Mathematical Methods in Engineering and Applied Science

Problem Set 4.

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(1). Explain:

(a) Why $A^T A$ is not singular when matrix A has independent columns;

Let $A = \begin{bmatrix} \dots\ a_1 \dots \\ \dots \\ \dots a_m \dots \end{bmatrix}$; $dim(R(A)) = dim(C(A)) = rank(A) = r$, it means that A has $a_1 \dots a_r$ independent rows, and $a_1^T \dots a_r^T$ are independent too.

Let $(A^T)_{r*m} A_{m*r} = B_{r*r}$, where B_{r*r} may be expressed as a combination of columns A^T: $a_1^T \dots a_r^T \dots a_m^T$, among which r independent columns. In other words, B_{r*r} consist of r independent columns => $rank(B) = r$, B is full rank => B is non-singular.

(b) Why A and $A^T A$ have the same nullspace.

Let's A is m by n matrix.

Let's prove that if $x \in N(A) => x \in N(A^T A)$:

If $x \in N(A), i.e. Ax = 0 => A^T Ax = A^T 0 = 0, i.e\ x \in N(A^T A)\ or\ N(A) \subset N(A^T A)$;

But for A it's true that: $dim(R(A)) + dim(N(A)) = n$

and for $(A^T A)$: $dim(R(A^T A)) + dim(N(A^T A)) = n$.

Since $dim\big(R(A^T A)\big) = dim\big(R(A)\big) = r$ (similar to prev. task) => $dim\big(N(A)\big) = dim\big(N(A^T A)\big)$; But we found that $N(A) \subset N(A^T A) => N(A) = N(A^T A)$

(2). A plane in R^3 is given by the equation $x_1 - 2x_2 + x_3 = 0$.

(a) Identify two orthonormal vectors u_1 and u_2 that span the plane.

Let's choose any two vectors of the plane $a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; a_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

Let's $u_1 = \frac{a_1}{|a_1|} = \frac{1}{\sqrt{3}} a_1$;

u_2 is supposed to be orthogonal to u_1: $u'_2 = (1 - P)a_2 = \left(1 - \frac{u_1 u_1^T}{u_1^T u_1}\right) a_2 = \left(1 - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} * [1,1,1]\right)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

After normalization: $u_2 = \frac{1}{\sqrt{2}} a_2$

$$u_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix};$$

(b) Find a projector matrix P that projects any vector x from R^3 to the plane and a projector $P_\perp$ that projects any vector to the direction normal to the plane.

From prev. task: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$ $\quad P = A(A^T A)^{-1} A^T = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$

$$P_\perp = 1 - P = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

(c) Using these projectors find the unit normal to the plane and verify that it agrees with a normal found by calculus methods (that use the gradient).

Normal of plane is $a'_\perp = grad(R) = grad(x_1 - 2x_2 + x_3) = (1, -2, 1)^T$;

$$a_\perp = \frac{a'_\perp}{|a'_\perp|} = \frac{1}{\sqrt{6}}(1, -2, 1)^T$$

Otherwise: choose vector that doesn't satisfy plane equation: [2,0,0] for instance and act on it by $P_\perp$

$$P_\perp[2,0,0] = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

And normalize it:

$$a_\perp = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(3). Let $M = span\{v1, v2\}$ where $v1 = [1\ 0\ 1\ 1]^T, v2 = [1 - 1\ 0 - 1]^T$.

(a) Find the orthogonal projector P_M on M.

Determine $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$; $P_M = A(A^TA)^{-1}A^T = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

(b) Find the kernel (nullspace) and range (column space) of P_M.

P_M – plane projector, $rank(P_M) = 2$, that's why column space is any 2 independent vectors

$$C(P_M) = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

For null space: Let $a_1, a_2, \ldots -$ columns of P_M, $a_1 = -a_2 + a_3$; $a_4 = a_2 + a_3$;

$$N(P_M) = \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}\right\}$$

(c) Find $x \in M$ which is closest in 2-norm to the vector $a = [1 - 1\ 1 - 1]^T$.

Suppose $a = a_\perp + a_=$; in other words a consists of $a_\perp$,perp to M and $a_=$ no perp to M.

To make $||a - x||_2 \to min$, $||a_\perp + a_= - x||_2 \Rightarrow x = a_=$.

$$x = P_M a = \frac{1}{3}\begin{bmatrix} 4 \\ -3 \\ 1 \\ -2 \end{bmatrix}$$

(4). he following problems look at tests of positive definiteness.

(a) Using the determinant test, find c and d that make the following matrices positive definite:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

Leading determinants must be positive:

$$\begin{cases} c > 0 \\ c^2 - 1 > 0 \\ c^3 - 3c + 2 > 0 \end{cases}$$
$$\Rightarrow c > 1$$

For second matrix:

$$\begin{cases} d - 4 > 0 \\ 12 - 4d > 0 \end{cases}$$

$\Rightarrow 4 < d < 3 \Rightarrow$impossible to make B positive defined

(b) positive definite matrix cannot have a zero (or a negative number) on its main diagonal. Show that the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Let's $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; then $x^T A x = 0$;

(5). Matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ is positive definite. Explain why and determine the minimum value of $z = x^T A x + 2b^T x + 1$, where $x = [x_1\ x_2\ x_3]^T$ and $b^T = [1 - 2\ \ 1]$.
A is positive defined because of leading determinants are positive.
The minimum is found from:

$$\frac{\partial z}{\partial x} = 0 \rightarrow x^T A + x^T A^T + 2b^T = 0 \rightarrow x_0 = -2(A^T + A)b = \begin{bmatrix} -8 \\ \frac{53}{11} \\ -\frac{5}{11} \end{bmatrix}$$

$$z_0 = z(x_0) = -17.(09)$$

The approach to this task the similar as to solving equation: $ax^2 + bx + 1$, where $a > 0$, because of A is positive defined. So, this equation describes a parabola with one extremum, and if $a > 0$ then it is minimum.`

(6). Explain these inequalities from the definition of the norms:

$$\|ABx\| \leq \|A\|\|Bx\| \leq \|A\|\|B\|\|x\|,$$

and deduce that $\|AB\| \leq \|A\|\|B\|$.
Let $z = Bx$

$$\|ABx\| = \|Az\| = \left||z|\right| \frac{\|Az\|}{||z||} \leq \left||z|\right| \max\left(\frac{\|Az\|}{\left||z|\right|}\right) = \left||z|\right| \left||A|\right| = \left||A|\right| \left||z|\right|$$

Proved that $\|Az\| \leq \left||A|\right| \left||z|\right| => \|ABx\| \leq \left||A|\right| \left||Bx|\right|$
It is easy to show that $\left||Bx|\right| \leq \left||B|\right| \left||x|\right|$, in the same way as why$\|Az\| \leq \left||A|\right| \left||z|\right|$
Finally, $\|ABx\| \leq \left||A|\right| \left||Bx|\right| \leq \left||A|\right| \left||B|\right| \left||x|\right|$.
$\|AB\| = \max\limits_{||x||=1} \|ABx\| \leq \left||A|\right| \left||B|\right|$ – (using prev. condition.)

(7). Compute by hand the norms and condition numbers of the following matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix};$$

Since A_1 is symmetric => from lectures $\left||A_1|\right| = \lambda_{max}$;

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3$$

$$\lambda_{1,2} = 3; 1$$

Therefor $\left||A_1|\right| = 3$;
Since A_1 is symmetric => from lectures $\left||A_1^{-1}|\right| = 1/\lambda_{min}$;
Therefor $\left||A_1^{-1}|\right| = 1$;
Condition number of A_1 $\kappa = \left||A_1|\right| \left||A_1^{-1}|\right| = 3$

For second matrix A_2 : $\left||A_2|\right| = \sigma_{max}$;

$$AA^T = U\Sigma^2 U^T$$

Find e-values Σ^2:$\det\left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \lambda I\right) = 0$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \lambda I\right) = 0$$

$$\lambda = 2;$$

Therefor $\sigma_{max} = \sigma_{min} = \sqrt{2}; => \left||A_2|\right| = \sqrt{2}$;

$$\left||A_2^{-1}|\right| = \frac{1}{\sigma_{min}} = \frac{1}{\sqrt{2}}$$

And condition number: $\kappa = ||A_2|| \, ||A_2^{-1}|| = 1$