Mathematical Methods in Engineering and Applied Science

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Problem Set 4. Due on Oct. 30 at 23:59.

(1) Explain:
  (a) why A^TA is not singular when matrix A has independent columns;
  (b) why A and A^TA have the same nullspace.

(2) A plane in $\mathbb{R}^3$ is given by the equation $x_1 - 2x_2 + x_3 = 0$.
  (a) Identify two orthonormal vectors u_1 and u_2 that span the plane.
  (b) Find a projector matrix P that projects any vector x from $\mathbb{R}^3$ to the plane and a projector $P_\perp$ that projects any vector to the direction normal to the plane.
  (c) Using these projectors find the unit normal to the plane and verify that it agrees with a normal found by calculus methods (that use the gradient).

(3) Let $M = span\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 1 & -1 & 0 & -1 \end{bmatrix}^T$.
  (a) Find the orthogonal projector P_M on M.
  (b) Find the kernel (nullspace) and range (column space) of P_M.
  (c) Find $x \in M$ which is closest in 2-norm to the vector $a = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

(4) The following problems look at tests of positive definiteness.
  (a) Using the determinant test, find c and d that make the following matrices positive definite:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

  (b) A positive definite matrix cannot have a zero (or a negative number) on its main diagonal. Show that the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

  is not positive definite by finding x such that $x^TAx \le 0$.

(5) Matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ is positive definite. Explain why and determine the minimum value of $z = x^TAx + 2b^Tx + 1$, where $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $b^T = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$.

(6) Explain these inequalities from the definition of the norms:

$$\|ABx\| \le \|A\| \, \|Bx\| \le \|A\| \, \|B\| \, \|x\|,$$

and deduce that $\|AB\| \le \|A\| \, \|B\|$.

(7) Compute by hand the norms and condition numbers of the following matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$