Mathematical Methods in Engineering and Applied Science
Problem Set 10.

(1). Solve the initial value problem for the advection equation

$$u_t + (1-t)u_x = 0, \qquad t > 0, x \in R$$

$$u(x,0) = \frac{1}{1+x^2}$$

Plot the characteristic curves as well as the solution u(x,t) at several different times.

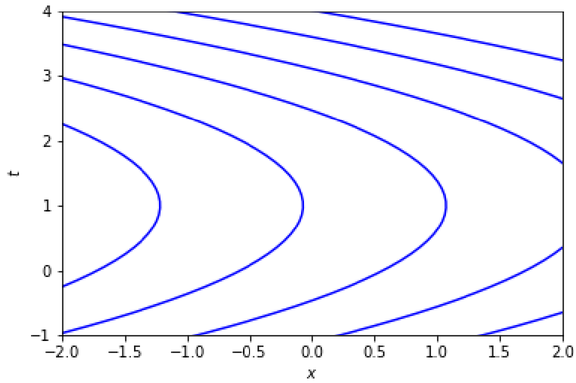
$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = 1-t, \qquad \frac{du}{ds} = 0$$

$$dt = ds, \qquad x = t - \frac{t^2}{2} + c$$

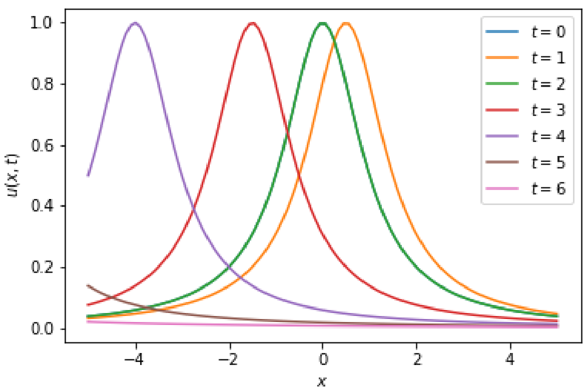
For initial: $x = p, \ \ t = 0, \ u(p,0) = \frac{1}{1+p^2}$:

$$x = t - \frac{t^2}{2} + p, \qquad u = \frac{1}{1+p^2} = \frac{1}{1+\left(x - t + \frac{t^2}{2}\right)^2}$$

Characteristics:



Solutions:



$u(x,0) = u(x,2)$, therefore $u(x,0)$ figure under $u(x,2)$

(2). Use the method of characteristics to solve the initial-boundary value problem:

$$u_t + u_x = t + x, \qquad t > 0, x > 0$$

$$u(x,0) = 0$$

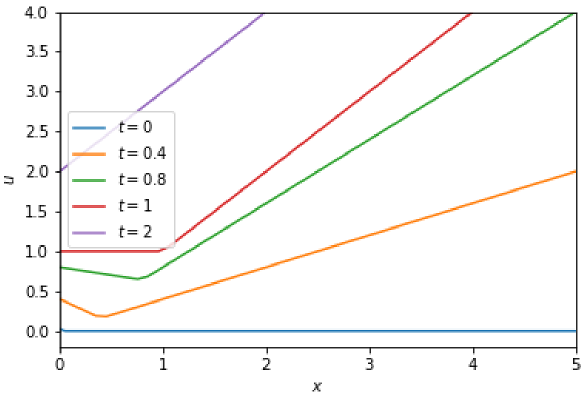
$$u(0,t) = t$$

Plot the characteristic curves as well as the solution u(x,t) at several different times.

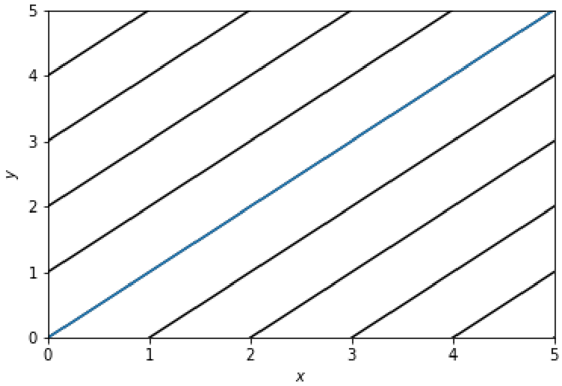
$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = 1, \qquad \frac{du}{ds} = t + x$$

$u(x,0) = 0,\ x > t, =>$initial: $t = 0, x = p, u = 0;$
$t = s, \qquad x = s + p$
$$\frac{du}{ds} = s + s + p => u = s^2 + ps = t^2 + (x - t)t = xt$$
$u = xt$

$u(0,t) = 0,\ x < t, =>$boundary: $t = p, x = 0, u = p;$
$t = s + p, \qquad x = s, => t = x + p$
$$\frac{du}{ds} = s + p + s => u = s^2 + ps + p = x^2 + (t - x)x + t - x = tx + t - x$$
$u = tx + t - x$
Summing up:
Solution:
$$u = \begin{matrix} xt + t - x, & x < t \\ xt, & x > t \end{matrix}$$



Characteristics:
$x = t + p, \qquad x > t$
$t = x + p, \qquad x < t$



(3). Solve the initial value problem for the Hopf equation:
$$u_t + uu_x = 0, \qquad x \in R, t > 0, \qquad u(x,0) = \begin{matrix} 1, & 0 \le x \le 1 \\ 0, & otherwise \end{matrix}$$

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = u, \qquad \frac{du}{ds} = 0$$
This equation means that u is speed.
From the initial cond. we have shock at $x = 1$, and fan at $x = 0$.
Shock speed:
$$\dot{s} = \frac{u_+ + u_-}{2} = \frac{1}{2} + \frac{0}{2} = \frac{1}{2} => s = \frac{1}{2}t + p$$

Position of shock at time =0 is 1, therefore $s = \frac{1}{2}t + 1$

The fan function: $u = \frac{x}{t}$, it intersects shock when: $u = 1$=>

For fan: $x = t$

For shock: $x = \frac{1}{2}t + 1$(- right side).

Solution is $x = 2, t = 2$

After this shock speed is $\dot{s} = \frac{u_+ + u_-}{2} = \frac{\frac{s}{t}+0}{2} = \frac{s}{2t} \Rightarrow \ln(s) = \ln(c\sqrt{t}) \Rightarrow s = c\sqrt{t}$

But when $t = 2$ $then$ $s = 2$,=>$s = \sqrt{2t}$

Summing up:

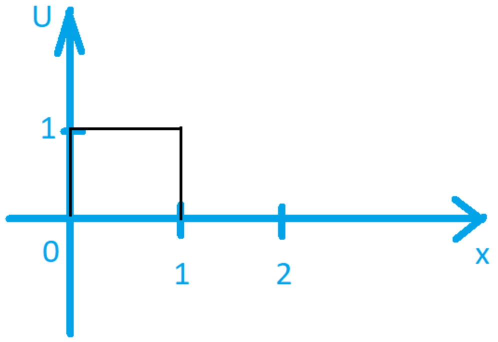
$u(x,t) = 0$: if $x \leq 0$ – before fan, $(x > \sqrt{2t}, t > 2)$ and $(x < 1 + \frac{t}{2}, t > 2)$- after shock.

$u(x,t) = 1$: only on the start: $(t < x < 1 + \frac{t}{2}, t < 2)$

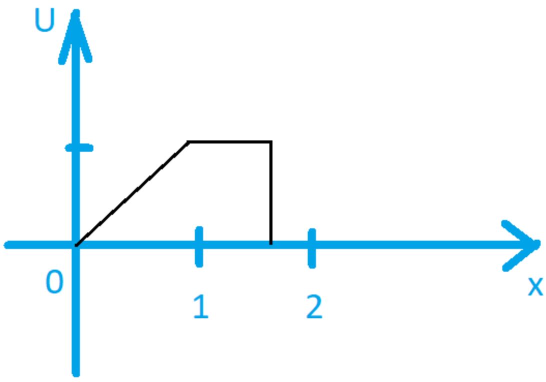
$u(x,t) = \frac{x}{t}$-otherwise

Let's Plot:

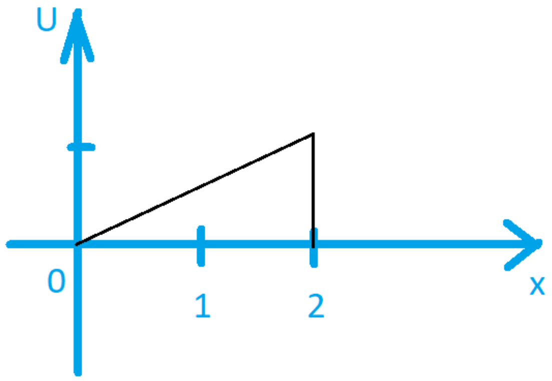
$t = 0$



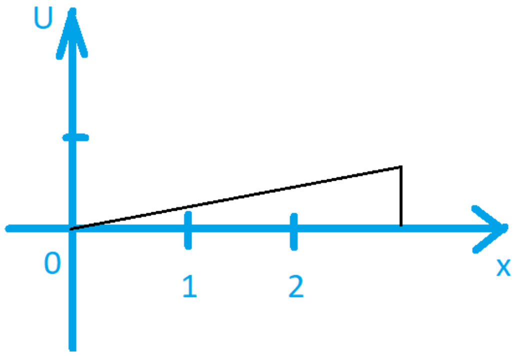
$t < 2$



$t = 2$



$t > 2$



(4). Determine the traveling wave solutions $u = U(z), z = x - ct$, of the problem

$$\left(u + \frac{u^2}{2}\right)_t = u_{xx} - u_x$$

with $U(-\infty) = 0, U(\infty) = 1, U(0) = \frac{1}{2}$.

Let's $u = U(z)$

$$\frac{d\left(U + \frac{U^2}{2}\right)}{dt} = \frac{d\left(U + \frac{U^2}{2}\right)}{dz}\frac{dz}{dt} = -c\frac{d}{dz}\left(U + \frac{U^2}{2}\right)$$

$$\frac{dx}{dz} = 1 \Rightarrow u_{xx} - u_x = U_{zz} - U_z$$

$$-c\frac{d}{dz}\left(U + \frac{U^2}{2}\right) = U_{zz} - U_z$$

$$-c\left(U + \frac{U^2}{2}\right) = U_z - U + C$$

as far as $U'(\pm\infty) = 0$ then:

since $U(-\infty) = 0 \Rightarrow -c(0) = 0 - 0 + C \Rightarrow C = 0$

since $U(\infty) = 1 \Rightarrow -c\left(1 + \frac{1}{2}\right) = 0 - 1 \Rightarrow c = \frac{2}{3}$

$$-\frac{2}{3}\left(U + \frac{U^2}{2}\right) = U_z - U$$

$$U_z = -\frac{2}{3}\left(U + \frac{U^2}{2}\right) + U = \frac{1}{3}U(1 - U)$$

$$\frac{dU}{U(1-U)} = \frac{1}{3}dz$$

$$\int \frac{dU}{U(1-U)} = \int \frac{1}{U} + \frac{1}{1-U}dU = \ln(U) - \ln(1-U) = \ln\left(\frac{U}{1-U}\right)$$

$$\ln\left(\frac{U}{1-U}\right) + \ln(A) = \frac{1}{3}z$$

$$\frac{U}{1-U}A = \exp\left(\frac{1}{3}z\right)$$

Since $U\,(0) = \frac{1}{2}$ then:

$$\frac{1/2}{1-1/2}A = \exp(0) \Rightarrow A = 1$$

$$\frac{U}{1-U} = \exp\left(\frac{1}{3}z\right)$$

Finally:

$$U = \frac{\exp\left(\frac{1}{3}z\right)}{1+\exp\left(\frac{1}{3}z\right)}$$

Where $z = x - ct, c = \frac{2}{3}$

(5). Consider the reaction-diffusion system

$$u_t = Du_{xx} + u + v, 0 < x < \pi$$
$$v_t = 3u - v,$$

with no-flux boundary conditions.

(a) Analyze first the spatially homogeneous case with D = 0.

$$u_t = u + v, \quad 0 < x < \pi$$
$$v_t = 3u - v,$$
$$J = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

Fixed point: (0,0)

$$\tau = 0, \Delta = -4 \Rightarrow sadle\ point$$

Nullclines: $(u = -v), (v = 3u)$

(b) Determine the growth rate σ of the normal modes, $w = [u, v]^T = a\,(t)\,cos\,nx, n = 0, 1, 2, \ldots$

Lecture formula:$\det(J - \sigma I - n^2 A)$, where $J = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, A = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$

$$\det(J - \sigma I - n^2 A) = det\begin{bmatrix} 1-\sigma-n^2 D & 1 \\ 3 & -1-\sigma \end{bmatrix} = (1-\sigma-n^2 D)(-1-\sigma) - 3$$
$$= \sigma^2 + \sigma n^2 D + n^2 D - 4 = 0$$

Find roots:

$$\sigma_{1,2} = \frac{-n^2 D \pm \sqrt{n^4 D^2 - 4n^2 D + 16}}{2}$$

(c) For a given D, which modes are unstable? Discuss the behavior at large D and at small D.
Unstable modes if $Re(\sigma) > 0$.
Consider:
$n^4 D^2 - 4n^2 D + 16$ which is always > 0 =>$-n^2 D - \sqrt{n^4 D^2 - 4n^2 D + 16} < 0$

$$Re(\sigma_2) < 0$$
$$-n^2 D + \sqrt{n^4 D^2 - 4n^2 D + 16} < 0\ if\ n > \frac{2}{\sqrt{D}}$$

Modes unstable if $n < \frac{2}{\sqrt{D}}$ and $Re(\sigma_1) > 0$.

For large D all modes are stable(except $n = 0$), because $n < \frac{2}{\sqrt{D}} \ll 1$

For small D number of unstable modes more, because $\frac{2}{\sqrt{D}} \gg 1$

(d) What is the largest value of D such that spatially non-uniform perturbations grow with time?
$\frac{2}{\sqrt{D}} = 1\ if\ D = 4 \Rightarrow$ spatially non-uniform perturbations grow with time if $D < 4$

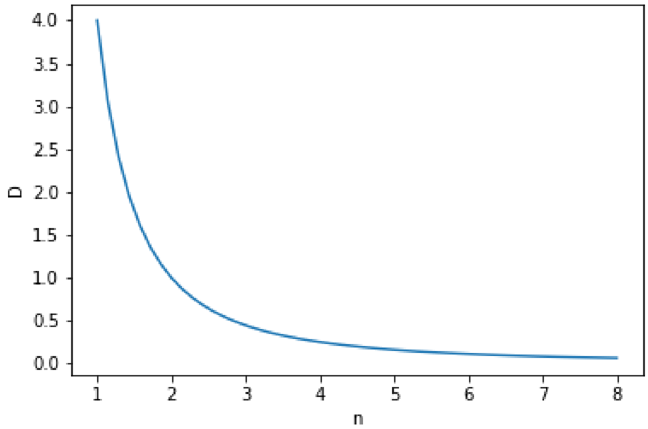
(e) Plot the neutral curve, i.e. D (n) dependence for zero growth rate, Re(σ) = 0, and indicate the regions in the D − n plane where the solution is stable and where it is unstable

As we found before $\sigma^2 + \sigma n^2 D + n^2 D - 4 = 0$

$$Re(\sigma) = 0 => \sigma = 0 => n^2 D - 4 = 0$$

Let's plot

$$D = \frac{4}{n^2}$$



Under line zone of instability

Above line zone of stability.

(6). (Extra credit). Equation $u_t + c(x,t)u_x = 0$ describes variable-speed advection in a certain non-uniform medium. Explain how to solve it by the method of characteristics for general $c(x,t)$ and initial data $u(x,0) = f(x)$. Next, specialize to $c = 1 + \epsilon \sin x$ with small parameter $\epsilon \to 0$ and find an explicit form of the solution including terms up to $O(\epsilon)$. Let $f = e^{-x^2}$ and plot the solution at $\epsilon = 0.1$ at several different t or as a surface in the (x,t)-plane. What happens if $c = 1 + \sin x$?

$$u_t + c(x,t)u_x = 0$$

$\dfrac{dt}{ds} = 1, \dfrac{dx}{ds} = c(x,t), \dfrac{du}{ds} = 0$

$u(x,0) = f(x)$=>initial: $t = 0, x = p, u = f(p)$

$\dfrac{dt}{ds} = 1 => t = s$

$\frac{dx}{ds} = c(x,s)$ =>solve this with respect to x, and we get $x = w(s) - w(0) + p$

$p = x - w(s) + w(0)$

$\dfrac{du}{ds} = 0 => u = const(s) = f(p)$

$u = f\big(x - w(t) + w(0)\big), where\ w(s)\ is\ the\ solution\ of\ \dfrac{dx}{ds} = c(x,s)$