3. Lectures 21, 22. Nov. 23, 24

Plan:

- ⋄ Derivation of basic PDE.
- ⋄ Random walk to diffusion.
- ⋄ Elastic waves and continuum limit of the spring-mass system.
- ⋄ Linear PDE. Initial-boundary-value problems. Separation of variables.

3.1. Main applied PDE.

⋄ Heat/diffusion equation

$$u_t = D\Delta u, \quad D > 0, \text{ the diffusion coefficient}$$

⋄ Wave equation

$$u_{tt} = c^2 \Delta u, \quad c \text{ the wave speed}$$

⋄ Laplace's equation

$$\Delta u = 0, \quad \text{equilibrium state}$$

⋄ Poisson's equation

$$\Delta u = f(x), \text{ forced equilibrium state}$$

⋄ Burgers equation (nonlinear advection-diffusion equation)

$$u_t + uu_x = \nu u_{xx}, \quad \nu \text{ the viscosity/diffusion coefficient.}$$

⋄ Korteweg-de Vries equation of solitons

$$u_t + uu_x + \nu u_{xxx} = 0.$$

⋄ Shrödinger's equation

$$i\hbar u_t = -\frac{\hbar^2}{2m}\Delta u + U(x)\, u.$$

⋄ Incompressible Navier-Stokes equations

$$\nabla \cdot u = 0, \qquad \text{incompressibility}$$

$$u_t + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + \nu \Delta u, \qquad \text{equation of motion}$$

⋄ Compressible Euler equations

$$\begin{aligned}
\rho_t + u \cdot \nabla \rho &= -\rho \nabla \cdot u \\
u_t + u \cdot \nabla u &= -\frac{1}{\rho}\nabla p \\
e_t + u \cdot \nabla e &= -\frac{p}{\rho}\nabla \cdot u,
\end{aligned}$$

where $e = e(p, \rho)$ is the equation of state, for example, $e = c_v RT = \frac{1}{\gamma-1}\frac{p}{\rho}$ for the ideal gas with specific heat ratio γ. Then the system is closed.

⋄ Maxwell's equations

$$\begin{aligned}
\nabla \cdot E &= \rho/\epsilon_0 \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times B &= \mu_0 \left(j + \epsilon_0 \frac{\partial E}{\partial t} \right)
\end{aligned}$$

◇ Linear elasticity

$$\begin{aligned}
\rho\ddot{u} &= \nabla\cdot\sigma+\mathbf{f}\\
\sigma &= 2\mu E+\lambda\left(\mathrm{Tr}E\right)I\\
E &= \frac{1}{2}\left(\nabla u+\nabla u^{T}\right)\\
\rho\ddot{u} &= \mu\Delta u+\left(\mu+\lambda\right)\nabla\left(\nabla\cdot u\right)+f.
\end{aligned}$$

◇ Reaction-diffusion systems in population dynamics or chemically reacting flows

$$u_t = D\Delta u + f\left(u\right).$$

3.2. **Derivation of the main PDE** . . . .

3.3. **Heat equation.** . . . Here we derive the heat equation considering heat conducting material

3.4. **Diffusion equation from random walk.** . . . . Here we derive the heat equation considering random walk of a particle in 1D. Let a particle hop on a lattice $x\pm nh$,$n=0,1,2,...$. Assume that a particle at site x can hop to the right with probability p, and to the left with probability $q=1-p$. The time interval between hops is τ.

Let $u\left(x,t\right)$ be the probability that the particle is at position x at time $t+\tau$. Then this probability is the sum of two probabilities:

◇ the particle was at $x+h$ at time t and hopped left: probability $u\left(x+h,t\right)q$.
◇ the particle was at $x-h$ at time t and hopped right: probability $u\left(x-h,t\right)p$.

Therefore,

$$u\left(x,t+\tau\right)=u\left(x+h,t\right)q+u\left(x-h,t\right)p.$$

Now Taylor expand with $\tau\to0$ and $h\to0$:

$$u+u_t\tau+...=\left(u+u_xh+\frac{1}{2}u_{xx}h^2+...\right)q+\left(u-u_xh+\frac{1}{2}u_{xx}h^2+...\right)p$$

$$u+u_t\tau+...=u-\left(p-q\right)u_xh+\frac{1}{2}u_{xx}h^2+...$$

Therefore, neglecting higher-order terms,

$$u_t+\frac{h}{\tau}\left(p-q\right)u_x=\frac{h^2}{2\tau}u_{xx}.$$

This is the advection-diffusion equation

$$u_t+cu_x=Du_{xx}.$$

The drift term is proportional to

$$c=\left(p-q\right)\frac{h}{\tau}.$$

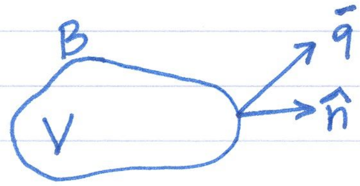
When it is positive, the drift is to the right. When $p=q$, there is no drift. The diffusion coefficient is

$$D=\frac{h^2}{2\tau}.$$

Lecture 2: Equations of heat conduction, diffusion, random walk, vibrating string.

~~The vibrating string~~: Heat conduction/diffusion.

Consider a solid ~~body~~ heat conducting body and a volume element V in it bounded by boundary B.



Main idea: Consider the balance of thermal energy.

$$\underbrace{\frac{d}{dt}\int_V \rho e \, dV}_{\text{Rate of change of energy in } V} = \underbrace{\int_V f \, dV}_{\text{Rate of production by body sources (e.g. chemistry)}} - \underbrace{\int_B \bar{q}\cdot\hat{n} \, dS}_{\text{Rate of heat flow in/out across the boundary } B}$$

e – thermal energy per unit mass

$\bar{q}$ – heat flux vector, $\bar{q}\cdot\hat{n}$ – the amount of thermal energy flowing across a unit surface per unit time.

Let $\rho = \rho(\bar{x})$, then using $\int_B \bar{q}\cdot\hat{n}\, ds = \int_V \nabla\cdot\bar{q}\, dV$, we obtain:

$$\int_V [\rho e_t + \nabla\cdot\bar{q} - f]\, dV = 0.$$

Since V is arbitrary and assuming continuity of $[\;]$, we find

$$\boxed{\rho e_t + \nabla\cdot\bar{q} - f = 0}$$

Still, too many unknowns: $e, \bar{q}, f$. Need constitutive laws.

$e = c\cdot u$, u – temperature, c – specific heat

$\bar{q} = -k\nabla u$, Fourier's law of heat conduction

$\Rightarrow$ assuming constant properties,

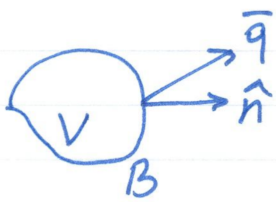
$$\boxed{u_t = a\Delta u + F(\bar{x}, t)}$$ Reaction-diffusion eqn

$\partial_t = 0$: $\Delta u = -F/a$ – Poisson

$F = 0$: $\Delta u = 0$ – Laplace

Diffusion & diffusion eq'n.

Let $C(\bar{x},t)$ represent the concentration (kg/m³) of one substance spreading in another, e.g. a pollutant in air. Similar to the heat balance, we now write down the eq's for the mass balance.



$$\frac{d}{dt}\int_V C\,dV = \int_V f\,dV - \int_B \bar{q}\cdot\hat{n}\,dS$$

$\bar{q}$ - the amount (mass) of C crossing B per unit area per unit time.

$$\Rightarrow \int_V \left[C_t + \nabla\cdot\bar{q} - f\right]dV = 0$$

$$\therefore \quad C_t + \nabla\cdot\bar{q} - f = 0 \quad \text{assuming continuity of } [\;]$$

$$\bar{q} = \underbrace{-D\nabla C}_{\substack{\text{diffusion}\\ \text{flux}\\ \text{(Fick's law)}}} + \underbrace{C\cdot\bar{v}}_{\substack{\text{bulk}\\ \text{flow}}}$$

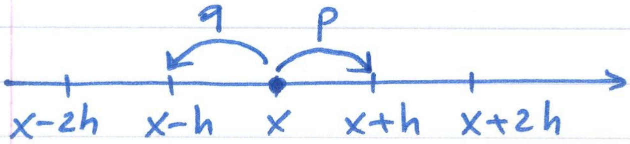
$$\Rightarrow \underbrace{C_t + \nabla\cdot C\bar{v}}_{\substack{\text{convection}\\ \text{(advection)}}} = \underbrace{\nabla(D\nabla C)}_{\text{diffusion}} + \underbrace{f}_{\text{reaction}}$$

Typically, $\bar{v}$ is given and not affected by diffusion and if, in addition, the flow is incompressible, then $\nabla\cdot\bar{v} = 0$, $\Rightarrow$

$$C_t + \bar{v}\cdot\nabla C = D\Delta C + f$$

If $\bar{v} = 0$, $f = 0$, $\Rightarrow$ $\boxed{C_t = D\Delta C}$ $\Leftarrow$ diffusion eq'n.

Random walk.



Consider a 1D lattice, $x \pm nh$, $n = 0, 1, 2, 3, \dots$

A particle at site x can jump to the right with probability p and left with probability q, each one over time τ. $p + q = 1$.
The process is called <u>random walk</u>.

Let $u(x,t)$ be the probability that the particle is at site x at time t. Our goal is to find an eq'n for u.

At time $t+\tau$ the particle can be at x due to:
1) jumping from $x+h$ to left with probability $u(x+h,t)\cdot q$

or 2) jumping from $x-h$ to right with probability $u(x-h,t)\cdot p$.

Then
$$\boxed{u(x, t+\tau) = u(x-h,t)\cdot p + u(x+h,t)\cdot q} \quad (*)$$

Now let $\tau, h \to 0$. Then expand in Taylor series:
$$u(x, t+\tau) = u(x,t) + u_t \cdot \tau + O(\tau^2)$$
$$u(x \pm h, t) = u(x,t) \pm u_x \cdot h + \tfrac{1}{2} u_{xx} h^2 + O(h^3)$$

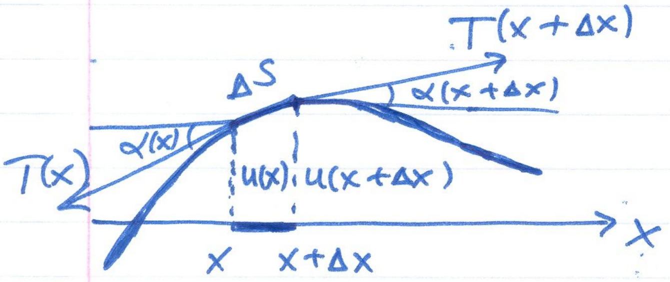
Put these into $(*)$ to obtain:
$$\boxed{u_t + \frac{h}{\tau}(p-q)\, u_x = \frac{h^2}{2\tau} u_{xx} + O(\tau) + O(h^3/\tau)}$$

1) $p = q$, $\Rightarrow$ $u_t = D u_{xx}$ with $D = \lim\limits_{\substack{h \to 0 \\ \tau \to 0}} h^2/2\tau$
pure diffusion

2) $p \neq q$, $\Rightarrow$ $u_t + c\, u_x = D u_{xx}$, $c = \lim \frac{h}{\tau}(p-q)$
advection-diffusion

for $c = O(1)$ we need $p - q = O(h)$

The vibrating string and wave eq'n.



Consider a string under tension $T(x,t)$.
Assume:
- no bending stiffness
- only transverse motion
- small displacement
- small slope.

Goal: Write down Newton's second law for the small element ΔS of mass Δm.

Let $\rho_0(x)$ be the linear density (mass/length) of the string in equilibrium, i.e. when horizontal. Since the motion is vertical the mass of the string element does not change during oscillations, i.e. $\boxed{\rho(x,t)\,ds = \rho_0(x)\,dx}$

No horizontal motion $\Rightarrow$

$$T(x+\Delta x,t)\cos\alpha(x+\Delta x,t) - T(x,t)\cos\alpha(x,t) = 0$$

i.e. $$\frac{\partial}{\partial x}\left[T(x,t)\cos\alpha(x,t)\right]\Delta x = 0 \quad \text{as} \quad \Delta x \to 0.$$

$\Rightarrow$ $T\cos\alpha$ depends only on t.
Since we assume $\alpha \to 0$, then $\cos\alpha \approx 1$, $\Rightarrow T = T(t)$.
For simplicity, take $T = \text{const}$.

In vertical direction:

$$\frac{d}{dt}\left(\int_S^{S+\Delta S} \rho u_t\, ds\right) = \sum \text{vertical forces}$$

$$\frac{d}{dt}\int_x^{x+\Delta x} \rho_0 u_t\, dx = T\sin\alpha(x+\Delta x,t) - T\sin\alpha(x,t) - \int_x^{x+\Delta x}\rho_0 g\, dx - \int_x^{x+\Delta x}\rho_0 \lambda u_t\, dx + \int_x^{x+\Delta x}\rho_0 f\, dx$$

where the last three terms are due to gravity, air resistance, and other forces.

Since $\sin\alpha \approx \tan\alpha = u_x$ as $\alpha \to 0$, we get

$$\int_x^{x+\Delta x} \rho_0 u_{tt}\, dx = T u_{xx} \cdot \Delta x + \int_x^{x+\Delta x} \rho_0 (f - g - \lambda u_t)\, dx$$

using $\int_x^{x+\Delta x} \phi\, dx = \phi(x)\Delta x$ as $\Delta x \to 0$, we obtain

$$\underbrace{\rho_0 u_{tt}}_{\text{mass} \times \text{acceleration}} = \underbrace{T u_{xx}}_{\text{tension}} - \underbrace{\rho_0 g}_{\text{gravity}} - \underbrace{\lambda \rho_0 u_t}_{\text{friction}} + \underbrace{\rho_0 f}_{\text{other forces.}}$$

$$\boxed{u_{tt} + \lambda u_t - c^2 u_{xx} = f - g} \qquad c^2 = T/\rho_0$$

Example: if $u_t = u_{tt} = 0$ and $f = 0$, then

$$u_{xx} = g/c^2 \Rightarrow u = \frac{g}{2c^2}x^2 + ax + b$$

If $u(0) = u(L) = 0$, $\Rightarrow \boxed{u(x) = -\frac{g}{2c^2}x(L-x)}$

Example: Let $u(x,t) = \sin kx \cdot y(t)$, $f = g = 0$.

$\Rightarrow \boxed{\ddot{y} + \lambda \dot{y} + c^2k^2 y = 0}$ ← damped oscillator eqn.

If $\lambda = 0$, $\Rightarrow$ time frequency for a wave of wavelength $2\pi/k$ is $\omega = ck = \frac{2\pi}{\lambda}\sqrt{\frac{T}{\rho_0}}$

3.5. **Wave equation from a chain of masses and springs. .** Recall the problem of a vertical chain of $n-1$ identical masses m and n identical springs with spring constants c. The length of the chain $l = na$ is assumed fixed, where a is the length of each spring (in a relaxed state). The forces on each mass are, for example, the forces of gravity, $f = mg$.

We now take the limit of $n \to \infty$ and $m \to 0$ to obtain a continuum version of the model. We will also look at the dynamic version, not just the static equilibrium.

Then, the equation of motion of i-th mass is

$$m_i \ddot{u}_i = k_{i+1}\left(u_{i+1} - u_i\right) - k_i\left(u_i - u_{i-1}\right) + m_i f_i, \tag{3.1}$$

where we have allowed for different masses and spring constants.

Suppose now that there is a function $u(x,t)$ such that $u(x_i,t) = u_i(t)$, that is $u(x,t)$ is an *interpolation* of u_i. Then, as $a \to 0$,

$$\underbrace{u(x_i + a, t)}_{u_{i+1}} = \underbrace{u(x_i, t)}_{u_i} + u_x(x_i,t)\,a + \frac{1}{2}u_{xx}(x_i,t)\,a^2 + \ldots$$

and

$$u(x_i - a, t) = u(x_i, t) - u_x(x_i,t)\,a + \frac{1}{2}u_{xx}(x_i,t)\,a^2 + \ldots.$$

Then

$$\begin{aligned} u_{i+1} - u_i &= u_x(x_i,t)\,a + \frac{1}{2}u_{xx}(x_i,t)\,a^2 + \ldots \\ u_i - u_{i-1} &= u_x(x_i,t)\,a - \frac{1}{2}u_{xx}(x_i,t)\,a^2 + \ldots. \end{aligned}$$

Putting this into 3.1 and neglecting higher order terms, we get

$$m_i u_{tt}(x_i,t) = a\left(k_{i+1} - k_i\right)u_x(x_i,t) + \frac{a^2}{2}\left(k_{i+1} + k_i\right)u_{xx}(x_i,t) + m_i f_i.$$

Now we divide by a and take the limit:

$$m_i \to 0, \quad a \to 0, \quad \text{so that } \frac{m_i}{a} \to \rho < \infty.$$

Here ρ is the linear density (mass per unit length).

Define

$$a k_i = E_i.$$

Then $x_i \to x$ and

$$\frac{m_i}{a} \to \rho, \quad k_{i+1} - k_i = \frac{E_{i+1} - E_i}{a} \to \frac{\partial E}{\partial x}(x,t), \quad a\frac{k_{i+1} + k_i}{2} \to E(x,t), \quad f_i \to f(x,t).$$

As a result

$$\rho u_{tt} = E_x u_x + E u_{xx} + f = \left(E u_x\right)_x + f.$$

And this is the wave equation for an elastic rod with possibly variable properties and body force f:

$$\rho u_{tt} = \left(E u_x\right)_x + f.$$

When $f = 0$ and ρ, E are constant, we get

$$u_{tt} = c^2 u_{xx}$$

with

$$c = \sqrt{\frac{E}{\rho}}.$$

Note that approximately, the Young's modulus in the spring-mass system is given by

$$E \approx ak$$

and the speed of sound is

$$c \approx \sqrt{\frac{ka^2}{m}}.$$

3.6. **Waves in an elastic rod.** .... Consider an elastic rod of cross section S that is subject to 1D longitudinal deformations along x axis. To investigate the deformations, which we assume small and time dependent, consider a small element dx between cross section x and $x+dx$. As the element deforms, its end at x will move to $x' = x + u(x,t)$, i.e. move by a distance u that depends on the chosen cross section x as well as on time. The other end of dx will move to $x' + dx' = x + dx + u(x+dx,t)$. Then the strain in the element, i.e. its relative deformation will be

$$\epsilon = \frac{dx' - dx}{dx} = \frac{u(x+dx,t) - u(x,t)}{dx} = \frac{\partial u}{\partial x}(x,t).$$

Hooke's law states that this strain is proportional to stress

$$\sigma = E\epsilon$$

with E denoting the Young's modulus of the material. Then the equation of motion of the material element dx becomes

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = S\sigma(x+dx,t) - S\sigma(x,t) = dx\frac{\partial}{\partial x}S\sigma(x,t).$$

Therefore

$$\rho S \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(ES\frac{\partial u}{\partial x}\right). \tag{3.2}$$

Here the density ρ, the cross section S, and Young's modulus E can depend on x, so that waves in a non-uniform rod are also described by (3.2).

For a uniform rod, all these parameters are constant and hence we obtain the standard wave equation

$$u_{tt} = c^2 u_{xx},$$

where

$$c = \sqrt{\frac{E}{\rho}}$$

is the longitudinal wave speed. It is the same speed that we obtained above in the continuum limit of the spring-chain system.

3.7. **Boundary conditions.** We formulate the main boundary conditions that arise in most applied problems. Their origin will be explained in the context of heat/diffusion problems and the wave equation.

3.7.1. *Dirichlet.* This is the condition requiring that the unknown is fixed at the boundary to be some given value.

For example, if we have a vibrating string between two end points, $x = 0$ and $x = l$. Then, various possible boundary conditions are:

- ◇ the string is tied at the ends. Then the displacement of the string from equilibrium must be 0 at the end points: $u(0,t) = u(l,t) = 0$.
- ◇ we move the end at $x = l$ as $u(l,t) = r(t)$ with a given function $r(t)$. It is non-homogeneous Dirichlet condition.

For heat conduction or diffusion problem, we fix the temperature or concentration:

- ◇ $T = f(x), x \in \partial D$ for $T_t = \kappa \Delta T$
- ◇ $c = f(x), x \in \partial D$ for $c_t = D\Delta T$

3.7.2. *Neumann.* This time, the condition is on the spatial derivative of the unknown.

For wave equation for a membrane $u_{tt} = c^2 \Delta u$, we could have some parts fixed (on ∂D_1), some free (∂D_2), in which case at the free boundary we have the Neumann condition

$$\mathbf{n} \cdot \nabla u = 0, \quad x \in \partial D_2.$$

For longitudinal elastic waves in a rod with one fixed (at $x = 0$) and one free end (at $x = l$), we find that

$$\begin{aligned} u(0,t) &= 0, \quad \text{fixed end} \\ u_x(l,t) &= 0, \quad \text{free end.} \end{aligned}$$

The free end condition coming from Hooke's law that says that the stress is $\sigma = E\epsilon = Eu_x$. And since the free end means the stress is absent, then we obtain that u_x must be zero at the free end.

For the heat equation, the Neumann condition represents no heat flux, or a prescribed heat flux,

$$\mathbf{n} \cdot \mathbf{q} = 0,$$

where the heat flux vector $\mathbf{q}$ (the amount of thermal energy crossing the boundary per unit time per unit area) is given by Fourier law, $\mathbf{q} = -\lambda \nabla T$, in which case the Neumann boundary condition becomes

$$\mathbf{n} \cdot \nabla T = 0.$$

For diffusion problems, similar condition arises due to Fick's law of diffusion, so that the diffusion flux (the mass of the diffusing substance crossing the boundary per unit time per unit area) is given by $\mathbf{q} = -D\nabla c$ and hence

$$\mathbf{n} \cdot \nabla c = 0.$$

3.7.3. *Robin.* More complicated boundary conditions do arise in applications. For example, if one has a body at high temperature and it loses thermal energy by radiation, then the flux of energy from the inner side of the boundary $-\lambda \mathbf{n} \cdot \nabla T$ must be balanced by the amount of the energy taken away by radiation $k\left(T^4 - T_0^4\right)$ via the Stefan-Boltzman formula, where T_0 is the outside temperature. Therefore

$$-\lambda \mathbf{n} \cdot \nabla T = k\left(T^4 - T_0^4\right).$$

Another possibility is to lose heat by Newton's law of cooling, which states that the heat flux between two bodies in contact is proportional to their temperature difference, $q = \mu(T - T_0)$, so that the boundary condition becomes

$$-\lambda \mathbf{n} \cdot \nabla T = \mu(T - T_0).$$

A boundary condition of this type is called a Robin condition. It includes both the unknown and its normal gradient at the boundary.

3.8. **Linear PDE. Initial-boundary-value problems.** .... Now we summarize the linear PDE, formulate initial and boundary conditions, and then try to seek their solutions. The focus is on the following basic equations:

$$
\begin{aligned}
u_t + a u_x &= 0, && \text{advection equation} \\
u_t &= D u_{xx}, && \text{heat/diffusion equation} \\
u_{tt} &= c^2 u_{xx}, && \text{wave equation} \\
\Delta u &= 0, && \text{Laplace's equation} \\
u_t &= \epsilon u_{xxx}. && \text{Airy equation}
\end{aligned}
$$

3.8.1. *Advection equation.* This is the most basic PDE,

$$u_t + a u_x = 0, \tag{3.3}$$

together with its initial condition

$$u(x,t) = f(x),$$

and it can be solved exactly using the following idea.

So our solution is a function of two variables, $u = u(x,t)$. We can calculate its derivatives with respect to x, u_x, to find out how the function varies with x if we keep t fixed, i.e. if we pretend that t is a constant. Or we can calculate u_t to see how u varies as t changes when x is kept fixed.

But now we ask a natural question: Why do we have to stick to these Cartesian axes of x and t to see how the function u varies? There is nothing special about these axes except for being so perpendicular. We can ask a more general question:

How does the function $u(x,t)$ vary if we follow some path $\Gamma : \{x = x(s), t = t(s)\}$ in the x,t plane which is parameterized by s?

Going along x axis or along t axis would be some special cases of this more general case.

To answer, we evaluate u on that path and calculate its derivative with respect to s:

$$\frac{d}{ds} u(x(s), t(s)) = u_t \frac{dt}{ds} + u_x \frac{dx}{ds}.$$

For further progress, we need to know what this path is, but we notice similarity with (3.3). If we chose the path such that

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = a, \tag{3.4}$$

then we get that the left-hand side of (3.3) is just a derivative of u along this path. And since it is zero, we find that

$$u = const, \quad \text{along (3.4).}$$

That is $u(x,t) = const$ along $dx/dt = a$, or $x = x_0 + at$. Hence

$$u(x,t) = u(x_0, 0) = f(x_0) = f(x - at).$$

So, the important thing to note is that:

$$u_t + au_x = \frac{d}{dt}u\left(x_0 + at, t\right)$$

$$\text{or: } u_t + au_x = \frac{du}{dt} \text{ following the path } \frac{dx}{dt} = a.$$

The path described by

$$\frac{dx}{dt} = a$$

is called a *characteristic*, and the method above is a case of the *method of characteristics.*

- ◇ In fact, we realize that we can go beyond this simple equation and generalize to

  $$u_t + a\left(u, x, t\right)u_x = b\left(u, x, t\right).$$

  Following the same geometric ideas, we can write this equation along the characteristic as

  $$\frac{du}{dt} = b\left(u, x, t\right) \qquad \text{on:} \qquad \frac{dx}{dt} = a\left(u, x, t\right)$$

  which will be recognized as a system of two generally nonlinear ODE for $x(t)$ and $u(t)$. One must supply appropriate initial conditions. These can be analyzed using the earlier methods. We will return to them later.
- ◇ Note the importance of the combination $u_t + au_x$ in continuum mechanics. Suppose we identify a small fluid element (call it fluid particle) and decide to follow its position $\mathbf{r}$ as it moves together with the fluid around. Let $\mathbf{u}(\mathbf{r}, t)$ be the velocity vector of the fluid particle when it is at position $\mathbf{r}$ at time t. Then its acceleration is given by

  $$\frac{d}{dt}\mathbf{u}\left(\mathbf{r}\left(t\right), t\right) = \mathbf{u}_t + \left(\dot{\mathbf{r}} \cdot \nabla\right)\mathbf{u} = \mathbf{u}_t + \left(\mathbf{u} \cdot \nabla\right)\mathbf{u},$$

  which is exactly of the form we considered above. In 1D, this is just $u_t + uu_x$. This derivative "following the particle" is called the Lagrangian derivative, as opposed to $\mathbf{u}_t$ which is the Eulerian derivative measuring the rate of change of the velocity field at a given fixed position $\mathbf{r}$.

  We could be interested in how the fluid particle density varies with time, in which case we calculate

  $$\frac{d}{dt}\rho\left(\mathbf{r}\left(t\right), t\right) = \rho_t + \mathbf{u} \cdot \nabla\rho,$$

  and if this is zero, then we have an *incompressible fluid.* Note that $\rho_t + \mathbf{u} \cdot \nabla\rho = \rho_t + \nabla\rho\mathbf{u} - \rho\nabla \cdot \mathbf{u}$ and, since $\rho_t + \nabla\rho\mathbf{u} = 0$ by conservation of mass, then incompressibility leads to $\nabla \cdot \mathbf{u} = 0$.
- ◇ Note concerning the boundary conditions on u. Suppose we solve $u_t + au_x = 0$ on a bounded domain $[0, l]$. What kind of boundary conditions can be imposed?

  The answer will depend on the sign of a. If $a > 0$, we can only impose boundary conditions at $x = 0$ and if $a < 0$ at $x = l$. The reason for this is the direction of propagation of the wave.

3.8.2. *Heat equation.* The 1D heat equation

$$u_t = ku_{xx}$$

with an initial condition

$$u\left(x, 0\right) = f\left(x\right)$$

and two boundary conditions is a prototypical equation describing diffusive processes (of heat, mass, etc.) that has the property of *smearing* any sharp or discontinuous structures in the solution. That is, if we input some $u_0(x)$ as an initial condition, then no matter how discontinuous that $u_0(x)$, after very short time the solution $u(x,t)$ of the equation subject to this initial condition will be nice and smooth.

Suppose we look at the problem on the whole real line $x \in \mathbb{R}$. We can solve the problem by Fourier transform and obtain that

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, f(y)\, dy \tag{3.5}$$

where

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$$

is the Green's function of the problem that solves the PDE subject to $f = \delta(x)$.

So far we assume that $k > 0$ on some physical grounds (positive diffusion coefficient or heat conduction coefficient).

Putting a mathematician's hat on, we can ask: What if $k < 0$? The answer is that the initial value problem will be *ill-posed*. This means that the solution will be so sensitive to the initial conditions that no matter how small the difference in the initial conditions, the corresponding solutions will become drastically different in an arbitrary short amount of time.

For example, consider the following problem

$$\begin{aligned} u_t &= -u_{xx}, \quad t > 0, x \in \mathbb{R} \\ u(x,0) &= u_0(x). \end{aligned}$$

If we take as an initial condition

$$u_0(x) = 1 \tag{3.6}$$

this problem has a solution

$$u_1(x,t) = 1. \tag{3.7}$$

If we take another initial condition

$$\bar{u}_0(x) = 1 + \frac{1}{n}\sin(nx) \tag{3.8}$$

with any constant n, then the corresponding solution is

$$\bar{u}_n = 1 + \frac{e^{n^2 t}}{n}\sin(nx) \tag{3.9}$$

Now notice that at $n \gg 1$, the initial conditions of these two problems (3.6) and (3.8) differ by very little, $O(1/n)$. However, the corresponding solutions (3.7) and (3.9) differ by a lot, $O\left(e^{n^2 t}/n\right)$ at any $t > 0$. This is ill-posedness.

So, the *backward heat equation is ill posed* ($k < 0$ is equivalent to changing the sign of time, $t \to -t$, hence the word "backward").

Similarly, one can show that an *initial value problem for Laplace's equation is also ill-posed.* Here's an example:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad x \geq 0 \\ u(0,y) &= 0, \\ u_x(0,y) &= \frac{1}{n}\sin(ny). \end{aligned}$$

This problem is solved by

$$u_n = \frac{1}{n^2}\sinh(nx)\sin(ny).$$

Notice now that as $n \gg 1$, the second boundary condition at $x = 0$ tends to 0, and so we expect the solution to be small as well. However, u_n is actually exponentially growing in x at large n at any $x > 0$.

Both of these examples are cases of what is called *Hadamard instability*. Or ill-posed in the sense of Hadamard.

3.8.3. *Wave equation.*

$$u_{tt} = c^2 u_{xx}.$$

The first thing to note is that the general solution of the wave equation is easy to obtain. Let $\xi = x - ct$ and $\eta = x + ct$. Then

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$$

and

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}.$$

Then

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = -4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

and this equation is solved by

$$u = f_1(\xi) + f_2(\eta)$$

with any smooth functions f_1 and f_2, which is then the general solution of the wave equation.

In order to solve the wave equation, we require two initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x),$$

and may also need boundary conditions. For an infinite space problem, the solution is given by d'Alembert's formula

$$u = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi.$$

If we have a finite domain, $a \le x \le b$ (a finite string or rod), then two basic possibilities:

(1) an end is fixed, then $u = 0$ at that end.
(2) an end is free, so $u_x = 0$.

One can consider more complicated boundary conditions, such as a mass attached to the end of the spring. In each case, one has to carefully pose the condition. For the infinite domain (in one or both directions), one does not need to explicitly impose any boundary conditions.

3.8.4. *Laplace's equation.* Equilibrium problems of heat conduction/diffusion or various other problems in fluid mechanics (potential flow), electromagnetism, and others lead to the Laplace equation

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0.$$

Fixed boundary condition is $u_{\partial D} = 0$ at all or part of the boundary, and
the free boundary condition is $\mathbf{n} \cdot \nabla u = 0$, where this is a derivative normal to the boundary.

3.8.5. *Airy equation. Dispersion.* If we look at the wave equation $u_{tt} = c^2 u_{xx}$ and seek solutions of the plane wave form (normal modes)

$$u = e^{i(kx-\omega t)},$$

then we find that the wavenumber k and frequency ω are related as

$$\omega^2 = c^2 k^2$$

which is called the dispersion relation as it tells us the frequency ω of the wave for any given wavenumber k, hence the phase velocity of the wave

$$c_p = \frac{\omega(k)}{k}.$$

For the wave equation, the dispersion relation gives

$$\omega = \pm ck$$

and hence

$$c_p = \pm c.$$

This means that the plane wave solutions represent waves that propagate to the right or to the left with a constant speed c. We say, the waves are *not dispersive*, as all the modes (i.e. all the wavenumbers), have the same velocity.

A different story is given by the Airy equation (which is a linearized version of the KdV equation):

$$u_t = \epsilon u_{xxx}. \tag{3.10}$$

Look again for plane wave solutions $u = e^{i(kx-\omega t)}$. Then we find

$$-i\omega = \epsilon (ik)^3$$
$$\omega = \epsilon k^3.$$

This means that the phase velocity

$$c_p = \frac{\omega}{k} = \epsilon k^2$$

does depend on k. So the consequence of this is that different waves (having different wavelength $2\pi/k$) propagate at different speeds. So they disperse. The Airy's equation (hence KdV) is a dispersive PDE.

3.9. **Separation of variables/Eigenfunction expansion/Fourier method.** ....

3.9.1. *The basic ideas leading to Fourier analysis.* Given a BVP

$$\hat{L}u = f(x), \quad x \in D$$
$$\hat{B}u = 0, \quad x \in \partial D$$

we look for the solution u as an expansion in the Fourier series in terms of the e-functions ϕ of $\hat{L}$:

$$\hat{L}\phi = \lambda\phi, \quad x \in D$$
$$\hat{B}\phi = 0, \quad x \in \partial D,$$

as

$$u = \sum a_n \phi_n(x).$$

Then

$$\sum a_n \lambda_n \phi_n = \sum f_n \phi_n,$$

from which using orthogonality of ϕ_n (will hold for good operators $\hat{L}$, e.g. self-adjoint, i.e. symmetric, like symmetric matrices), we get

$$a_n = \frac{f_n}{\lambda_n}.$$

3.9.2. *Heat equation.* Consider the example of 1D heat conduction:

$$\begin{aligned} u_t = ku_{xx}, \quad 0 < x < l, \quad t > 0 \\ u(x,0) = f(x) \\ u(0,t) = u(l,t) = 0. \end{aligned}$$

Consider the PDE as $-u_{xx} = -\frac{1}{k}u_t$ with the spatial operator $\hat{L} = -\frac{d^2}{dx^2}$. We look for eigenfunctions of $\hat{L}$ with the homogeneous Dirichlet conditions:

$$\begin{aligned} -\phi'' &= \lambda\phi \\ \phi(0) &= 0 \\ \phi(l) &= 0. \end{aligned}$$

That λ is positive that can be seen from the following calculation:

$$\begin{aligned} -\phi\phi'' &= \lambda\phi^2 \\ -\int_0^l \phi\phi'' dx &= \lambda\int_0^l \phi^2 dx \\ \underbrace{-\phi\phi'|_0^l}_{=0 \text{ because of BC}} + \int_0^l (\phi')^2 dx &= \lambda\int_0^l \phi^2 dx. \end{aligned}$$

Then

$$\lambda = \frac{\int_0^l (\phi')^2 dx}{\int_0^l \phi^2 dx} \quad \text{(Rayleigh quotient)}$$

is actually positive. It can only be zero if $\phi' \equiv 0$, but this is only possible for $\phi \equiv 0$ because of the BC. We are looking for nontrivial solutions.

Then, with $\lambda > 0$, we obtain

$$\phi = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x.$$

The BC at $x = 0$: $\phi(0) = 0 = c_2$. The BC at $x = l$: $\phi(l) = 0 = c_1 \sin\sqrt{\lambda}l$. Now nontrivial solutions ($c_1 \neq 0$) are possible provided $\sqrt{\lambda}l = n\pi$, $n = 1, 2, 3, ...$ That is if

$$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2.$$

Thus we have found the eigenvalues and eigenfunctions as

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad \phi_n = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots$$

Now we expand the solution in this basis as

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x),$$

substitute into the PDE

$$\sum_{n=1}^{\infty} \dot{a}_n(t)\,\phi_n(x) = k\sum_{n=1}^{\infty} a_n(t)\,\phi_n''(x) = -k\sum_{n=1}^{\infty} a_n(t)\,\lambda_n\phi_n(x)\,.$$

Therefore

$$\dot{a}_n = -k\lambda_n a_n$$

and

$$a_n(t) = c_n e^{-k\lambda_n t}.$$

The solution is now

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n t}\phi_n(x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}\phi_n(x)\,.$$

To find c_n, we use the initial condition $u(x,0) = f(x)$, which gives that c_n are the Fourier coefficients of $f(x)$

$$c_n = \frac{2}{l}\int_0^l f(x)\,\phi_n(x)\,dx.$$

A note on the nonhomogeneous heat equation.
Consider the example of 1D heat conduction:

$$\begin{aligned} u_t = k u_{xx}, \quad 0 < x < l, \quad t > 0 \\ u(x,0) =& f(x) \\ u(0,t) =& 0 \\ u_x(l,t) =& r(t)\,. \end{aligned}$$

Note: we cannot directly apply separation of variables, because the second boundary condition is not homogeneous.

First, we homogenize the BC. Let $u = v(x,t) + xr(t)$. Then $u(0,t) = 0$ gives $v(0,t) = 0$ and $u_x(l,t) = r(t)$ gives $v_x(l,t) = 0$. So, the boundary conditions are captured by $xr(t)$.

Then for $v(x,t)$ we get the following problem with homogeneous BC:

$$\begin{aligned} v_t = k v_{xx} - x\dot{r}(t)\,, \quad 0 < x < l, \quad t > 0 \\ v(x,0) =& f(x) - xr(0)\,, \\ v(0,t) =& 0, \\ v_x(l,t) =& 0. \end{aligned}$$

3.10. **Examples of application of Fourier analysis to solution of IBVP..** The following problems will be discussed:

(1) 1D wave equation with damping. How do different modes decay?
(2) 2D wave equation on a rectangle. Modes, nodes, and frequencies. Some discussion of damping of the modes.
(3) *Heat conduction in a cylinder. Cylindrical coordinates.*

Problem 13. Consider the one-dimensional damped wave equation for a finite string:

$$\begin{aligned}
&u_{tt} + \mu u_t = c^2 u_{xx}, \quad 0 < x < l,\ t > 0\\
&u(x,0) = f(x), \quad u_t(x,0) = 0\\
&u(0,t) = u(l,t) = 0.
\end{aligned}$$

Questions:

- ⋄ How do the modes decay?
- ⋄ What happens to the total energy of the string, $\mathcal{E} = \int_0^l \left(\rho u_t^2 + \rho c^2 u_x^2\right) dx$?

Solution. The e-functions and e-values we use here are the same as with the undamped equation with the same homogeneous Dirichlet conditions:

$$\begin{aligned}
\phi_n &= \sin\left(\frac{n\pi x}{l}\right),\\
\lambda_n &= \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, ...
\end{aligned}$$

Then

$$u = \sum_{n=1}^{\infty} u_n(t)\,\phi_n(x),$$

and plugging this into the PDE we find that

$$\ddot{u}_n + \mu \dot{u}_n + c^2 \lambda_n u_n = 0$$

or letting $\omega_n = c\sqrt{\lambda_n} = n\pi c/l$,

$$\ddot{u}_n + \mu \dot{u}_n + \omega_n^2 u_n = 0$$

which is solved by $u_n(t) = e^{\alpha t}$ with

$$\alpha^2 + \mu\alpha + \omega_n^2 = 0.$$

From here, we obtain

$$\alpha = -\frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} - \omega_n^2}.$$

We will assume that the damping is small enough so that all ω_n are larger than $\mu/2$, that is require $\pi c/l > \mu/2$. In that case, define

$$\Omega_n = \sqrt{\omega_n^2 - \frac{\mu^2}{4}}$$

so that

$$\alpha = -\frac{\mu}{2} \pm i\Omega_n.$$

The solution for u_n becomes

$$u_n(t) = e^{-\mu t/2}\left[a_n \cos(\Omega_n t) + b_n \sin(\Omega_n t)\right]$$

with parameters a_n and b_n.

Then the full solution is

$$u = \sum_{n=1}^{\infty} e^{-\mu t/2}\left[a_n \cos(\Omega_n t) + b_n \sin(\Omega_n t)\right]\phi_n(x).$$

The parameters a_n and b_n are found from initial conditions. For example

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x),$$

so that

$$a_n = \frac{2}{l}\int_0^l f(x)\,\phi_n(x)\,dx.$$

With $u_t(x,0) = 0$, we find

$$0 = \sum_{n=1}^{\infty}\left[\frac{-\mu}{2}a_n + b_n\Omega_n\right]\phi_n(x),$$

so that

$$b_n = \frac{\mu\Omega_n}{2}a_n.$$

Note that all of these modes decay in time at the same rate. This is unlike the heat equation where high wavenumbers decay very fast.

The energy of the vibrations is obtained by multiplying the PDE by u_t and integrating over x from $x = 0$ to $x = l$.

$$\int_0^l \left(u_t u_{tt} + \mu u_t^2\right)dx = c^2\int_0^l u_{xx}u_t dx.$$

Rearrange:

$$\frac{d}{dt}\int_0^l \frac{u_t^2}{2}dx - c^2\int_0^l u_{xx}u_t dx = -\mu\int_0^l u_t^2 dx.$$

The second term on the left can be integrated by parts once

$$\int_0^l u_{xx}u_t dx = \underbrace{u_x u_t|_0^l}_{=0 \text{ due to the BC}} - \int_0^l u_x u_{xt} dx = -\frac{d}{dt}\int_0^l \frac{u_x^2}{2}dx.$$

Then

$$\frac{d}{dt}\int_0^l\left[\frac{u_t^2}{2} + \frac{c^2u_x^2}{2}\right]dx = -\mu\int_0^l u_t^2 dx.$$

The integral on the left is exactly the total energy (up to a factor of density ρ) of the vibrating string, that includes the kinetic energy density $u_t^2/2$ and the density of the elastic energy $c^2u_x^2/2$. You may recall from basic elasticity theory that the elastic energy density (per unit volume) is $E\epsilon^2/2$, and what we have here is exactly that.

We conclude that the vibration energy always decays as

$$\frac{d\mathcal{E}}{dt} = -\mu\int_0^l u_t^2 dx < 0.$$

Since $\int_0^l u_t^2 dx \le \int_0^l \left(u_t^2 + c^2u_x^2\right)dx = 2\mathcal{E}$, then we find that $\dot{\mathcal{E}} \le -2\mu\mathcal{E}$ and hence $\mathcal{E}(t) \le Ce^{-2\mu t}$.

Problem 14. Suppose we have a rectangular elastic membrane with sides a and b with all of its sides fixed. We want to know how the membrane will vibrate in response to some initial displacement or velocity. The IBVP is

$$\begin{aligned}
&u_{tt} = c^2\left(u_{xx} + u_{yy}\right), \quad t > 0, (x,y) \in D = (0,a)\times(0,b),\\
&u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y),\\
&u(x,0,t) = u(x,b,t) = 0,\\
&u(0,y,t) = u(a,y,t) = 0.
\end{aligned}$$

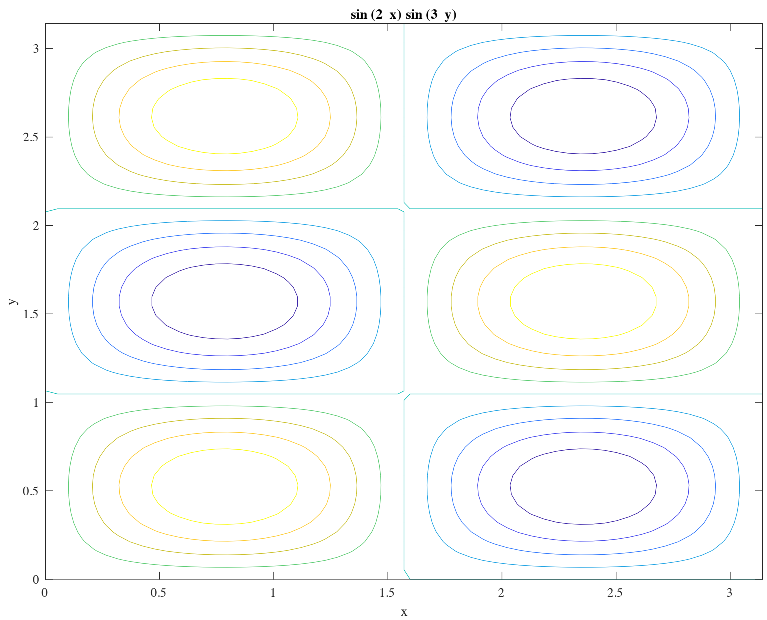


FIGURE 3.1. Nodes of the eigenfunctions of the 2D wave equation

Solution. This time we need the eigenfunctions of the Laplacian in 2D. The e-value problem

$$-\Delta\phi = \lambda\phi,$$
$$\phi(x,0) = \phi(x,b) = 0,$$
$$\phi(0,y) = \phi(a,y) = 0,$$

has eigenfunctions

$$\phi_{nm}(x,y) = \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

and eigenvalues

$$\lambda_{nm} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2, \quad n, m = 1, 2, 3...$$

The solution can be expanded in these e-functions as

$$u = \sum_{n,m} u_{nm}(t)\,\phi_{nm}(x,y)$$

where $u_{nm}(t)$ solves

$$\ddot{u}_{nm} + c^2\lambda_{nm}u_{nm} = 0$$

so that

$$u_{nm} = a_{nm}\cos(\omega_{nm}t) + b_{nm}\sin(\omega_{nm}t)$$

where

$$\omega_{nm} = c\sqrt{\lambda_{nm}} = \sqrt{\left(\frac{n\pi c}{a}\right)^2 + \left(\frac{m\pi c}{b}\right)^2}$$

is the frequency of mode nm.

Coefficients a_{nm} and b_{nm} are found from the initial conditions.

Note the existence of nodal lines – modes nm have lines on the membrane that have $\phi_{nm} = 0$.