A practical introduction to the kinetic simulation of plasmas

Supplemental Material

This supplemental material provides some theoretical support to the practicals. It relies purely on the linear theory of either wave or instabilities. The nonlinear saturation stage will not be addressed here even though it will be discussed in the practicals.

Nota bene: Below are given mainly technical points. The physical interpretation of the different processes studied will be investigated in the practical, or can be found in most text book on plasma physics.

A. General framework

For the sake of generality, all electromagnetic fields will be described by Maxwell's equations, and Maxwell-Ampère and Maxwell Faraday equations in particular:

$$\frac{1}{c^2}\partial_t \mathbf{E} = -\mu_0 \mathbf{J} + \nabla \times \mathbf{B}\,, \tag{1a}$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}\,, \tag{1b}$$

with $\mathbf{J} = \sum_s \mathbf{J}_s$ the total current density provided by the various species s in the plasma.

Throughout this *supplemental material*, we will focus on linear waves and/or the linear phase of instabilities in the absence of any external electric or magnetic fields $\left[\mathbf{E}^{(0)} = \mathbf{B}^{(0)} = 0\right]$. Furthermore, looking at first order quantities in the form

$$\phi^{(1)} = \tilde{\phi}^{(1)} \, \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$$

leads for the perturbed electric and magnetic fields:

$$\omega\, \mathbf{E}^{(1)} = -\frac{i}{\epsilon_0}\, \mathbf{J}^{(1)} - c^2\, \mathbf{k} \times \mathbf{B}^{(1)}\,, \tag{2a}$$

$$\omega\, \mathbf{B}^{(1)} = \mathbf{k} \times \mathbf{E}^{(1)}\,. \tag{2b}$$

From Eqs. (2a) and (2b), we obtain:

$$\omega^2 \mathbf{E}^{(1)} + c^2\, \mathbf{k} \times \mathbf{k} \times \mathbf{E}^{(1)} = -i\, \frac{\omega}{\epsilon_0}\, \mathbf{J}^{(1)}\,, \tag{3}$$

and making use of the vector identity $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$:

$$\left[(\omega^2 - c^2\, \mathbf{k}^2)\, \mathbb{I} + c^2\, \mathbf{k} \otimes \mathbf{k}\right] \mathbf{E}^{(1)} = -i\, \frac{\omega}{\epsilon_0}\, \mathbf{J}^{(1)}\,, \tag{4}$$

with $\mathbb{I}$ the identity matrix and $\mathbf{u} \otimes \mathbf{v}$ the dyadic product of vector $\mathbf{u}$ and $\mathbf{v}$ that returns the matrix $\underline{\mathbf{M}}$ with elements $M_{ij} = u_i v_j$.

Now, all the trick is to express all (first order) current densities $\mathbf{J}_s^{(1)}$ as a function of $\mathbf{E}^{(1)}$. In a linear theory, one will always be able to rewrite the current density in the form:

$$\mathbf{J}^{(1)} = \underline{\boldsymbol{\sigma}}\,\mathbf{E}^{(1)}$$

where $\underline{\boldsymbol{\sigma}}$ is the conductivity tensor, leading:

$$\left[\omega^2 \underline{\boldsymbol{\epsilon}} + c^2\,\left(\mathbf{k}\otimes\mathbf{k} - k^2\,\mathbb{I}\right)\right]\mathbf{E}^{(1)} = 0\,, \tag{5}$$

where we have introduced the permittivity tensor:

$$\underline{\boldsymbol{\epsilon}} = \mathbb{I} + \frac{i}{\epsilon_0\omega}\underline{\boldsymbol{\sigma}}\,. \tag{6}$$

Without loss of generality, we now consider that the wavevector[1] $\mathbf{k} = (k_\parallel, k_\perp, 0)$, and Eq. (5) can be rewritten in the matrix form:

$$\underline{\mathbf{P}}\,\mathbf{E}^{(1)} = 0\,, \tag{7}$$

with:

$$\underline{\mathbf{P}} = \begin{bmatrix} \omega^2\epsilon_{11} - c^2k_\perp^2 & \omega^2\epsilon_{12} - c^2k_\parallel k_\perp & \omega^2\epsilon_{13} \\ \omega^2\epsilon_{21} - c^2k_\parallel k_\perp & \omega^2\epsilon_{22} - c^2k_\parallel^2 & \omega^2\epsilon_{23} \\ \omega^2\epsilon_{31} & \omega^2\epsilon_{32} & \omega^2\epsilon_{33} - c^2k^2 \end{bmatrix} \tag{8}$$

To solve Eq. (7), one needs to cancel the determinant of the *propagator* $\underline{\mathbf{P}}$. Doing so will provide us with the *dispersion relations* for the various waves and/or instabilities at play.

Now, the permittivity tensor needs to be computed for each species so that the total permittivity tensor can be obtained by summing the contribution of each species. This tensor contains all the information of the plasma species. In what follows we will do it first in the framework of a relativistic cold fluid model, then in the framework of the (kinetic) Vlasov equation.

B. Weibel & two-stream instabilities driven by two counter-streaming cold electron plasmas

In this Section, we focus on two instabilities that develop in the presence of counter-streaming flows. For the sake of simplicity, we here consider the case of two counter-streaming electron plasmas, each with density $n_0/2$ and opposite velocities $\pm\mathbf{v}_0$, and a neutralizing background of immobile ions with density n_0. We will consider the first spatial dimension as that given by $\mathbf{v}_0$, i.e. $\mathbf{v}_0 = (v_0, 0, 0)$.

Both electron species (labeled $s = \pm$) are treated in the framework of a relativistic cold fluid model:

$$\partial_t n_s + \nabla\cdot(n_s\mathbf{v}_s) = 0\,, \tag{9a}$$

$$\partial_t\mathbf{p}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{p}_s = q_s\,\left(\mathbf{E} + \mathbf{v}_s\times\mathbf{B}\right)\,. \tag{9b}$$

We linearize the fluid Eqs. (9a)-(9b) considering all initial (zero-order) quantities homogeneous (and no external fields, as previously assumed). Looking for first order quantities in the form $\phi^{(1)} = \tilde{\phi}^{(1)}\,\exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$, leads:

$$n_s^{(1)} = n_s^{(0)}\,\frac{\mathbf{k}\cdot\mathbf{v}_s^{(1)}}{\omega_{ds}}\,, \tag{10a}$$

$$\mathbf{p}_s^{(1)} = i\,\frac{q_s}{\omega_{ds}}\,\left(\mathbf{E}^{(1)} + \mathbf{v}_s^{(0)}\times\mathbf{B}^{(1)}\right)\,, \tag{10b}$$

[1]Later on, we will allow the plasma species to flow in the direction $(\pm1, 0, 0)$.

where we have introduced:

$$\omega_{ds} = \omega - \mathbf{v}_s^{(0)} \cdot \mathbf{k}\,.$$

As previously stated, we wish to derive the equation for the species (first order) density current:

$$\mathbf{J_s}^{(1)} = q_s\, n_s^{(0)}\, \mathbf{v}_s^{(1)} + q_s\, n_s^{(1)}\, \mathbf{v}_s^{(0)}\,. \tag{11}$$

Using Eq. (10a), we get:

$$\mathbf{J_s}^{(1)} = q_s\, n_s^{(0)}\, \left[\mathbb{I} + \frac{\mathbf{v}_s^{(0)} \otimes \mathbf{k}}{\omega_{ds}}\right]\, \mathbf{v}_s^{(1)} = q_s n_s^{(0)} \begin{bmatrix} 1 + \frac{v_s^{(0)} k_\parallel}{\omega_{ds}} & \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v}_s^{(1)}\,. \tag{12}$$

To get $\mathbf{v}_s^{(1)}$ we Taylor expand $\mathbf{v}_s$ expressed as a function of the species momentum $\mathbf{p}_s$ around $\mathbf{v}_s^{(0)}$, which leads:

$$\mathbf{v}_s^{(1)} = \frac{\mathbf{p}_s^{(1)}}{m_s \gamma_{s0}} - \frac{\mathbf{p}_s^{(0)}}{m_s \gamma_{s0}^3} \frac{\mathbf{p}_s^{(0)} \cdot \mathbf{p}_s^{(1)}}{m_s^2 c^2} = \frac{1}{m_s \gamma_{s0}}\, \left[\mathbb{I} - \mathbf{p}_s^{(0)} \otimes \mathbf{p}_s^{(0)}\right]\, \mathbf{p}_s^{(1)}\,, \tag{13}$$

with $\gamma_{s0} = \sqrt{1 + \mathbf{p}_s^{(0)2}/(m_s c)^2}$. We rewrite in the matrix form:

$$\mathbf{v}_s^{(1)} = \frac{1}{m_s \gamma_{s0}} \begin{bmatrix} \gamma_{s0}^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_s^{(1)}\,. \tag{14}$$

One then computes $\mathbf{p}_s^{(1)}$ from Eq. (10b) (using Faraday's equation to rewrite the magnetic field in terms of the electric field) as:

$$\mathbf{p}_s^{(1)} = i\, \frac{q_s}{\omega}\, \left[\mathbb{I} + \frac{\mathbf{k} \otimes \mathbf{v}_s^{(0)}}{\omega_{ds}}\right]\, \mathbf{E}^{(1)} = i\, \frac{q_s}{\omega} \begin{bmatrix} 1 + \frac{v_s^{(0)} k_\parallel}{\omega_{ds}} & 0 & 0 \\ \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)}\,. \tag{15}$$

Combining Eqs. (12), (14) and (15), one finally gets for the current density:

$$\mathbf{J}_s^{(1)} = i\, \epsilon_0 \frac{\omega_{ps,0}^2}{\gamma_{s0}\omega} \begin{bmatrix} \frac{1}{\gamma_{s0}^2} \left(1 + \frac{v_s^{(0)} k_\parallel}{\omega_{ds}}\right)^2 + \frac{v_s^{(0)2} k_\perp^2}{\omega_{ds}^2} & \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 0 \\ \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)}\,, \tag{16}$$

where we have introduced the species plasma frequency

$$\omega_{ps,0} = \sqrt{\frac{q_s^2 n_s^{(0)}}{\epsilon_0 m_s}}\,. \tag{17}$$

B.1 Purely transverse instability: the Weibel or filamentation instability

Let us first consider the case where the wavevector $\mathbf{k} = (0, k_\perp = k, 0)$ is perpendicular to the electron flow velocity, so that $\forall s,\ \mathbf{v}_s^{(0)} \cdot \mathbf{k} = 0$ and $\omega_{ds} = \omega$. The total current density (at first order) is obtained by summing over the two ($s = \pm$) species which, we recall, have opposite drift-velocity $\mathbf{v}_\pm^{(0)} = \pm\mathbf{v}_0$. From Eq. (16), one gets[2]:

$$\mathbf{J}^{(1)} = i\,\epsilon_0 \frac{\omega_{p0}^2}{\gamma_0\omega} \begin{bmatrix} \frac{1}{\gamma_0^2} + \frac{v_0^2k^2}{\omega^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)}\,. \tag{18}$$

This corresponds to the (total) permittivity:

$$\underline{\boldsymbol{\epsilon}} = \begin{bmatrix} 1 - \frac{\omega_{p0}^2}{\gamma_0\omega^2}\left(\frac{1}{\gamma_0^2} + \frac{v_0^2k^2}{\omega^2}\right) & 0 & 0 \\ 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0\omega^2} \end{bmatrix} \tag{19}$$

Injecting Eq. (19) in Eq. (8) leads:

$$\begin{vmatrix} \omega^2 - c^2k^2 - \frac{\omega_{p0}^2}{\gamma_0}\left(\frac{1}{\gamma_0^2} + \frac{v_0^2k^2}{\omega^2}\right) & 0 & 0 \\ 0 & \omega^2 - \frac{\omega_{p0}^2}{\gamma_0} & 0 \\ 0 & 0 & \omega^2 - c^2k^2 - \frac{\omega_{p0}^2}{\gamma_0} \end{vmatrix} = 0\,. \tag{20}$$

Now, the propagator is a diagonal matrix so that cancelling its determinant is straightforward. Two solutions correspond to the standard electrostatic and electromagnetic waves in a plasma. Of particular importance is the mode corresponding to the electric field aligned with the flow velocity, for which:

$$\omega^2 - \frac{\omega_{p0}^2}{\gamma_0}\left(\frac{1}{\gamma_0^2} + \frac{v_0^2k^2}{\omega^2}\right) - c^2k^2 = 0\,. \tag{21}$$

Indeed, this equation, which is the *dispersion relation for the Weibel or filamentation instability*, allows for purely imaginary solutions $\omega(k) = i\Gamma_\perp(k)$ (with $\Gamma_\perp(k) > 0$) so that the (flow aligned) electric field grows exponentially at a rate:

$$\Gamma_\perp(k) = \frac{1}{\sqrt{2}}\left[\sqrt{\left(c^2k^2 + \frac{\omega_{p0}^2}{\gamma_0^3}\right)^2 + 4\frac{\omega_{pe}^2}{\gamma_0}\,v_0^2k^2} - \left(c^2k^2 + \frac{\omega_{p0}^2}{\gamma_0^3}\right)\right]^{1/2}. \tag{22}$$

Note that this growth rate corresponds to the electric field component aligned with the flow velocity. Hence (see e.g. Faraday's equation), the growth of this electric field will be accompanied by a growth of a magnetic field that is transverse to both, the wavevector $\mathbf{k}$ and $\mathbf{v}_0$. This instability is thus of an electromagnetic nature.

Exercise – Show that in the small k limit, the growth rate of the instability increases linearly with k as $\Gamma(k \ll 1) \to \gamma_0\,v_0\,k$. Then, in the large k limit, show that the growth rate asymptotically reaches the maximum value $\Gamma(k \gg 1) = v_0\omega_{p0}/(c\sqrt{\gamma_0})$.

B.2 Purely longitudinal instability: the two-stream instability

Let us now consider the case in which the wavevector $\mathbf{k} = (k_\parallel = k, 0, 0)$ is aligned with the electron flow velocity, so that $\forall s,\ \mathbf{v}_s^{(0)} \cdot \mathbf{k} = \pm v_0 k$ and $\omega_{d\pm} = \omega \pm v_0 k$. The total current density (at first order)

[2] Note that $\omega_{p0} = \sqrt{2}\,\omega_{p\pm,0}$ corresponds to the overall plasma frequency computed at n_0.

is obtained by summing over the two ($s = \pm$) species, leading to:

$$\mathbf{J}^{(1)} = i\epsilon_0 \frac{\omega_{p0}^2}{\gamma_0 \omega} \begin{bmatrix} \frac{1}{2\gamma_0^2}\left[\frac{\omega^2}{(\omega - v_0 k)^2} + \frac{\omega^2}{(\omega + v_0 k)^2}\right] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)} \tag{23}$$

and to the (total) permittivity:

$$\underline{\boldsymbol{\epsilon}} = \begin{bmatrix} 1 - \frac{\omega_{p0}^2}{2\gamma_0^3}\left[\frac{1}{(\omega - v_0 k)^2} + \frac{1}{(\omega + v_0 k)^2}\right] & 0 & 0 \\ 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0 \omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0 \omega^2} \end{bmatrix} . \tag{24}$$

Injecting Eq. (24) in Eq. (8) leads to:

$$\begin{vmatrix} \omega^2 - \frac{\omega_{p0}^2}{2\gamma_0^3}\left[\frac{\omega^2}{(\omega - v_0 k)^2} + \frac{\omega^2}{(\omega + v_0 k)^2}\right] & 0 & 0 \\ 0 & \omega^2 - c^2 k^2 - \frac{\omega_{p0}^2}{\gamma_0} & 0 \\ 0 & 0 & \omega^2 - c^2 k^2 - \frac{\omega_{p0}^2}{\gamma_0} \end{vmatrix} = 0\,. \tag{25}$$

Here again, we find some trivial (propagating wave) solutions for the y and z components of the electric field. In addition, cancelling the first (1,1) term in Eq. (25) provides the *dispersion relation for the two-stream instability*:

$$\omega^4 - \left(2v_0^2 k^2 + \frac{\omega_{p0^2}}{\gamma_0^3}\right)\omega^2 + \left(v_0^2 k^2 - \frac{\omega_{p0}^2}{\gamma_0^3}\right) v_0^2 k^2 = 0\,. \tag{26}$$

Note that this instability involves the electric field parallel to the initial flow velocity only. It is accompanied by a charge density perturbation (through e.g. Poisson equation) but is not associated to any magnetic field perturbation. The two-stream instability is therefore of a purely electrostatic nature.

Solving for ω^2 gives:

$$\omega^2 = \frac{1}{2}\left[\frac{\omega_{p0}^2}{\gamma_0^3} + 2v_0^2 k^2 \pm \frac{\omega_{p0}}{\gamma_0^{3/2}}\sqrt{\frac{\omega_{p0}^2}{\gamma_0^3} + 8v_0^2 k^2}\right] . \tag{27}$$

It turns out that, whenever $l_0 k < 1$, with $l_0 = \gamma_0^{3/2} v_0/\omega_{p0}$, $\omega = i\,\Gamma_{\parallel}(k)$, where $\Gamma_{\parallel}(k) > 0$ is the growth rate for the two-stream instability.

Exercise – (i) Show that the maximum growth rate for the two-stream instability is maximum for $l_0 k = \sqrt{6}/4$, and is then $\Gamma_{\parallel,\max} = \omega_{p0}/(2\sqrt{2}\gamma_0^{3/2})$. (ii) Compare the growth rate of the two-stream and Weibel/filamentation instabilities at small ($\gamma_0 \sim 1$) and large ($\gamma_0 \ll 1$). Which instability do you expect to dominate in the presence of relativistic flows?

C. An example of kinetic waves & instabilities: the Landau damping of Langmuir waves and the bump-on-tail instability

Let us now investigate the case of kinetic waves and instabilities. To do so, the plasma species are modelled using the Vlasov equation (for simplicity we restrict ourselves to the non-relativistic limit):

$$\partial_t f_s + \mathbf{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_s = 0\,. \tag{28}$$

In the absence of any external electromagnetic fields, and considering all perturbed quantities $\phi^{(1)} = \tilde{\phi}^{(1)} \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$, one gets at first order:

$$f_s^{(1)} = -i\frac{q_s}{m_s}\left(\mathbf{E}^{(1)} + \mathbf{v}\times\mathbf{B}^{(1)}\right)\cdot\frac{\nabla_v f_s^{(0)}}{\omega - \mathbf{k}\cdot\mathbf{v}}\,, \tag{29}$$

where $f_s^{(0)}$ is the initial distribution function which we will assume to be a Maxwellian with temperature T_s and mean velocity $\mathbf{v}_s^{(0)}$:

$$f_s^{(0)}(\mathbf{v}) = n_s^{(0)}\left(\frac{m_s}{2\pi T_s}\right)^{\frac{3}{2}}\exp\left(-\frac{m_s\,(\mathbf{v}-\mathbf{v}_s^{(0)})^2}{2T_s}\right) \tag{30}$$

In what follows, we will further restrict our analysis to (i) purely electrostatic perturbations, $\mathbf{B}^{(1)} = 0$ in Eq. (29), and (ii) species drift (mean) velocities $\mathbf{v}_s^{(0)}$ aligned with the wavevector $\mathbf{k} = (k,0,0)$. It follows that only the (first order) electric field and current densities aligned with $\mathbf{k}$, $E_\parallel^{(1)}$ and $J_{s,\parallel}^{(1)}$, are of importance for the problem at hand. One thus focus on expressing the current density perturbation in the form:

$$J_{s,\parallel}^{(1)} = q_s\int d^3v\,v_1 f_s^{(1)} = -i\frac{q_s^2}{m_s}\int d^3v\frac{v_1\,\partial f_s^{(0)}/\partial v_1}{\omega - kv_1}\,E_\parallel^{(1)}\,. \tag{31}$$

Introducing the reduced distribution function:

$$\bar{f}_s^{(0)} = \frac{1}{n_s^{(0)}}\int dv_2 dv_3 f_s^{(0)} = \sqrt{\frac{m_s}{2\pi T_s}}\exp\left(-\frac{m_s\,(v_1 - v_s^{(0)})^2}{2T_s}\right)\,, \tag{32}$$

one finally gets for the current density:

$$J_{s,\parallel}^{(1)} = -i\frac{q_s^2 n_s^{(0)}}{m_s}\int dv_1\frac{v_1\,\partial \bar{f}_s^{(0)}/\partial v_1}{\omega - kv_1}\,E_\parallel^{(1)} = -i\epsilon_0\omega\frac{\omega_{ps,0}^2}{k^2}\int dv_1\frac{\partial \bar{f}_s^{(0)}/\partial v_1}{\omega/k - v_1}\,E_\parallel^{(1)}\,. \tag{33}$$

One then gets the permittivity:

$$\epsilon_s^\parallel = 1 + \frac{\omega_{ps,0}^2}{k^2}\int dv_1\frac{\partial \bar{f}_s^{(0)}/\partial v_1}{\omega/k - v_1}\,. \tag{34}$$

C.1 Landau damping of Langmuir waves

Langmuir waves are electrostatic perturbations propagating in a plasma. Let us thus consider a plasma with (immobile ions and) an homogeneous density n_0, zero mean velocity, and temperature T_0. The plasma permittivity then reads:

$$\epsilon^\parallel(\omega,k) = 1 + \frac{\omega_{p0}^2}{k^2}\int dv_1\frac{\partial \bar{f}^{(0)}/\partial v_1}{\omega/k - v_1}\,. \tag{35}$$

The real part of the permittivity provides us with the dispersion relation for the Langmuir waves. It can be easily computed in the limit $kv_{th}/\omega \ll 1$, and after some algebra, leads to the (approximated) *dispersion relation for the Langmuir waves*:

$$\omega(k) \simeq \omega_{p0}\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right)\,. \tag{36}$$

Exercise – Compute the phase and group velocity for the Langmuir waves? What happens in the limit $T_0 \to 0$?

In addition, the presence of a singularity in the integral in Eq. (35) leads to an imaginary contribution to the permittivity:

$$\operatorname{Im}\epsilon^{\|}(\omega, k) = -\pi \frac{\omega_{p0}}{k^2} \int dv_1 \partial_{v1} \bar{f}^{(0)} \, \delta(\omega/k - v_1) = -\pi \frac{\omega_{p0}}{k^2} \left. \frac{\partial \bar{f}^{(0)}}{\partial v_1} \right|_{v_1 = \omega/k} . \tag{37}$$

which gives rise to the damping of the Langmuir wave at a rate:

$$\gamma_L(k) = \frac{\operatorname{Im}\epsilon^{\|}}{\partial_\omega \operatorname{Re}\epsilon^{\|}} = \sqrt{\frac{8}{\pi}} \frac{\omega_{p0}}{k^3 \lambda_{De}^3} \exp\left(-\frac{3}{2} - \frac{1}{2k^2\lambda_{De}^2}\right) . \tag{38}$$

C.2 Bump on Tail instability

The bump on tail instability refers to the (electrostatic) instability that develops in the presence of (i) a background plasma with zero (or close to zero) mean velocity and temperature T_0, and of (ii) a beam, with density $n_b \ll n_0$, non zero velocity v_b (in the direction parallel to the instability wavevector) and temperature T_b.

The presence of the electron beam will drive a Langmuir wave supported by the background plasma. Its dispersion relation is thus given by:

$$\omega^2 \simeq \omega_{p0}^2 \left(1 + 3k^2 \lambda_{De}^2\right) . \tag{39}$$

This wave will be damped by Landau damping (trapping of the background plasma electron), but at the same time can pick up energy from the beam electrons, which gives rise to the *bump in tail instability*. For the instability to develop, its growth rate $\propto \partial_{v_1} f_b^{(0)}|_{v_1=\omega/k} > 0$ needs to be larger than the rate of Landau damping $\propto \partial_{v_1} f_0^{(0)}|_{v_1=\omega/k} < 0$. This leads to a threshold for the instability that can be cast in the form:

$$\frac{n_b}{n_0} \left(\frac{v_b}{v_{th,b}}\right)^2 > \left(\frac{v_b}{v_{th,0}}\right)^3 \exp\left(-\frac{m v_b^2}{2T_0}\right) . \tag{40}$$

Exercise – Following Sec. C.1, derive in the limit $\operatorname{Im}\omega \ll \operatorname{Re}\omega$ the dispersion relation for this instability.