Problem set 2

(1)

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 2 \\ x_2 + 2x_3 &= 3 \end{aligned} \Rightarrow \quad A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$Ax = b$$

(a) LU fact.:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} row_1 \\ row_2 - \frac{1}{2} row_1 \\ row_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} row_1 \\ row_2 \\ row_3 - \frac{2}{3} row_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = \underset{L}{\underset{\parallel}{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}}} \underset{U}{\underset{\parallel}{\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}}$$

Show that $U = DL^T$:

$$U = IU = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{2} & 1 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{4}{3} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 2 \cdot \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{3}{2} \begin{bmatrix} 0 & 1 & \frac{2}{3} \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{4}{3} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \underset{D}{\underset{\parallel}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = L^T$$

Find solution $LUx=b$.

$$Ax=b \Rightarrow LUx=b \Rightarrow Ux = L^{-1}b = y$$

$$\cancel{L^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ \ \end{bmatrix}}$$

from (a)

$$L^{-1} = \left( \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}^{-1} \right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ +\frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$$y = L^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 2 \end{bmatrix}$$

solve: $Ux = y$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} x = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 2 \end{bmatrix} \to \begin{bmatrix} row_1 \\ row_2 \\ row_3 \cdot \frac{3}{4} \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & \frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right]$$

$$\to \begin{bmatrix} row_1 \\ row_2 - row_3 \\ row_3 \end{bmatrix} \to \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \to \begin{aligned} x_1 &= \tfrac{1}{2} \\ x_2 &= 0 \\ x_3 &= \tfrac{3}{2}. \end{aligned}$$

(b) Solve the system using Jacobi and Gauss-Seidel iterations. How many iterations are needed to reduce the relative error of the solution to $10^{-8}$?

(c) Plot in semilog scales the relative errors by both methods as a function of the number of iterations.

(d) Explain the convergence rate. Which of the methods is better and why?

$$\text{Gauss} - \text{Seidel}\ method:$$

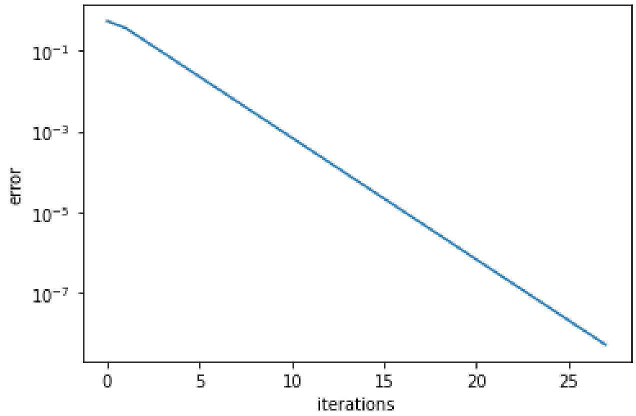
$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}; A_2 = A - A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$Let\ x_0 = [0,0,0];$$

$$\mathrm{x}_1 = \mathrm{A}_1^{-1}(\mathrm{b} - \mathrm{A}_2\mathrm{x}_0);$$

$$\dots$$

$$\mathrm{x}_\mathrm{k} = \mathrm{A}_1^{-1}(\mathrm{b} - \mathrm{A}_2\mathrm{x}_{\mathrm{k}-1});$$

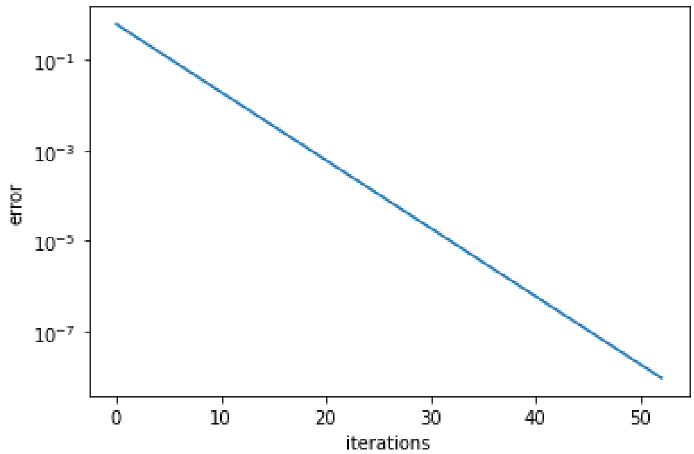



$\text{Gauss} - \text{Seidel}\ method; iterations: 28$
$error < 10^{-8}$

$$Jacoby\ method:$$

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; A_2 = A - A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$Let\ x_0 = [0,0,0];$$




$\text{Gauss} - \text{Seidel}\ method; iterations: 53$
$error < 10^{-8}$

convergence rate:

$$\lim_{k \to \infty} \frac{\| x_{k+1} - x \|}{\| x_k - x \|^q} = a$$

or

$$\log \| x_{k+1} - x \| < \log a + q \log \| x_k - x^* \|$$

$$\log \| error_{k+1} \| < \log a + q \log \| error_k \|$$

for $a \in (0; 1)$ I got $q = 1$

for both methods

Of course Gauss-Seidel method is better for this task, because

$\| B \| = \| -A_1^{-1} A_2 \|$ is smaller

(2) $A$ into $S \Lambda S^{-1}$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) = 0 \Rightarrow \lambda = 1 \quad \lambda = 3$$

For $\lambda_1 = 1$:

$$\begin{bmatrix} 1-1 & 2 \\ 0 & 3-1 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 3$:

$$\begin{bmatrix} 1-3 & 2 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$S^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

(a) $$A^3 = S \Lambda^3 S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 27 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$$

(b) $$A^{-1} = (S \Lambda S^{-1})^{-1} = S \Lambda^{-1} S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 3 = \lambda^2 - 4\lambda = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 4$$

For $\lambda_1 = 0$:

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} x = 0. \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 4$:

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

$$S^{-1} = \frac{1}{-3-1} \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix}^T = \frac{1}{4} \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$$

(a) $$A^3 = S\Lambda^3 S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 64 \end{bmatrix} \frac{1}{4} \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 16 \\ 0 & 48 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 16 \\ 48 & 48 \end{bmatrix}$$

(b) $A^{-1}$ doesn't exist, because $\text{rank}(A) = 1 \Rightarrow$
$\Rightarrow \det A = 0$.
$\det \Lambda = 0$ correspondingly $\Lambda^{-1}$ doesn't exist.

Problem № 3

$$Ax = b \ : \ A = \begin{bmatrix} 1 & -1 & -3 \\ 2 & 3 & 4 \\ -2 & 1 & 4 \end{bmatrix} ; \ b = \begin{bmatrix} 3 \\ a \\ -1 \end{bmatrix}$$

$Ax = a_1 x_1 + a_2 x_2 + a_3 x_3 \in C(A) \Rightarrow$

~~x is a solution => $b \in C(A)$~~

$\Rightarrow$ $b$ must $\in C(A)$, otherwise there is no solutions.

$$\begin{bmatrix} 1 & -1 & -3 \\ 2 & 3 & 4 \\ -2 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} col_1 & col_2 + col_1 & col_3 + 3col_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 10 \\ -2 & -1 & -2 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ -2 & -1 & 0 \end{bmatrix} \Rightarrow C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} y_1 + \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} y_2 : \forall y_1, y_2 \right\}$$

we suppose $b \in C(A) \Rightarrow$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & 5 & a \\ -2 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 5 & a-6 \\ 0 & -1 & -1+6 \end{array}\right] \to \begin{cases} y_1 = 3 \\ 5y_2 = a - 6 \\ y_2 = -5 \end{cases}$$

$-25 = a - 6 \Rightarrow \boxed{a = -19}$

General sol. of $Ax = b$ will find as a
combination sol. of homogeneous
system $Ax = 0$ and partial sol. of $Ax = b$

From the top we know that: $col_3 + 3col_1 - 2(col_2 + col_1) = 0 \Rightarrow col_1 - 2col_2 + col_3 = 0$,

considering $Ax = a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \Rightarrow$

$\Rightarrow x_0 = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix}$ - sol. of homogen. sys.

Partial sol. we have already received

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -2 & -1 \end{bmatrix} \cdot y = \begin{bmatrix} 3 \\ -19 \\ 1 \end{bmatrix}, \text{ where } y = [3, -5]^T \Rightarrow$$

$$a_1 \cdot 3 - 5a_2' = a_1 \cdot 3 - 5(a_1 + a_2) = -2a_1 - 5a_2 -$$

- partial sol $\Rightarrow x_n = \begin{bmatrix} -2 \\ -5 \\ 0 \end{bmatrix}$

$x_0 + x_n$ - is a general solution.

$$X = \begin{bmatrix} z - 2 \\ -2z - 5 \\ z \end{bmatrix}$$

Problem №4

$$\dot{u} = Au \qquad A = S\Lambda S^{-1}$$

$$\dot{u} = S\Lambda S^{-1}u$$

$$\frac{d}{dt}(S^{-1}u) = \Lambda S^{-1}u \qquad \text{Let } v = S^{-1}u$$

$$\dot{v} = \Lambda v$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(\lambda - 3)(\lambda - 1)\lambda = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = 3$$

For $\lambda_2 = 1$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}$$

Сканировано с CamScanner

For $\lambda_1 = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_3 = 3$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) general sol of system:

$$\dot{v} = \Lambda v$$

$$\dot{v}_i = \lambda_i v_i \Rightarrow v_i = c_i e^{\lambda_i t}$$

$$\dot{v}_0 = 0 \Rightarrow v_0 = c_0$$

$$\dot{v}_1 = v_1 \Rightarrow v_1 = c_1 e^{t}$$

$$\dot{v}_2 = 3v_2 \Rightarrow v_2 = c_2 e^{3t}$$

$$v = S^{-1} u \Rightarrow u = S v$$

$$u = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 e^{t} \\ c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} c_0 - c_1 e^{t} + c_2 e^{3t} \\ c_0 - 2 c_2 e^{3t} \\ c_0 + c_1 e^{t} + c_2 e^{3t} \end{bmatrix}$$

(b) $u(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad v_0 = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} v_0 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & -1 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow$$

$$\rightarrow \begin{bmatrix} row_1 + row_3 \\ row_2 \\ row_3 \end{bmatrix} \rightarrow$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 2 \\ 1 & 0 & -2 & -1 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{c} row_1 - 2row_2 \\ row_2 \\ row_3 - row_2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 6 & 4 \\ 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 2 \end{array}\right] \rightarrow$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_0 = \frac{1}{3} \\ c_1 = 0 \\ c_2 = \frac{2}{3} \end{array}$$

at large time $c_0 \ll e^{3t}$, therefore we ~~can~~ may neglect $c_0$

$$u = \begin{bmatrix} \frac{2}{3} e^{3t} \\ -\frac{4}{3} e^{2t} \\ \frac{2}{3} e^{3t} \end{bmatrix}$$

## Problem № 5

$$A = \begin{bmatrix} 2021 & 20 & 0 \\ 20 & 2021 & 21 \\ 0 & 21 & 2021 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

What is the most likely direction $x = A^{2021} b$?

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$$\det(A - I\lambda_i) = 0 \Rightarrow \begin{array}{l} \lambda_1 = 1992 \\ \lambda_2 = 2021 \\ \lambda_3 = 2050 \end{array}$$

a lot of calculations and papers:

$$\Lambda = \begin{bmatrix} 1992 & 0 & 0 \\ 0 & 2021 & 0 \\ 0 & 0 & 2050 \end{bmatrix}$$

$$S = \begin{bmatrix} 20 & -21 & 20 \\ -29 & 0 & 29 \\ 21 & 20 & 21 \end{bmatrix} \quad S^{-1} = \frac{1}{1682}\begin{bmatrix} 20 & -29 & 21 \\ -42 & 0 & 40 \\ 20 & 29 & 21 \end{bmatrix}$$

$$x = A^{2021} b = S \Lambda^{2021} S^{-1} b \;=$$

$$S^{-1} b = b' = \frac{1}{1682}\begin{bmatrix} 11 \\ -84 \\ 69 \end{bmatrix}$$

$$= S \Lambda^{2021} \cdot b' = \cancel{S\, \lambda_i^{2021}} = S \begin{bmatrix} \lambda_1^{2021} b_1' \\ \lambda_2^{2021} b_2' \\ \lambda_3^{2021} b_3' \end{bmatrix} =$$

$$= \sum S_i \cdot \lambda_i^{2021} b_i' =$$

$$= \lambda_1^{2021} \cdot \frac{11}{1682}\begin{bmatrix} 20 \\ -29 \\ 21 \end{bmatrix} + \lambda_2^{2021} \frac{-84}{1682}\begin{bmatrix} -21 \\ 0 \\ 20 \end{bmatrix} +$$

$$+ \lambda_3^{2021} \cdot \frac{69}{1682}\begin{bmatrix} 20 \\ 29 \\ 21 \end{bmatrix}$$

~~Let $\lambda_1 \approx \lambda_2 \approx \lambda_3 \approx \lambda$, very very rough estimation / approximation in this case!~~

~~$x \approx \lambda^{2021} \frac{1}{1682}\begin{bmatrix} 3364 \\ 1682 \\ 0 \end{bmatrix} \Rightarrow x \approx C \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$~~

where $C$ - some const.

approximately $\lambda_1 \approx \lambda_2 \approx \lambda_3$, but

$\lambda_1^{2021} << \lambda_2^{2021} << \lambda_3^{2021} \Rightarrow$ x will in direction of vector with $\lambda_3^{2021}$ coefficient

$$x \approx \lambda_3^{2021} \frac{69}{1682} \begin{bmatrix} 20 \\ 29 \\ 21 \end{bmatrix} \Rightarrow \text{direction of } x \text{ is } \bar{v}$$

$$\bar{v} = \begin{bmatrix} 20 \\ 29 \\ 21 \end{bmatrix}$$

Problem № 6

$$\ddot{u} = + K_4 u, \text{ where } K_4 = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

$$u = [\varphi_1, \varphi_2, \varphi_3, \varphi_4]^T$$

(a) since $\det(K_4) = 0$ then $K_4$ is singular.
I think physical meaning that.
$m\ddot{\varphi}_j$ - force is a result of action of other unit masses. I mean if we have values of $\varphi_i$; $\ddot{\varphi}_i$ : $i = 1,2,3 \Rightarrow$ we must be able to find $\varphi_4$; $\ddot{\varphi}_4$ because they are dependent.
(because $\det(K_4) = 0$)

(b)

$$\det(K_4 - \lambda I) = 0 \Rightarrow \lambda^4 + 8\lambda^3 + 20\lambda^2 + 16\lambda =$$

$$= \lambda(\lambda+4)(\lambda+2)^2 = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = -4 \quad \lambda_{3,4} = -2$$

For $\lambda_1 = 0$ $\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} x = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -4$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} x = 0 \Rightarrow x_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda_{3,4} = -2$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} x = 0 \Rightarrow x_{3,4} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad S = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\ddot{u} = S\Lambda S^{-1} u$$

$$S^{-1}\ddot{u} = \Lambda S^{-1} u \qquad \text{Let } v = S^{-1} u \Rightarrow$$

$$\ddot{v} = \Lambda v \Rightarrow \ddot{v}_i = \lambda_i v_i$$

$$\ddot{v}_i = \lambda_i v$$

$$\ddot{v}_i - \lambda_i v = 0$$

$$\gamma^2 - \lambda_i \cancel{\gamma} = 0$$

$$\gamma_{1,2} = \pm\sqrt{\lambda_i}$$

$$v_i = C_1 e^{\sqrt{\lambda_i}t} + C_2 e^{-\sqrt{\lambda_i}t} = C_1 e^{-i^2\sqrt{\lambda_i}t} + C_2 e^{i^2\sqrt{\lambda_i}t} =$$

$$= C_3\cos(i\sqrt{\lambda_i}\cancel{t}) + C_4\sin(i\sqrt{\lambda_i}\cancel{t})$$

if $\lambda_i = 0$

$$\ddot{v}_i = 0 \Rightarrow v_i = C_1 t + C_2.$$

$$v = \begin{bmatrix} C_1 t + C_2 \\ C_3\cos\sqrt{4}\,t + C_4\sin\sqrt{4}\,t \\ C_5\cos\sqrt{2}\,t + C_6\sin\sqrt{2}\,t \\ C_7\cos\sqrt{2}\,t + C_8\sin\sqrt{2}\,t \end{bmatrix}$$

$$v = S^{-1}u \Rightarrow u = Sv$$

$$u = Sv = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}v_1 + \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}v_2 + \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}v_3 + \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}v_4 =$$

$$= \begin{bmatrix} C_1 t + C_2 + C_3\cos\sqrt{4}\,t + C_4\sin\sqrt{4}\,t - C_5\cos\sqrt{2}\,t + C_6\sin\sqrt{2}\,t \\ \dots \\ \dots \\ \dots \end{bmatrix}$$

(c) root of eigenvalue is a frequency

$|\lambda_i| = \omega_i^2$ : eigenvalues describe available frequencies.

eigenvectors describe phases in which points with corresponding eigenval. may be.

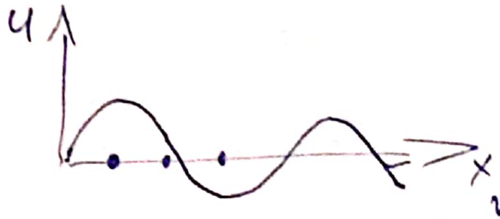
example:

$$\lambda_0 = 0 = \omega_0^2 \; ; \; S_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$$

=> all points ~~move~~ rotate in one direction. as $\omega_0$ is smallest freq.

Such problem may be described by wave equation:

$u = c e^{i(\omega t - kx)}$ like a wave moving forward:




wave is moving with some velocity. it is clear: the more wave crests the more freq. max count of crests are received by antiphase of next point. - it is $S_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ and max freq $\omega^2 = 4$ correspondingly.

д) $u(0) = [1 \ 0 \ -1 \ 0]$

$\dot{u}(0) = 0$

$u = S v \Rightarrow u(0) = S v(0) \Rightarrow$

$\Rightarrow S^{-1} u(0) = v(0)$

$$v(0) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$v_0(0) = C_2 = 0$

$v_1(0) = C_3 = 0$

$v_2(0) = C_5 = -1$

$v_3(0) = C_7 = 0$

$\dot{u}(0) = 0 \Rightarrow S^{-1} \cdot \dot{u}(0) = \dot{v}(0)$

$$\dot{v}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dot{v}_0(0) = C_1 = 0$

$\dot{v}_1(0) = \sqrt{4} C_4 = 0$

$\dot{v}_2(0) = \sqrt{2}\, C_6 = 0$

$v_3(0) = \sqrt{2}\, C_6 = 0$

$$v = \begin{bmatrix} 0 \\ 0 \\ -\cos\sqrt{2}\,t \\ 0 \end{bmatrix} \Rightarrow u = S v = +\cos\sqrt{2}\,t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$