Seminar 9

Limit cycles, bifurcations

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Limit cycles

Isolated closed trajectory
Self-sustained oscillations

No closed orbits

Gradient systems $\dot{\mathbf{x}} = -\nabla V$, then

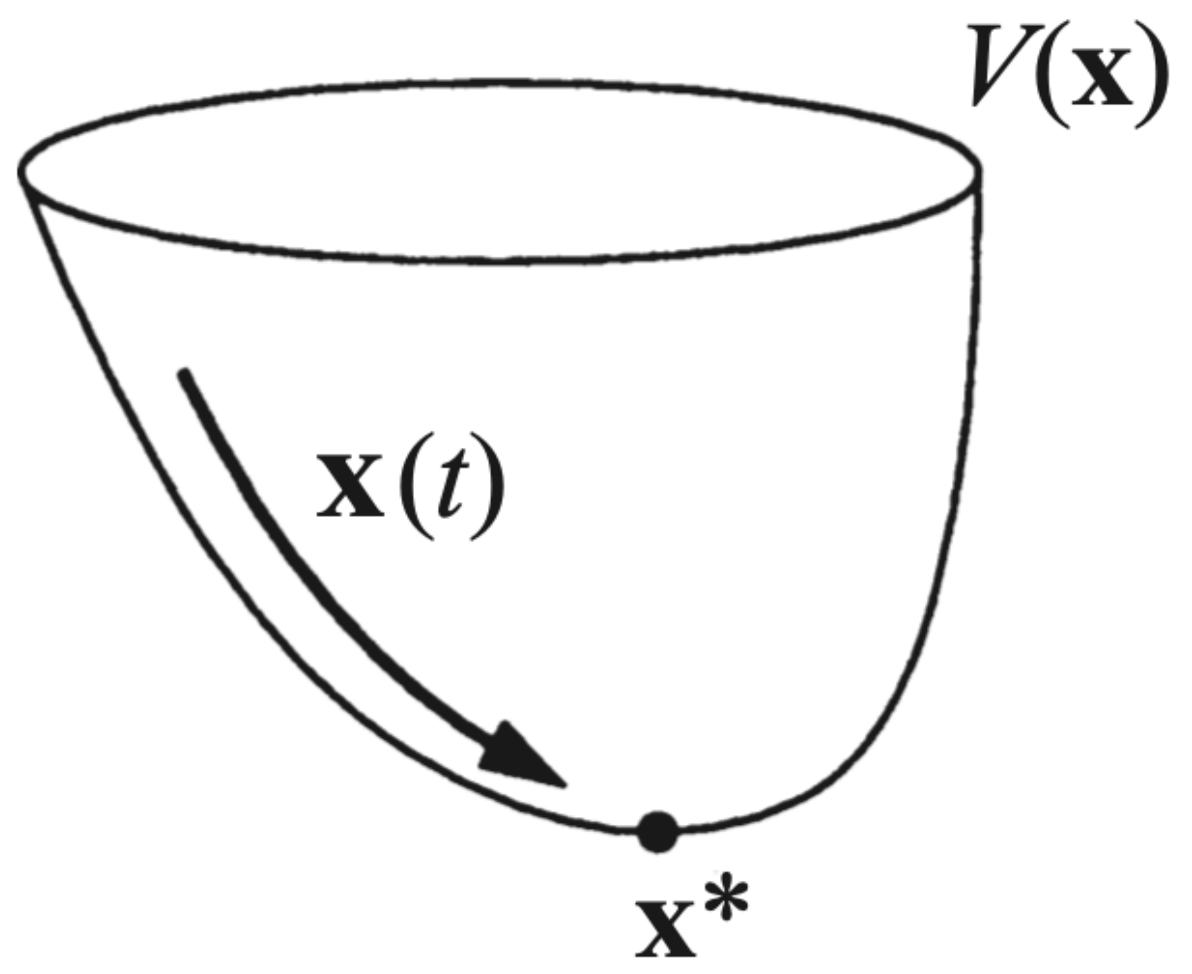
$$\Delta V = \int_0^T \frac{dV}{dt} dt = \int_0^T (\nabla V \cdot \dot{\mathbf{x}})\, dt = -\int_0^T \|\dot{\mathbf{x}}\|^2 dt < 0.$$

If you find such $E(\mathbf{x})$ that $\int_0^T \dot{E} dt \neq 0$, than there is no closed orbits

Lyapunov function $V(\mathbf{x}) \in C^1$ for a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

1. $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$
2. $\dot{V} < 0$ for $\mathbf{x} \neq \mathbf{x}^*$

Then, $\forall \mathbf{x}_0 \to \mathbf{x}^*$, and no closed orbits



Example: typical Lyapunov

$$\begin{cases} \dot{x} = -x + 4y \\ \dot{y} = -x - y^3 \end{cases}$$

Search for $V(x, y) = x^2 + ay^2$

$$\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x\left(-x + 4y\right) + 2ay\left(-x - y^3\right) =$$

$$= -2x^3 + (8 - 2a)\,xy - 2ay^4$$

For $a = 4$**,** $V = x^2 + 4y^2 > 0$ **and** $\dot{V} < 0$**, so no closed orbits**

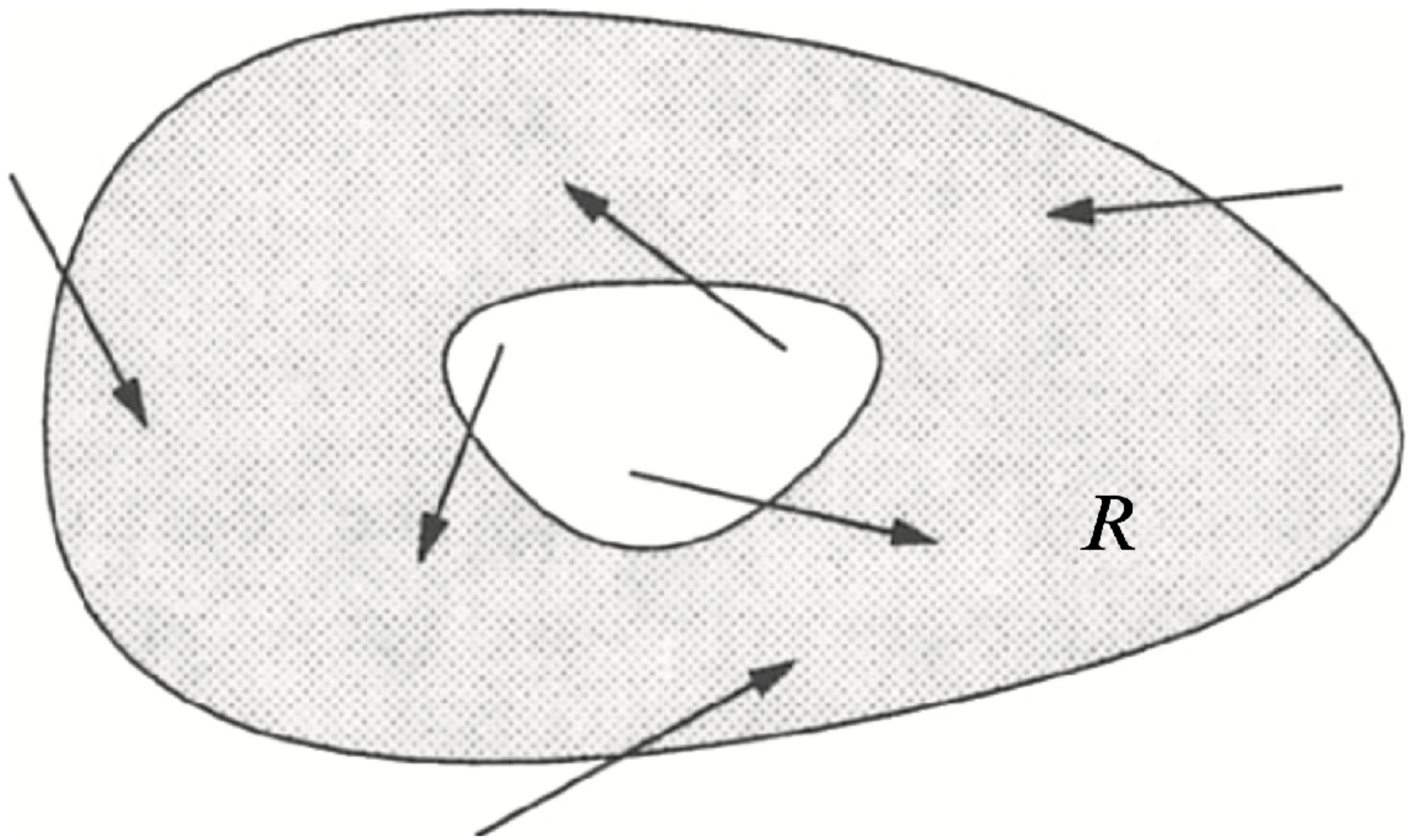
What if...?

Poincare-Bendixson theorem

1. R is a closed, bounded subset of the plane
2. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is C^1 on an open set containing R
3. R does not contain any fixed points
4. There exist a trajectory C that starts in R and stays in R for all the future time.

$$\Downarrow$$

Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$.
So, R contains a closed orbit.



Standard trick: to construct a trapping region R

NO CHAOS in the phase plane.

Example: using PBT

$$\begin{cases} \dot{r} = r\left(1 - r^2\right) + \mu r \cos\theta \\ \dot{\theta} = 1 \end{cases}$$

$\mu = 0$?

$\Rightarrow$ a stable limit cycle at $r = 1$.

Is there a closed orbit for $0 < \mu \ll 1$?

Let's find a region R with $\dot{r} < 0$ at the outer circle with radius $r_{\max}$ and $\dot{r} > 0$ at the inner circe with radius $r_{\min}$. Fixed points in R?

$\dot{\theta} > 0 \Rightarrow$ No. Possible $r_{\min}$ and $r_{\max}$?

$$r\left(1 - r^2\right) + \mu r \ \cos\theta > 0 \Rightarrow 1 - r^2 - \mu > 0 \Rightarrow$$

any $r_{\min} < \sqrt{1 - \mu}$ with $\mu < 1$

Similarly, $r_{\max} > \sqrt{1 + \mu}$.

$R: \ r_{\min} < r < r_{\max}$ or $0.999\sqrt{1 - \mu} < r < 1.001\sqrt{1 + \mu}$.

Therefore the closed orbit exists somewhere in R for all $\mu < 1$.

Main nonlinear oscillators

- Mathieu (stable and unstable regions, reversed pendulum):

$$\ddot{x} + \left(\alpha + \beta \cos t\right) x = 0$$

- Van der Pol (relaxation for $\mu \gg 1$, phase locking by external force):

$$\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0$$

- Weakly nonlinear Duffing or VdP for $\mu \ll 1$
(multiple scales, subharmonic resonances)

$$\ddot{x} + \mu x^3 + x = 0$$

Lienard equations

$\ddot{x} + f(x)\,\dot{x} + g(x) = 0$:

nonlinear damping $-f(x)\,\dot{x}$**,**

nonlinear restoring $-g(x)$**.**

If $f(x)$**,** $g(x) \in C^1(\mathbb{R})$**;** $g(x) > 0$ **for all** $x > 0$**;**

$g(-x) = -g(x)$ **for all** x**;** $f(-x) = f(x)$ **for all** x**;**

$F(x) = \int_0^x f(u)\,\mathrm{d}u$ **is odd and has exactly one positive zero at** $x = a$**, is negative for** $0 < x < a$**, is positive and nondecreasing for** $x > a$**, and** $F(x) \to \infty$ **as** $x \to \infty$**.**

$\Downarrow$

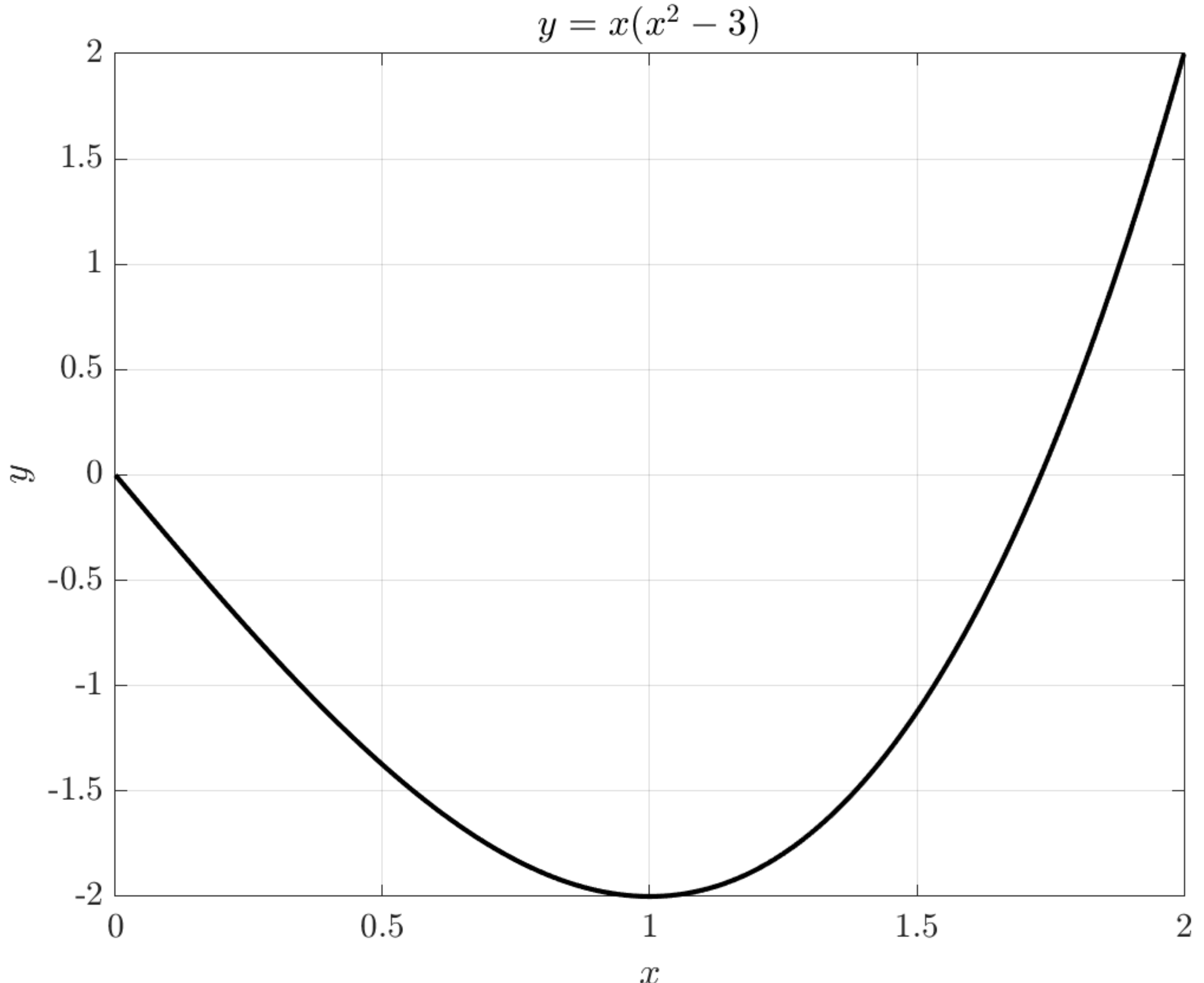
Then the system has a unique, stable limit cycle surrounding the origin in phase plane.

Van der Pol equation $\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$:

$f(x) = \mu\left(x^2 - 1\right), g(x) = x \Rightarrow$ **(1)-(4) are satisfied.**

$$F(x) = \int_0^x f(u)\,\mathrm{d}u = \mu\left(\frac{x^3}{3} - x\right) = \frac{1}{3}\mu x\left(x^2 - 3\right)$$

There is a unique stable limit cycle



Bifurcations

Phase portrait changes its topological structure as a parameter is varied (changes in the number or stability of fixed points, closed orbits, collisions of fixed points as a parameter is varied)

Example: genetic control system

x, y are the concentrations of protein and RNA; $a, b > 0$ are the rates of degradations

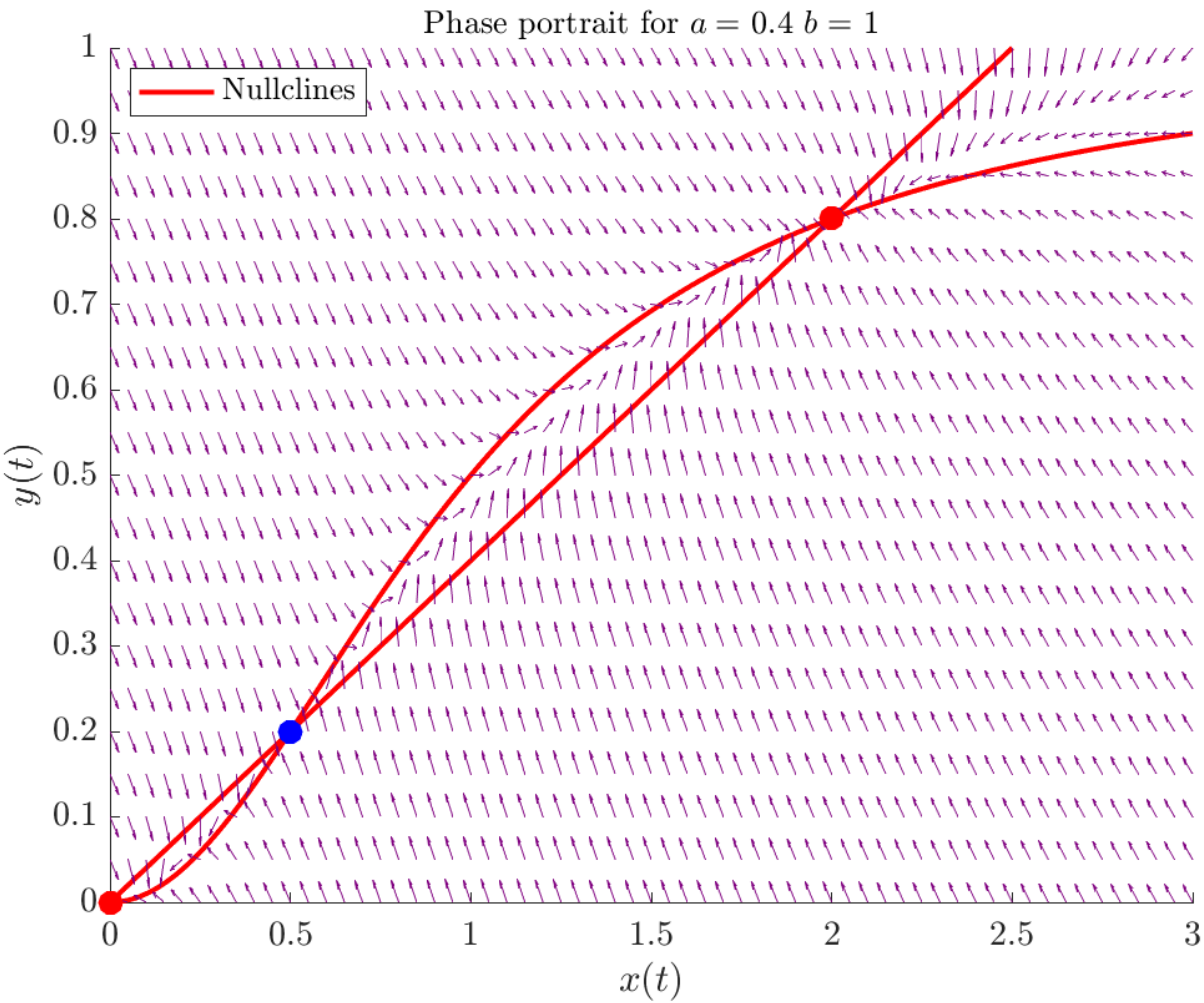
$$\begin{cases} \dot{x} = -ax + y \\ \dot{y} = \dfrac{x^2}{1+x^2} - by \end{cases}$$

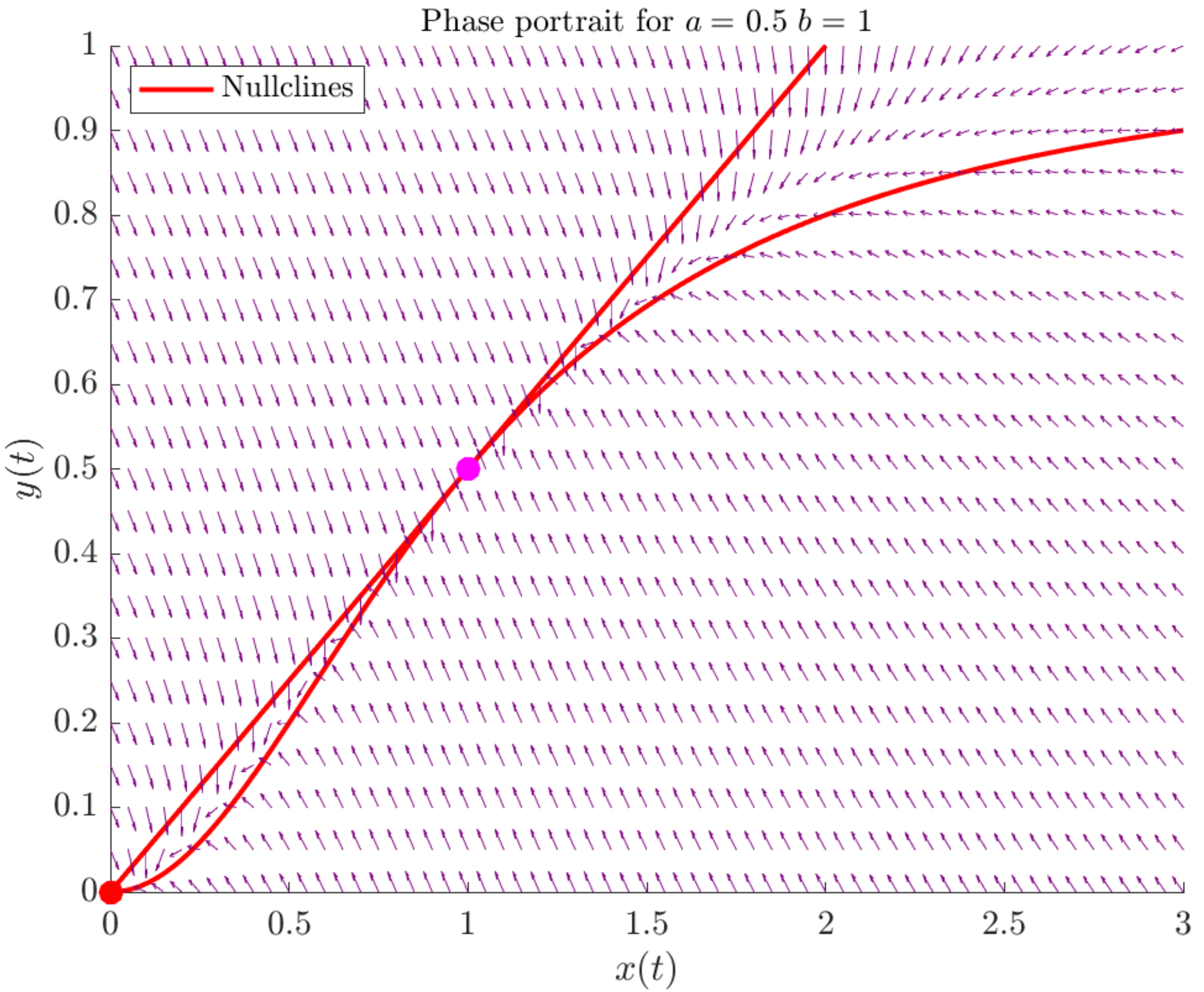
$$ax = \frac{x^2}{b\left(1+x^2\right)} \Rightarrow x_1 = 0,\ x_{2,3} = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}$$

$$J\left(x, y\right) = \begin{pmatrix} -a & 1 \\ \dfrac{2x}{\left(1+x^2\right)^2} & -b \end{pmatrix} \Rightarrow \begin{matrix} \tau = -a-b < 0 \\ \Delta = ab\,\dfrac{x^2-1}{1+x^2} \text{ at } x_{2,3},\ ab \text{ at } x_1 \end{matrix} \Rightarrow$$

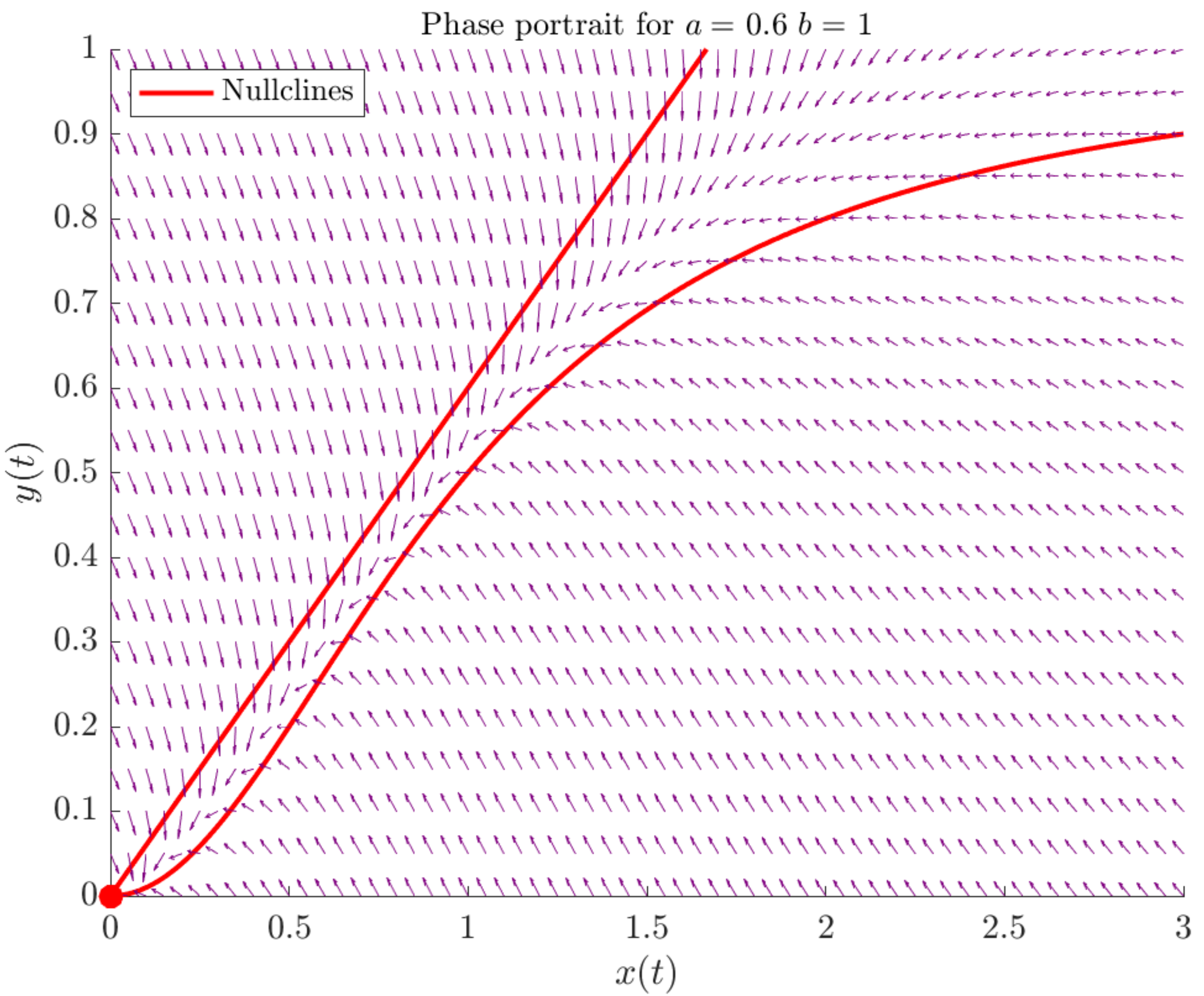
$(0,0)$ **is a stable node,** $\left(x_2, y_2\right), x_2 < 1$ **is a saddle,** $\left(x_3, y_3\right), x_3 > 1$ **is a stable node.**

Let's fix $b = 1$ and increase $0.4 < a < 0.6$. What happens?


Phase portrait for $a = 0.4$ $b = 1$
Nullclines
$y(t)$
$x(t)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3



Phase portrait for $a = 0.5$ $b = 1$
Nullclines
$y(t)$
$x(t)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3



Phase portrait for $a = 0.6$ $b = 1$
Nullclines
$y(t)$
$x(t)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3

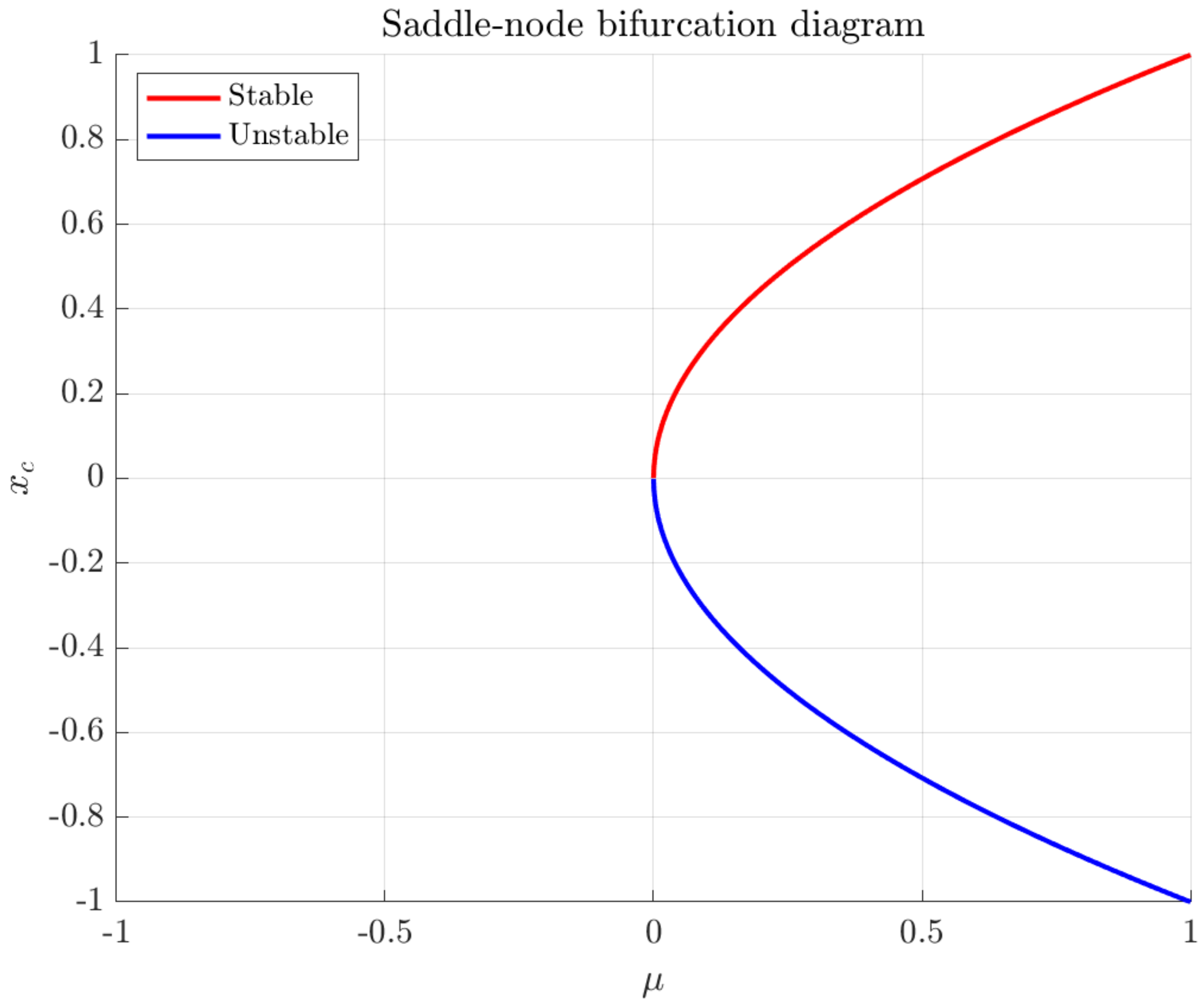
Saddle-node

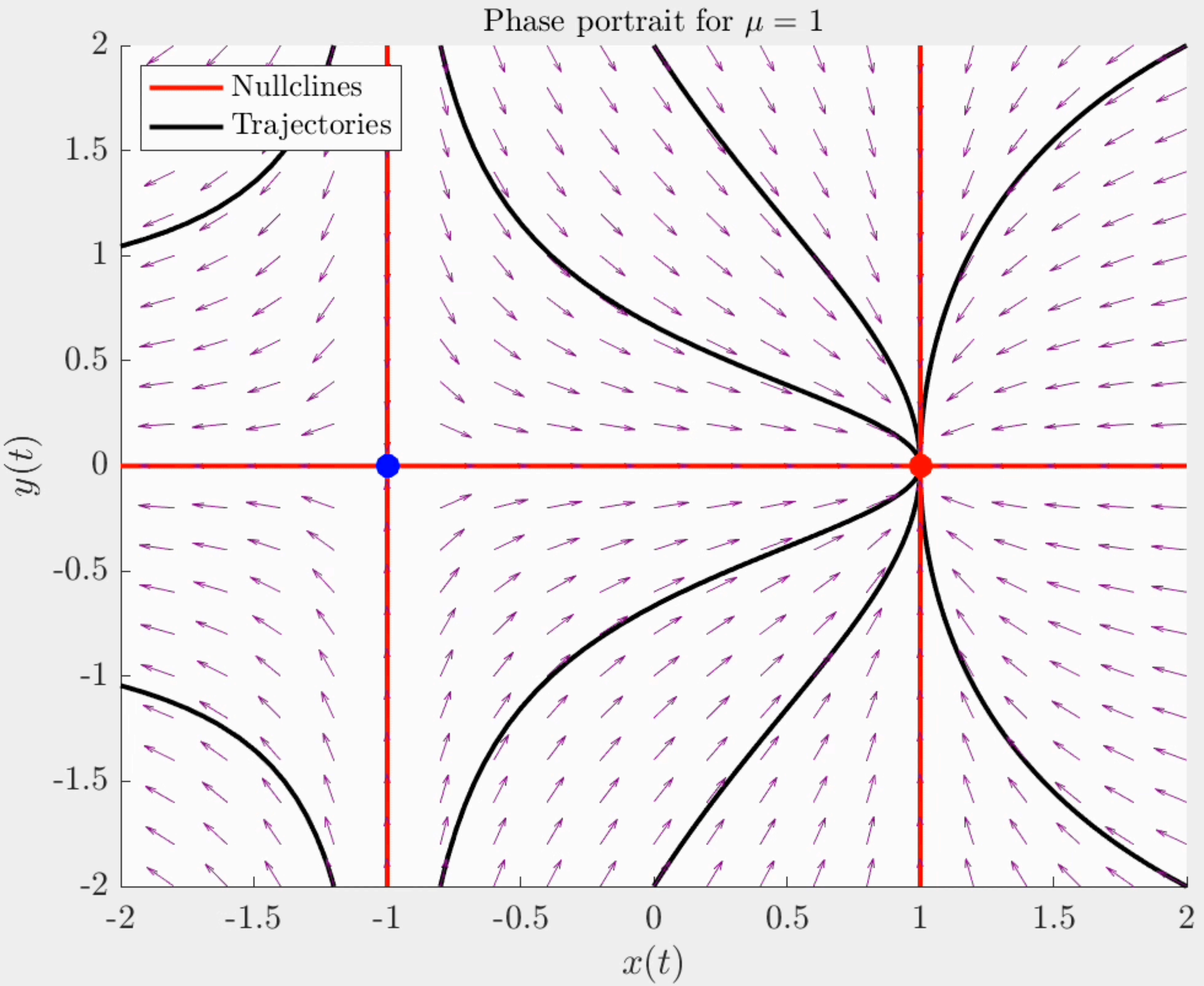
2D example:

$$\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = -y \end{cases} \Rightarrow \begin{cases} x_c = \pm\sqrt{\mu} & ,\mu \geq 0 \\ \text{no } x_c & ,\mu < 0 \end{cases}$$

The basic mechanism for the creation and destruction of fixed points.

Blue sky bifurcation.



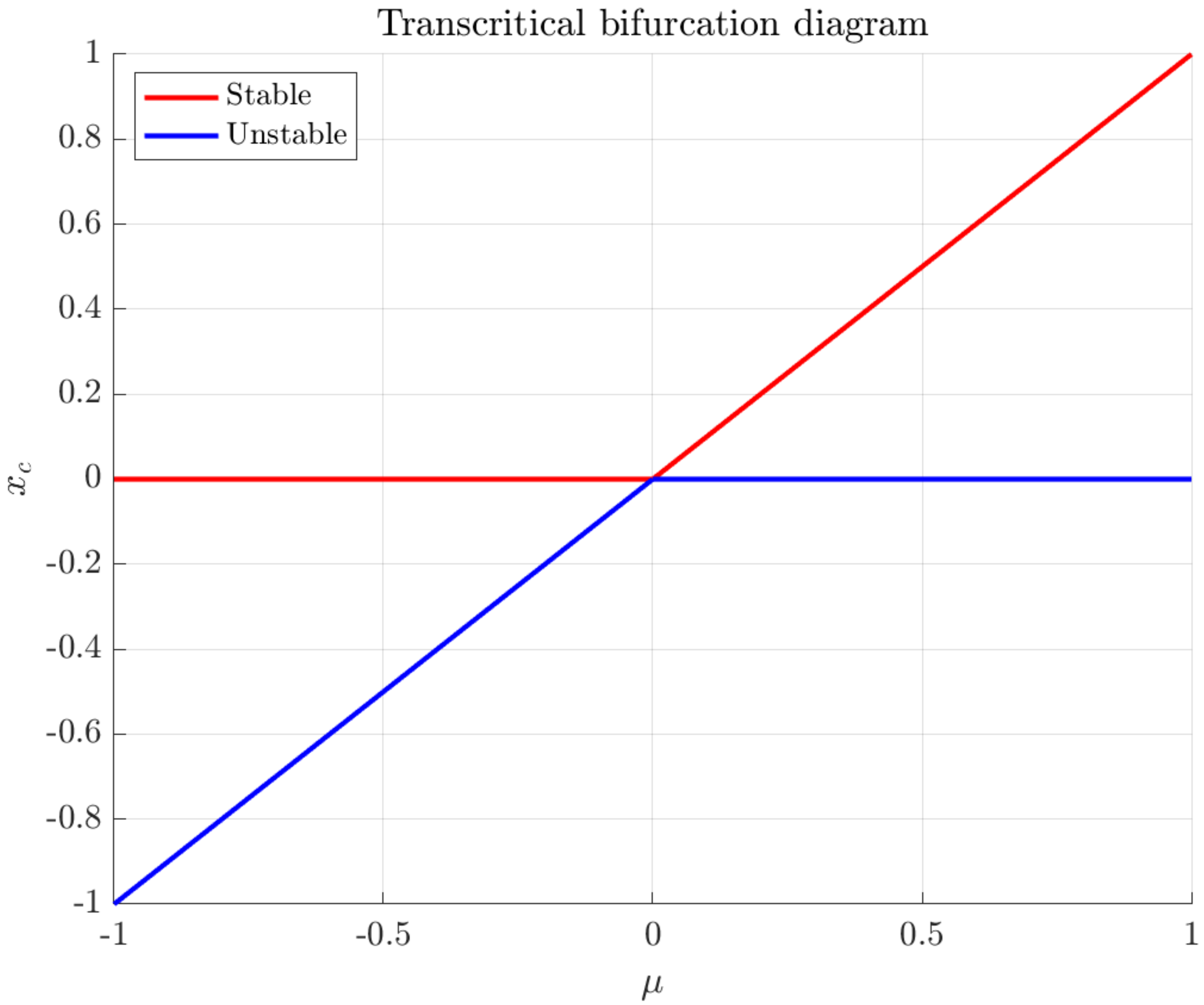


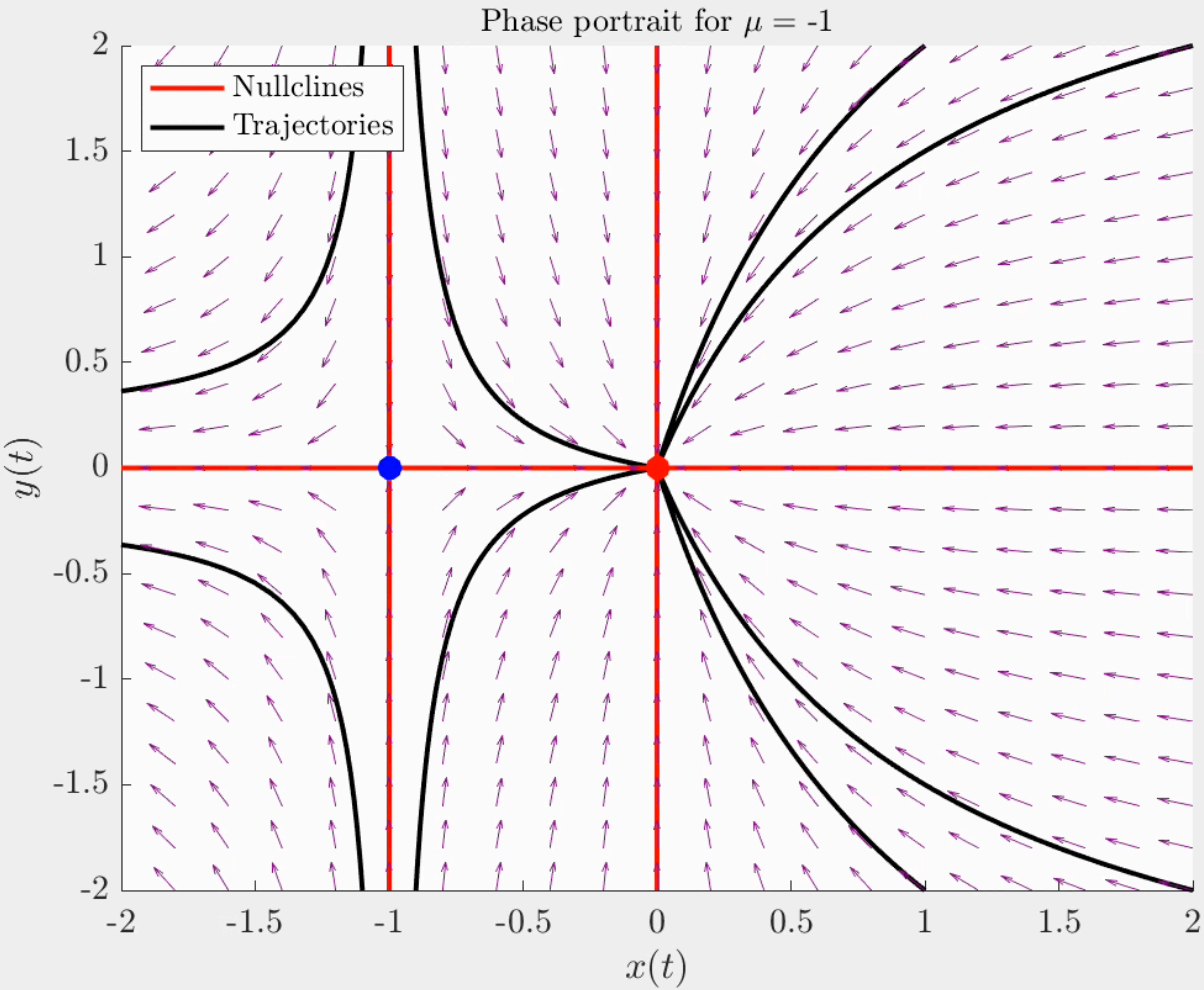
Phase portrait for $\mu = 1$
Nullclines
Trajectories
$y(t)$
$x(t)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Transcritical

2D example:

$$\begin{cases} \dot{x} = \mu x - x^2 \\ \dot{y} = -y \end{cases}$$





Phase portrait for μ = -1
Nullclines
Trajectories
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
$y(t)$
$x(t)$

Example

$$\begin{cases} \dot{x} = \mu x + y + \sin x \\ \dot{y} = x - y \end{cases}$$

$(0,0)$ **is a fixed point with** $J\left(x, y\right) = \begin{pmatrix} \mu + 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{cases} \tau = \mu \\ \Delta = -\left(\mu + 2\right) \end{cases} \Rightarrow$

$(0,0)$ **is a saddle if** $\mu > -2$ **and is a stable fixed point for** $\mu < -2$

Most likely, the bifurcation occurs when $\mu = -2$**.**

$$\begin{cases} x = y \\ \mu x + x + \sin x = 0 \end{cases}$$

$\mu x + x + x - \dfrac{x^3}{3} \approx 0 \Rightarrow \mu + 2 - \dfrac{x^2}{3!} \approx 0 \Rightarrow x_{\pm} \approx \pm\sqrt{6\left(\mu + 2\right)}$ **for** $\mu > -2$

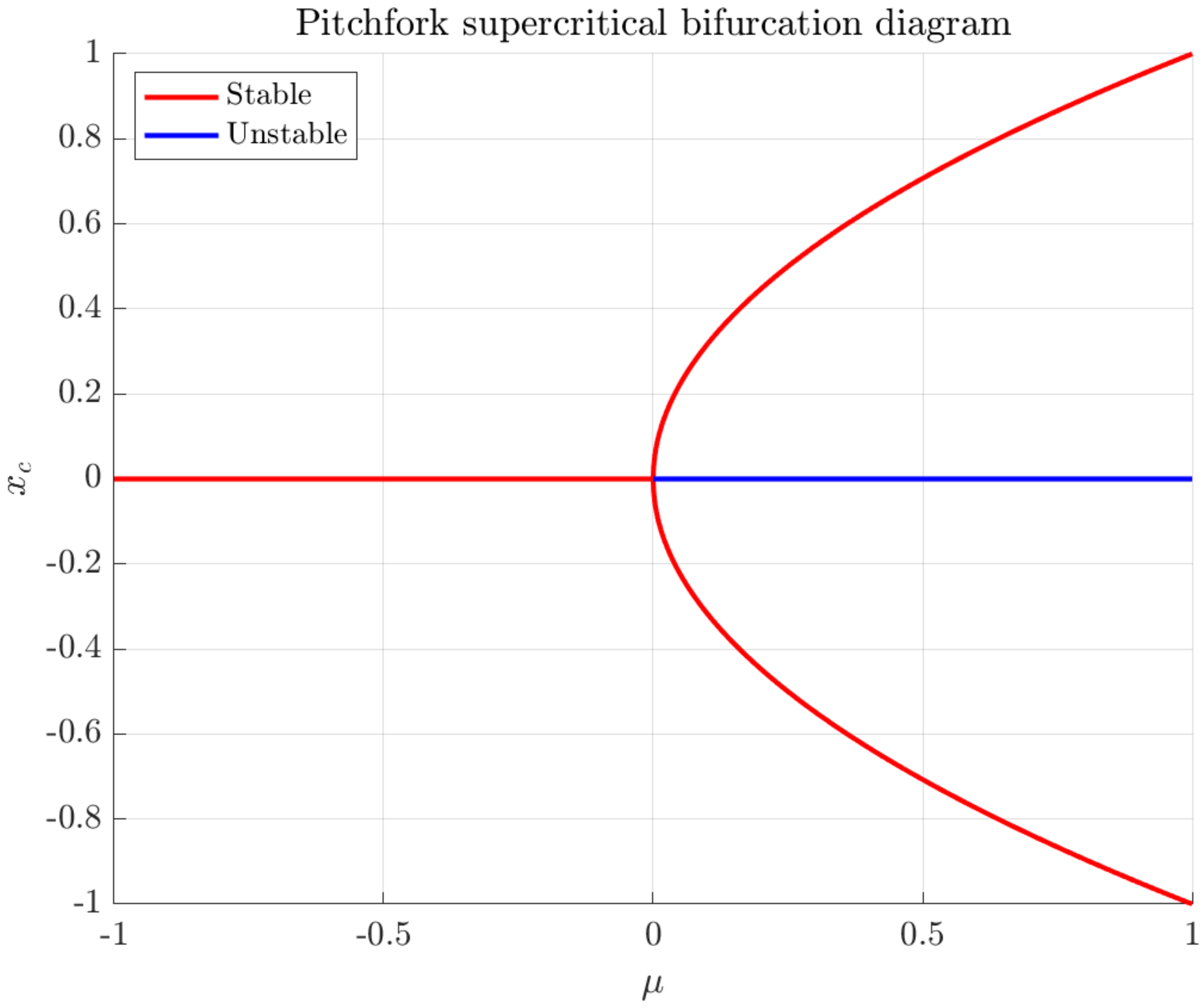
One stable $(0,0) \rightarrow$ **One unstable** $(0,0)$ **+ Two new** (x_+, x_+) **and** (x_-, x_-)

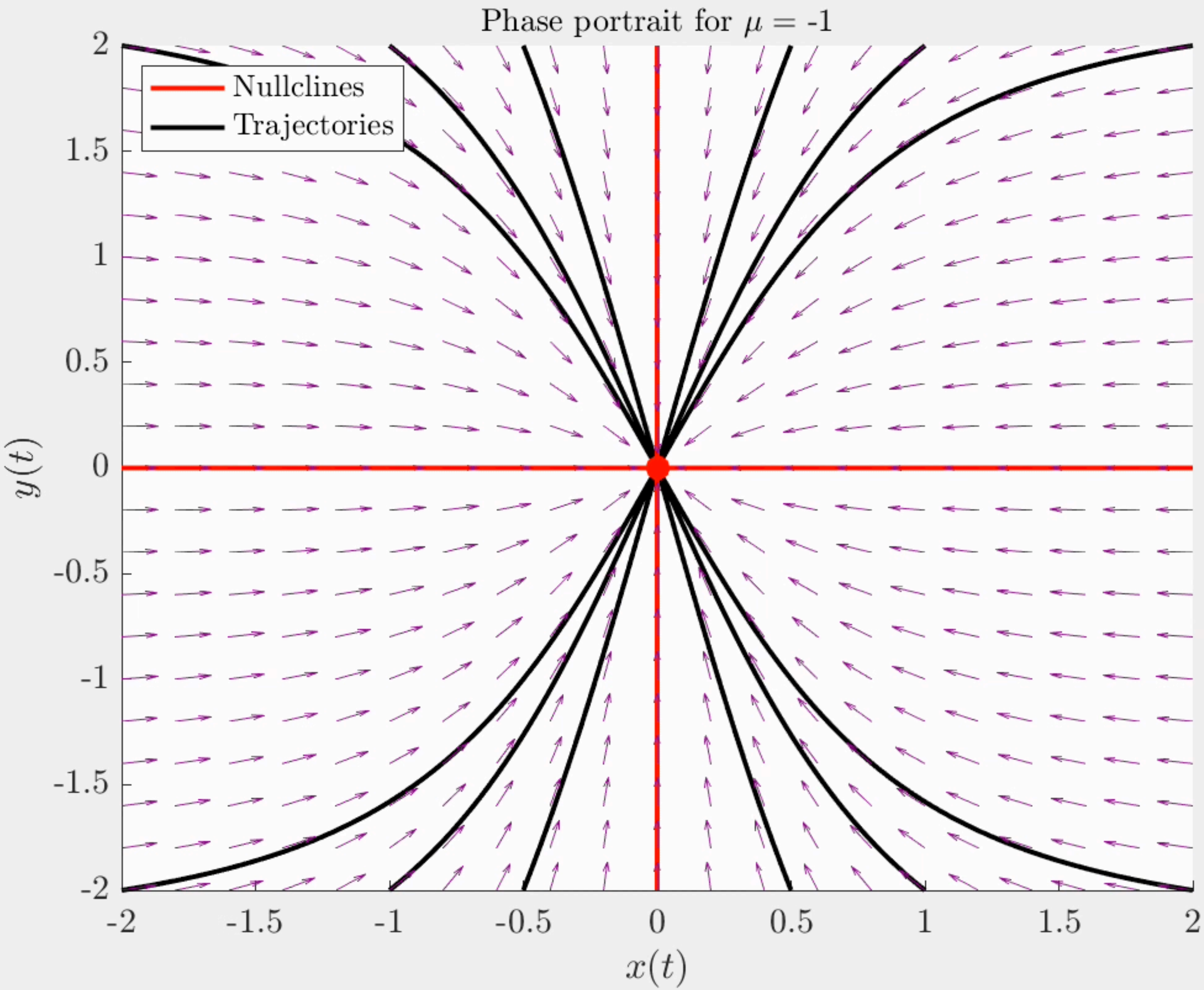
From one stable f.p. to three f.p. with the change of stability.

Supercritical pitchfork

2D example:

$$\begin{cases} \dot{x} = \mu x - x^3 \\ \dot{y} = -y \end{cases}$$



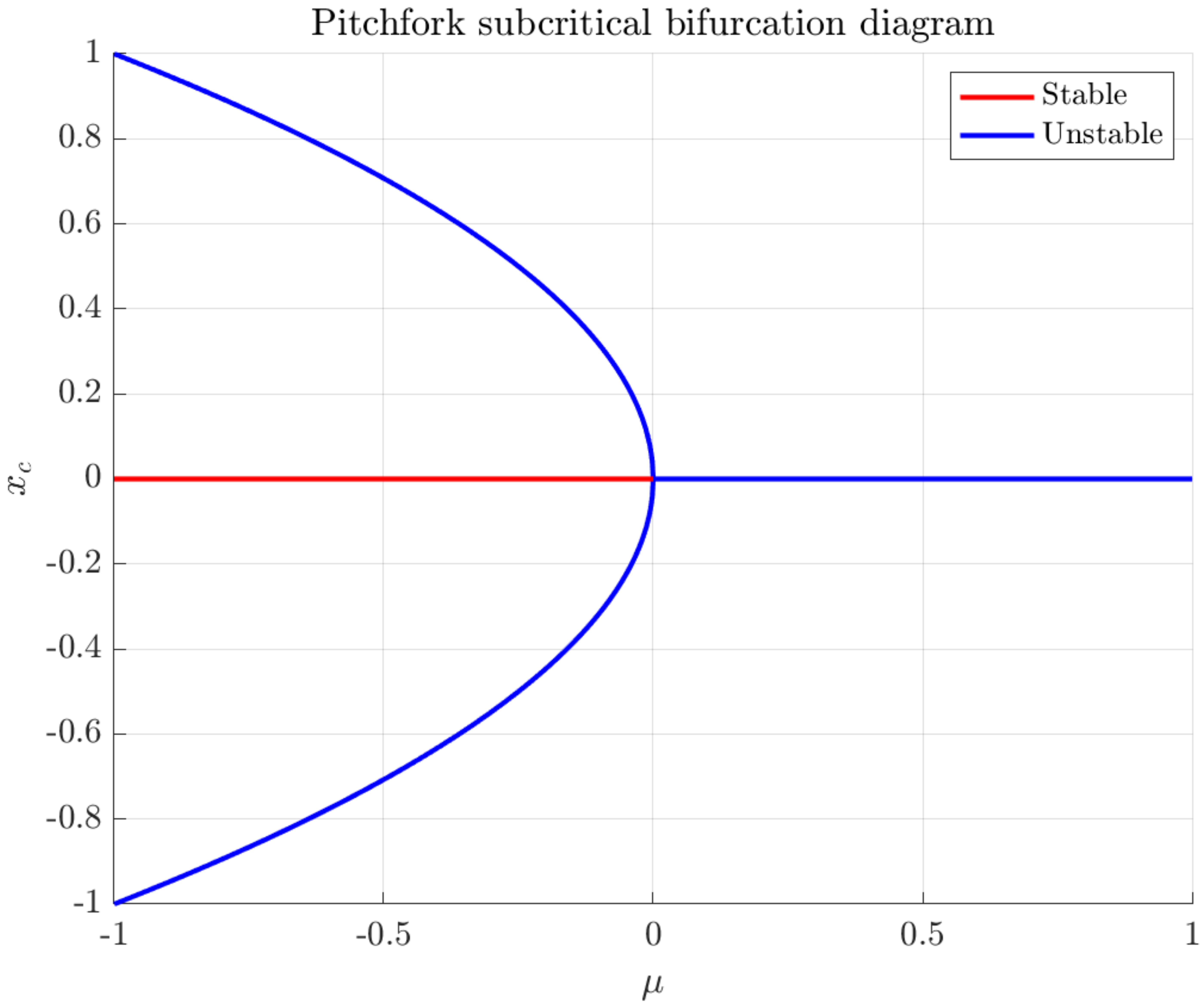


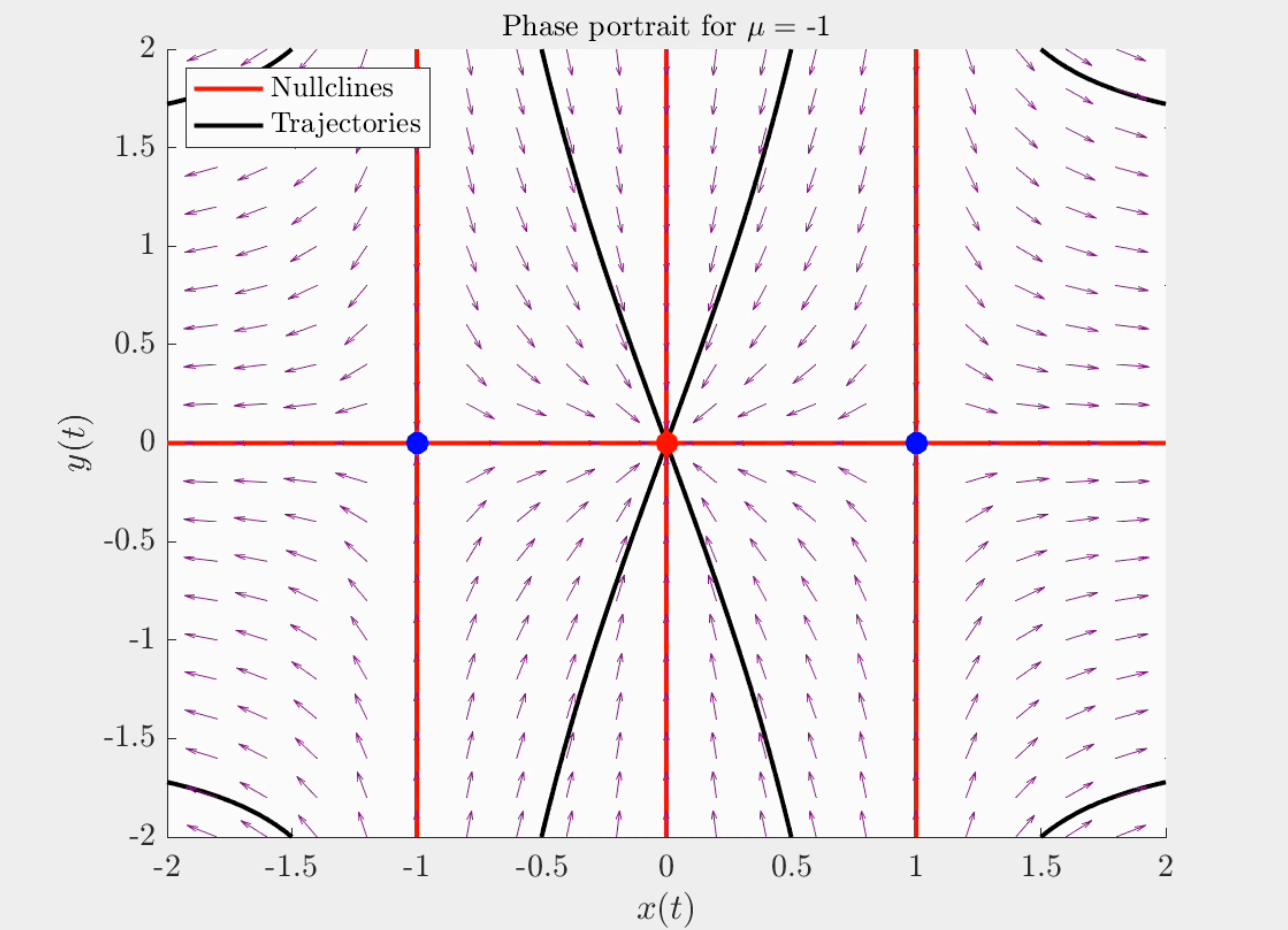
Phase portrait for μ = -1
Nullclines
Trajectories
$y(t)$
$x(t)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Subcritical pitchfork

2D example:

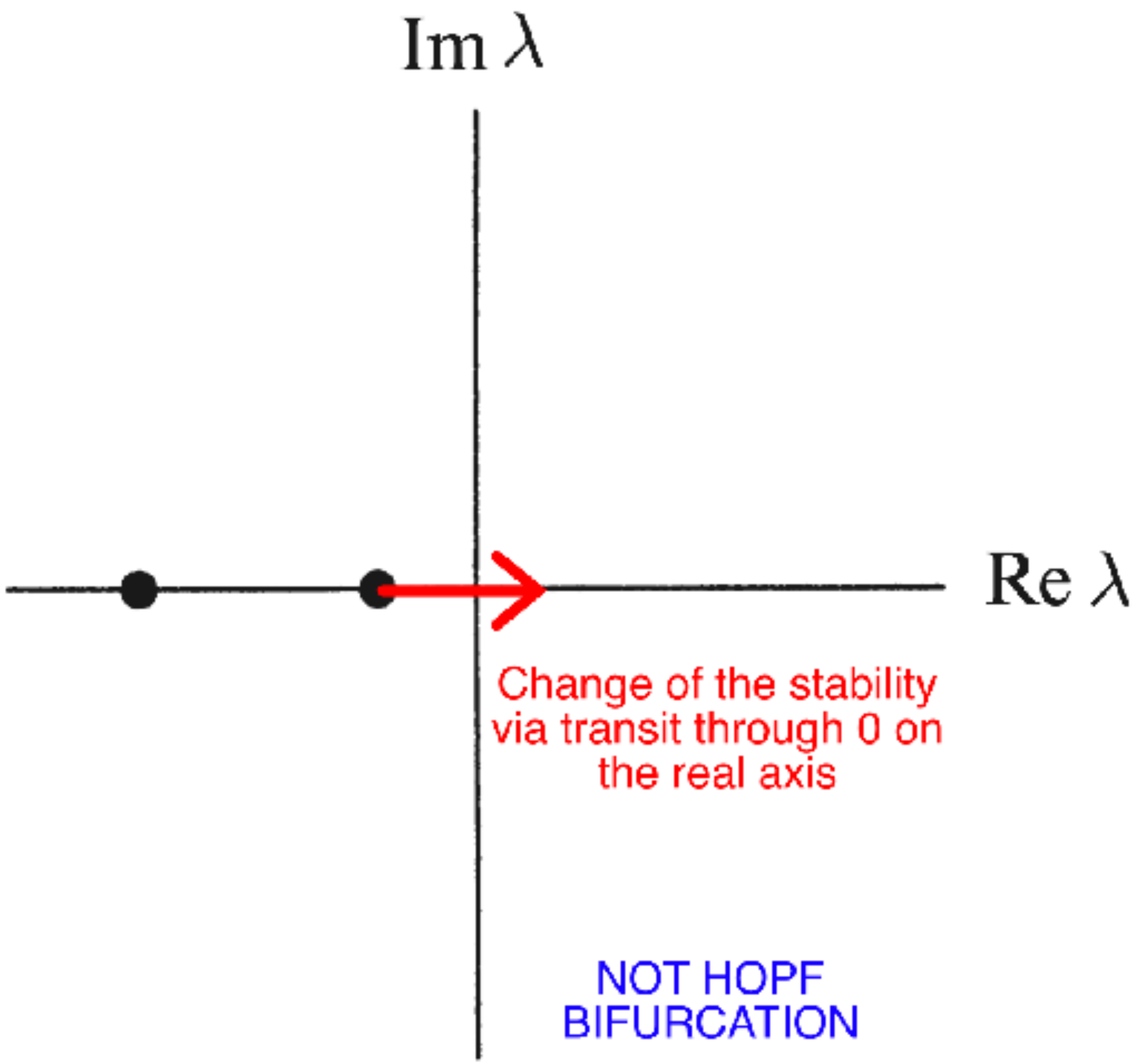
$$\begin{cases} \dot{x} = \mu x + x^3 \\ \dot{y} = -y \end{cases}$$



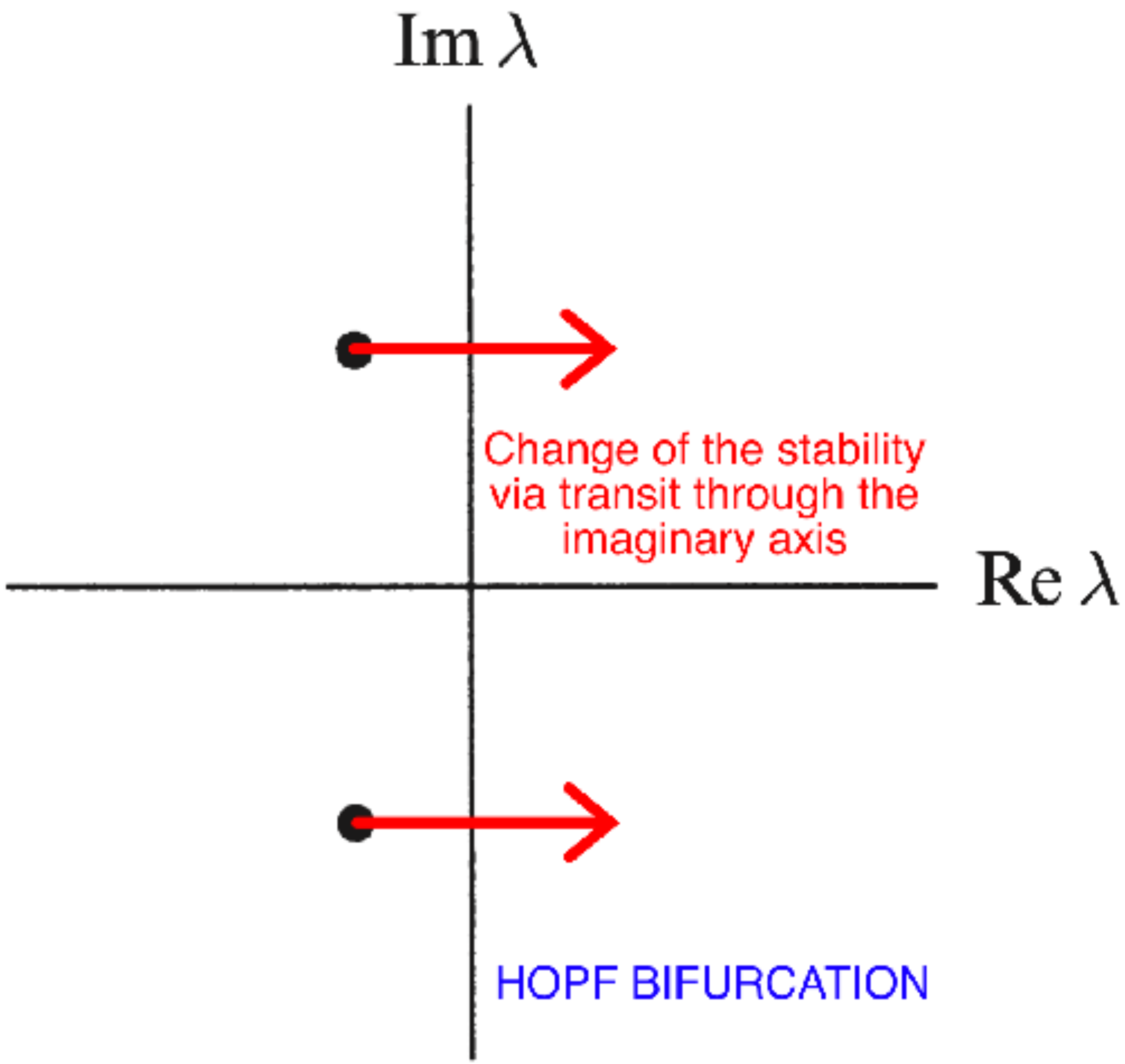


Phase portrait for μ = -1
Nullclines
Trajectories
$y(t)$
$x(t)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2

What with eigenvalues?



Saddle-node, transcritical, pitchfork, can be in 1D



Hopf, involves limit cycles only in 2D

Supercritical Hopf

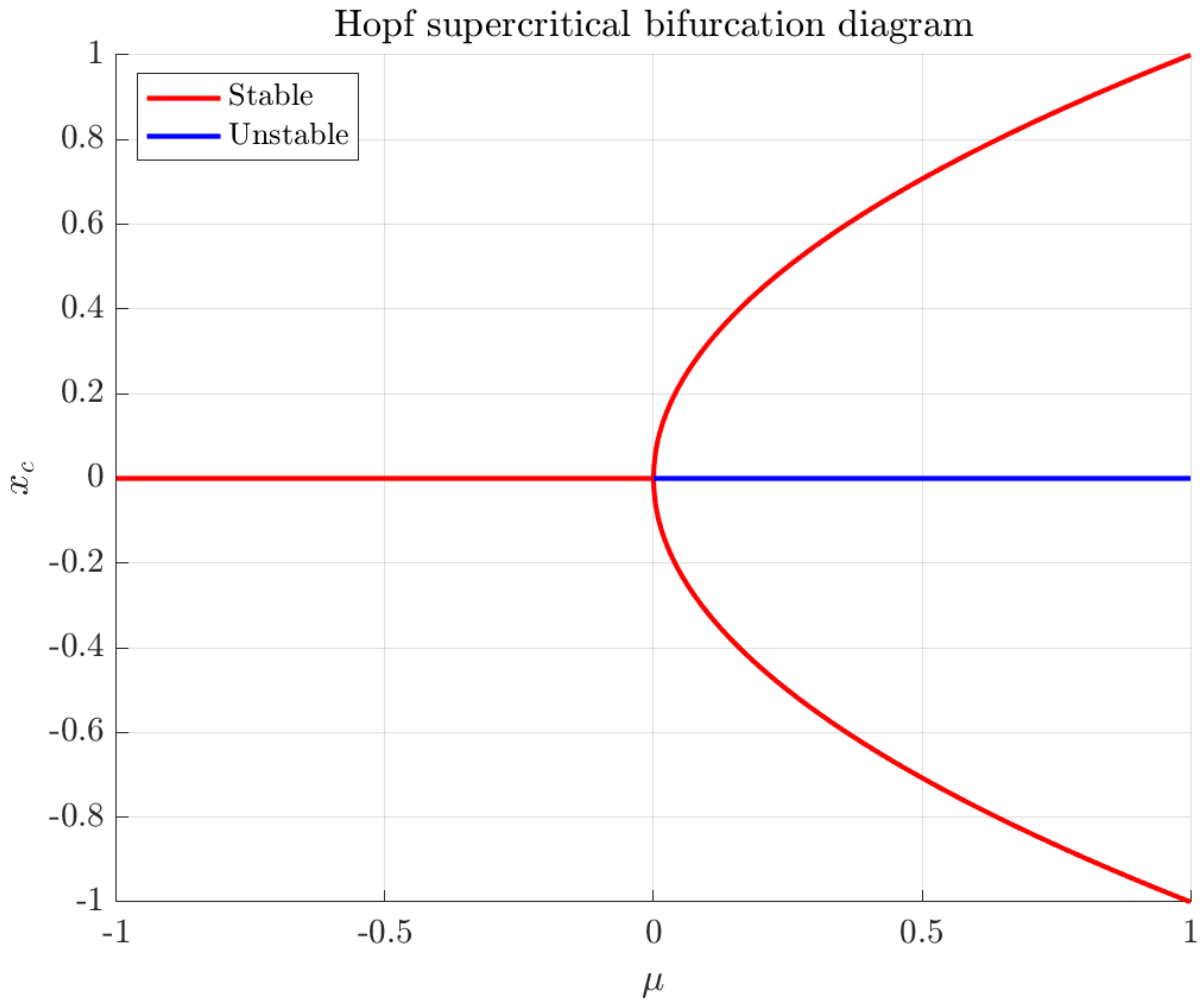
Roots transit through imaginary axis

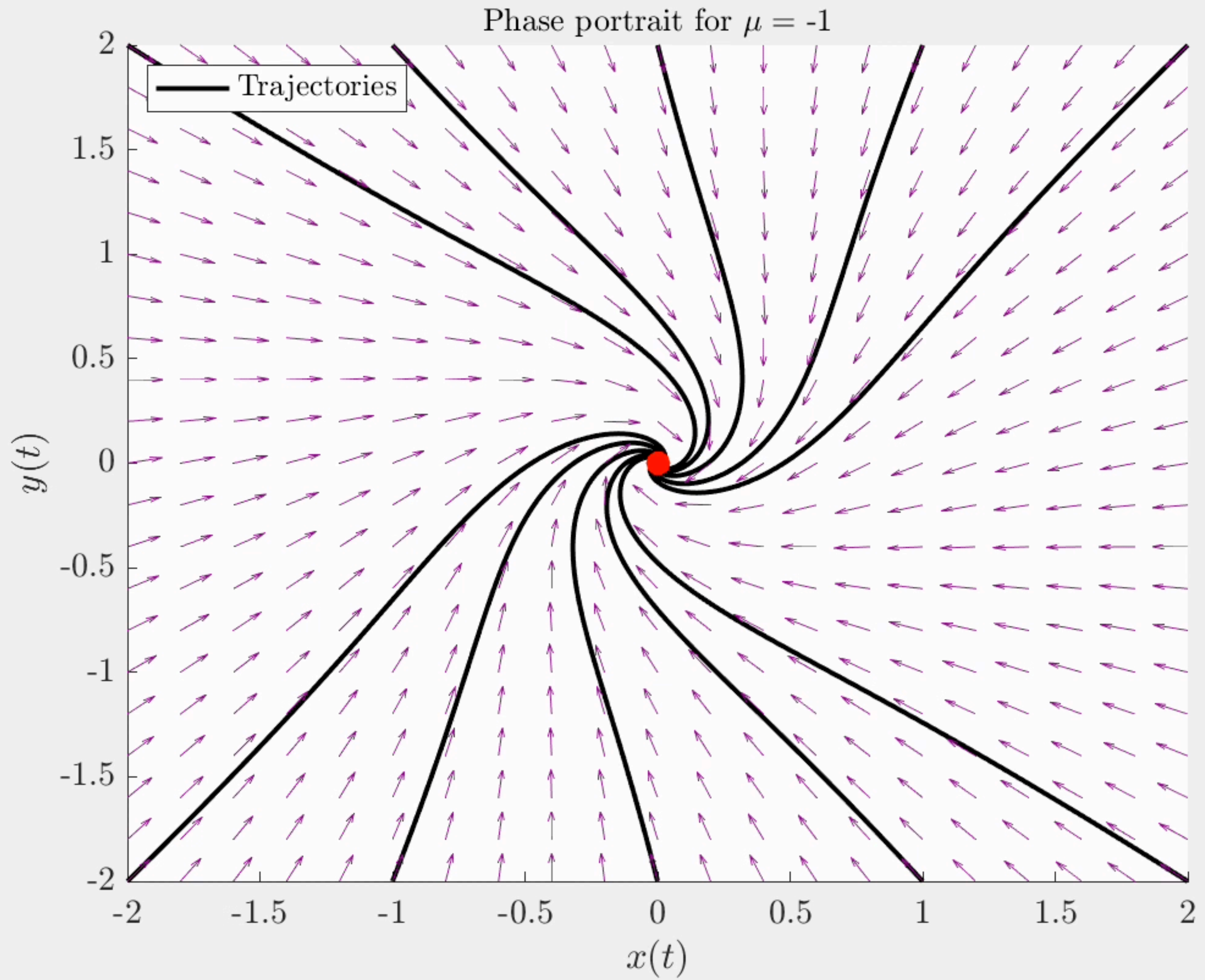
2D example: $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega \end{cases} \Rightarrow$

$$J = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}, \ \lambda = \mu \pm i\omega$$

$\mu < 0$ **stable spiral,** $\mu > 0$ **unstable spiral and limit cycle.**

1. **Radius of the limit cycle grows from zero proportional to** $\sqrt{\mu - \mu_c}$ **for** $\left|\mu - \mu_c\right| \ll 1$
2. **Frequency of the limit cycle is given approximately by** $\mathrm{Im}\lambda\,|_{\mu=\mu_c}$





Phase portrait for μ = -1
Trajectories
$y(t)$
$x(t)$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

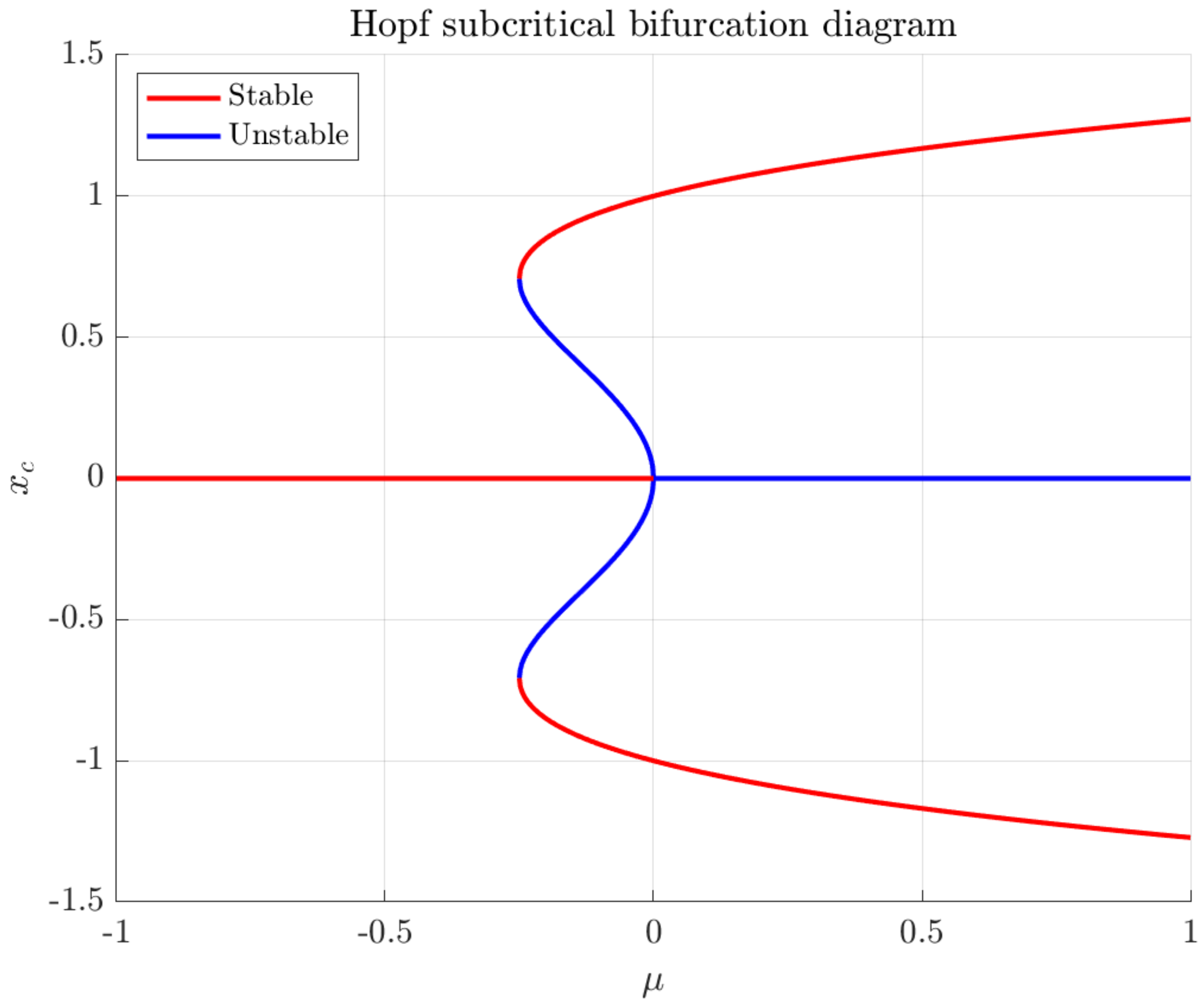
Subcritical Hopf

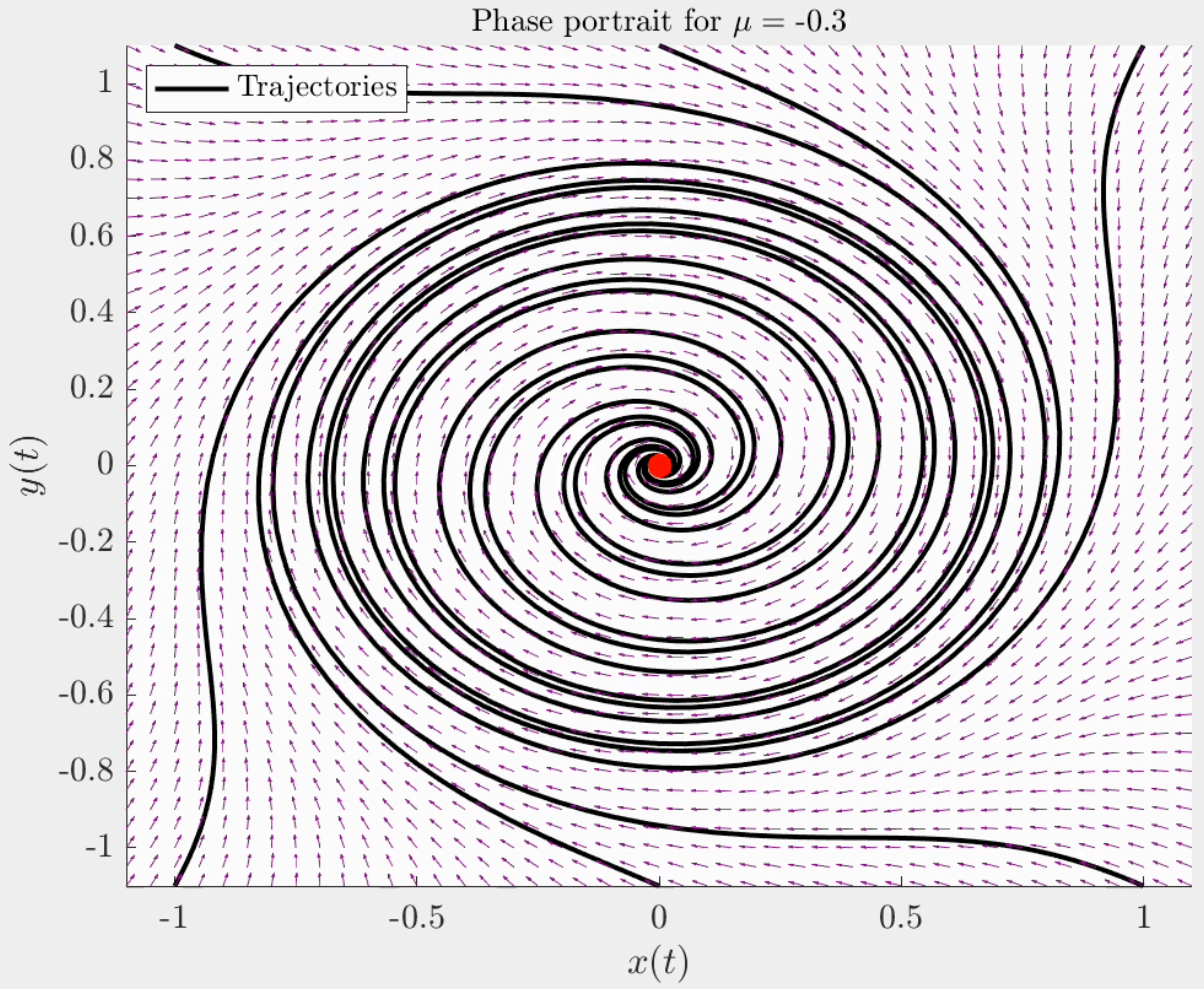
2D example:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega \end{cases}$$

r^3 **is destabilizing.**

Origin: stable spiral to the unstable spiral, unstable cycle appears and decreases near the origin.
After the bifurcation, the trajectories must jump to a distant attractor: another fixed point, limit cycle, infinity.





Phase portrait for μ = -0.3
Trajectories
$y(t)$
$x(t)$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1

Example: Hopf

Show that Hopf bifurcation occurs and determine its type

$$\begin{cases} \dot{x} = \mu x - y + xy^2 \\ \dot{y} = x + \mu y + y^3 \end{cases}$$

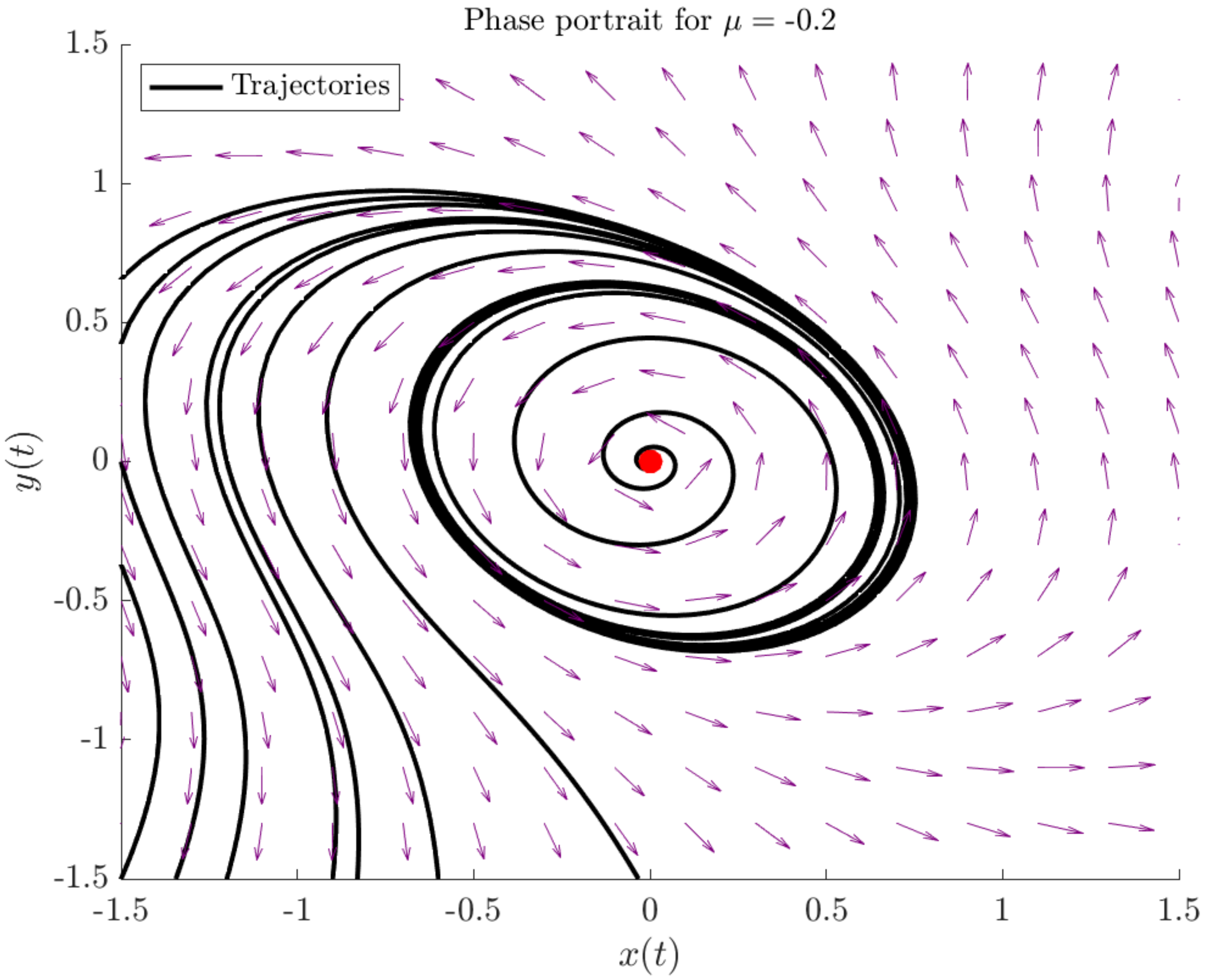
$$J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix},\ \lambda_{1,2} = \mu \pm i\textbf{,}\ \begin{cases} \mu < 0, & \textbf{stable spiral} \\ \mu > 0, & \textbf{unstable spiral} \end{cases}$$

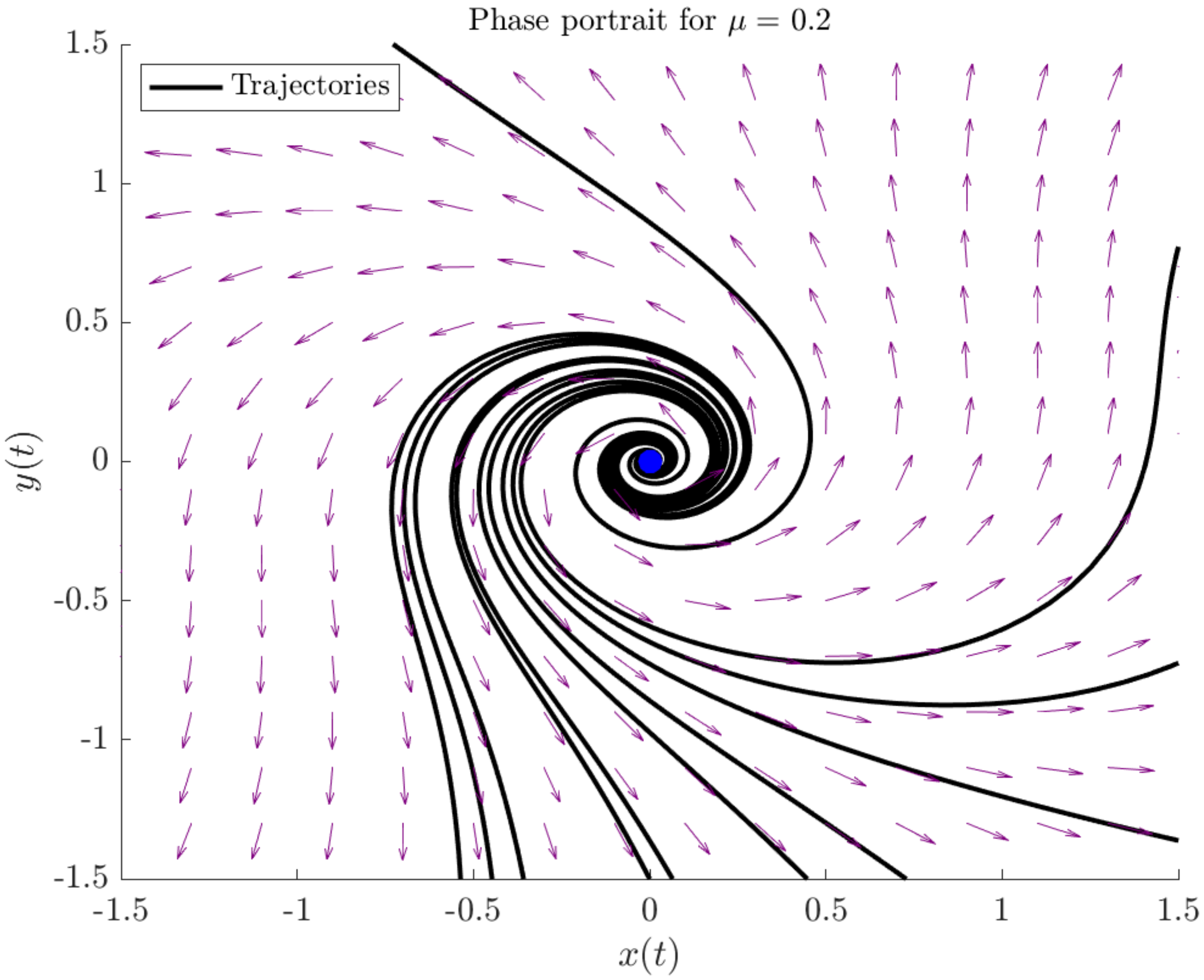
E-vals transit the imaginary axis $\Rightarrow$ Hopf

$\dot{r} = \mu r + ry^2 \Rightarrow \dot{r} \geq \mu r \Rightarrow r$ **grows as** $r_0 e^{\mu t}$ **if** $\mu > 0 \Rightarrow$ **orbits get repelled**

The unstable spiral is not surrounded by a stable limit cycle; hence the bifurcation cannot be supercritical.

Example: Hopf





Unstable cycle near the stable origin

No closed orbits, trajectories $\rightarrow \infty$