5

LINEAR SYSTEMS

5.0 Introduction

As we've seen, in one-dimensional phase spaces the flow is extremely confined—all trajectories are forced to move monotonically or remain constant. In higher-dimensional phase spaces, trajectories have much more room to maneuver, and so a wider range of dynamical behavior becomes possible. Rather than attack all this complexity at once, we begin with the simplest class of higher-dimensional systems, namely *linear systems in two dimensions*. These systems are interesting in their own right, and, as we'll see later, they also play an important role in the classification of fixed points of *nonlinear* systems. We begin with some definitions and examples.

5.1 Definitions and Examples

A ***two-dimensional linear system*** is a system of the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a, b, c, d are parameters. If we use boldface to denote vectors, this system can be written more compactly in matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Such a system is ***linear*** in the sense that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Notice that $\dot{\mathbf{x}} = \mathbf{0}$ when $\mathbf{x} = \mathbf{0}$, so $\mathbf{x}^* = \mathbf{0}$ is always a fixed point for any choice of A.

The solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualized as trajectories moving on the (x, y) plane, in this context called the ***phase plane***. Our first example presents the phase plane analysis of a familiar system.

EXAMPLE 5.1.1:

As discussed in elementary physics courses, the vibrations of a mass hanging from a linear spring are governed by the linear differential equation

$$m\ddot{x} + kx = 0 \tag{1}$$

where m is the mass, k is the spring constant, and x is the displacement of the mass from equilibrium (Figure 5.1.1). Give a phase plane analysis of this ***simple harmonic oscillator***.

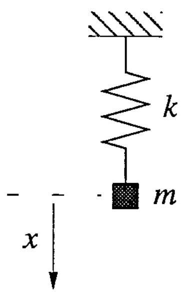


Figure 5.1.1

Solution: As you probably recall, it's easy to solve (1) analytically in terms of sines and cosines. But that's precisely what makes linear equations so special! For the *nonlinear* equations of ultimate interest to us, it's usually impossible to find an analytical solution. We want to develop methods for deducing the behavior of equations like (1) *without actually solving them.*

The motion in the phase plane is determined by a vector field that comes from the differential equation (1). To find this vector field, we note that the ***state*** of the system is characterized by its current position x and velocity v; if we know the values of *both* x and v, then (1) uniquely determines the future states of the system. Therefore we rewrite (1) in terms of x and v, as follows:

$$\dot{x} = v \tag{2a}$$
$$\dot{v} = -\tfrac{k}{m}x. \tag{2b}$$

Equation (2a) is just the definition of velocity, and (2b) is the differential equation (1) rewritten in terms of v. To simplify the notation, let $\omega^2 = k/m$. Then (2) becomes

$$\dot{x} = v \tag{3a}$$
$$\dot{v} = -\omega^2 x. \tag{3b}$$

The system (3) assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point (x, v), and therefore represents a ***vector field*** on the phase plane.

For example, let's see what the vector field looks like when we're on the x-axis. Then $v = 0$ and so $(\dot{x}, \dot{v}) = (0, -\omega^2 x)$. Hence the vectors point vertically downward for positive x and vertically upward for negative x (Figure 5.1.2). As x gets larger in magnitude, the vectors $(0, -\omega^2 x)$ get longer. Similarly, on the v-axis, the vector field is $(\dot{x}, \dot{v}) = (v, 0)$, which points to the right when $v > 0$ and to the left when $v < 0$. As we move around in phase space, the vectors change direction as shown in Figure 5.1.2.

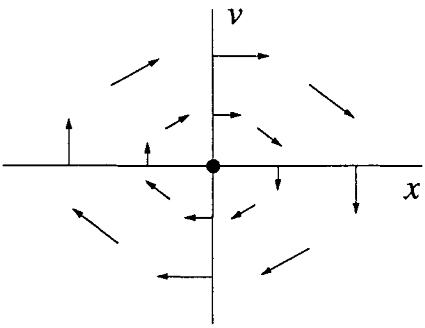


Figure 5.1.2

Just as in Chapter 2, it is helpful to visualize the vector field in terms of the motion of an imaginary fluid. In the present case, we imagine that a fluid is flowing steadily on the phase plane with a local velocity given by $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$. Then, to find the trajectory starting at (x_0, v_0), we place an imaginary particle or ***phase point*** at (x_0, v_0) and watch how it is carried around by the flow.

The flow in Figure 5.1.2 swirls about the origin. The origin is special, like the eye of a hurricane: a phase point placed there would remain motionless, because $(\dot{x}, \dot{v}) = (0, 0)$ when $(x, v) = (0, 0)$; hence the origin is a ***fixed point***. But a phase point starting anywhere else would circulate around the origin and eventually return to its starting point. Such trajectories form ***closed orbits***, as shown in Figure 5.1.3. Figure 5.1.3 is called the ***phase portrait*** of the system—it shows the overall picture of trajectories in phase space.

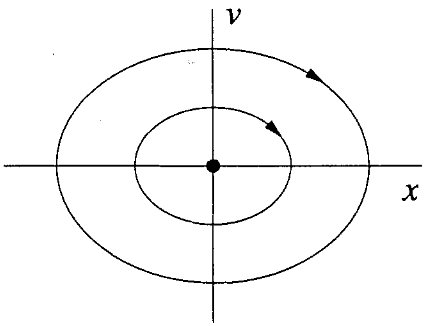


Figure 5.1.3

What do fixed points and closed orbits have to do with the original problem of a mass on a spring? The answers are beautifully simple. The fixed point $(x, v) = (0, 0)$ corresponds to static equilibrium of the system: the mass is at rest at its equilibrium position and will remain there forever, since the spring is relaxed. The closed orbits have a more interesting interpretation: they correspond to periodic motions, i.e., oscillations of the mass. To see this, just look at some points on a closed orbit (Figure 5.1.4). When the displacement x is most negative, the velocity v is zero; this corresponds to one extreme of the oscillation, where the spring is most compressed (Figure 5.1.4).

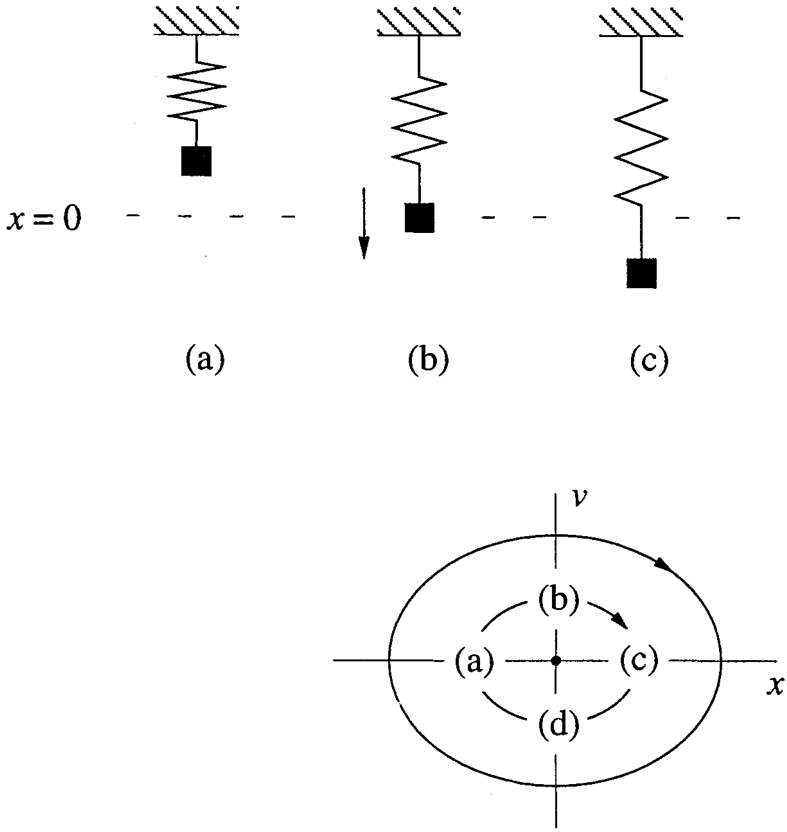


Figure 5.1.4

In the next instant as the phase point flows along the orbit, it is carried to points where x has increased and v is now positive; the mass is being pushed back toward its equilibrium position. But by the time the mass has reached $x = 0$, it has a large positive velocity (Figure 5.1.4b) and so it overshoots $x = 0$. The mass eventually comes to rest at the other end of its swing, where x is most positive and v is zero again (Figure 5.1.4c). Then the mass gets pulled up again and eventually completes the cycle (Figure 5.1.4d).

The shape of the closed orbits also has an interesting physical interpretation. The orbits in Figures 5.1.3 and 5.1.4 are actually *ellipses* given by the equation $\omega^2 x^2 + v^2 = C$, where $C \geq 0$ is a constant. In Exercise 5.1.1, you are asked to derive this geometric result, and to show that it is equivalent to conservation of energy. ■

EXAMPLE 5.1.2:

Solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$. Graph the phase portrait as a varies from $-\infty$ to $+\infty$, showing the qualitatively different cases.

Solution: The system is

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}a & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Matrix multiplication yields

$$\dot{x} = ax$$
$$\dot{y} = -y$$

which shows that the two equations are ***uncoupled***; there's no x in the y-equation and vice versa. In this simple case, each equation may be solved separately. The solution is

$$x(t) = x_0 e^{at} \tag{1a}$$
$$y(t) = y_0 e^{-t}. \tag{1b}$$

The phase portraits for different values of a are shown in Figure 5.1.5. In each case, $y(t)$ decays exponentially. When $a < 0$, $x(t)$ also decays exponentially and so all trajectories approach the origin as $t \to \infty$. However, the direction of approach depends on the size of a compared to -1.

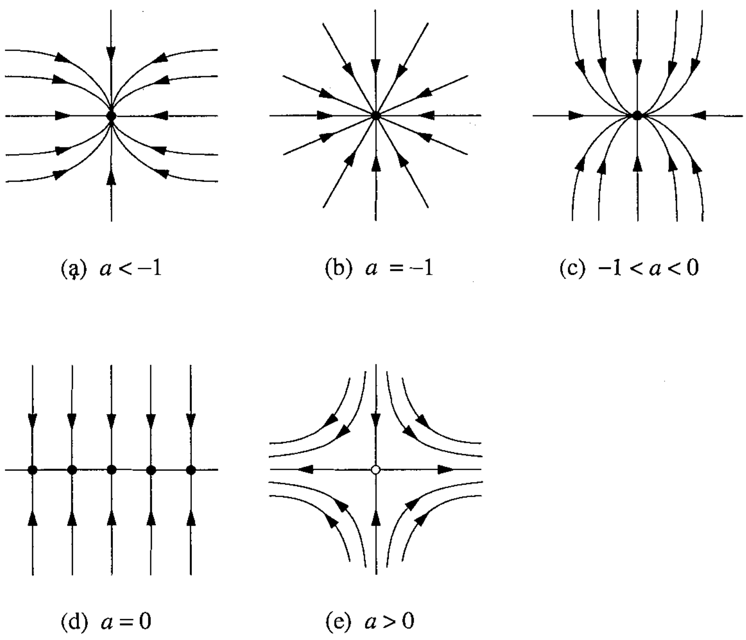


Figure 5.1.5

In Figure 5.1.5a, we have $a < -1$, which implies that $x(t)$ decays more rapidly than $y(t)$. The trajectories approach the origin tangent to the *slower* direction (here, the y-direction). The intuitive explanation is that when a is very negative, the trajectory slams horizontally onto the y-axis, because the decay of $x(t)$ is almost instantaneous. Then the trajectory dawdles along the y-axis toward the origin, and so the approach is tangent to the y-axis. On the other hand, if we look *backwards* along a trajectory ($t \to -\infty$), then the trajectories all become parallel to the faster decaying direction (here, the x-direction). These conclusions are easily proved by looking at the slope $dy/dx = \dot{y}/\dot{x}$ along the trajectories; see Exercise 5.1.2. In Figure 5.1.5a, the fixed point $\mathbf{x}^* = \mathbf{0}$ is called a ***stable node***.

Figure 5.1.5b shows the case $a = -1$. Equation (1) shows that $y(t)/x(t) = y_0/x_0 =$ constant, and so all trajectories are straight lines through the origin. This is a very special case—it occurs because the decay rates in the two directions are precisely equal. In this case, $\mathbf{x}^*$ is called a symmetrical node or ***star***.

When $-1 < a < 0$, we again have a node, but now the trajectories approach $\mathbf{x}^*$ along the x-direction, which is the more slowly decaying direction for this range of a (Figure 5.1.5c).

Something dramatic happens when $a = 0$ (Figure 5.1.5d). Now (1a) becomes $x(t) \equiv x_0$ and so there's an entire ***line of fixed points*** along the x-axis. All trajectories approach these fixed points along vertical lines.

Finally when $a > 0$ (Figure 5.1.5e), $\mathbf{x}^*$ becomes unstable, due to the exponential growth in the x-direction. Most trajectories veer away from $\mathbf{x}^*$ and head out to infinity. An exception occurs if the trajectory starts on the y-axis; then it walks a tightrope to the origin. In forward time, the trajectories are asymptotic to the x-axis; in backward time, to the y-axis. Here $\mathbf{x}^* = \mathbf{0}$ is called a ***saddle point***. The y-axis is called the ***stable manifold*** of the saddle point $\mathbf{x}^*$, defined as the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. Likewise, the ***unstable manifold*** of $\mathbf{x}^*$ is the set of initial conditions such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to -\infty$. Here the unstable manifold is the x-axis. Note that a typical trajectory asymptotically approaches the unstable manifold as $t \to \infty$, and approaches the stable manifold as $t \to -\infty$. This sounds backwards, but it's right! ■

Stability Language

It's useful to introduce some language that allows us to discuss the stability of different types of fixed points. This language will be especially useful when we analyze fixed points of *nonlinear* systems. For now we'll be informal; precise definitions of the different types of stability will be given in Exercise 5.1.10.

We say that $\mathbf{x}^* = \mathbf{0}$ is an ***attracting*** fixed point in Figures 5.1.5a–c; all trajectories that start near $\mathbf{x}^*$ approach it as $t \to \infty$. That is, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In fact $\mathbf{x}^*$ attracts *all* trajectories in the phase plane, so it could be called ***globally attracting***.

There's a completely different notion of stability which relates to the behavior of trajectories for *all* time, not just as $t \to \infty$. We say that a fixed point $\mathbf{x}^*$ is ***Liapunov stable*** if all trajectories that start sufficiently close to $\mathbf{x}^*$ remain close to it for all time. In Figures 5.1.5a–d, the origin is Liapunov stable.

Figure 5.1.5d shows that a fixed point can be Liapunov stable but not attracting. This situation comes up often enough that there is a special name for it. When a fixed point is Liapunov stable but not attracting, it is called ***neutrally stable***. Nearby trajectories are neither attracted to nor repelled from a neutrally stable point. As a second example, the equilibrium point of the simple harmonic oscillator (Figure 5.1.3) is neutrally stable. Neutral stability is commonly encountered in mechanical systems in the absence of friction. Conversely, it's possible for a fixed point to be attracting but not Liapunov stable; thus, neither notion of stability implies the other. An example is given by the following vector field on the circle: $\dot{\theta} = 1 - \cos\theta$ (Figure 5.1.6). Here $\theta^* = 0$ attracts all trajectories as $t \to \infty$, but it is not Liapunov stable; there are trajectories that start infinitesimally close to θ^* but go on a very large excursion before returning to θ^*.

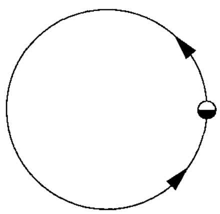


Figure 5.1.6

However, in practice the two types of stability often occur together. If a fixed point is *both* Liapunov stable and attracting, we'll call it ***stable***, or sometimes ***asymptotically stable***.

Finally, $\mathbf{x}^*$ is ***unstable*** in Figure 5.1.5e, because it is neither attracting nor Liapunov stable.

A graphical convention: we'll use open dots to denote unstable fixed points, and solid black dots to denote Liapunov stable fixed points. This convention is consistent with that used in previous chapters.

5.2 Classification of Linear Systems

The examples in the last section had the special feature that two of the entries in the matrix A were zero. Now we want to study the general case of an arbitrary 2×2 matrix, with the aim of classifying all the possible phase portraits that can occur.

Example 5.1.2 provides a clue about how to proceed. Recall that the x and y axes played a crucial geometric role. They determined the direction of the trajectories as $t \to \pm\infty$. They also contained special ***straight-line trajectories***: a trajectory starting on one of the coordinate axes stayed on that axis forever, and exhibited simple exponential growth or decay along it.

For the general case, we would like to find the analog of these straight-line trajectories. That is, we seek trajectories of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}, \qquad (2)$$

where $\mathbf{v} \neq \mathbf{0}$ is some fixed vector to be determined, and λ is a growth rate, also to be determined. If such solutions exist, they correspond to exponential motion along the line spanned by the vector $\mathbf{v}$.

To find the conditions on $\mathbf{v}$ and λ, we substitute $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ into $\dot{\mathbf{x}} = A\mathbf{x}$, and obtain $\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$. Canceling the nonzero scalar factor $e^{\lambda t}$ yields

$$A\mathbf{v} = \lambda\mathbf{v}, \tag{3}$$

which says that the desired straight line solutions exist if $\mathbf{v}$ is an ***eigenvector*** of A with corresponding ***eigenvalue*** λ. In this case we call the solution (2) an ***eigensolution***.

Let's recall how to find eigenvalues and eigenvectors. (If your memory needs more refreshing, see any text on linear algebra.) In general, the eigenvalues of a matrix A are given by the ***characteristic equation*** $\det(A - \lambda I) = 0$, where I is the identity matrix. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic equation becomes

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0.$$

Expanding the determinant yields

$$\lambda^2 - \tau\lambda + \Delta = 0 \tag{4}$$

where

$$\tau = \operatorname{trace}(A) = a + d,$$
$$\Delta = \det(A) = ad - bc.$$

Then

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \tag{5}$$

are the solutions of the quadratic equation (4). In other words, the eigenvalues depend only on the trace and determinant of the matrix A.

The typical situation is for the eigenvalues to be distinct: $\lambda_1 \neq \lambda_2$. In this case, a theorem of linear algebra states that the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and hence span the entire plane (Figure 5.2.1). In particular, any initial condition $\mathbf{x}_0$ can be written as a linear combination of eigenvectors, say $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

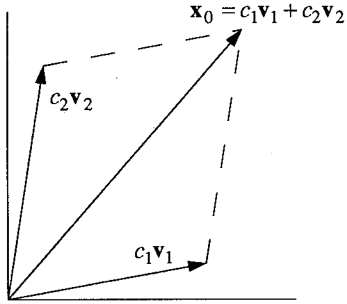


Figure 5.2.1

This observation allows us to write down the general solution for $\mathbf{x}(t)$—it is simply

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 . \tag{6}$$

Why is this the general solution? First of all, it is a linear combination of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, and hence is itself a solution. Second, it satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and so by the existence and uniqueness theorem, it is the *only* solution. (See Section 6.2 for a general statement of the existence and uniqueness theorem.)

EXAMPLE 5.2.1:

Solve the initial value problem $\dot{x} = x + y$, $\dot{y} = 4x - 2y$, subject to the initial condition $(x_0, y_0) = (2, -3)$.

Solution: The corresponding matrix equation is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

First we find the eigenvalues of the matrix A. The matrix has $\tau = -1$ and $\Delta = -6$, so the characteristic equation is $\lambda^2 + \lambda - 6 = 0$. Hence

$$\lambda_1 = 2, \quad \lambda_2 = -3.$$

Next we find the eigenvectors. Given an eigenvalue λ, the corresponding eigenvector $\mathbf{v} = (v_1, v_2)$ satisfies

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\lambda_1 = 2$, this yields $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a nontrivial solution

$(v_1, v_2) = (1,1)$, or any scalar multiple thereof. (Of course, any multiple of an eigenvector is always an eigenvector; we try to pick the simplest multiple, but any one will do.) Similarly, for $\lambda_2 = -3$, the eigenvector equation becomes $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a nontrivial solution $(v_1, v_2) = (1, -4)$. In summary,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Next we write the general solution as a linear combination of eigensolutions. From (6), the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}. \tag{7}$$

Finally, we compute c_1 and c_2 to satisfy the initial condition $(x_0, y_0) = (2, -3)$. At $t = 0$, (7) becomes

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

which is equivalent to the algebraic system

$$\begin{aligned} 2 &= c_1 + c_2 , \\ -3 &= c_1 - 4c_2 . \end{aligned}$$

The solution is $c_1 = 1$, $c_2 = 1$. Substituting back into (7) yields

$$\begin{aligned} x(t) &= e^{2t} + e^{-3t}, \\ y(t) &= e^{2t} - 4e^{-3t} \end{aligned}$$

for the solution to the initial value problem. ■

Whew! Fortunately we don't need to go through all this to draw the phase portrait of a linear system. All we need to know are the eigenvectors and eigenvalues.

EXAMPLE 5.2.2:

Draw the phase portrait for the system of Example 5.2.1.

Solution: The system has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -3$. Hence the first eigensolution grows exponentially, and the second eigensolution decays. This means the origin is a *saddle point*. Its stable manifold is the line spanned by the eigenvector $\mathbf{v}_2 = (1, -4)$, corresponding to the decaying eigensolution. Similarly, the unstable manifold is the line spanned by $\mathbf{v}_1 = (1,1)$. As with all saddle points, a typical trajectory approaches the unstable manifold as $t \to \infty$, and the stable manifold as $t \to -\infty$. Figure 5.2.2 shows the phase portrait. ■

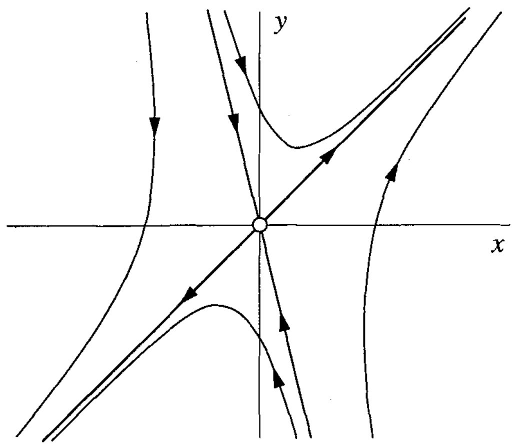


Figure 5.2.2

EXAMPLE 5.2.3:

Sketch a typical phase portrait for the case $\lambda_2 < \lambda_1 < 0$.

Solution: First suppose $\lambda_2 < \lambda_1 < 0$. Then both eigensolutions decay exponentially. The fixed point is a stable node, as in Figures 5.1.5a and 5.1.5c, except now the eigenvectors are not mutually perpendicular, in general. Trajectories typically approach the origin tangent to the ***slow eigendirection***, defined as the direction spanned by the eigenvector with the smaller $|\lambda|$. In backwards time ($t \to -\infty$), the trajectories become parallel to the fast eigendirection. Figure 5.2.3 shows the phase portrait. (If we reverse all the arrows in Figure 5.2.3, we obtain a typical phase portrait for an ***unstable node***.) ■

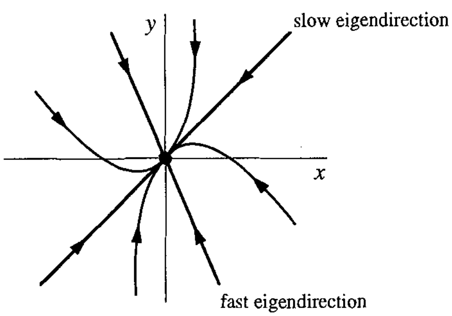


Figure 5.2.3

EXAMPLE 5.2.4:

What happens if the eigenvalues are *complex* numbers?

Solution: If the eigenvalues are complex, the fixed point is either a ***center*** (Figure 5.2.4a) or a ***spiral*** (Figure 5.2.4b). We've already seen an example of a center in the simple harmonic oscillator of Section 5.1; the origin is surrounded by a family of closed orbits. Note that centers are *neutrally stable*, since nearby trajectories are neither attracted to nor repelled from the fixed point. A spiral would occur if the harmonic oscillator were lightly damped. Then the trajectory would just fail to close, because the oscillator loses a bit of energy on each cycle.

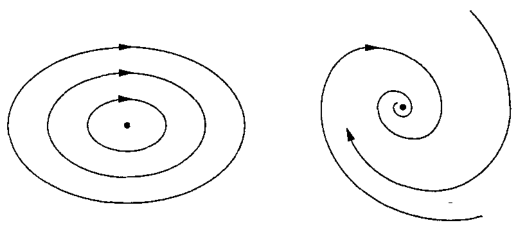

(a) center (b) spiral

Figure 5.2.4

To justify these statements, recall that the eigenvalues are $\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$. Thus complex eigenvalues occur when

$$\tau^2 - 4\Delta < 0.$$

To simplify the notation, let's write the eigenvalues as

$$\lambda_{1,2} = \alpha \pm i\omega$$

where

$$\alpha = \tau/2, \qquad \omega = \tfrac{1}{2}\sqrt{4\Delta - \tau^2}.$$

By assumption, $\omega \neq 0$. Then the eigenvalues are distinct and so the general solution is still given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

But now the c's and $\mathbf{v}$'s are *complex*, since the λ's are. This means that $\mathbf{x}(t)$ involves linear combinations of $e^{(\alpha \pm i\omega)t}$. By Euler's formula, $e^{i\omega t} = \cos\omega t + i\sin\omega t$. Hence $\mathbf{x}(t)$ is a combination of terms involving $e^{\alpha t}\cos\omega t$ and $e^{\alpha t}\sin\omega t$. Such terms represent exponentially *decaying oscillations* if $\alpha = \mathrm{Re}(\lambda) < 0$ and *growing oscillations* if $\alpha > 0$. The corresponding fixed points are ***stable*** and ***unstable spirals***, respectively. Figure 5.2.4b shows the stable case.

If the eigenvalues are pure imaginary ($\alpha = 0$), then all the solutions are periodic with period $T = 2\pi/\omega$. The oscillations have fixed amplitude and the fixed point is a center.

For both centers and spirals, it's easy to determine whether the rotation is clockwise or counterclockwise; just compute a few vectors in the vector field and the sense of rotation should be obvious. ■

EXAMPLE 5.2.5:

In our analysis of the general case, we have been assuming that the eigenvalues are distinct. What happens if the eigenvalues are *equal*?

Solution: Suppose $\lambda_1 = \lambda_2 = \lambda$. There are two possibilities: either there are two independent eigenvectors corresponding to λ, or there's only one.

If there are two independent eigenvectors, then they span the plane and so *every vector is an eigenvector with this same eigenvalue* λ. To see this, write an arbitrary vector $\mathbf{x}_0$ as a linear combination of the two eigenvectors: $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then

$$A\mathbf{x}_0 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda\mathbf{v}_1 + c_2\lambda\mathbf{v}_2 = \lambda\mathbf{x}_0$$

so $\mathbf{x}_0$ is also an eigenvector with eigenvalue λ. Since multiplication by A simply stretches every vector by a factor λ, the matrix must be a multiple of the identity:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Then if $\lambda \neq 0$, all trajectories are straight lines through the origin ($\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$) and the fixed point is a ***star node*** (Figure 5.2.5).

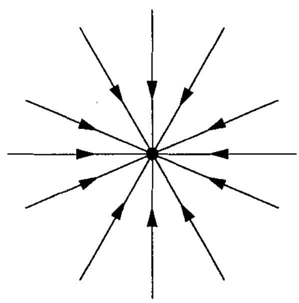


Figure 5.2.5

On the other hand, if $\lambda = 0$, the whole plane is filled with fixed points! (No surprise—the system is $\dot{\mathbf{x}} = \mathbf{0}$.)

The other possibility is that there's only one eigenvector (more accurately, the eigenspace corresponding to λ is one-dimensional.) For example, any matrix of the form $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, with $b \neq 0$ has only a one-dimensional eigenspace (Exercise 5.2.11).

When there's only one eigendirection, the fixed point is a ***degenerate node***. A

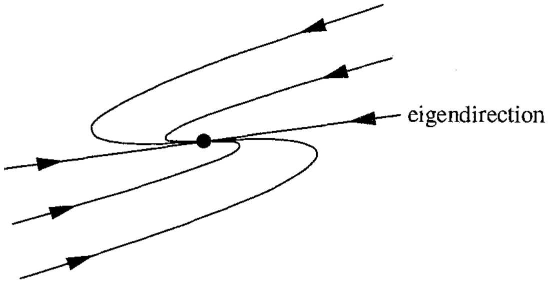


Figure 5.2.6

typical phase portrait is shown in Figure 5.2.6. As $t \to +\infty$ and also as $t \to -\infty$, all trajectories become parallel to the one available eigendirection.

A good way to think about the degenerate node is to imagine that it has been created by deforming an ordinary node. The ordinary node has two independent eigendirections; all trajectories are parallel to the slow eigendirection as $t \to \infty$, and to the fast eigendirection as $t \to -\infty$ (Figure 5.2.7a).

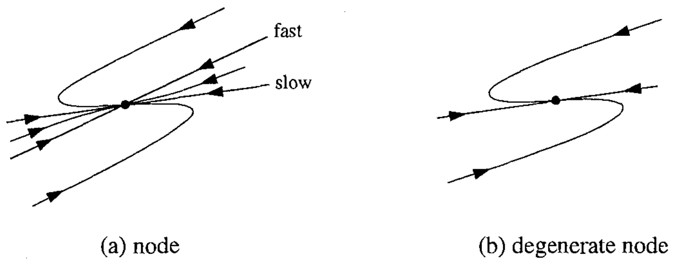


Figure 5.2.7

Now suppose we start changing the parameters of the system in such a way that the two eigendirections are scissored together. Then some of the trajectories will get squashed in the collapsing region between the two eigendirections, while the surviving trajectories get pulled around to form the degenerate node (Figure 5.2.7b).

Another way to get intuition about this case is to realize that the degenerate node is on the *borderline between a spiral and a node*. The trajectories are trying to wind around in a spiral, but they don't quite make it. ■

Classification of Fixed Points

By now you're probably tired of all the examples and ready for a simple classification scheme. Happily, there is one. We can show the type and stability of all the different fixed points on a single diagram (Figure 5.2.8).

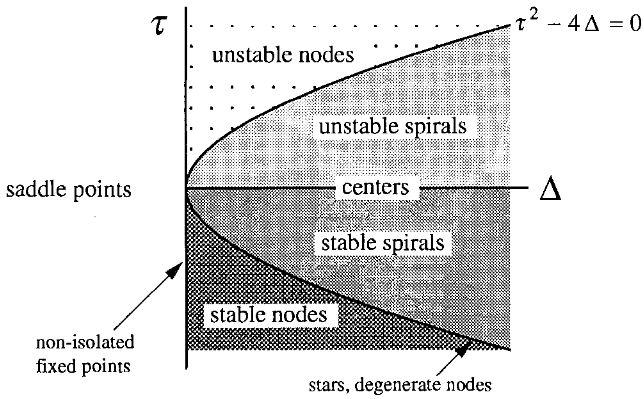


Figure 5.2.8

The axes are the trace τ and the determinant Δ of the matrix A. All of the information in the diagram is implied by the following formulas:

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2.$$

The first equation is just (5). The second and third can be obtained by writing the characteristic equation in the form $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \tau\lambda + \Delta = 0$.

To arrive at Figure 5.2.8, we make the following observations:

If $\Delta < 0$, the eigenvalues are real and have opposite signs; hence the fixed point is a *saddle point.*

If $\Delta > 0$, the eigenvalues are either real with the same sign (*nodes*), or complex conjugate (*spirals* and *centers*). Nodes satisfy $\tau^2 - 4\Delta > 0$ and spirals satisfy $\tau^2 - 4\Delta < 0$. The parabola $\tau^2 - 4\Delta = 0$ is the borderline between nodes and spirals; star nodes and degenerate nodes live on this parabola. The stability of the nodes and spirals is determined by τ. When $\tau < 0$, both eigenvalues have negative real parts, so the fixed point is stable. Unstable spirals and nodes have $\tau > 0$. Neutrally stable centers live on the borderline $\tau = 0$, where the eigenvalues are purely imaginary.

If $\Delta = 0$, at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point. There is either a whole line of fixed points, as in Figure 5.1.5d, or a plane of fixed points, if $A = 0$.

Figure 5.2.8 shows that saddle points, nodes, and spirals are the major types of fixed points; they occur in large open regions of the (Δ, τ) plane. Centers, stars, degenerate nodes, and non-isolated fixed points are ***borderline cases*** that occur along curves in the (Δ, τ) plane. Of these borderline cases, centers are by far the most important. They occur very commonly in frictionless mechanical systems where energy is conserved.

EXAMPLE 5.2.6:

Classify the fixed point $\mathbf{x}^* = \mathbf{0}$ for the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution: The matrix has $\Delta = -2$; hence the fixed point is a saddle point. ■

EXAMPLE 5.2.7:

Redo Example 5.2.6 for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

Solution: Now $\Delta = 5$ and $\tau = 6$. Since $\Delta > 0$ and $\tau^2 - 4\Delta = 16 > 0$, the fixed point is a node. It is unstable, since $\tau > 0$. ■

5.3 Love Affairs

To arouse your interest in the classification of linear systems, we now discuss a simple model for the dynamics of love affairs (Strogatz 1988). The following story illustrates the idea.

Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let

$R(t)$ = Romeo's love/hate for Juliet at time t

$J(t)$ = Juliet's love/hate for Romeo at time t.

Positive values of R, J signify love, negative values signify hate. Then a model for their star-crossed romance is

$$\dot{R} = aJ$$
$$\dot{J} = -bR$$

where the parameters a and b are positive, to be consistent with the story.

The sad outcome of their affair is, of course, a neverending cycle of love and hate; the governing system has a center at $(R, J) = (0, 0)$. At least they manage to achieve simultaneous love one-quarter of the time (Figure 5.3.1).

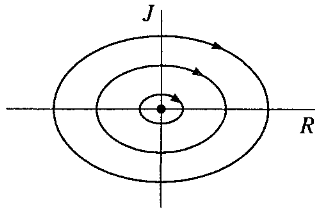


Figure 5.3.1

Now consider the forecast for lovers governed by the general linear system

$$\dot{R} = aR + bJ$$
$$\dot{J} = cR + dJ$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles. As named by one of my students, the choice $a > 0$, $b > 0$ means that Romeo is an "eager beaver"—he gets excited by Juliet's love for him, and is further spurred on by his own affectionate feelings for her. It's entertaining to name the other three romantic styles, and to predict the outcomes for the various pairings. For example, can a "cautious lover" ($a < 0$, $b > 0$) find true love with an eager beaver? These and other pressing questions will be considered in the exercises.

EXAMPLE 5.3.1:

What happens when two identically cautious lovers get together?

Solution: The system is

$$\dot{R} = aR + bJ$$
$$\dot{J} = bR + aJ$$

with $a < 0$, $b > 0$. Here a is a measure of cautiousness (they each try to avoid throwing themselves at the other) and b is a measure of responsiveness (they both get excited by the other's advances). We might suspect that the outcome depends on the relative size of a and b. Let's see what happens.

The corresponding matrix is

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

which has

$$\tau = 2a < 0\,, \qquad \Delta = a^2 - b^2\,, \qquad \tau^2 - 4\Delta = 4b^2 > 0\,.$$

Hence the fixed point $(R,J)=(0,0)$ is a saddle point if $a^2<b^2$ and a stable node if $a^2>b^2$. The eigenvalues and corresponding eigenvectors are

$$\lambda_1 = a+b\,, \qquad \mathbf{v}_1=(1,1), \qquad \lambda_2=a-b\,, \qquad \mathbf{v}_2=(1,-1).$$

Since $a+b>a-b$, the eigenvector $(1,1)$ spans the unstable manifold when the origin is a saddle point, and it spans the slow eigendirection when the origin is a stable node. Figure 5.3.2 shows the phase portrait for the two cases.

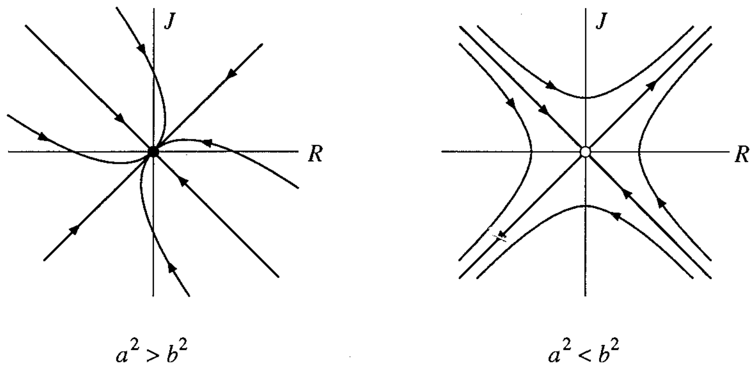


Figure 5.3.2

If $a^2>b^2$, the relationship always fizzles out to mutual indifference. The lesson seems to be that excessive caution can lead to apathy.

If $a^2<b^2$, the lovers are more daring, or perhaps more sensitive to each other. Now the relationship is explosive. Depending on their feelings initially, their relationship either becomes a love fest or a war. In either case, all trajectories approach the line $R=J$, so their feelings are eventually mutual. ■

EXERCISES FOR CHAPTER 5

5.1 Definitions and Examples

5.1.1 (Ellipses and energy conservation for the harmonic oscillator) Consider the harmonic oscillator $\dot{x}=v$, $\dot{v}=-\omega^2 x$.

a) Show that the orbits are given by ellipses $\omega^2x^2+v^2=C$, where C is any non-negative constant. (Hint: Divide the $\dot{x}$ equation by the $\dot{v}$ equation, separate the v's from the x's, and integrate the resulting separable equation.)

b) Show that this condition is equivalent to conservation of energy.

5.1.2 Consider the system $\dot{x} = ax$, $\dot{y} = -y$, where $a < -1$. Show that all trajectories become parallel to the y-direction as $t \to \infty$, and parallel to the x-direction as $t \to -\infty$.

(Hint: Examine the slope $dy/dx = \dot{y}/\dot{x}$.)

Write the following systems in matrix form.

5.1.3 $\dot{x} = -y,\ \dot{y} = -x$

5.1.4 $\dot{x} = 3x - 2y,\ \dot{y} = 2y - x$

5.1.5 $\dot{x} = 0,\ \dot{y} = x + y$

5.1.6 $\dot{x} = x,\ \dot{y} = 5x + y$

Sketch the vector field for the following systems. Indicate the length and direction of the vectors with reasonable accuracy. Sketch some typical trajectories.

5.1.7 $\dot{x} = x,\ \dot{y} = x + y$

5.1.8 $\dot{x} = -2y,\ \dot{y} = x$

5.1.9 Consider the system $\dot{x} = -y,\ \dot{y} = -x$.

a) Sketch the vector field.
b) Show that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$. (Hint: Show that the governing equations imply $x\dot{x} - y\dot{y} = 0$ and then integrate both sides.)
c) The origin is a saddle point; find equations for its stable and unstable manifolds.
d) The system can be decoupled and solved as follows. Introduce new variables u and v, where $u = x + y,\ v = x - y$. Then rewrite the system in terms of u and v. Solve for $u(t)$ and $v(t)$, starting from an arbitrary initial condition (u_0, v_0).
e) What are the equations for the stable and unstable manifolds in terms of u and v?
f) Finally, using the answer to (d), write the general solution for $x(t)$ and $y(t)$, starting from an initial condition (x_0, y_0).

5.1.10 (Attracting and Liapunov stable) Here are the official definitions of the various types of stability. Consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

We say that $\mathbf{x}^*$ is ***attracting*** if there is a $\delta > 0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. In other words, any trajectory that starts within a distance δ of $\mathbf{x}^*$ is guaranteed to converge to $\mathbf{x}^*$ *eventually*. As shown schematically in Figure 1, trajectories that start nearby are allowed to stray from $\mathbf{x}^*$ in the short run, but they must approach $\mathbf{x}^*$ in the long run.

In contrast, Liapunov stability requires that nearby trajectories remain close for *all* time. We say that $\mathbf{x}^*$ is ***Liapunov stable*** if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$ whenever $t \geq 0$ and $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ remain within ε of $\mathbf{x}^*$ for all positive time (Figure 1).

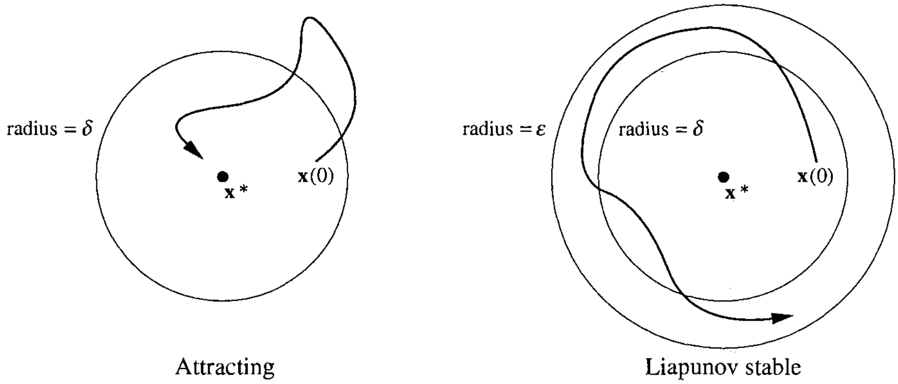


Figure 1

Finally, $\mathbf{x}^*$ is ***asymptotically stable*** if it is both attracting and Liapunov stable.

For each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

a) $\dot{x} = y$, $\dot{y} = -4x$.
b) $\dot{x} = 2y$, $\dot{y} = x$
c) $\dot{x} = 0$, $\dot{y} = x$
d) $\dot{x} = 0$, $\dot{y} = -y$
e) $\dot{x} = -x$, $\dot{y} = -5y$
f) $\dot{x} = x$, $\dot{y} = y$

5.1.11 (Stability proofs) Prove that your answers to 5.1.10 are correct, using the definitions of the different types of stability. (You must produce a suitable δ to prove that the origin is attracting, or a suitable $\delta(\varepsilon)$ to prove Liapunov stability.)

5.1.12 (Closed orbits from symmetry arguments) Give a simple proof that orbits are closed for the simple harmonic oscillator $\dot{x} = v$, $\dot{v} = -x$, using *only* the symmetry properties of the vector field. (Hint: Consider a trajectory that starts on the v-axis at $(0, -v_0)$, and suppose that the trajectory intersects the x-axis at $(x, 0)$. Then use symmetry arguments to find the subsequent intersections with the v-axis and x-axis.)

5.1.13 Why do you think a "saddle point" is called by that name? What's the connection to real saddles (the kind used on horses)?

5.2 Classification of Linear Systems

5.2.1 Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

a) Write the system as $\dot{\mathbf{x}} = A\mathbf{x}$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors of A.
b) Find the general solution of the system.
c) Classify the fixed point at the origin.
d) Solve the system subject to the initial condition $(x_0, y_0) = (3, 4)$.

5.2.2 (Complex eigenvalues) This exercise leads you through the solution of a linear system where the eigenvalues are complex. The system is $\dot{x} = x - y$, $\dot{y} = x + y$.

a) Find A and show that it has eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$, with eigenvectors $\mathbf{v}_1 = (i, 1)$, $\mathbf{v}_2 = (-i, 1)$. (Note that the eigenvalues are complex conjugates, and so are the eigenvectors—this is always the case for real A with complex eigenvalues.)

b) The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. So in one sense we're done! But this way of writing $\mathbf{x}(t)$ involves complex coefficients and looks unfamiliar. Express $\mathbf{x}(t)$ purely in terms of real-valued functions. (Hint: Use $e^{i\omega t} = \cos \omega t + i \sin \omega t$ to rewrite $\mathbf{x}(t)$ in terms of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.)

Plot the phase portrait and classify the fixed point of the following linear systems. If the eigenvectors are real, indicate them in your sketch.

5.2.3 $\dot{x} = y$, $\dot{y} = -2x - 3y$

5.2.4 $\dot{x} = 5x + 10y$, $\dot{y} = -x - y$

5.2.5 $\dot{x} = 3x - 4y$, $\dot{y} = x - y$

5.2.6 $\dot{x} = -3x + 2y$, $\dot{y} = x - 2y$

5.2.7 $\dot{x} = 5x + 2y$, $\dot{y} = -17x - 5y$

5.2.8 $\dot{x} = -3x + 4y$, $\dot{y} = -2x + 3y$

5.2.9 $\dot{x} = 4x - 3y$, $\dot{y} = 8x - 6y$

5.2.10 $\dot{x} = y$, $\dot{y} = -x - 2y$.

5.2.11 Show that any matrix of the form $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, with $b \neq 0$, has only a one-dimensional eigenspace corresponding to the eigenvalue λ. Then solve the system $\dot{\mathbf{x}} = A\mathbf{x}$ and sketch the phase portrait.

5.2.12 (*LRC* circuit) Consider the circuit equation $L\ddot{I} + R\dot{I} + I/C = 0$, where $L, C > 0$ and $R \geq 0$.

a) Rewrite the equation as a two-dimensional linear system.

b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.

c) Classify the fixed point at the origin, depending on whether $R^2C - 4L$ is positive, negative, or zero, and sketch the phase portrait in all three cases.

5.2.13 (Damped harmonic oscillator) The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.

a) Rewrite the equation as a two-dimensional linear system.

b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

5.2.14 (A project about random systems) Suppose we pick a linear system at random; what's the probability that the origin will be, say, an unstable spiral? To be more specific, consider the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose we pick the entries a, b, c, d independently and at random from a uniform distribution on the interval $[-1,1]$. Find the probabilities of all the different kinds of fixed points.

To check your answers (or if you hit an analytical roadblock), try the *Monte Carlo method.* Generate millions of random matrices on the computer and have the machine count the relative frequency of saddles, unstable spirals, etc.

Are the answers the same if you use a normal distribution instead of a uniform distribution?

5.3 Love Affairs

5.3.1 (Name-calling) Suggest names for the four romantic styles, determined by the signs of a and b in $\dot{R} = aR + bJ$.

5.3.2 Consider the affair described by $\dot{R} = J$, $\dot{J} = -R + J$.

a) Characterize the romantic styles of Romeo and Juliet.
b) Classify the fixed point at the origin. What does this imply for the affair?
c) Sketch $R(t)$ and $J(t)$ as functions of t, assuming $R(0) = 1$, $J(0) = 0$.

In each of the following problems, predict the course of the love affair, depending on the signs and relative sizes of a and b.

5.3.3 (Out of touch with their own feelings) Suppose Romeo and Juliet react to each other, but not to themselves: $\dot{R} = aJ$, $\dot{J} = bR$. What happens?

5.3.4 (Fire and water) Do opposites attract? Analyze $\dot{R} = aR + bJ$, $\dot{J} = -bR - aJ$.

5.3.5 (Peas in a pod) If Romeo and Juliet are romantic clones ($\dot{R} = aR + bJ$, $\dot{J} = bR + aJ$), should they expect boredom or bliss?

5.3.6 (Romeo the robot) Nothing could ever change the way Romeo feels about Juliet: $\dot{R} = 0$, $\dot{J} = aR + bJ$. Does Juliet end up loving him or hating him?

6

PHASE PLANE

6.0 Introduction

This chapter begins our study of two-dimensional *nonlinear* systems. First we consider some of their general properties. Then we classify the kinds of fixed points that can arise, building on our knowledge of linear systems (Chapter 5). The theory is further developed through a series of examples from biology (competition between two species) and physics (conservative systems, reversible systems, and the pendulum). The chapter concludes with a discussion of index theory, a topological method that provides global information about the phase portrait.

This chapter is mainly about fixed points. The next two chapters will discuss closed orbits and bifurcations in two-dimensional systems.

6.1 Phase Portraits

The general form of a vector field on the phase plane is

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

where f_1 and f_2 are given functions. This system can be written more compactly in vector notation as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$. Here $\mathbf{x}$ represents a point in the phase plane, and $\dot{\mathbf{x}}$ is the velocity vector at that point. By flowing along the vector field, a phase point traces out a solution $\mathbf{x}(t)$, corresponding to a trajectory winding through the phase plane (Figure 6.1.1).

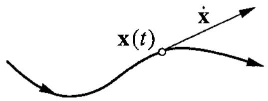


Figure 6.1.1

Furthermore, the entire phase plane is filled with trajectories, since each point can play the role of an initial condition.

For nonlinear systems, there's typically no hope of finding the trajectories analytically. Even when explicit formulas are available, they are often too complicated to provide much insight. Instead we will try to determine the *qualitative* behavior of the solutions. Our goal is to find the system's phase portrait directly from the properties of $\mathbf{f}(\mathbf{x})$. An enormous variety of phase portraits is possible; one example is shown in Figure 6.1.2.

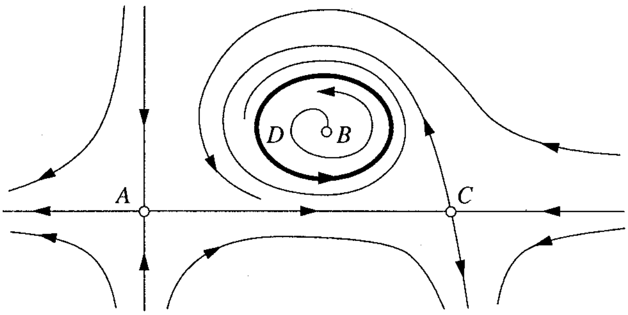


Figure 6.1.2

Some of the most salient features of any phase portrait are:

1. The ***fixed points***, like A, B, and C in Figure 6.1.2. Fixed points satisfy $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, and correspond to steady states or equilibria of the system.
2. The ***closed orbits***, like D in Figure 6.1.2. These correspond to periodic solutions, i.e., solutions for which $\mathbf{x}(t+T) = \mathbf{x}(t)$ for all t, for some $T > 0$.
3. The arrangement of trajectories near the fixed points and closed orbits. For example, the flow pattern near A and C is similar, and different from that near B.
4. The stability or instability of the fixed points and closed orbits. Here, the fixed points A, B, and C are unstable, because nearby trajectories tend to move away from them, whereas the closed orbit D is stable.

Numerical Computation of Phase Portraits

Sometimes we are also interested in *quantitative* aspects of the phase portrait. Fortunately, numerical integration of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is not much harder than that of $\dot{x} = f(x)$. The numerical methods of Section 2.8 still work, as long as we replace the numbers x and $f(x)$ by the vectors $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$. We will always use the Runge-Kutta method, which in vector form is

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \tfrac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$$

where

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_n)\Delta t \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{x}_n + \tfrac{1}{2}\mathbf{k}_1)\Delta t \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{x}_n + \tfrac{1}{2}\mathbf{k}_2)\Delta t \\
\mathbf{k}_4 &= \mathbf{f}(\mathbf{x}_n + \mathbf{k}_3)\Delta t.
\end{aligned}$$

A stepsize $\Delta t = 0.1$ usually provides sufficient accuracy for our purposes.

When plotting the phase portrait, it often helps to see a grid of representative vectors in the vector field. Unfortunately, the arrowheads and different lengths of the vectors tend to clutter such pictures. A plot of the ***direction field*** is clearer: short line segments are used to indicate the local direction of flow.

EXAMPLE 6.1.1:

Consider the system $\dot{x} = x + e^{-y}$, $\dot{y} = -y$. First use qualitative arguments to obtain information about the phase portrait. Then, using a computer, plot the direction field. Finally, use the Runge–Kutta method to compute several trajectories, and plot them on the phase plane.

Solution: First we find the fixed points by solving $\dot{x} = 0$, $\dot{y} = 0$ simultaneously. The only solution is $(x^*, y^*) = (-1, 0)$. To determine its stability, note that $y(t) \to 0$ as $t \to \infty$, since the solution to $\dot{y} = -y$ is $y(t) = y_0 e^{-t}$. Hence $e^{-y} \to 1$ and so in the long run, the equation for x becomes $\dot{x} \approx x + 1$; this has exponentially growing solutions, which suggests that the fixed point is unstable. In fact, if we restrict our attention to initial conditions on the x-axis, then $y_0 = 0$ and so $y(t) = 0$ for all time. Hence the flow on the x-axis is governed *strictly* by $\dot{x} = x + 1$. Therefore the fixed point is unstable.

To sketch the phase portrait, it is helpful to plot the ***nullclines***, defined as the curves where either $\dot{x} = 0$ or $\dot{y} = 0$. The nullclines indicate where the flow is purely horizontal or vertical (Figure 6.1.3). For example, the flow is horizontal where $\dot{y} = 0$, and since $\dot{y} = -y$, this occurs on the line $y = 0$. Along this line, the flow is to the right where $\dot{x} = x + 1 > 0$, that is, where $x > -1$.

Similarly, the flow is vertical where $\dot{x} = x + e^{-y} = 0$, which occurs on the curve shown in Figure 6.1.3. On the upper part of the curve where $y > 0$, the flow is downward, since $\dot{y} < 0$.

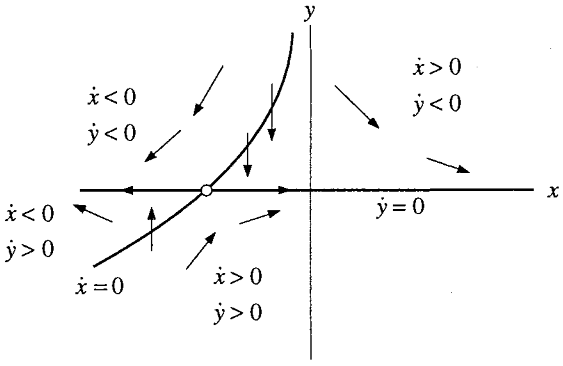


Figure 6.1.3

The nullclines also partition the plane into regions where $\dot{x}$ and $\dot{y}$ have various signs. Some of the typical vectors are sketched above in Figure 6.1.3. Even with the limited information obtained so far, Figure 6.1.3 gives a good sense of the overall flow pattern.

Now we use the computer to finish the problem. The direction field is indicated by the line segments in Figure 6.1.4, and several trajectories are shown. Note how the trajectories always follow the local slope.

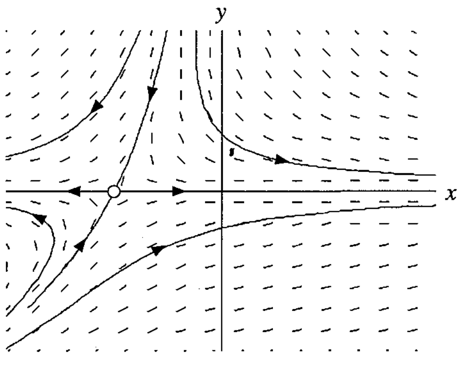


Figure 6.1.4

The fixed point is now seen to be a nonlinear version of a saddle point. ■

6.2 Existence, Uniqueness, and Topological Consequences

We have been a bit optimistic so far—at this stage, we have no guarantee that the general nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ even *has* solutions! Fortunately the existence and uniqueness theorem given in Section 2.5 can be generalized to two-dimensional systems. We state the result for n-dimensional systems, since no extra effort is involved:

Existence and Uniqueness Theorem: Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Suppose that $\mathbf{f}$ is continuous and that all its partial derivatives $\partial f_i/\partial x_j$, $i, j = 1, \ldots, n$, are continuous for $\mathbf{x}$ in some open connected set $D \subset \mathbf{R}^n$. Then for $\mathbf{x}_0 \in D$, the initial value problem has a solution $\mathbf{x}(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

In other words, existence and uniqueness of solutions are guaranteed if $\mathbf{f}$ is continuously differentiable. The proof of the theorem is similar to that for the case $n = 1$, and can be found in most texts on differential equations. Stronger versions of the theorem are available, but this one suffices for most applications.

From now on, we'll assume that all our vector fields are smooth enough to ensure the existence and uniqueness of solutions, starting from any point in phase space.

The existence and uniqueness theorem has an important corollary: *different trajectories never intersect.* If two trajectories *did* intersect, then there would be two solutions starting from the same point (the crossing point), and this would violate the uniqueness part of the theorem. In more intuitive language, a trajectory can't move in two directions at once.

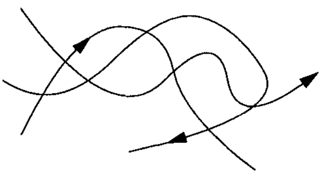


Figure 6.2.1

Because trajectories can't intersect, phase portraits always have a well-groomed look to them. Otherwise they might degenerate into a snarl of criss-crossed curves (Figure 6.2.1). The existence and uniqueness theorem prevents this from happening.

In two-dimensional phase spaces (as opposed to higher-dimensional phase spaces), these results have especially strong topological consequences. For example, suppose there is a closed orbit C in the phase plane. Then any trajectory starting inside C is trapped in there forever (Figure 6.2.2).

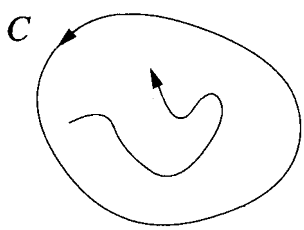


Figure 6.2.2

What is the fate of such a bounded trajectory? If there are fixed points inside C, then of course the trajectory might eventually approach one of them. But what if there *aren't* any fixed points? Your intuition may tell you that the trajectory can't meander around forever—if so, you're right. For vector fields on the plane, the ***Poincaré–Bendixson theorem*** states that if a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must eventually approach a closed orbit. We'll discuss this important theorem in Section 7.3.

But that part of our story comes later. First we must become better acquainted with fixed points.

6.3 Fixed Points and Linearization

In this section we extend the ***linearization*** technique developed earlier for one-dimensional systems (Section 2.4). The hope is that we can approximate the phase portrait near a fixed point by that of a corresponding linear system.

Linearized System

Consider the system

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

and suppose that (x^*, y^*) is a fixed point, i.e.,

$$f(x^*,y^*) = 0, \qquad g(x^*,y^*) = 0.$$

Let

$$u = x - x^*, \qquad v = y - y^*$$

denote the components of a small disturbance from the fixed point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v. Let's do the u-equation first:

$$
\begin{aligned}
\dot{u} &= \dot{x} && \text{(since } x^* \text{ is a constant)}\\
&= f(x^*+u,\, y^*+v) && \text{(by substitution)}\\
&= f(x^*,y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(Taylor series expansion)}\\
&= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(since } f(x^*,y^*)=0\text{)}.
\end{aligned}
$$

To simplify the notation, we have written $\partial f/\partial x$ and $\partial f/\partial y$, but please remember that these partial derivatives are to be evaluated *at the fixed point* (x^*, y^*); thus they are *numbers*, not functions. Also, the shorthand notation $O(u^2, v^2, uv)$ denotes ***quadratic terms*** in u and v. Since u and v are small, these quadratic terms are *extremely* small.

Similarly we find

$$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + O(u^2, v^2, uv).$$

Hence the disturbance (u, v) evolves according to

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms.} \tag{1}$$

The matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

is called the ***Jacobian matrix*** at the fixed point (x^*, y^*). It is the multivariable analog of the derivative $f'(x^*)$ seen in Section 2.4.

Now since the quadratic terms in (1) are tiny, it's tempting to neglect them altogether. If we do that, we obtain the ***linearized system***

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2}$$

whose dynamics can be analyzed by the methods of Section 5.2.

The Effect of Small Nonlinear Terms

Is it really safe to neglect the quadratic terms in (1)? In other words, does the linearized system give a qualitatively correct picture of the phase portrait near (x^*, y^*)? The answer is *yes, as long as the fixed point for the linearized system is not one of the borderline cases* discussed in Section 5.2. In other words, if the linearized system predicts a saddle, node, or a spiral, then the fixed point *really is* a saddle, node, or spiral for the original nonlinear system. See Andronov et al. (1973) for a proof of this result, and Example 6.3.1 for a concrete illustration.

The borderline cases (centers, degenerate nodes, stars, or non-isolated fixed points) are much more delicate. They can be altered by small nonlinear terms, as we'll see in Example 6.3.2 and in Exercise 6.3.11.

EXAMPLE 6.3.1:

Find all the fixed points of the system $\dot{x} = -x + x^3$, $\dot{y} = -2y$, and use linearization to classify them. Then check your conclusions by deriving the phase portrait for the full nonlinear system.

Solution: Fixed points occur where $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Hence we need $x = 0$ or $x = \pm 1$, and $y = 0$. Thus, there are three fixed points: $(0,0)$, $(1,0)$, and $(-1,0)$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Next we evaluate A at the fixed points. At $(0,0)$, we find $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, so $(0,0)$ is a stable node. At $(\pm 1,0)$, $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, so both $(1,0)$ and $(-1,0)$ are saddle points.

Now because stable nodes and saddle points are not borderline cases, we can be certain that the fixed points for the full nonlinear system have been predicted correctly.

This conclusion can be checked explicitly for the nonlinear system, since the x and y equations are *uncoupled*; the system is essentially two independent first-order systems at right angles to each other. In the y-direction, all trajectories decay exponentially to $y = 0$. In the x-direction, the trajectories are attracted to $x = 0$ and repelled from $x = \pm 1$. The vertical lines $x = 0$ and $x = \pm 1$ are *invariant*, because $\dot{x} = 0$ on them; hence any trajectory that starts on these lines stays on them forever. Similarly, $y = 0$ is an invariant horizontal line. As a final observation, we note that the phase portrait must be symmetric in both the x and y axes, since the equations are invariant under the transformations $x \to -x$ and $y \to -y$. Putting all this information together, we arrive at the phase portrait shown in Figure 6.3.1.

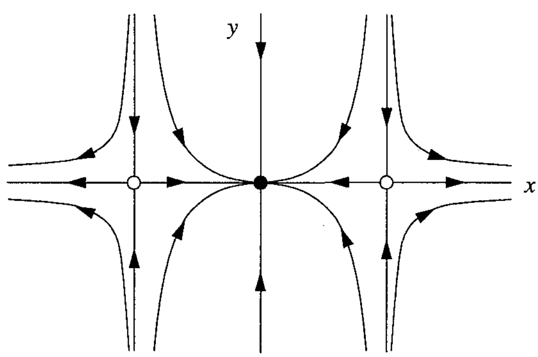


Figure 6.3.1

This picture confirms that $(0,0)$ is a stable node, and $(\pm 1, 0)$ are saddles, as expected from the linearization. ■

The next example shows that small nonlinear terms can change a center into a spiral.

EXAMPLE 6.3.2:

Consider the system

$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

where a is a parameter. Show that the linearized system *incorrectly* predicts that the origin is a center for all values of a, whereas in fact the origin is a stable spiral if $a < 0$ and an unstable spiral if $a > 0$.

Solution: To obtain the linearization about $(x^*, y^*) = (0,0)$, we can either compute the Jacobian matrix directly from the definition, or we can take the following shortcut. For any system with a fixed point at the origin, x and y represent deviations from the fixed point, since $u = x - x^* = x$ and $v = y - y^* = y$; hence we can linearize by simply omitting nonlinear terms in x and y. Thus the linearized system is $\dot{x} = -y$, $\dot{y} = x$. The Jacobian is

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which has $\tau = 0$, $\Delta = 1 > 0$, so the origin is always a center, according to the linearization.

To analyze the nonlinear system, we change variables to ***polar coordinates***. Let $x = r\cos\theta$, $y = r\sin\theta$. To derive a differential equation for r, we note $x^2 + y^2 = r^2$, so $x\dot{x} + y\dot{y} = r\dot{r}$. Substituting for $\dot{x}$ and $\dot{y}$ yields

$$\begin{aligned} r\dot{r} &= x\left(-y + ax(x^2 + y^2)\right) + y\left(x + ay(x^2 + y^2)\right) \\ &= a(x^2 + y^2)^2 \\ &= ar^4. \end{aligned}$$

Hence $\dot{r} = ar^3$. In Exercise 6.3.12, you are asked to derive the following differential equation for θ:

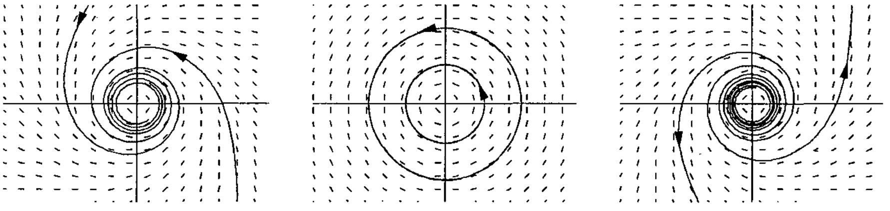
$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}.$$

After substituting for $\dot{x}$ and $\dot{y}$ we find $\dot{\theta} = 1$. Thus in polar coordinates the original system becomes

$$\dot{r} = ar^3$$
$$\dot{\theta} = 1.$$

The system is easy to analyze in this form, because the radial and angular motions are independent. All trajectories rotate about the origin with constant angular velocity $\dot{\theta} = 1$.

The radial motion depends on a, as shown in Figure 6.3.2.



$a<0$ $a=0$ $a>0$

Figure 6.3.2

If $a < 0$, then $r(t) \to 0$ monotonically as $t \to \infty$. In this case, the origin is a stable spiral. (However, note that the decay is extremely slow, as suggested by the computer-generated trajectories shown in Figure 6.3.2.) If $a = 0$, then $r(t) = r_0$ for all t and the origin is a center. Finally, if $a > 0$, then $r(t) \to \infty$ monotonically and the origin is an unstable spiral.

We can see now why centers are so delicate: all trajectories are required to close *perfectly* after one cycle. The slightest miss converts the center into a spiral. ■

Similarly, stars and degenerate nodes can be altered by small nonlinearities, but unlike centers, *their stability doesn't change.* For example, a stable star may be changed into a stable spiral (Exercise 6.3.11) but not into an unstable spiral. This is plausible, given the classification of linear systems in Figure 5.2.8: stars and degenerate nodes live squarely in the stable or unstable region, whereas centers live on the razor's edge between stability and instability.

If we're only interested in *stability*, and not in the detailed geometry of the trajectories, then we can classify fixed points more coarsely as follows:

Robust cases:

Repellers (also called *sources*): both eigenvalues have positive real part.

Attractors (also called *sinks*): both eigenvalues have negative real part.

Saddles: one eigenvalue is positive and one is negative.

Marginal cases:

Centers: both eigenvalues are pure imaginary.

Higher-order and non-isolated fixed points: at least one eigenvalue is zero.

Thus, from the point of view of stability, the marginal cases are those where at least one eigenvalue satisfies $\text{Re}(\lambda) = 0$.

Hyperbolic Fixed Points, Topological Equivalence, and Structural Stability

If $\text{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is often called ***hyperbolic***. (This is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard.) Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. Nonhyperbolic fixed points are the fragile ones.

We've already seen a simple instance of hyperbolicity in the context of vector fields on the line. In Section 2.4 we saw that the stability of a fixed point was accurately predicted by the linearization, *as long as* $f'(x^*) \neq 0$. This condition is the exact analog of $\text{Re}(\lambda) \neq 0$.

These ideas also generalize neatly to higher-order systems. A fixed point of an nth-order system is *hyperbolic* if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $\text{Re}(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. The important *Hartman–Grobman theorem* states that the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here ***topologically equivalent*** means that there is a ***homeomorphism*** (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

Hyperbolic fixed points also illustrate the important general notion of structural stability. A phase portrait is ***structurally stable*** if its topology cannot be changed by an arbitrarily small perturbation to the vector field. For instance, the phase portrait of a saddle point is structurally stable, but that of a center is not: an arbitrarily small amount of damping converts the center to a spiral.

6.4 Rabbits versus Sheep

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. Furthermore, ignore all other complications, like predators, seasonal effects, and other sources of food. Then there are two main effects we should consider:

1. Each species would grow to its carrying capacity in the absence of the other. This can be modeled by assuming logistic growth for each species (recall Section 2.3). Rabbits have a legendary ability to reproduce, so perhaps we should assign them a higher intrinsic growth rate.

2. When rabbits and sheep encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually the sheep nudges the rabbit aside and starts nibbling (on the grass, that is). We'll assume that these conflicts occur at a rate proportional to the size of each population. (If there were twice as many sheep, the odds of a rabbit encountering a sheep would be twice as great.) Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for the rabbits.

A specific model that incorporates these assumptions is

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

where

$$x(t) = \text{population of rabbits,}$$
$$y(t) = \text{population of sheep}$$

and $x, y \geq 0$. The coefficients have been chosen to reflect this scenario, but are otherwise arbitrary. In the exercises, you'll be asked to study what happens if the coefficients are changed.

To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Now consider the four fixed points in turn:

$$(0,0): \text{ Then } A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues are $\lambda = 3, 2$ so $(0,0)$ is an *unstable node.* Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near $(0,0)$ looks like Figure 6.4.1.

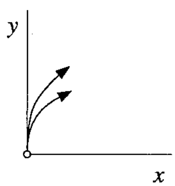


Figure 6.4.1

$$(0,2): \text{ Then } A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since the matrix is triangular. Hence the fixed point is a *stable node.* Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (1, -2)$. Figure 6.4.2 shows the phase portrait near the fixed point $(0, 2)$.

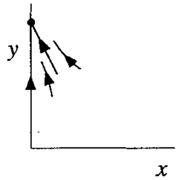


Figure 6.4.2

$(3,0)$: Then $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node.* The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3, -1)$, as shown in Figure 6.4.3.

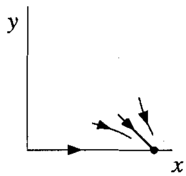


Figure 6.4.3

$(1,1)$: Then $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$. Hence this is a *saddle point.* As you can check, the phase portrait near $(1,1)$ is as shown in Figure 6.4.4.

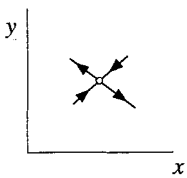


Figure 6.4.4

Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x} = 0$ when $x = 0$, and $\dot{y} = 0$ when $y = 0$.

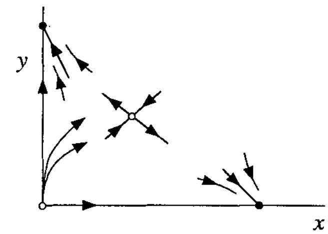


Figure 6.4.5

Now we use common sense to fill in the rest of the phase portrait (Figure 6.4.6). For example, some of the trajectories starting near the origin must go to the stable node on the x-axis, while others must go to the stable node on the y-axis. In between, there must be a special trajectory that can't decide which way to turn, and so it dives into the saddle point. This trajectory is part of the ***stable manifold*** of the saddle, drawn with a heavy line in Figure 6.4.6.

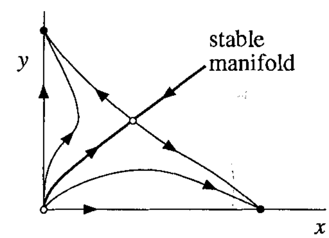


Figure 6.4.6

The other branch of the stable manifold consists of a trajectory coming in "from infinity." A computer-generated phase portrait (Figure 6.4.7) confirms our sketch.

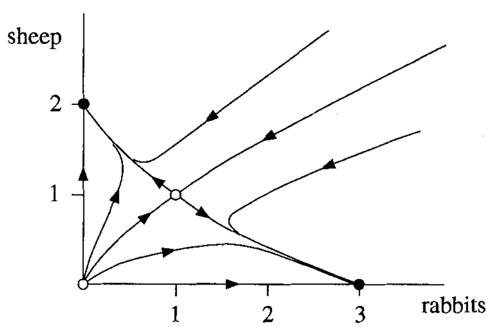


Figure 6.4.7

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other models of competition and has led biologists to formulate the *principle of competitive exclusion*, which states that two species competing for the same limited resource typically cannot coexist. See Pianka (1981) for a biological discussion, and

Pielou (1969), Edelstein–Keshet (1988), or Murray (1989) for additional references and analysis.

Our example also illustrates some general mathematical concepts. Given an attracting fixed point $\mathbf{x}^*$, we define its ***basin of attraction*** to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. For instance, the basin of attraction for the node at $(3,0)$ consists of all the points lying below the stable manifold of the saddle. This basin is shown as the shaded region in Figure 6.4.8.

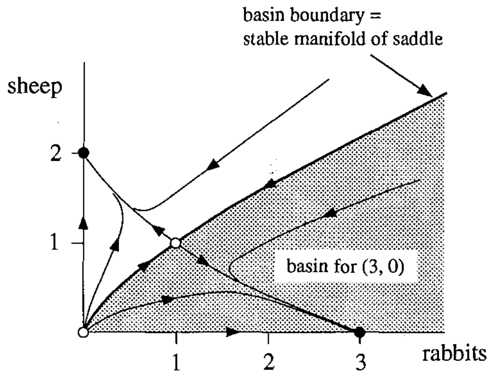


Figure 6.4.8

Because the stable manifold separates the basins for the two nodes, it is called the ***basin boundary***. For the same reason, the two trajectories that comprise the stable manifold are traditionally called ***separatrices***. Basins and their boundaries are important because they partition the phase space into regions of different long-term behavior.

6.5 Conservative Systems

Newton's law $F = ma$ is the source of many important second-order systems. For example, consider a particle of mass m moving along the x-axis, subject to a nonlinear force $F(x)$. Then the equation of motion is

$$m\ddot{x} = F(x).$$

Notice that we are assuming that F is independent of both $\dot{x}$ and t; hence there is no damping or friction of any kind, and there is no time-dependent driving force.

Under these assumptions, we can show that *energy is conserved*, as follows. Let $V(x)$ denote the ***potential energy***, defined by $F(x) = -dV/dx$. Then

$$m\ddot{x} + \frac{dV}{dx} = 0. \tag{1}$$

Now comes a trick worth remembering: multiply both sides by $\dot{x}$ and notice that the left-hand side becomes an exact time-derivative!

$$m\dot{x}\,\ddot{x}+\frac{dV}{dx}\dot{x}=0 \;\Rightarrow\; \frac{d}{dt}\left[\tfrac{1}{2}m\dot{x}^2+V(x)\right]=0$$

where we've used the chain rule

$$\frac{d}{dt}V(x(t))=\frac{dV}{dx}\frac{dx}{dt}$$

in reverse. Hence, for a given solution $x(t)$, the total ***energy***

$$E=\tfrac{1}{2}m\dot{x}^2+V(x)$$

is constant as a function of time. The energy is often called a conserved quantity, a constant of motion, or a first integral. Systems for which a conserved quantity exists are called ***conservative systems***.

Let's be a bit more general and precise. Given a system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, a ***conserved quantity*** is a real-valued continuous function $E(\mathbf{x})$ that is constant on trajectories, i.e. $dE/dt=0$. To avoid trivial examples, we also require that $E(\mathbf{x})$ be nonconstant on every open set. Otherwise a constant function like $E(\mathbf{x})\equiv 0$ would qualify as a conserved quantity for every system, and so *every* system would be conservative! Our caveat rules out this silliness.

The first example points out a basic fact about conservative systems.

EXAMPLE 6.5.1:

Show that *a conservative system cannot have any attracting fixed points.*

Solution: Suppose $\mathbf{x}^*$ were an attracting fixed point. Then all points in its basin of attraction would have to be at the same energy $E(\mathbf{x}^*)$ (because energy is constant on trajectories and all trajectories in the basin flow to $\mathbf{x}^*$). Hence $E(\mathbf{x})$ must be a *constant function* for $\mathbf{x}$ in the basin. But this contradicts our definition of a conservative system, in which we required that $E(\mathbf{x})$ be nonconstant on all open sets. ■

If attracting fixed points can't occur, then what kind of fixed points *can* occur? One generally finds saddles and centers, as in the next example.

EXAMPLE 6.5.2:

Consider a particle of mass $m=1$ moving in a double-well potential $V(x)=-\frac{1}{2}x^2+\frac{1}{4}x^4$. Find and classify all the equilibrium points for the system. Then plot the phase portrait and interpret the results physically.

Solution: The force is $-dV/dx=x-x^3$, so the equation of motion is

$$\ddot{x} = x - x^3.$$

This can be rewritten as the vector field

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

where y represents the particle's velocity. Equilibrium points occur where $(\dot{x}, \dot{y}) = (0,0)$. Hence the equilibria are $(x^*, y^*) = (0,0)$ and $(\pm 1, 0)$. To classify these fixed points we compute the Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}.$$

At (0,0), we have $\Delta = -1$, so the origin is a saddle point. But when $(x^*, y^*) = (\pm 1, 0)$, we find $\tau = 0$, $\Delta = 2$; hence these equilibria are predicted to be centers.

At this point you should be hearing warning bells—in Section 6.3 we saw that small nonlinear terms can easily destroy a center predicted by the linear approximation. But that's not the case here, because of energy conservation. The trajectories are closed curves defined by the ***contours*** of constant energy, i.e.,

$$E = \tfrac{1}{2} y^2 - \tfrac{1}{2} x^2 + \tfrac{1}{4} x^4 = \text{constant}.$$

Figure 6.5.1 shows the trajectories corresponding to different values of E. To decide which way the arrows point along the trajectories, we simply compute the vector $(\dot{x}, \dot{y})$ at a few convenient locations. For example, $\dot{x} > 0$ and $\dot{y} = 0$ on the positive y-axis, so the motion is to the right. The orientation of neighboring trajectories follows by continuity.

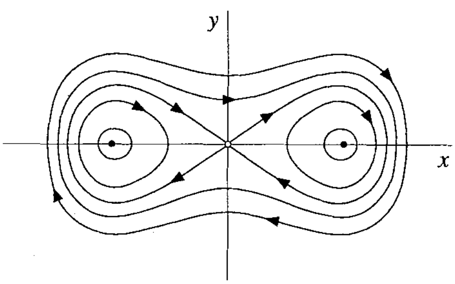


Figure 6.5.1

As expected, the system has a saddle point at $(0,0)$ and centers at $(1,0)$ and $(-1,0)$. Each of the neutrally stable centers is surrounded by a family of small closed orbits. There are also large closed orbits that encircle all three fixed points.

Thus solutions of the system are typically *periodic*, except for the equilibrium solutions and two very special trajectories: these are the trajectories that appear to start and end at the origin. More precisely, these trajectories approach the origin as $t \to \pm\infty$. Trajectories that start and end at the same fixed point are called ***homoclinic orbits***. They are common in conservative systems, but are rare otherwise. Notice that a homoclinic orbit does *not* correspond to a periodic solution, because the trajectory takes forever trying to reach the fixed point.

Finally, let's connect the phase portrait to the motion of an undamped particle in a double-well potential (Figure 6.5.2).

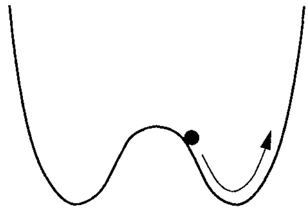


Figure 6.5.2

The neutrally stable equilibria correspond to the particle at rest at the bottom of one of the wells, and the small closed orbits represent small oscillations about these equilibria. The large orbits represent more energetic oscillations that repeatedly take the particle back and forth over the hump. Do you see what the saddle point and the homoclinic orbits mean physically? ■

EXAMPLE 6.5.3:

Sketch the graph of the energy function $E(x, y)$ for Example 6.5.2.

Solution: The graph of $E(x, y)$ is shown in Figure 6.5.3. The energy E is plotted above each point (x, y) of the phase plane. The resulting surface is often called the ***energy surface*** for the system.

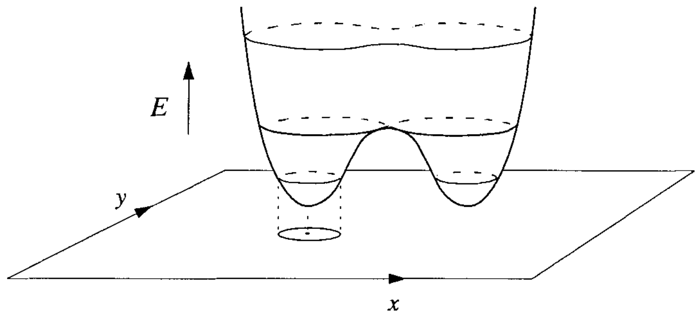


Figure 6.5.3

Figure 6.5.3 shows that the local minima of E project down to centers in the phase plane. Contours of slightly higher energy correspond to the small orbits surrounding the centers. The saddle point and its homoclinic orbits lie at even higher energy, and the large orbits that encircle all three fixed points are the most energetic of all.

It's sometimes helpful to think of the flow as occurring on the energy surface itself, rather than in the phase plane. But notice—the trajectories must maintain a constant height E, so they would run *around* the surface, not down it. ■

Nonlinear Centers

Centers are ordinarily very delicate but, as the examples above suggest, they are much more robust when the system is conservative. We now present a theorem about nonlinear centers in second-order conservative systems.

The theorem says that centers occur at the local minima of the energy function. This is physically plausible—one expects neutrally stable equilibria and small oscillations to occur at the bottom of *any* potential well, no matter what its shape.

Theorem 6.5.1: (Nonlinear centers for conservative systems) Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbf{R}^2$, and $\mathbf{f}$ is continuously differentiable. Suppose there exists a conserved quantity $E(\mathbf{x})$ and suppose that $\mathbf{x}^*$ is an isolated fixed point (i.e., there are no other fixed points in a small neighborhood surrounding $\mathbf{x}^*$). If $\mathbf{x}^*$ is a local minimum of E, then all trajectories sufficiently close to $\mathbf{x}^*$ are closed.

Ideas behind the proof: Since E is constant on trajectories, each trajectory is contained in some contour of E. Near a local maximum or minimum, the contours are *closed.* (We won't prove this, but Figure 6.5.3 should make it seem obvious.) The only remaining question is whether the trajectory actually goes all the way around the contour or whether it stops at a fixed point on the contour. But because we're assuming that $\mathbf{x}^*$ is an *isolated* fixed point, there cannot be any fixed points on contours sufficiently close to $\mathbf{x}^*$. Hence all trajectories in a sufficiently small neighborhood of $\mathbf{x}^*$ are closed orbits, and therefore $\mathbf{x}^*$ is a center. ■

Two remarks about this result:

1. The theorem is valid for local *maxima* of E also. Just replace the function E by $-E$, and maxima get converted to minima; then Theorem 6.5.1 applies.
2. We need to assume that $\mathbf{x}^*$ is isolated. Otherwise there are counterexamples due to fixed points on the energy contour—see Exercise 6.5.12.

Another theorem about nonlinear centers will be presented in the next section.

6.6 Reversible Systems

Many mechanical systems have ***time-reversal symmetry***. This means that their dynamics look the same whether time runs forward or backward. For example, if you were watching a movie of an undamped pendulum swinging back and forth, you wouldn't see any physical absurdities if the movie were run backward.

In fact, any mechanical system of the form $m\ddot{x} = F(x)$ is symmetric under time reversal. If we make the change of variables $t \to -t$, the second derivative $\ddot{x}$ stays the same and so the equation is unchanged. Of course, the velocity $\dot{x}$ would be reversed. Let's see what this means in the phase plane. The equivalent system is

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \tfrac{1}{m}F(x)\end{aligned}$$

where y is the velocity. If we make the change of variables $t \to -t$ and $y \to -y$, both equations stay the same. Hence if $(x(t),\ y(t))$ is a solution, then so is $(x(-t),\ -y(-t))$. Therefore every trajectory has a twin: they differ only by time-reversal and a reflection in the x-axis (Figure 6.6.1).

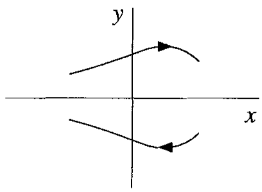


Figure 6.6.1

The trajectory above the x-axis looks just like the one below the x-axis, except the arrows are reversed.

More generally, let's define a ***reversible system*** to be *any* second-order system that is invariant under $t \to -t$ and $y \to -y$. For example, any system of the form

$$\begin{aligned}\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y),\end{aligned}$$

where f is *odd* in y and g is *even* in y (i.e., $f(x,-y) = -f(x,y)$ and $g(x,-y) = g(x,y)$) is reversible.

Reversible systems are different from conservative systems, but they have many of the same properties. For instance, the next theorem shows that centers are robust in reversible systems as well.

Theorem 6.6.1: (Nonlinear centers for reversible systems) Suppose the origin $\mathbf{x}^* = \mathbf{0}$ is a linear center for the continuously differentiable system

$$\begin{aligned}\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y),\end{aligned}$$

and suppose that the system is reversible. Then sufficiently close to the origin, all trajectories are closed curves.

Ideas behind the proof: Consider a trajectory that starts on the positive x-axis near the origin (Figure 6.6.2). Sufficiently near the origin, the flow swirls around the origin, thanks to the dominant influence of the linear center, and so the trajectory eventually intersects the *negative* x-axis. (This is the step where our proof lacks rigor, but the claim should seem plausible.)

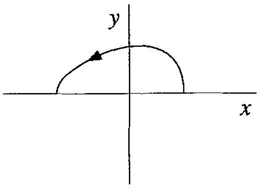


Figure 6.6.2

Now we use reversibility. By reflecting the trajectory across the x-axis, and changing the sign of t, we obtain a twin trajectory with the same endpoints but with its arrow reversed (Figure 6.6.3).

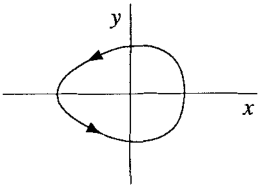


Figure 6.6.3

Together the two trajectories form a closed orbit, as desired. Hence all trajectories sufficiently close to the origin are closed. ∎

EXAMPLE 6.6.1:

Show that the system

$$\dot{x} = y - y^3$$
$$\dot{y} = -x - y^2$$

has a nonlinear center at the origin, and plot the phase portrait.

Solution: We'll show that the hypotheses of the theorem are satisfied. The Jacobian at the origin is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This has $\tau = 0$, $\Delta > 0$, so the origin is a linear center. Furthermore, the system is reversible, since the equations are invariant under the transformation $t \to -t$, $y \to -y$. By Theorem 6.6.1, the origin is a *nonlinear* center.

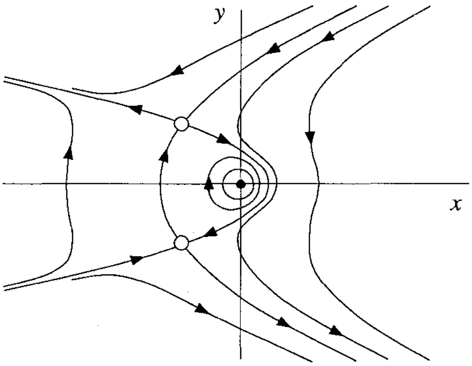


Figure 6.6.4

The other fixed points of the system are $(-1,1)$ and $(-1,-1)$. They are saddle points, as is easily checked by computing the linearization. A computer-generated phase portrait is shown in Figure 6.6.4. It looks like some exotic sea creature, perhaps a manta ray. The reversibility symmetry is apparent. The trajectories above the x-axis have twins below the x-axis, with arrows reversed.

Notice that the twin saddle points are joined by a pair of trajectories. They are called ***heteroclinic trajectories*** or ***saddle connections***. Like homoclinic orbits, heteroclinic trajectories are much more common in reversible or conservative systems than in other types of systems. ■

Although we have relied on the computer to plot Figure 6.6.4, it can be sketched on the basis of qualitative reasoning alone. For example, the existence of the heteroclinic trajectories can be deduced rigorously using reversibility arguments (Exercise 6.6.6). The next example illustrates the spirit of such arguments.

EXAMPLE 6.6.2:

Using reversibility arguments alone, show that the system

$$\dot{x} = y$$
$$\dot{y} = x - x^2$$

has a homoclinic orbit in the half-plane $x \ge 0$.

Solution: Consider the unstable manifold of the saddle point at the origin. This manifold leaves the origin along the vector $(1,1)$, since this is the unstable eigendirection for the linearization. Hence, close to the origin, part of the unstable manifold lies in the first quadrant $x, y > 0$. Now imagine a phase point with coordinates $(x(t), y(t))$ moving along the unstable manifold, starting from x, y small and positive. At first, $x(t)$ must increase since $\dot{x} = y > 0$. Also, $y(t)$ increases initially, since $\dot{y} = x - x^2 > 0$ for small x. Thus the phase point moves up and to the right. Its horizontal velocity is continually increasing, so at some time it must cross the vertical line $x=1$. Then $\dot{y}<0$ so $y(t)$ decreases, eventually reaching $y=0$. Figure 6.6.5 shows the situation.

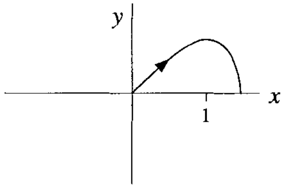


Figure 6.6.5

Now, *by reversibility*, there must be a twin trajectory with the same endpoints but with arrow reversed (Figure 6.6.6). Together the two trajectories form the desired homoclinic orbit. ■

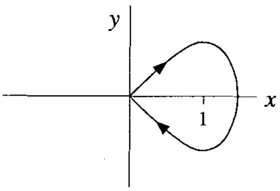


Figure 6.6.6

There is a more general definition of reversibility which extends nicely to higher-order systems. Consider any mapping $R(\mathbf{x})$ of the phase space to itself that satisfies $R^2(\mathbf{x})=\mathbf{x}$. In other words, if the mapping is applied *twice*, all points go back to where they started. In our two-dimensional examples, a reflection about the x-axis (or any axis through the origin) has this property. Then the system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ is ***reversible*** if it is invariant under the change of variables $t \to -t$, $\mathbf{x} \to R(\mathbf{x})$.

Our next example illustrates this more general notion of reversibility, and also highlights the main difference between reversible and conservative systems.

EXAMPLE 6.6.3:

Show that the system

$$\dot{x} = -2\cos x - \cos y$$
$$\dot{y} = -2\cos y - \cos x$$

is reversible, but *not* conservative. Then plot the phase portrait.

Solution: The system is invariant under the change of variables $t \to -t$, $x \to -x$, and $y \to -y$. Hence the system is reversible, with $R(x,y)=(-x,-y)$ in the preceding notation.

To show that the system is not conservative, it suffices to show that it has an attracting fixed point. (Recall that a conservative system can never have an attracting fixed point—see Example 6.5.1.)

The fixed points satisfy $2\cos x = -\cos y$ and $2\cos y = -\cos x$. Solving these equations simultaneously yields $\cos x^* = \cos y^* = 0$. Hence there are four fixed points, given by $(x^*, y^*) = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$.

We claim that $(x^*, y^*) = (-\frac{\pi}{2}, -\frac{\pi}{2})$ is an attracting fixed point. The Jacobian there is

$$A = \begin{pmatrix} 2\sin x^* & \sin y^* \\ \sin x^* & 2\sin y^* \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix},$$

which has $\tau = -4$, $\Delta = 3$, $\tau^2 - 4\Delta = 4$. Therefore the fixed point is a stable node. This shows that the system is not conservative.

The other three fixed points can be shown to be an unstable node and two saddles. A computer-generated phase portrait is shown in Figure 6.6.7.

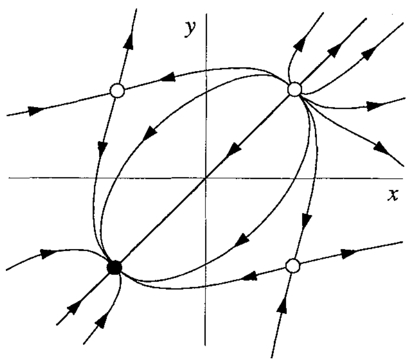


Figure 6.6.7

To see the reversibility symmetry, compare the dynamics at any two points (x, y) and $R(x, y) = (-x, -y)$. The trajectories look the same, but the arrows are reversed. In particular, the stable node at $(-\frac{\pi}{2}, -\frac{\pi}{2})$ is the twin of the unstable node at $(\frac{\pi}{2}, \frac{\pi}{2})$. ■

The system in Example 6.6.3 is closely related to a model of two superconducting Josephson junctions coupled through a resistive load (Tsang et al. 1991). For further discussion, see Exercise 6.6.9 and Example 8.7.4. Reversible, nonconservative systems also arise in the context of lasers (Politi et al. 1986) and fluid flows (Stone, Nadim, and Strogatz 1991 and Exercise 6.6.8).

6.7 Pendulum

Do you remember the first nonlinear system you ever studied in school? It was probably the pendulum. But in elementary courses, the pendulum's essential nonlinearity is sidestepped by the small-angle approximation $\sin\theta \approx \theta$. Enough of that! In this section we use phase plane methods to analyze the pendulum, even in the dreaded large-angle regime where the pendulum whirls over the top.

In the absence of damping and external driving, the motion of a pendulum is governed by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{1}$$

where θ is the angle from the downward vertical, g is the acceleration due to gravity, and L is the length of the pendulum (Figure 6.7.1).

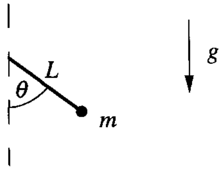


Figure 6.7.1

We nondimensionalize (1) by introducing a frequency $\omega = \sqrt{g/L}$ and a dimensionless time $\tau = \omega t$. Then the equation becomes

$$\ddot{\theta} + \sin\theta = 0 \tag{2}$$

where the overdot denotes differentiation with respect to τ. The corresponding system in the phase plane is

$$\dot{\theta} = v \tag{3a}$$
$$\dot{v} = -\sin\theta \tag{3b}$$

where v is the (dimensionless) angular velocity.

The fixed points are $(\theta^*, v^*) = (k\pi, 0)$, where k is any integer. There's no physical difference between angles that differ by 2π, so we'll concentrate on the two fixed points $(0,0)$ and $(\pi, 0)$. At $(0,0)$, the Jacobian is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so the origin is a linear center.

In fact, the origin is a *nonlinear* center, for two reasons. First, the system (3) is *reversible*: the equations are invariant under the transformation $\tau \to -\tau$, $v \to -v$. Then Theorem 6.6.1 implies that the origin is a nonlinear center.

Second, the system is also *conservative*. Multiplying (2) by $\dot{\theta}$ and integrating yields

$$\dot{\theta}(\ddot{\theta} + \sin\theta) = 0 \Rightarrow \tfrac{1}{2}\dot{\theta}^2 - \cos\theta = \text{constant}.$$

The energy function

$$E(\theta, v) = \tfrac{1}{2} v^2 - \cos\theta \tag{4}$$

has a local minimum at (0,0), since $E \approx \frac{1}{2}(v^2 + \theta^2) - 1$ for small (θ, v). Hence Theorem 6.5.1 provides a second proof that the origin is a nonlinear center. (This argument also shows that the closed orbits are approximately *circular*, with $\theta^2 + v^2 \approx 2(E+1)$.)

Now that we've beaten the origin to death, consider the fixed point at $(\pi, 0)$. The Jacobian is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 - 1 = 0$. Therefore $\lambda_1 = -1$, $\lambda_2 = 1$; the fixed point is a saddle. The corresponding eigenvectors are $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, 1)$.

The phase portrait near the fixed points can be sketched from the information obtained so far (Figure 6.7.2).

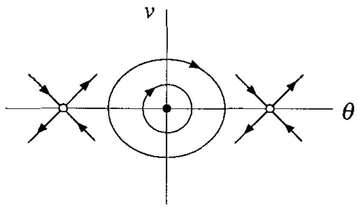


Figure 6.7.2

To fill in the picture, we include the energy contours $E = \frac{1}{2} v^2 - \cos\theta$ for different values of E. The resulting phase portrait is shown in Figure 6.7.3. The picture is periodic in the θ-direction, as we'd expect.

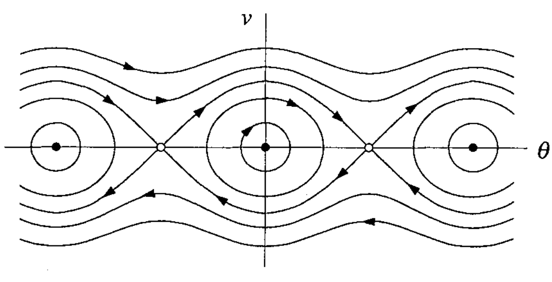


Figure 6.7.3

Now for the physical interpretation. The center corresponds to a state of neutrally stable equilibrium, with the pendulum at rest and hanging straight down. This is the lowest possible energy state ($E = -1$). The small orbits surrounding the center represent small oscillations about equilibrium, traditionally called ***librations***. As E increases, the orbits grow. The critical case is $E = 1$, corresponding to the heteroclinic trajectories joining the saddles in Figure 6.7.3. The saddles represent an *inverted* pendulum at rest;

hence the heteroclinic trajectories represent delicate motions in which the pendulum slows to a halt precisely as it approaches the inverted position. For $E > 1$, the pendulum whirls repeatedly over the top. These ***rotations*** should also be regarded as periodic solutions, since $\theta = -\pi$ and $\theta = +\pi$ are the same physical position.

Cylindrical Phase Space

The phase portrait for the pendulum is more illuminating when wrapped onto the surface of a cylinder (Figure 6.7.4). In fact, a cylinder is the *natural* phase space for the pendulum, because it incorporates the fundamental geometric difference between v and θ: the angular velocity v is a real number, whereas θ is an *angle*.

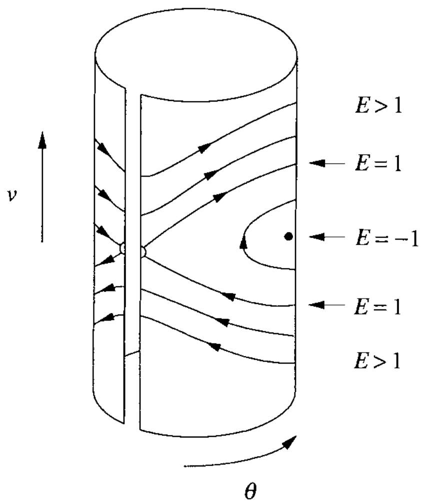


Figure 6.7.4

There are several advantages to the cylindrical representation. Now the periodic whirling motions *look* periodic—they are the closed orbits that encircle the cylinder for $E > 1$. Also, it becomes obvious that the saddle points in Figure 6.7.3 are all the same physical state (an inverted pendulum at rest). The heteroclinic trajectories of Figure 6.7.3 become homoclinic orbits on the cylinder.

There is an obvious symmetry between the top and bottom half of Figure 6.7.4. For example, both homoclinic orbits have the same energy and shape. To highlight this symmetry, it is interesting (if a bit mind-boggling at first) to plot the *energy* vertically instead of the angular velocity v (Figure 6.7.5). Then the orbits on the cylinder remain at constant height, while the cylinder gets bent into a ***U-tube***. The two arms of the tube are distinguished by the sense of rotation of the pendulum, either clockwise or counterclock-

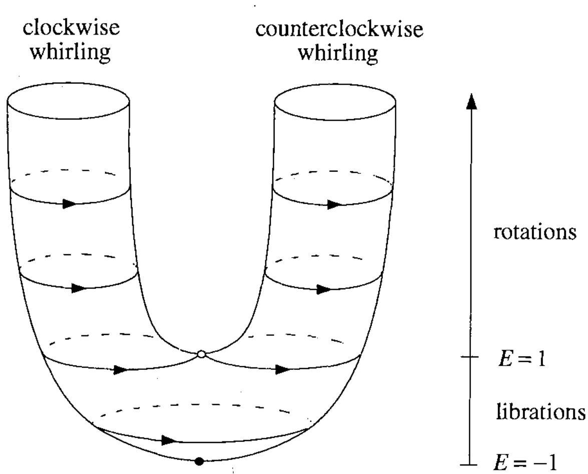


Figure 6.7.5

wise. At low energies, this distinction no longer exists; the pendulum oscillates to and fro. The homoclinic orbits lie at $E = 1$, the borderline between rotations and librations.

At first you might think that the trajectories are drawn incorrectly on one of the arms of the U-tube. It might seem that the arrows for clockwise and counterclockwise motions should go in *opposite* directions. But if you think about the coordinate system shown in Figure 6.7.6, you'll see that the picture is correct.

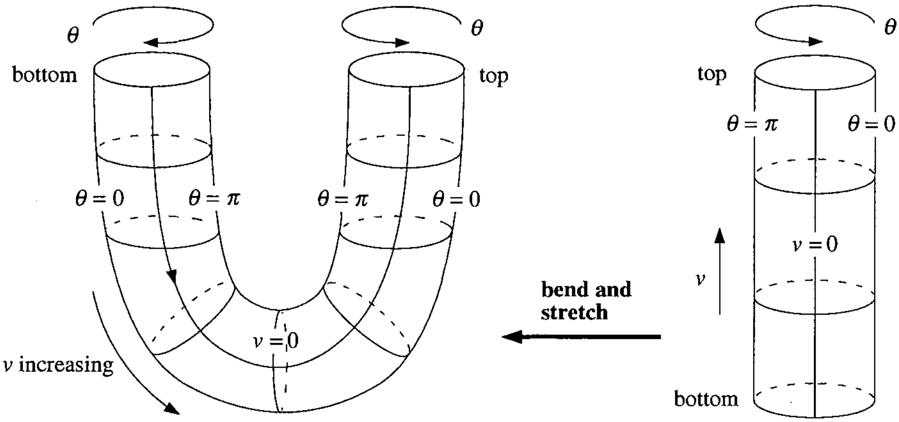


Figure 6.7.6

The point is that the direction of increasing θ has reversed when the bottom of the cylinder is bent around to form the U-tube. (Please understand that Figure 6.7.6 shows the coordinate system, not the actual trajectories; the trajectories were shown in Figure 6.7.5.)

Damping

Now let's return to the phase plane, and suppose that we add a small amount of linear damping to the pendulum. The governing equation becomes

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$$

where $b > 0$ is the damping strength. Then centers become stable spirals while saddles remain saddles. A computer-generated phase portrait is shown in Figure 6.7.7.

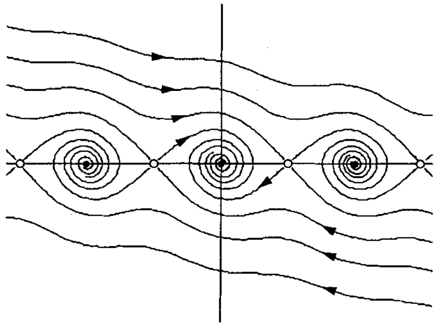


Figure 6.7.7

The picture on the U-tube is clearer. *All trajectories continually lose altitude,* except for the fixed points (Figure 6.7.8).

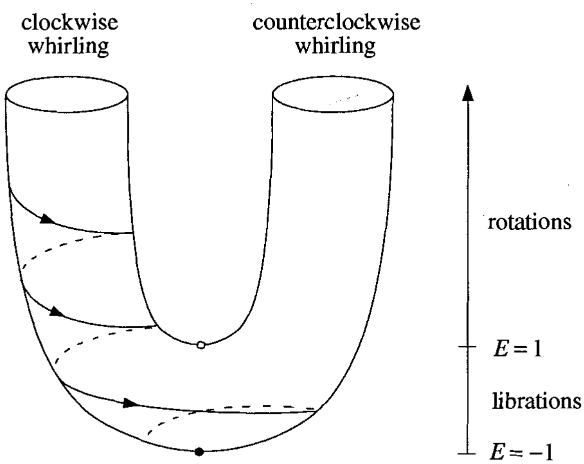


Figure 6.7.8

We can see this explicitly by computing the change in energy along a trajectory:

$$\frac{dE}{d\tau}=\frac{d}{d\tau}\left(\tfrac{1}{2}\dot{\theta}^2-\cos\theta\right)=\dot{\theta}\left(\ddot{\theta}+\sin\theta\right)=-b\dot{\theta}^2\le 0.$$

Hence E decreases monotonically along trajectories, except at fixed points where $\dot{\theta}\equiv 0$.

The trajectory shown in Figure 6.7.8 has the following physical interpretation: the pendulum is initially whirling clockwise. As it loses energy, it has a harder time rotating over the top. The corresponding trajectory spirals down the arm of the U-tube until $E<1$; then the pendulum doesn't have enough energy to whirl, and so it settles down into a small oscillation about the bottom. Eventually the motion damps out and the pendulum comes to rest at its stable equilibrium.

This example shows how far we can go with pictures—without invoking any difficult formulas, we were able to extract all the important features of the pendulum's dynamics. It would be much more difficult to obtain these results analytically, and much more confusing to interpret the formulas, even if we *could* find them.

6.8 Index Theory

In Section 6.3 we learned how to linearize a system about a fixed point. Linearization is a prime example of a *local* method: it gives us a detailed microscopic view of the trajectories near a fixed point, but it can't tell us what happens to the trajectories after they leave that tiny neighborhood. Furthermore, if the vector field starts with quadratic or higher-order terms, the linearization tells us nothing.

In this section we discuss index theory, a method that provides *global* information about the phase portrait. It enables us to answer such questions as: Must a closed trajectory always encircle a fixed point? If so, what types of fixed points are permitted? What types of fixed points can coalesce in bifurcations? The method also yields information about the trajectories near higher-order fixed points. Finally, we can sometimes use index arguments to rule out the possibility of closed orbits in certain parts of the phase plane.

The Index of a Closed Curve

The index of a closed curve C is an integer that measures the winding of the vector field on C. The index also provides information about any fixed points that might happen to lie inside the curve, as we'll see.

This idea may remind you of a concept in electrostatics. In that subject, one often introduces a hypothetical closed surface (a "Gaussian surface") to probe a configuration of electric charges. By studying the behavior of the electric field on the surface, one can determine the total amount of charge *inside* the surface. Amazingly, the behavior *on* the surface tells us what's happening far away *inside* the surface! In the present context, the electric field is analogous to our vector field, the Gaussian surface is analogous to the curve C, and the total charge is analogous to the index.

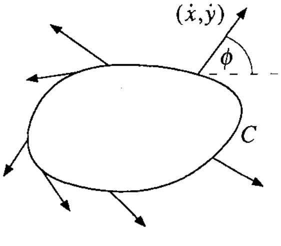


Figure 6.8.1

Now let's make these notions precise. Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a smooth vector field on the phase plane. Consider a closed curve C (Figure 6.8.1). This curve is *not* necessarily a trajectory—it's simply a loop that we're putting in the phase plane to probe the behavior of the vector field. We also assume that C is a "simple closed curve" (i.e., it doesn't intersect itself) and that it doesn't pass through any fixed points of the system. Then at each point $\mathbf{x}$ on C, the vector field $\dot{\mathbf{x}} = (\dot{x}, \dot{y})$ makes a well-defined angle

$$\phi = \tan^{-1}(\dot{y}/\dot{x})$$

with the positive x-axis (Figure 6.8.1).

As $\mathbf{x}$ moves counterclockwise around C, the angle ϕ changes *continuously* since the vector field is smooth. Also, when $\mathbf{x}$ returns to its starting place, ϕ returns to its original direction. Hence, over one circuit, ϕ has changed by an *integer* multiple of 2π. Let $[\phi]_C$ be the net change in ϕ over one circuit. Then the ***index of the closed curve*** C with respect to the vector field $\mathbf{f}$ is defined as

$$I_C = \tfrac{1}{2\pi}[\phi]_C \,.$$

Thus, I_C is the net number of counterclockwise revolutions made by the vector field as $\mathbf{x}$ moves once counterclockwise around C.

To compute the index, we do not need to know the vector field everywhere; we only need to know it along C. The first two examples illustrate this point.

EXAMPLE 6.8.1:

Given that the vector field varies along C as shown in Figure 6.8.2, find I_C.

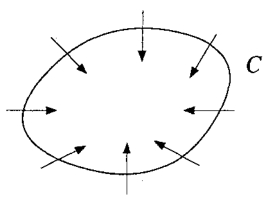


Figure 6.8.2

Solution: As we traverse C once counterclockwise, the vectors rotate through one full turn in the same sense. Hence $I_C = +1$.

If you have trouble visualizing this, here's a foolproof method. Number the vectors in counterclockwise order, starting anywhere on C (Figure 6.8.3a). Then transport these vectors *(without rotation!)* such that their tails lie at a common origin (Figure 6.8.3b). The index equals the net number of counterclockwise revolutions made by the numbered vectors.

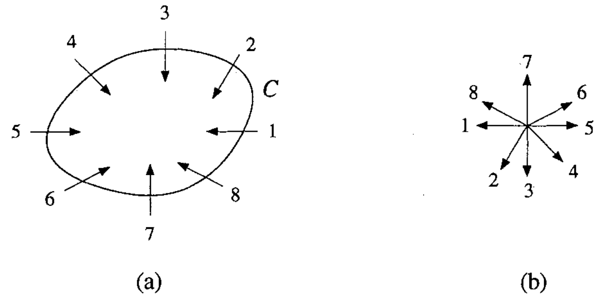


Figure 6.8.3

As Figure 6.8.3b shows, the vectors rotate once counterclockwise as we go in increasing order from vector #1 to vector #8. Hence $I_C = +1$. ■

EXAMPLE 6.8.2:

Given the vector field on the closed curve shown in Figure 6.8.4a, compute I_C.

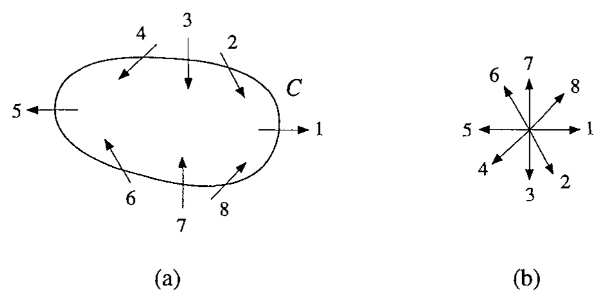


Figure 6.8.4

Solution: We use the same construction as in Example 6.8.1. As we make one circuit around C, the vectors rotate through one full turn, but now in the *opposite* sense. In other words, the vectors on C rotate *clockwise* as we go around C counterclockwise. This is clear from Figure 6.8.4b; the vectors rotate clockwise as we go in increasing order from vector #1 to vector #8. Therefore $I_C = -1$. ■

In many cases, we are given equations for the vector field, rather than a picture of it. Then we have to draw the picture ourselves, and repeat the steps above. Sometimes this can be confusing, as in the next example.

EXAMPLE 6.8.3:

Given the vector field $\dot{x} = x^2 y$, $\dot{y} = x^2 - y^2$, find I_C, where C is the unit circle $x^2 + y^2 = 1$.

Solution: To get a clear picture of the vector field, it is sufficient to consider a few conveniently chosen points on C. For instance, at $(x, y) = (1,0)$, the vector is $(\dot{x}, \dot{y}) = (x^2 y,\ x^2 - y^2) = (0,1)$. This vector is labeled #1 in Figure 6.8.5a. Now we move counterclockwise around C, computing vectors as we go. At $(x, y) = \frac{1}{\sqrt{2}}(1,1)$, we have $(\dot{x}, \dot{y}) = (x, y) = \frac{1}{2\sqrt{2}}(1,0)$, labeled #2. The remaining vectors are found similarly. Notice that different points on the circle may be associated with the same vector; for example, vector #3 and #7 are both $(0,-1)$.

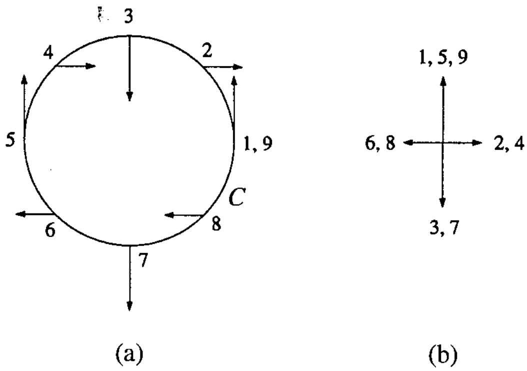


Figure 6.8.5

Now we translate the vectors over to Figure 6.8.5b. As we move from #1 to #9 in order, the vectors rotate 180° clockwise between #1 and #3, then swing back 360° counterclockwise between #3 and #7, and finally rotate 180° clockwise again between #7 and #9 as we complete the circuit of C. Thus $[\phi]_C = -\pi + 2\pi - \pi = 0$ and therefore $I_C = 0$. ■

We plotted nine vectors in this example, but you may want to plot more to see the variation of the vector field in finer detail.

Properties of the Index

Now we list some of the most important properties of the index.

1. Suppose that C can be continuously deformed into C' without passing through a fixed point. Then $I_C = I_{C'}$.

   This property has an elegant proof: Our assumptions imply that as we deform C into C', the index I_C varies *continuously*. But I_C is an integer—hence it can't change without jumping! (To put it more formally, if an integer-valued function is continuous, it must be *constant.*)

   As you think about this argument, try to see where we used the assumption that the intermediate curves don't pass through any fixed points.
2. If C doesn't enclose any fixed points, then $I_C = 0$.

   Proof: By property (1), we can shrink C to a tiny circle without changing the index. But ϕ is essentially constant on such a circle, because all the vectors point in nearly the same direction, thanks to the assumed smoothness of the vector field (Figure 6.8.6). Hence $[\phi]_C = 0$ and therefore $I_C = 0$.

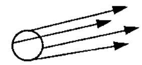


Figure 6.8.6

3. If we reverse all the arrows in the vector field by changing $t \to -t$, the index is unchanged.

   Proof: All angles change from ϕ to $\phi + \pi$. Hence $[\phi]_C$ stays the same.
4. Suppose that the closed curve C is actually a *trajectory* for the system, i.e., C is a closed orbit. Then $I_C = +1$.

   We won't prove this, but it should be clear from geometric intuition (Figure 6.8.7).

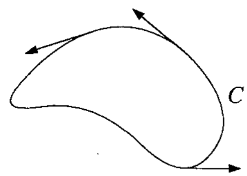


Figure 6.8.7

Notice that the vector field is everywhere tangent to C, because C is a trajectory. Hence, as $\mathbf{x}$ winds around C once, the tangent vector also rotates once in the same sense.

Index of a Point

The properties above are useful in several ways. Perhaps most importantly, they allow us to define the index of a fixed point, as follows.

Suppose $\mathbf{x}^*$ is an isolated fixed point. Then the ***index*** I of $\mathbf{x}^*$ is defined as I_C, where C is *any* closed curve that encloses $\mathbf{x}^*$ and no other fixed points. By property (1) above, I_C is independent of C and is therefore a property of $\mathbf{x}^*$ alone. Therefore we may drop the subscript C and use the notation I for the index of a point.

EXAMPLE 6.8.4:

Find the index of a stable node, an unstable node, and a saddle point.

Solution: The vector field near a stable node looks like the vector field of Example 6.8.1. Hence $I = +1$. The index is also $+1$ for an unstable node, because the only difference is that all the arrows are reversed; by property (3), this doesn't change the index! (This observation shows that *the index is not related to stability,* per se.) Finally, $I=-1$ for a saddle point, because the vector field resembles that discussed in Example 6.8.2. ■

In Exercise 6.8.1, you are asked to show that spirals, centers, degenerate nodes and stars all have $I=+1$. Thus, a saddle point is truly a different animal from all the other familiar types of isolated fixed points.

The index of a curve is related in a beautifully simple way to the indices of the fixed points inside it. This is the content of the following theorem.

Theorem 6.8.1: If a closed curve C surrounds n isolated fixed points $\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*$, then

$$I_C = I_1 + I_2 + \ldots + I_n$$

where I_k is the index of $\mathbf{x}_k^*$, for $k = 1, \ldots, n$.

Ideas behind the proof: The argument is a familiar one, and comes up in multivariable calculus, complex variables, electrostatics, and various other subjects. We think of C as a balloon and suck most of the air out it, being careful not to hit any of the fixed points. The result of this deformation is a new closed curve Γ, consisting of n small circles $\gamma_1, \ldots, \gamma_n$ about the fixed points, and two-way bridges connecting these circles (Figure 6.8.8). Note that $I_\Gamma = I_C$, by property (1), since we didn't cross any fixed points during the deformation. Now let's compute I_Γ by considering $[\phi]_\Gamma$. There are contributions to $[\phi]_\Gamma$ from the small circles and from the two-way bridges. The key point is that *the contributions from the bridges cancel out*: as we move around Γ, each bridge is traversed once in one direction, and later in the opposite direction. Thus we only need to consider the contributions from the small circles. On γ_k, the angle ϕ changes by $[\phi]_{\gamma_k} = 2\pi I_k$, by definition of I_k. Hence

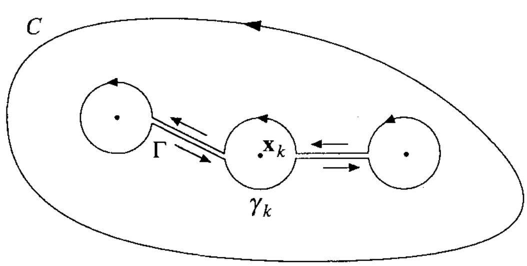


Figure 6.8.8

$$I_\Gamma = \tfrac{1}{2\pi}[\phi]_\Gamma = \tfrac{1}{2\pi}\sum_{k=1}^{n}[\phi]_{\gamma_k} = \sum_{k=1}^{n} I_k$$

and since $I_\Gamma = I_C$, we're done. ■

This theorem is reminiscent of Gauss's law in electrostatics, namely that the electric flux through a surface is proportional to the total charge enclosed. See Exercise 6.8.12 for a further exploration of this analogy between index and charge.

Theorem 6.8.2: Any closed orbit in the phase plane must enclose fixed points whose indices sum to +1.

Proof: Let C denote the closed orbit. From property (4) above, $I_C = +1$.

Then Theorem 6.8.1 implies $\sum_{k=1}^{n} I_k = +1$. ■

Theorem 6.8.2 has many practical consequences. For instance, it implies that there is always at least one fixed point inside any closed orbit in the phase plane (as you may have noticed on your own). If there is *only* one fixed point inside, it cannot be a saddle point. Furthermore, Theorem 6.8.2 can sometimes be used to rule out the possible occurrence of closed trajectories, as seen in the following examples.

EXAMPLE 6.8.5:

Show that closed orbits are impossible for the "rabbit vs. sheep" system

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

studied in Section 6.4. Here $x, y \geq 0$.

Solution: As shown previously, the system has four fixed points: $(0,0)$ = unstable node; $(0,2)$ and $(3,0)$ = stable nodes; and $(1,1)$ = saddle point. The index at each of these points is shown in Figure 6.8.9. Now suppose that the system had a closed trajectory. Where could it lie? There are three qualitatively different locations, indicated by the dotted curves C_1, C_2, C_3. They can be ruled out as follows: orbits like C_1 are impossible because they don't enclose any fixed points, and orbits like C_2 violate the requirement that the indices inside must sum to $+1$. But what is wrong with orbits like C_3, which satisfy the index requirement? The trouble is that such orbits always cross the x-axis or the y-axis, and these axes contain straight-line trajectories. Hence C_3 violates the rule that trajectories can't cross (recall Section 6.2). ■

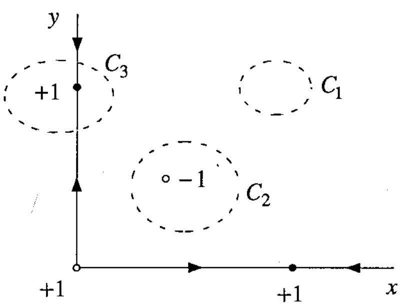


Figure 6.8.9

EXAMPLE 6.8.6:

Show that the system $\dot{x} = xe^{-x}$, $\dot{y} = 1 + x + y^2$ has no closed orbits.

Solution: This system has no fixed points: if $\dot{x} = 0$, then $x = 0$ and so $\dot{y} = 1 + y^2 \neq 0$. By Theorem 6.8.2, closed orbits cannot exist. ■

EXERCISES FOR CHAPTER 6

6.1 Phase Portraits

For each of the following systems, find the fixed points. Then sketch the nullclines, the vector field, and a plausible phase portrait.

6.1.1 $\dot{x} = x - y$, $\dot{y} = 1 - e^x$

6.1.2 $\dot{x} = x - x^3$, $\dot{y} = -y$

6.1.3 $\dot{x} = x(x - y)$, $\dot{y} = y(2x - y)$

6.1.4 $\dot{x} = y$, $\dot{y} = x(1 + y) - 1$

6.1.5 $\dot{x} = x(2 - x - y)$, $\dot{y} = x - y$

6.1.6 $\dot{x} = x^2 - y$, $\dot{y} = x - y$

6.1.7 (Nullcline vs. stable manifold) There's a confusing aspect of Example 6.1.1. The nullcline $\dot{x} = 0$ in Figure 6.1.3 has a similar shape and location as the stable manifold of the saddle, shown in Figure 6.1.4. But they're not the same curve! To clarify the relation between the two curves, sketch both of them on the same phase portrait.

(Computer work) Plot computer-generated phase portraits of the following systems. As always, you may write your own computer programs or use any ready-made software, e.g., *MacMath* (Hubbard and West 1992).

6.1.8 (van der Pol oscillator) $\dot{x} = y$, $\dot{y} = -x + y(1 - x^2)$

6.1.9 (Dipole fixed point) $\dot{x} = 2xy$, $\dot{y} = y^2 - x^2$

6.1.10 (Two-eyed monster) $\dot{x} = y + y^2$, $\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borrelli and Coleman 1987, p. 385.)

6.1.11 (Parrot) $\dot{x} = y + y^2$, $\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borrelli and Coleman 1987, p. 384.)

6.1.12 (Saddle connections) A certain system is known to have exactly two fixed points, both of which are saddles. Sketch phase portraits in which

a) there is a single trajectory that connects the saddles;

b) there is no trajectory that connects the saddles.

6.1.13 Draw a phase portrait that has exactly three closed orbits and one fixed point.

6.1.14 (Series approximation for the stable manifold of a saddle point) Recall the system $\dot{x} = x + e^{-y}$, $\dot{y} = -y$ from Example 6.1.1. We showed that this system has one fixed point, a saddle at $(-1,0)$. Its unstable manifold is the x-axis, but its stable manifold is a curve that is harder to find. The goal of this exercise is to approximate this unknown curve.

a) Let (x, y) be a point on the stable manifold, and assume that (x, y) is close to $(-1,0)$. Introduce a new variable $u = x + 1$, and write the stable manifold as $y = a_1 u + a_2 u^2 + O(u^3)$. To determine the coefficients, derive two expressions for dy/du and equate them.

b) Check that your analytical result produces a curve with the same shape as the stable manifold shown in Figure 6.1.4.

6.2 Existence, Uniqueness, and Topological Consequences

6.2.1 We claimed that different trajectories can never intersect. But in many phase portraits, different trajectories appear to intersect at a fixed point. Is there a contradiction here?

6.2.2 Consider the system $\dot{x} = y$, $\dot{y} = -x + (1 - x^2 - y^2)y$.

a) Let D be the open disk $x^2 + y^2 < 4$. Verify that the system satisfies the hypotheses of the existence and uniqueness theorem throughout the domain D.

b) By substitution, show that $x(t) = \sin t$, $y(t) = \cos t$ is an exact solution of the system.

c) Now consider a different solution, in this case starting from the initial condition $x(0) = \frac{1}{2}$, $y(0) = 0$. Without doing any calculations, explain why this solution *must* satisfy $x(t)^2 + y(t)^2 < 1$ for all $t < \infty$.

6.3 Fixed Points and Linearization

For each of the following systems, find the fixed points, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

6.3.1 $\dot{x} = x - y$, $\dot{y} = x^2 - 4$

6.3.2 $\dot{x} = \sin y$, $\dot{y} = x - x^3$

6.3.3 $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$

6.3.4 $\dot{x} = y + x - x^3$, $\dot{y} = -y$

6.3.5 $\dot{x} = \sin y$, $\dot{y} = \cos x$

6.3.6 $\dot{x} = xy - 1$, $\dot{y} = x - y^3$

6.3.7 For each of the nonlinear systems above, plot a computer-generated phase portrait and compare to your approximate sketch.

6.3.8 (Gravitational equilibrium) A particle moves along a line joining two stationary masses, m_1 and m_2, which are separated by a fixed distance a. Let x denote the distance of the particle from m_1.

a) Show that $\ddot{x} = \dfrac{Gm_2}{(x-a)^2} - \dfrac{Gm_1}{x^2}$, where G is the gravitational constant.

b) Find the particle's equilibrium position. Is it stable or unstable?

6.3.9 Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.

a) Find all the fixed points and classify them.

b) Show that the line $x = y$ is invariant, i.e., any trajectory that starts on it stays on it.

c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all other trajectories. (Hint: Form a differential equation for $x - y$.)

d) Sketch the phase portrait.

e) If you have access to a computer, plot an accurate phase portrait on the square domain $-20 \le x, y \le 20$. (To avoid numerical instability, you'll need to use a fairly small step size, because of the strong cubic nonlinearity.) Notice the trajectories seem to approach a certain curve as $t \to -\infty$; can you explain this behavior intuitively, and perhaps find an approximate equation for this curve?

6.3.10 (Dealing with a fixed point for which linearization is inconclusive) The goal of this exercise is to sketch the phase portrait for $\dot{x} = xy$, $\dot{y} = x^2 - y$.

a) Show that the linearization predicts that the origin is a non-isolated fixed point.

b) Show that the origin is in fact an isolated fixed point.

c) Is the origin repelling, attracting, a saddle, or what? Sketch the vector field along the nullclines and at other points in the phase plane. Use this information to sketch the phase portrait.

d) Plot a computer-generated phase portrait to check your answer to (c).

(Note: This problem can also be solved by a method called *center manifold theory*, as explained in Wiggins (1990) and Guckenheimer and Holmes (1983).)

6.3.11 (Nonlinear terms can change a star into a spiral) Here's another example that shows that borderline fixed points are sensitive to nonlinear terms. Consider the system in polar coordinates given by $\dot{r} = -r$, $\dot{\theta} = 1/\ln r$.

a) Find $r(t)$ and $\theta(t)$ explicitly, given an initial condition (r_0, θ_0).

b) Show that $r(t) \to 0$ and $|\theta(t)| \to \infty$ as $t \to \infty$. Therefore the origin is a stable spiral for the nonlinear system.

c) Write the system in x, y coordinates.

d) Show that the linearized system about the origin is $\dot{x} = -x$, $\dot{y} = -y$. Thus the origin is a stable star for the linearized system.

6.3.12 (Polar coordinates) Using the identity $\theta = \tan^{-1}(y/x)$, show that $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

6.3.13 (Another linear center that's actually a nonlinear spiral) Consider the system $\dot{x} = -y - x^3$, $\dot{y} = x$. Show that the origin is a spiral, although the linearization predicts a center.

6.3.14 Classify the fixed point at the origin for the system $\dot{x} = -y + ax^3$, $\dot{y} = x + ay^3$, for all real values of the parameter a.

6.3.15 Consider the system $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1 - \cos\theta$, where r, θ represent polar coordinates. Sketch the phase portrait and thereby show that the fixed point $r^* = 1$, $\theta^* = 0$ is attracting but not Liapunov stable.

6.3.16 (Saddle switching and structural stability) Consider the system $\dot{x} = a + x^2 - xy$, $\dot{y} = y^2 - x^2 - 1$, where a is a parameter.

a) Sketch the phase portrait for $a = 0$. Show that there is a trajectory connecting two saddle points. (Such a trajectory is called a *saddle connection.*)
b) With the aid of a computer if necessary, sketch the phase portrait for $a < 0$ and $a > 0$.

Notice that for $a \neq 0$, the phase portrait has a different topological character: the saddles are no longer connected by a trajectory. The point of this exercise is that the phase portrait in (a) is *not structurally stable*, since its topology can be changed by an arbitrarily small perturbation a.

6.3.17 (Nasty fixed point) The system $\dot{x} = xy - x^2y + y^3$, $\dot{y} = y^2 + x^3 - xy^2$ has a nasty higher-order fixed point at the origin. Using polar coordinates or otherwise, sketch the phase portrait.

6.4 Rabbits versus Sheep

Consider the following "rabbits vs. sheep" problems, where $x, y \geq 0$. Find the fixed points, investigate their stability, draw the nullclines, and sketch plausible phase portraits. Indicate the basins of attraction of any stable fixed points.

6.4.1 $\dot{x} = x(3 - x - y)$, $\dot{y} = y(2 - x - y)$

6.4.2 $\dot{x} = x(3 - 2x - y)$, $\dot{y} = y(2 - x - y)$

6.4.3 $\dot{x} = x(3 - 2x - 2y)$, $\dot{y} = y(2 - x - y)$

The next three exercises deal with competition models of increasing complexity. We assume $N_1, N_2 \geq 0$ in all cases.

6.4.4 The simplest model is $\dot{N}_1 = r_1N_1 - b_1N_1N_2$, $\dot{N}_2 = r_2N_2 - b_2N_1N_2$.

a) In what way is this model less realistic than the one considered in the text?
b) Show that by suitable rescalings of N_1, N_2, and t, the model can be nondimensionalized to $x' = x(1-y)$, $y' = y(\rho - x)$. Find a formula for the dimensionless group ρ.
c) Sketch the nullclines and vector field for the system in (b).
d) Draw the phase portrait, and comment on the biological implications.
e) Show that (almost) all trajectories are curves of the form $\rho \ln x - x = \ln y - y + C$. (Hint: Derive a differential equation for dx/dy, and separate the variables.) Which trajectories are not of the stated form?

6.4.5 Now suppose that species #1 has a finite carrying capacity K_1. Thus

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2 .$$

Nondimensionalize the model and analyze it. Show that there are two qualitatively different kinds of phase portrait, depending on the size of K_1. (Hint: Draw the nullclines.) Describe the long-term behavior in each case.

6.4.6 Finally, suppose that both species have finite carrying capacities:

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2 .$$

a) Nondimensionalize the model. How many dimensionless groups are needed?
b) Show that there are four qualitatively different phase portraits, as far as long-term behavior is concerned.
c) Find conditions under which the two species can stably coexist. Explain the biological meaning of these conditions. (Hint: The carrying capacities reflect the competition *within* a species, whereas the b's reflect the competition *between* species.)

6.4.7 (Two-mode laser) According to Haken (1983, p. 129), a two-mode laser produces two different kinds of photons with numbers n_1 and n_2. By analogy with the simple laser model discussed in Section 3.3, the rate equations are

$$\dot{n}_1 = G_1 N n_1 - k_1 n_1$$
$$\dot{n}_2 = G_2 N n_2 - k_2 n_2$$

where $N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2$ is the number of excited atoms. The parameters $G_1, G_2, k_1, k_2, \alpha_1, \alpha_2, N_0$ are all positive.
a) Discuss the stability of the fixed point $n_1{*} = n_2{*} = 0$.
b) Find and classify any other fixed points that may exist.
c) Depending on the values of the various parameters, how many qualitatively different phase portraits can occur? For each case, what does the model predict about the long-term behavior of the laser?

6.5 Conservative Systems

6.5.1 Consider the system $\ddot{x} = x^3 - x$.
a) Find all the equilibrium points and classify them.
b) Find a conserved quantity.
c) Sketch the phase portrait.

6.5.2 Consider the system $\ddot{x} = x - x^2$.
a) Find and classify the equilibrium points.

b) Sketch the phase portrait.
c) Find an equation for the homoclinic orbit that separates closed and nonclosed trajectories.

6.5.3 Find a conserved quantity for the system $\ddot{x} = a - e^x$, and sketch the phase portrait for $a < 0$, $a = 0$, and $a > 0$.

6.5.4 Sketch the phase portrait for the system $\ddot{x} = ax - x^2$ for $a < 0$, $a = 0$, and $a > 0$.

6.5.5 Investigate the stability of the equilibrium points of the system $\ddot{x} = (x-a)(x^2 - a)$ for all real values of the parameter a. (Hints: It might help to graph the right-hand side. An alternative is to rewrite the equation as $\ddot{x} = -V'(x)$ for a suitable potential energy function V and then use your intuition about particles moving in potentials.)

6.5.6 (Epidemic model revisited) In Exercise 3.7.6, you analyzed the Kermack–McKendrick model of an epidemic by reducing it to a certain first-order system. In this problem you'll see how much easier the analysis becomes in the phase plane. As before, let $x(t) \geq 0$ denote the size of the healthy population and $y(t) \geq 0$ denote the size of the sick population. Then the model is

$$\dot{x} = -kxy, \qquad \dot{y} = kxy - \ell y$$

where $k, \ell > 0$. (The equation for $z(t)$, the number of deaths, plays no role in the x, y dynamics so we omit it.)

a) Find and classify all the fixed points.
b) Sketch the nullclines and the vector field.
c) Find a conserved quantity for the system. (Hint: Form a differential equation for dy/dx. Separate the variables and integrate both sides.)
d) Plot the phase portrait. What happens as $t \to \infty$?
e) Let (x_0, y_0) be the initial condition. An *epidemic* is said to occur if $y(t)$ increases initially. Under what condition does an epidemic occur?

6.5.7 (General relativity and planetary orbits) The relativistic equation for the orbit of a planet around the sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$$

where $u = 1/r$ and r, θ are the polar coordinates of the planet in its plane of motion. The parameter α is positive and can be found explicitly from classical Newtonian mechanics; the term εu^2 is Einstein's correction. Here ε is a very small positive parameter.

a) Rewrite the equation as a system in the (u, v) phase plane, where $v = du/d\theta$.

b) Find all the equilibrium points of the system.
c) Show that one of the equilibria is a center in the (u, v) phase plane, according to the linearization. Is it a *nonlinear* center?
d) Show that the equilibrium point found in (c) corresponds to a circular planetary orbit.

Hamiltonian systems are fundamental to classical mechanics; they provide an equivalent but more geometric version of Newton's laws. They are also central to celestial mechanics and plasma physics, where dissipation can sometimes be neglected on the time scales of interest. The theory of Hamiltonian systems is deep and beautiful, but perhaps too specialized and subtle for a first course on nonlinear dynamics. See Arnold (1978), Lichtenberg and Lieberman (1992), Tabor (1989), or Hénon (1983) for introductions.

Here's the simplest instance of a Hamiltonian system. Let $H(p,q)$ be a smooth, real-valued function of two variables. The variable q is the "generalized coordinate" and p is the "conjugate momentum." (In some physical settings, H could also depend explicitly on time t, but we'll ignore that possibility.) Then a system of the form

$$\dot{q} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial q$$

is called a ***Hamiltonian system*** and the function H is called the ***Hamiltonian***. The equations for $\dot{q}$ and $\dot{p}$ are called Hamilton's equations.

The next three exercises concern Hamiltonian systems.

6.5.8 (Harmonic oscillator) For a simple harmonic oscillator of mass m, spring constant k, displacement x, and momentum p, the Hamiltonian is $H = \frac{p^2}{2m} + \frac{kx^2}{2}$. Write out Hamilton's equations explicitly. Show that one equation gives the usual definition of momentum and the other is equivalent to $F = ma$. Verify that H is the total energy.

6.5.9 Show that for any Hamiltonian system, $H(x,p)$ is a conserved quantity. (Hint: Show $\dot{H} = 0$ by applying the chain rule and invoking Hamilton's equations.) Hence the trajectories lie on the contour curves $H(x,p) = C$.

6.5.10 (Inverse-square law) A particle moves in a plane under the influence of an inverse-square force. It is governed by the Hamiltonian $H(p,r) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r}$ where $r > 0$ is the distance from the origin and p is the radial momentum. The parameters h and k are the angular momentum and the force constant, respectively.
a) Suppose $k > 0$, corresponding to an attractive force like gravity. Sketch the

phase portrait in the (r, p) plane. (Hint: Graph the "effective potential" $V(r) = h^2/2r^2 - k/r$ and then look for intersections with horizontal lines of height E. Use this information to sketch the contour curves $H(p, r) = E$ for various positive and negative values of E.)

b) Show that the trajectories are closed if $-k^2/2h^2 < E < 0$, in which case the particle is "captured" by the force. What happens if $E > 0$? What about $E = 0$?

c) If $k < 0$ (as in electric repulsion), show that there are no periodic orbits.

6.5.11 (Basins for damped double-well oscillator) Suppose we add a small amount of damping to the double-well oscillator of Example 6.5.2. The new system is $\dot{x} = y$, $\dot{y} = -by + x - x^3$, where $0 < b << 1$. Sketch the basin of attraction for the stable fixed point $(x^*, y^*) = (1, 0)$. Make the picture large enough so that the global structure of the basin is clearly indicated.

6.5.12 (Why we need to assume *isolated* minima in Theorem 6.5.1) Consider the system $\dot{x} = xy$, $\dot{y} = -x^2$.

a) Show that $E = x^2 + y^2$ is conserved.

b) Show that the origin is a fixed point, but not an isolated fixed point.

c) Since E has a local minimum at the origin, one might have thought that the origin has to be a center. But that would be a misuse of Theorem 6.5.1; the theorem does not apply here because the origin is *not* an isolated fixed point. Show that in fact the origin is not surrounded by closed orbits, and sketch the actual phase portrait.

6.5.13 (Nonlinear centers)

a) Show that the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$ has a nonlinear center at the origin for all $\varepsilon > 0$.

b) If $\varepsilon < 0$, show that all trajectories near the origin are closed. What about trajectories that are far from the origin?

6.5.14 (Glider) Consider a glider flying at speed v at an angle θ to the horizontal. Its motion is governed approximately by the dimensionless equations

$$\dot{v} = -\sin\theta - Dv^2$$
$$v\dot{\theta} = -\cos\theta + v^2$$

where the trigonometric terms represent the effects of gravity and the v^2 terms represent the effects of drag and lift.

a) Suppose there is no drag ($D = 0$). Show that $v^3 - 3v\cos\theta$ is a conserved quantity. Sketch the phase portrait in this case. Interpret your results physically—what does the flight path of the glider look like?

b) Investigate the case of positive drag ($D > 0$).

In the next four exercises, we return to the problem of a bead on a rotating hoop, discussed in Section 3.5. Recall that the bead's motion is governed by

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi .$$

Previously, we could only treat the overdamped limit. The next four exercises deal with the dynamics more generally.

6.5.15 (Frictionless bead) Consider the undamped case $b = 0$.

a) Show that the equation can be nondimensionalized to $\phi'' = \sin\phi(\cos\phi - \gamma^{-1})$, where $\gamma = r\omega^2/g$ as before, and prime denotes differentiation with respect to dimensionless time $\tau = \omega t$.
b) Draw all the qualitatively different phase portraits as γ varies.
c) What do the phase portraits imply about the physical motion of the bead?

6.5.16 (Small oscillations of the bead) Return to the original dimensional variables. Show that when $b = 0$ and ω is sufficiently large, the system has a symmetric pair of stable equilibria. Find the approximate frequency of small oscillations about these equilibria. (Please express your answer with respect to t, not τ.)

6.5.17 (A puzzling constant of motion for the bead) Find a conserved quantity when $b = 0$. You might think that it's essentially the bead's total energy, but it isn't! Show explicitly that the bead's kinetic plus potential energy is *not* conserved. Does this make sense physically? Can you find a physical interpretation for the conserved quantity? (Hint: Think about reference frames and moving constraints.)

6.5.18 (General case for the bead) Finally, allow the damping b to be arbitrary. Define an appropriate dimensionless version of b, and plot all the qualitatively different phase portraits that occur as b and γ vary.

6.5.19 (Rabbits vs. foxes) The model $\dot{R} = aR - bRF$, $\dot{F} = -cF + dRF$ is the ***Lotka–Volterra predator-prey model***. Here $R(t)$ is the number of rabbits, $F(t)$ is the number of foxes, and $a, b, c, d > 0$ are parameters.

a) Discuss the biological meaning of each of the terms in the model. Comment on any unrealistic assumptions.
b) Show that the model can be recast in dimensionless form as $x' = x(1-y)$, $y' = \mu y(x-1)$.
c) Find a conserved quantity in terms of the dimensionless variables.
d) Show that the model predicts *cycles* in the populations of both species, for almost all initial conditions.

This model is popular with many textbook writers because it's simple, but some are beguiled into taking it too seriously. Mathematical biologists dismiss the Lotka–Volterra model because it is not structurally stable, and because real predator-prey cycles typically have a characteristic amplitude. In other words, realistic models should predict a *single* closed orbit, or perhaps finitely many, but not a continuous family of neutrally stable cycles. See the discussions in May (1972), Edelstein–Keshet (1988), or Murray (1989).

6.6 Reversible Systems

Show that each of the following systems is reversible, and sketch the phase portrait.

6.6.1 $\dot{x} = y(1-x^2)$, $\dot{y} = 1 - y^2$

6.6.2 $\dot{x} = y$, $\dot{y} = x\cos y$

6.6.3 (Wallpaper) Consider the system $\dot{x} = \sin y$, $\dot{y} = \sin x$.

a) Show that the system is reversible.
b) Find and classify all the fixed points.
c) Show that the lines $y = \pm x$ are invariant (any trajectory that starts on them stays on them forever).
d) Sketch the phase portrait.

6.6.4 (Computer explorations) For each of the following reversible systems, try to sketch the phase portrait by hand. Then use a computer to check your sketch. If the computer reveals patterns you hadn't anticipated, try to explain them.

a) $\ddot{x} + (\dot{x})^2 + x = 3$ b) $\dot{x} = y - y^3$, $\dot{y} = x\cos y$ c) $\dot{x} = \sin y$, $\dot{y} = y^2 - x$

6.6.5 Consider equations of the form $\ddot{x} + f(\dot{x}) + g(x) = 0$, where f is an even function, and both f and g are smooth.

a) Show that the equation is invariant under the pure time-reversal symmetry $t \to -t$.
b) Show that the equilibrium points cannot be stable nodes or spirals.

6.6.6 (Manta ray) Use qualitative arguments to deduce the "manta ray" phase portrait of Example 6.6.1.

a) Plot the nullclines $\dot{x} = 0$ and $\dot{y} = 0$.
b) Find the sign of $\dot{x}$, $\dot{y}$ in different regions of the plane.
c) Calculate the eigenvalues and eigenvectors of the saddle points at $(-1, \pm 1)$.
d) Consider the unstable manifold of $(-1,-1)$. By making an argument about the signs of $\dot{x}$, $\dot{y}$, prove that this unstable manifold intersects the negative x-axis. Then use reversibility to prove the existence of a heteroclinic trajectory connecting $(-1,-1)$ to $(-1,1)$.
e) Using similar arguments, prove that another heteroclinic trajectory exists, and sketch several other trajectories to fill in the phase portrait.

6.6.7 (Oscillator with both positive and negative damping) Show that the system $\ddot{x} + x\dot{x} + x = 0$ is reversible and plot the phase portrait.

6.6.8 (Reversible system on a cylinder) While studying chaotic streamlines inside a drop immersed in a steady Stokes flow, Stone et al. (1991) encountered the system

$$\dot{x} = \tfrac{\sqrt{2}}{4}x(x-1)\sin\phi, \qquad \dot{\phi} = \tfrac{1}{2}\left[\beta - \tfrac{1}{\sqrt{2}}\cos\phi - \tfrac{1}{8\sqrt{2}}x\cos\phi\right]$$

where $0 \le x \le 1$ and $-\pi \le \phi < \pi$.

Since the system is 2π-periodic in ϕ, it may be considered as a vector field on a *cylinder*. (See Section 6.7 for another vector field on a cylinder.) The x-axis runs along the cylinder, and the ϕ-axis wraps around it. Note that the cylindrical phase space is finite, with edges given by the circles $x = 0$ and $x = 1$.

a) Show that the system is reversible.

b) Verify that for $\frac{9}{8\sqrt{2}} > \beta > \frac{1}{\sqrt{2}}$, the system has three fixed points on the cylinder, one of which is a saddle. Show that this saddle is connected to itself by a homoclinic orbit that winds around the waist of the cylinder. Using reversibility, prove that there is a *band of closed orbits* sandwiched between the circle $x = 0$ and the homoclinic orbit. Sketch the phase portrait on the cylinder, and check your results by numerical integration.

c) Show that as $\beta \to \frac{1}{\sqrt{2}}$ from above, the saddle point moves toward the circle $x = 0$, and the homoclinic orbit tightens like a noose. Show that all the closed orbits disappear when $\beta = \frac{1}{\sqrt{2}}$.

d) For $0 < \beta < \frac{1}{\sqrt{2}}$, show that there are two saddle points on the edge $x = 0$. Plot the phase portrait on the cylinder.

6.6.9 (Josephson junction array) As discussed in Exercises 4.6.4 and 4.6.5, the equations

$$\frac{d\phi_k}{d\tau} = \Omega + a\sin\phi_k + \frac{1}{N}\sum_{j=1}^{N}\sin\phi_j, \quad \text{for } k = 1, 2,$$

arise as the dimensionless circuit equations for a resistively loaded array of Josephson junctions.

a) Let $\theta_k = \phi_k - \frac{\pi}{2}$, and show that the resulting system for θ_k is reversible.

b) Show that there are four fixed points (mod 2π) when $|\Omega/(a+1)| < 1$, and none when $|\Omega/(a+1)| > 1$.

c) Using the computer, explore the various phase portraits that occur for $a = 1$, as Ω varies over the interval $0 \le \Omega \le 3$.

For more about this system, see Tsang et al. (1991).

6.6.10 Is the origin a nonlinear center for the system $\dot{x} = -y - x^2$, $\dot{y} = x$?

6.6.11 (Rotational dynamics and a phase portrait on a sphere) The rotational dynamics of an object in a shear flow are governed by

$$\dot{\theta} = \cot\phi \cos\theta\,, \qquad \dot{\phi} = (\cos^2\phi + A\sin^2\phi)\sin\theta\,,$$

where θ and ϕ are spherical coordinates that describe the orientation of the object. Our convention here is that $-\pi < \theta \le \pi$ is the "longitude," i.e., the angle around the z-axis, and $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ is the "latitude," i.e., the angle measured northward from the equator. The parameter A depends on the shape of the object.

a) Show that the equations are reversible in two ways: under $t \to -t$, $\theta \to -\theta$ and under $t \to -t$, $\phi \to -\phi$.
b) Investigate the phase portraits when A is positive, zero, and negative. You may sketch the phase portraits as Mercator projections (treating θ and ϕ as rectangular coordinates), but it's better to visualize the motion on the sphere, if you can.
c) Relate your results to the tumbling motion of an object in a shear flow. What happens to the orientation of the object as $t \to \infty$?

6.7 Pendulum

6.7.1 (Damped pendulum) Find and classify the fixed points of $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ for all $b > 0$, and plot the phase portraits for the qualitatively different cases.

6.7.2 (Pendulum driven by constant torque) The equation $\ddot{\theta} + \sin\theta = \gamma$ describes the dynamics of an undamped pendulum driven by a constant torque, or an undamped Josephson junction driven by a constant bias current.

a) Find all the equilibrium points and classify them as γ varies.
b) Sketch the nullclines and the vector field.
c) Is the system conservative? If so, find a conserved quantity. Is the system reversible?
d) Sketch the phase portrait on the plane as γ varies.
e) Find the approximate frequency of small oscillations about any centers in the phase portrait.

6.7.3 (Nonlinear damping) Analyze $\ddot{\theta} + (1 + a\cos\theta)\dot{\theta} + \sin\theta = 0$, for all $a \ge 0$.

6.7.4 (Period of the pendulum) Suppose a pendulum governed by $\ddot{\theta} + \sin\theta = 0$ is swinging with an amplitude α. Using some tricky manipulations, we are going to derive a formula for $T(\alpha)$, the period of the pendulum.

a) Using conservation of energy, show that $\dot{\theta}^2 = 2(\cos\theta - \cos\alpha)$ and hence that

$$T = 4\int_0^{\alpha} \frac{d\theta}{\left[2(\cos\theta - \cos\alpha)\right]^{1/2}}.$$

b) Using the half-angle formula, show that $T = 4\displaystyle\int_0^{\alpha} \frac{d\theta}{\left[4(\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta)\right]^{1/2}}$.
c) The formulas in parts (a) and (b) have the disadvantage that α appears in both the integrand and the upper limit of integration. To remove the α-dependence

from the limits of integration, we introduce a new angle ϕ that runs from 0 to $\frac{\pi}{2}$ when θ runs from 0 to α. Specifically, let $(\sin\frac{1}{2}\alpha)\sin\phi = \sin\frac{1}{2}\theta$. Using this substitution, rewrite (b) as an integral with respect to ϕ. Thereby derive the exact result

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\cos\frac{1}{2}\theta} = 4K(\sin^2\tfrac{1}{2}\alpha),$$

where the *complete elliptic integral of the first kind* is defined as

$$K(m) = \int_0^{\pi/2} \frac{d\phi}{(1-m\sin^2\phi)^{1/2}}, \quad \text{for } 0 \le m < 1.$$

d) By expanding the elliptic integral using the binomial series and integrating term-by-term, show that

$$T(\alpha) = 2\pi\left[1+\tfrac{1}{16}\alpha^2 + O(\alpha^4)\right] \text{ for } \alpha \ll 1.$$

Note that larger swings take longer.

6.7.5 (Numerical solution for the period) Redo Exercise 6.7.4 using either numerical integration of the differential equation, or numerical evaluation of the elliptic integral. Specifically, compute the period $T(\alpha)$, where α runs from 0 to 180° in steps of 10°.

6.8 Index Theory

6.8.1 Show that each of the following fixed points has an index equal to +1.
a) stable spiral b) unstable spiral c) center d) star e) degenerate node

(Unusual fixed points) For each of the following systems, locate the fixed points and calculate the index. (Hint: Draw a small closed curve C around the fixed point and examine the variation of the vector field on C.)

6.8.2 $\dot{x} = x^2,\ \dot{y} = y$

6.8.3 $\dot{x} = y - x,\ \dot{y} = x^2$

6.8.4 $\dot{x} = y^3,\ \dot{y} = x$

6.8.5 $\dot{x} = xy,\ \dot{y} = x + y$

6.8.6 A closed orbit in the phase plane encircles S saddles, N nodes, F spirals, and C centers, all of the usual type. Show that $N + F + C = 1 + S$.

6.8.7 (Ruling out closed orbits) Use index theory to show that the system $\dot{x} = x(4 - y - x^2)$, $\dot{y} = y(x-1)$ has no closed orbits.

6.8.8 A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, say C_1 and C_2, lie inside the third cycle C_3. However, C_1 does not lie inside C_2, nor vice-versa.
a) Sketch the arrangement of the three cycles.

b) Show that there must be at least one fixed point in the region bounded by C_1, C_2, C_3.

6.8.9 A smooth vector field on the phase plane is known to have exactly two closed trajectories, one of which lies inside the other. The inner cycle runs clockwise, and the outer one runs counterclockwise. True or False: There must be at least one fixed point in the region between the cycles. If true, prove it. If false, provide a simple counterexample.

6.8.10 (Open-ended question for the topologically minded) Does Theorem 6.8.2 hold for surfaces other than the plane? Check its validity for various types of closed orbits on a torus, cylinder, and sphere.

6.8.11 (Complex vector fields) Let $z = x + iy$. Explore the complex vector fields $\dot{z} = z^k$ and $\dot{z} = (\bar{z})^k$, where $k > 0$ is an integer and $\bar{z} = x - iy$ is the complex conjugate of z.

a) Write the vector fields in both Cartesian and polar coordinates, for the cases $k = 1, 2, 3$.
b) Show that the origin is the only fixed point, and compute its index.
c) Generalize your results to arbitrary integer $k > 0$.

6.8.12 ("Matter and antimatter") There's an intriguing analogy between bifurcations of fixed points and collisions of particles and anti-particles. Let's explore this in the context of index theory. For example, a two-dimensional version of the saddle-node bifurcation is given by $\dot{x} = a + x^2$, $\dot{y} = -y$, where a is a parameter.

a) Find and classify all the fixed points as a varies from $-\infty$ to $+\infty$.
b) Show that the sum of the indices of all the fixed points is conserved as a varies.
c) State and prove a generalization of this result, for systems of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a)$, where $\mathbf{x} \in \mathbf{R}^2$ and a is a parameter.

6.8.13 (Integral formula for the index of a curve) Consider a smooth vector field $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ on the plane, and let C be a simple closed curve that does not pass through any fixed points. As usual, let $\phi = \tan^{-1}(\dot{y}/\dot{x})$ as in Figure 6.8.1.

a) Show that $d\phi = (f\,dg - g\,df)/(f^2 + g^2)$.
b) Derive the integral formula

$$I_C = \frac{1}{2\pi} \oint_C \frac{f\,dg - g\,df}{f^2 + g^2}.$$

6.8.14 Consider the family of linear systems $\dot{x} = x\cos\alpha - y\sin\alpha$, $\dot{y} = x\sin\alpha + y\cos\alpha$, where α is a parameter that runs over the range $0 \le \alpha \le \pi$. Let C be a simple closed curve that does not pass through the origin.

a) Classify the fixed point at the origin as a function of α.

b) Using the integral derived in Exercise 6.8.13, show that I_C is *independent* of α.

c) Let C be a circle centered at the origin. Compute I_C explicitly by evaluating the integral for any convenient choice of α.