Von Neumann stability analysis for the central-time central-space scheme for the wave equation.

We consider the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

in the infinite domain $-\infty < x < \infty$ and with periodic initial conditions. The celerity c is constant.

We solve this problem numerically using a central-time central-space scheme

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k} - c^2 \frac{U_{j+1}^n - 2U_j^n + U_{j+1}^n}{h} = 0 \tag{2}$$

where k and h are the time and space discretization step sizes, respectively.

$$U_j^{n+1} = 2U_j^n - U_j^{n-1} + c^2\frac{k^2}{h^2}(U_{j-1}^n - 2U_j^n + U_{j+1}^n).$$

For the Von Neumann analysis, we assume

$$U_j^n = e^{\iota jh\xi} \quad \text{and} \quad U_j^{n+1} = g(\xi)e^{\iota jh\xi}, \quad U_j^{n-1} = \frac{1}{g(\xi)}e^{\iota jh\xi}.$$

We substitute U_j^n, U_j^{n+1} and U_j^{n-1} in the scheme,

$$g(\xi)e^{\iota jh\xi} - 2e^{\iota jh\xi} + \frac{1}{g(\xi)}e^{\iota jh\xi} = c^2\frac{k}{h^2}\left(e^{\iota(j-1)h\xi} - 2e^{\iota jh\xi} + e^{\iota(j+1)h\xi},\right) \tag{3}$$

then,

$$\left(g(\xi) - 2 + \frac{1}{g(\xi)}\right)e^{\iota jh\xi} = c^2\frac{k^2}{h^2}\left(e^{-\iota h\xi} - 2 + e^{\iota h\xi}\right)e^{\iota jh\xi}, \tag{4}$$

and we obtain

$$g(\xi) - 2 + \frac{1}{g(\xi)} = 2c^2\frac{k^2}{h^2}(\cos(\xi h) - 1).$$

We obtain a quadratic equation

$$g^2 - 2\beta g + 1 = 0, \tag{5}$$

where

$$\beta = 1 - \alpha\frac{c^2k^2}{h^2} \tag{6}$$

and

$$\alpha = 1 - \cos(\xi h). \tag{7}$$

The latter implies $0 \leq \alpha \leq 2$.
Equation (5) has two solutions, g_1 and g_2, which are either for real-valued or both complex. But we know from the equation that $g_1 g_2 = 1$, therefore, if both roots are real, they cannot be both less than one. If $g_1 < 1$ then $g_2 > 1$ or if $g_2 < 1$ then $g_1 > 1$, both situations entail lack of stability. The remaining real-valued case $g_1 = g_2 = 1$ requires that $\alpha\frac{c^2k^2}{h^2} = 0$, which can occur if and only if $\alpha = 0$, and the latter is not satisfied unless $\xi = 0$. We conclude that real-valued solutions g_1 and g_2 cannot ensure $|g| \leq 1$ for all $\xi \in [-\pi/h \leq \xi \leq \pi/h]$.

The only remaining possibility is that g_1 and g_2 are complex conjugate and $|g_1| = |g_2| = 1$. This situation happens when $\beta^2 - 1 < 0$, i.e., $-1 \leq \beta \leq 1$. This gives the condition

$$-2 \leq -\alpha c^2 \frac{k^2}{h^2} \leq 0, \tag{8}$$

which can be rewritten as

$$\alpha c^2 \frac{k^2}{h^2} \geq 0 \quad \text{and} \quad \alpha c^2 \frac{k^2}{h^2} \leq 2 \tag{9}$$

Remembering that $0 \leq \alpha \leq 2$, we see that the first condition above is satisfied always, but the second condition is satisfied only when

$$c^2 \frac{k^2}{h^2} \leq 1. \tag{10}$$

Since c, h and k are positive, we obtain the *Courant-Friedrichs-Lewy* condition

$$\frac{ck}{h} \leq 1. \tag{11}$$