Homework 2. Continuum mechanics

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(1). Problem 1

A Lagrangian description of continuum motion is

$x_1 = k_1\xi_2^2t^2 + \xi_1, \qquad x_2 = k_2\xi_2 t + \xi_2, \qquad x_3 = \xi_3 t.$

where $k1$ and $k2$ are constants. Find the velocity and acceleration fields using

(a) Lagrangian description

$\dot{x}_1 = 2k_1\xi_2^2 t, \qquad \dot{x}_2 = k_2\xi_2, \qquad \dot{x_3} = \xi_3$

$\ddot{x}_1 = 2k_1\xi_2^2, \qquad \ddot{x}_2 = 0, \qquad \ddot{x}_3 = 0$

(b) Eulerian description

$$\xi_1 = x_1 - k_1\xi_2^2t^2, \qquad \xi_2 = \frac{x_2}{1+k_2t}, \qquad \xi_3 = \frac{x_3}{t}$$

$$\xi_1 = x_1 - k_1t^2\left(\frac{x_2}{1+k_2t}\right)^2$$

$$\dot{x}_1 = 2k_1\left(\frac{x_2}{1+k_2t}\right)^2 t, \qquad \dot{x}_2 = k_2\frac{x_2}{1+k_2t}, \qquad \dot{x_3} = \frac{x_3}{t}$$

$$\ddot{x}_1 = 2k_1\left(\frac{x_2}{1+k_2t}\right)^2, \qquad \ddot{x}_2 = 0, \qquad \ddot{x}_3 = 0$$

(2). Problem 2

Introduce Lagrangian coordinates and find a Lagrangian description of continuum motion if its velocity field is represented with the following equations:

$$v_1 = \frac{x_1}{t+r}, \qquad v_2 = \frac{2tx_2}{t^2+r^2}, \qquad v_3 = \frac{3t^2x_3}{t^3+r^3}, \qquad r = const > 0$$

$$\frac{dx_1}{dt} = \frac{x_1}{t+r}, \qquad \frac{dx_1}{x_1} = \frac{dt}{t+r}, \qquad \int\frac{dx_1}{x_1} = \int\frac{dt}{t+r}, \qquad \ln(x_1) = \ln(t+r) + \ln(C_1),$$

$$x_1 = (t+r)C_1$$

$$\int\frac{dx_2}{x_2} = \int\frac{2tdt}{t^2+r^2}, \qquad \begin{bmatrix} u = t^2+r^2 \\ du = 2tdt \end{bmatrix}, \ln(x_2) = \int\frac{du}{u}, \qquad \ln(x_2) = \ln(t^2+r^2) + \ln(C_2)$$

$$x_2 = (t^2+r^2)C_2$$

$$\int\frac{dx_3}{x_3} = \int\frac{3t^2dt}{t^3+r^3}, \qquad \begin{bmatrix} u = t^3+r^3 \\ du = 3t^2dt \end{bmatrix}, \ln(x_3) = \int\frac{du}{u}, \qquad \ln(x_3) = \ln(t^3+r^3) + \ln(C_3)$$

$$x_3 = (t^3+r^3)C_3$$

subjected to initial conditions:

$$x_1(0) = rC_1 = \xi_1 \Rightarrow C_1 = \frac{\xi_1}{r}$$

$$x_2(0) = r^2C_2 = \xi_2 \Rightarrow C_2 = \frac{\xi_2}{r^2}$$

$$x_3(0) = r^3C_3 = \xi_3 \Rightarrow C_3 = \frac{\xi_3}{r^3}$$

$$x_1 = (t+r)\frac{\xi_1}{r}$$

$$x_2 = (t^2+r^2)\frac{\xi_2}{r^2}$$

$$x_3 = (t^3+r^3)\frac{\xi_3}{r^3}$$

(3). Problem 3

The position of the particles in continuum medium can be found using:

$x_1 = 2\xi_1 + k\xi_3^2 t, \qquad x_2 = \xi_2 + 2k\xi_2 t, \qquad x_3 = \xi_3, \qquad k = 10^{-4}$

Find the unit elongations and the decreases in angles between the basis directions of the particles.

At time t = 2, find the unit elongation of a particle initially in the direction of $2\boldsymbol{e_1} + \boldsymbol{e_2}$.

$$\varepsilon_{ij} = \varepsilon_{ij}^{\circ} = \frac{1}{2}\left(\frac{\partial x_k}{\partial x_i^{\circ}}\frac{\partial x_k}{\partial x_j^{\circ}} - \delta_{ij}\right)$$

$$\varepsilon_{11} = \frac{1}{2}(2^2 - 1) = \frac{3}{2}, \qquad \varepsilon_{22} = \frac{1}{2}((1 + 2kt)^2 - 1) = 2kt + 2k^2t^2,$$

$$\varepsilon_{33} = \frac{1}{2}((2k\xi_3 t)^2 + 1^2 - 1) = 2k^2\xi_3^2 t^2$$

$$\varepsilon_{12} = \frac{1}{2}(0) = 0, \qquad \varepsilon_{13} = \frac{1}{2}(2 * 2k\xi_3 t) = 2k\xi_3 t, \qquad \varepsilon_{23} = 0$$

$$(\varepsilon_{ij}) = \begin{bmatrix} \frac{3}{2} & 0 & 2k\xi_3 t \\ 0 & 2kt + 2k^2t^2 & 0 \\ 2k\xi_3 t & 0 & 2k^2\xi_3^2 t^2 \end{bmatrix}$$

ε_{11}- unit elongation in $x_1$direction
ε_{22}- unit elongation in $x_2$direction
ε_{33}- unit elongation in $x_3$direction
$2\varepsilon_{12} -$ angle between the x_1, x_2 directions
$2\varepsilon_{13} -$ angle between the x_1, x_3 directions
$2\varepsilon_{23} -$ angle between the x_2, x_3 directions

$$X' = 2\boldsymbol{e_1} + \boldsymbol{e_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad X = \frac{X'}{|X|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$|[e_{ij}]X| = \left|\frac{1}{\sqrt{5}}\begin{bmatrix} 3 \\ 2kt + 2k^2t^2 \\ 4k\xi_3 t \end{bmatrix}\right| = \frac{1}{\sqrt{5}}\sqrt{9 + (2kt + 2k^2t^2)^2 + (4k\xi_3 t)^2} = \begin{bmatrix} k = 10^{-4} \\ t = 2 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{5}}\sqrt{9 + k^2(\ldots)} \approx \frac{3}{\sqrt{5}}, \qquad because\ k^2(\ldots) \ll 9, suppose\ \xi_3 < k$$

(4). Problem 4

Strains in the medium are characterized with the strain tensor:

$$[E] = \begin{bmatrix} 2k\xi_2^2 & 1 & 2\xi_1\xi_2 \\ 1 & 4k^2\xi_1\xi_2 & 2k\xi_1 + \xi_2 \\ 2\xi_1\xi_2 & 2k\xi_1 + \xi_2 & \xi_1(2k\xi_1 + \xi_2) \end{bmatrix}$$

Find the Lagrangian coordinates of particles whose volume does not change.

$$\frac{dV}{dV_0} = 1 + \varepsilon_{ii}, \qquad must\ be\ \ equal\ \ to\ 1$$

$$1 + \varepsilon_{ii} = 1 => \varepsilon_{ii} = 0$$

$$\varepsilon_{ii} = 2k\xi_2^2 + 4k^2\xi_1\xi_2 + \xi_1(2k\xi_1 + \xi_2) = 2k\xi_2(\xi_2 + 2k\xi_1) + \xi_1(2k\xi_1 + \xi_2) =$$

$$= (2k\xi_2 + \xi_1)(2k\xi_1 + \xi_2) = 0$$

Finally:

$$\xi_1 = -2k\xi_2, \qquad \xi_2, \xi_3 - are\ arbitrary$$

$$\xi_2 = -2k\xi_1, \qquad \xi_1, \xi_3 - are\ arbitrary$$

(5). Problem 5

Given the displacement field of a continuum in Eulerian coordinates:

$$\omega_1 = 2(k-1)x_2 t - x_3 t, \qquad \omega_2 = x_3 t - \frac{2}{3}kx_1 t, \qquad \omega_3 = \left(k^2 t - \frac{5}{4}t\right)(x_1 - x_2)$$

Find for which k the field is describing the displacements of a rigid body.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right)$$

$$\varepsilon_{11} = 0, \qquad \varepsilon_{22} = 0, \qquad \varepsilon_{33} = 0$$

$$\varepsilon_{12} = \frac{1}{2}\left(2(k-1)t - \frac{2}{3}kt\right), \qquad \varepsilon_{13} = \frac{1}{2}\left(-t + \left(k^2 t - \frac{5}{4}t\right)\right), \qquad \varepsilon_{23} = \frac{1}{2}\left(t - \left(k^2 t - \frac{5}{4}t\right)\right)$$

A rigid body is a non-deformable body, which means that all components of the strain tensor are equal to zero.

$$\varepsilon_{12} = \frac{1}{2}\left(2(k-1)t - \frac{2}{3}kt\right) = 0 => \ 2(k-1)t = \frac{2}{3}kt$$

$$2kt - 2t - \frac{2}{3}kt = k\left(\frac{6}{3}t - \frac{2}{3}t\right) - 2t = k\frac{4}{3}t - 2t = 0 => k = \frac{6}{4}$$

$$\varepsilon_{13} = \frac{1}{2}\left(-t + \left(k^2 t - \frac{5}{4}t\right)\right) = \frac{1}{2}\left(-t + \left(\frac{36}{16}t - \frac{5}{4}t\right)\right) = \frac{1}{2}\left(-t + \left(\frac{36}{16}t - \frac{20}{16}t\right)\right) = 0$$

$$\varepsilon_{23} = \frac{1}{2}\left(t - \left(\frac{36}{16}t - \frac{5}{4}t\right)\right) = 0$$

$$k = \frac{6}{4}$$