Rao IIT Academy

Symbol of Excellence and Perfection

JEE | MEDICAL-UG | BOARDS | KVPY | NTSE | OLYMPIADS

Rao IIT Academy JEE - Main Level

ONLINE - 2018

Mathematics

Solutions

1.

Difficulty : Difficult

Topics :

Definite integrals,

$x > x^2 \qquad x \in (0, 1)$

$\Rightarrow -x < -x^2$

$\Rightarrow e^{-x} < e^{-x^2}$

$\Rightarrow I_1 < I_2$

$x^3 < x^2$

$\Rightarrow -x^3 < -x^2$

$\Rightarrow e^{-x^3} > e^{-x^2}$

$\Rightarrow e^{-x^3} > e^{-x^2} > e^{-x^2} \cos^2 x$

$\Rightarrow e^{-x^3} > e^{-x^2} \cos^2 x$

$I_3 > I_2$

$I_3 > I_2 > I_1$

2.

Difficulty : Medium

Topics :

Trigonometric Equations,

$$\sin 3x = \cos 2x$$

$$\Rightarrow \cos 2x = \cos \left(\frac{\pi}{2} - 3x \right)$$

$$\Rightarrow 2x = 2n\pi \pm \frac{\pi}{2} \pm 3x$$

$$\Rightarrow 5x = \left(2n + \frac{1}{2} \right) \pi, -x = \left(2n - \frac{\pi}{2} \right) \pi$$

$$\Rightarrow x = (4n+1) \frac{\pi}{10} \quad x = -(4n-1)\frac{\pi}{2}$$

but $x \in \left(\frac{\pi}{2}, \pi \right)$

$$x = \frac{9\pi}{10}, 1$$

$$\therefore 1 \, solution$$

3.

Difficulty : Medium

Topics :

Differentiation,

Let $3^x = t$

$$f(x) = \sin^{-1} \left(\frac{2t}{1 + t^2} \right)$$

Let $t = \tan \theta$

$$f(x) = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1} 3^x$$

$$f'(x) = \frac{2}{1 + (3^x)^2} \times 3^x \times \ln 3$$

$$f'\left(-\frac{1}{2} \right) = \frac{2}{1 + 3^{-1}} \times \frac{1}{\sqrt{3}} \times \log 3$$

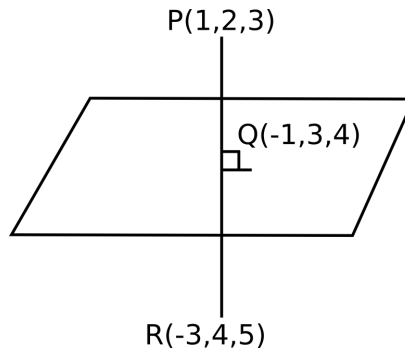$$= \frac{2}{4} \times 3 \times \frac{1}{\sqrt{3}} \times \log 3$$

$$= \sqrt{3} \log \sqrt{3}$$

4.

Difficulty : Medium

Topics :

GEOMETRY,

P(1,2,3)

Q(-1,3,4)

R(-3,4,5)

$Q(-1, 3, 4)$

$\vec{n} = \vec{PQ} = 2\hat{i} - \hat{j} - \hat{k}$

Equation of plane $2x - y - z + d = 0$

$(-1, 3, 4)$ satisfies it

$2x - y - z + 9 = 0$

$(-3, 2, 1)$ satisfies it.

5.

Difficulty : Medium

Topics :

Determinants & Matrices,

$x + ay + z = 3$

$x + 2y + 2z = 6$

$x + 5y + 3z = b$

$\begin{vmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{vmatrix} = 0 \begin{vmatrix} 3 & a & 1 \\ 6 & 2 & 2 \\ b & 5 & 3 \end{vmatrix} \neq 0$

$a = -1 \qquad b \neq 9$

6.

Difficulty : Medium

Topics :

Indefinite integration,

$$\int \frac{2x+5}{\sqrt{7-6x-x^2}}.dx$$

Let $7 - 6x - x^2 = t^2$

$(-6, -2x).dx = 2t.dt$

$$-\int \frac{2t.dt}{t} - \int \frac{1}{\sqrt{16-(x+3)^2}}dx$$

$$-2t - \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

$A = -2, B = -1$

7.

Difficulty : Easy

Topics :

Limits,
Maxima & Minima,

$f(x) = ax^4 + bx^3 + cx^2$

$f'(x) = 12ax^2 + 6bx + 2c$

$\lim_{x \to 0} \dfrac{f''(x)}{2} = 2$

$2c = 4$

$c = 2$

$f'(x) = 4ax^3 + 3bx^2 + 2cx$

$f'(1) = 0 \quad f'(2) = 0$

$4a + 3b + 4 = 0$

$32a + 12b + 8 = 0$

$8a + 3b + 2 = 0 ..... (i)$

$4a + 3b + 4 = 0 ..... (ii)$

$(i) - (ii) \times (ii)$

$-3b - 6 = 0$

$b = -2 ; a = \dfrac{1}{2}$

$f(-1) = a - b + c$

$= \dfrac{1}{2} + 2 + 2$

$= \dfrac{9}{2}$

8.

Difficulty : Medium

Topics :

Definite integration,

$$I = \int_{\pi/4}^{3\pi/4} \frac{x}{1 + \sin x} dx$$

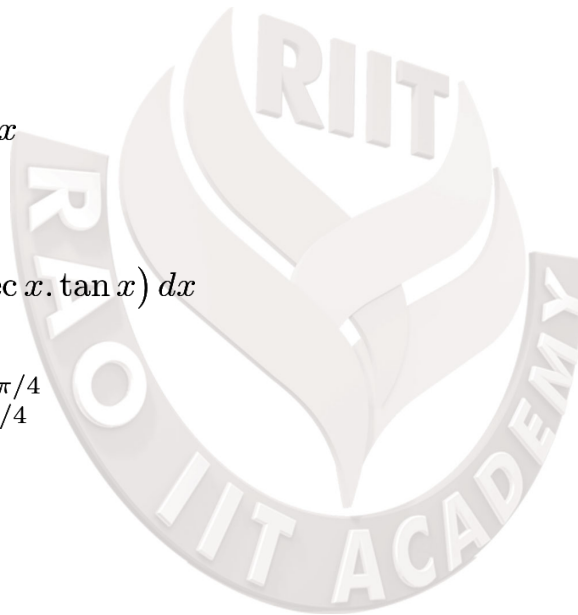$$I = \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1 + \sin x} dx$$

$$2I = \pi \int_{\pi/4}^{3\pi/4} \frac{1}{1 + \sin x} dx$$

$$I = \frac{\pi}{2} \int_{\pi/4}^{3\pi/4} \frac{1 - \sin x}{\cos^2 x} . dx$$

$$= \pi/2 \int_{\pi/4}^{3\pi/4} \left(\sec^2 x - \sec x . \tan x \right) dx$$

$$= \pi/2 \left(\tan x - \sec x \right)_{\pi/4}^{3\pi/4}$$
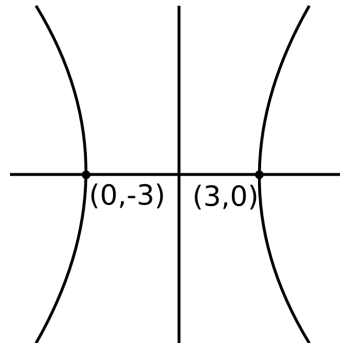
$$= \pi \left(\sqrt{2} - 1 \right)$$

9.

Difficulty : Medium

Topics :

Hyperbola,

(0,-3)   (3,0)

$$\frac{3x}{\sec\theta} + \frac{2y}{\tan\theta} = 13$$

$$h = \frac{13\sec\theta}{3} \qquad k = \frac{13\tan\theta}{2}$$

$$9x^2 - 4y^2 = 169$$

10.

B

Difficulty : Medium

Topics :

Mathematics,

$$\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta} = \left|\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right| - \left|\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right|$$

$$\frac{\theta}{2} = 120^0$$

$$\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} + \cos\frac{\theta}{2} - \sin\frac{\theta}{2}$$

$$= 2\cos\frac{\theta}{2}$$

statement P is false.

Statement Q :

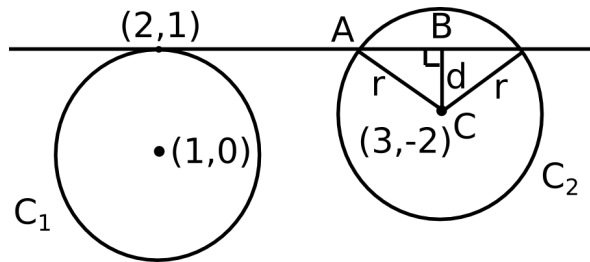$$cos\left(\frac{A+C}{2}\right) + cos\left(\frac{2\pi - (A+C)}{2}\right)$$

$$= 0$$

statement Q is true.

11.

Difficulty : Medium

Topics :

Mathematics,

Equation of tangent at point $(2, 1)$ for circle C_1

$2x + y - (x + 2) - 1 = 0$

$\Rightarrow x + y = 3$

r distance of centre $(3, -2)$ from line $x + y - 3 = 0$

$d = \left| \dfrac{3 - 2 - 3}{\sqrt{2}} \right| = \sqrt{2}$

In $\Delta ABC \ r^2 = d^2 + 2^2$
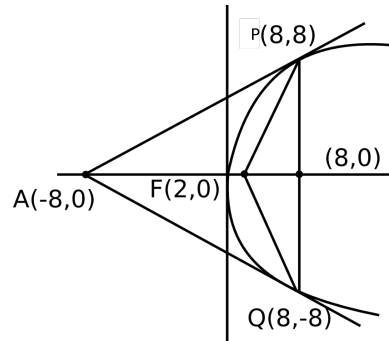
$\Rightarrow r^2 = 2 + 4$

$r = \sqrt{6}$

12.

Difficulty : Medium

Topics :

Parabola,

Equation of tangent for parabola $y^2 = 8x$

$$y = mx + \frac{2}{m}$$

tangent passing through $(-8, 0)$

$$0 = -8m + \frac{2}{m}$$

$$m^2 = \frac{1}{4}$$

$$m = \pm \frac{1}{2}$$

for point $P\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{2}{1/2}, \frac{4}{1/2}\right) = (8, 8)$

$$Q\left(\frac{2}{1/4}, \frac{4}{-1/2}\right) = (8, -8)$$

Area of $\Delta PFQ = \frac{1}{2} \times 16 \times 6 = 48 sq. units.$

13.

Difficulty : Easy

Topics :

STATISTICS,

$$Mean = 8 = \frac{7 + 8 + 9 + 7 + 8 + 7 + \lambda + 8}{8}$$

$$64 = 54 + \lambda$$

$$\lambda = 10$$

variance $= r^2 = \sum \frac{(x - M)^2}{N}$

$$= \frac{3(7 - 8)^2 + 3(8 - 8)^2 + (9 - 8)^2 + (10 - 8)^2}{8}$$

$$= \frac{3 + 0 + 1 + 4}{8}$$

$$= \frac{8}{8} = 1$$

14. B

Difficulty : Difficult

Topics :

Sequence & Series,

$$A_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 \ldots\ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$$

$$A_n = \frac{\frac{3}{4}\left(\left(\frac{3}{4}\right)^n - 1\right)}{-\frac{7}{4}} = \frac{3}{7}\left(1 - \left(\frac{-3}{4}\right)^n\right)$$

$$B_n = \frac{4}{7} + \frac{3}{7}\left(\frac{-3}{4}\right)^n$$

$$B_n > A_n$$

$$\frac{4}{7} + \frac{3}{7}\left(-\frac{3}{4}\right)^n > \frac{3}{7} - \frac{3}{7}\left(-\frac{3}{4}\right)^n$$

$$6\left(-\frac{3}{4}\right)^n > -1$$

$$\left(-\frac{3}{4}\right)^n > -\frac{1}{6}$$

Least odd natural number $= 7$

15.

Difficulty : Easy

Topics :

MATRICES,

Given $(A - 3I)(A - 5I) = 0$

$A^2 - 8A + 15I = 0.....(i)$

$\alpha A + \beta A^{-1} = 4I$

multiplying by A

$\alpha A^2 + \beta I = 4A$

$\Rightarrow \alpha A^2 - 4A + \beta I = 0$

$A^2 - \left(\dfrac{4}{\alpha}\right) A + \left(\dfrac{\beta}{\alpha}\right) I = 0.......(ii)$

by comparing (i) and (ii)

$\alpha = \dfrac{1}{2}; \beta = \dfrac{15}{2}$

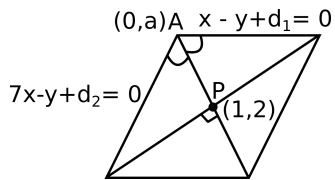$\alpha + \beta = \dfrac{1}{2} + \dfrac{15}{2} = 8$

16.

Difficulty : Medium

Topics :

Straight lines,

$$x - y + 2 = 0$$
$$7x - y + 3 = 0$$

Let two sides be

$$x - y + d_1 = 0$$
$$7x - y + d_2 = 0$$

Slope of AP $= m$

$$\left|\frac{m - 1}{1 + m}\right| = \left|\frac{m - 7}{1 + 7m}\right|$$

$$m = \frac{-1}{2}, 2$$

$$m_{np} = \frac{2 - a}{1 - 0}$$

$$2 - a = \frac{-1}{2}$$

$$a = \frac{5}{2}$$

17.

Difficulty : Difficult

Topics :

Three Dimensional Geometry,

$$l + 3m + 5n = 0$$
$$l = -3m - 5n$$
$$(5m + 6n)(3m + 5n) + 2mn = 0$$
$$15m^2 + 30n^2 + 45mn = 0$$
$$m = -n \quad m = -2n$$

Put in $l + 3m + 5n = 0$

$$l = -2n \quad l = n$$
$$l^2 + m^2 + n^2 = 1$$
$$n = \pm\frac{1}{\sqrt{6}} \qquad n = \pm\frac{1}{\sqrt{6}}$$
$$l = \pm\frac{2}{\sqrt{6}} \qquad l = \pm\frac{1}{\sqrt{6}}$$
$$m = \pm\frac{1}{\sqrt{6}} \qquad m = \pm\frac{2}{\sqrt{6}}$$
$$\cos\theta = \frac{1}{6} - \frac{2}{6} + \frac{2}{6}$$
$$\theta = \cos^{-1}\left(\frac{1}{6}\right)$$

18.

Difficulty : Easy

Topics :

Limits,

$$\lim_{x \to 0} \frac{\left[x \times \frac{2\tan x}{1 - \tan^2 x} - 2x\tan x\right]}{(2\sin^2 x)^2}$$

$$\Rightarrow \lim_{x \to 0} \frac{2x\tan x(1 - 1 + \tan^2 x)}{(1 - \tan^2 x)4\sin^4 x}$$

$$\Rightarrow \lim_{x \to 0} \frac{\tan^2 x}{2x^2}$$

$$= \frac{1}{2}$$

19.

Difficulty : Medium

Topics :

Quadratic Equations,

$$f(x) = (ax + b)(x + 1)$$
$$f(1) + f(2) = 0$$
$$\Rightarrow (a + b)2 + (2a + b)3 = 0$$
$$\Rightarrow 8a + 5b = 0$$
$$f(x) = \left(ax - \frac{8a}{5}\right)(x + 1)$$
$$\quad = a\left(x - \frac{8}{5}\right)(x + 1)$$

20.

Difficulty : Medium

Topics :

Permutations & Combinations,

$$B'S - 1 \quad A'S - 2$$
$$R'S - 2 \quad C'S - 1$$
$$K'S - 1$$

2 Ideatical, 2 Identical or 2 Identical, 2 different or 4 different letters can be taken

$$\Rightarrow {}^2C_2 \frac{4!}{2!2!} + {}^2C_1 \, {}^4C_2 \frac{4!}{2!} + {}^5C_4 4!$$
$$\Rightarrow 6 + 24 \frac{24}{4} + 5.24 \Rightarrow 6 + 11.24 = 270$$

21.

Difficulty : Easy

Topics :

Equation of continuty,

$$f(2) = \lim_{x+2}(x - 1)^{\frac{1}{2-x}}$$
$$k = \lim_{x+2}\left([1 + (x - 2)]^{\frac{1}{x-2}}\right)^{-1}$$
$$\quad = e^{-1}$$

22.

Difficulty : Easy

Topics :

Functions,

$$yx - 2y = x - 1$$
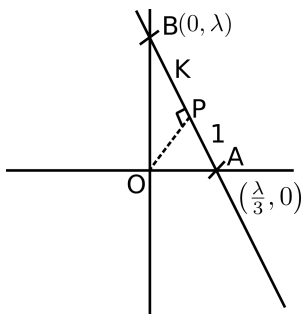$$-x + yx = 2y - 1$$
$$x = \frac{2y - 1}{y - 1}$$

23.

Difficulty : Medium
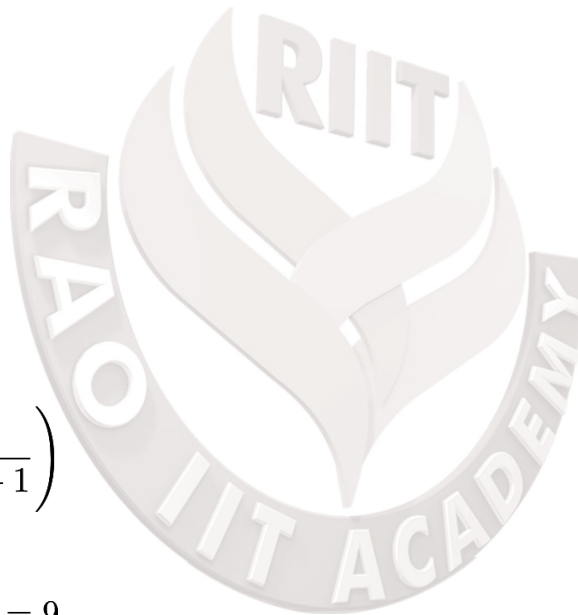
Topics :

Straight lines,

Let $BP : PA = K : 1$

$$P = \left(\frac{\left(\frac{K\lambda}{3} + 0\right)}{K + 1}, \frac{\lambda}{K + 1} \right)$$

$$m_{OP} \, m_{AB} = -1$$

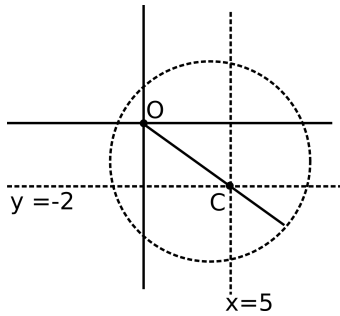$$\Rightarrow -3\frac{3}{K} = -1 \quad \Rightarrow K = 9$$

24.

Difficulty : Medium

Topics :

Complex Numbers,
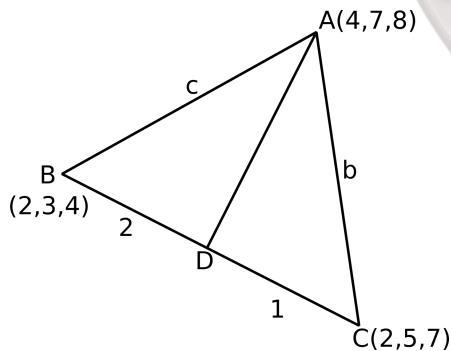

$$p(z) \quad c(3, -2) \quad r = 4$$
$$|z|_{\min} = 0$$
$$|z|_{\max} = OC + r$$
$$= \sqrt{9 + 4} + 4$$
$$= \sqrt{13} + 4$$

25.

Difficulty : Medium

Topics :

Vectors,

$$b = AC = \sqrt{4 + 4 + 1}$$
$$= 3$$
$$c = AB = \sqrt{4 + 16 + 16}$$
$$= 6$$
$$D = x = \frac{4 + 2}{2 + 1} \quad y = \frac{10 + 3}{3} \quad z = \frac{14 + 4}{3}$$
$$= 2 \qquad = \frac{13}{3} \qquad = 6$$

26.

Difficulty : Medium

Topics :

Height & Distance,
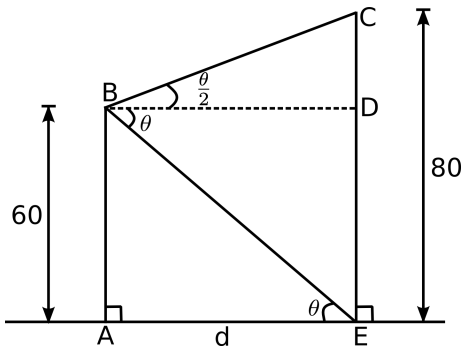
$$\tan \theta = \frac{60}{d}$$

$$\tan \left(\frac{\theta}{2} \right) = \frac{20}{d}$$

$$\frac{2}{1 - \tan^2 \frac{\theta}{2}} = 3 \qquad \Rightarrow \frac{2}{3} = 1 - \tan^2 \frac{\theta}{2}$$

$$\Rightarrow \tan^2 \frac{\theta}{2} = \frac{1}{3}$$

$$\Rightarrow \frac{\theta}{2} = 30$$

$$\theta = 60$$

Now $d = \dfrac{60}{\tan 60} = \dfrac{60}{r_3} = 20\sqrt{3}$

27.

Difficulty : Medium

Topics :

Differential Equations,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \text{ put } y = ux$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{u^2 - 1}{2u}$$

$$\Rightarrow x\frac{du}{dx} = \frac{u^2 - 1 - 2u^2}{2u}$$

$$\Rightarrow \quad = \frac{-1 - u^2}{2u}$$

$$\Rightarrow \int \frac{2u}{1 + u^2}du = -\int \frac{dx}{x}$$

$$\Rightarrow In(1 + u^2) = -Inx + Inc$$

$$\Rightarrow In\left(\frac{x^2 + y^2}{x^2}\right) = In\left(\frac{c}{x}\right)$$

$$\Rightarrow x^2 + y^2 = cx$$

at $(1, 1)$ $c = 2$

$$x^2 + y^2 = 2x$$

28.

Difficulty : Medium

Topics :

Probability,

$$P(X) = P + (1 - P)\left((1 - P)\frac{1}{2}\right)^2 P + \dots\infty$$

$$= \frac{P}{1 - (1 - P)\frac{1}{2}} = \frac{2P}{1 + P}$$

$$P(Y) = (1 - P)\frac{1}{2} + \left((1 - P)\frac{1}{2}\right)^2 + \dots\infty$$

$$= \frac{(1 - P)\frac{1}{2}}{1 - (1 - P)\frac{1}{2}} = \frac{1 - P}{1 + P}$$

$$P(X) = P(Y) = 2P = 1 - P \Rightarrow \frac{1}{3}$$

29.

Difficulty : Medium

Topics :

Sequence & Series,

$$3b = \frac{3}{4} \Rightarrow b = \frac{1}{4} \quad b^4 = (ac)^2$$

$$\Rightarrow \frac{1}{4^4} = ((b-d)(b+d))^2$$

$$\Rightarrow \pm\frac{1}{4^2} = b^2 - d^2$$

$$\Rightarrow \pm\frac{1}{16} = \frac{1}{16} - d^2$$

$$\Rightarrow d = 0, d^2 = \frac{2}{16} \quad d = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow a = \frac{1}{4} - \frac{1}{2\sqrt{2}}$$

30.

Difficulty : Difficult

Topics :

Binomial Theorem,

$$\left({}^2C_0 + {}^2C_1 x + {}^2C_2 X^2\right) \times \left({}^3C_0 + {}^3C_1 X^2 + {}^3C_2 X^4 + {}^3C_3 X^6\right)$$
$$\times \left({}^4C_0 + {}^4C_1 X^3 + {}^4C_2 X^6 + {}^4C_3 X^9 + {}^4C_4 X^{12}\right)$$

Coefficient of $X^{10} = {}^2C_0\,{}^3C_0 0 + {}^2C_0\,{}^3C_1 0 + {}^2C_0\,{}^3C_2\,{}^4C_2 + {}^2C_0\,{}^3C_3 0 + {}^2C_1\,{}^3C_0\,{}^4C_3$

$$+ {}^2C_1\,{}^3C_3\,{}^4C_1 + {}^2C_2\,{}^3C_1\,{}^4C_2$$

$$= \frac{3.4.3}{2} + 2.4 + 2.4 + \frac{3.4.3}{2}$$

$$= 18 + 8 + 8 + 18 = 36 + 16$$

$$= 52$$