist *unstable* limit cycles, where neighbouring phase paths, on one side or the other are repelled from the limit cycle.

An important example of the significance of a stable limit cycle is the pendulum clock (for details see Section 1.6(iii)). Energy stored in a hanging weight is gradually supplied to the system by means of the escapement; this keeps the pendulum swinging. A balance is *automatically* set up between the rate of supply of energy and the loss due to friction in the form of a stable limit cycle which ensures strict periodicity, and recovery from any sudden disturbances.

An equation of the form

$$\ddot{x} = f(x),$$

which is the 'conservative' type treated in Section 1.3, cannot lead to a limit cycle. From the argument in that section (see Fig. 1.12), there is no shape for $\mathcal{V}(x)$ that could produce an *isolated* closed phase path.

We conclude this section by illustrating several approaches to equations having the form

$$\ddot{x} + h(x, \dot{x}) + g(x) = 0, \tag{1.57}$$

which do not involve any necessary reference to mechanical models or energy.

(i) Polar coordinates

We shall repeat Example 1.9 using polar coordinates. The structure of the phase diagram is made clearer, and other equations of similar type respond to this technique. Let r, θ be polar coordinates, where $x = r\cos\theta$, $y = r\sin\theta$, so that

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

Then, differentiating these equations with respect to time,

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}, \qquad \dot{\theta}\sec^2\theta = \frac{x\dot{y} - \dot{x}y}{x^2}$$

so that

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}, \qquad \dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2}. \tag{1.58}$$

We then substitute

$$x = r\cos\theta, \qquad \dot{x} = y = r\sin\theta$$

into these expressions, using the form for $\dot{y}$ obtained from the given differential equation.

Example 1.10 *Express the equation (see Example 1.9)*

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$$

on the phase plane, in terms of polar coordinates r, θ.

We have $x = r\cos\theta$ and $\dot{x} = y = r\sin\theta$, and

$$\dot{y} = -(x^2 + \dot{x}^2 - 1)\dot{x} - x = -(r^2 - 1)r\sin\theta - r\cos\theta.$$

By substituting these functions into (1.58) we obtain

$$\dot{r} = -r(r^2 - 1)\sin^2\theta,$$
$$\dot{\theta} = -1 - (r^2 - 1)\sin\theta\cos\theta.$$

One particular solution is

$$r = 1, \qquad \theta = -t,$$

corresponding to the limit cycle, $x = \cos t$, $y = -\sin t$, observed in Example 1.9. Also (except when $\sin\theta = 0$; that is, except on the x axis)

$$\dot{r} > 0 \quad \text{when } 0 < r < 1$$
$$\dot{r} < 0 \quad \text{when } r > 1,$$

showing that the paths approach the limit cycle $r = 1$ from both sides. The equation for $\dot{\theta}$ also shows a steady clockwise spiral motion for the representative points, around the limit cycle. ●

(ii) Topographic curves

We shall introduce topographic curves through an example.

Example 1.11 *Investigate the trend of the phase paths for the differential equation*

$$\ddot{x} + |\dot{x}|\dot{x} + x^3 = 0.$$

The system has only one equilibrium point, at (0,0). Write the equation in the form

$$\ddot{x} + x^3 = -|\dot{x}|\dot{x},$$

and multiply through by $\dot{x}$:

$$\dot{x}\ddot{x} + x^3\dot{x} = -|\dot{x}|\dot{x}^2.$$

In terms of the phase plane variables x, y this becomes

$$y\frac{dy}{dt} + x^3\frac{dx}{dt} = -|y|y^2.$$

Consider a phase path that passes through an arbitrary point A at time t_A and arrives at a point B at time $t_B > t_A$. By integrating this last equation from t_A to t_B we obtain

$$\left[\tfrac{1}{2}y^2 + \tfrac{1}{4}x^4\right]_{t=t_A}^{t_B} = -\int_{t_A}^{t_B} |y|y^2\, dt.$$

The right-hand side is negative everywhere, so

$$\left(\tfrac{1}{2}y^2 + \tfrac{1}{4}x^4\right)_{t=t_B} > \left(\tfrac{1}{2}y^2 + \tfrac{1}{4}x^4\right)_{t=t_A}$$

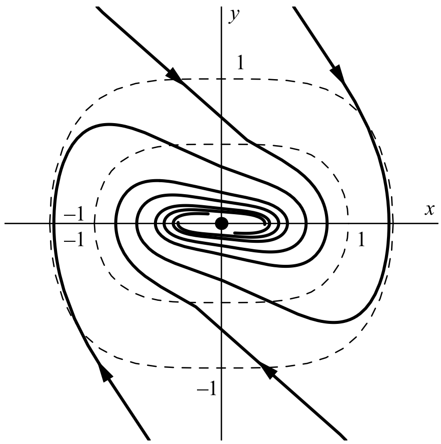

Figure 1.20 Phase diagram for $\dot{x} = y$, $\dot{y} = -|y|y - x^3$: the broken lines represent constant level curves.

along the phase path. Therefore the values of the bracketed expression, $\frac{1}{2}y^2 + \frac{1}{4}x^4$, constantly diminishes along every phase path. But the family of curves given by

$$\tfrac{1}{2}y^2 + \tfrac{1}{4}x^4 = \text{constant}$$

is a family of ovals closing in on the origin as the constant diminishes. The paths cross these ovals successively in an inward direction, so that the phase paths all move towards the origin as shown in Fig. 1.20. In mechanical terms, the ovals are curves of constant energy. ●

Such familes of closed curves, which can be used to track the paths to a certain extent, are called **topographic curves**, and are employed extensively in Chapter 10 to determine stability. The 'constant energy' curves, or **energy levels**, in the example constitute a special case.

(iii) Equations of motion in generalized coordinates

Suppose we have a conservative mechanical system, which may be in one, two, or three dimensions, and may contain solid elements as well as particles, but such that its configuration is completely specified by the value of a certain single variable x. The variable need not represent a displacement; it might, for example, be an angle, or even the reading on a dial forming part of the system. It is called a **generalized coordinate**.

Generally, the kinetic and potential energies $\mathcal{T}$ and $\mathcal{V}$ will take the forms

$$\mathcal{T} = p(x)\dot{x}^2 + q(x), \qquad \mathcal{V} = \mathcal{V}(x),$$

where $p(x) > 0$. The equation of motion can be derived using Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{T}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{T}}{\partial x} = -\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}x}.$$

Upon substituting for $\mathcal{T}$ and $\mathcal{V}$ we obtain the equation of motion in terms of x:

$$2p(x)\ddot{x} + p'(x)\dot{x}^2 + (\mathcal{V}'(x) - q'(x)) = 0. \tag{1.59}$$

This equation is not of the form $\ddot{x} = f(x)$ discussed in Section 1.3. To reduce to this form substitute u for x from the definition

$$u = \int p^{1/2}(x)\,\mathrm{d}x. \tag{1.60}$$

Then

$$\dot{u} = p^{1/2}(x)\dot{x} \quad \text{and} \quad \ddot{u} = \tfrac{1}{2}p^{-1/2}(x)p'(x)\dot{x}^2 + p^{1/2}(x)\ddot{x}.$$

After obtaining $\dot{x}$ and $\ddot{x}$ from these equations and substituting in eqn (1.59) we have

$$\ddot{u} + g(u) = 0,$$

where

$$g(u) = \tfrac{1}{2}p^{-1/2}(x)(\mathcal{V}'(x) - q'(x)).$$

This is of the conservative type discussed in Section 1.3.

Exercise 1.6
Find the equation of the limit cycle of

$$\ddot{x} + (4x^2 + \dot{x}^2 - 4)\dot{x} + 4x = 0.$$

What is its period?

1.6 Some applications

(i) Dry friction

Dry (or Coulomb) friction occurs when the surfaces of two solids are in contact and in relative motion without lubrication. The model shown in Fig. 1.21 illustrates dry friction. A continuous belt is driven by rollers at a constant speed v_0. A block of mass m connected to a fixed support by a spring of stiffness c rests on the belt. If F is the frictional force between the block and the belt and x is the extension of the spring, then the equation of motion is

$$m\ddot{x} + cx = F.$$

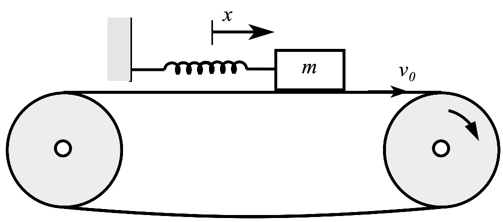

Figure 1.21 A device illustrating dry friction.

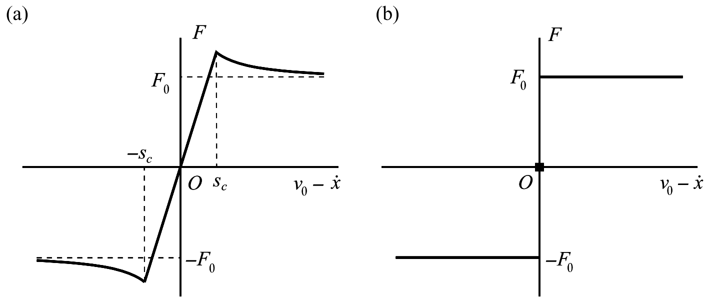

Figure 1.22 (a) Typical dry friction/slip velocity profile. (b) Idealized approximation with discontinuity.

Assume that F depends on the **slip velocity**, $v_0 - \dot{x}$; a typical relation is shown in Fig. 1.22(a). For small slip velocities the frictional force is proportional to the slip velocity. At a fixed small value of the slip speed s_c the magnitude of the frictional force peaks and then gradually approaches a constant F_0 or $-F_0$ for large slip speeds. We will replace this function by a simpler one having a discontinuity at the origin:

$$F = F_0 \operatorname{sgn}(v_0 - \dot{x})$$

where F_0 is a positive constant (see Fig. 1.22(b)) and the sgn (signum) function is defined by

$$\operatorname{sgn}(u) = \begin{cases} 1, & u > 0, \\ 0, & u = 0, \\ -1, & u < 0. \end{cases}$$

The equation of motion becomes

$$m\ddot{x} + cx = F_0 \operatorname{sgn}(v_0 - \dot{x}).$$

The term on the right is equal to F_0 when $v_0 > \dot{x}$, and $-F_0$ when $v_0 < \dot{x}$, and we obtain the following solutions for the phase paths in these regions:

$$y = \dot{x} > v_0: \quad my^2 + c\left(x + \frac{F_0}{c}\right)^2 = \text{constant},$$

$$y = \dot{x} < v_0: \quad my^2 + c\left(x - \frac{F_0}{c}\right)^2 = \text{constant}.$$

These are families of displaced ellipses, the first having its centre at $(-F_0/c, 0)$ and the second at $(F_0/c, 0)$. Figure 1.23 shows the corresponding phase diagram, in non-dimensional form with $x' = x\sqrt{c}$ and $y' = y\sqrt{m}$. In terms of these variables the paths are given by

$$y' > v_0\sqrt{m}: \quad y'^2 + \left(x' + \frac{F_0}{\sqrt{c}}\right)^2 = \text{constant},$$

$$y' < v_0\sqrt{m}: \quad y'^2 + \left(x' - \frac{F_0}{\sqrt{c}}\right)^2 = \text{constant},$$

which are arcs of displaced circles.

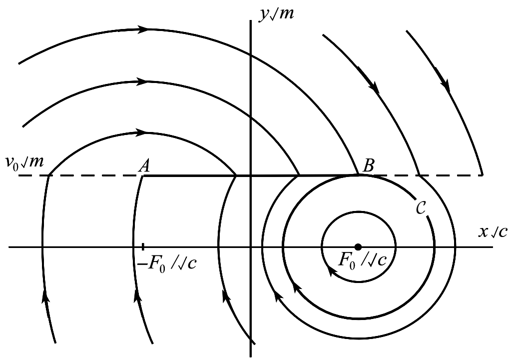

Figure 1.23 Phase diagram for the stick–slip dry friction oscillator. (Note that the axes are scaled as $x\sqrt{c}$, $y\sqrt{m}$.)

There is a single equilibrium point, at $(F_0/c, 0)$, which is a centre. Points on y (or $\dot{x}$) $= v_0$ are not covered by the differential equation since this is where F is discontinuous, so the behaviour must be deduced from other physical arguments. On encountering the state $\dot{x} = v_0$ for $|x| < F_0/c$, the block will move with the belt along AB until the maximum available friction, F_0, is insufficient to resist the increasing spring tension. This is at B when $x = F_0/c$; the block then goes into an oscillation represented by the closed path $\mathcal{C}$ through $(F_0/c, v_0)$. In fact, for any initial conditions lying outside this ellipse, the system ultimately settles into this oscillation. A computed phase diagram corresponding to a more realistic frictional force as in Fig. 1.21(a) is displayed in Problem 3.50, Fig 3.32. This kind of motion is often described as a **stick–slip oscillation**.

(ii) The brake

Consider a simple brake shoe applied to the hub of a wheel as shown in Fig. 1.24. The friction force will depend on the pressure and the angular velocity of the wheel, $\dot{\theta}$. We assume again a simplified dry-friction relation corresponding to constant pressure

$$F = -F_0 \operatorname{sgn}(\dot{\theta})$$

so if the wheel is otherwise freely spinning its equation of motion is

$$I\ddot{\theta} = -F_0 a \operatorname{sgn}(\dot{\theta})$$

where I is the moment of inertia of the wheel and a the radius of the brake drum. The phase paths are found by rewriting the differential equation, using the transformation (1.2):

$$I\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} = -F_0 a \operatorname{sgn}(\dot{\theta}),$$

whence for $\dot{\theta} > 0$

$$\tfrac{1}{2}I\dot{\theta}^2 = -F_0 a\theta + C$$

and for $\dot{\theta} < 0$

$$\tfrac{1}{2}I\dot{\theta}^2 = F_0 a\theta + C.$$

These represent two families of parabolas as shown in Fig. 1.25. $(\theta, 0)$ is an equilibrium point for every θ.

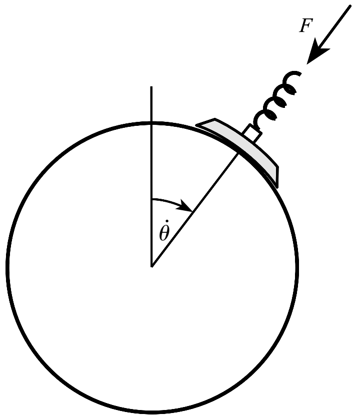

Figure 1.24 A brake model.

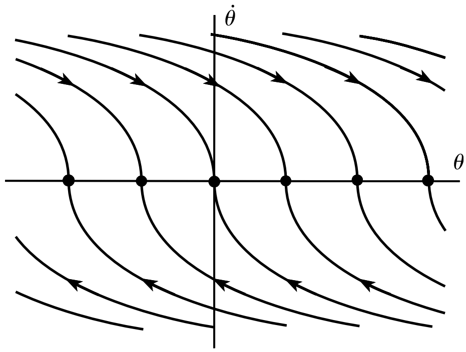

Figure 1.25 Phase diagram for the brake model in Fig. 1.24.

(iii) The pendulum clock: a limit cycle

Figure 1.26 shows the main features of the pendulum clock. The 'escape wheel' is a toothed wheel, which drives the hands of the clock through a succession of gears. It has a spindle around which is wound a wire with a weight at its free end. The escape wheel is intermittently arrested by the 'anchor', which has two teeth. The anchor is attached to the shaft of the pendulum and rocks with it, controlling the rotation of the escape wheel. The anchor and teeth on the escape wheel are so designed that as one end of the anchor loses contact with a tooth, the other end engages a tooth but allows the escape wheel to turn through a small angle, which turns the hands of the clock. Every time this happens the anchor receives small impulses, which is heard as the 'tick' of the clock. These impulses maintain the oscillation of the pendulum, which would otherwise die away. The loss of potential energy due to the weight's descent is therefore fed periodically into the pendulum via the anchor mechanism. Although the impulses push the pendulum in the same direction at each release of the anchor, the anchor shape ensures that they are slightly different in magnitude.

It can be shown that the system will settle into steady oscillations of fixed amplitude independently of sporadic disturbance and of initial conditions. If the pendulum is swinging with

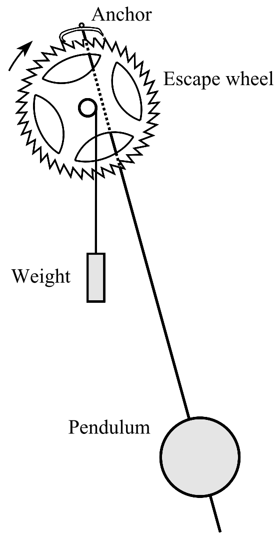

Figure 1.26 The driving mechanism (escapement) of weight-driven pendulum clock.

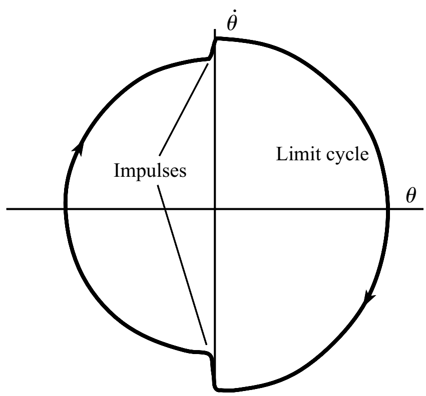

Figure 1.27 Phase diagram for a damped impulse-driven oscillator model for a pendulum clock.

too great an amplitude, its loss of energy per cycle due to friction is large, and the impulse supplied by the escapement is insufficient to offset this. The amplitude consequently decreases. If the amplitude is too small, the frictional loss is small; the impulses will over-compensate and the amplitude will build up. A balanced state is therefore approached, which appears in the θ, $\dot{\theta}$ plane (Fig. 1.27) as an isolated closed curve $\mathcal{C}$. Such an isolated periodic oscillation, or **limit cycle** (see Section 1.5) can occur only in systems described by nonlinear equations, and the following simple model shows where the nonlinearity is located.

The motion can be approximated by the equation

$$I\ddot{\theta} + k\dot{\theta} + c\theta = f(\theta, \dot{\theta}), \tag{1.61}$$

where I is the moment of inertia of the pendulum, k is a small damping constant, c is another constant determined by gravity, θ is the angular displacement, and $f(\theta, \dot{\theta})$ is the moment, supplied twice per cycle by the escapement mechanism. The moment $f(\theta, \dot{\theta})$ will be a nonlinear function in θ and $\dot{\theta}$. The model function

$$f(\theta, \dot{\theta}) = \tfrac{1}{2}[(k_1 + k_2) + (k_1 - k_2)\,\mathrm{sgn}(\dot{\theta})]\delta(\theta),$$

where $\delta(\theta)$ is the Dirac delta, or impulse, function, and k_1, k_2 are positive constraints, delivers impulses to the pendulum when $\theta = 0$. If $\dot{\theta} > 0$, then $f(\theta, \dot{\theta}) = k_1\delta(\theta)$, and if $\dot{\theta} < 0$, then $f(\theta, \dot{\theta}) = k_2\delta(\theta)$. The pendulum will be driven to overcome damping if $k_2 > k_1$. The pendulum clock was introduced by Huyghens in 1656. The accurate timekeeping of the pendulum represented a great advance in clock design over the earlier weight-driven clocks (see Baker and Blackburn (2005)).

Such an oscillation, generated by an energy source whose input is not regulated externally, but which *automatically* synchronizes with the existing oscillation, is called a **self-excited oscillation**. Here the build-up is limited by the friction.

Exercise 1.7
A smooth wire has the shape of a cycloid given parametrically by $x = a(\phi + \sin\phi)$, $y = a(1 - \cos\phi)$, $(-\pi < \phi < \pi)$. A bead is released from rest where $\phi = \phi_0$. Using conservation of energy confirm that

$$a\dot{\phi}^2 \cos^2 \tfrac{1}{2}\phi = g(\sin^2 \tfrac{1}{2}\phi_0 - \sin^2 \tfrac{1}{2}\phi).$$

Hence show that the period of oscillation of the bead is $4\pi\sqrt{(a/g)}$ (i.e., independent of ϕ_0). This is known as the **tautochrone**.

1.7 Parameter-dependent conservative systems

Suppose $x(t)$ satisfies

$$\ddot{x} = f(x, \lambda) \tag{1.62}$$

where λ is a parameter. The equilibrium points of the system are given by $f(x, \lambda) = 0$, and in general their location will depend on the parameter λ. In mechanical terms, for a particle of unit mass with displacement x, $f(x, \lambda)$ represents the force experienced by the particle. Define a function $\mathcal{V}(x, \lambda)$ such that $f(x, \lambda) = -\partial\mathcal{V}/\partial x$ for each value of λ; then $\mathcal{V}(x, \lambda)$ is the potential energy per unit mass of the equivalent mechanical system and equilibrium points correspond to stationary values of the potential energy. As indicated in Section 1.3, we expect a minimum of potential energy to correspond to a stable equilibrium point, and other stationary values (the maximum and point of inflexion) to be unstable. In fact, $\mathcal{V}$ is a minimum at $x = x_1$ if $\partial\mathcal{V}/\partial x$ changes from negative to positive on passing through x_1; this implies that $f(x, \lambda)$ changes sign from positive to negative as x increases through $x = x_1$. It acts as a restoring force.

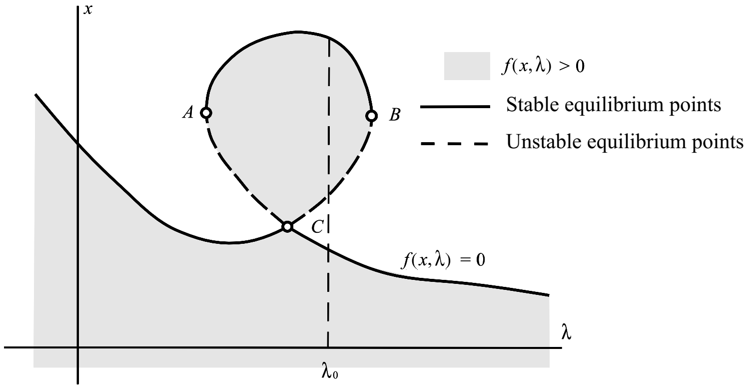

Figure 1.28 Representative stability diagram showing stability curves for the equilibrium points of $\ddot{x} = f(x, \lambda)$.

There exists a simple graphical method of displaying the stability of equilibrium points for a parameter-dependent system, in which both the number and stability of equilibrium points may vary with λ. We assume $f(x, \lambda)$ to be continuous in both x and λ. Plot the curve $f(x, \lambda) = 0$ in the λ, x plane; this curve represents the equilibrium points. Shade the domains in which $f(x, \lambda) > 0$ as shown in Fig. 1.28: If a segment of the curve has shading below it, the corresponding equilibrium points are stable, since for fixed λ, f changes from positive to negative as x increases.

The solid line between A and B corresponds to stable equilibrium points. A and B are unstable: C is also unstable since f is positive on both sides of C. The nature of the equilibrium points for a given value of λ can easily be read from the figure; for example when $\lambda = \lambda_0$ as shown, the system has three equilibrium points, two of which are stable. A, B and C are known as **bifurcation points**. As λ varies through such points the equilibrium point may split into two or more, or several equilibrium points may merge into a single one. More information on bifurcation can be found in Chapter 12.

Example 1.12 *A bead slides on a smooth circular wire of radius a which is constrained to rotate about a vertical diameter with constant angular velocity ω. Analyse the stability of the bead.*

The bead has a velocity component $a\dot{\theta}$ tangential to the wire and a component $a\omega \sin\theta$ perpendicular to the wire, where θ is the inclination of the radius to the bead to the downward vertical as shown in Fig. 1.29. The kinetic energy $\mathcal{T}$ and potential energy $\mathcal{V}$ are given by

$$\mathcal{T} = \tfrac{1}{2} m a^2 (\dot{\theta}^2 + \omega^2 \sin^2\theta), \quad \mathcal{V} = -mga\cos\theta.$$

Since the system is subject to a moving constraint (that is, the angular velocity of the wire is imposed), the usual energy equation does not hold. Lagrange's equation for the system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{T}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{T}}{\partial \theta} = -\frac{\partial \mathcal{V}}{\partial \theta},$$

which gives

$$a\ddot{\theta} = a\omega^2 \sin\theta\cos\theta - g\sin\theta.$$

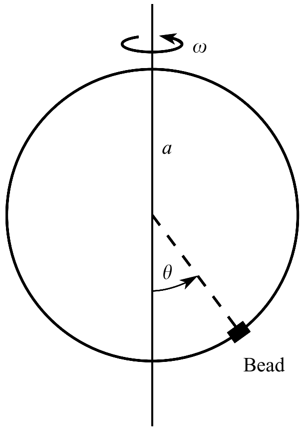

Figure 1.29 Bead on a rotating wire.

Set $a\omega^2/g = \lambda$. Then

$$a\ddot{\theta} = a\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} = g\sin\theta(\lambda\cos\theta - 1)$$

which, after integration, becomes

$$\tfrac{1}{2}a\dot{\theta}^2 = g(1 - \tfrac{1}{2}\lambda\cos\theta)\cos\theta + C,$$

the equation of the phase paths.

In the notation of eqn (1.62), we have from (i):

$$f(\theta, \lambda) = \frac{g\sin\theta(\lambda\cos\theta - 1)}{a}.$$

The equilibrium points are given by $f(\theta, \lambda) = 0$, which is satisfied when $\sin\theta = 0$ or $\cos\theta = \lambda^{-1}$. From the periodicity of the problem, $\theta = \pi$ and $\theta = -\pi$ correspond to the same state of the system.

The regions where $f < 0$ and $f > 0$ are separated by curves where $f = 0$, and can be located, therefore, by checking the sign at particular points; for example, $f(\frac{1}{2}\pi, 1) = -g/a < 0$. Figure 1.30 shows the stable and unstable equilibrium positions of the bead. The point A is a **bifurcation point**, and the equilibrium there is stable. It is known as a **pitchfork bifurcation** because of its shape.

Phase diagrams for the system may be constructed as in Section 1.3 for fixed values of λ. Two possibilities are shown in Fig. 1.31. Note that they confirm the stability predictions of Fig. 1.30. ●

Exercise 1.8

Sketch the stability diagram for the parameter-dependent equation

$$\ddot{x} = \lambda^3 + \lambda^2 - x^2,$$

and discuss the stability of the equilibrium points.

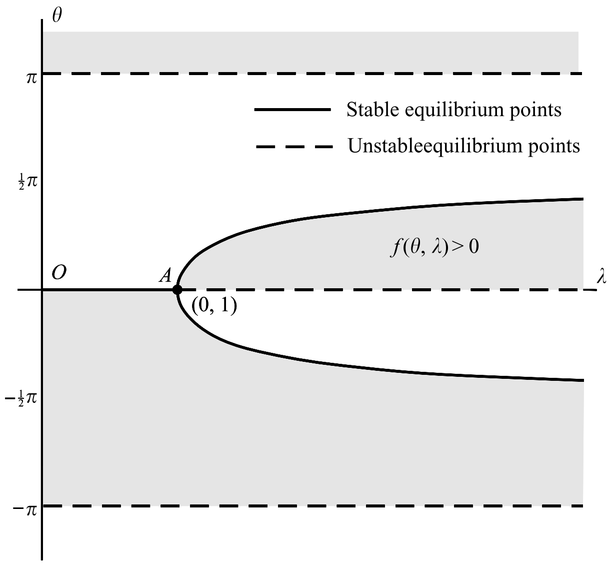

Figure 1.30 Stability diagram for a bead on a rotating wire.

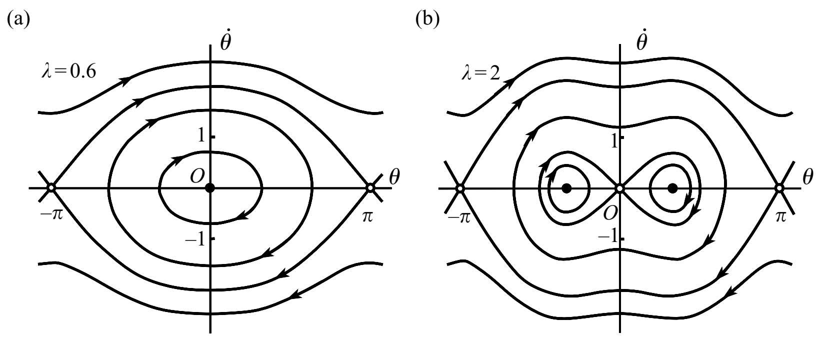

Figure 1.31 Typical phase diagrams for the rotating bead equation $\ddot{\theta} = (g/a)\sin\theta(\lambda\cos\theta - 1)$ for the cases (a) $\lambda < 0$; (b) $\lambda > 0$, with $a = g$ in both cases.

1.8 Graphical representation of solutions

Solutions and phase paths of the system

$$\dot{x} = y, \qquad \dot{y} = f(x, y)$$

can be represented graphically in a number of ways. As we have seen, the solutions of $dy/dx = f(x, y)/y$ can be displayed as paths in the phase plane (x, y). Different ways of viewing paths and solutions of the pendulum equation $\ddot{x} = -\sin x$ are shown in Fig. 1.32. Figure 1.32(a) shows typical phase paths including separatrices.

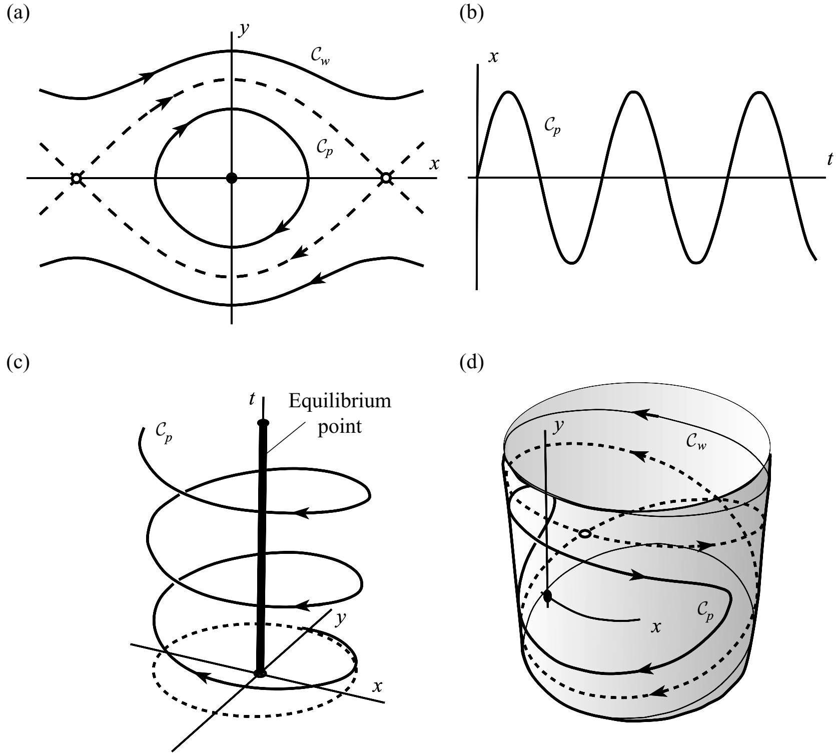

Figure 1.32 Different views of solutions of the pendulum equation $\ddot{x} + \sin x = 0$. In particular, the periodic solution $\mathcal{L}_p$ and a whirling solution $\mathcal{L}_w$ are shown (a) in the phase plane; (b) as an (x, t) solution; (c) in the solution space (x, y, t); (d) on a cylindrical phase surface which can be used for differential equations periodic in x.

If the solutions of $\ddot{x} = f(x, \dot{x})$ are known, either exactly or numerically, then the behaviour of x in terms of t can be shown in an (x, t) graph as in Fig. 1.32(b) for a periodic solution of the pendulum. Alternatively time (t) can be added as a third axis to the phase plane, so that solutions can be plotted parametrically as $(x(t), y(t), t)$ in three dimensions: solutions of the pendulum equation are shown in Fig. 1.32(c). This representation is particularly appropriate for the general phase plane (Chapter 2) and for forced systems.

If $f(x, \dot{x})$ is periodic in x, that is if there exists a number C such that $f(x + C, \dot{x}) = f(x, \dot{x})$ for all x, then phase paths on any x-interval of length C are repeated on any prior or succeeding intervals of length C. Hence solutions can be wrapped round a cylinder of circumference C. Figure 1.32(d) shows the phase paths in Fig. 1.32(a) plotted on to the cylinder, the x-axis now being wrapped round the cylinder.

Exercise 1.9
Sketch phase paths and solution of the damped oscillations
(i) $\ddot{x} + 2\dot{x} + 2x = 0$, (ii) $\ddot{x} - 3\dot{x} + 2x = 0$,
as in Fig. 1.32(a)–(c).

Problems

1.1 Locate the equilibrium points and sketch the phase diagrams in their neighbourhood for the following equations:

(i) $\ddot{x} - k\dot{x} = 0$.

(ii) $\ddot{x} - 8x\dot{x} = 0$.

(iii) $\ddot{x} = k(|x| > 1), \quad \ddot{x} = 0(|x| < 1)$.

(iv) $\ddot{x} + 3\dot{x} + 2x = 0$.

(v) $\ddot{x} - 4\dot{x} + 40x = 0$.

(vi) $\ddot{x} + 3|\dot{x}| + 2x = 0$.

(vii) $\ddot{x} + k\,\mathrm{sgn}(\dot{x}) + c\,\mathrm{sgn}(x) = 0, c > k$. Show that the path starting at $(x_0, 0)$ reaches $((c-k)^2 x_0/(c+k)^2, 0)$ after one circuit of the origin. Deduce that the origin is a spiral point.

(viii) $\dot{x} + x\,\mathrm{sgn}(x) = 0$.

1.2 Sketch the phase diagram for the equation $\ddot{x} = -x - \alpha x^3$, considering all values of α. Check the stability of the equilibrium points by the method of Section 1.7.

1.3 A certain dynamical system is governed by the equation $\ddot{x} + \dot{x}^2 + x = 0$. Show that the origin is a centre in the phase plane, and that the open and closed paths are separated by the path $2y^2 = 1 - 2x$.

1.4 Sketch the phase diagrams for the equation $\ddot{x} + \mathrm{e}^x = a$, for $a < 0$, $a = 0$, and $a > 0$.

1.5 Sketch the phase diagram for the equation $\ddot{x} - \mathrm{e}^x = a$, for $a < 0$, $a = 0$, and $a > 0$.

1.6 The potential energy $\mathcal{V}(x)$ of a conservative system is continuous, and is strictly increasing for $x < -1$, zero for $|x| \leq 1$, and strictly decreasing for $x > 1$. Locate the equilibrium points and sketch the phase diagram for the system.

1.7 Figure 1.33 shows a pendulum striking an inclined wall. Sketch the phase diagram of this 'impact oscillator', for α positive and α negative, when (i) there is no loss of energy at impact, (ii) the magnitude of the velocity is halved on impact.

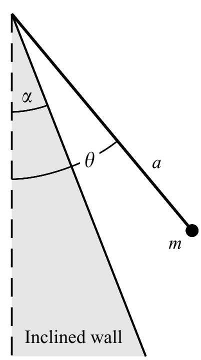

Figure 1.33 Impact pendulum.

1.8 Show that the time elapsed, T, along a phase path $\mathcal{C}$ of the system $\dot{x} = y, \dot{y} = f(x, y)$ is given, in a form alternative to (1.13), by

$$T = \int_{\mathcal{C}} (y^2 + f^2)^{1/2}\,\mathrm{d}s,$$

where $\mathrm{d}s$ is an element of distance along $\mathcal{C}$.

By writing $\delta s \simeq (y^2 + f^2)^{1/2}\delta t$, indicate, very roughly, equal time intervals along the phase paths of the system $\dot{x} = y,\ \dot{y} = 2x$.

1.9 On the phase diagram for the equation $\ddot{x} + x = 0$, the phase paths are circles. Use (1.13) in the form $\delta t \simeq \delta x/y$ to indicate, roughly, equal time steps along several phase paths.

1.10 Repeat Problem 1.9 for the equation $\ddot{x} + 9x = 0$, in which the phase paths are ellipses.

1.11 The pendulum equation, $\ddot{x} + \omega^2 \sin x = 0$, can be approximated for moderate amplitudes by the equation $\ddot{x} + \omega^2(x - \frac{1}{6}x^3) = 0$. Sketch the phase diagram for the latter equation, and explain the differences between it and Fig. 1.2.

1.12 The displacement, x, of a spring-mounted mass under the action of Coulomb dry friction is assumed to satisfy $m\ddot{x} + cx = -F_0 \operatorname{sgn}(\dot{x})$, where m, c, and F_0 are positive constants (Section 1.6). The motion starts at $t = 0$, with $x = x_0 > 3F_0/c$ and $\ddot{x} = 0$. Subsequently, whenever $x = -\alpha$, where $2F_0/c - x_0 < -\alpha < 0$ and $\dot{x} > 0$, a trigger operates, to increase suddenly the forward velocity so that the kinetic energy increases by a constant amount E. Show that if $E > 8F_0^2/c$, a periodic motion exists, and show that the largest value of x in the periodic motion is equal to $F_0/c + E/4F_0$.

1.13 In Problem 1.12, suppose that the energy is increased by E at $x = -\alpha$ for both $\ddot{x} < 0$ and $\ddot{x} > 0$; that is, there are two injections of energy per cycle. Show that periodic motion is possible if $E > 6F_0^2/c$, and find the amplitude of the oscillation.

1.14 The 'friction pendulum' consists of a pendulum attached to a sleeve, which embraces a close-fitting cylinder (Fig. 1.34). The cylinder is turned at a constant rate Ω. The sleeve is subject to Coulomb dry friction through the couple $G = -F_0 \operatorname{sgn}(\dot{\theta} - \Omega)$. Write down the equation of motion, find the equilibrium states, and sketch the phase diagram.

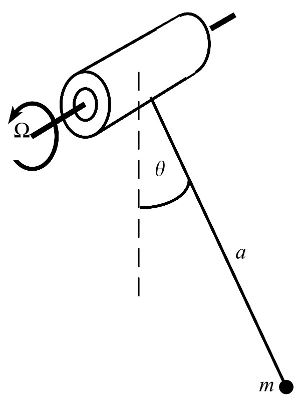

Figure 1.34 Friction-driven pendulum.

1.15 By plotting the 'potential energy' of the nonlinear conservative system $\ddot{x} = x^4 - x^2$, construct the phase diagram of the system. A particular path has the initial conditions $x = \frac{1}{2}, \dot{x} = 0$ at $t = 0$. Is the subsequent motion periodic?

1.16 The system $\ddot{x} + x = -F_0 \operatorname{sgn}(\dot{x})$, $F_0 > 0$, has the initial conditions $x = x_0 > 0, \dot{x} = 0$. Show that the phase path will spiral exactly n times before entering equilibrium (Section 1.6) if $(4n-1)F_0 < x_0 < (4n+1)F_0$.

1.17 A pendulum of length a has a bob of mass m which is subject to a horizontal force $m\omega^2 a \sin\theta$, where θ is the inclination to the downward vertical. Show that the equation of motion is $\ddot{\theta} = \omega^2(\cos\theta - \lambda)\sin\theta$, where $\lambda = g/\omega^2 a$. Investigate the stability of the equilibrium states by the method of Section 1.7 for parameter-dependent systems. Sketch the phase diagrams for various λ.

1.18 Investigate the stability of the equilibrium points of the parameter-dependent system $\ddot{x} = (x-\lambda)(x^2-\lambda)$.

1.19 If a bead slides on a smooth parabolic wire rotating with constant angular velocity ω about a vertical axis, then the distance x of the particle from the axis of rotation satisfies $(1+x^2)\ddot{x} + (g - \omega^2 + \dot{x}^2)x = 0$. Analyse the motion of the bead in the phase plane.

1.20 A particle is attached to a fixed point O on a smooth horizontal plane by an elastic string. When unstretched, the length of the string is $2a$. The equation of motion of the particle, which is constrained to move on a straight line through 0, is

$$\ddot{x} = -x + a\ \mathrm{sgn}(x), \quad |x| > a \text{ (when the string is stretched)},$$
$$\ddot{x} = 0 \quad |x| \le a \text{ (when the string is slack)},$$

x being the displacement from 0. Find the equilibrium points and the equations of the phase paths, and sketch the phase diagram.

1.21 The equation of motion of a conservative system is $\ddot{x} + g(x) = 0$, where $g(0) = 0$, $g(x)$ is strictly increasing for all x, and

$$\int_0^x g(u)\,\mathrm{d}u \to \infty \qquad \text{as } x \to \pm\infty. \tag{a}$$

Show that the motion is always periodic.

By considering $g(x) = x\mathrm{e}^{-x^2}$, show that if (a) does not hold, the motions are not all necessarily periodic.

1.22 The wave function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u^3 + \gamma\frac{\partial u}{\partial t} = 0,$$

where α, β, and γ are positive constants. Show that there exist travelling wave solutions of the form $u(x,t) = U(x - ct)$ for any c, where $U(\zeta)$ satisfies

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\zeta^2} + (\alpha - \gamma c)\frac{\mathrm{d}U}{\mathrm{d}\zeta} + \beta U^3 = 0.$$

Using Problem 1.21, show that when $c = \alpha/\gamma$, all such waves are periodic.

1.23 The linear oscillator $\ddot{x} + \dot{x} + x = 0$ is set in motion with initial conditions $x = 0$, $\dot{x} = v$, at $t = 0$. After the first and each subsequent cycle the kinetic energy is instantaneously increased by a constant, E, in such a manner as to increase $\dot{x}$. Show that if $E = \frac{1}{2}v^2(1 - \mathrm{e}^{4\pi/\sqrt{3}})$, a periodic motion occurs. Find the maximum value of x in a cycle.

1.24 Show how phase paths of Problem 1.23 having arbitrary initial conditions spiral on to a limit cycle. Sketch the phase diagram.

1.25 The kinetic energy, $\mathcal{T}$, and the potential energy, $\mathcal{V}$, of a system with one degree of freedom are given by

$$\mathcal{T} = T_0(x) + \dot{x}T_1(x) + \dot{x}^2 T_2(x), \quad \mathcal{V} = \mathcal{V}(x).$$

Use Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{T}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{T}}{\partial x} = -\frac{\partial\mathcal{V}}{\partial x}$$

to obtain the equation of motion of the system. Show that the equilibrium points are stationary points of $T_0(x) - \mathcal{V}(x)$, and that the phase paths are given by the energy equation

$$T_2(x)\dot{x}^2 - T_0(x) + \mathcal{V}(x) = \text{constant}.$$

1.26 Sketch the phase diagram for the equation $\ddot{x} = -f(x + \dot{x})$, where

$$f(u) = \begin{cases} f_0, & u \geq c, \\ f_0 u/c, & |u| \leq c, \\ -f_0, & u \leq -c, \end{cases}$$

where f_0, c are constants, $f_0 > 0$, and $c > 0$. How does the system behave as $c \to 0$?

1.27 Sketch the phase diagram for the equation $\ddot{x} = u$, where

$$u = -\operatorname{sgn}\left(\sqrt{2}|x|^{1/2}\operatorname{sgn}(x) + \dot{x}\right).$$

(u is an elementary control variable which can switch between $+1$ and -1. The curve $\sqrt{2}|x|^{1/2}\operatorname{sgn}(x)+y = 0$ is called the switching curve.)

1.28 The relativistic equation for an oscillator is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{m_0\dot{x}}{\sqrt{[1-(\dot{x}/c)^2]}}\right\} + kx = 0, \quad |\dot{x}| < c$$

where m_0, c, and k are positive constants. Show that the phase paths are given by

$$\frac{m_0 c^2}{\sqrt{[1-(y/c)^2]}} + \tfrac{1}{2}kx^2 = \text{constant}.$$

If $y = 0$ when $x = a$, show that the period, T, of an oscillation is given by

$$T = \frac{4}{c\sqrt{\varepsilon}}\int_0^a \frac{[1+\varepsilon(a^2-x^2)]\mathrm{d}x}{\sqrt{(a^2-x^2)}\sqrt{[2+\varepsilon(a^2-x^2)]}}, \qquad \varepsilon = \frac{k}{2m_0c^2}.$$

The constant ε is small; by expanding the integrand in powers of ε show that

$$T \approx \frac{\pi\sqrt{2}}{c}\left(\frac{1}{\varepsilon^{1/2}} + \frac{3}{8}\varepsilon^{1/2}a^2\right).$$

1.29 A mass m is attached to the mid-point of an elastic string of length $2a$ and stiffness λ (Fig. 1.35). There is no gravity acting, and the tension is zero in the equilibrium position. Obtain the equation of motion for transverse oscillations and sketch the phase paths.

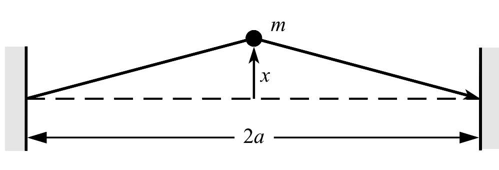

Figure 1.35 Transverse oscillations.

1.30 The system $\ddot{x} + x = F(v_0 - \dot{x})$ is subject to the friction law

$$F(u) = \begin{cases} 1, & u > \varepsilon, \\ u/\varepsilon, & -\varepsilon \leq u \leq \varepsilon, \\ -1, & u < -\varepsilon, \end{cases}$$

where $u = v_0 - \dot{x}$ is the slip velocity and $v_0 > \varepsilon > 0$. Find explicit equations for the phase paths in the $(x, y = \dot{x})$ plane. Compute a phase diagram for $\varepsilon = 0.2, v_0 = 1$ (say). Explain using the phase diagram that the equilibrium point at (1,0) is a centre, and that all paths which start outside the circle $(x-1)^2 + y^2 = (v_0 - \varepsilon)^2$ eventually approach this circle.

1.31 The system $\ddot{x} + x = F(\dot{x})$ where

$$F(\dot{x}) = \begin{cases} k\dot{x} + 1, & \dot{x} < v_0, \\ 0, & \dot{x} = v_0, \\ -k\dot{x} - 1, & \dot{x} > v_0, \end{cases}$$

and $k > 0$, is a possible model for Coulomb dry friction with damping. If $k < 2$, show that the equilibrium point is an unstable spiral. Compute the phase paths for, say, $k = 0.5$, $v_0 = 1$. Using the phase diagram discuss the motion of the system, and describe the limit cycle.

1.32 A pendulum with a magnetic bob oscillates in a vertical plane over a magnet, which repels the bob according to the inverse square law, so that the equation of motion is (Fig. 1.36)

$$ma^2\ddot{\theta} = -mga \sin\theta + Fh \sin\phi,$$

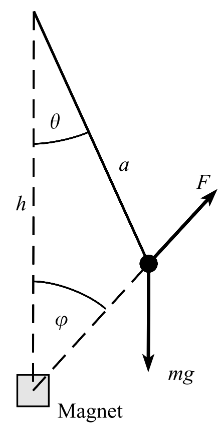

Figure 1.36 Magnetically repelled pendulum.

where $h > a$ and $F = c/(a^2 + h^2 - 2ah\cos\theta)$ and c is a constant. Find the equilibrium positions of the bob, and classify them as centres and saddle points according to the parameters of the problem. Describe the motion of the pendulum.

1.33 A pendulum with equation $\ddot{x} + \sin x = 0$ oscillates with amplitude a. Show that its period, T, is equal to $4K(\beta)$, where $\beta = \sin^2 \frac{1}{2}a$ and

$$K(\beta) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{(1 - \beta \sin^2 \phi)}}.$$

The function $K(\beta)$ has the power series representation

$$K(\beta) = \tfrac{1}{2}\pi \left[1 + \left(\tfrac{1}{2}\right)^2 \beta + \left(\tfrac{1.3}{2.4}\right)^2 \beta^2 + \cdots\right], \quad |\beta| < 1.$$

Deduce that, for small amplitudes,

$$T = 2\pi\left(1 + \tfrac{1}{16}a^2 + \tfrac{11}{3072}a^4\right) + O(a^6).$$

1.34 Repeat Problem 1.33 with the equation $\ddot{x} + x - \varepsilon x^3 = 0$ ($\varepsilon > 0$), and show that

$$T = \frac{4\sqrt{2}}{\sqrt{(2 - \varepsilon a^2)}} K(\beta), \quad \beta = \frac{\varepsilon a^2}{2 - \varepsilon a^2},$$

and that

$$T = 2\pi\left(1 + \tfrac{3}{8}\varepsilon a^2 + \tfrac{57}{256}\varepsilon^2 a^4\right) + O(\varepsilon^3 a^6),$$

as $\varepsilon a^2 \to 0$.

1.35 Show the equation of the form $\ddot{x} + g(x)\dot{x}^2 + h(x) = 0$ are effectively conservative. (Find a transformation of x which puts the equations into the usual conservative form. Compare with eqn (1.59).)

1.36 Sketch the phase diagrams of the following: (i) $\dot{x} = y$, $\dot{y} = 0$, (ii) $\dot{x} = y$, $\dot{y} = 1$, (iii) $\dot{x} = y$, $\dot{y} = y$.

1.37 Show that the phase plane for the equation $\ddot{x} - \varepsilon x\dot{x} + x = 0$ has a centre at the origin, by finding the equation of the phase paths.

1.38 Show that the equation $\ddot{x} + x + \varepsilon x^3 = 0$ ($\varepsilon > 0$) with $x(0) = a, \dot{x}(0) = 0$ has phase paths given by

$$\dot{x}^2 + x^2 + \tfrac{1}{2}\varepsilon x^4 = \left(1 + \tfrac{1}{2}\varepsilon a^2\right)a^2.$$

Show that the origin is a centre. Are all phase paths closed, and hence all solutions periodic?

1.39 Locate the equilibrium points of the equation $\ddot{x} + \lambda + x^3 - x = 0$, in the x, λ plane. Show that the phase paths are given by

$$\tfrac{1}{2}\dot{x}^2 + \lambda x + \tfrac{1}{4}\lambda x^4 - \tfrac{1}{2}x^2 = \text{ constant.}$$

Investigate the stability of the equilibrium points.

1.40 Burgers' equation

$$\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi}{\partial x} = c\frac{\partial^2\phi}{\partial x^2}$$

shows diffusion and nonlinear effects in fluid mechanics (see Logan (1994)). Find the equation for permanent waves by putting $\phi(x, t) = U(x - ct)$, where c is the constant wave speed. Find the equilibrium points and the phase paths for the resulting equation and interpret the phase diagram.

1.41 A uniform rod of mass m and length L is smoothly pivoted at one end and held in a vertical position of equilibrium by two unstretched horizontal springs, each of stiffness k, attached to the other end as shown in Fig. 1.37. The rod is free to oscillate in a vertical plane through the springs and the rod. Find the potential energy $\mathcal{V}(\theta)$ of the system when the rod is inclined at an angle θ to the upward vertical. For

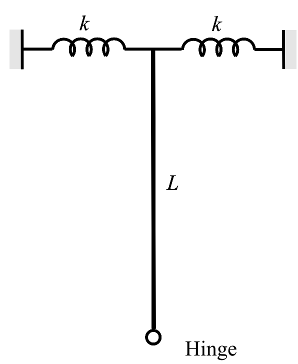

Figure 1.37 Spring restrained inverted pendulum.

small θ confirm that

$$\mathcal{V}(\theta) \approx (kL - \tfrac{1}{4}mg)L\theta^2.$$

Sketch the phase diagram for small $|\theta|$, and discuss the stability of this inverted pendulum.

1.42 Two stars, each with gravitational mass μ, are orbiting each other under their mutual gravitational forces in such a way that their orbits are circles of radius a. A satellite of relatively negligible mass is moving on a straight line through the mass centre G such that the line is perpendicular to the plane of the mutual orbits of this binary system. Explain why the satellite will continue to move on this line. If z is the displacement of the satellite from G, show that

$$\ddot{z} = -\frac{2\mu z}{(a^2 + z^2)^{3/2}}.$$

Obtain the equations of the phase paths. What type of equilibrium point is $z = 0$?

1.43 A long wire is bent into the shape of a smooth curve with equation $z = f(x)$ in a fixed vertical (x, z) plane (assume that $f'(x)$ and $f''(x)$ are continuous). A bead of mass m can slide on the wire: assume friction is negligible. Find the kinetic and potential energies of the bead, and write down the equation of the phase paths. Explain why the method of Section 1.3 concerning the phase diagrams for stationary values of the potential energy still holds.

1.44 In the previous problem suppose that friction between the bead and the wire is included. Assume linear damping in which motion is opposed by a frictional force proportional (factor k) to the velocity. Show that the equation of motion of the bead is given by

$$m(1 + f'(x)^2)\ddot{x} + mf''(x)\dot{x}^2 + k\dot{x}(1 + f'(x)^2) + mgf'(x) = 0,$$

where m is its mass.

Suppose that the wire has the parabolic shape given by $z = x^2$ and that dimensions are chosen so that $k = m$ and $g = 1$. Compute the phase diagram in the neighbourhood of the origin, and explain general features of the diagram near and further away from the origin. (Further theory and experimental work on motion on tracks can be found in the book by Virgin (2000).)

2 Plane autonomous systems and linearization

Chapter 1 describes the application of phase-plane methods to the equation $\ddot{x} = f(x, \dot{x})$ through the equivalent first-order system $\dot{x} = y$, $\dot{y} = f(x, y)$. This approach permits a useful line of argument based on a mechanical interpretation of the original equation. Frequently, however, the appropriate formulation of mechanical, biological, and geometrical problems is not through a second-order equation at all, but directly as a more general type of first-order system of the form $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$. The appearance of these equations is an invitation to construct a phase plane with x, y coordinates in which solutions are represented by curves $(x(t), y(t))$ where $x(t)$, $y(t)$ are the solutions. The constant solutions are represented by equilibrium points obtained by solving the equations $X(x, y) = 0$, $Y(x, y) = 0$, and these may now occur anywhere in the plane. Near the equilibrium points we may make a linear approximation to $X(x, y)$, $Y(x, y)$, solve the simpler equations obtained, and so determine the local character of the paths. This enables the stability of the equilibrium states to be settled and is a starting point for global investigations of the solutions.

2.1 The general phase plane

Consider the general autonomous first-order system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{2.1}$$

of which the type considered in Chapter 1,

$$\dot{x} = y, \qquad \dot{y} = f(x, y),$$

is a special case. Assume that the functions $X(x, y)$ and $Y(x, y)$ are smooth enough to make the system **regular** (see Appendix A) in the region of interest. As in Section 1.2, the system is called **autonomous** because the time variable t does not appear in the right-hand side of (2.1). We shall give examples later of how such systems arise.

The solutions $x(t)$, $y(t)$ of (2.1) may be represented on a plane with Cartesian axes x, y. Then as t increases $(x(t), y(t))$ traces out a directed curve in the plane called a **phase path**.

The appropriate form for the initial conditions of (2.1) is

$$x = x_0, \quad y = y_0 \qquad \text{at } t = t_0 \tag{2.2}$$

where x_0 and y_0 are the **initial values** at time t_0; by the existence and uniqueness theorem (Appendix A) there is one and only one solution satisfying this condition when (x_0, y_0) is an 'ordinary point'. This does not at once mean that there is one and only one phase path through the point (x_0, y_0) on the phase diagram, because this same point could serve as the initial conditions for other starting times. Therefore it might seem that other phase paths through the same point could result: the phase diagram would then be a tangle of criss-crossed curves. We may see that this is not so by forming the differential equation for the phase paths. Since $\dot{y}/\dot{x} = \mathrm{d}y/\mathrm{d}x$ on a path the required equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Y(x,y)}{X(x,y)}. \tag{2.3}$$

Equation (2.3) does not give any indication of the **direction** to be associated with a phase path for increasing t. This must be settled by reference to eqns (2.1). The signs of X and Y at any particular point determine the direction through the point, and generally the directions at all other points can be settled by the requirement of *continuity of direction* of adjacent paths.

The diagram depicting the phase paths is called the **phase diagram**. A typical point (x, y) is called a **state** of the system, as before. The phase diagram shows the evolution of the states of the system, starting from arbitrary initial states.

Points where the right-hand side of (2.3) satisfy the conditions for regularity (Appendix A) are called the **ordinary points** of (2.3). There is one and only one phase path through any ordinary point (x_0, y_0), no matter at what time t_0 the point (x_0, y_0) is encountered. Therefore infinitely many solutions of (2.1), differing only by time displacements, produce the same phase path.

However, eqn (2.3) may have singular points where the conditions for regularity do not hold, even though the time equations (2.1) have no peculiarity there. Such singular points occur where $X(x, y) = 0$. Points where $X(x, y)$ and $Y(x, y)$ are both zero,

$$X(x,y) = 0, \qquad Y(x,y) = 0 \tag{2.4}$$

are called **equilibrium points**. If x_1, y_1 is a solution of (2.4), then

$$x(t) = x_1, \qquad y(t) = y_1$$

are **constant solutions** of (2.1), and are degenerate phase paths. The term **fixed point** is also used.

Since $\mathrm{d}y/\mathrm{d}x = Y(x, y)/X(x, y)$ is the differential equation of the phase paths, phase paths which cut the curve defined by the equation $Y(x, y) = cX(x, y)$ will do so with the same slope c: such curves are known as **isoclines**. The two particular isoclines $Y(x, y) = 0$, which paths cut with zero slope, and $X(x, y) = 0$, which paths cut with infinite slope, are helpful in phase diagram sketching. The points where these isoclines intersect define the equilibrium points. Between the isoclines, $X(x, y)$ and $Y(x, y)$ must each be of one sign. For example, in a region in the (x, y) plane in which $X(x, y) > 0$ and $Y(x, y) > 0$, the phase paths must have positive slopes. This will also be the case if $X(x, y) < 0$ and $Y(x, y) < 0$. Similarly, if $X(x, y)$ and $Y(x, y)$ have opposite signs in a region, then the phase paths must have negative slopes.

Example 2.1 *Locate the equilibrium points, and sketch the phase paths of*

$$\dot{x} = y(1 - x^2), \qquad \dot{y} = -x(1 - y^2).$$

The equilibrium points occur at the simultaneous solutions of

$$y(1 - x^2) = 0, \qquad x(1 - y^2) = 0.$$

The solutions of these equations are, respectively, $x = \pm 1$, $y = 0$ and $x = 0$, $y = \pm 1$, so that there are five solution pairs, $(0, 0), (1, 1), (1, -1), (-1, 1)$ and $(-1, -1)$ which are equilibrium points.

The phase paths satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x(1 - y^2)}{y(1 - x^2)},$$

which is a first-order separable equation. Hence

$$-\int \frac{x\mathrm{d}x}{1 - x^2} = \int \frac{y\mathrm{d}y}{1 - y^2},$$

so that

$$\tfrac{1}{2}\ln|1 - x^2| = -\tfrac{1}{2}\ln|1 - y^2| + C,$$

which is equivalent to

$$(1 - x^2)(1 - y^2) = A, \text{a constant},$$

(the modulus signs are no longer necessary). Notice that there are special solutions along the lines $x = \pm 1$ and $y = \pm 1$ where $A = 0$. These solutions and the locations of the equilibrium points help us to plot the phase diagram, which is shown in Fig. 2.1. Paths cross the axis $x = 0$ with zero slope, and paths cross $y = 0$ with

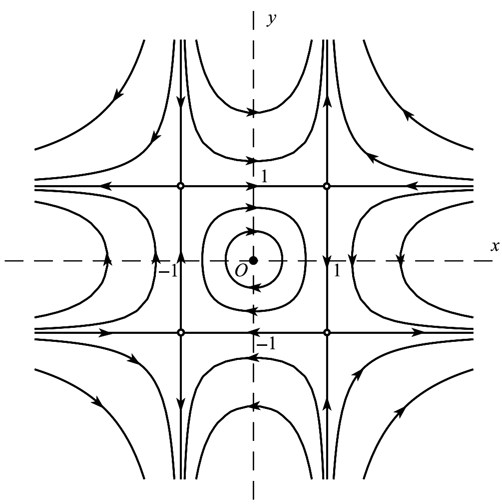

Figure 2.1 Phase diagram for $\dot{x} = y(1 - x^2)$; $\dot{y} = -x(1 - y^2)$; the dashed lines are isoclines of zero slope and infinite slope.

infinite slope. The **directions** of the paths may be found by continuity, starting at the point (0, 1), say, where $\dot{x} > 0$, and the phase path therefore runs from left to right. ●

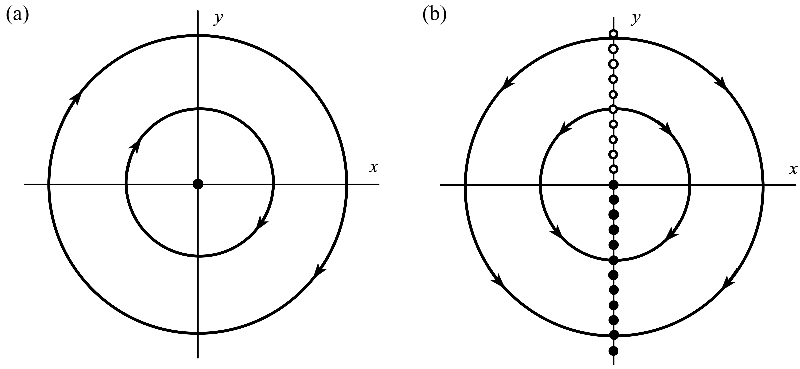

Figure 2.2 Phase diagrams for (a) $\dot{x} = y, \dot{y} = -x$; (b) $\dot{x} = xy, \dot{y} = -x^2$.

Example 2.2 *Compare the phase diagrams of the systems*
(a) $\dot{x} = y, \dot{y} = -x$; (*b*) $\dot{x} = xy, \dot{y} = -x^2$.
The equation for the paths is the same for both, namely

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

(strictly, for $y \neq 0$, and for $x \neq 0$ in the second case), giving a family of circles in both cases (Fig. 2.2). However, in case (a) there is an equilibrium point only at the origin, but in case (b) every point on the y axis is an equilibrium point. The directions, too, are different. By considering the signs of $\dot{x}, \dot{y}$ in the various quadrants the phase diagram of Fig. 2.2(b) is produced. ●

A second-order differential equation can be reduced to the general form (2.1) in an arbitrary number of ways, and this occasionally has advantages akin to changing the variable to simplify a differential equation. For example the straightforward reduction $\dot{x} = y$ applied to $x\ddot{x} - \dot{x}^2 - x^3 = 0$ leads to the system

$$\dot{x} = y, \qquad \dot{y} = \frac{y^2}{x} + x^2. \tag{2.5}$$

Suppose, instead of y, we use another variable y_1 given by $y_1(t) = \dot{x}(t)/x(t)$, so that

$$\dot{x} = xy_1. \tag{2.6}$$

Then from (2.6) $\ddot{x} = x\dot{y}_1 + \dot{x}y_1 = x\dot{y}_1 + xy_1^2$ (using (2.6) again). But from the differential equation, $\ddot{x} = (\dot{x}^2/x) + x^2 = xy_1^2 + x^2$. Therefore,

$$\dot{y}_1 = x. \tag{2.7}$$

The pair of eqns (2.6) and (2.7) provide a representation alternative to (2.5). The phase diagram using x, y_1 will, of course, be different in appearance from the x, y diagram.

Returning to the general equation (2.1), the time T elapsing along a segment $\mathcal{C}$ of a phase path connecting two states (compare Fig. 1.4(a) and eqn (1.13)) is given by

$$T = \int_{\mathcal{C}} \mathrm{d}t = \int_{\mathcal{C}} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) \mathrm{d}t = \int_{\mathcal{C}} \frac{\mathrm{d}x}{X(x,y)}. \tag{2.8}$$

Alternatively, let $\mathrm{d}s$ be a length element of $\mathcal{C}$. Then $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ on the path, and

$$T = \int_{\mathcal{C}} \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1} \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) \mathrm{d}t = \int_{\mathcal{C}} \frac{\mathrm{d}s}{(X^2+Y^2)^{1/2}}. \tag{2.9}$$

The integrals above depend only on X and Y and the geometry of the phase path; therefore the time scale is implicit in the phase diagram. **Closed paths** represent periodic solutions.

Exercise 2.1
Locate all equilibrium points of $\dot{x} = \cos y$, $\dot{y} = \sin x$. Find the equation of all phase paths.

2.2 Some population models

In the following examples systems of the type (2.1) arise naturally. Further examples from biology can be found in Pielou (1969), Rosen (1973) and Strogatz (1994).

Example 2.3 *A predator–prey problem* (Volterra's model)
In a lake there are two species of fish: A, which lives on plants of which there is a plentiful supply, and B (the predator) which subsists by eating A (the prey). We shall construct a crude model for the interaction of A and B.

Let $x(t)$ be the population of A and $y(t)$ that of B. We assume that A is relatively long-lived and rapidly breeding if left alone. Then in time δt there is a population increase given by

$$ax\delta t, \quad a > 0$$

due to births and 'natural' deaths, and 'negative increase'

$$-cxy\delta t, \quad c > 0$$

owing to A's being eaten by B (the number being eaten in this time being assumed proportional to the number of encounters between A and B). The net population increase of A, δx, is given by

$$\delta x = ax\delta t - cxy\delta t,$$

so that in the limit $\delta t \to 0$

$$\dot{x} = ax - cxy. \tag{2.10}$$

Assume that, in the absence of prey, the starvation rate of B predominates over the birth rate, but that the compensating growth of B is again proportional to the number of encounters with A. This gives

$$\dot{y} = -by + xyd \tag{2.11}$$

with $b > 0, d > 0$. Equations (2.10) and (2.11) are a pair of simultaneous nonlinear equations of the form (2.1).

We now plot the phase diagram in the x, y plane. Only the quadrant

$$x \geq 0, \qquad y \geq 0$$

is of interest. The equilibrium points are where

$$X(x, y) \equiv ax - cxy = 0, \qquad Y(x, y) \equiv -by + xyd = 0;$$

that is at (0, 0) and $(b/d, a/c)$. The phase paths are given by $\mathrm{d}y/\mathrm{d}x = Y/X$, or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(-b + xd)y}{(a - cy)x},$$

which is a separable equation leading to

$$\int \frac{(a - cy)}{y}\,\mathrm{d}y = \int \frac{(-b + xd)}{x}\,\mathrm{d}x.$$

or

$$a \log_e y + b \log_e x - cy - xd = C, \tag{2.12}$$

where C is an arbitrary constant, the parameter of the family. Writing (2.12) in the form $(a \log_e y - cy) + (b \log_e x - xd) = C$, the result of Problem 2.25 shows that this is a system of closed curves centred on the equilibrium point $(b/d,\ a/c)$.

Figure 2.3 shows the phase paths for a particular case. The direction on the paths can be obtained from the sign of $\dot{x}$ at a single point, even on $y = 0$. This determines the directions at all points by continuity. From (2.11) and (2.10) the isoclines of zero slope occur on $\dot{y} = 0$, that is, on the lines $y = 0$ and $y = xd/b$, and those of infinite slope on $\dot{x} = 0$, that is, on the lines $x = 0$ and $y = ax/c$.

Since the paths are closed, the fluctuations of $x(t)$ and $y(t)$, starting from any initial population, are periodic, the maximum population of A being about a quarter of a period behind the maximum population of B. As A gets eaten, causing B to thrive, the population x of A is reduced, causing eventually a drop in that of B. The shortage of predators then leads to a resurgence of A and the cycle starts again. A sudden change in state

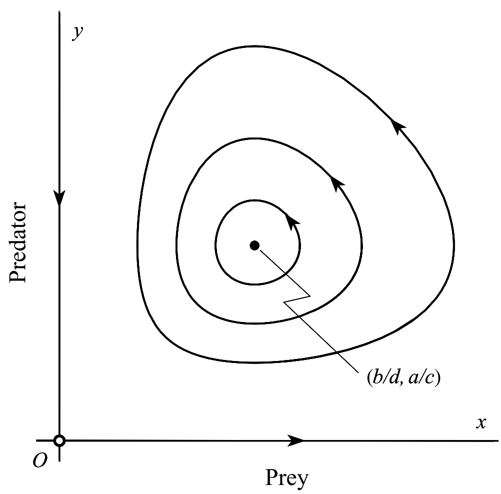

Figure 2.3 Typical phase diagram for the predator–prey model.

due to external causes, such as a bad season for the plants, puts the state on to another closed curve, but no tendency to an equilibrium population, nor for the population to disappear, is predicted. If we expect such a tendency, then we must construct a different model (see Problems 2.12 and 2.13). ●

Example 2.4 *A general epidemic model*
Consider the spread of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic. At time t suppose the population consists of

$x(t)$ *susceptibles*: those so far uninfected and therefore liable to infection;

$y(t)$ *infectives*: those who have the disease and are still at large;

$z(t)$ who are isolated, or who have recovered and are therefore immune.

Assume there is a steady contact rate between susceptibles and infectives and that a constant proportion of these contacts result in transmission. Then in time $\delta t, \delta x$ of the susceptibles become infective, where

$$\delta x = -\beta x y \delta t,$$

and β is a positive constant.

If $\gamma > 0$ is the rate at which current infectives become isolated, then

$$\delta y = \beta x y \delta t - \gamma y \delta t.$$

The number of new isolates δz is given by

$$\delta z = \gamma y \delta t.$$

Now let $\delta t \to 0$. Then the system

$$\dot{x} = -\beta x y, \qquad \dot{y} = \beta x y - \gamma y, \qquad \dot{z} = \gamma y, \tag{2.13}$$

with suitable initial conditions, determines the progress of the disease. Note that the result of adding the equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}(x + y + z) = 0;$$

that is to say, the assumption of a constant population is built into the model. x and y are defined by the first two equations in (2.13). With the restriction $x \geq 0$, $y \geq 0$, equilibrium occurs for $y = 0$ (all $x \geq 0$). The analysis of this problem in the phase plane is left as an exercise (Problem 2.29). ●

We shall instead look in detail at a more complicated epidemic:

Example 2.5 *Recurrent epidemic*

Suppose that the problem is as before, except that the stock of susceptibles $x(t)$ is being added to at a constant rate μ per unit time. This condition could be the result of fresh births in the presence of a childhood disease such as measles in the absence of vaccination. In order to balance the population in the simplest way we shall assume that deaths occur naturally and only among the immune, that is, among the $z(t)$ older people most of whom have had the disease. For a constant population the equations become

$$\dot{x} = -\beta x y + \mu, \tag{2.14}$$

$$\dot{y} = \beta x y - \gamma y, \tag{2.15}$$

$$\dot{z} = \gamma y - \mu \tag{2.16}$$

(note that $(\mathrm{d}/\mathrm{d}t)(x + y + z) = 0$: the population size is steady).

Consider the variation of x and y, the active participants, represented on the x, y phase plane. We need only (2.14) and (2.15), which show an equilibrium point $(\gamma/\beta, \mu/\gamma)$.

Instead of trying to solve the equation for the phase paths we shall try to get an idea of what the phase diagram is like by forming linear approximations to the right-hand sides of (2.14), (2.15) in the neighbourhood of the equilibrium point. Near the equilibrium point we write

$$x = \gamma/\beta + \xi, \qquad y = \mu/\gamma + \eta$$

(ξ, η small) so that $\dot{x} = \dot{\xi}$ and $\dot{y} = \dot{\eta}$. Retaining only the *linear terms* in the expansion of the right sides of (2.14), (2.15), we obtain

$$\dot{\xi} = -\frac{\beta\mu}{\gamma}\xi - \gamma\eta, \tag{2.17}$$

$$\dot{\eta} = \frac{\beta\mu}{\gamma}\xi. \tag{2.18}$$

We are said to have **linearized** (2.14) and (2.15) near the equilibrium point. Elimination of ξ gives

$$\gamma\ddot{\eta} + (\beta\mu)\dot{\eta} + (\beta\mu\gamma)\eta = 0. \tag{2.19}$$

This is the equation for the damped linear oscillator (Section 1.4), and we may compare (2.19) with eqn (1.36) of Chapter 1, but it is necessary to remember that eqn (2.19) only holds as an approximation close to the equilibrium point of (2.14) and (2.15). When the 'damping' is light ($\beta\mu/\gamma^2 < 4$) the phase path is a spiral. Figure 2.4 shows some phase paths for a particular case. All starting conditions lead to the stable equilibrium point E: this point is called the **endemic state** for the disease. ●

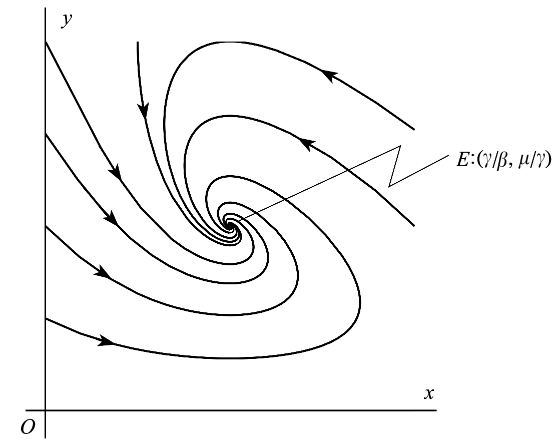

Figure 2.4 Typical spiral phase paths for the recurrent epidemic.

Exercise 2.2
The populations x and y of two species satisfy the equations

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(3 - 2x - y), \quad (x, y \geq 0),$$

(after scaling). Find the equilibrium points of the system. Confirm that $y = x$ is a phase path. Sketch the phase diagram. What happens to the species in the cases with initial populations (a) $x = 10$, $y = 9$, (b) $x = 9$, $y = 10$?

2.3 Linear approximation at equilibrium points

Approximation to a nonlinear system by linearizing it at near equilibrium point, as in the last example, is a most important and generally useful technique. If the geometrical nature of the equilibrium points can be settled in this way the broad character of the phase diagram often becomes clear. Consider the system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y). \tag{2.20}$$

Suppose that the *equilibrium point to be studied has been moved to the origin* by a translation of axes, if necessary, so that

$$X(0, 0) = Y(0, 0) = 0.$$

We can therefore write, by a Taylor expansion,

$$X(x, y) = ax + by + P(x, y), \qquad Y(x, y) = cx + dy + Q(x, y),$$

where

$$a = \frac{\partial X}{\partial x}(0, 0), \quad b = \frac{\partial X}{\partial y}(0, 0), \quad c = \frac{\partial Y}{\partial x}(0, 0), \quad d = \frac{\partial Y}{\partial y}(0, 0) \tag{2.21}$$

and $P(x, y)$, $Q(x, y)$ are of lower order of magnitude than the linear terms as (x, y) approaches the origin (0, 0). The **linear approximation** to (2.21) in the neighbourhood at the origin is defined as the system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy. \tag{2.22}$$

We expect that the solutions of (2.22) will be geometrically similar to those of (2.20) near the origin, an expectation fulfilled in most cases (but see Problem 2.7: a centre may be an exception).

Over the next two sections we shall show how simple relations between the coefficients a, b, c, d enable the equilibrium point of the system (2.22) to be classified.

Exercise 2.3
Find the linear approximations of

$$\dot{x} = \sin x + 2y, \qquad \dot{y} = xy + 3ye^x + x$$

near the origin.

2.4 The general solution of linear autonomous plane systems

The following two examples illustrate techniques for solving the system of linear differential equations with constant coefficients:

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy \tag{2.23}$$

for $x(t), y(t)$. They are followed by a more general treatment.

Example 2.6 *Solve the differential equations*

$$\dot{x} = x - 2y, \qquad \dot{y} = -3x + 2y.$$

Look for two **linearly independent solutions** (that is, solutions which are not simply constant multiples of each other) each of which takes the form of the pair

$$x = r\mathrm{e}^{\lambda t}, \qquad y = s\mathrm{e}^{\lambda t} \tag{i}$$

where r, s, λ are certain constants. Suppose that the two solutions are $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$; then the general solution will be given by

$$x(t) = C_1 x_1(t) + C_2 x_2(t), \qquad y(t) = C_1 y_1(t) + C_2 y_2(t). \tag{ii}$$

To obtain the basic solutions, substitute (i) into the given differential equations. After cancelling the common factor $\mathrm{e}^{\lambda t}$ and rearranging the terms, we obtain the pair of algebraic equations for the three unknowns λ, r, and s:

$$(1 - \lambda)r - 2s = 0, \qquad -3r + (2 - \lambda)s = 0. \tag{iii}$$

Regarding these as linear simultaneous equations for r and s, it is known that the determinant of the coefficients must be zero, or else the only solution is $r = 0, s = 0$. Therefore we require

$$\det \begin{bmatrix} 1 - \lambda & -2 \\ -3 & 2 - \lambda \end{bmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0.$$

There are two permissible values of λ:

$$\lambda = \lambda_1 = 4 \quad \text{and} \quad \lambda = \lambda_2 = -1. \tag{iv}$$

For each of these values in turn we have to solve (iii) for r and s:

The case $\lambda = \lambda_1 = 4$. Equations (iii) become

$$-3r - 2s = 0, \qquad -3r - 2s = 0. \tag{iv}$$

The equations are identical in this case; and in *every* case it turns out that one is simply a multiple of the other, so that essentially the two equations become a single equation. Choose *any* nonzero solution of (iv), say

$$r = r_1 = -2, \qquad s = s_1 = 3,$$

and we have found a solution having the form (i):

$$x(t) = x_1(t) = -2\mathrm{e}^{4t}, \qquad y(t) = y_1(t) = 3\mathrm{e}^{4t}. \tag{v}$$

The case $\lambda = \lambda_2 = -1$. Equations (iii) become

$$2r - 2s = 0, \qquad -3r + 3s = 0. \tag{vi}$$

These are both equivalent to $r - s = 0$. We take the simplest solution

$$r = r_2 = 1, \qquad s = s_2 = 1.$$

We now have as the second, independent solution of the differential equations:

$$x(t) = x_2(t) = \mathrm{e}^{-t}, \qquad y(t) = y_2(t) = \mathrm{e}^{-1}. \tag{vii}$$

Finally, from (ii), (v), and (vii), the general solution is given by

$$x(t) = -2C_1\mathrm{e}^{4t} + C_2\mathrm{e}^{-t}, \qquad y(t) = 3C_1\mathrm{e}^{4t} + C_2\mathrm{e}^{-t},$$

where C_1 and C_2 are arbitrary constants. ●

In the following example the exponents λ_1 and λ_2 are complex numbers.

Example 2.7 *Obtain the general solution of the system*

$$\dot{x} = x + y, \qquad \dot{y} = -5x - 3y.$$

Proceed exactly as in Example 2.6. Substitute

$$x = r\mathrm{e}^{\lambda t}, \qquad y = s\mathrm{e}^{\lambda t}$$

into the differential equations specified, obtaining

$$(1 - \lambda)r + s = 0, \qquad -5r - (3 + \lambda)s = 0. \tag{i}$$

There exist nonzero solutions (r, s) only if

$$\det\begin{bmatrix} 1-\lambda & 1 \\ -5 & -3-\lambda \end{bmatrix} = \lambda^2 + 2\lambda + 2 = 0. \tag{ii}$$

The permitted values of λ are therefore

$$\lambda_1 = -1 + \mathrm{i}, \qquad \lambda_2 = -1 - \mathrm{i}. \tag{iii}$$

These are complex numbers, and since (ii) is a quadratic equation, they are complex conjugate:

$$\lambda_2 = \bar{\lambda}_1. \tag{iv}$$

The case $\lambda = \lambda_1 = -1 + \mathrm{i}$. Equations (i) become

$$(2 - i)r + s = 0, \qquad -5r - (2 + \mathrm{i})s = 0;$$

(as always, these equations are actually multiples of each other). A particular solution of the differential equations associated with these is

$$x_1(t) = \mathrm{e}^{(-1+\mathrm{i})t}, \qquad y_1(t) = (-2 + \mathrm{i})\mathrm{e}^{(-1+\mathrm{i})t}. \tag{v}$$

The case $\lambda = \lambda_2 = \bar{\lambda}_1$. There is no need to rework the equations for r and s: eqn (i) shows that since $\lambda_2 = \bar{\lambda}$, we may take

$$r_2 = \bar{r}_1 = 1, \qquad s_2 = \bar{s}_1 = -2 - \mathrm{i}.$$

The corresponding solution of the differential equations is

$$x_2(t) = \mathrm{e}^{(-1-\mathrm{i})t}, \qquad y_2(t) = (-2 - \mathrm{i})\mathrm{e}^{(-1-\mathrm{i})t},$$

(which are the complex conjugates of $x_1(t), y_1(t)$). The general solution of the system is therefore

$$\begin{aligned} x(t) &= C_1 e^{(-1+i)t} + C_2 e^{(-1-i)t}, \\ y(t) &= C_1(-2+i)e^{(-1+i)t} + C_2(-2-i)e^{(-1-i)t}. \end{aligned} \tag{iv}$$

If we allow C_1 and C_2 to be arbitrary *complex* numbers, then (vi) gives us all the real and complex solutions of the equations. We are interested only in real solutions, but we must be sure that we extract all of them from (vi). This is done by allowing C_1 to be arbitrary and complex, and requiring that

$$C_2 = \bar{C}_1.$$

The terms on the right of eqn (vi) are then complex conjugate, so their sums are real; we obtain

$$\begin{aligned} x(t) &= 2\mathrm{Re}\{C_1 e^{(-1+i)t}\}, \\ y(t) &= 2\mathrm{Re}\{C_1(-2+i)e^{(-1+i)t}\}, \end{aligned}$$

By putting

$$2C_1 = c_1 + ic_2,$$

where c_1 and c_2 are *real* arbitrary constants, we obtain the general real solution in the form

$$\begin{aligned} x(t) &= e^{-t}(c_1 \cos t - c_2 \sin t), \\ y(t) &= -e^{-t}\{(2c_1 + c_2)\cos t + (c_1 - 2c_2)\sin t\}. \end{aligned}$$ ●

The general linear autonomous case is more manageable (particularly for higher order systems) when the algebra is expressed in matrix form. Define the column vectors

$$\boldsymbol{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \dot{\boldsymbol{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}.$$

The system to be solved is

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy,$$

which may be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \quad \text{with } \boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.24}$$

We shall only consider cases where there is a single equilibrium point, at the origin, the condition for this being

$$\det \boldsymbol{A} = ad - bc \neq 0. \tag{2.25}$$

(If $\det \boldsymbol{A} = 0$, then one of its rows is a multiple of the other, so that $ax + by = 0$ (or $cx + dy = 0$) consists of a line of equilibrium points.)

We seek a **fundamental solution** consisting of two linearly independent solutions of (2.24), having the form

$$\boldsymbol{x}_1(t) = \boldsymbol{v}_1 \mathrm{e}^{\lambda_1 t}, \qquad \boldsymbol{x}_2(t) = \boldsymbol{v}_2 \mathrm{e}^{\lambda_2 t}, \tag{2.26}$$

where $\lambda_1,\ \lambda_2$ are constants, and $\boldsymbol{v}_1,\ \boldsymbol{v}_2$ are constant vectors. It is known (see Chapter 8) that the general solution is given by

$$\boldsymbol{x}(t) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t), \tag{2.27}$$

where C_1 and C_2 are arbitrary constants.

To determine $\lambda_1, \boldsymbol{v}_1, \lambda_2,\ \boldsymbol{v}_2$ in (2.26) substitute

$$\boldsymbol{x}(t) = \boldsymbol{v}\mathrm{e}^{\lambda t} \tag{2.28}$$

into the system equations (2.24). After cancelling the common factor $\mathrm{e}^{\lambda t}$, we obtain

$$\lambda \boldsymbol{v} = \boldsymbol{A}\boldsymbol{v},$$

or

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}, \tag{2.29}$$

where $\boldsymbol{I}$ is the identity matrix. If we put

$$\boldsymbol{v} = \begin{bmatrix} r \\ s \end{bmatrix}, \tag{2.30}$$

eqn (2.29) represents the pair of scalar equations

$$(a-\lambda)r + bs = 0, \qquad cr + (d-\lambda)s = 0, \tag{2.31}$$

for λ, r, s.

It is known from algebraic theory that eqn (2.29) has nonzero solutions for $\boldsymbol{v}$ only if the determinant of the matrix of the coefficients in eqns (2.31) is zero. Therefore

$$\det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0, \tag{2.32}$$

or

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{2.33}$$

This is called the **characteristic equation**, and its solutions, λ_1 and λ_2, the **eigenvalues** of the matrix A, or the **characteristic exponents** for the problem. For the purpose of classifying the solutions of the characteristic equation (2.33) it is convenient to use the following notations:

$$\lambda^2 - p\lambda + q = 0,$$

where

$$p = a + d, \qquad q = ad - bc. \tag{2.34}$$

Also put

$$\Delta = p^2 - 4q. \tag{2.35}$$

The eigenvalues $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are given by

$$\lambda_1,\ \lambda_2 = \tfrac{1}{2}(p \pm \Delta^{1/2}). \tag{2.36}$$

These are to be substituted successively into (2.31) to obtain corresponding values for the constants r and s.

There are two main classes to be considered; when the eigenvalues are real and different, and when they are complex. These cases are distinguished by the sign of the **discriminant** Δ (we shall not consider the special case when $\Delta = 0$). If $\Delta > 0$ the eigenvalues are real, and if $\Delta < 0$ they are complex. We assume also that $q \neq 0$ (see (2.25)).

Time solutions when $\Delta > 0, q \neq 0$

In this case λ_1 and λ_2 are real and distinct. When $\lambda = \lambda_1$ eqns (2.31) for r and s become

$$(a - \lambda_1)r + bs = 0, \qquad cr + (d - \lambda_1)s = 0. \tag{2.37}$$

Since the determinant (2.32) is zero, its rows are linearly dependent. Therefore one of these eqns (2.37) is simply a multiple of the other; effectively we have only one equation connecting r and s. Let $r = r_1,\ s = s_1$ be any (nonzero) solution of (2.37), and put (in line with (2.30))

$$\boldsymbol{v}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} \neq 0. \tag{2.38}$$

This is called an **eigenvector** of $\boldsymbol{A}$ corresponding to the eigenvalue λ_1. We have then obtained one of the two basic time solutions having form (2.26).

This process is repeated starting with $\lambda = \lambda_2$, giving rise to an eigenvector

$$\boldsymbol{v}_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}.$$

The general solution is then given by (2.27):

$$\boldsymbol{x}(t) = C_1 \boldsymbol{v}_1 \mathrm{e}^{\lambda_1 t} + C_2 \boldsymbol{v}_2 \mathrm{e}^{\lambda_2 t} \tag{2.39}$$

in vector form, where C_1 and C_2 are arbitrary constants.

Time solutions when $\Delta < 0, q \neq 0$

In this case λ_1 and λ_2, obtained from (2.36), are complex, given by

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}\{p + \mathrm{i}(-\Delta)^{1/2}\} = \alpha + \mathrm{i}\beta, \\ \lambda_2 &= \tfrac{1}{2}\{p - \mathrm{i}(-\Delta)^{1/2}\} = \alpha - \mathrm{i}\beta, \end{aligned} \tag{2.40}$$

where $\alpha = \frac{1}{2}p$ and $\beta = \frac{1}{2}(-\Delta)^{1/2}$ are real numbers. Therefore λ_1 and λ_2 are complex conjugates.

Obtain an eigenvector corresponding to λ_1 from (2.31),

$$\boldsymbol{v} = \boldsymbol{v}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \qquad (2.41)$$

exactly as before, where r_1 and s_1 are now complex. Since a, b, c, d are all real numbers, a suitable eigenvector corresponding to λ_2 $(= \bar{\lambda}_1)$ is given by taking $r_2 = \bar{r}_1, s_2 = \bar{s}_1$ as solutions of (2.31):

$$\boldsymbol{v}_2 = \bar{\boldsymbol{v}}_1 = \begin{bmatrix} \bar{r}_1 \\ \bar{s}_1 \end{bmatrix}.$$

Therefore, two basic **complex** time solutions taking the form (2.26) are

$$\boldsymbol{v}\mathrm{e}^{(\alpha+\mathrm{i}\beta)t}, \qquad \bar{\boldsymbol{v}}\mathrm{e}^{(\alpha-\mathrm{i}\beta)t},$$

where $\boldsymbol{v}$ is given by (2.41). The general (complex) solution of (2.24) is therefore

$$\boldsymbol{x}(t) = C_1\boldsymbol{v}\mathrm{e}^{(\alpha+\mathrm{i}\beta)t} + C_2\bar{\boldsymbol{v}}\mathrm{e}^{(\alpha-\mathrm{i}\beta)t}, \qquad (2.42)$$

where C_1 and C_2 are arbitrary constants which are in general complex.

We are interested only in *real solutions*. These are included among those in (2.42); the expression is real if and only if

$$C_2 = \bar{C}_1,$$

in which case the second term is the conjugate of the first, and we obtain

$$\boldsymbol{x}(t) = 2\mathrm{Re}\{C_1\boldsymbol{v}\mathrm{e}^{(\alpha+\mathrm{i}\beta)t}\},$$

or

$$\boldsymbol{x}(t) = \mathrm{Re}\{C\boldsymbol{v}\mathrm{e}^{(\alpha+\mathrm{i}\beta)t}\}, \qquad (2.43)$$

where $C(=2C_1)$ is an arbitrary complex constant.

Exercise 2.4
Find the eigenvalues, eigenvectors and general solution of

$$\dot{x} = -4x + y, \qquad \dot{y} = -2x - y$$

2.5 The phase paths of linear autonomous plane systems

For the system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy, \qquad (2.44)$$

the general character of the phase paths can be obtained from the time solutions (2.39) and (2.43). It might be thought that an easier approach would be to obtain the phase paths directly by solving their differential equation

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by},$$

but the standard methods for solving these equations lead to implicit relations between x and y which are difficult to interpret geometrically. If eqn (2.44) is a linear approximation near the origin then the following phase diagrams will generally approximate to the phase diagram of the nonlinear system. Linearization is an important tool in phase plane analysis.

The phase diagram patterns fall into three main classes depending on the eigenvalues, which are the solutions λ_1, λ_2 of the characteristic equation (2.34):

$$\lambda^2 - p\lambda + q = 0, \tag{2.45}$$

with $p = a + d$ and $q = ad - bc \neq 0$. The three classes are

(A) λ_1, λ_2 real, distinct and having the same sign;
(B) λ_1, λ_2 real, distinct and having opposite signs;
(C) λ_1, λ_2 are complex conjugates.

These cases are now treated separately.

(A) The eigenvalues real, distinct, and having the same sign

Call the greater of the two eigenvalues λ_1, so that

$$\lambda_2 < \lambda_1. \tag{2.46}$$

In component form the general solution (2.39) for this case becomes

$$x(t) = C_1 r_1 e^{\lambda_1 t} + C_2 r_2 e^{\lambda_2 t}, \quad y(t) = C_1 s_1 e^{\lambda_1 t} + C_2 s_2 e^{\lambda_2 t}, \tag{2.47}$$

where C_1, C_2 are arbitrary constants and r_1, s_1 and r_2, s_2 are constants obtained by solving (2.37) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. From (2.47) we obtain also

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{C_1 s_1 \lambda_1 e^{\lambda_1 t} + C_2 s_2 \lambda_2 e^{\lambda_2 t}}{C_1 r_1 \lambda_1 e^{\lambda_1 t} + C_2 r_2 \lambda_2 e^{\lambda_2 t}} \tag{2.48}$$

Suppose firstly that λ_1 and λ_2 are *negative*, so that

$$\lambda_2 < \lambda_1 < 0. \tag{2.49}$$

From (2.49) and (2.47), along any phase path

$$\left.\begin{array}{l} x \text{ and } y \text{ approach the origin as } t \to \infty, \\ x \text{ and } y \text{ approach infinity as } t \to -\infty. \end{array}\right\} \tag{2.50}$$

Also there are four radial phase paths, which lie along a pair of straight lines as follows:

$$\left.\begin{aligned} &\text{if } C_2 = 0, \quad \frac{y}{x} = \frac{s_1}{r_1}; \\ &\text{if } C_1 = 0, \quad \frac{y}{x} = \frac{s_2}{r_2}. \end{aligned}\right\} \tag{2.51}$$

From (2.48), the dominant terms being those involving $e^{\lambda_1 t}$ for large positive t, and $e^{\lambda_2 t}$ for large negative t, we obtain

$$\left.\begin{aligned} &\frac{dy}{dx} \to \frac{s_1}{r_1} \quad \text{as } t \to \infty, \\ &\frac{dy}{dx} \to \frac{s_2}{r_2} \quad \text{as } t \to -\infty. \end{aligned}\right\} \tag{2.52}$$

Along with (2.50) and (2.51), this shows that every phase path is tangential to $y = (s_1/r_1)x$ at the origin, and approaches the direction of $y = (s_2/r_2)x$ at infinity. The radial solutions (2.51) are called **asymptotes** of the family of phase paths. These features can be seen in Fig. 2.5(a).

If the eigenvalues λ_1, λ_2 are both positive, with $\lambda_1 > \lambda_2 > 0$, the phase diagram has similar characteristics (Fig. 2.5(b)), but all the phase paths are directed outward, running from the origin to infinity.

These patterns show a new feature called a **node**. Figure 2.5(a) shows a **stable node** and Fig. 2.5(b) an **unstable node**. The conditions on the coefficients which correspond to these cases are:

$$\left.\begin{aligned} &\textbf{stable node: } \Delta = p^2 - 4q > 0, \quad q > 0,\ p < 0; \\ &\textbf{unstable node: } \Delta = p^2 - 4q > 0, \quad q > 0,\ p > 0. \end{aligned}\right\} \tag{2.53}$$

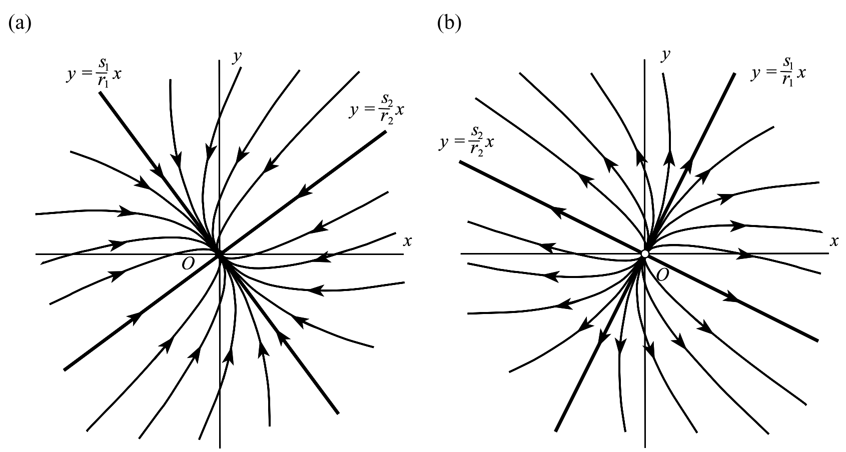

Figure 2.5 (a) Stable node; (b) unstable node.

(B) The eigenvalues are real, distinct, and of opposite signs

In this case

$$\lambda_2 < 0 < \lambda_1,$$

and the solution (2.47) and the formulae (2.48) still apply. In the same way as before, we can deduce that four of the paths are straight lines radiating from the origin, two of them lying along each of the lines

$$\frac{y}{x} = \frac{s_1}{r_1} \quad \text{and} \quad \frac{y}{x} = \frac{s_2}{r_2} \tag{2.54}$$

which are broken by the equilibrium point at the origin.

In this case however there are only *two paths* which *approach the origin*. From (2.47) it can be seen that these are the straight-line paths which lie along $y/x = s_2/r_2$, obtained by putting $C_1 = 0$. The other pair of straight-line paths go to infinity as $t \to \infty$, as do all the other paths. Also, every path (except for the two which lie along $y/x = s_2/r_2$) starts at infinity as $t \to -\infty$. The pattern is like a family of hyperbolas together with its asymptotes, as illustrated in Fig. 2.6.

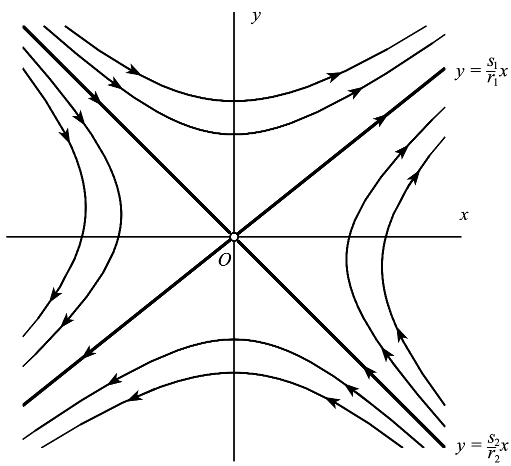

Figure 2.6 Saddle point.

The equilibrium point at the origin is a **saddle**. From (2.45), the conditions on the coefficients of the characteristic equation are

$$\textbf{saddle point}\ \Delta = p^2 - 4q > 0, \quad q < 0. \tag{2.55}$$

A saddle is always *unstable*.

The following example illustrates certain short-cuts which are useful for cases when the system coefficients a, b, c, d are given numerically.

Example 2.8 *Sketch the phase diagram and obtain the time solutions of the system*

$$\dot{x} = 3x - 2y, \qquad \dot{y} = 5x - 4y \tag{i}$$

The characteristic equation is $\lambda^2 - p\lambda + q = 0$, where $p = a + d = -1$ and $q = ad - bc = -2$. Therefore

$$\lambda^2 + \lambda - 2 = 0 = (\lambda - 1)(\lambda + 2),$$

so that

$$\lambda_1 = 1, \qquad \lambda_2 = -2. \tag{ii}$$

Since these are of opposite sign the origin is a saddle. (If all we had needed was the phase diagram, we could have checked (2.55) instead:

$$q = -2 < 0 \quad \text{and} \quad p^2 - 4q = 9 > 0;$$

but we need λ_1 and λ_2 for the time solution.)

We know that the asymptotes are straight lines, and therefore have the form $y = mx$. We can find m by substituting $y = mx$ into the equation for the paths:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{cx + dy}{ax + by} = \frac{5x - 4y}{3x - 2y},$$

which implies that $m = (5 - 4m)/(3 - 2m)$. Therefore

$$2m^2 - 7m + 5 = 0,$$

from which $m = \frac{5}{2}$ and 1, so the asymptotes are

$$y = \tfrac{5}{2}x \quad \text{and} \quad y = x. \tag{iii}$$

The pattern of the phase diagram is therefore as sketched in Fig. 2.7. The directions may be found by continuity, starting at any point. For example, at C: (1, 0), eqn (i) gives $\dot{y} = 5 > 0$, so the path through C follows the direction of increasing y. This settles the directions of all other paths.

The general time solution of (i) is

$$\left.\begin{aligned} x(t) &= C_1 r_1 \mathrm{e}^{\lambda_1 t} + C_2 r_2 \mathrm{e}^{\lambda_2 t} = C_1 r_1 \mathrm{e}^{t} + C_2 r_2 \mathrm{e}^{-2t} \\ y(t) &= C_1 s_1 \mathrm{e}^{\lambda_1 t} + C_2 s_2 \mathrm{e}^{\lambda_2 t} = C_1 s_1 \mathrm{e}^{t} + C_2 s_2 \mathrm{e}^{-2t}. \end{aligned}\right\} \tag{iv}$$

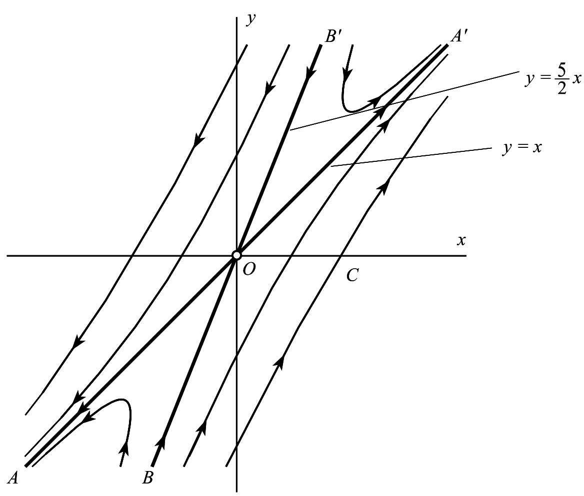

Figure 2.7 Saddle point of $\dot{x} = 3x - 2y$, $\dot{y} = 5x - 4y$.

The paths $B'O, BO$ correspond to solutions with $C_1 = 0$, since they enter the origin as $t \to \infty$. On these paths

$$\frac{y}{x} = \frac{C_2 s_2 e^{-2t}}{C_2 r_2 e^{-2t}} = \frac{s_2}{r_2} = \frac{5}{2}$$

from eqn (iii). Similarly, on OA and OA'

$$\frac{y}{x} = \frac{s_1}{r_1} = 1.$$

We may therefore choose $s_1 = r_1 = 1$, and $s_2 = 5,\ r_2 = 2$. Putting these values into (iv) we obtain the general solution

$$x(t) = C_1 e^{t} + 2C_2 e^{-2t},$$
$$y(t) = C_1 e^{t} + 5C_2 e^{-2t},$$

where C_1 and C_2 are arbitrary constants. ●

(C) The eigenvalues are complex

Complex eigenvalues of real matrices always occur as complex conjugate pairs, so put

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta \qquad (\alpha, \beta \text{ real}). \tag{2.56}$$

By separating the components of (2.43) we obtain for the general solution

$$x(t) = e^{\alpha t}\mathrm{Re}\{C r_1 e^{i\beta t}\}, \quad y(t) = e^{\alpha t}\mathrm{Re}\{C s_1 e^{i\beta t}\}, \tag{2.57}$$

where C, r_1, and s_1 are all complex in general.

Suppose firstly that $\alpha = 0$. Put C, r_1, s_1 into polar form:

$$C = |C|e^{i\gamma}, \quad r_1 = |r_1|e^{i\rho}, \quad s_1 = |s_1|e^{i\sigma}.$$

Then (2.57), with $\alpha = 0$, becomes

$$x(t) = |C||r_1|\cos(\beta t + \gamma + \rho), \quad y(t) = |C||s_1|\cos(\beta t + \gamma + \sigma). \tag{2.58}$$

The motion of the representative point $(x(t), y(t))$ in the phase plane consists of two simple harmonic components of equal circular frequency β, in the x and y directions, but they have different phase and amplitude. The phase paths therefore form a family of geometrically similar ellipses which, in general, is inclined at a constant angle to the axes. (The construction is similar to the case of elliptically polarized light; the proof involves eliminating $\cos(\beta t + \gamma)$ and $\sin(\beta t + \gamma)$ between the equations in (2.58).)

This case is illustrated in Fig. 2.8. The algebraic conditions corresponding to the centre at the origin are

$$\textbf{centre}\ p = 0, \quad q > 0. \tag{2.59}$$

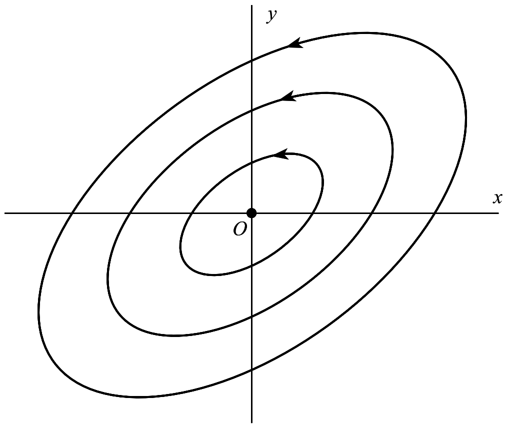

Figure 2.8 Typical centre: rotation may be in either sense.

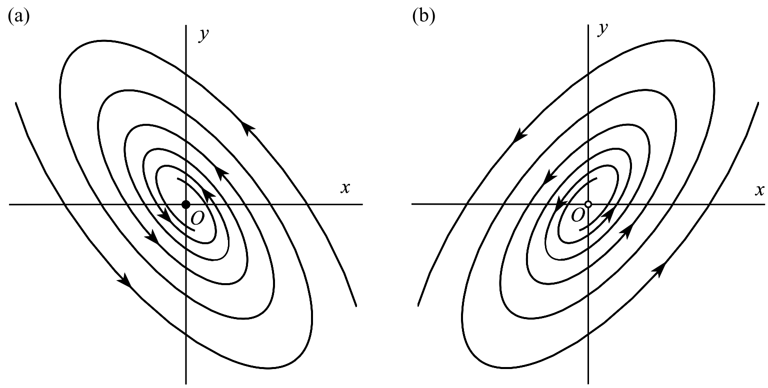

Figure 2.9 (a) Stable spiral; (b) unstable spiral.

Now suppose that $\alpha \neq 0$. As t increases in eqns (2.57), the elliptical paths above are modified by the factor $e^{\alpha t}$. This prevents them from closing, and each ellipse turns into a spiral; a contracting spiral if $\alpha < 0$, and an expanding spiral if $\alpha > 0$ (see Fig. 2.9). The equilibrium point is then called a **spiral** or **focus**, stable if $\alpha < 0$, unstable if $\alpha > 0$. The directions may be *clockwise* or *counterclockwise*.

The algebraic conditions are

$$\left.\begin{array}{ll}\textbf{stable spiral:} & \Delta = p^2 - 4q < 0,\ \ q > 0,\ \ p < 0; \\ \textbf{unstable spiral:} & \Delta = p^2 - 4q < 0,\ \ q > 0,\ \ p > 0.\end{array}\right\} \tag{2.60}$$

Example 2.9 *Determine the nature of the equilibrium point of the system $\dot{x} = -x - 5y, \dot{y} = x + 3y$.*

We have $a = -1, b = -5, c = 1, d = 3$. Therefore

$$p = a + d = 2 > 0, \qquad q = ad - bc = 2 > 0,$$

so that $\Delta = p^2 - 4q = -4 < 0$. These are the conditions (2.60) for an unstable spiral. By putting, say $x > 0, y = 0$ into the equation for $\dot{y}$, we obtain $\dot{y} > 0$ for phase paths as they cross the positive x axis. The spiral paths therefore unwind in the counterclockwise direction. ●

In addition to the cases discussed there are several **degenerate cases**. These occur when there is a repeated eigenvalue, or when an eigenvalue is zero.

If $q = \det A = 0$, then the eigenvalues are $\lambda_1 = p,\ \lambda_2 = 0$. If $p \neq 0$, then as in the case (2.39), with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ the eigenvectors,

$$\boldsymbol{x}(t) = C_1\boldsymbol{v}_1 e^{pt} + C_2\boldsymbol{v}_2.$$

There is a line of equilibrium points given by

$$ax + by = 0$$

(which is effectively the same equation as $cx + dy = 0$; the two expressions are linearly dependent). The phase paths form a family of parallel straight lines as shown in Fig. 2.10. A further special case arises if $q = 0$ and $p = 0$.

If $\Delta = 0$, then eigenvalues are real and equal with $\lambda = \frac{1}{2}p$. If $p \neq 0$, it can be shown that the equilibrium point becomes a degenerate node (see Fig. 2.10), in which the two asymptotes have converged.

Figure 2.10 summarizes the results of this section as a pictorial diagram, whilst the table classifies the equilibrium points in terms of the parameters p, q and Δ.

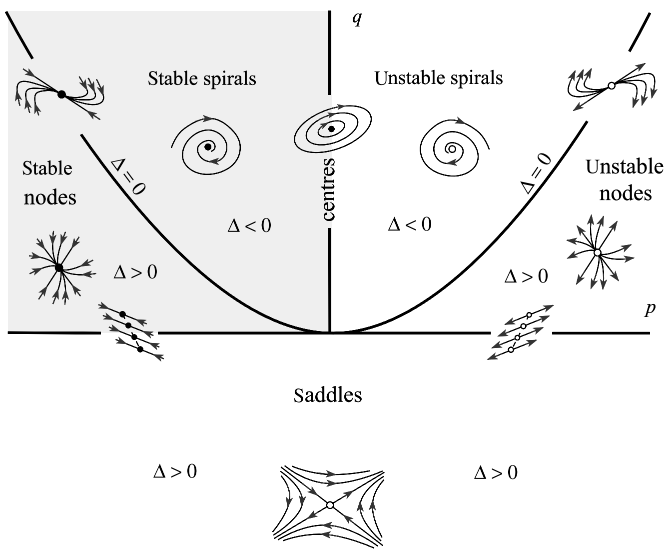

Figure 2.10 Classification for the linear system in the (p, q) plane: $\dot{x} = ax + by,\ \dot{y} = cx + dy$ with $p = a + d,\ q = ad - bc,\ \Delta = p^2 - 4q$.

Classification of equilibrium points of $\dot{x} = ax + by, \dot{y} = cx + dy$

	$p = a + d$	$q = ad - bc$	$\Delta = p^2 - 4q$
Saddle	—	$q < 0$	$\Delta > 0$
Stable node	$p < 0$	$q > 0$	$\Delta > 0$
Stable spiral	$p < 0$	$q > 0$	$\Delta < 0$
Unstable node	$p > 0$	$q > 0$	$\Delta > 0$
Unstable spiral	$p > 0$	$q > 0$	$\Delta < 0$
centre	$p = 0$	$q > 0$	$\Delta < 0$
Degenerate stable node	$p < 0$	$q > 0$	$\Delta = 0$
Degenerate unstable node	$p > 0$	$q > 0$	$\Delta = 0$

(2.61)

A **centre** may be regarded as a degenerate case, forming a transition between stable and unstable spirals. The existence of a centre depends on there being a particular *exact* relation, namely $a + d = 0$, between coefficients of the system, so a centre is rather a fragile feature. Consequently, if the linear approximation to a nonlinear system predicts a centre it cannot be reliably concluded that the original system has a centre: it might have a stable, or worse, an unstable spiral (see, e.g. Problem 2.7). The same applies to all the *degenerate* cases indicated: if they are used as linear approximations then, taken alone, they are unreliable indicators.

If there exists a neighbourhood of an equilibrium point such that every phase path starting in the neighbourhood ultimately approaches the equilibrium point, the point is known as an **attractor**. (The term is used both for linear and nonlinear systems.) The stable node and stable spiral are attractors. An attractor with all path directions reversed is a **repellor**. Unstable nodes and spirals are repellors, but a saddle point is not. The terms attractor and repellor can also be applied to limit cycles, and to less well defined attracting sets, such as the strange attractor of Chapter 13, from which paths cannot escape.

If the eigenvalues of the linearized equation have nonzero real parts then the equilibrium point is said to be **hyperbolic**. It is shown in Chapter 10 that at hyperbolic points the phase diagrams of the nonlinear equations and the linearized equations are, locally, qualitatively the same. Spirals, nodes, and saddles are hyperbolic but the centre is not.

Exercise 2.5

Using Fig. 2.10 classify the equilibrium points of:

(a) $\dot{x} = -4x + 2y, \ \dot{y} = 4x + 3y$;

(b) $\dot{x} = -6x + 5y, \ \dot{y} = -5x + 2y$;

(b) $\dot{x} = 11x + 6y, \ \dot{y} = -6x - 2y$.

2.6 Scaling in the phase diagram for a linear autonomous system

Consider the system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy. \tag{2.62}$$

In Fig. 2.11, $\mathcal{P}$ represents a segment of a phase path, passing through $A : (x_A, y_A)$ at (say) $t = 0$, and $P : (x_P(t),\ y_P(t))$ is the representative point on $\mathcal{P}$ at time t.

The segment $\mathcal{C}_k$ is a scaled copy of $\mathcal{P}$, constructed in the following way. Choose any constant k; and the two points $B : (x_B, y_B)$ and $Q : (x_Q, y_Q)$ such that

$$(x_B, y_B) = (kx_A, ky_A), \qquad (x_Q, y_Q) = (kx_P, ky_P). \tag{2.63}$$

Then B lies on the radius OA and Q on the radius OP, extended as necessary. Points B and Q are on the same side or opposite sides of the origin according to whether $k > 0$ or $k < 0$, respectively (in Fig. 2.11, $k > 1$). As the representative point P traces the phase path $\mathcal{P}$, Q traces the curve $\mathcal{C}_k$.

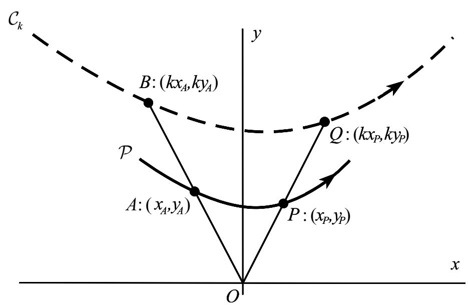

Figure 2.11 $\mathcal{P}$ is a phase path segment. $\mathcal{C}_k$ is the image of $\mathcal{P}$, expanded by a factor k. It is also a phase path.

Since P is the representative point on the phase path $\mathcal{P}$, the system equations (2.62) give

$$k\dot{x}_P = a(kx_P) + b(ky_P), \qquad k\dot{y}_P = c(kx_P) + d(ky_P).$$

Therefore, from (2.63)

$$\dot{x}_Q = ax_Q + by_Q, \qquad \dot{y}_Q = cx_Q + dy_Q. \tag{2.64}$$

The functions $x_Q(t), y_Q(t)$ therefore satisfy the system equations, with Q passing through B at time $t = 0$. Hence *given any value of k, $\mathcal{C}_k$ is another phase path, with Q its representative point.*

Various facts follow easily from this result.

(i) Any phase path segment spanning a sector centred on the origin determines all the rest within the sector, and in the opposite sector. A region consisting of a circle of radius r, centred on the origin, contains the same geometrical pattern of phase paths no matter what the value of r may be.

(ii) All the phase paths spanning a two-sided sector are geometrically similar. They are similarly positioned and directed if $k > 0$, and are reflected in the origin if $k < 0$.

(iii) Any half cycle of a spiral (that is any segment of angular width π) generates the complete spiral structure of the phase diagram.

(iv) All path segments spanning a two-sided radial sector are traversed by the representative points in the same time. In particular, all closed paths have the same period. All complete loops of any spiral path (that is, in a sectorial angle 2π) have the same transit time.

(v) A linear system has no limit cycles (i.e., no *isolated* closed paths).

2.7 Constructing a phase diagram

Suppose that the given system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{2.65}$$

has an equilibrium point at (x_0, y_0):

$$X(x_0, y_0 = 0, \qquad Y(x_0, y_0) = 0. \tag{2.66}$$

The pattern of phase paths close to (x_0, y_0) may be investigated by linearizing the equations at this point, retaining only linear terms of the Taylor series for X and Y there. It is simplest to use the method leading up to eqn (2.22) to obtain the coefficients. If local coordinates are defined by

$$\xi = x - x_0, \qquad \eta = y - y_0,$$

then, approximately,

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{2.67}$$

where the coefficients are given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \dfrac{\partial X}{\partial x}(x_0, y_0) & \dfrac{\partial X}{\partial y}(x_0, y_0) \\ \dfrac{\partial Y}{\partial x}(x_0, y_0) & \dfrac{\partial Y}{\partial y}(x_0, y_0) \end{bmatrix}. \tag{2.68}$$

The equilibrium point is then classified using the methods of Section 2.4. This is done for each equilibrium point in turn, and it is then possible to make a fair guess at the complete pattern of the phase paths, as in the following example.

Example 2.10 *Sketch the phase diagram for the nonlinear system*

$$\dot{x} = x - y, \qquad \dot{y} = 1 - xy. \tag{i}$$

The equilibrium points are at $(-1, -1)$ and $(1, 1)$. The matrix for linearization, to be evaluated at each equilibrium point in turn, is

$$\begin{bmatrix} \dfrac{\partial X}{\partial x} & \dfrac{\partial X}{\partial y} \\ \dfrac{\partial Y}{\partial x} & \dfrac{\partial Y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -y & -x \end{bmatrix}. \tag{ii}$$

At $(-1, -1)$ eqns (2.67) becomes

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{iii}$$

where $\xi = x + 1$, $\eta = y + 1$. The eigenvalues of the coefficient matrix are found to be λ_1, $\lambda_2 = 1 \pm \mathrm{i}$, implying an unstable spiral. To obtain the direction of rotation, it is sufficient to use the linear equations (iii) (or the original equations may be used): putting $\eta = 0$, $\xi > 0$ we find $\dot{\eta} = \xi > 0$, indicating that the rotation is *counterclockwise* as before.

At $(1, 1)$, we find that

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{iv}$$

where $\xi = x - 1$, $\eta = y - 1$. The eigenvalues are given by λ_1, $\lambda_2 = \pm\sqrt{2}$, which implies a saddle. The directions of the 'straight-line' paths from the saddle (which become curved separatrices when away from the equilibrium point), are resolved by the technique of Example 2.8: from (iv)

$$\frac{\dot{\eta}}{\dot{\xi}} = \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{-\xi - \eta}{\xi - \eta}. \tag{v}$$

We know that two solutions of this equation have the form $\eta = m\xi$ for some values of m. By substituting in (v) we obtain $m^2 - 2m - 1 = 0$, so that $m = 1 \pm \sqrt{2}$.

Finally the phase diagram is put together as in Fig. 2.12, where the phase paths in the neighbourhoods of the equilibrium points are now known. The process can be assisted by sketching in the direction fields on the lines $x = 0$, $x = 1$, etc., also on the curve $1 - xy = 0$ on which the phase paths have zero slopes, and the line $y = x$ on which the paths have infinite slopes. ●

Exercise 2.6

Find and classify the equilibrium points of

$$\dot{x} = \tfrac{1}{8}(x + y)^3 - y, \qquad \dot{y} = \tfrac{1}{8}(x + y)^3 - x.$$

Verify that lines $y = x$, $y = 2 - x$, $y = -2 - x$, are phase paths. Finally sketch the phase diagram of the system.

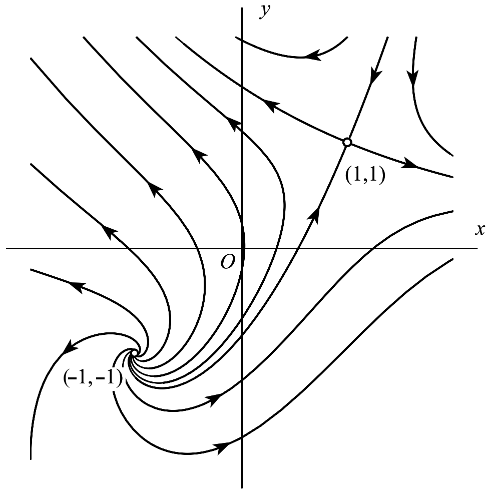

Figure 2.12 Phase diagram for $\dot{x} = x - y,\ \dot{y} = 1 - xy$.

Exercise 2.7
Classify the equilibrium points of the following systems:
(a) $\dot{x} = x^2 + y^2 - 2, \quad \dot{y} = y - x^2$;
(b) $\dot{x} = x^2 - y^2 + 1, \quad \dot{y} = y - x^2 + 5$;
Using isoclines draw a rough sketch of their phase diagrams. Compare your diagrams with computed phase diagrams.

2.8 Hamiltonian systems

By analogy with the form of Hamilton's canonical equations in mechanics, a system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{2.69}$$

is called a **Hamiltonian system** if there exists a function $H(x, y)$ such that

$$X = \frac{\partial H}{\partial y} \quad \text{and} \quad Y = \frac{\partial H}{\partial x}. \tag{2.70}$$

Then H is called the **Hamiltonian function** for the system. A necessary and sufficient condition for (2.69) to be Hamiltonian is that

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0. \tag{2.71}$$

Let $x(t), y(t)$ represent a particular time solution. Then along the corresponding phase path,

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\partial H}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial H}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= -YX + XY \quad \text{(from (2.69), (2.70))}\\ &= 0.\end{aligned}$$

Therefore,

$$H(x, y) = \text{constant} \tag{2.72}$$

along any phase path. From (2.72), the phase paths are the **level curves**, or **contours**,

$$H(x, y) = C \tag{2.73}$$

of the surface

$$z = H(x, y)$$

in three dimensions.

Suppose that the system has an equilibrium point at (x_0, y_0) so that

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0 \quad \text{at } (x_0, y_0). \tag{2.74}$$

Then $H(x, y)$ has a stationary point at (x_0, y_0). Sufficient conditions for the three main types of stationary point are given by standard theory; we condense the standard criteria as follows. Put

$$q_0 = \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right) \tag{2.75}$$

evaluated at (x_0, y_0). Then

(a) $H(x, y)$ has a maximum or minimum at (x_0, y_0) if

$$q_0 > 0; \tag{2.76}$$

(b) $H(x, y)$ has a saddle at (x_0, y_0) if

$$q_0 < 0. \tag{2.77}$$

(We shall not consider cases where $q_0 = 0$, although similar features may still be present.)

Since the phase paths are the contours of $z = H(x, y)$, we expect that in the case (2.76), the equilibrium point at (x_0, y_0) will be a centre, and that in case (2.77) it will be a saddle point.

There is no case corresponding to a node or spiral: a Hamiltonian system contains only centres and various types of saddle point.

The same prediction is obtained by linearizing the equations at the equilibrium point. From (2.46) the linear approximation at (x_0, y_0) is

$$\dot{x} = a(x - x_0) + b(y - y_0), \quad \dot{y} = c(x - x_0) + d(y - y_0), \tag{2.78}$$

where, in the Hamiltonian case, the coefficients become

$$a = \frac{\partial^2 H}{\partial x \partial y}, \quad b = \frac{\partial^2 H}{\partial y^2}, \quad c = -\frac{\partial^2 H}{\partial x^2}, \quad d = -\frac{\partial^2 H}{\partial x \partial y}, \tag{2.79}$$

all evaluated at (x_0, y_0).

The classification of the equilibrium point is determined by the values of p and q defined in eqn (2.34), and it will be seen that the parameter q is exactly the same as the parameter q_0 defined in (2.75). We have, from (2.45) and (2.79),

$$p = a + d = 0, \quad q = ad - bc = -\left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 + \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2}, \tag{2.80}$$

at (x_0, y_0) (therefore $q = q_0$, as defined in (2.75)). The conditions (2.60) for a centre are that $p = 0$ (automatically satisfied in (2.80)) and that $q > 0$. This is the same as the requirement (2.76), that H should have a maximum or minimum at (x_0, y_0).

Note that the criterion (2.80) for a **centre**, based on the complete geometrical character of $H(x, y)$, is conclusive whereas, as we have pointed out several times, the linearization criterion is not always conclusive.

If $q = 0$ maxima and minima of H still correspondent to centres, but more complicated types of saddle are possible.

Example 2.11 *For the equations*

$$\dot{x} = y(13 - x^2 - y^2), \qquad \dot{y} = 12 - x(13 - x^2 - y^2):$$

(a) show that the system is Hamiltonian and obtain the Hamiltonian function $H(x, y)$; (b) obtain the equilibrium points and classify them; (c) sketch the phase diagram.

(a) We have

$$\begin{aligned}\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} &= \frac{\partial}{\partial x}\{y(13 - x^2 - y^2)\} + \frac{\partial}{\partial y}\{12 - x(13 - x^2 - y^2)\}\\ &= -2xy + 2xy = 0.\end{aligned}$$

Therefore, by (2.71), this is a Hamiltonian system. From (2.70)

$$\frac{\partial H}{\partial x} = -Y = -12 + x(13 - x^2 - y^2), \tag{i}$$

$$\frac{\partial H}{\partial y} = X = y(13 - x^2 - y^2). \tag{ii}$$

Integrate (i) with respect to x keeping y constant, and (ii) with respect to y keeping x constant: we obtain

$$H = -12x + \tfrac{13}{2}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2y^2 + u(y), \tag{iii}$$

$$H = \tfrac{13}{2}y^2 - \tfrac{1}{2}x^2y^2 - \tfrac{1}{4}y^4 + v(x), \tag{iv}$$

respectively, where $u(y)$ and $v(x)$ are *arbitrary* functions of y and x only, but subject to the consistency of eqns (iii) and (iv). The two equations will match only if we choose

$$u(y) = \tfrac{13}{2}y^2 - \tfrac{1}{4}y^4 - C,$$
$$v(x) = -12x + \tfrac{13}{2}x^2 - \tfrac{1}{4}x^4 - C,$$

where C is any constant (to see why, subtract (iv) from (iii): the resulting expression must be identically zero). Therefore the phase paths are given by

$$H(x, y) = -12x + \tfrac{13}{2}(x^2 + y^2) - \tfrac{1}{4}(x^4 + y^4) - \tfrac{1}{4}(x^4 + y^4) - \tfrac{1}{4} - \tfrac{1}{2}x^2y^2 = C, \tag{v}$$

where C is a parameter.
(b) the equilibrium points occur where

$$y(13 - x^2 - y^2) = 0, \quad 12 - x(13 - x^2 - y^2) = 0.$$

Solutions exist only at points where $y = 0$, and

$$x^3 - 13x + 12 = (x - 1)(x - 3)(x + 4) = 0.$$

Therefore, the coordinates of the equilibrium points are

$$(1, 0), \quad (3, 0), \quad (-4, 0). \tag{vi}$$

The second derivatives of $H(x, y)$ are

$$\frac{\partial^2 H}{\partial x^2} = 13 - 3x^2 - y^2, \quad \frac{\partial^2 H}{\partial y^2} = 13 - x^2 - 3y^2, \quad \frac{\partial^2 H}{\partial x \partial y} = -2xy.$$

We need only compute q (see (2.75)) at the equilibrium points. The results are

Equilibrium points	(1, 0)	(3, 0)	(−4, 0)
Value of q	$120 > 0$	$-56 < 0$	$105 > 0$
Classification	centre	Saddle	centre

(c) A computed phase diagram is shown in Fig. 2.13. ●

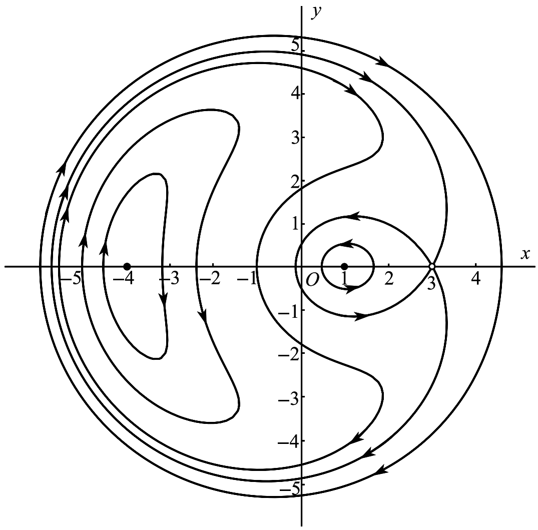

Figure 2.13 Phase diagram for the Hamiltonian system $\dot{x} = y(13 - x^2 - y^2)$, $\dot{y} = 12 - x(13 - x^2 - y^2)$.

Exercise 2.8
Shows that the system

$$\dot{x} = (x^2 - 1)(3y^2 - 1), \qquad \dot{y} = -2xy(y^2 - 1)$$

is Hamiltonian. Find the coordinates of the 8 equilibrium points. Using the obvious exact solutions and the Hamiltonian property draw a rough sketch of the phase diagram.

Problems

2.1 Sketch phase diagrams for the following linear systems and classify the equilibrium point:

(i) $\dot{x} = x - 5y, \quad \dot{y} = x - y;$
(ii) $\dot{x} = x + y, \quad \dot{y} = x - 2y;$
(iii) $\dot{x} = -4x + 2y, \quad \dot{y} = 3x - 2y;$
(iv) $\dot{x} = x + 2y, \quad \dot{y} = 2x + 2y;$
(v) $\dot{x} = 4x - 2y, \quad \dot{y} = 3x - y;$
(vi) $\dot{x} = 2x + y, \quad \dot{y} = -x + y.$

2.2 Some of the following systems either generate a single eigenvalue, or a zero eigenvalue, or in other ways vary from the types illustrated in Section 2.5. Sketch their phase diagrams.

(i) $\dot{x} = 3x - y, \quad \dot{y} = x + y;$
(ii) $\dot{x} = x - y, \quad \dot{y} = 2x - 2y;$
(iii) $\dot{x} = x, \quad \dot{y} = 2x - 3y;$
(iv) $\dot{x} = x, \quad \dot{y} = x + 3y;$
(v) $\dot{x} = -y, \quad \dot{y} = 2x - 4y;$
(vi) $\dot{x} = x, \quad \dot{y} = y;$
(vii) $\dot{x} = 0, \quad \dot{y} = x.$

2.3 Locate and classify the equilibrium points of the following systems. Sketch the phase diagrams: it will often be helpful to obtain isoclines and path directions at other points in the plane.

(i) $\dot{x} = x - y, \quad \dot{y} = x + y - 2xy;$
(ii) $\dot{x} = ye^y, \quad \dot{y} = 1 - x^2;$
(iii) $\dot{x} = 1 - xy, \quad \dot{y} = (x - 1)y;$
(iv) $\dot{x} = (1 + x - 2y)x, \quad \dot{y} = (x - 1)y;$
(v) $\dot{x} = x - y, \quad \dot{y} = x^2 - 1;$
(vi) $\dot{x} = -6y + 2xy - 8, \quad \dot{y} = y^2 - x^2;$
(vii) $\dot{x} = 4 - 4x^2 - y^2, \quad \dot{y} = 3xy;$
(viii) $\dot{x} = -y\sqrt{(1 - x^2)}, \quad \dot{y} = x\sqrt{(1 - x^2)}$ for $|x| \leq 1;$
(ix) $\dot{x} = \sin y, \quad \dot{y} = -\sin x;$
(x) $\dot{x} = \sin x \cos y, \quad \dot{y} = \sin y \cos x.$

2.4 Construct phase diagrams for the following differential equations, using the phase plane in which $y = \dot{x}$.

(i) $\ddot{x} + x - x^3 = 0;$
(ii) $\ddot{x} + x + x^3 = 0;$
(iii) $\ddot{x} + \dot{x} + x - x^3 = 0;$
(iv) $\ddot{x} + \dot{x} + x + x^3 = 0;$
(v) $\ddot{x} = (2\cos x - 1)\sin x.$

2.5 Confirm that system $\dot{x} = x - 5y,\ \dot{y} = x - y$ consists of a centre. By substitution into the equation for the paths or otherwise show that the family of ellipses given by

$$x^2 - 2xy + 5y^2 = \text{constant}$$

describes the paths. Show that the axes are inclined at about 13.3° (the major axis) and −76.7° (the minor axis) to the x direction, and that the ratio of major to minor axis length is about 2.62.

2.6 The family of curves which are orthogonal to the family described by the equation $(\mathrm{d}y/\mathrm{d}x) = f(x, y)$ is given by the solution of $(\mathrm{d}y/\mathrm{d}x) = -[1/f(x, y)]$. (These are called orthogonal trajectories of the first family.) Prove that the family which is orthogonal to a centre that is associated with a linear system is a node.

2.7 Show that the origin is a spiral point of the system $\dot{x} = -y - x\sqrt{(x^2 + y^2)}$, $\dot{y} = x - y\sqrt{(x^2 + y^2)}$ but a centre for its linear approximation.

2.8 Show that the systems $\dot{x} = y,\ \dot{y} = -x - y^2$, and $\dot{x} = x + y_1,\ \dot{y}_1 = -2x - y_1 - (x + y_1)^2$, both represent the equation $\ddot{x} + \dot{x}^2 + x = 0$ in different (x, y) and (x, y_1) phase planes. Obtain the equation of the phase paths in each case.

2.9 Use eqn (2.9) in the form $\delta s \simeq \delta t\sqrt{(X^2 + Y^2)}$ to mark off approximately equal time steps on some of the phase paths of $\dot{x} = xy,\ \dot{y} = xy - y^2$.

2.10 Obtain approximations to the phase paths described by eqn (2.12) in the neighbourhood of the equilibrium point $x = b/d, y = a/c$ for the predator-prey problem $\dot{x} = ax - cxy,\ \dot{y} = -by + dxy$ (see Example 2.3). (Write $x = b/d + \xi, y = a/c + \eta$, and expand the logarithms to second-order terms in ξ and η.)

2.11 For the system $\dot{x} = ax + by, \dot{y} = cx + dy$, where $ad - bc = 0$, show that all points on the line $cx + dy = 0$ are equilibrium points. Sketch the phase diagram for the system $\dot{x} = x - 2y, \dot{y} = 2x - 4y$.

2.12 The interaction between two species is governed by the deterministic model $\dot{H} = (a_1 - b_1 H - c_1 P)H$, $\dot{P} = (-a_2 + c_2 H)P$, where H is the population of the host (or prey), and P is that of the parasite (or predator), all constants being positive. (Compare Example 2.3: the term $-b_1 H^2$ represents interference with the host population when it gets too large.) Assuming that $a_1 c_2 - b_1 a_2 > 0$, find the equilibrium states for the populations, and find how they vary with time from various initial populations.

2.13 With the same terminology as in Problem 2.12, analyse the system $\dot{H} = (a_1 - b_1 H - c_1 P)H, \dot{P} = (a_2 - b_2 P + c_2 H)P$, all the constants being positive. (In this model the parasite can survive on alternative food supplies, although the prevalence of the host encourages growth in population.) Find the equilibrium states. Confirm that the parasite population can persist even if the host dies out.

2.14 Consider the host–parasite population model $\dot{H} = (a_1 - c_1 P)H, \dot{P} = (a_2 - c_2(P/H))P$, where the constants are positive. Analyse the system in the H, P plane.

2.15 In the population model $\dot{F} = -\alpha F + \beta\mu(M)F, \dot{M} = -\alpha M + \gamma\mu(M)F$, where $\alpha > 0, \beta > 0, \gamma > 0$, F and M are the female and male populations. In both cases the death rates are α. The birth rate is governed by the coefficient $\mu(M) = 1 - e^{-kM}$, $k > 0$, so that for large M the birth rate of females is βF and that for males is γF, the rates being unequal in general. Show that if $\beta > \alpha$ then there are two equilibrium states, at $(0, 0)$ and at $([-\beta/(\gamma k)]\log[(\beta - \alpha)/\beta], [-1/k]\log[(\beta - \alpha)/\beta])$.

Show that the origin is stable and that the other equilibrium point is a saddle point, according to their linear approximations. Verify that $M = \gamma F/\beta$ is a particular solution. Sketch the phase diagram and discuss the stability of the populations.

2.16 A rumour spreads through a closed population of constant size $N + 1$. At time t the total population can be classified into three categories:

x persons who are ignorant of the rumour;

y persons who are actively spreading the rumour;

z persons who have heard the rumour but have stopped spreading it: if two persons who are spreading the rumour meet then they stop spreading it.

The contact rate between any two categories is a constant, μ.

Show that the equations

$$\dot{x} = -\mu xy, \qquad \dot{y} = \mu[xy - y(y - 1) - yz]$$

give a deterministic model of the problem. Find the equations of the phase paths and sketch the phase diagram.

Show that, when initially $y = 1$ and $x = N$, the number of people who ultimately never hear the rumour is x_1, where

$$2N + 1 - 2x_1 + N\log(x_1/N) = 0.$$

2.17 The one-dimensional steady flow of a gas with viscosity and heat conduction satisfies the equations

$$\frac{\mu_0}{\rho c_1}\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{(2v)}[2v - \sqrt{(2v)} + \theta],$$

$$\frac{k}{gR\rho c_1}\frac{d\theta}{dx} = \sqrt{(2v)}\left[\frac{\theta}{\gamma - 1} - v + \sqrt{(2v)} - c\right],$$

where $v = u^2/(2c_1^2)$, $c = c_2^2/c_1^2$ and $\theta = gRT/c_1^2 = p/(\rho c_1^2)$. In this notation, x is measured in the direction of flow, u is the velocity, T is the temperature, ρ is the density, p the pressure, R the gas constant, k the coefficient of thermal conductivity, μ_0 the coefficient of viscosity, γ the ratio of the specific heats, and c_1, c_2 are arbitrary constants. Find the equilibrium states of the system.

2.18 A particle moves under a central attractive force γ/r^α per unit mass, where r, θ are the polar coordinates of the particle in its plane of motion. Show that

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\gamma}{h^2}u^{\alpha-2},$$

where $u = r^{-1}$, h is the angular momentum about the origin per unit mass of the particle, and γ is a constant. Find the non-trival equilibrium point in the $u, \mathrm{d}u/\mathrm{d}\theta$ plane and classify it according to its linear approximation. What can you say about the stability of the circular orbit under this central force?

2.19 The relativistic equation for the central orbit of a planet is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = k + \varepsilon u^2,$$

where $u = 1/r$, and $r,\ \theta$ are the polar coordinates of the planet in the plane of its motion. The term εu^2 is the 'Einstein correction', and k and ε are positive constants, with ε very small. Find the equilibrium point which corresponds to a perturbation of the Newtonian orbit. Show that the equilibrium point is a centre in the $u, \mathrm{d}u/\mathrm{d}\theta$ plane according to the linear approximation. Confirm this by using the potential energy method of Section 1.3.

2.20 A top is set spinning at an axial rate n radians / sec about its pivotal point, which is fixed in space. The equations for its motion, in terms of the angles θ and μ are (see Fig. 2.14)

$$A\dot{\theta} - A(\Omega + \dot{\mu})^2 \sin\theta\cos\theta + Cn(\Omega + \dot{\mu})\sin\theta - Mgh\sin\theta = 0,$$

$$A\dot{\theta}^2 + A(\Omega + \dot{\mu})^2 \sin^2\theta + 2Mgh\cos\theta = E;$$

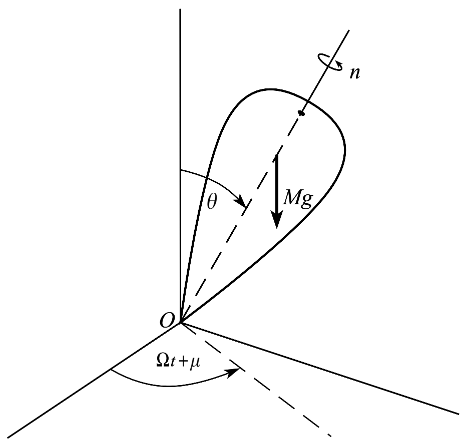

Figure 2.14 Spinning top.

where (A, A, C) are the principal moments of inertia about O, M is the mass of the top, h is the distance between the mass centre and the pivot, and E is a constant. Show that an equilibrium state is given by $\theta = \alpha$, after elimination of Ω between

$$A\Omega^2\cos\alpha - Cn\Omega + Mgh = 0, \quad A\Omega^2\sin^2\alpha + 2Mgh\ \cos\alpha = E.$$

Suppose that $E = 2Mgh$, so that $\theta = 0$ is an equilibrium state. Show that, close to this state, θ satisfies

$$A\ddot{\theta} + [(C - A)\Omega^2 - Mgh]\theta = 0.$$

For what condition on Ω is the motion stable?

2.21 Three freely gravitating particles with gravitational masses $\mu_1,\ \mu_2,\ \mu_3$, move in a plane so that they always remain at the vertices of an equilateral triangle $P_1,\ P_2,\ P_3$ with varying side-length $a(t)$ as shown in Figure 2.15. The triangle rotates in the plane with spin $\Omega(t)$ about the combined mass-centre G. If the position vectors of the particles $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, relative to G, show that the equations of motion are

$$\ddot{\mathbf{r}}_i = -\frac{\mu_1 + \mu_2 + \mu_3}{a^3}\mathbf{r}_i \quad (i = 1, 2, 3).$$

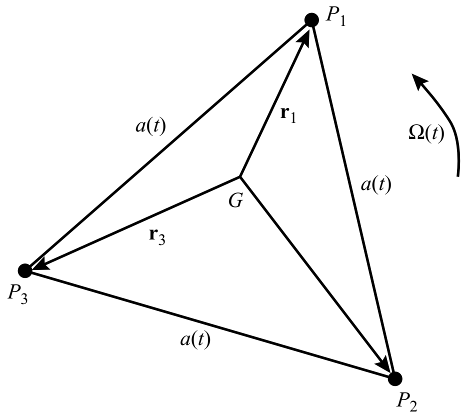

Figure 2.15 Lagrange equilateral configuration for a three-body problem with $P_1P_2 = P_2P_3 = P_3P_1 = a(t)$.

If $|\mathbf{r}_i| = r_i$, deduce the polar equations

$$\ddot{r}_i - r_i\Omega^2 = -\frac{\mu_1 + \mu_2 + \mu_3}{a^3} r_i, \quad r_i^2\Omega = \text{constant} \qquad (i = 1, 2, 3).$$

Explain why a satisfies

$$\ddot{a} - a\Omega^2 = -\frac{\mu_1 + \mu_2 + \mu_3}{a^2}, \; a^2\Omega = \text{constant} = K,$$

say, and that solutions of these equations completely determine the position vectors. Express the equation in non-dimensionless form by the substitutions $a = K^2/(\mu_1 + \mu_2 + \mu_3)$, $t = K^3\tau/(\mu_1+\mu_2+\mu_3)^2$, sketch the phase diagram for the equation in μ obtained by eliminating Ω, and discuss possible motions of this Lagrange configuration.

2.22 A disc of radius a is freely pivoted at its centre A so that it can turn in a vertical plane. A spring, of natural length $2a$ and stiffness λ connects a point B on the circumference of the disc to a fixed point O, distance $2a$ above A. Show that θ satisfies

$$I\ddot{\theta} = -Ta\sin\phi, \qquad T = \lambda a[(5 - 4\cos\theta)^{1/2} - 2],$$

where T is the tension in the spring, I is the moment of inertia of the disc about A, $\widehat{OAB} = \theta$ and $\widehat{ABO} = \phi$. Find the equilibrium states of the disc and their stability.

2.23 A man rows a boat across a river of width a occupying the strip $0 \le x \le a$ in the x, y plane, always rowing towards a fixed point on one bank, say (0, 0). He rows at a constant speed u relative to the water, and the river flows at a constant speed v. Show that

$$\dot{x} = -ux/\sqrt{(x^2 + y^2)}, \quad \dot{y} = v - uy/\sqrt{(x^2 + y^2)},$$

where (x, y) are the coordinates of the boat. Show that the phase paths are given by $y + \sqrt{(x^2 + y^2)} = Ax^{1-\alpha}$, where $\alpha = v/u$. Sketch the phase diagram for $\alpha < 1$ and interpret it. What kind of point is the origin? What happens to the boat if $\alpha > 1$?

2.24 In a simple model of a national economy, $\dot{I} = I - \alpha C$, $\dot{C} = \beta(I - C - G)$, where I is the national income, C is the rate of consumer spending, and G the rate of government expenditure; the constants α and β satisfy $1 < \alpha < \infty$, $1 \le \beta < \infty$. Show that if the rate of government expenditure G is constant G_0 there is an equilibrium state. Classify the equilibrium state and show that the economy oscillates when $\beta = 1$.

Consider the situation when government expenditure is related to the national income by the rule $G = G_0 + kI$, where $k > 0$. Show that there is no equilibrium state if $k \geq (\alpha - 1)/\alpha$. How does the economy then behave?

Discuss an economy in which $G = G_0 + kI^2$, and show that there are two equilibrium states if $G_0 < (\alpha - 1)^2/(4k\alpha^2)$.

2.25 Let $f(x)$ and $g(y)$ have local minima at $x = a$ and $y = b$ respectively. Show that $f(x) + g(y)$ has a minimum at (a, b). Deduce that there exists a neighbourhood of (a, b) in which all solutions of the family of equations

$$f(x) + g(y) = \text{constant}$$

represent closed curves surrounding (a, b).

Show that $(0, 0)$ is a centre for the system $\dot{x} = y^5$, $\dot{y} = -x^3$, and that all paths are closed curves.

2.26 For the predator–prey problem in Section 2.2, show by using Problem 2.25 that all solutions in $y > 0$, $x > 0$ are periodic.

2.27 Show that the phase paths of the Hamiltonian system $\dot{x} = -\partial H/\partial y$, $\dot{y} = \partial H/\partial x$ are given by $H(x, y) = constant$. Equilibrium points occur at the stationary points of $H(x, y)$. If (x_0, y_0) is an equilibrium point, show that (x_0, y_0) is stable according to the linear approximation if $H(x, y)$ has a maximum or a minimum at the point. (Assume that all the second derivatives of H are nonzero at x_0, y_0.)

2.28 The equilibrium points of the nonlinear parameter-dependent system $\dot{x} = y$, $\dot{y} = f(x, y, \lambda)$ lie on the curve $f(x, 0, \lambda) = 0$ in the x, λ plane. Show that an equilibrium point (x_1, λ_1) is stable and that all neighbouring solutions tend to this point (according to the linear approximation) if $f_x(x_1, 0, \lambda_1) < 0$ and $f_y(x_1, 0, \lambda_1) < 0$.

Investigate the stability of $\dot{x} = y$, $\dot{y} = -y + x^2 - \lambda x$.

2.29 Find the equations for the phase paths for the general epidemic described (Section 2.2) by the system

$$\dot{x} = -\beta xy, \qquad \dot{y} = \beta xy - \gamma y, \qquad \dot{z} = \gamma y.$$

Sketch the phase diagram in the x, y plane. Confirm that the number of infectives reaches its maximum when $x = \gamma/\beta$.

2.30 Two species x and y are competing for a common food supply. Their growth equations are

$$\dot{x} = x(1 - x - y), \quad \dot{y} = y(3 - x - \tfrac{3}{2}y), \qquad (x, y > 0).$$

Classify the equilibrium points using linear approximations. Draw a sketch indicating the slopes of the phase paths in $x \geq 0$, $y \geq 0$. If $x = x_0 > 0$, $y = y_0 > 0$ initially, what do you expect the long term outcome of the species to be? Confirm your conclusions numerically by computing phase paths.

2.31 Sketch the phase diagram for the competing species x and y for which

$$\dot{x} = (1 - x^2 - y^2)x, \qquad \dot{y} = (\tfrac{5}{4} - x - y)y.$$

2.32 A space satellite is in free flight on the line joining, and between, a planet (mass m_1) and its moon (mass m_2), which are at a fixed distance a apart. Show that

$$-\frac{\gamma m_1}{x^2} + \frac{\gamma m_2}{(a - x)^2} = \ddot{x},$$

where x is the distance of the satellite from the planet and γ is the gravitational constant. Show that the equilibrium point is unstable according to the linear approximation.

2.33 The system

$$\dot{V}_1 = -\sigma V_1 + f(E - V_2), \quad \dot{V}_2 = -\sigma V_2 + f(E - V_1), \qquad \sigma > 0, E > 0$$

represents (Andronov and Chaikin 1949) a model of a triggered sweeping circuit for an oscilloscope. The conditions on $f(u)$ are: $f(u)$ continuous on $-\infty < u < \infty$, $f(-u) = -f(u)$, $f(u)$ tends to a limit as $u \to \infty$, and $f'(u)$ is monotonic decreasing (see Fig. 3.20).

Show by a geometrical argument that there is always at least one equilibrium point, (v_0, v_0) say, and that when $f'(E - v_0) < \sigma$ it is the only one; and deduce by taking the linear approximation that it is a stable node. (Note that $f'(E - v) = -\,\mathrm{d}f(E - v)/\mathrm{d}v$.)

Show that when $f'(E - v_0) > \sigma$ there are two others, at $(V', (1/\sigma)f(E - V'))$ and $((1/\sigma)f(E - V'), V')$ respectively for some V'. Show that these are stable nodes, and that the one at (v_0, v_0) is a saddle point.

2.34 Investigate the equilibrium points of $\dot{x} = a - x^2$, $\dot{y} = x - y$. Show that the system has a saddle and a stable node for $a > 0$, but no equilibrium points if $a < 0$. The system is said to undergo a **bifurcation** as a increases through $a = 0$. This bifurcation is an example of a **saddle-node bifurcation**. This will be discussed in more detail in Section 12.4. Draw phase diagrams for $a = 1$ and $a = -1$.

2.35 Figure 2.16 represents a circuit for activating an electric arc A which has the voltage–current characteristic shown. Show that $L\dot{I} = V - V_a(I)$, $RC\dot{V} = -RI - V + E$ where $V_a(I)$ has the general shape shown in Fig. 2.16. By forming the linear approximating equations near the equilibrium points find the conditions on E, L, C, R, and V_a' for stable working assuming that $V = E - RI$ meets the curve $V = V_a(I)$ in three points of intersection.

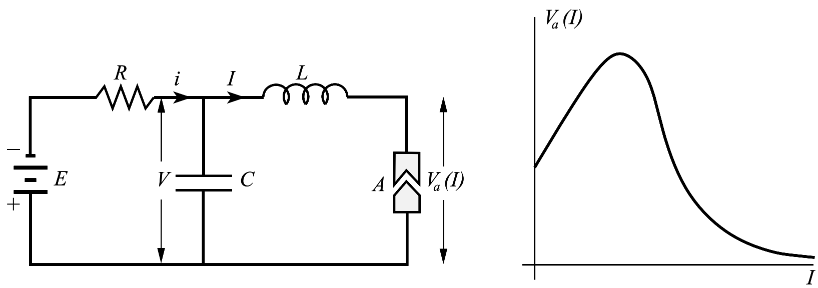

Figure 2.16

2.36 The equation for the current x in the circuit of Fig. 2.17(a) is

$$LC\ddot{x} + RC\dot{x} + x = I.$$

Neglect the grid current, and assume that I depends only on the relative grid potential e_g : $I = I_s$ (saturation current) for $e_g > 0$ and $I = 0$ for $e_g < 0$ (see Fig. 2.17(b)). Assume also that the mutual inductance $M > 0$, so that $e_g \gtrless 0$ according as $\dot{x} \gtrless 0$. Find the nature of the phase paths. By considering their successive intersections with the x axis show that a limit cycle is approached from all initial conditions (Assume $R^2C < 4L$).

2.37 For the circuit in Fig. 2.17(a) assume that the relation between I and e_g is as in Fig. 2.18; that is $I = f(e_g + ke_p)$, where e_g and e_p are the relative grid and plate potentials, $k > 0$ is a constant, and in the neighbourhood of the point of inflection, $f(u) = I_0 + au - bu^3$, where $a > 0$, $b > 0$. Deduce the equation for x when the D.C. source E is set so that the operating point is the point of inflection. Find when the origin is a stable or an unstable point of equilibrium. (A form of Rayleigh's equation, Example 4.6, is obtained, implying an unstable or a stable limit cycle respectively.)

2.38 Figure 2.19(a) represents two identical D.C. generators connected in parallel, with inductance and resistance L, r. Here R is the resistance of the load. Show that the equations for the currents are

$$L\frac{\mathrm{d}i_1}{\mathrm{d}t} = -(r + R)i_1 - Ri_2 + E(i_1),\; L\frac{\mathrm{d}i_2}{\mathrm{d}t} = -Ri_1 - (r + R)i_2 + E(i_2).$$

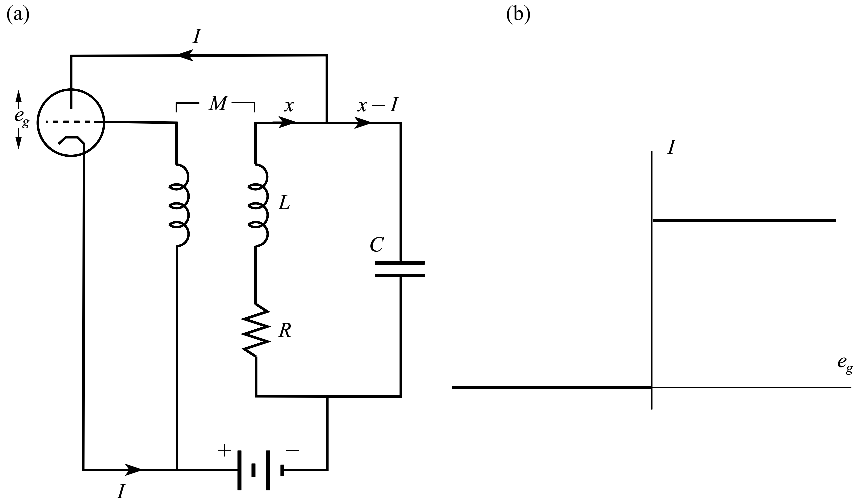

Figure 2.17

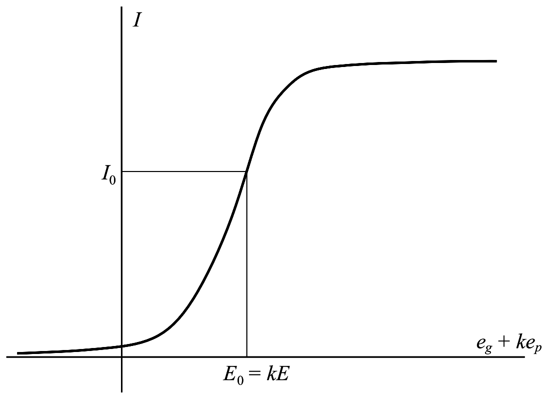

Figure 2.18

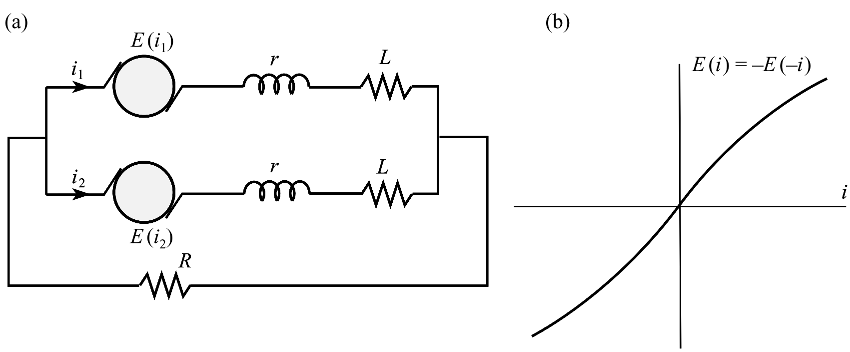

Figure 2.19

Assuming that $E(i)$ has the characteristics indicated by Fig. 2.19(b) show that
(i) when $E'(0) < r$ the state $i_1 = i_2 = 0$ is stable and is otherwise unstable;
(ii) when $E'(0) < r$ there is a stable state $i_1 = -i_2$ (no current flows to R);
(iii) when $E'(0) > r + 2R$ there is a state with $i_1 = i_2$, which is unstable.

2.39 Show that the Emden–Fowler equation of astrophysics

$$(\xi^2\eta')' + \xi^\lambda\eta^n = 0$$

is equivalent to the predator–prey model

$$\dot{x} = -x(1 + x + y), \qquad \dot{y} = y(\lambda + 1 + nx + y)$$

after the change of variable $x = \xi\eta'/\eta$, $y = \xi^{\lambda-1}\eta^n/\eta'$, $t = \log|\xi|$.

2.40 Show that Blasius' equation $\eta''' + \eta\eta'' = 0$ is transformed by $x = \eta\eta'/\eta''$, $y = \eta'^2/\eta\eta''$, $t = \log|\eta'|$ into

$$\dot{x} = x(1 + x + y), \qquad \dot{y} = y(2 + x - y).$$

2.41 Consider the family of linear systems

$$\dot{x} = X\cos\alpha - Y\sin\alpha, \qquad \dot{y} = X\sin\alpha + Y\cos\alpha$$

where $X = ax + by$, $Y = cx + dy$, and a, b, c, d are constants and α is a parameter. Show that the parameters (Table (2.62)) are

$$p = (a + d)\cos\alpha + (b - c)\sin\alpha, \quad q = ad - bc.$$

Deduce that the origin is a saddle point for all α if $ad < bc$.

If $a = 2$, $b = c = d = 1$, show that the origin passes through the sequence stable node, stable spiral, centre, unstable spiral, unstable node, as α varies over range π.

2.42 Show that, given $X(x, y)$, the system equivalent to the equation $\ddot{x} + h(x, \dot{x}) = 0$ is

$$\dot{x} = X(x, y), \qquad \dot{y} = -\left\{h(x, X) + x\frac{\partial X}{\partial x}\right\}\bigg/\frac{\partial X}{\partial y}.$$

2.43 The following system models two species with populations N_1 and N_2 competing for a common food supply:

$$\dot{N}_1 = \{a_1 - d_1(bN_1 + cN_2)\}N_1, \qquad \dot{N}_2 = \{a_2 - d_2(bN_1 + cN_2)\}N_2.$$

Classify the equilibrium points of the system assuming that all coefficients are positive. Show that if $a_1d_2 > a_2d_1$ then the species N_2 dies out and the species N_1 approaches a limiting size (Volterra's Exclusion Principle).

2.44 Show that the system

$$\dot{x} = X(x, y) = -x + y, \qquad \dot{y} = Y(x, y) = \frac{4x^2}{1 + 3x^2} - y$$

has three equilibrium points at (0,0), $(\frac{1}{3}, \frac{1}{3})$ and (1, 1). Classify each equilibrium point. Sketch the isoclines $X(x, y) = 0$ and $Y(x, y) = 0$, and indicate the regions where $\mathrm{d}y/\mathrm{d}x$ is positive, and where $\mathrm{d}y/\mathrm{d}x$ is negative. Sketch the phase diagram of the system.

2.45 Show that the systems (A) $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ and (B) $\dot{x} = Q(x, y)$, $\dot{y} = P(x, y)$ have the same equilibrium points. Suppose that system (A) has three equilibrium points which, according to their linear approximations are, (a) a stable spiral, (b) an unstable node, (c) a saddle point. To what extent can the equilibrium points in (B) be classified from this information?

2.46 The system defined by the equations

$$\dot{x} = a + x^2 y - (1+b)x, \quad \dot{y} = bx - yx^2 \qquad (a \neq 0, b \neq 0)$$

is known as the **Brusselator** and arises in a mathematical model of a chemical reaction (see Jackson (1990)). Show that the system has one equilibrium point at $(a, b/a)$. Classify the equilibrium point in each of the following cases:

(a) $a = 1, b = 2$;

(b) $a = \frac{1}{2}, b = \frac{1}{4}$.

In case (b) draw the isoclines of zero and infinite slope in the phase diagram. Hence sketch the phase diagram.

2.47 A Volterra model for the population size $p(t)$ of a species is, in reduced form,

$$\kappa \frac{\mathrm{d}p}{\mathrm{d}t} = p - p^2 - p \int_0^t p(s)\,\mathrm{d}s, \quad p(0) = p_0,$$

where the integral term represents a toxicity accumulation term (see Small (1989)). Let $x = \log p$, and show that x satisfies

$$\kappa \ddot{x} + \mathrm{e}^x \dot{x} + \mathrm{e}^x = 0.$$

Put $y = \dot{x}$, and show that the system is also equivalent to

$$\dot{y} = -(y+1)p/\kappa, \quad \dot{p} = yp.$$

Sketch the phase diagram in the (y, p) plane. Also find the exact equation of the phase paths.