MAT205

Fourier Analysis

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Now we on...

Start-UP

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Now we on. . .

Basic Beyond

What is Fourier series

Fourier Series

FS is a special trigonometric periodic series derived by Joseph Fourier$(1768 \to 1830)$.

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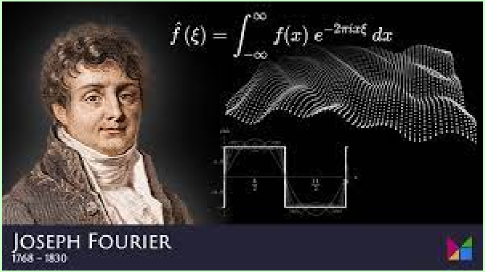

Figure: Joseph Fourier

Basic Beyond

Fourier series: Mathematical Representation

Our basic trigonometric series is:

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

It will called **Fourier Series** if the terms A_0, A_n, B_n is:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot dx$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos nx \cdot dx$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx \cdot dx$$

Where $f(x)$ is any single-valued function defined in interval $(-\pi, \pi)$.

Basic Beyond

What is Fourier transform

Fourier Transform

FT Is a mathematical transform that decomposes functions depending on time or space into functions depending on spatial or temporal frequency such as the expression of a musical chord.

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The Fourier transform of a function due to respect of time is a complex valued function of frequency whose magnitude represents the amount of that frequency present in the original function.

Basic Beyond

Some helpful equations

1. $\sin 0^\circ = \sin \pi = 0$

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7. $\int \cos\theta = \sin\theta$
8. $\int UV \cdot dx = U\int V \cdot dx + \int \frac{dU}{dx}(\int V \cdot dx) \cdot dx$

Now we on...

Question 1

1

Use the method of Fourier transform to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x), -\infty < x < \infty$. Also show that the solution can be put in the form:

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

Question 1

Trying to solve

[Trying to solve:]
Here we have to solve the one-dimensional wave equation

$$\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2} \qquad [-\infty < x < \infty, t > 0]$$

subject to the following initial conditions

$$\begin{aligned} y(x,0) &= \text{Initial displacement} = f(x) \\ \text{and } y_t(x,0) &= \text{initial velocity} = 0 \end{aligned}$$

The given partial differential equation is

$$\frac{\delta^2 y}{\delta t^2} = c^2 \frac{\delta^2 y}{\delta x^2} \tag{1}$$

Question 1

Trying to solve

Taking the complex Fourier transform of both sides of (1), we have

$$\int_{-\infty}^{+\infty} \frac{\delta^2 y}{\delta t^2} e^{-iux} dx = c^2 \int_{-\infty}^{+\infty} \frac{\delta^2 y}{\delta x^2} e^{-iux} dx \qquad (2)$$

By the Fourier transform of the derivative of a function we have if $F^n(x)$ is the nth-derivative of $F(x)$ and the first $(n-1)$ derivatives of F(x) vanish as $x \to \pm\infty$ then $F\{F^n(x)\} = (-iu)^n F\{F(x)\}$.
Thus from (2) we have

$$\frac{d^2}{dt^2} \int_{-\infty}^{+\infty} y e^{-iux} dx = c^2 (-iu)^2 F\{y(x,t)\}$$

$$\text{or, } \frac{d^2}{dt^2} \int_{-\infty}^{+\infty} y(x,t) e^{-iux} dx = c^2 (-u^2) F\{y(x,t)\}$$

$$\text{or, } \frac{d^2}{dt^2} F\{y(x,t)\} = -c^2 u^2 F\{y(x,t)\}$$

$$\text{or, } \frac{d^2 \overline{y}}{dt^2} = -c^2 u^2 \overline{y} \qquad \left[\text{where } \overline{y} = \overline{y}(u,t) = F\{y(x,t)\} = \int_{-\infty}^{+\infty} y(x,t) e^{-iux} dx\right]$$

Question 1

Trying to solve

$$\therefore \qquad \frac{d^2\overline{y}}{dt^2} + c^2u^2\overline{y} = 0 \tag{3}$$

Which is ordinary second order differential equation whose solution is

$$\overline{y} = \overline{y}(u,t) = A\cos(cut) + B\sin(cut) \tag{4}$$

Differentiating with both sides with respect to t

$$\text{we get, } \overline{y}_t(u,t) = -Acu\sin(cut) + Bcu\cos(cut) \tag{5}$$

Also from the initial given conditions, we have

$$y(x,0) = f(x) \tag{6}$$

$$y_t(x,0) = 0 \tag{7}$$

Question 1

Trying to solve

Taking the Fourier transform of (6) and (7) we get,

$$\overline{y}(u,0) = \int_{-\infty}^{+\infty} y(x,0)e^{-iux} \cdot dx = \int_{-\infty}^{+\infty} f(x)e^{-iux} \cdot dx = \overline{f}(u) \quad \text{(say)}$$

$$\therefore \quad \overline{y}(u,0) = \overline{f}(u) \tag{8}$$

$$\begin{aligned} \text{and } \overline{y}_t(u,0) &= \int_{-\infty}^{+\infty} y_t(x,0)e^{-iux} \cdot dx \\ &= \int_{-\infty}^{+\infty} 0(e^{-iux}) \cdot du = 0 \end{aligned}$$

$$\therefore \quad \overline{y}_t(u,0) = 0 \tag{9}$$

Putting t = 0 in (5), we have $\overline{y}_t(u,0) = Bcu$
or, $Bcu = 0$ using (9)

$$\therefore \quad B = 0 \qquad \text{Since, } cu \neq 0$$

Question 1

Trying to solve

Again putting t = 0 in (4), we get

$$\overline{y}(u,0) = A$$
$$\therefore \quad \overline{f}(u) = A \qquad \text{using (8)}$$
$$\text{or, } A = \overline{f}(u)$$

Putting the values of A and B in (4), we get

$$\overline{y} = \overline{y}(u,t) = \overline{f}(u)\cos(cut) \tag{10}$$

Taking the inverse Fourier transform of (10) we have

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{f}(u)\cos(cut)e^{iux}\cdot du$$
$$\text{or, } y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{f}(u)\left(\frac{e^{icut}+e^{-icut}}{2}\right)e^{iux}\cdot du$$
$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{f}(u)e^{iu(x+ct)}\cdot du + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{f}(u)e^{iu(x-ct)}\cdot du\right]$$
$$= \tfrac{1}{2}\left[f(x+ct)+f(x-ct)\right] \text{ (Using the definition of inverse Fourier transform)}$$

Question 1

Trying to solve: Over

Finally

$$\therefore \qquad y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right].$$

Now we on...

Question 2

2

A thin membrane of great extent is released from rest in the position $z = f(x, y)$, show that the displacement at any subsequent time is given by

$$z(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u, v) \cos ct\sqrt{(u^2 + v^2)} \cdot e^{-i(ux+vy)} \cdot du \cdot dv$$

where $F(u, v)$ is the double Fourier transform of $f(x, y)$.

Question 2

Trying to proof

[Trying to proof:]

Here the displacement of the membrane is governed by two dimensional wave equation

$$\frac{\delta^2 z}{\delta t^2} = c^2 \left(\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} \right) \qquad \text{where } c^2 = \frac{T}{\rho} \tag{1}$$

Taking the double Fourier transform of both sides of (1) we get

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\delta^2 z}{\delta t^2} e^{i(ux=vy)} \cdot dx \cdot dy = \frac{c^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} \right) e^{i(ux+vy)} \cdot dx \cdot$$

$$\text{or, } \frac{d^2}{dt^2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} z e^{i(ux+vy)} \cdot dx \cdot dy = \frac{c^2}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} \right) e^{i(ux+vy)} \cdot dx \cdot dy$$

Question 2

Trying to proof

$$\text{or}, \frac{d^2\overline{z}}{dt^2} = c^2\{(-iu)^2+(-iv)^2\}F\{z(x,y,t)\} \text{ where } \overline{z} = \overline{z}(u,v,t) = F\{z(x,y,t)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ze^{i(ux+vy)} \cdot dx \cdot dy$$

$$\text{or}, \frac{d^2\overline{z}}{dt^2} = -c^2(u^2+v^2)\overline{z}$$

$$\therefore \quad \frac{d^2\overline{z}}{dt^2} + c^2(u^2+v^2)\overline{z} = 0 \tag{2}$$

Question 2

Trying to proof

which is an ordinary differential equation whose solution is

$$\overline{z} = A\cos\{c\sqrt{(u^2+v^2)}t\} + B\sin\{c\sqrt{(u^2+v^2)}t\} \tag{3}$$

The given initial conditions are $\overline{z} = f(x,y)$ and $\frac{\delta z}{\delta t} = 0$ at $t = 0$
Taking the Fourier transform of these initial conditions, we get

$$\overline{z} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{i(ux+vy)} \cdot dx \cdot dy = F(u,v) \tag{4}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\delta z}{\delta t} e^{i(ux+vy)} \cdot dx \cdot dy$$

$$= \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z e^{i(ux+vy)} \cdot dx \cdot dy$$

$$= \frac{d\overline{z}}{dt} \text{ since } \overline{z} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z e^{i(ux+vy)} \cdot dx \cdot dy$$

Question 2

Trying to proof

$$\therefore \quad \frac{d\bar{z}}{dt} = 0 \text{ at } t = 0$$

When t=0, combining (3) and (4), we get $A = F(u, v)$

$$\text{Also } \frac{d\bar{z}}{dt} = -Ac\sqrt{(u^2+v^2)}\sin c\sqrt{(u^2+v^2)}t + Bc\sqrt{(u^2+v^2)}\cos c\sqrt{(u^2+v^2)}t$$

$$\therefore \; 0 = \left(\frac{d\bar{z}}{dt}\right)_{t=0} = Bc\sqrt{(u^2+v^2)}$$

$$\text{or, } B = 0$$

Putting the values of A and B in (3), we get

$$\bar{z} = F(u, v)cos\{c\sqrt{(u^2+v^2)t}\} \tag{5}$$

Question 2

Trying to proof: Over

Ultimately

Now applying the inversion formula for double Foruier transform, we have

$$z(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)\cos\{c\sqrt{(u^2+v^2)}t\}e^{-i(ux+vy)}\cdot du\cdot dv$$

Which is the required displacement at any subsequent time t.

Now we on. . .

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