SOLUTIONS TO PROBLEM SET #4

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Problem 1. Explain:

(a): why A^TA is not singular when matrix A has independent columns;
(b): why A and A^TA have the same nullspace.

Solution. (a) Assume that A has independent columns, but A^TA is singular. Then, there exist a non-zero vector x such that $A^TAx = 0$. Multiplying with x^T from the left, we have $x^TA^TAx = \|Ax\|_2^2 = 0$ that is possible if and only if $Ax = 0$. This contradicts independency of the columns of A, because x gives non-zero coefficients of a zero linear combination of the columns.

(b)

(1)

$$x \in N(A) \iff Ax = 0 \Rightarrow A^TAx = 0 \iff x \in N\left(A^TA\right) \Rightarrow N(A) \subset N\left(A^TA\right)$$

(2)

$$\begin{aligned} x \in N\left(A^TA\right) &\iff A^TAx = 0 \Rightarrow x^TA^TAx = 0 \iff \|Ax\|_2^2 = 0 \\ &\iff Ax = 0 \iff x \in N(A) \Rightarrow N\left(A^TA\right) \subset N(A) \end{aligned}$$

This means that $N(A) = N\left(A^TA\right)$.

Problem 2. A plane in $\mathbb{R}^3$ is given by the equation $x_1 - 2x_2 + x_3 = 0$.

(a): Identify two orthonormal vectors u_1 and u_2 that span the plane.
(b): Find a projector matrix P that projects any vector x from $\mathbb{R}^3$ to the plane and a projector $P_\perp$ that projects any vector to the direction normal to the plane.
(c): Using these projectors find the unit normal to the plane and verify that it agrees with a normal found by calculus methods (that use the gradient).

Solution. (a) We can start with two linearly independent vectors from a plane, say $v_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and $v_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$, and then normalize the first one while projecting the second vector orthogonally to the first and normalizing it:

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\left(I - u_1u_1^T\right)v_2 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

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(b) Since vectors u_1 and u_2 form an orthonormal set, we can find the projector onto the plane using the matrix with these vectors as columns:

$$P = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

$$P_{\perp} = I - P = \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

(c) If we take a function $F(x) = x_1 - 2x_2 + x_3$, then the plane is the zero level set of this function (plane $= \{x : F(x) = 0\}$). The gradient is normal to the level sets and

$$n_1 = \nabla F / \|\nabla F\| = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

is the unit normal to the plane. Another way to find the unit normal is to take an arbitrary vector that does not belong to the plane and project it on the direction perpendicular to the plane:

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_{\perp} v = \frac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad n_2 = \frac{P_{\perp} v}{\|P_{\perp} v\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

So, we get that $n_1 = n_2$, but note that, if we chose a vector v pointing to the other side of the plane, we would have $n_2 = -n_1$.

Problem 3. Let $M = \operatorname{span}\{v_1, v_2\}$ where $v_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}^T$, $v_2 = \begin{pmatrix} 1 & -1 & 0 & -1 \end{pmatrix}^T$.

(a): Find the orthogonal projector P_M on M.
(b): Find the kernel (nullspace) and range (column space) of P_M .
(c): Find $x \in M$ which is closest in 2-norm to the vector $a = \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$.

Solution. (a) We see that $v_1 \perp v_2$ and we can normalize them to construct the projector easily:

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix},$$

$$P_M = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

(b) The range of any projector is a space onto which it projects, so $C(P_M) = M$. From the fact that $N(P_M) = C(I - P_M)$, we can derive the following

$$I - P_M = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \Rightarrow C(I - P_M) = N(P_M) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

(c) From the seminar notes, $x = P_M a = \frac{1}{3} \begin{pmatrix} 4 & -3 & 1 & -2 \end{pmatrix}^T$.

Problem 4. The following problems look at tests of positive definiteness

(a) Using the determinant test, find c and d that make the following matrices positive definite:

$$A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

(b) A positive definite matrix cannot have a zero (or a negative number) on its main diagonal. Show that the matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

is not positive definite by finding x such that $x^T Ax \leq 0$.

Solution. (a) For A to be positive definite, the following conditions should be satisfied simultaneously

$$c > 0, \quad \begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} = c^2 - 1 > 0, \quad \begin{vmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{vmatrix} = c^3 - 3c + 2 > 0.$$

Solving this inequalities, we have that $A > 0$ for any $c > 1$.

For B to be positive definite, the following conditions should be satisfied simultaneously

$$\begin{vmatrix} 1 & 2 \\ 2 & d \end{vmatrix} = d - 4 > 0, \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{vmatrix} = -4d + 12 > 0.$$

Solving this inequalities, we see that there is no d such that $B > 0$.

(b) Taking $v = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, we get $v^T Av = 0$.

Problem 5. Matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ is positive definite. Explain why and determine the minimum value of $z = x^T Ax + 2b^T x + 1$, where $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ and $b^T = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$.

Solution. $A > 0$ because $1 > 0$, $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 > 0$ and $\det A = 4 > 0$. To find minimum, we write z with a symmetric matrix $A^{sym} = \dfrac{A + A^T}{2}$ such that

$$z = x^T Ax + 2b^T x + 1 = x^T A^{sym} x + 2b^T x + 1,$$

and in this case

$$\frac{\partial z}{\partial x} = 2A^{sym} x + 2b = 0 \Rightarrow x_{min} = -\left(A^{sym}\right)^{-1} b, \; z_{min} = 1 + b^T x_{min}.$$

So,

$$A^{sym} = \begin{pmatrix} 1 & 3/2 & 1/2 \\ 3/2 & 3 & 1 \\ 1/2 & 1 & 4 \end{pmatrix}, \quad x_{min} = \frac{1}{11}\begin{pmatrix} -88 \\ 53 \\ -5 \end{pmatrix}, \quad z_{min} = -\frac{188}{11}.$$

Problem 6. Explain these inequalities from the definition of the norms:

$$\|ABx\| \leq \|A\|\|Bx\| \leq \|A\|\|B\|\|x\|,$$

and deduce that $\|AB\| \leq \|A\|\|B\|$.

Solution. If $x = 0$, then the inequalities are trivial. Assume now that $x \neq 0$. From the definitions of the matrix norm of A and B

$$\|A\| = \sup_{y \neq 0} \frac{\|Ay\|}{\|y\|}, \ \|B\| = \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|}$$

we state that for any $y \neq 0$ $\|A\| \geq \|Ay\|/\|y\|$ and for any $x \neq 0$ $\|B\| \geq \|Bx\|/\|x\|$. Now take $y = Bx$. If $y = 0$, then the inequalities are trivial. If $y \neq 0$, then

$$\|ABx\| \leq \|A\|\|Bx\|$$

and since $\|Bx\| \leq \|B\|\|x\|$, we obtain the necessary chain of inequalities. Finally, we see that for any $x \neq 0$

$$\|A\|\|B\| \geq \frac{\|ABx\|}{\|x\|}$$

and taking the supremum of the both sides we deduce that $\|AB\| \leq \|A\|\|B\|$.

Problem 7. Compute by hand the norms and condition numbers of the following matrices:

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution. Let's use 2-norm. For A_1 we have:

$$|A_1 - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

$$A_1^T = A_1 \Rightarrow \|A_1\|_2 = |\lambda_{max}| = 3, \ \kappa_2 = \frac{|\lambda_{max}|}{|\lambda_{min}|} = \frac{3}{1} = 3.$$

For A_2 we have:

$$A_2^T A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$

$$A_2^T \neq A_2 \Rightarrow \|A_2\|_2 = \sigma_1 = \sqrt{2}, \ \kappa_2 = \frac{\sigma_1}{\sigma_2} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$