4. Lectures 23, 24. Nov. 30, Dec. 1

Plan:

- ◇ Burgers equation, characteristics, shocks and fans.
- ◇ Reaction-diffusion equations: decease propagation, porous-media equation, nonlinear heat conduction, traveling waves and patterns.

4.1. **Burgers equation.**

4.2. **Hopf/inviscid Burgers equation.** Equation

$$u_t + uu_x = \nu u_{xx}$$

arises in great many contexts (fluid mechanics, traffic modeling, any weakly nonlinear waves, etc.). The role of u_{xx} term is to dissipate energy as in the diffusion equation. The role of $uu_x$is expected to be the same as of au_x in the advection equation $u_t + au_x = 0$, i.e. to represent translation of the solution with some speed. The problem is that the speed is now u, that depends on the solution. So larger values of u will propagate faster than smaller values. This is the nonlinear effect that we will analyze below.

It is useful to look at the energy of the solution that is defined as

$$E(t) = \int_{-\infty}^{\infty} u^2(x,t)\, dx.$$

The rate of change of this energy is

$$\begin{aligned}
\dot{E} &= 2\int_{-\infty}^{\infty} uu_t dx = 2\int_{-\infty}^{\infty} u\left(\nu u_{xx} - uu_x\right) dx = \\
&= -2\int_{-\infty}^{\infty} u^2 u_x dx + 2\nu \int_{-\infty}^{\infty} uu_{xx} dx = \\
&= -2\frac{u^3}{3}|_{-\infty}^{\infty} + 2\nu uu_x|_{-\infty}^{\infty} - 2\nu \int_{-\infty}^{\infty} u_x^2 dx = -2\nu \int_{-\infty}^{\infty} u_x^2 dx < 0.
\end{aligned}$$

We have assumed that u and u_x vanish at infinity. Therefore, this energy E (L_2 norm of u) decays with time, and this is what we mean by dissipation. Notice that the advection term uu_x plays no role here, so this dissipation is that of the diffusion equation.

When $\nu = 0$, we get the Hopf equation

$$u_t + uu_x = 0.$$

We will analyze this equation next in some detail in order to understand the role of uu_x term. Recall, this term or more generally a term like

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{u} = \left(u\partial_x + v\partial_y + w\partial_z\right)\mathbf{u} = \\
\left(\underline{uu_x} + vu_y + wu_z\right)\mathbf{i} \\
\left(uv_x + \underline{vv_y} + wv_z\right)\mathbf{j} \\
\left(uw_x + vw_y + \underline{ww_z}\right)\mathbf{k}
\end{aligned}$$

is common in equations of fluid mechanics, among others.

4.2.1. *Solution by the MOC:.* Let the initial condition be $u(x,0) = f(x)$. The MOC is:

$$\frac{du}{dt} = 0$$
$$\text{on: } \frac{dx}{dt} = u.$$

Therefore, $u = const$, hence $x = x_0 + ut$. Then $u(x,t) = u(x_0,0) = f(x_0) = f(x - ut)$ gives the solution

$$u = f(x - ut)\,. \tag{4.1}$$

4.2.2. *Gradient catastrophe and shock waves.* Note that in general f will be some complicated nonlinear function of x, and so when solving $u = f(x-ut)$ for u one should expect multiple solutions and be ready to deal with it. Multiple solutions at the same (x,t) is not something easily acceptable.

To see how problems with u arise, take x derivative of (4.1):

$$u_x = f'(x-ut)(1-u_x t) = f'(x_0)(1-u_x t)\,,$$
$$u_x = \frac{f'(x_0)}{1+f'(x_0)\,t}.$$

Hence, for any initial point x_0 with $f'(x_0) < 0$ the gradient of the solution will blow up at some finite time $t_b = -1/f'(x_0)$. This is called *gradient catastrophe*. The time t_b is called the *breaking time*.

If we take two nearby characteristics from x_0 and $x_1 > x_0$, then the solutions emanating from them are

$$u_0 = f(x - u_0 t) = f(x_0)$$
$$u_1 = f(x - u_1 t) = f(x_1)\,.$$

If $f'(x_0) < 0$, then $f(x_0) > f(x_1)$, and so the slopes satisfy

$$\frac{dx}{dt}|_0 = f(x_0) > \frac{dx}{dt}|_1 = f(x_1)\,.$$

These two straight lines will intersect at some time. That time is t_b.

We can look another way to the same trouble. Let $x = x_0 + u(x_0,0)\,t = x_0 + f(x_0)\,t$ be the equation of characteristic from point x_0. When can we find x_0 from this equation? The *implicit function theorem* says that to be able to solve $g(x_0) = x - x_0 - f(x_0)\,t = 0$ for x_0, we must satisfy the condition that $\partial g/\partial x_0 \neq 0$, i.e.

$$-1 - f'(x_0)\,t \neq 0.$$

This is the same condition as the breaking condition above.

The main question: How to extend the solution beyond t_b?

Answer: Introduce a generalized solution that admits discontinuities in $u(x,t)$.

Example 15. As an example, consider

$$f(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}.$$

Rewrite $u_t + uu_x = 0$ as

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$
$$u_t + (F(u))_x = 0,$$

where $F = u^2/2$ is the *flux* function. The MOC construction gives a discontinuity at $t > 0$. Let the shock path be given by the jump condition (Rankine-Hugoniot condition, to be explained below)

$$\dot{s} = \frac{[F]}{[u]} = \frac{u_+^2/2 - u_-^2/2}{u_+ - u_-} = \frac{u_+ + u_-}{2} = \frac{1}{2}.$$

Note: This is a case of a *Riemann problem*, i.e. an initial value problem with discontinuous initial conditions.

4.2.3. *Rarefaction waves/fans.* If in the previous example we change the initial condition to

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases},$$

then we run into another difficulty – no characteristics come into $0 < x < t$. We cannot have a void in the solution. Something must happen in that region. There are several ways to resolve this. One is that we fill in the region with a *fan* or a *rarefaction wave:*

$$u(x,t) = \frac{x}{t}, \quad 0 < x < t.$$

This is a *similarity solution* of $u_t + uu_x = 0$ that depends only on $\xi = x/t$. If we let $u = u(\xi)$, then $u_t = u'\xi_t = -u'x/t^2$ and $u_x = u'\xi_x = u'/t$, hence

$$u'\left(-x/t^2 + u/t\right) = 0$$

and since $u' = 0$ is not acceptable we get $u = x/t$.

Note: Even though this fan is the correct solution, one can construct other seemingly acceptable solutions, and one has to have a way to choose among them.

For example, another possible solution with the same initial condition is a shock-wave solution

$$u = \begin{cases} 1, & x \geq s(t) \\ 0, & x < s(t) \end{cases},$$

where

$$s = \frac{u_+ + u_-}{2} = \frac{1}{2}.$$

Why such a solution was acceptable in the previous example and is not acceptable here? The answer is in what is called an *entropy condition.* In this particular example, the Lax entropy condition requiring that characteristics must run into a shock wave rather than emanate from it is

$$F'(u_+) < \dot{s} < F'(u_-)$$

which becomes for $F = u^2/2$:

$$u_+ < \dot{s} < u_-.$$

That is the shock speed $\dot{s}$ must be larger than the characteristic speed ahead and smaller than that behind. This condition is satisfied by a *compression shock* of the first example, but not by *rarefaction shock* of the second.

Example 16. Solve the following initial value problem:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$
$$u(x,0) = \begin{cases} 1, & x \le 0 \\ -1, & 0 < x \le 1 \\ 0, & x > 1 \end{cases}.$$

Solution. Solution proceeds as follows. We sketch the initial condition and construct the characteristics. There will be a shock starting at $x = 0$ and a fan starting at $x = 1$.

The shock speed is given by

$$\dot{s} = \frac{u_+ + u_-}{2} = 0$$

before the fan reaches the shock. The fan is given by

$$u = \frac{x-1}{t}, \quad \text{on: } \frac{x-1}{t} = k = \textit{constant}.$$

Here k changes from $k = -1$ for the leftmost characteristic to $k = 0$ for the rightmost one. The fan $x - 1 = -t$ reaches the shock at $x = 0$ at time $t = 1$. Beyond this time, the shock motion is controlled by a new jump condition with new

$$u_+ = \frac{s-t}{t}.$$

$$\dot{s} = \frac{u_+ + u_-}{2} = \frac{(s-1)/t + 1}{2} = \frac{s-1+t}{2t}.$$

We rewrite this equation as

$$\dot{s} - \frac{1}{2t}s = \frac{1}{2} - \frac{1}{2t}.$$

It is solved by using an integrating factor and the solution is

$$s(t) = t + 1 - 2\sqrt{t}.$$

So the shock will move forward and this solution will hold until the shock meets the rightmost edge of the fan, which is located at $x = 1$. There $u_+ = 0$, hence the shock condition will change to $\dot{s} = 1/2$ and the shock will move with constant speed after that. The meeting of the shock and the right edge of the fan happens when $s = t + 1 - 2\sqrt{t} = 1$, i.e. at $t = 4$.

Thus, we have the following solution:

$t \in (0,1)$:

$$u(x,t) = \begin{cases} 1, & x \le 0 \\ \frac{x-1}{t}, & -1 < \frac{x-1}{t} \le 0 \\ 0, & \frac{x-1}{t} > 0 \end{cases}$$

$t \in [1,4]$:

$$u(x,t) = \begin{cases} 1, & x \le s(t) \\ \frac{x-1}{t}, & s(t) < x \le 1 \\ 0, & x > 1 \end{cases}, \quad s(t) = t + 1 - 2\sqrt{t}$$

$t \in [4,\infty)$:

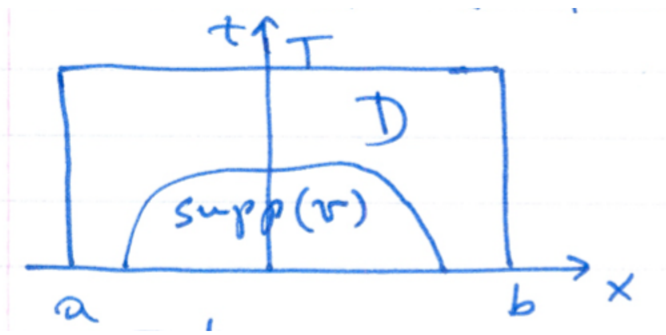

FIGURE 4.1. For derivation of weak solution.

$$u(x,t) = \begin{cases} 1, & x \le t/2 \\ 0, & x > t/2 \end{cases}.$$

4.3. **Weak solutions.** Discontinuous solutions of PDE are made sense of by generalizing the concept of a solution as a weak solution. Instead of requiring that a function u solves a PDE, we require a weaker condition, that it satisfies some integral condition. This integral condition is motivated by the following reasoning.

Suppose we have a conservation law

$$u_t + F(u)_x = 0 \tag{4.2}$$

with some initial condition $u(x,0) = g(x)$ and $x \in \mathbb{R}$, $t > 0$. Assume u is a classical solution that is C^1 in both t and x (i.e. continuously differentiable in t and x). Introduce a smooth function v that has compact support (i.e. is zero outside of some finite domain in x, t plane). This v is called a *test function.* Multiply (4.2) by v and integrate it over some domain D that contains the support of v.

$$\begin{aligned}
&\int_0^T dt \int_a^b dx\,(vu_t + vF_x) = \text{ (integrate by parts to move the derivatives over to } v) \\
&\int_a^b \left[vu|_0^T - \int_0^T uv_t dt \right] dx + \int_0^T \left[vF|_a^b - \int_a^b F v_x dx \right] dt = \\
&\quad - \int_a^b u(x,0)\,v(x,0)\,dx - \int_0^T \int_a^b uv_t dx dt - \int_0^T \int_a^b dt dx F v_x = 0.
\end{aligned}$$

Therefore

$$\int_0^T \int_a^b (uv_t + Fv_x)\,dxdt + \int_a^b g(x)\,v(x,0)\,dx = 0, \tag{4.3}$$

which means that any classical solution of (4.2) satisfies this equation for any test function v. Now we use (4.3) as *a definition of a weak solution* that allows for non-smooth functions u and $F(u)$. It will contain classical solutions as a special case.

How does this definition treat discontinuities? In particular, what are the jump conditions? They better agree with the Rankine-Hugoniot conditions used above.

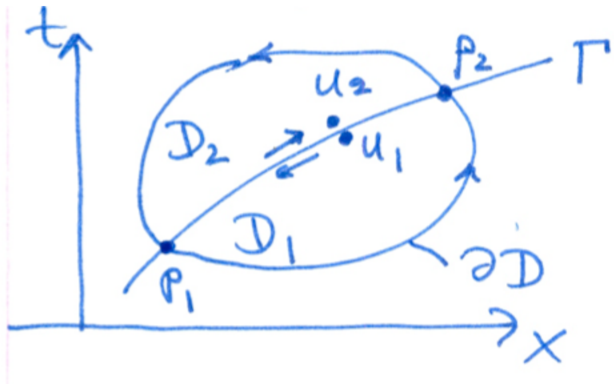

FIGURE 4.2. For derivation of the Rankine-Hugoniot condition.

To derive the jump conditions, consider the domain shown with Γ denoting the shock path. Let v be a test function such that $v = 0$ on ∂D. Then

$$\int_D (uv_t + Fv_x)\, dxdt = 0 = \int_{D_1} + \int_{D_2}.$$

$$\begin{aligned}
\int_{D_1} &= \int_{D_1} [uv_t + Fv_x + v\,(u_t + F_x)]\, dxdt = \\
&= \int_{D_1} [(uv)_t + (Fv)_x]\, dxdt = \text{(using Green's theorem)} \\
&= \int_{\partial D_1} (-uvdx + Fvdt) = \\
&= \int_{p_2}^{p_1} v\,(-u_1 dx + F_1 dt)\,.
\end{aligned}$$

Similarly

$$\int_{D_2} = \int_{p_1}^{p_2} v\,(-u_2 dx + F_2 dt)\,.$$

Combining the two,

$$\int_{p_1}^{p_2} v\,[(u_1 - u_2)\, dx - (F_1 - F_2)\, dt] = 0.$$

Since v is arbitrary, then along Γ we must have

$$(u_1 - u_2)\, dx - (F_1 - F_2)\, dt = 0,$$

that is

$$\frac{dx}{dt}|_\Gamma = \dot{s} = \frac{F(u_2) - F(u_1)}{u_2 - u_1} = \frac{[F]}{[u]}.$$

This is exactly the Rankine-Hugoniot condition.

Note: Weak solutions depend on the specific form of the conservation law. For example, starting with $u_t + uu_x = 0$, we could derive two different conservation laws:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \text{ and}$$
$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0.$$

These two would be equivalent for smooth solutions, but they give different weak solutions! Indeed, if we consider the same initial value problem as above with $f = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0, \end{cases}$ then the shock speeds in these two conservation laws are $\dot{s} = \frac{1}{2}$ and

$$\dot{s} = \frac{2u_+^3/3 - 2u_-^3/3}{u_+ - u_-} = \frac{-2/3}{-1} = \frac{2}{3}.$$

And this means different solutions.

This problem is resolved by physics – we derive PDE from integral conservation laws, not the other way around.

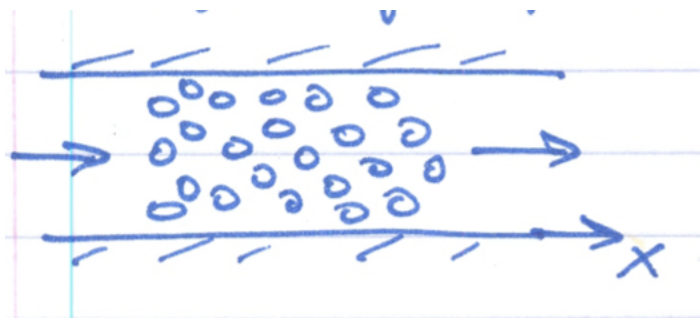

FIGURE 4.3. Gas flow in a porous medium.

4.4. **Reaction-diffusion equations.** Next, we consider equations of the following type:

$u_t = (D(u)\, u_x)_x$ nonlinear diffusion equation

$u_t + uu_x = \nu u_{xx}$ Burgers equation

$u_t = (u^n)_{xx}$ porous media equation

$u_t = Du_{xx} + f(u)$ reaction-diffusion equation

$u_t = \nabla \cdot D\nabla u + f(u)$ reaction-diffusion system

The porous media equation arises from the following considerations.

Let α denote the fraction of space occupied by gas, so that $1-\alpha$ is the fraction occupied by solid phase (e.g. densely packed particles). Consider the 1D case for simplicity. Then conservation of mass for a volume between positions x_1 and x_2 is given by

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\alpha dx = \phi(x_1,t) - \phi(x_2,t),$$

where $\phi(x,t)$ is the flux function equal to the mass crossing the given section at x per unit time per unit area. Localizing the previous balance, we write

$$(\alpha\rho)_t + \phi_x = 0.$$

The flux is given by

$$\phi = \alpha\rho v$$

where v is the gas velocity. Often the latter is taken to follow Darcy's law

$$v = -Kp_x,$$

where K is called a permeability of the porous medium, typically a function of porosity α, p is pressure. If we further assume the so-called polytropic gas, then $p = a\rho^\gamma$, then with α, a, K taken as constants, we arrive at

$$\rho_t = (\rho^\gamma \rho_x)_x$$

or

$$\rho_t = (\rho^n)_{xx}.$$

4.5. **Similarity solutions.** Sometimes, one can transform a given PDE into an ODE in terms of new variables that are combinations of the old. We have seen an example of that in the rarefaction solution of the Hopf equation. The solution u depended only on $\xi = x/t$, which was the similarity variable.

Consider a general first order PDE

$$G(x,t,u,p,q) = 0,$$

where $p = u_x$ and $q = u_t$.

Then a one-parameter family of *stretching transformations* is a scaling of uxt space as:

$$\bar{x} = \epsilon^a x, \quad \bar{t} = \epsilon^b t, \quad \bar{u} = \epsilon^c u,$$

where a, b, c are constants, ϵ is a real parameter in an interval I containing $\epsilon = 1$.

PDE $G = 0$ is said to be invariant under the stretching transformation if there is a smooth $f(\epsilon)$ such that

$$G(\bar{x}, \bar{t}, \bar{u}, \bar{p}, \bar{q}) = f(\epsilon)\, G(x, t, u, p, q)$$

for any $\epsilon \in I$ and $f(1) = 1$.

Theorem. *If the PDE $G = 0$ is invariant under the above stretching transformation, then*

$$u = t^{c/b} y(z), \quad z = \frac{x}{t^{a/b}} \quad \textit{(similarity transformation)}$$

reduces the PDE to an ODE of the form

$$g(z, y, y') = 0.$$

Here z is called a *similarity variable* and $u = t^{c/b} y(z)$ - *self-similar (similarity) solution.*

Actually, the same transformation converts a more general PDE of the form $G(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}) = 0$ to a second order ODE of the form $g(z, y, y', y'') = 0$.

Example 17. Consider the Hopf equation $u_t + uu_x = 0$.

Invariance: $\bar{x} = \epsilon^a x$, $\bar{t} = \epsilon^b t$, $\bar{u} = \epsilon^c u$, hence

$$\epsilon^{c-b} \bar{u}_{\bar{t}} + \epsilon^{2c-a} \bar{u}\bar{u}_{\bar{x}} = 0, \quad \text{invariant if } c - b = 2c - a.$$

Similarity transformation:

$$z = \frac{x}{t^{a/b}} = \frac{x}{t^{a/(a-c)}}, \quad u = t^{\frac{c}{a-c}} y(z)$$

leads to

$$u_t + uu_x = t^{-\frac{a}{a-c}} \left[\frac{c}{a-a} y - \frac{a}{a-c} zy' + yy' \right] = 0,$$

hence

$$\frac{c}{a-a} y - \frac{a}{a-c} zy' + yy' = 0.$$

Note that if $c = 0$, then we find $z = x/t$ and $u = y(z)$ as in the fan solution before.

Example 18. Consider the diffusion equation $u_t = Du_{xx}$.

Invariance: $\bar{x} = \epsilon^a x$, $\bar{t} = \epsilon^{2a} t$, $\bar{u} = \epsilon^c u$ leads to

$$u_t - Du_{xx} = \epsilon^{c-b} \left(\bar{u}_{\bar{t}} - D\bar{u}_{\bar{x}\bar{x}} \right) = 0.$$

Similarity transformation:

$$z = \frac{x}{t^{a/b}} = \frac{x}{\sqrt{t}}, \quad u = t^{\frac{c}{2a}} y(z)$$

leads to

$$Dy'' + \frac{z}{2} y' - \frac{c}{2a} y = 0.$$

Example 19. Consider the nonlinear diffusion equation

$$u_t = (uu_x)_x$$

subject to $u(\pm\infty, t) = 0$ and $\int_{-\infty}^{\infty} u\,dx = 1$.

Invariance: $\bar{x} = \epsilon^a x$, $\bar{t} = \epsilon^b t$, $\bar{u} = \epsilon^c u$, hence

$$\epsilon^{c-b} \bar{u}_{\bar{t}} = \epsilon^{2c-2a} \left(\bar{u}\bar{u}_{\bar{x}}\right)_{\bar{x}}, \quad \text{invariant if } c - b = 2c - 2a.$$

Similarity transformation:

$$z = \frac{x}{t^{a/b}}, \quad u = t^{\frac{c}{b}} y\left(z\right) = t^{\frac{2a}{b}-1} y\left(z\right).$$

From $\int u dx = 1$ we get $a/b = 1/3$ hence

$$u = t^{-1/3} y\left(z\right), \quad z = t^{-1/3} x.$$

The resultant ODE is

$$\left(yy'\right)' + \frac{1}{3}\left(y + zy'\right) = 0.$$

Integrate once to get

$$yy' + \frac{1}{3} zy = const.$$

For further progress, one needs to note that the PDE is invariant wrt $x \to -x$, hence $u\left(-x, t\right) = u\left(x, t\right)$, so that $u_x\left(0, t\right) = 0$. Therefore, $y'\left(0\right) = 0$ and $const = 0$. Then

$$y\left(y' + \frac{1}{3} z\right) = 0.$$

A nontrivial solution of this is

$$y = C - \frac{z^2}{6}.$$

The trouble with this solution is that it does not vanish at $z \to \pm\infty$.

A way out is to let $y = 0$ outside of some $|z| = A$, so that

$$y = \begin{cases} \frac{A^2 - z^2}{6}, & |z| \le A, \\ 0, & |z| > A. \end{cases}$$

This A is determined by $\int_{-A}^{A} y dz = 1$: $A = (9/2)^{1/3}$. Therefore, the full solution is

$$u\left(x, t\right) = \begin{cases} \frac{1}{6t^{1/3}}\left[\left(\frac{9}{2}\right)^{2/3} - \frac{x^2}{t^{2/3}}\right], & |x| < \left(\frac{9}{2}\right)^{1/3} t^{1/3} \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

The front of this diffusion wave propagates along

$$x_f = \left(\frac{9}{2}\right)^{1/2} t^{1/3}.$$

Note that this front is sharp and propagates at finite speed in contrast to the linear diffusion equation.

4.6. **Traveling-wave solutions.** Here we investigate special solutions representing waves traveling at a constant speed - solutions of the type $u\left(x - ct\right)$ with constant c.

For example, consider

$$u_t = Du_{xx} + r\left(u - \frac{1}{k}u^2\right) \qquad \text{Kolmogorov-Petrovskii-Piskunov or Fisher equation.}$$

With $\bar{t} = rt$, $\bar{x} = x/\sqrt{D/r}$ and $\bar{u} = u/k$, this equation can be cleaned up (dropping overbars afterwards):

$$u_t - u_{xx} = u\left(1 - u\right).$$

FIGURE 4.4. Similarity solutions (4.4).

We seek TWS of this equation

$$u = U(z), \quad \text{where } z = x - ct,$$

subject to far-field conditions

$$U(\pm\infty) = u_{\pm} = const.$$

Then

$$\begin{aligned} -cU' - U'' &= U(1-U) \\ U(+\infty) &= u_+ \\ U(-\infty) &= u_-. \end{aligned}$$

We convert this equation to a system of first order equations for $w = [U, V]^T$, where $V = U'$.

$$w' = F(w) = \begin{bmatrix} V \\ -cV - U(1-U) \end{bmatrix}.$$

This is a system of standard type we have analyzed before. Its fixed points are $P = (0,0)$ and $Q = (1,0)$. The Jacobian is

$$A = \frac{\partial F}{\partial w} = \begin{bmatrix} 0 & 1 \\ -1 + 2U & -c \end{bmatrix}$$

and at P it is

$$A_P = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}, \quad \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - 1},$$

while at Q it is

$$A_Q = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}, \quad \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 1}.$$

Therefore, P is a stable spiral if $|c| < 2$ and a stable node if $|c| > 2$, while Q is a saddle point for any c.

Now, since U must remain positive (as the density of a population), the spiral must be excluded. Therefore, the wave speed must be constrained as $|c| > 2$. For any $c > 2$, we can connect the stable node and the saddle with a trajectory, hence can find the TWS of the problem. The solution profile is found by integrating the ODE system subject to $U(\pm\infty) = u_\pm$ for any given c.

Following similar reasoning, one can obtain the TWS of Burgers equation

$$u_t + uu_x = \nu u_{xx}.$$

If we assume $U(-\infty) = u_2$ and $U(+\infty) = u_1 < u_2$, then $U(z)$ solves

$$\begin{aligned} -cU' + UU' - \nu U'' &= 0 \\ -cU + \frac{U^2}{2} - \nu U' &= A \\ U' &= \frac{1}{\nu}\left[\frac{U^2}{2} - cU - A\right] \end{aligned}$$

with $U' \to 0$ as $|z| \to \infty$. We obtain the TWS of the form

$$U(z) = u_1 + \frac{u_2 - u_1}{1 + e^{z/\delta}}, \quad z = x - ct$$

where

$$\delta = \frac{2\nu}{u_2 - u_1}$$

and the wave speed

$$c = \frac{u_1 + u_2}{2},$$

which is exactly the same as that obtained via the Rankine-Hugoniot conditions in the inviscid case.

4.7. **Instabilities and Turing patterns.** Finally, we look at systems of reaction-diffusion PDE of the type that arise in population dynamics and reacting mixtures.

$$\begin{aligned} u_t &= D_1 u_{xx} + au - buv, \quad \text{prey} \\ v_t &= D_2 v_{xx} - cv + duv, \quad \text{predator} \end{aligned}$$

$$\begin{aligned} a_t &= Da_{xx} - f(a, T), \quad \text{chemical reaction equation} \\ T_t &= kT_{xx} + qf(a, T), \quad \text{energy equation} \end{aligned}$$

Typically,

$$f = Ka^m e^{-\frac{E}{RT}}, \quad \text{Arrhenius law.}$$

One can look for TWS of such systems and they are of great importance. Their analysis proceeds in principle as before, but may be technically quite involved in practice.

We next look at a simple example in which a rather counterintuitive phenomenon occurs – instability due to diffusion. We usually expect diffusion to play a stabilizing role. However, it may act a destabilizer.

Here is the problem.

Let

$$w_t = Dw_{xx} + f(w), \tag{4.5}$$

where w is a vector, for example, $w = [u, v]^T$, D is a matrix of diffusion coefficients, and f is some source function. We will assume that $x \in [0, L]$ and the boundary conditions are $u_x = 0$ at both ends.

Suppose w_0 is a uniform (i.e. with no x-dependence) solution of the system, then $f(w_0) = 0$.

The question we ask is:

Is this equilibrium solution w_0 stable?

That is, if we take $w(x,0) = w_0 + \epsilon w_1(x)$, $\epsilon \ll 1$ as an initial condition for (4.5), will the solution $w(x,t)$ tend to w_0 as $t \to \infty$ for any $w_1(x)$ and $\epsilon \to 0$?

To analyze the situation, substitute $w(x,t) = w_0 + \epsilon w_1(x,t)$ into (4.5) and linearize by dropping all terms smaller than $O(\epsilon)$. Then

$$\begin{aligned} w_{1,t} &= D w_{1,xx} + J w_1, \\ w_{1,x}(0) &= 0, w_{1,x}(L) = 0, \end{aligned} \tag{4.6}$$

where

$$J = \frac{\partial f}{\partial w}(w_0).$$

Since D and J are constant matrices, we can look for separable solutions (components of the Fourier cosine series) of the form

$$w_1 = c e^{\sigma t} \cos\left(\frac{n\pi x}{L}\right).$$

Putting this into (4.6):

$$\left(J - \sigma I - \left(\frac{n\pi}{L}\right)^2 D\right) c = 0.$$

For nontrivial solutions

$$\det\left(J - \sigma I - \left(\frac{n\pi}{L}\right)^2 D\right) = 0. \tag{4.7}$$

The roots of this equation give the *growth rate* σ for a given mode n (Fourier component). And if $Re(\sigma) > 0$ for any n, then that mode will be unstable. This kind of instability is called *Turing instability*.

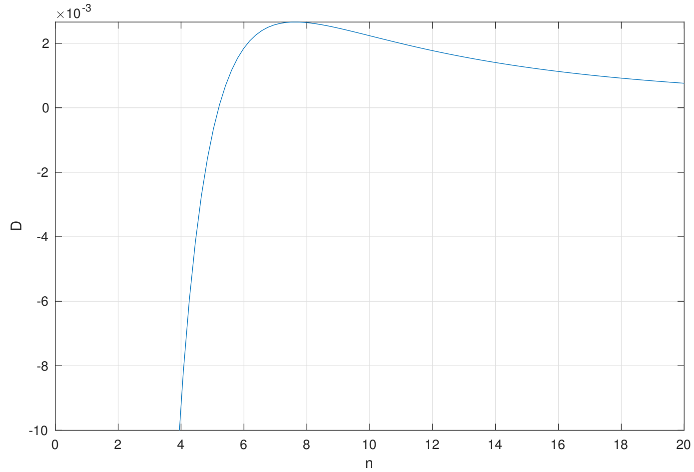

FIGURE 4.5. Function $D = \frac{n^2-27}{3n^2(n^2+9)}$.

Example 20. Consider the following example.

$$
\begin{aligned}
&u_t = Du_{xx} + 1 - u + u^2 v, \\
&v_t = v_{xx} + 2 - u^2 v, \\
&x \in (0,\pi)\,, \quad u_x = v_x = 0 \text{ at } x = 0, \pi.
\end{aligned} \tag{4.8}
$$

This system has a fixed point $w_0 = [3, 2/9]^T$. If there is no space dependence, then the system is

$$
\begin{aligned}
&u_t = 1 - u + u^2 v, \\
&v_t = 2 - u^2 v,
\end{aligned}
$$

with the Jacobian

$$
J = \begin{bmatrix} -1+2uv & u^2 \\ -2uv & -u^2 \end{bmatrix}
$$

which at the fixed point is

$$
J(w_0) = \begin{bmatrix} 1/3 & 9 \\ -1/3 & -9 \end{bmatrix}.
$$

It has $\tau = -26/3 < 0$ and $\Delta = 0$, so the fixed point is stable. Next, we consider the spatial dependence and see if there is instability at any D.

Analysis of (4.7) shows:

- ⋄ stability at all D if $n \leq 5$;
- ⋄ instability if $D < D_c \approx 0.0026$ for all n satisfying $D < \frac{n^2-27}{3n^2(n^2+9)}$. Stability otherwise.

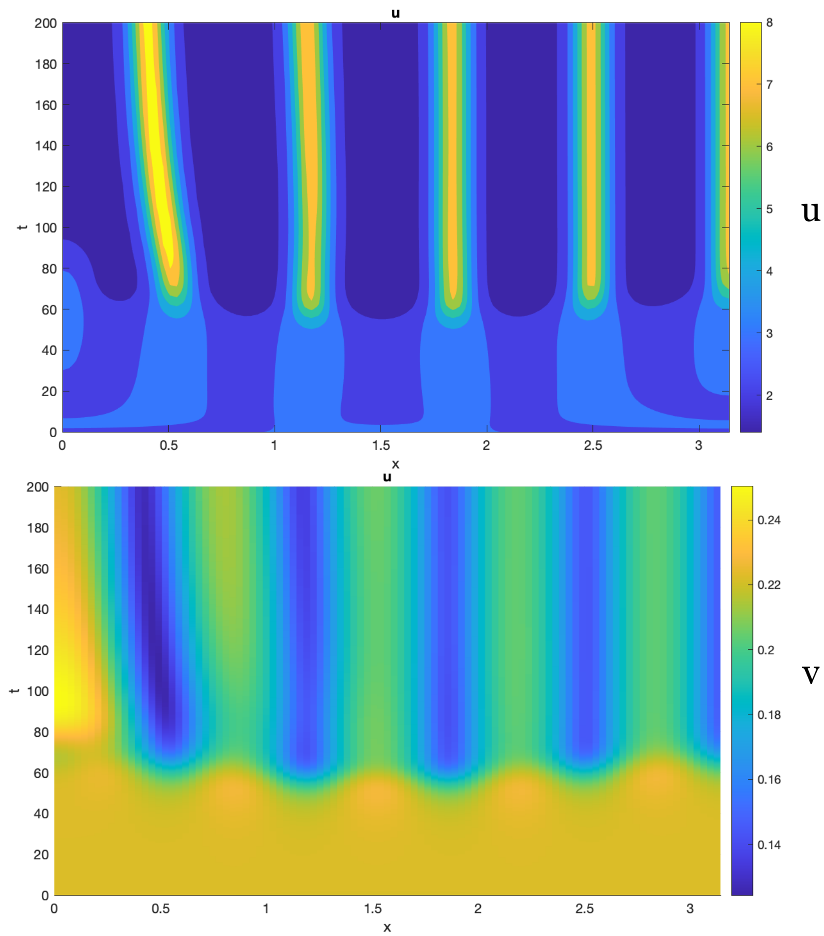

FIGURE 4.6. Patterns that form with $D = 0.001$ and initial condition $u(x,0) = 3 + 0.01 \cdot (1 < x < 2)$, $v(x,0) = 2/9$. These are obtained by solving system (4.8) numerically. Note that these patterns at large t are steady-state solutions of (4.8), and are analogs of fixed points of ODE. With PDE we have patterns (sometimes steady as here, but sometimes unsteady) instead of just fixed points in the phase plane.