Seminar 6: Fourier

Series and transform

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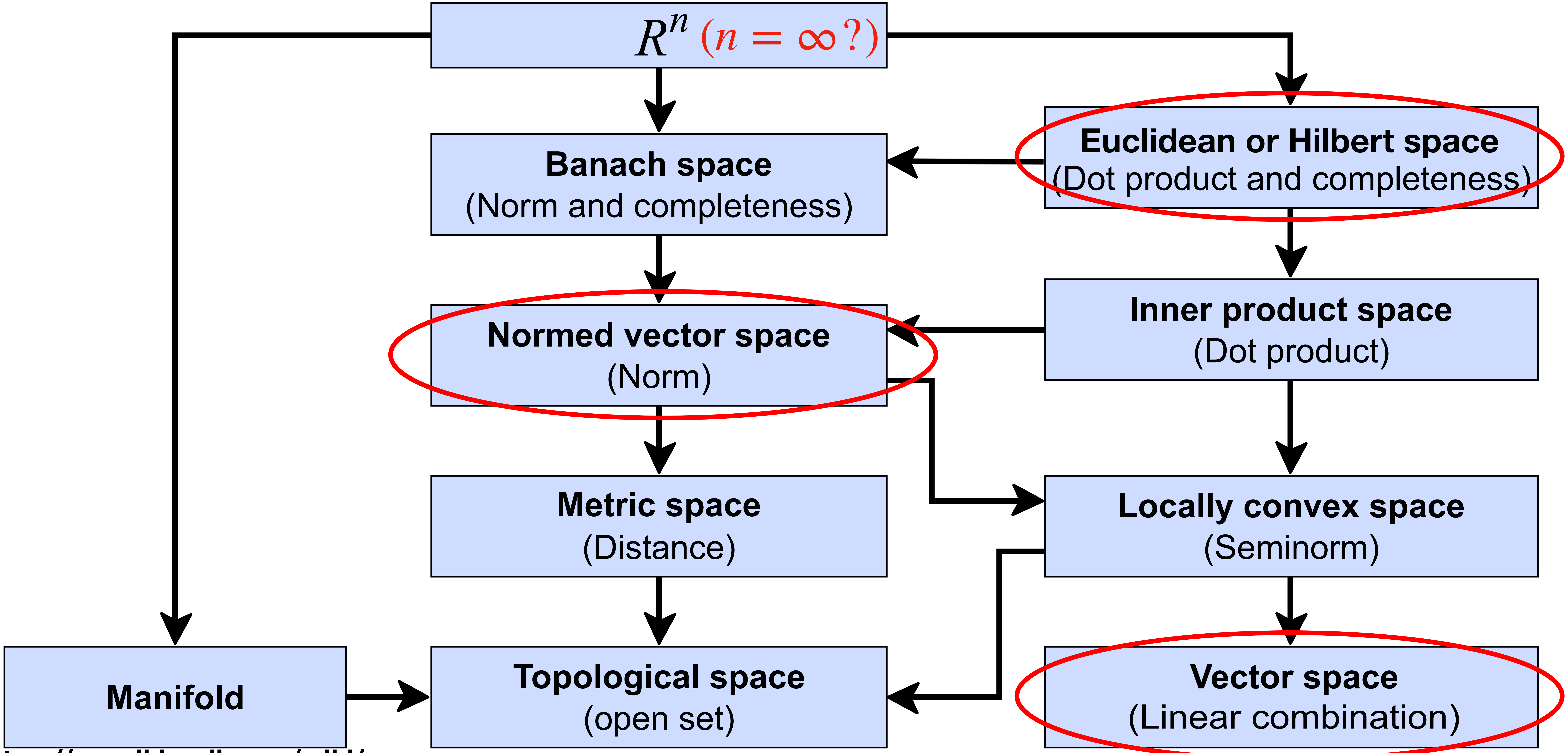
Some changes

- Lecture - 22 November
- Seminar/Office hours — 23 November
- Problem set 7 — 23 November, next Tuesday
- Midterm 2 — 25 November, next Thursday

Abstract spaces



https://en.wikipedia.org/wiki/Space_(mathematics)

Space = Set + Structure

We know spaces with $\dim V < \infty$...

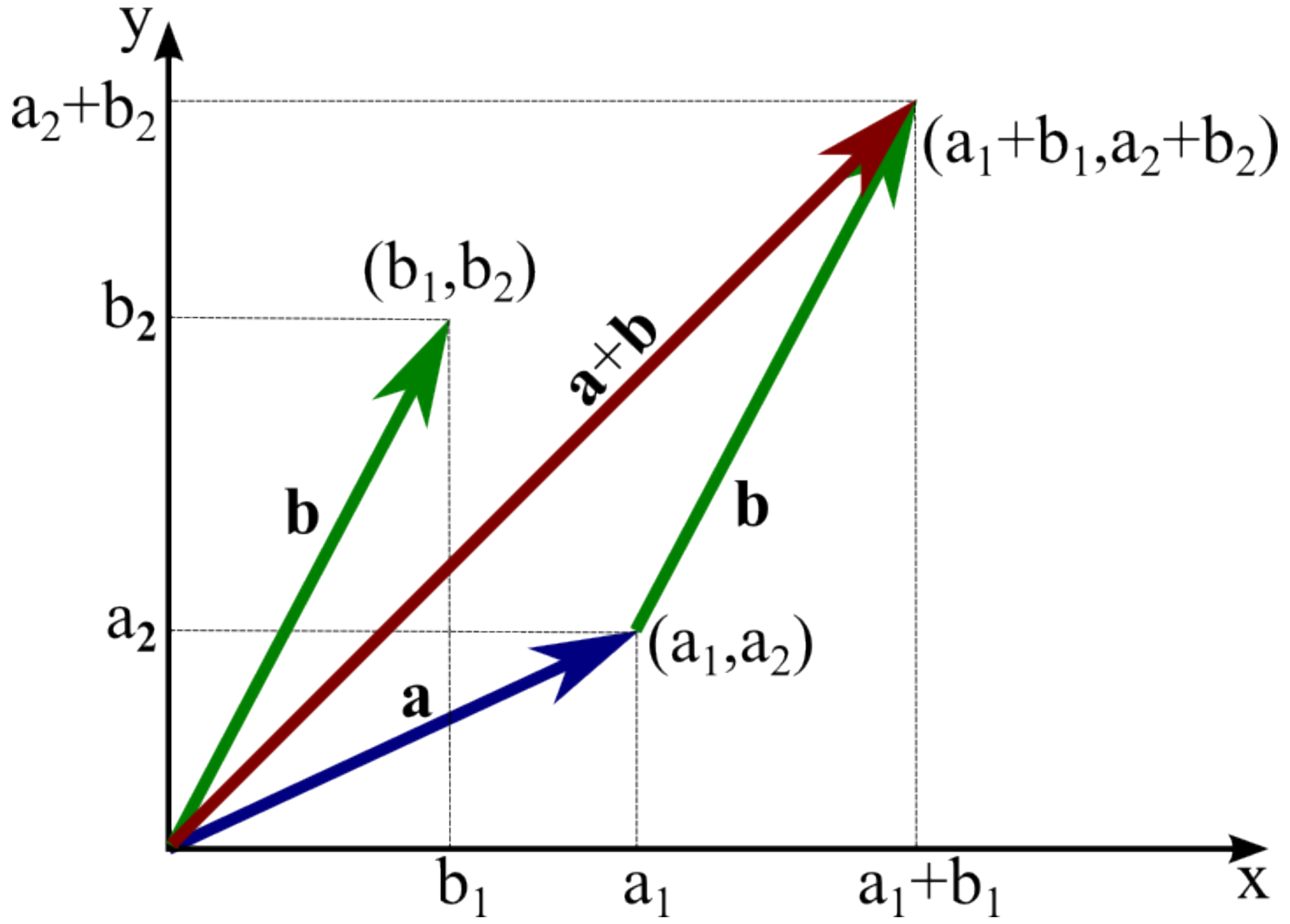
Vector space = Set of vectors + Linear combination

$$a, b \in V, \alpha, \beta \in \mathbb{R} \Rightarrow \alpha a + \beta b \in V$$

$\{e_1, e_2, \ldots, e_n\}$ **is basis of** V **and** $\dim V = n$ **iff**

a. $V = \mathrm{span}\{e_1, e_2, \ldots, e_n\} = \{x : x = \alpha_1 e_1 + \ldots + \alpha_n e_n,\ \alpha_i \in \mathbb{R}\}$

b. Linear independent



Polynomials of degree ≤ 2**:**

$$f = 3x^2 - 2x + 7$$

$$f = 3(x-1)^2 + 4(x-1)$$

Possible bases: $1,\ x,\ x^2$ **or** $1,\ x-1,\ (x-1)^2$ **or others**

But what if $\dim V = \infty$?

Space of ALL the polynomials. What basis can be used to decompose ANY polynomial?

$3x^2 - 2x + 7,\ 25x^7 + 6x^{15},\ x^{100} + \pi\, x^{314},\ x^{1234} + 1234x, \ldots$

One choice is $1, x, x^2, x^3, x^4, \ldots, x^n, \ldots$ **where** $n \in \mathbb{N}$.

Differentiable functions: $C^k\left[a, b\right],\ C^k(\mathbb{R})$

Integrable functions: $L^p\left[a, b\right],\ L^p(\mathbb{R})$ **.**

The norms are

$$\| f \|_{C^k} = \sum_{l=0}^{k} \max \left| f^{(l)}(x) \right|,\ \| f \|_{L^p} = \left(\int_a^b \left| f(x) \right|^p dx \right)^{1/p}$$

Scalar product for $f,\ g \in L^2\left[a, b\right]$**:** $\langle f, g \rangle = \int_a^b f(x)\, g^*(x)\, dx$

$$\langle f,f \rangle = \int_a^b f(x) f^*(x)\, dx = \int_a^b \left| f(x) \right|^2 dx = \| f \|_{L^2}^2$$

Possible bases:

- **Trigonometric functions**
- **Polynomial families**
- **Wavelets and so on**

Generalised series

If $f \in L^2\left[a, b\right]$ **and** x_n **is the orthonormal system, then** $f = \sum_{n=1}^{\infty} a_n x_n$

We can work with approximations $f \approx \sum_{n=1}^{N} a_n x_n$ **for** $N < \infty$

What is the best values for coefficients a_n **if** x_n **is the orthonormal basis in** $L^2\left[a, b\right]$**?**

$$\left\| f - \sum_{n=1}^{N} a_n x_n \right\|_{L^2}^2 = \left\langle f - \sum_{n=1}^{N} a_n x_n, f - \sum_{n=1}^{N} a_n x_n, \right\rangle = \left\| f \right\|_{L^2}^2 - 2\sum_{k=1}^{N} a_n \left\langle f, x_n \right\rangle + \sum_{n=1}^{N} a_n^2 =$$

$$\left\| f \right\|_{L^2}^2 - \sum_{n=1}^{N} \left\langle f, x_n \right\rangle^2 + \sum_{n=1}^{N} \left\langle f, x_n \right\rangle^2 - 2\sum_{k=1}^{N} a_n \left\langle f, x_n \right\rangle + \sum_{n=1}^{N} a_n^2 =$$

$$= \left\| f \right\|_{L^2}^2 - \sum_{n=1}^{N} \left\langle f, x_n \right\rangle^2 + \sum_{n=1}^{N} \left(a_n - \left\langle f, x_n \right\rangle\right)^2$$

$$f \approx \sum_{n=1}^{N} \left\langle f, x_n \right\rangle x_n$$

Let's fix a basis!

$\left\{e^{ikx}\right\},\ k \in \mathbb{Z}$ **or** $\left\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos\left(kx\right), \sin\left(kx\right), \ldots\right\}$

2π**-periodic function** $f \in L_2[-\pi, \pi]$ **and**

partial sums $S_N(x) = \dfrac{a_0}{2} + \displaystyle\sum_{k=1}^{N} a_k \cos\left(kx\right) + b_k \sin\left(kx\right)$ **with**

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(kx\right)dx,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(kx\right)dx$$

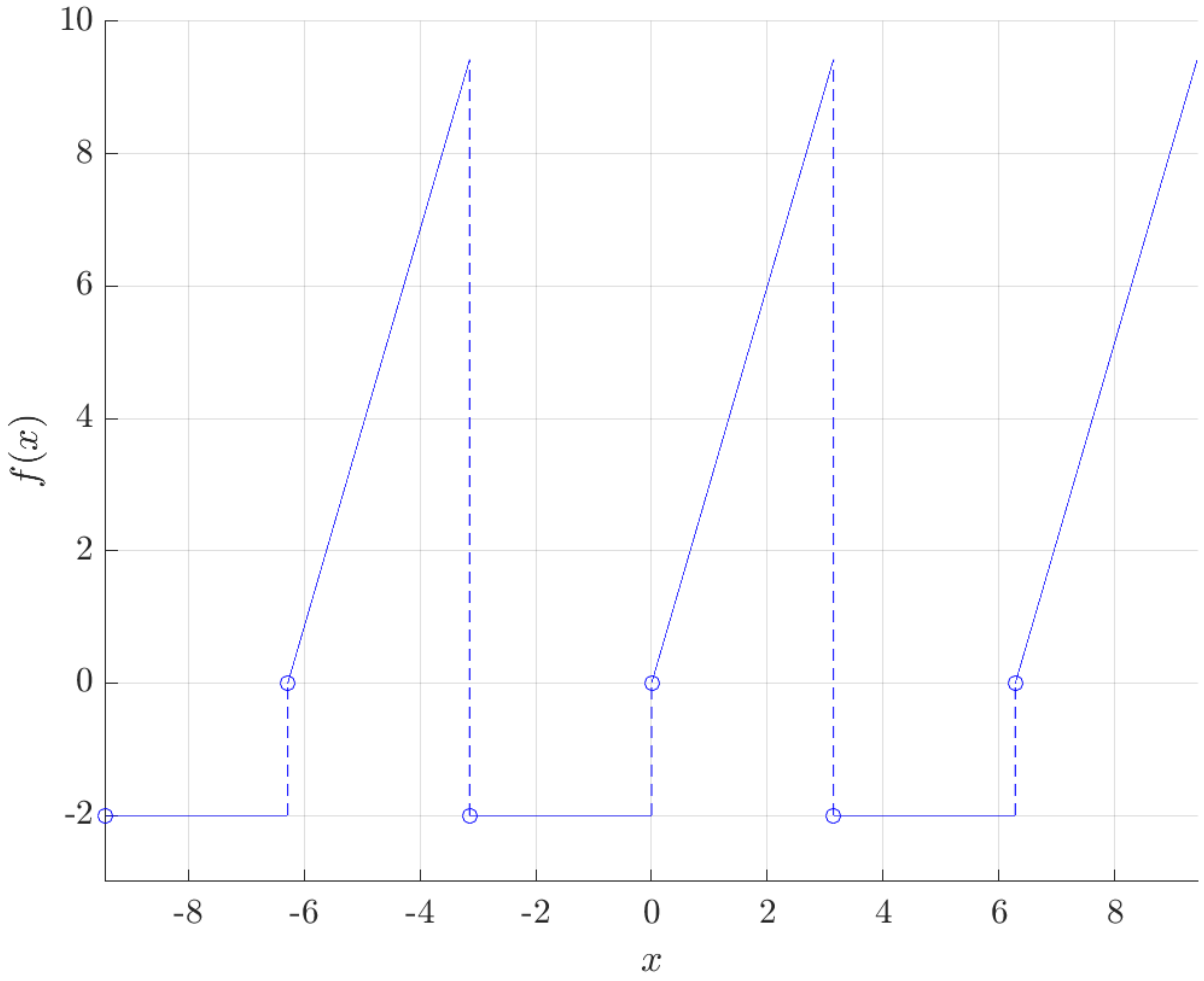
that can be viewed as projections to the finite-dimensional subspace of $L_2[-\pi, \pi]$

Dirichlet theorem: $\left\| f - S_N\left(f\right) \right\|_2 \to 0$

Example: Series

Find a Fourier decomposition of the function

$$f(x) = \begin{cases} -2, & -\pi < x \leq 0 \\ 3x, & 0 < x \leq \pi \end{cases}$$



Example: Series

1. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = -\frac{2}{\pi}\int_{-\pi}^{0} dx + \frac{3}{\pi}\int_{0}^{\pi} xdx = -2 + \frac{3}{2}\pi$

2. $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(kx\right) dx = -\frac{2}{\pi}\int_{-\pi}^{0} \cos\left(kx\right) dx + \frac{3}{\pi}\int_{0}^{\pi} x\ \cos\left(kx\right) dx$

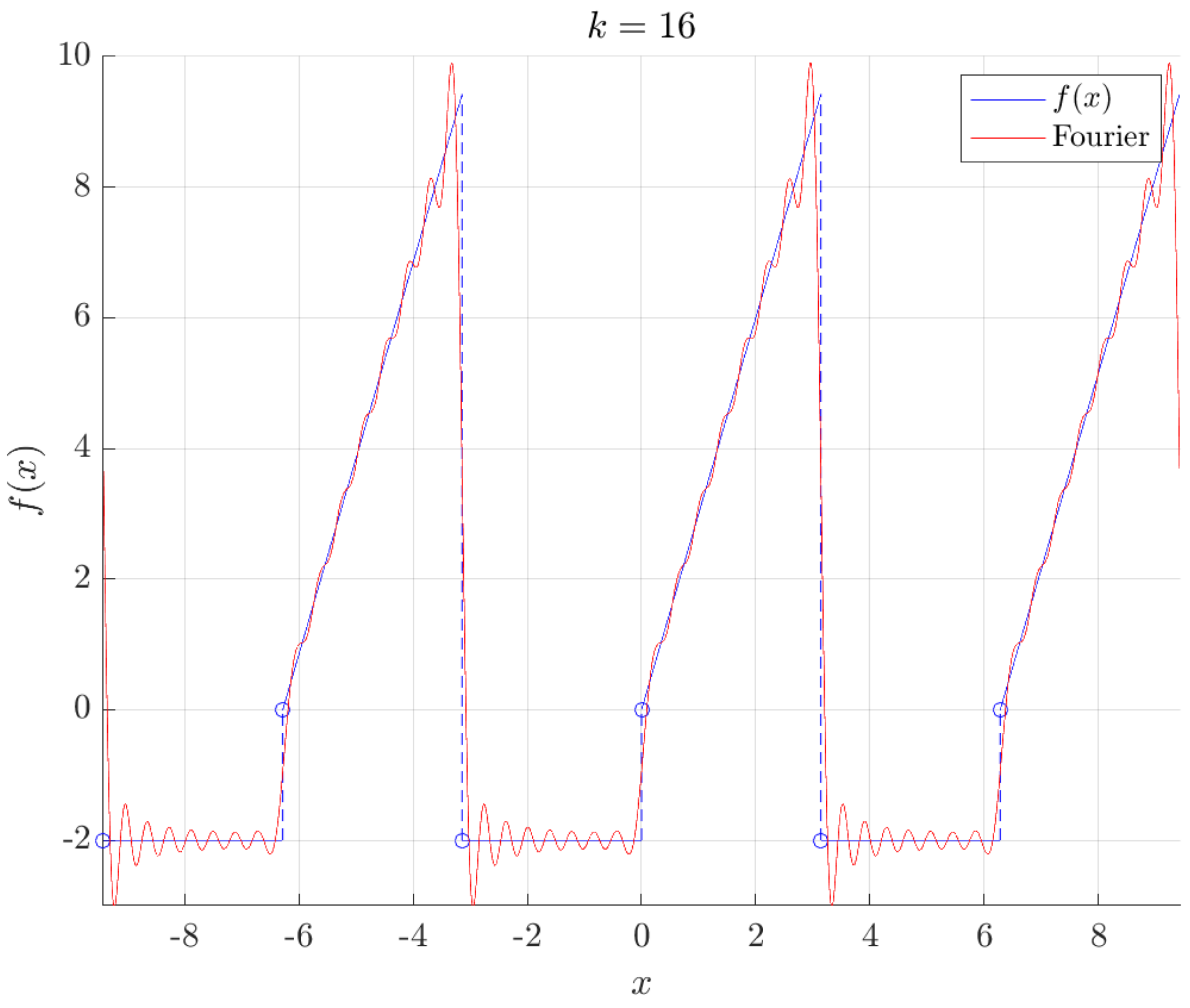
$$= \frac{3}{\pi}\left(\frac{x\ \sin\left(kx\right)}{k}\Bigg|_{x=0}^{x=\pi} - \frac{3}{\pi k}\int_{0}^{\pi} \sin\left(kx\right) dx\right) = -\frac{3\left(1-\cos\left(\pi k\right)\right)}{\pi k^2} = -\frac{6\sin^2\left(\frac{\pi k}{2}\right)}{\pi\ k^2}$$

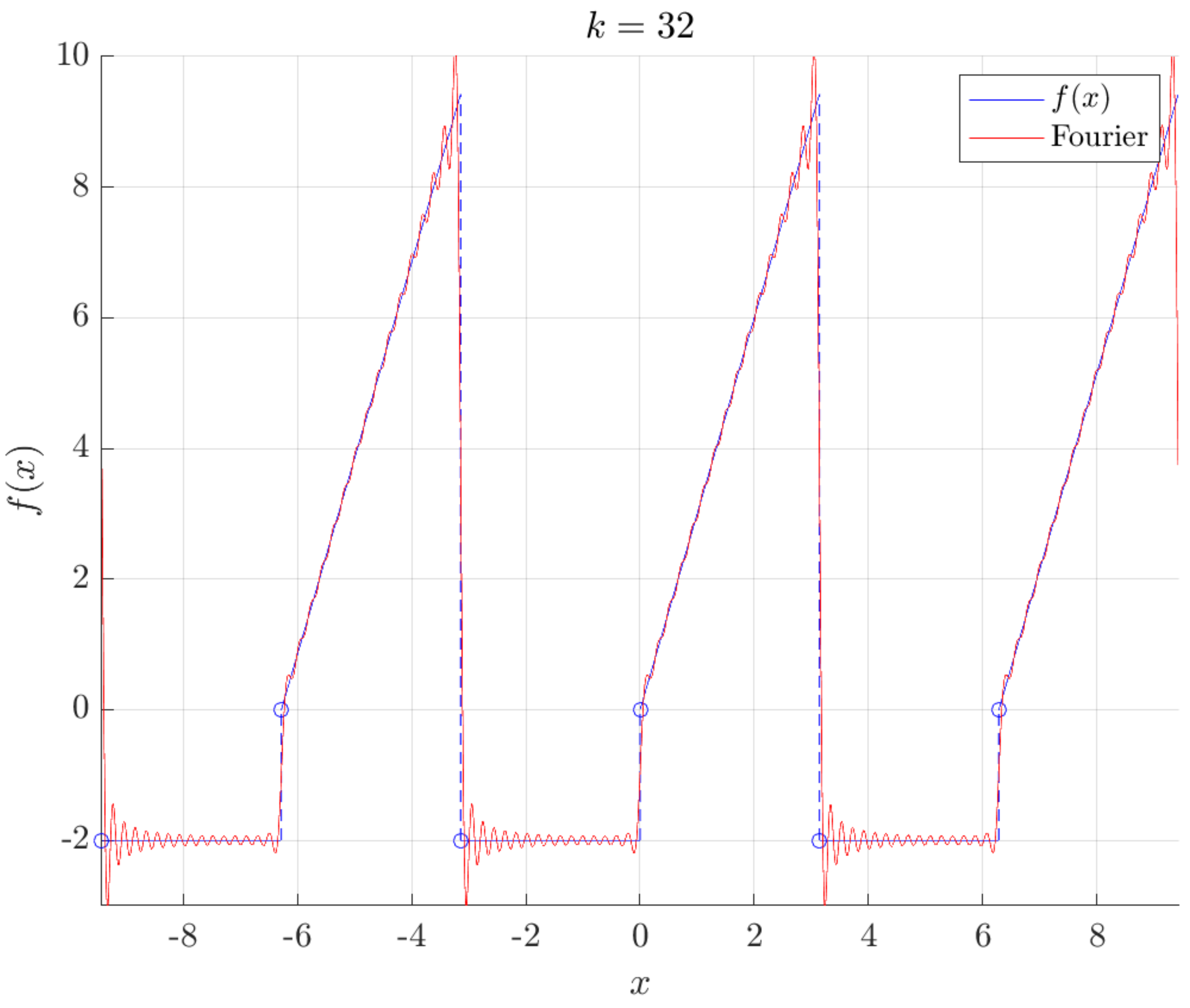
3. $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(kx\right) dx = -\frac{2}{\pi}\int_{-\pi}^{0} \sin\left(kx\right) dx + \frac{3}{\pi}\int_{0}^{\pi} x\ \sin\left(kx\right) dx =$

$$-\frac{2}{\pi}\left(-\frac{\cos\left(kx\right)}{k}\right)\Bigg|_{x=-\pi}^{x=0} + \frac{3}{\pi}\left(-\frac{x\ \cos\left(kx\right)}{k}\Bigg|_{x=0}^{x=\pi} + \frac{1}{k}\int_{0}^{\pi} \cos\left(kx\right) dx\right)$$

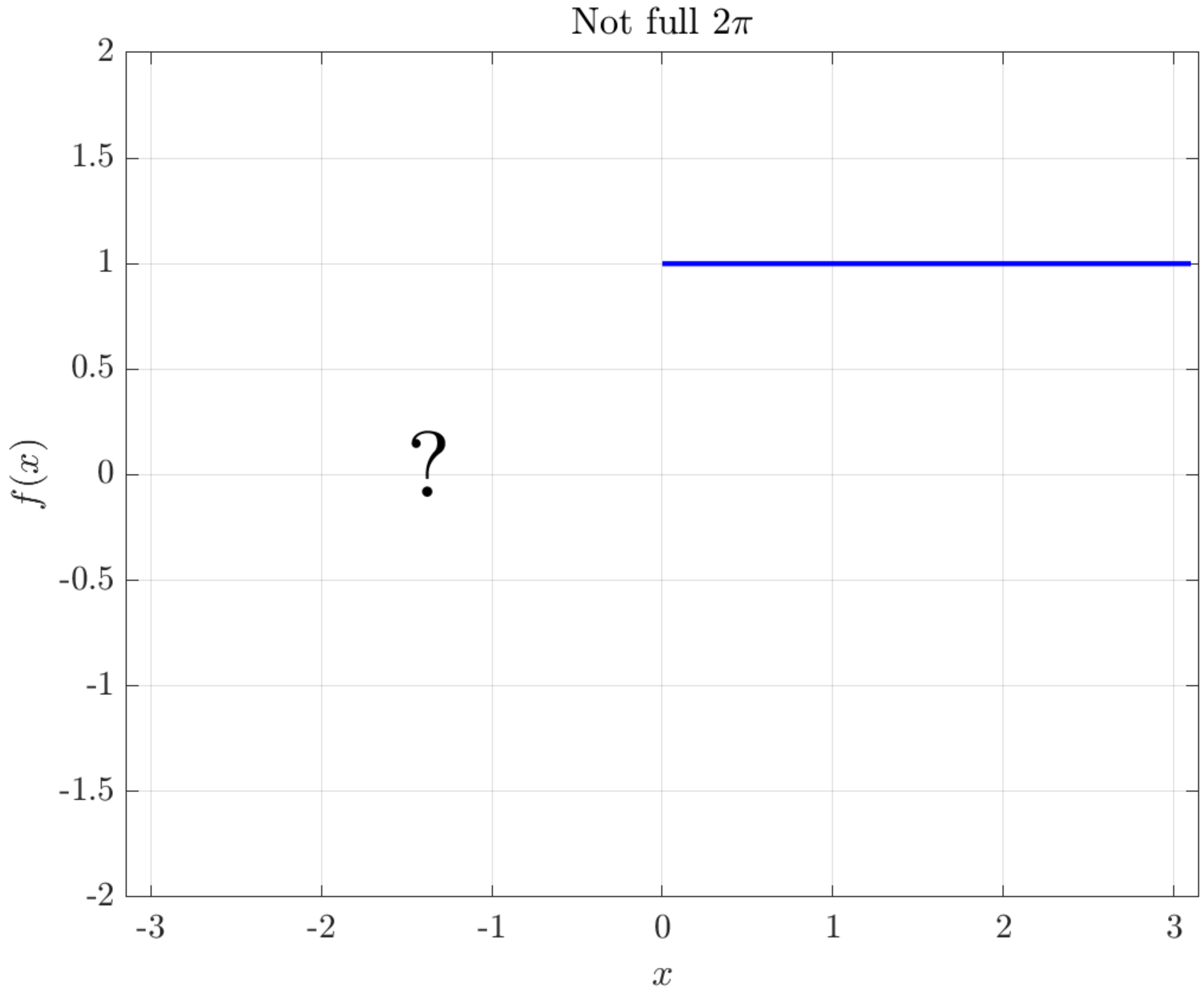
$$= \frac{2\left(1-\cos\left(\pi k\right)\right)}{\pi k} - \frac{3\cos\left(\pi k\right)}{k} = \frac{2-(2+3\pi)\cos\left(\pi k\right)}{\pi k}$$

Example: Series





Example: not 2π



1. Adjusted formulas

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos\left(\frac{2\pi}{T}kx\right) + b_k \sin\left(\frac{2\pi}{T}kx\right)$$

$$a_k = \frac{2}{T}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{2\pi}{T}kx\right)dx,$$

$$b_k = \frac{2}{T}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{2\pi}{T}kx\right)dx$$

2. Continuation
Define the function on the remaining part and use the usual formulas for 2π periodic functions

Example: not 2π

Example: complex series

Decompose $f(x) = e^{-|x|},\ x \in [-\pi, \pi]$ **with respect to the basis** $\left\{e^{ikx}\right\}, k \in \mathbb{Z}$**.**

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx},\ \ c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx} dx$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-ikx} dx = \frac{1}{2\pi}\int_{-\pi}^{0} e^{x} e^{-ikx} dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-x} e^{-ikx} dx = \frac{1}{2\pi}\int_{-\pi}^{0} e^{x-ikx} dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-ikx-x} dx =$$

$$\frac{1}{2\pi}\int_{-\pi}^{0} e^{(1-ik)x} dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-(ik+1)x} dx =$$

$$\frac{1}{2\pi}\frac{1}{1-ik}\left(1 - e^{(ik-1)\pi}\right) - \frac{1}{2\pi}\frac{1}{ik+1}\left(e^{-(ik+1)\pi} - 1\right) =$$

$$\frac{1}{2\pi}\frac{(ik+1)\left(1 - e^{(ik-1)\pi}\right) - (1-ik)\left(e^{-(ik+1)\pi} - 1\right)}{(1-ik)(ik+1)} =$$

$$\frac{1}{2\pi}\frac{ik + 1 - ike^{(ik-1)\pi} - e^{(ik-1)\pi} - e^{-(ik+1)\pi} + 1 + ike^{-(ik+1)\pi} - ik}{1+k^2} =$$

Example: complex series

$$= \frac{1}{2\pi} \frac{1 + ike^{-\left(1+ik\right)\pi} - e^{-\left(1+ik\right)\pi} - e^{\left(ik-1\right)\pi} + 1 - ike^{\left(ik-1\right)\pi}}{1+k^2} =$$

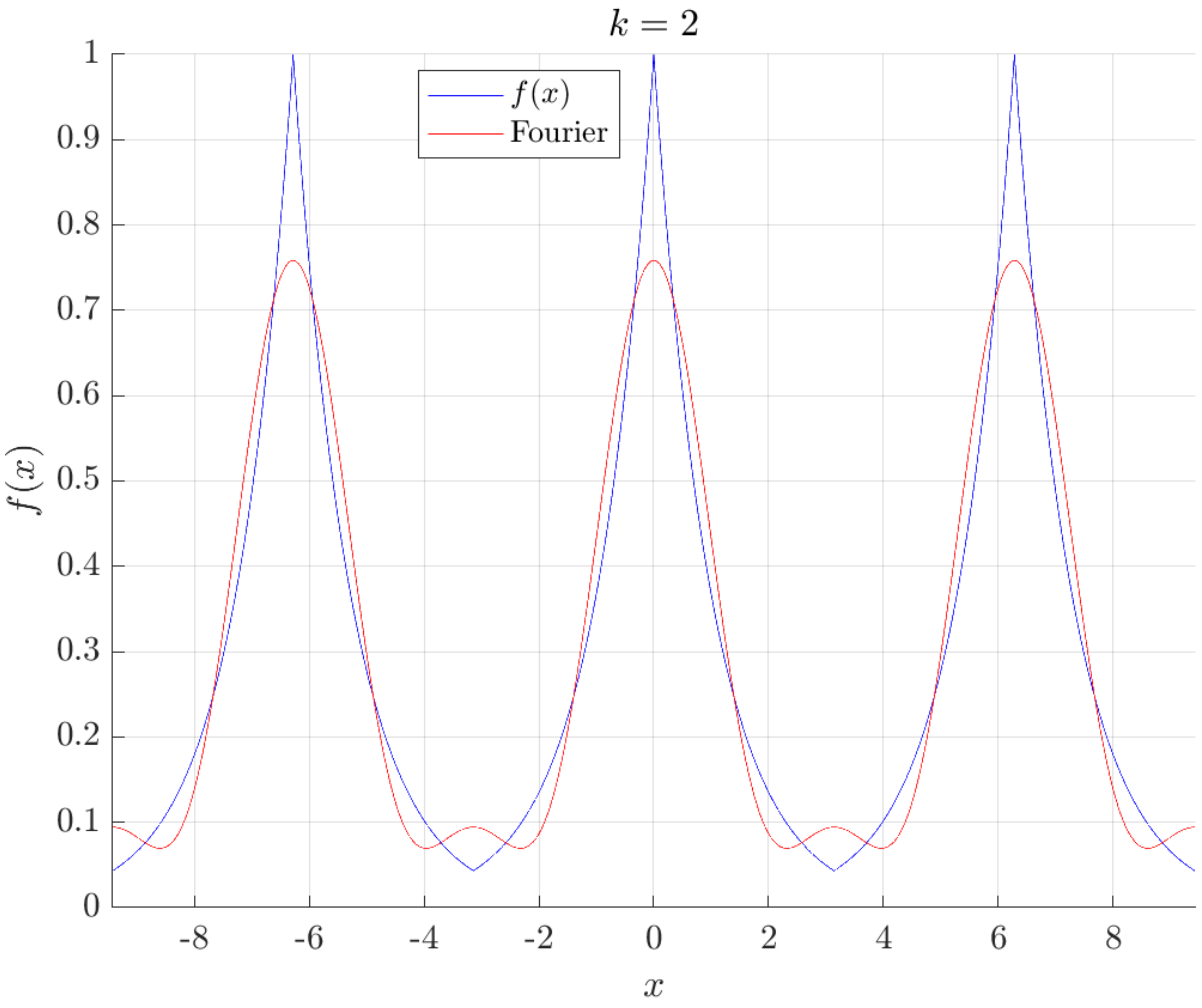
$$= \frac{1}{2\pi} \frac{2 + e^{-\pi}\left(ike^{-ik\pi} - e^{-ik\pi} - e^{ik\pi} - ike^{ik\pi}\right)}{1+k^2} = \frac{1}{2\pi} \frac{2 - e^{-\pi}\left(ike^{ik\pi} - ike^{-ik\pi} + e^{-ik\pi} + e^{ik\pi}\right)}{1+k^2}$$

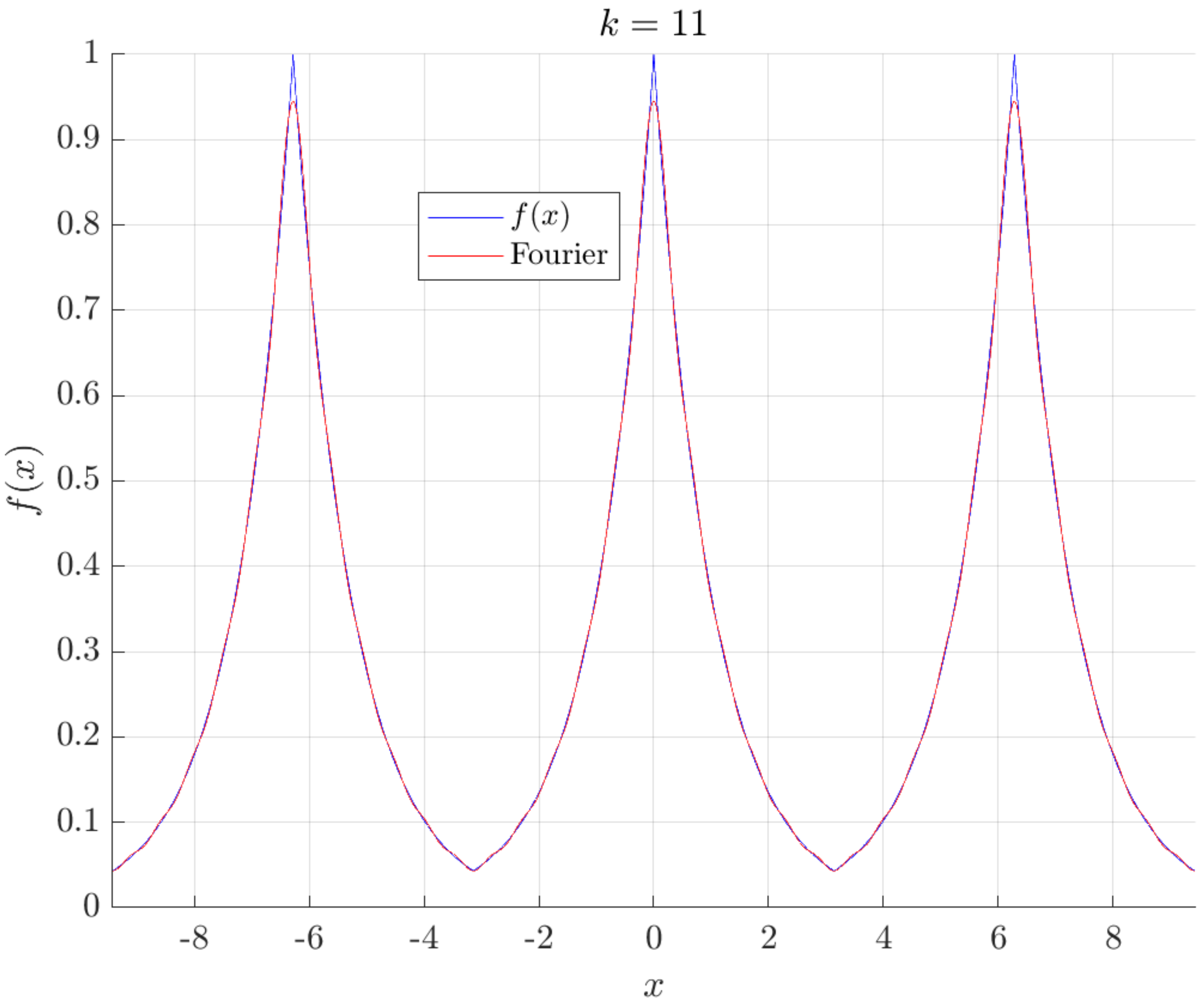
$$= \frac{1}{2\pi} \frac{2 - e^{-\pi}\left(-2k\sin\left(\pi k\right) + 2\cos\left(\pi k\right)\right)}{1+k^2} = \frac{1 - e^{-\pi}\cos\left(\pi k\right)}{\pi\left(1+k^2\right)}$$

Use some basic properties:

$$e^{ix} = \cos(x) + i\sin(x), \qquad \cos(x) = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}, \qquad e^{z+2\pi ik} = e^z, k \in \mathbb{Z}$$

Example: complex series





Example: ODE solution

Using Fourier series find the solution to the following initial value problem

$$\frac{d^2x}{dt^2} + 4x = 3\cos(3t) + 5\sin(t),\ x(0) = 0,\ \frac{dx}{dt}(0) = 0$$

The solution of the homogeneous problem is simple to found

$$\frac{d^2x}{dt^2} + 4x = 0 \Rightarrow x_h(t) = C_1\cos(2t) + C_2\sin(2t),$$ **where C_1 and C_2 are arbitrary.**

Assuming that $x(t)$ is a periodic function, we write it down as a Fourier series

$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\left(kt\right) + b_k\sin\left(kt\right)$ **and substitute it into the equation**

$$\frac{d^2}{dt^2}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\left(kt\right) + b_k\sin\left(kt\right)\right) + 4\left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\left(kt\right) + b_k\sin\left(kt\right)\right) = 3\cos(3t) + 5\sin(t)$$

$$-\left(\sum_{k=1}^{\infty} a_k k^2\cos\left(kt\right) + b_k k^2\sin\left(kt\right)\right) + 4\left(\sum_{k=1}^{\infty} a_k\cos\left(kt\right) + b_k\sin\left(kt\right)\right) = 3\cos(3t) + 5\sin(t)$$

Example: ODE solution

$$3\cos(3t) + 5\sin(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(kt\right) + b_k \sin\left(kt\right) \Rightarrow a_3 = 3, b_1 = 5, \textbf{ others} = 0$$

$$\sum_{k=1}^{\infty} \left(4 - k^2\right) a_k \cos\left(kt\right) = 3\cos(3t) \quad \textbf{and} \quad \sum_{k=1}^{\infty} \left(4 - k^2\right) b_k \sin\left(kt\right) = 5\sin(t)$$

$$\left(4 - 3^2\right) a_3 = 3, \quad \left(4 - 1^2\right) b_1 = 5 \Rightarrow a_3 = -\frac{3}{5}, b_1 = \frac{5}{3}$$

Particular solution: $x_p(t) = -\frac{3}{5}\cos(3t) + \frac{5}{3}\sin(t)$

The general solution: $x(t) = x_h(t) + x_p(t) = C_1\cos(2t) + C_2\sin(2t) - \frac{3}{5}\cos(3t) + \frac{5}{3}\sin(t)$

Finally, determine the arbitrary constants from the initial condition:

$$x(t) = \frac{3}{5}\cos(2t) - \frac{5}{6}\sin(2t) - \frac{3}{5}\cos(3t) + \frac{5}{3}\sin(t)$$

Fourier transform

$$F\left[f\right] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx,\ F^{-1}\left[\hat{f}\right] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i\omega x} d\omega$$

Linearity: $F\left[f+g\right] = F\left[f\right] + F\left[g\right]$

Shift: $F\left[f(x-a)\right] = e^{i\omega a} F\left[f\right]$

Derivative: $F\left[f'\right] = -i\omega F\left[f\right]$

Convolution: $F\left[f * g\right] = F\left[f\right] F\left[g\right],\ f * g = \displaystyle\int_{-\infty}^{\infty} f(x-y) g(y) dy$

$$F\left[f'\right] = \int_{-\infty}^{\infty} f'(x)\, e^{i\omega x} dx = \int_{-\infty}^{\infty} e^{i\omega x} d\left(f'(x)\right) = f(x)\, e^{i\omega x}\, \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, d\left(e^{i\omega x}\right) =$$

$$-i\omega \int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx = -i\omega F\left[f\right]$$

Example: properties

Find Fourier transform for

$$f(x) = \left(-\frac{x}{2} + 3\right) \exp\left(-\frac{(x-6)^2}{4}\right)$$

We know that $F\left[e^{-ax^2}\right] = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$**. How can we use this?**

$$f(x) = g'(x) \cdot \exp\left(g(x)\right) \Rightarrow F\left[f\right] = F\left[\left(\exp\ \left(g(x)\right)\right)'\right] = -\,i\omega\, F\left[\exp\left(g(x)\right)\right]$$

$$= -\,i\omega\, F\left[\exp\left(-\frac{1}{4}\left(x-6\right)^2\right)\right] = -\,i\omega\, e^{6i\omega}\, F\left[\exp\left(-\frac{1}{4}x^2\right)\right] = -\sqrt{4\pi}\, i\omega\, e^{6i\omega}\, e^{-\omega^2}$$

Example: PDE to ODE

Solve an initial value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \; u(x,0) = f(x), \; u_t(x,0) = 0$$

Applying the Fourier transform in x:

$$\hat{u}_{tt} = F\left[u_{tt}(x,t)\right] = c^2 F\left[u_{xx}(x,t)\right] = c^2(-i\omega)^2 F\left[u(x,t)\right] = -c^2\omega^2\hat{u}$$

Solution of this ODE

$$\hat{u}(\omega,t) = Ae^{-i\omega ct} + Be^{i\omega ct}$$

Using initial conditions

$$F\left[f\right] = F\left[u(x,0)\right] = \hat{u}(\omega,0) = A + B,$$

$$F\left[0\right] = F\left[u_t(x,0)\right] = \hat{u}(\omega,0)_t = -i\omega cA + i\omega cB = 0$$

$$\Rightarrow A = B = \frac{1}{2}F\left[f\right](\omega)$$

Argument

Example: PDE to ODE

Inverse Fourier transform

$$u(x,t) = F^{-1}\left[\hat{u}(\omega,t)\right] = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(F\left[f\right]e^{-i\omega ct} + F\left[f\right]e^{i\omega ct}\right)e^{-i\omega x}dx =$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(F\left[f\right]e^{-i\omega(x+ct)} + F\left[f\right]e^{-i\omega(x-ct)}\right)d\omega =$$

$$\frac{1}{2}F^{-1}\left[F\left[f\right]\underset{\text{Argument}}{(x+ct)} + F\left[f\right]\underset{\text{Argument}}{(x-ct)}\right] = \frac{f(x+ct)+f(x-ct)}{2}$$

Next seminar

- Discrete and fast Fourier transform
- Spectral method for the quantum oscillator problem