1. Lectures 17, 18. Nov. 9,10

Plan:

- ⋄ Modeling by differential equations: oscillations, waves, diffusion, reactions, steady states.
- ⋄ Variational principles. Euler-Lagrange equations.
- ⋄ Linear vs nonlinear equations. Linear systems.
- ⋄ General stability discussion.
- ⋄ The role of nonlinear terms.

1.1. General overview of applied ODE and PDE

What kinds of ODE/PDE arise in applications and what are their basic properties?

1.1.1. *ODE and their phenomena.*

- ⋄ An ODE is a relationship between the independent variable x, the unknown $u(x)$ and its derivatives u', u'', etc.
$$f\left(x, u, u', u'', \ldots\right) = 0.$$
- ⋄ First and second order ODE are most frequent, e.g.:
$$u' = f(x, u)$$
$$u'' = f\left(x, u, u'\right).$$
- ⋄ A second order ODE can be written as a first order system of two ODE:
$$u' = v$$
$$v' = f(x, u, v).$$
Similarly, any high order ODE can be written as a first order system.

Some representative and frequently arising ODE are:

- ⋄ Damped linear oscillator
$$\ddot{u} + \nu\dot{u} + \omega_0^2 u = 0.$$
- ⋄ Nonlinear oscillator
$$\ddot{u} + \omega_0^2 u + a u^3 = 0.$$
- ⋄ Forced damped oscillator
$$\ddot{u} + \nu\dot{u} + \omega_0^2 u = f_0 \cos(\omega t).$$
- ⋄ Parametric oscillator
$$\ddot{u} + (\alpha + \beta\cos(t))\, u = 0. \quad \text{Mathieu equation}$$
- ⋄ Logistic model
$$\dot{u} = au - bu^2, \quad a, b > 0$$
- ⋄ Predator-prey systems (Volterra model)
$$\dot{u} = au - buv, \quad \text{prey}$$
$$\dot{v} = -cu + duv, \quad \text{predator}$$
- ⋄ Coupled linear oscillators
$$M\ddot{u} = -Ku, \quad u \in \mathbb{R}^n, \quad K = A^T C A \quad s.p.d.$$

Main phenomena:

- ⋄ Equilibrium states – stable or unstable

- ⋄ Multiple equilibria – bifurcations from one to multiple solutions
- ⋄ Oscillations – periodic, aperiodic
- ⋄ Existence of limit cycles
- ⋄ Chaotic attractors

1.1.2. *PDE and their phenomena.* PDE relate the unknown function $u(x,t)$ of two or more independent variables (x,t) with the partial derivatives of u: u_t, u_{tt},..., u_x, u_{xx}, u_{xt}, ...

Common basic PDE:

- ⋄ Heat/diffusion equation

$$u_t = D\Delta u, \quad D > 0, \text{ the diffusion coefficient}$$

- ⋄ Wave equation

$$u_{tt} = c^2 \Delta u, \quad c \text{ the wave speed}$$

- ⋄ Laplace's equation

$$\Delta u = 0, \quad \text{equilibrium state}$$

- ⋄ Poisson's equation

$$\Delta u = f(x), \text{ forced equilibrium state}$$

- ⋄ Burgers equation (nonlinear advection-diffusion equation)

$$u_t + u u_x = \nu u_{xx}, \quad \nu \text{ the viscosity/diffusion coefficient.}$$

- ⋄ Korteweg-de Vries equation of solitons

$$u_t + u u_x + \nu u_{xxx} = 0.$$

More general PDE from physics and applied sciences:

- ⋄ Shrödinger's equation

$$i\hbar u_t = -\frac{\hbar^2}{2m}\Delta u + U(x)\,u.$$

- ⋄ Incompressible Navier-Stokes equations

$$\begin{aligned} \nabla \cdot u &= 0, && \text{incompressibility} \\ u_t + u \cdot \nabla u &= -\frac{1}{\rho}\nabla p + \nu \Delta u, && \text{equation of motion} \end{aligned}$$

- ⋄ Compressible Euler equations

$$\begin{aligned} \rho_t + u \cdot \nabla \rho &= -\rho \nabla \cdot u \\ u_t + u \cdot \nabla u &= -\frac{1}{\rho}\nabla p \\ e_t + u \cdot \nabla e &= -\frac{p}{\rho}\nabla \cdot u, \end{aligned}$$

  where $e = e(p,\rho)$ is the equation of state, for example, $e = pv/(\gamma - 1) = \frac{1}{\gamma-1}\frac{p}{\rho}$ for the ideal gas with specific heat ratio γ. Then the system is closed.

- ⋄ Maxwell's equations

$$\begin{aligned} \nabla \cdot E &= \rho/\epsilon_0 \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \times B &= \mu_0 \left(j + \epsilon_0 \frac{\partial E}{\partial t} \right) \end{aligned}$$

◇ Linear elasticity

$$
\begin{aligned}
\rho\ddot{u} &= \nabla\cdot\sigma + \mathbf{f} \\
\sigma &= 2\mu E + \lambda\,(\mathrm{Tr}E)\,I \\
E &= \frac{1}{2}\left(\nabla u + \nabla u^T\right) \\
\rho\ddot{u} &= \mu\Delta u + (\mu+\lambda)\,\nabla\,(\nabla\cdot u) + f.
\end{aligned}
$$

◇ Reaction-diffusion systems in population dynamics or chemically reacting flows

$$u_t = D\Delta u + f\,(u)\,.$$

Main phenomena:

◇ All the phenomena of ODE and
- Traveling wave solutions – fronts, solitons, kinks, etc.
- Pattern formation

1.2. **Nonlinear ODE**

1.2.1. *Variational principles. Euler-Lagrange equations.* Recall that if a dynamical system has a Lagrangian

$$L = T\,(q,\dot{q},t) - U\,(q,t)$$

with kinetic energy T and potential energy U, then the least action principle

$$S = \int_{t_1}^{t_2} L dt = min, \quad \delta S = 0,$$

results in the Euler-Lagrange equations of mechanics

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

Note that the generalized coordinate q here may be a vector, in which case there are as many equations as there are degrees of freedom.

In the simplest case of a single particle of mass m in a potential field $U\,(q)$, we get

$$L = \frac{m\dot{q}^2}{2} - U\,(q)$$

and so

$$m\ddot{q} = -\frac{\partial U}{\partial q} = F$$

is the Newton's second law.

Example 1. Consider a pendulum of length l with a mass m whose point of support oscillates vertically with frequency ω and amplitude a according to $y_0 = a\cos\omega t$.

Let y axis point downward and x to the right. The coordinates of the mass are given then by

$$
\begin{aligned}
x &= l\sin\phi \\
y &= l\cos\phi + a\cos\omega t.
\end{aligned}
$$

The kinetic energy of the mass is

$$T = \frac{m\left(\dot{x}^2+\dot{y}^2\right)}{2} = \frac{m}{2}\left(l^2\dot{\phi}^2\cos^2\phi + l^2\dot{\phi}^2\sin^2\phi + 2la\dot{\phi}\omega\sin\phi\sin\omega t + a^2\omega^2\sin^2\omega t\right),$$

while the potential energy is

$$U = -mgl\cos\phi + mga\cos\omega t.$$

Keeping in mind that we can remove any full derivative in time from the Lagrangian, the final Lagrangian becomes

$$L = \frac{m}{2}\left(l^2\dot{\phi}^2 + 2la\dot{\phi}\omega\sin\phi\sin\omega t\right) + mgl\cos\phi.$$

Then the equations of motion are

$$\frac{d}{dt}\left(ml^2\dot{\phi} + mla\omega\sin\omega t\sin\phi\right) - mla\dot{\phi}\omega\cos\phi\sin\omega t + mgl\sin\phi = 0$$
$$ml^2\ddot{\phi} + mla\omega^2\cos\omega t\sin\phi + mgl\sin\phi = 0.$$

Hence

$$\ddot{\phi} + \left(\omega_0^2 + \frac{a\omega^2}{l}\cos\omega t\right)\sin\phi = 0$$

where

$$\omega_0 = \sqrt{\frac{g}{l}}$$

is the natural frequency of the pendulum.

Now, if we let $\omega t = \tau$, upon dividing by ω^2, this equation becomes

$$\frac{d^2\phi}{d\tau^2} + \left(\frac{\omega_0^2}{\omega^2} + \frac{a}{l}\cos\tau\right)\sin\phi = 0.$$

Let

$$\alpha = \frac{\omega_0^2}{\omega^2}, \quad \beta = \frac{a}{l},$$

and assume further small oscillations, so that $\sin\phi \approx \phi$, then we obtain the Mathieu equation

$$\ddot{\phi} + (\alpha + \beta\cos\tau)\,\phi = 0.$$

It describes the phenomenon of *parametric resonance*. One interesting fact about such an equation is that it is exactly the same as Bloch's equation in the solid state physics, which describes electronic states in a periodic lattice of atoms. Only in the Bloch equation $\ddot{u}$ is the spatial derivative of the wave function and $\omega(t)$ is the spatially-periodic potential of the lattice. But mathematically, the equations are the same and therefore their solutions are the same. We will look at some of the properties of the solutions of Mathieu's equation later.

Matlab calculations: see what happens numerically when $\beta = 1$ and α varies from 0 to 2.

More generally, an equation of the type

$$\ddot{u} + \omega(t)\,u = 0$$

with a periodic $\omega(t)$ is called *Hill's equation*.

Example 2. If we consider a double pendulum, then its Lagrangian is

$$L = \frac{m_1+m_2}{2}l_1^2\dot{\phi}_1^2 + \frac{m_2}{2}l_2^2\dot{\phi}_2^2 + m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) +$$
$$+ (m_1+m_2)\,gl_1\cos\phi_1 + m_2gl_2\cos\phi_2.$$

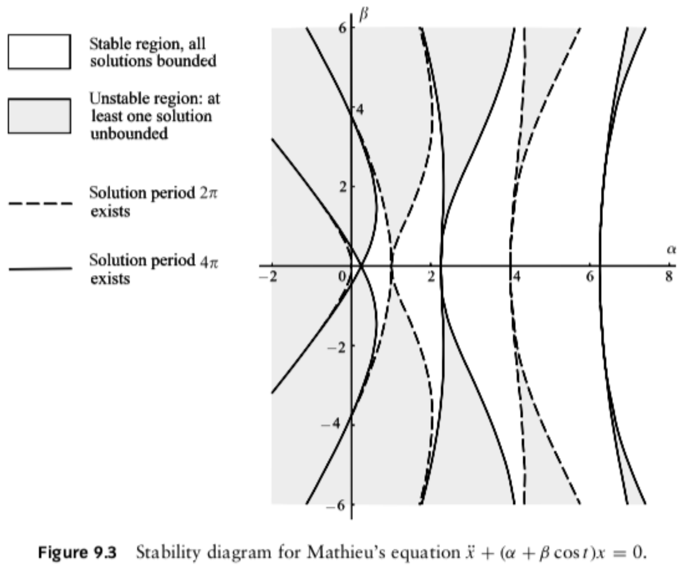

FIGURE 1.1. Stability diagram for Mathieu equation at various parameter α, β (Jordan and Smith).

The equations of motion are

$$\ddot{\phi}_1 + \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}\left[\cos(\phi_1-\phi_2)\,\ddot{\phi}_2 + \sin(\phi_1-\phi_2)\,\dot{\phi}_2^2\right] + \frac{g}{l_1}\sin\phi_1 = 0,$$
$$\ddot{\phi}_2 + \frac{l_1}{l_2}\left[\cos(\phi_1-\phi_2)\,\ddot{\phi}_1 - \sin(\phi_1-\phi_2)\,\dot{\phi}_1^2\right] + \frac{g}{l_2}\sin\phi_2 = 0.$$

This is a nice example of a system with complicated, sometimes chaotic, motion.

1.2.2. *Nonlinear oscillators.* For example, the basic *nonlinear pendulum* is described by the equation

$$\ddot{\phi} + \omega_0^2\sin\phi = 0.$$

Let

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix}.$$

Then

$$\dot{u} = F(u), \quad F = \begin{bmatrix} u_2 \\ -\omega_0^2\sin u_1 \end{bmatrix}.$$

The infinitely many fixed points of the system are $u_2 = 0$ and $u_1 = n\pi$, $n = \mathbb{Z}$.

Weakly nonlinear oscillators are described by an approximation $\sin\phi \approx \phi - \frac{1}{3}\phi^3$. Then the oscillator equation becomes, neglecting higher order terms and also absorbing ω_0 into time

$$\ddot{\phi} + \phi - \frac{1}{3}\phi^3 = 0.$$

This is a version of the *Duffing equation.*

1.3. **Behavior of linear ODE**....... Now we look at the very basic system of ODE on a plane

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy,$$

or written as a system

$$\frac{du}{dt} = Au, \quad u = \begin{bmatrix} x \\ y \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Clearly, the fixed point is $u = u_0 = 0$, i.e. $x = y = 0$. The main question is: What happens to the solution that begins in some neighborhood of u_0? The answer will depend on A as well as u_0, and this is what we will figure out next.

Basically, options depend on the eigenvalues and eigenvectors of A, whether they are real or not, distinct signs or not, two eigenvectors or one.

Example 3. Let $A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$. Then $\dot{x} = ax$ and $\dot{y} = -y$, which is solved by

$$x = x_0 e^{at}, \quad y = y_0 e^{-t}$$

Then we obtain the following behavior depending on a:

- $a < -1$ - stable node. The solution tends to 0 as $t \to \infty$, but faster in x than in y. For example, if $a = -2$, we get $x = cy^2$.
- $a = -1$ - star. Now both x and y tend to 0 at the same rate, $x \sim y$ as $t \to \infty$.
- $-1 < a < 0$ - stable node. This time y goes to 0 faster than x, otherwise similar to the first case.
- $a = 0$ - line of fixed points. Now the whole x axis are fixed points, as $x = x_0$ remains constant in time, while $y \to 0$ as $t \to \infty$.
- $a > 0$ - saddle point. Here x grows exponentially while y tends to 0 as $t \to \infty$.

Example 4. Let $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$ and the initial condition is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. E-values and e-vectors of A are

$$\lambda_1 = 2, s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda_2 = -3, s_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Then

$$u = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} e^{2t} + e^{-3t} \\ e^{2t} - 4e^{-3t} \end{bmatrix}.$$

The phase plane shows growth in s_1 direction and decay in s_2. This is a saddle point.

The e-values of A are best expressed via the trace $\tau = a + d$ and the determinant $\Delta = ad - bc$. From $det\,(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = 0$, we find that $\lambda^2 - (a + d)\lambda + ad - bc = 0$, i.e. $\lambda^2 - \tau\lambda + \Delta = 0$ and therefore

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right).$$

Now different cases arise depending on the relationship between τ and Δ. They are depicted in the figure above from Strogatz. In terms of the eigenvalues, we have the following:

(1) $\lambda_{1,2}$ different signs – saddle. Here $\Delta < 0$.

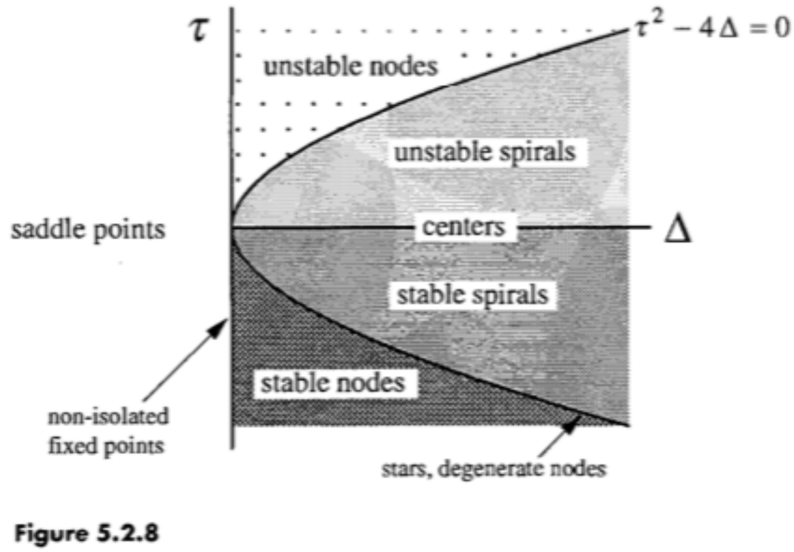

FIGURE 1.2. Classification of fixed points (Strogatz).

(2) λ pure imaginary – center. Now $\tau = 0$, $\Delta > 0$.
(3) $\lambda_2 = \lambda_1^*$ and $Re\,(\lambda) \neq 0$ – spiral (stable or unstable).
(4) $\lambda_1 > \lambda_2 > 0$ or $\lambda_1 < \lambda_2 < 0$ – node (stable or unstable).
(5) $\lambda_1 = \lambda_2$ and real with one eigenvector – degenerate node.

Example 5. Love affairs (Strogatz). Let R be Romeo's love (>0) /hate (<0) of Juliet and J be the same for Juliet. The problem is to find out the fate of their relationship depending on their attitudes and initial conditions.

(1) *Opposites.* Suppose Romeo gets excited when Juliet is positive towards him: $\dot{R} = aJ$. However, Juliet's reaction is opposite: $\dot{J} = -bR$. Here a, b are positive. What will happen with time?

    To answer, note that $A = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix}$ and therefore eigenvalues are $\lambda = \pm i\sqrt{ab}$, i.e. purely imaginary. This means the equilibrium is a center and that they will be stuck on a cycle of love-hate relationship forever.
(2) *Cautious.* Suppose now that

$$\dot{R} = -aR + bJ$$
$$\dot{J} = bR - aJ,$$

    with a, b positive. Again, Romeo gets excited when Juliet is positive towards him (bJ), however, now he is cautious about his feelings and restrains himself ($-aR$). Juliet behaves the same towards Romeo.
    To see what happens now, note that the e-values and e-vectors of $A = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix}$ are $\lambda_1 = -a + b$, $s_1 = [1\ 1]^T$, and $\lambda_2 = -a - b$, $s_2 = [1\ -1]^T$. This means that there are two possibilities:

(a) $b < a$. Then both e-values are negative and therefore the origin is a stable node. Whatever the initial excitement between R and J, it is going to disappear exponentially fast. Reason – too much caution (a large compared to b).

(b) $b > a$. Now the eigenvalues are of different sign, hence the origin is a saddle point. The attracting direction is s_2, which corresponds to the quadrants where Romeo is positive, but Juliet is negative or vice versa. This will try to end the relationship. However, this being a saddle, the other direction will take over. If they start on the more positive side, they will eventually end up in harmony on the direction of s_1 in the first quadrant. Otherwise, will end up hating each other in the third quadrant.

1.4. **Linearization of nonlinear ODE.....** Now we look at the very basic system of ODE on a plane

$$\frac{dx}{dt} = f(x, y),$$
$$\frac{dy}{dt} = g(x, y).$$

Suppose (x_0, y_0) is a fixed point, i.e. it solves $f = 0$, $g = 0$. Then, linearize about it:

$$x = x_0 + x'$$
$$y = y_0 + y'.$$

Then

$$\frac{du}{dt} = Au, \quad A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_0 \text{ Jacobian at fixed point.}$$

Behavior (almost always, except for borderline cases) depends on A – if the f.p. is a saddle, node or a spiral in the linear case, it is so in the nonlinear case as well. In all the other cases, nonlinear terms will be important. We will come back to this later.

Example 6. Nonlinear pendulum: $\ddot{x} + \omega_0^2 \sin x = 0$. Its fixed points are the roots of $\sin x = 0$: $x = n\pi$, $n \in \mathbb{Z}$.

To understand the nature of the fixed points, we linearize the ODE about $x_n = n\pi$. Let $x = n\pi + x'$, then $\sin(n\pi + x') = (-1)^n \sin x' = (-1)^n x' + ...$, and

$$\ddot{x}' + (-1)^n \omega_0^2 x' = 0.$$

Therefore, the fixed points with even n are (stable) centers, while those with odd n are saddle points. This is seen in the figure below.

Example 7. Predator-prey system (rabbits-sheep, Strogatz):

$$\frac{dx}{dt} = x(3 - x - 2y),$$
$$\frac{dy}{dt} = y(2 - x - y).$$

The analysis of this system proceeds in several steps. The goal is always to find out what the solutions do depending on the initial conditions. For this purpose, the phase plane is the main instrument.

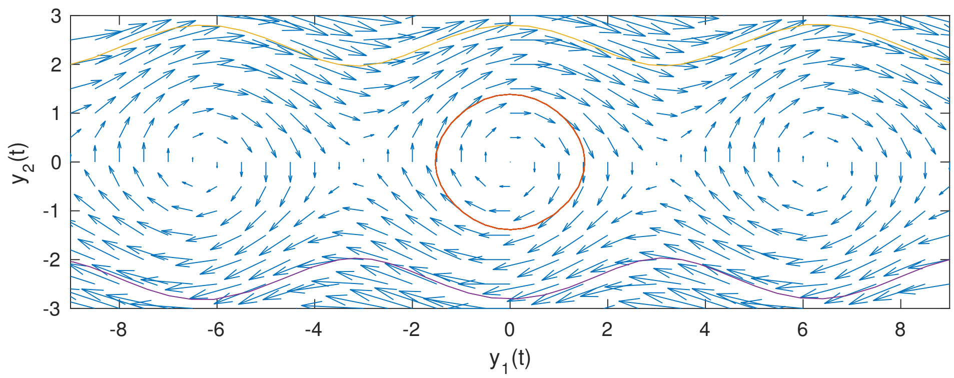

FIGURE 1.3. The phase portrait of the nonlinear pendulum $\ddot{x} + \omega_0^2 \sin x = 0$.

Step 1. Figure out all the fixed points. These are all the solutions of the system

$$x\,(3 - x - 2y) = 0$$
$$y\,(2 - x - y) = 0.$$

And the fixed points are: $u_1 = (0, 0)$, $u_2 = (0, 2)$, $u_3 = (3, 0)$, and $u_4 = (1, 1)$.

Step 2. Linearize the system about the fixed points as above and calculate the Jacobian matrix at the fixed points. For this example, the general Jacobian is

$$J = \begin{bmatrix} 3 - 2x - 2y, & -2x \\ -y, & 2 - x - 2y \end{bmatrix}.$$

The matrix A is the Jacobian at u_i: $A = J\,(u_i)$. The type of the fixed point is determined based on the eigenvalues.

Fixed point	A	$\lambda_1 \lambda_2$	type
$(1) : (0, 0)$	$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$	3,2	unstable node
$(2) : (0, 2)$	$\begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$	$-1, -2$	stable node
$(3) : (3, 0)$	$\begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$	$-3, -1$	stable node
$(4) : (1, 1)$	$\begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$	$\frac{-1+\sqrt{5}}{2}\frac{-1-\sqrt{5}}{2}$	saddle

Step 3. Then we plot the integral curves in the neighborhood of each fixed point and connect the trajectories. The result is shown in the figure. It is important to look at the eigenvectors as well when sketching the integral curves near fixed points.

1.5. **Some general stability terms.** . . . .

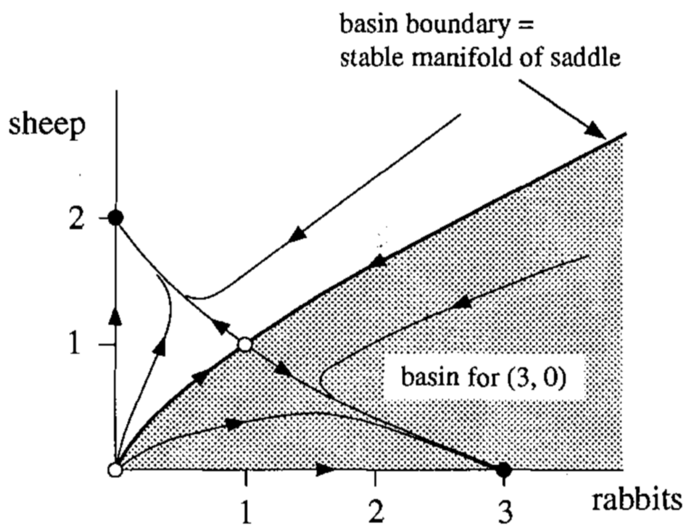

FIGURE 1.4. The phase portrait of the rabbits-sheep system (Strogatz).

- ◇ A fixed point x^* is called *attracting* if all trajectories that start near x^* approach x^* as $t \to \infty$.
- ◇ Fixed point x^* is called *Lyapunov stable* if all trajectories that start sufficiently close to x^* remain close to it at all times.
- ◇ If a fixed point is Lyapunov stable, but not attracting, then it is called *neutrally stable*.
- ◇ If a f.p. is both Lyapunov stable and attracting, then it is called *stable* or *asymptotically stable*.
- ◇ A f.p. is *unstable* if it is neither attracting nor Lyapunov stable.

1.6. **The role of nonlinear terms ...** Here we look at what happens when fixed points are not robust. Saddles, nodes and spirals are robust, other fixed points are fragile.

Example 8. Let

$$\dot{x} = -y + ax\left(x^2 + y^2\right)$$
$$\dot{y} = x + ay\left(x^2 + y^2\right).$$

The fixed point is $(0,0)$ and linearizing about it, we get $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which shows that the f.p. is a center, which is stable. However, it is a borderline case, and so this linear result may be misleading.

To linearize, let $x = x_0 + \epsilon x'$, $y = y_0 + \epsilon y'$ where (x_0, y_0) is the fixed point, $\epsilon\,(x', y')$ is a perturbation about the fixed point, and $\epsilon \ll 1$ is the small size of the perturbation. Substitute these expressions for x and y and then neglect all terms of order ϵ^2 or smaller. What remains is the linearized system,

$$\begin{bmatrix} \dot{x}' \\ \dot{y}' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

To see what happens when we retain the nonlinear terms, change to polar coordinates:

$$r^2 = x^2 + y^2, \quad x = r\cos\vartheta, \quad y = r\sin\vartheta.$$

Then

$$\dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = -r\sin\vartheta + ar^3\cos\vartheta$$
$$\dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = r\cos\vartheta + ar^3\sin\vartheta.$$

Multiply the first equation by $\sin\vartheta$ and subtract from the second multiplied by $\cos\vartheta$:

$$r\dot{\vartheta} = r, \longrightarrow \dot{\vartheta} = 1.$$

Using this, it follows from the first equation that

$$\dot{r} = ar^3.$$

Therefore, we have that ϑ increases linearly with time while $r(t)$ either grows to infinity if $a > 0$ or tends to 0 if $a < 0$. Only if $a = 0$ the conclusion of the linearization is valid. But then the system is not nonlinear.