Mathematical Methods in Engineering and Applied Science

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Problem Set 1. Due on Oct. 9 at 23:59. ....

(1) Some basic problems on matrix/vector multiplication.

(a) Calculate by hand the following matrix/vector products:

(i) $\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$;

(ii) $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}$ as a combination of columns of the left matrix as well as a combination of rows of the right matrix.

(b) Write down a permutation matrix P_4 that exchanges row 1 with row 3 and row 2 with row 4. What is the connection of this matrix with the permutation matrices that exchange only row 1 and row 3, and only row 2 and row 4?

(2) Given a 3×3 matrix $A = [a_1\, a_2\, a_3]$ with columns a_i, find a matrix B that when multiplied with A, either from left or right, performs the following operations with A:

(a) exchanges row 1 and row 2;

(b) exchanges columns 1 and 2;

(c) doubles the first row;

(d) subtracts twice row 1 from row 2. Also find the inverse of this matrix. What does the inverse of this B do?

(3) For matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$, determine the following:

(a) rank;

(b) eigenvalues and eigenvectors;

(c) nullspace and left nullspace;

(d) column space and row space;

(e) write A as a sum of rank-1 matrices in at least two different ways.

(4) The columns of matrix $C = \begin{bmatrix} 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$ represent vertices of a cube. Describe transformations of the cube that result from the action on C of the following three matrices:

$$A_1 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Relate the results to the ranks of A_k and to the dimensions and bases of the four fundamental subspaces of A_k. Is there a 3×3 matrix A that can transform a cube into a tetrahedron? Explain.

(5) This problem explores some properties of eigenvalues and eigenvectors. For matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ determine which unit vector x_M is stretched the most and which x_m the least and by how much. That is, find x such that $y = Ax$ has the largest (or smallest) possible Euclidian length. You can do this by calculus methods, e.g. using Lagrange multipliers. Relate your findings to eigenvalues and eigenvectors of A.

(6) Find eigenvalues and eigenvectors of the following matrices:

(a) $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If x is any real vector, how is $y = A_1 x$ related to x geometrically?

(b) $A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. What is the rank of A_2? How many eigenvectors are there?