Example 3·1. *What is the chance that a leap year selected at random will contain 53 Sundays ?*

Solution. In a leap year (which consists of 366 days), there are 52 complete weeks and 2 days over. The following are the possible combinations for these two '*over*' days : (*i*) Sunday and Monday, (*ii*) Monday and Tuesday, (*iii*) Tuesday and Wednesday, (*iv*) Wednesday and Thursday, (*v*) Thursday and Friday, (*vi*) Friday and Saturday, and (*vii*) Saturday and Sunday.

In order that a leap year selected at random should contain 53 Sundays, one of the two 'over' days must be Sunday. Since out of the above 7 possibilities, 2, *viz.*, (*i*) and (*vii*), are favourable to this event.

$$\therefore \qquad \text{Required probability} = \frac{2}{7} \cdot$$

Example 3·2. *Two unbiased dice are thrown. Find the probability that :*

(*i*) *both the dice show the same number,*

(*ii*) *the first die shows 6,*

(iii) the total of the numbers on the dice is 8.

(iv) the total of the numbers on the dice is greater than 8,

(v) the total of the numbers on the dice is 13, and

(vi) the total of the numbers on the dice is any number from 2 to 12, both inclusive.

Solution. In a random throw of two dice, since each of the six faces of one die can be associated with each of six faces of the other die, the total number of cases is $6 \times 6 = 36$, as given below :

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Here, the expression, say, (i, j) means that the first die shows the number i and the second die shows the number j. Obviously, $(i, j) \neq (j, i)$ if $i \neq j$.

$\therefore$ Exhaustive number of cases $(n) = 36$.

(*a*) The favourable cases that both the dice show the same number are :

$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$ and $(6, 6)$, *i.e.*, $m = 6$.

$\therefore$ Probability that the two dice show the same number $= \frac{6}{36} = \frac{1}{6}$.

(*b*) The favourable cases that the first die shows 6 are :

$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5)$ and $(6, 6)$, *i.e.*, 6 in all.

$\therefore$ Probability that the first die shows '6' $= \frac{6}{36} = \frac{1}{6}$.

(*c*) The cases favourable to getting a total of 8 on the two dice are :

$(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)$, *i.e.*, $m = 5$.

$\therefore$ Probability that total of numbers on two dice is 8 $= \frac{5}{36}$.

(*d*) The cases favourable to getting a total of more than 8 are :

$(3, 6), (6, 3), (4, 5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)$, *i.e.*, $m = 10$.

$\therefore$ Probability that the total of numbers on two dice is greater than 8 $= \frac{10}{36} = \frac{5}{18}$.

(*e*) This is an example of an *impossible event*, since the maximum total can be $6 + 6 = 12$. Therefore, the required probability is 0.

(*f*) The probability is 1, as the total of the numbers on the two dice certainly ranges from 2 to 12. The given event is called a *certain event*.

Example 3·3. (*a*) *Among the digits 1, 2, 3, 4, 5 at first one is chosen and then a second selection is made among the remaining four digits. Assuming that all twenty possible outcomes have equal probabilities, find the probability that an odd digit will be selected*

(i) the first time, (ii) the second time, and (iii) both times.

(*b*) *From 25 tickets, marked with first 25 numerals, one is drawn at random. Find the chance that (i) it is multiple of 5 or 7, and (ii) it is a multiple of 3 or 7.*

Solution. (*a*) Total number of cases $= 5 \times 4 = 20$.

(*i*) Now there are 12 cases in which the first digit drawn is odd, *viz.*, (1, 2), (1, 3), (1, 4), (1, 5), (3, 1), (3, 2), (3, 4), (3, 5), (5, 1), (5, 2), (5, 3) and (5, 4).

$\therefore$ The probability that the first digit drawn is odd $= \frac{12}{20} = \frac{3}{5}$.

(*ii*) Also there are 12 cases in which the second digit drawn is odd, *viz.*, (2, 1), (3, 1), (4, 1), (5, 1), (1, 3), (2, 3), (4, 3), (5, 3), (1, 5), (2, 5), (3, 5) and (4, 5).

$\therefore$ The probability that the second digit drawn is odd $= \frac{12}{20} = \frac{3}{5}$.

(*iii*) There are six cases in which both the digits drawn are odd, *viz.*, (1, 3), (1, 5), (3, 1), (3, 5), (5, 1) and (5, 3).

$\therefore$ The probability that both the digits drawn are odd $= \frac{6}{20} = \frac{3}{10}$.

(*b*) (*i*) Numbers (out of the first 25 numerals) which are multiples of 5 are 5, 10, 15, 20 and 25, *i.e.*, 5 in all and the numbers which are multiples of 7 are 7, 14 and 21, *i.e.*, 3 in all. Hence required number of favourable cases are $5 + 3 = 8$.

$\therefore$ Required probability $= \frac{8}{25}$.

(*ii*) Numbers (among the first 25 numerals) which are multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, *i.e.*, 8 in all ; and the numbers, which are multiples of 7 are 7, 14, 21, *i.e.*, 3 in all. Since the number 21 is common in both the cases, the required number of distinct favourable cases is $8 + 3 - 1 = 10$.

$\therefore$ Required probability $= \frac{10}{25} = \frac{2}{5}$.

Example 3·4. (*a*) *Four cards are drawn at random from a pack of 52 cards. Find the probability that*

(*i*) *They are a king, a queen, a jack and an ace.*

(*ii*) *Two are kings and two are queens.*

(*iii*) *Two are black and two are red.*

(*iv*) *There are two cards of hearts and two cards of diamonds.*

(*b*) *In shuffling a pack of cards, four are accidentally dropped, find the chance that the missing cards should be one from each suit.*

Solution. Four cards can be drawn from a well-shuffled pack of 52 cards in ${}^{52}C_4$ ways, which gives the exhaustive number of cases.

(*i*) 1 king can be drawn out of the 4 kings in ${}^4C_1$ ways. Similarly, 1 queen, 1 jack and an ace can each be drawn in ${}^4C_1 = 4$ ways. Since any one of the ways of drawing a king can be associated with any one of the ways of drawing a queen, a jack and an ace, the favourable number of cases are ${}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1$.

$$\text{Hence the required probability } = \frac{{}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1}{{}^{52}C_4} = \frac{256}{{}^{52}C_4}.$$

(*ii*) $$\text{Required probability } = \frac{{}^4C_2 \times {}^4C_2}{{}^{52}C_4}.$$

(*iii*) Since there are 26 black cards (of spades and clubs) and 26 red cards (of diamonds and hearts) in a pack of cards, the required probability $= \frac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$.

(*iv*) $$\text{Required probability } = \frac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4}.$$

(b) There are ${}^{52}C_4$ possible ways in which four cards can slip while shuffling a pack of cards. The favourable number of cases in which the four cards can be one from each suit is : ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$.

$$\therefore \quad \text{The required probability} = \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4} = \frac{2197}{20825}.$$

Example 3·5. *What is the probability of getting 9 cards of the same suit in one hand at a game of bridge ?*

Solution. Since one hand in a bridge game consists of 13 cards, the exhaustive number of cases is ${}^{52}C_{13}$.

The number of ways in which 9 cards of a suit can come out of 13 cards of the suit $= {}^{13}C_9$. The number of ways in which balance $13 - 9 = 4$ cards can come in one hand out of a balance of 39 cards of other suits is ${}^{39}C_4$.

Since there are four different suits and 9 cards of any suit can come, by the principle of counting, the total number of favourable cases of getting 9 cards of suit $= {}^{13}C_4 \times {}^{39}C_4 \times 4$.

$$\therefore \quad \text{Required probability} = \frac{{}^{13}C_9 \times {}^{39}C_4 \times 4}{{}^{52}C_{13}}.$$

Example 3·6. *A man is dealt 4 spade cards from an ordinary pack of 52 cards. If he is given three more cards, find the probability p that at least one of the additional cards is also a spade.*

Solution. After a man has dealt 4 spade cards from an ordinary pack of 52 cards, there are $52 - 4 = 48$ cards left in the pack, out of which 9 are spade cards and 39 are non-spade cards.

Since 3 more cards can be dealt to the same man out of the 48 cards in ${}^{48}C_3$ ways, the exhaustive number of outcomes $= {}^{48}C_3$.

If none of these 3 additional cards is a spade card, then the 3 additional cards must be drawn out of the 39 non-spade cards, which can be done in ${}^{39}C_3$ ways. The probability that none of the three additional cards dealt to the man is a spade card is given by ${}^{39}C_3/{}^{48}C_3$.

Hence, the probability p that at least one of the three additional cards is also a spade is given by :

$$p = 1 - P\,[\text{None of the three additional cards is a spade.}]$$

$$= 1 - \frac{{}^{39}C_3}{{}^{48}C_3} = 1 - \frac{39 \times 38 \times 37}{3\,!} \times \frac{3\,!}{48 \times 47 \times 46} = 1 - \frac{13 \times 19 \times 37}{16 \times 47 \times 23} = 0{\cdot}4718.$$

Example 3·7. *A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, 2 officers of the sales department and 1 chartered accountant. Find the probability of forming the committee in the following manner :*

(i) There must be one from each category.

(ii) It should have at least one from the purchase department.

(iii) The chartered accountant must be in the committee.

Solution. There are $3 + 4 + 2 + 1 = 10$ persons in all and a committee of 4 people can formed out of them in ${}^{10}C_4$ ways. Hence, exhaustive number of cases is :

$${}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4\,!} = 210$$

(i) Favourable number of cases for the committee to consist of 4 members, one from each category, is : ${}^4C_1 \times {}^3C_1 \times {}^2C_1 \times 1 = 4 \times 3 \times 2 = 24$

$$\therefore \quad \text{Required probability} = \frac{24}{210} = \frac{4}{35}.$$

(ii) P [Committee has at least one purchase officer]

$$= 1 - P \text{ (Committee has no purchase officer)}$$

In order that the committee has no purchase officer, all the 4 members are to be selected from amongst officers of production department, sales department and chartered accountant, *i.e.*, out of 3 + 2 + 1 = 6 members and this can be done in ${}^6C_4 = \dfrac{6 \times 5}{1 \times 2} = 15$ ways. Hence

$$P \text{ (Committee has no purchase officer)} = \frac{15}{210} = \frac{1}{14}$$

$$\therefore \quad P \text{ (Committee has at least one purchase officer)} = 1 - \frac{1}{14} = \frac{13}{14}.$$

(iii) Favourable number of cases that the committee consists of a chartered accountant as a member and three others are : $1 \times {}^9C_3 = \dfrac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84$ ways, since a chartered accountant can be selected out of one chartered accountant in only 1 way and the remaining 3 members can be selected out of the remaining $10 - 1 = 9$ persons in ${}^9C_3$ ways. Hence the required probability $= \dfrac{84}{210} = \dfrac{2}{5}$.

Example 3·8. *An urn contains 6 white, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that : (i) two of the balls drawn are white, (ii) one is of each colour, (iii) none is red, (iv) at least one is white.*

Solution. Total number of balls in the urn is 6 + 4 + 9 = 19. Since 3 balls can be drawn out of 19 in ${}^{19}C_3$ ways, the exhaustive number of cases are ${}^{19}C_3$.

(i) If 2 balls of the 3 drawn balls are to be white, these two balls should be drawn out of 6 white balls which can be done in ${}^6C_2$ ways, and the third ball can be drawn out of the remaining $19 - 6 = 13$ balls, which can be done in ${}^{13}C_1$ ways. Since any of the former ways can be associated with any one of the later ways, the number of favourable cases $= {}^6C_2 \times {}^{13}C_1$.

$$\text{Hence, required probability} = \frac{{}^6C_2 \times {}^{13}C_1}{{}^{19}C_3}.$$

(ii) Since the number of favourable cases of getting one ball of each colour is ${}^6C_1 \times {}^4C_1 \times {}^9C_1$, the required probability $= \dfrac{{}^6C_1 \times {}^4C_1 \times {}^9C_1}{{}^{19}C_3}$.

(iii) If none of the drawn balls is red, then all the 3 balls must be out of the white and black balls, *viz.*, out of 6 + 9 = 15 balls. Hence the number of favourable cases for this event is ${}^{15}C_3$.

$$\therefore \quad \text{Required probability} = \frac{{}^{15}C_3}{{}^{19}C_3}.$$

(iv) P (at least one ball is white) $= 1 - P$ (none of the three balls is white) ... (*)

In order that none of the three balls is white, all the three balls must be drawn out of the red and black balls, *i.e.*, out of 4 + 9 = 13 balls and this can be done in ${}^{13}C_3$ ways.

$$\text{Hence} \quad P \text{ (none of the three balls is white)} = \frac{{}^{13}C_3}{{}^{19}C_3}.$$

Substituting in (*), we obtain $\quad P$ (at least one ball is white) $= 1 - \dfrac{^{13}C_3}{^{19}C_3}$.

Example 3·9. *In a random arrangement of the letters of the word 'COMMERCE', find the probability that all the vowels come together.*

Solution. The total number of permutations of the letters of the word 'COMMERCE' are $(8\,!)/(2\,!\,2\,!\,2\,!)$, because it contains 8 letters of which 2 are C's, 2 M's, and 2 E's, and remaining are all different.

The word COMMERCE contains 3 vowels, *viz.*, OEE (2 E's being identical). To obtain the total number of arrangements in which these 3 vowels come together, we regard them as tied together, forming only one letter so that total number of letters in COMMERCE may be taken as $8 - 2 = 6$, out of which 2 are C's, 2 are M's and rest distinct and, therefore, their number of arrangement is given by $(6\,!)/(2\,!\,2\,!)$.

Further, the three vowels OEE, two of which are identical, can be arranged among themselves in $3\,!/2\,!$ ways. Hence, the total number of arrangements favourable to getting all vowels together $= \dfrac{6\,!}{2\,!\,2\,!} \times \dfrac{3\,!}{2\,!}$.

Hence, the required probability $= \dfrac{6\,!\,3\,!}{2\,!\,2\,!\,2} \div \dfrac{8\,!}{2\,!\,2\,!\,2\,!} = \dfrac{3}{28}$.

Example 3·10. *(a) If the letters of the word 'REGULATIONS' be arranged at random, what is the chance that there will be exactly 4 letters between R and E ?*

(b) What is the probability that four S's come consecutively in the word 'MISSISSIPPI' ?

Solution. (*a*) The word 'REGULATIONS' consists of 11 letters. The two letters R and E can occupy $^{11}P_2$, *i.e.*, $11 \times 10 = 110$ positions.

The number of ways in which there will be exactly 4 letters between R and E are enumerated below :

(*i*) R is in the 1st place and E is in the 6th place.

(*ii*) R is in the 2nd place and E is in the 7th place.

... ...

... ...

... ...

(*vi*) R is in the 6th place and E is in the 11th place.

Since R and E can interchange their positions, the required number of favourable cases is $2 \times 6 = 12$.

$\therefore$ The required probability $= \dfrac{12}{110} = \dfrac{6}{55}$.

(*b*) Total number of permutations of the 11 letters of the word '*MISSISSIPPI*' in which 4 are of one kind (*viz.*, S), 4 of other kind (*viz.*, I), 2 of third kind (*viz.*, P) and 1 of fourth kind (*viz.*, M) are $11\,!/4\,!\,4\,!\,2\,!\,1\,!$.

Following are the 8 possible combinations of 4 S's coming consecutively :

(*i*)	S	S	S	S							
(*ii*)	—	S	S	S	S						
(*iii*)	—	—	S	S	S	S					
⋮			⋮			⋮					
(*viii*)	—	—	—	—	—	—	—	S	S	S	S

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, *viz.*, *MIIIPPI* of which 4 are of one kind, 2 of other kind and one of third kind are $\frac{7!}{4!2!1!}$, the required number of favourable cases $= \frac{8 \times 7!}{4!2!1!}$.

$$\therefore \quad \text{Required probability} = \frac{8 \times 7!}{4!2!1!} \div \frac{11!}{4!4!2!1!} = \frac{8 \times 7! \times 4!}{11!} = \frac{4}{165}.$$

Example 3·11. *Twenty-five books are placed at random in a shelf. Find the probability that a particular pair of books shall be : (i) Always together, and (ii) Never together.*

Solution. Since 25 books can be arranged among themselves in 25 ! ways, the exhaustive number of cases is 25 !

(*i*) Let us now regard that the two particular books are tagged together so that we shall regard them as a single book. Thus, now we have $(25 - 1) = 24$ books which can be arranged among themselves in 24 ! ways. But the two books which are fastened together can be arranged among themselves in 2 ! ways. Hence, associating these two operations, the number of favourable cases for getting a particular pair of books always together is $24! \times 2!$.

$$\therefore \quad \text{Required probability} = \frac{24! \times 2!}{25!} = \frac{2}{25}.$$

(*ii*) Total number of arrangements of 25 books among themselves is 25 ! and the total number of arrangements that a particular pair of books will always be together is $24! \times 2$. Hence, the number of arrangements in which a particular pair of books is never together is : $25! - 2 \times 24! = (25 - 2) \times 24! = 23 \times 24!$

$$\therefore \quad \text{Required probability} = \frac{23 \times 24!}{25!} = \frac{23}{25}.$$

Aliter :

P [A particular pair of books shall never be together.]

$= 1 - P$ [A particular pair of books is always together.]

$$= 1 - \frac{2}{25} = \frac{23}{25}.$$

Example 3·12. *n persons are seated on n chairs at a round table. Find the probability that two specified persons are sitting next to each other.*

Solution. Since n persons can be seated in n chairs at a round table in $(n - 1)!$ ways, the exhaustive number of cases $= (n - 1)!$.

Assuming the two specified persons A and B who sit together as one, we get $(n - 1)$ persons in all, who can be seated at a round table in $(n - 2)!$ ways. Further, since A and B can interchange their positions in 2 ! ways, total number of favourable cases of getting A and B together is $(n - 2)! \times 2!$.

$$\therefore \quad \text{Required probability} = \frac{(n-2)! \times 2!}{(n-1)!} = \frac{2}{n-1}.$$

Example 3·13. *A five-figure number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4.*

Solution. The total number of ways in which the five digits 0, 1, 2, 3, 4 can be arranged among themselves is 5 !. Out of these, the number of arrangements which begin with 0 (and, therefore, will give only 4-digited numbers) is 4 !. Hence the total number of five-digited numbers that can be formed from the digits 0, 1, 2, 3, 4 is :

$$5! - 4! = 120 - 24 = 96.$$

The number formed will be divisible by 4 if number formed by the two digits on extreme right (*i.e.*, the digits in the unit and tens places) is divisible by 4. Such numbers are :

$$04, \quad 12, \quad 20, \quad 24, \quad 32, \quad \text{and} \quad 40.$$

If the numbers end in 04, the remaining three digits, *viz.*, 1, 2 and 3 can be arranged among themselves in 3 ! ways. Similarly, the number of arrangements of the numbers ending with 20 and 40 is 3 ! in each case.

If the numbers end with 12, the remaining three digits 0, 3, 4 can be arranged in 3 ! ways. Out of these we shall reject those numbers which start with 0 (*i.e.*, have 0 as the first digit). There are $(3 - 1)! = 2\,!$ such cases. Hence, the number of five-digited numbers ending with 12 is $3\,! - 2\,! = 6 - 2 = 4$.

Similarly, the number of five digited numbers ending with 24 and 32 each is 4. Hence the total number of favourable cases is $3 \times 3\,! + 3 \times 4 = 18 + 12 = 30$.

Hence, the required probability $= \dfrac{30}{96} = \dfrac{5}{16}$.

Example 3·14. *(a) Twelve balls are distributed at random among three boxes. What is the probability that the first box will contain 3 balls ?*

(b) If n biscuits be distributed among N persons, find the chance that a particular person receives r (< n) biscuits.

Solution. (*a*) Since each ball can go to any one of the there boxes, there are 3 ways in which a ball can go to any one of the three boxes. Hence there are 3^{12} ways in which 12 balls can be placed in the three boxes.

Number of ways in which 3 balls out of 12 can go to the first box is ${}^{12}C_3$. Now the remaining 9 balls are to be placed in remaining 2 boxes and this can be done in 2^9 ways. Hence, the total number of favourable cases $= {}^{12}C_3 \times 2^9$.

$$\therefore \quad \text{Required probability} = \frac{{}^{12}C_3 \times 2^9}{3^{12}}.$$

(*b*) Take any one biscuit. This can be given to any one of the *N* beggars so that there are *N* ways of distributing any one biscuit. Hence the total number of ways in which n biscuits can be distributed at random among *N* beggars $= N.\ N \ldots (n \text{ times}) = N^n$.

'*r*' biscuits can be given to any particular beggar in ${}^nC_r$ ways. Now we are left with $(n - r)$ biscuits which are to be distributed among the remaining $(N - 1)$ beggars and this can be done in $(N - 1)^{n-r}$ ways.

$$\therefore \quad \text{Number of favourable cases} = {}^nC_r \cdot (N - 1)^{n-r}$$

$$\text{Hence, required probability} = \frac{{}^nC_r\,(N - 1)^{n-r}}{N^n}.$$

Example 3·15. *A car is parked among N cars in a row, not at either end. On his return the owner finds that exactly r of the N places are still occupied. What is the probability that both neighbouring places are empty ?*

Solution. Since the owner finds on return that exactly r of the N places (including owner's car) are occupied, the exhaustive number of cases for such an arrangement is ${}^{N-1}C_{r-1}$ [since the remaining $r - 1$ cars are to be parked in the remaining $N - 1$ places and this can be done in ${}^{N-1}C_{r-1}$ ways].

Let *A* denote the event that both the neighbouring places to owner's car are empty. This requires the remaining $(r - 1)$ cars to be parked in the remaining $N - 3$ places and hence the number of cases favourable to *A* is ${}^{N-3}C_{r-1}$. Hence,

$$P(A) = \frac{{}^{N-3}C_{r-1}}{{}^{N-1}C_{r-1}} = \frac{(N-r)(N-r-1)}{(N-1)(N-2)}.$$

Example 3·16. *What is the probability that at least two out of n people have the same birthday ? Assume 365 days in a year and that all days are equally likely.*

Solution. Since the birthday of any person can fall on any one of the 365 days, the exhaustive number of cases for the birthdays of n persons is 365^n.

If the birthdays of all n persons fall on different days, then the number of favourable cases is : $365(365-1)(365-2)\ldots[365-(n-1)]$,

because in this case the birthday of the first person can fall on any one of 365 days, the birthday of the second person can fall on any one of the remaining 364 days, and so on. Hence, the probability (p) that birthdays of all the n persons are different is given by :

$$p = \frac{365(365-1)(365-2)\ldots\{365-(n-1)\}}{365^n}$$

$$= \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\ldots\left(1-\frac{n-1}{365}\right)$$

Hence, the required probability that at least two persons have same birthday is :

$$1-p = 1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\ldots\left(1-\frac{n-1}{365}\right).$$

Example 3·17. *Compare the chances of throwing 4 with one die, 8 with two dice and 12 with three dice.*

Solution. *(i) Probability of throwing 4 with one die* : There are 6 possible ways in which the die can fall, and of these one is favourable to the required event.

$\therefore$ Required probability $(p_1) = \frac{1}{6}$.

(ii) Probability of throwing 8 with two dice : Exhaustive number of cases in single throw with two dice is $6^2 = 36$. Now the sum of '8' can be obtained on the two dice in the following ways : (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), *i.e.*, 5 cases in all, where the first and second number in the brackets ( ) refer to the numbers on the 1st and 2nd die respectively.

$\therefore$ Required probability $(p_2) = \frac{5}{36}$.

(iii) Probability of throwing 12 with three dice : The exhaustive number of ways in a single throw of three dice $= 6 \times 6 \times 6 = 216$.

To make a throw of 12, the three dice must show the faces either (6, 1, 5) or (6, 2, 4) or (6, 3, 3) or (5, 2, 5) or (5, 3, 4) or (4, 4, 4). The first two of these arrangements can occur in $3! = 6$ ways each, the second two (*i.e*, third and fourth arrangement) in $\frac{3!}{2!\,1!} = 3$ ways each, the fifth in $3! = 6$ ways and the last in one way only. Thus, the total number of favourable cases $= 6 + 6 + 3 + 3 + 6 + 1 = 25$.

$\therefore$ Required probability $(p_3) = \frac{25}{216}$.

Hence the chances of throwing 4 with one die, 8 with two dice, and 12 with three dice are :

$$p_1 : p_2 : p_3 :: \frac{1}{6} : \frac{5}{36} : \frac{25}{216} \quad \text{or} \quad 36 : 30 : 25.$$

Example 3·18. *A and B throw with three dice ; if A throws 14, find B's chance of throwing a higher number.*

Solution. To throw higher number than A, B must throw either 15 or 16 or 17 or 18.

Now a throw amounting to 18 must be made up of (6, 6, 6), which can occur in one way ; 17 can be made up of (6, 6, 5), which can occur in $\frac{3\,!}{2\,!\,1\,!} = 3$ ways ; 16 may be made up of (6, 6, 4), and (6, 5, 5), each of which arrangements can occur in $\frac{3\,!}{2\,!\,1\,!} = 3$ ways ; 15 can be made up of (6, 4, 5), or (6, 3, 6), or (5, 5, 5), which can occur in 3 !, 3 and 1 way respectively.

$\therefore$ The number of favourable cases. $= 1 + 3 + 3 + 3 + 6 + 3 + 1 = 20.$

In a random throw of 3 dice, the exhaustive number of cases $= 6^3 = 6 \times 6 \times 6 = 216$.

Hence, the required probability $= \frac{20}{216} = \frac{5}{54}$.

Example 3·19. *Each coefficient in the equation $ax^2 + bx + c = 0$, is determined by throwing an ordinary die. Find the probability that the equation will have real roots.*

Solution. The roots of the equation $ax^2 + bx + c = 0$ will be real if its discriminant is non-negative, *i.e.*, if $b^2 - 4ac \geq 0 \Rightarrow b^2 \geq 4ac$.

Since each coefficient in equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die, each of the coefficients a, b and c can take the values from 1 to 6.

$\therefore$ Total number of possible outcomes (all being equally likely) $= 6 \times 6 \times 6 = 216$.

The number of favourable cases can be enumerated as follows :

ac	*a*	*c*	*4ac*	*b* (so that $b^2 \geq 4ac$)	*No. of cases*
1	1	1	4	2, 3, 4, 5, 6	$1 \times 5 = 5$
2	(*i*) 1 (*ii*) 2	2 1	8	3, 4, 5, 6	$2 \times 4 = 8$
3	(*i*) 1 (*ii*) 3	3 1	12	4, 5, 6	$2 \times 3 = 6$
4	(*i*) 1 (*ii*) 4 (*iii*) 2	4 1 2	16	4, 5, 6	$3 \times 3 = 9$
5	(*i*) 1 (*ii*) 5	5 1	20	5, 6	$2 \times 2 = 4$
6	(*i*) 1 (*ii*) 6 (*iii*) 3 (*iv*) 2	6 1 2 3	24	5, 6	$4 \times 2 = 8$
7	($ac = 7$ *is not possible*)				
8	(*i*) 2 (*ii*) 4	4 2	32	6	$2 \times 1 = 2$
9	3	3	36	6	1
					Total = 43

Since $b^2 \geq 4ac$ and since the maximum value of b^2 is 36, $ac = 10, 11, 12, \ldots$ etc. is not possible. Hence total number of favourable cases = 43.

$\therefore$ Required probability $= \frac{43}{216}$.

Remark. The probability p that in the above case, the equation $ax^2 + bx + c = 0$, will have imaginary roots is given by :

$$p = 1 - P \text{ (That equation } ax^2 + bx + c = 0\text{, has real roots.)}$$
$$= 1 - \frac{43}{216} = \frac{173}{216}.$$

Example 3·20. *The sum of two non-negative quantities is equal to 2n. Find the chance that their product is not less than $\frac{3}{4}$ times their greatest product.*

Solution. Let $x > 0$ and $y > 0$, be the given quantities so that $x + y = 2n$. ... (*)

We know that the product of two positive quantities whose sum is constant (fixed) is greatest when the quantities are equal. Thus the product of x and y is maximum when $x = y = n$.

$\therefore$ Maximum product $= n.n = n^2$

Now $P\left(xy \not< \frac{3}{4}n^2\right) = P\left(xy \geq \frac{3}{4}n^2\right) = P\left(x(2n - x) \geq \frac{3}{4}n^2\right)$ [From (*)]

$$= P\left[(4x^2 - 8nx + 3n^2) \leq 0\right] = P\left[(2x - 3n)(2x - n) \leq 0\right]$$
$$= P\left(x \text{ lies between } \frac{n}{2} \text{ and } \frac{3n}{2}\right)$$

$\therefore$ Favourable range $= \frac{3n}{2} - \frac{n}{2} = n,$ Total range $= 2n$

Hence, the required probability $= \dfrac{\text{Favourable range}}{\text{Total range}} = \dfrac{n}{2n} = \dfrac{1}{2}.$

Example 3·21. *Out of (2n + 1) tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.*

Solution. Since out of $(2n + 1)$ tickets, 3 tickets can be drawn in ${}^{2n+1}C_3$ ways,

Exhaustive number of cases $= {}^{2n+1}C_3 = \dfrac{(2n+1)\,2n\,(2n-1)}{3\,!} = \dfrac{n(4n^2 - 1)}{3}$

To find the favourable number of cases we are to enumerate all the cases in which the numbers on the drawn tickets are in A.P with common difference, (say $d = 1, 2, 3, \ldots, n-1, n$).

If $d = 1$, the possible cases are as follows :

$$\left.\begin{array}{ccc} 1, & 2, & 3, \\ 2, & 3, & 4, \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 2n-1, & n, & 2n+1, \end{array}\right\}$$

i.e., $(2n - 1)$ cases in all

If $d = 2$, the possible cases are as follows :

$$\left.\begin{array}{ccc} 1, & 3, & 5 \\ 2, & 4, & 6 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 2n-3, & 2n-1, & 2n+1 \end{array}\right\}$$

i.e., $(2n - 3)$ cases in all ; and so on

If $d = n - 1$, the possible cases are :

$1, n, 2n - 1$; $\quad 2, n + 1, 2n$; $\quad 3, n + 2, 2n + 1$, *i.e.*, 3 cases in all.

If $d = n$, there is only one case, *viz.*, $(1, n + 1, 2n + 1)$.

Thus total number of favourable cases $= (2n - 1) + (2n - 3) + \ldots + 5 + 3 + 1$
$$= 1 + 3 + 5 + \ldots + (2n - 1),$$

which is a series in A.P. with $a = 1$, $d = 2$ and n terms.

$\therefore$ Number of favourable cases $= \frac{n}{2}\left\{1 + (2n-1\right\} = n^2$

Hence, the required probability $= \dfrac{n^2}{n\,(4n^2-1)/3} = \dfrac{3n}{(4n^2-1)}$.

Example 3·22. *If 6n tickets numbered 0, 1, 2, ..., 6n – 1 are placed in a bag and three are drawn out, show that the chance that the sum of the numbers on them is equal to 6n is 3n/(6n – 1) (6n – 2).*

Solution. The total number of ways of drawing 3 tickets out of $6n$ is given by :

$$^{6n}C_3 = n\ (6n-1)\,(6n-2).$$

Favourable cases for obtaining a sum of $6n$ on the three drawn tickets are given below :

$$\begin{array}{lllll}
(0, 1, 6n-1)\,; & (0, 2, 6n-2)\,; & \ldots\,; & (0, 3n-1, 3n+1), & i.e., (3n-1) \text{ cases} \\
(1, 2, 6n-3)\,; & (1, 3, 6n-4)\,; & \ldots\,; & (1, 3n-1, 3n), & i.e., (3n-2) \text{ cases} \\
(2, 3, 6n-5)\,; & (2, 4, 6n-6)\,; & \ldots\,; & (2, 3n-2, 3n), & i.e., (3n-4) \text{ cases} \\
(3, 4, 6n-7)\,; & (3, 5, 6n-8)\,; & \ldots\,; & (3, 3n-2, 3n-1), & i.e., (3n-5) \text{ cases} \\
\vdots & \vdots & & \vdots & \vdots \\
(2n-2, 2n-1, 2n+3)\,; & (2n-2\,; 2n, 2n+2) & & & i.e., 2 \text{ cases} \\
 & (2n-1, 2n, 2n+1) & & & i.e., 1 \text{ case}
\end{array}$$

$\therefore$ Total number of favourable cases

$$= \left\{(3n-1) + (3n-4) + \ldots + 5 + 2\right\} + \left\{(3n-2) + (3n-5) + \ldots + 4 + 1\right\} \quad \ldots(*)$$

The expression in each bracket of (*) is the sum of n terms of an Arithmetic Progression (A.P.) series.

$\therefore$ Total number of favourable cases

$$= \left\{2 + 5 + \ldots + (3n-4) + (3n-1)\right\} + \left\{1 + 4 + \ldots + (3n-5) + (3n-2)\right\}$$

$$= \frac{n}{2}\left\{2 \times 2 + (n-1)\,3\right\} + \frac{n}{2}\left\{2 \times 1 + (n-1) \times 3\right\}$$

$$= \frac{n}{2}\left(4 + 3n - 3 + 2 + 3n - 3\right) = 3n^2$$

Hence, the required probability $= \dfrac{3n^2}{n\,(6n-1)\,(6n-2)} = \dfrac{3n}{(6n-1)\,(6n-2)}$.

Example 3·23. *A, B and C are three arbitrary events. Find expression for the events noted below, in the context of A, B and C :*

(i) Only A occurs,	*(ii) Both A and B, but not C, occur,*
(iii) All three events occur,	*(iv) At least one occurs,*
(v) At least two occur,	*(vi) One and no more occurs,*
(vii) Two and no more occur,	*(viii) None occurs.*

Solution.

(i) $A \cap \overline{B} \cap \overline{C}$, (ii) $A \cap B \cap \overline{C}$, (iii) $A \cap B \cap C$ (iv) $A \cup B \cup C$,

(v) $A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$

(vii) $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$

(viii) $(\overline{A} \cap \overline{B} \cap \overline{C})$ or $\overline{A \cup B \cup C}$

3·9. SOME THEOREMS ON PROBABILITY

In this section, we shall prove a few simple theorems which help us evaluate the probabilities of some complicated events in a rather simple way. In proving these theorems, we shall follow the axiomatic approach, based on the three axioms, discussed in § 3·8·5.

Theorem 3·2. *Probability of the impossible event is zero, i.e.,* $P(\phi) = 0$. ... (3·8)

Proof. Impossible event contains no sample point and hence the certain event S and the impossible event ϕ are mutually exclusive.

$$\therefore \quad S \cup \phi = S \quad \Rightarrow \quad P(S \cup \phi) = P(S)$$

Hence, using Axiom 2 of probability, *i.e.*, Axiom of Additivity, we get

$$P(S) + P(\phi) = P(S) \quad \Rightarrow \quad P(\phi) = 0$$

Remark. It may be noted $P(A) = 0$, does not imply that A is necessarily an empty set. In practice, probability '0' is assigned to the events which are so rare that they happen only once in a lifetime. For example, if a person who does not know typing is asked to type one page of the manuscript of a book, the probability of the event that he will type it correctly without any mistake is 0.

As another illustration, let us consider the random tossing of a coin. The event that the coin will stand erect on its edge, is assigned the probability 0.

The study of continuous random variable provides another illustration to the fact that $P(A) = 0$, does not imply $A = \phi$, because in case of continuous random variable X, the probability at a point is always zero, *i.e.*, $P(X = c) = 0$ [See Chapter 5].

Theorem 3·3. *Probability of the complementary event $\bar{A}$ of A is given by*

$$P(\bar{A}) = 1 - P(A). \qquad \ldots (3{\cdot}9)$$

Proof. A and $\bar{A}$ are mutually disjoint events, so that

$$A \cup \bar{A} = S \quad \Rightarrow \quad P(A \cup \bar{A}) = P(S)$$

Hence, from Axioms 2 and 3 of probability, we have

$$P(A) + P(\bar{A}) = P(S) = 1 \quad \Rightarrow \quad P(\bar{A}) = 1 - P(A)$$

Cor. 1. We have $P(A) = 1 - P(\bar{A}) \le 1$ $\quad$ [$\because P(\bar{A}) \ge 0$, by Axiom 1]

Further, since $P(A) \ge 0$ (Axiom 1)

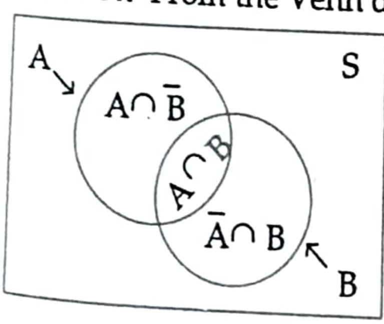
$$\therefore \qquad 0 \le P(A) \le 1. \qquad \ldots (3{\cdot}9a)$$

Cor. 2. $P(\phi) = 0$, since $\phi = \bar{S}$ and $P(\phi) = P(\bar{S}) = 1 - P(S) = 1 - 1 = 0$.

Theorem 3·4. *For any two events A and B, we have*

(*i*) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$ $\quad$ (*ii*) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Proof. From the Venn diagram, we get $\quad B = (A \cap B) \cup (\bar{A} \cap B)$,



where $\bar{A} \cap B$ and $A \cap B$ are disjoint events.

Hence by Axiom (3), we get

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\Rightarrow \quad P(\bar{A} \cap B) = P(B) - P(A \cap B) \qquad \ldots (3{\cdot}10)$$

(*ii*) Similarly, we have $\qquad A = (A \cap B) \cup (A \cap \bar{B})$,

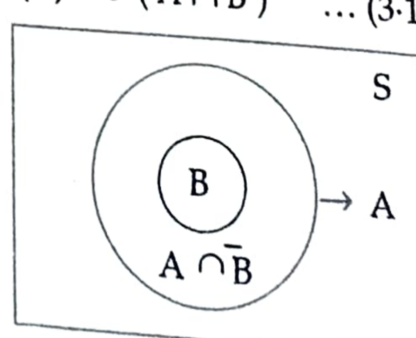
where $(A \cap B)$ and $A \cap \bar{B}$ are disjoint events. Hence, by Axiom 3, we get

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) \quad \Rightarrow \quad P(A \cap \bar{B}) = P(A) - P(A \cap B) \qquad \ldots (3{\cdot}11)$$

Theorem 3·5. *If $B \subset A$, then*

(*i*) $P(A \cap \bar{B}) = P(A) - P(B)$,

(*ii*) $P(B) \le P(A)$



Proof. (*i*) When $B \subset A$, B and $A \cap \bar{B}$ are mutually exclusive events so that $A = B \cup (A \cap \bar{B})$

$$\Rightarrow \qquad P(A) = P[B \cup (A \cap \bar{B})]$$

$$= P(B) + P(A \cap \bar{B}) \qquad \text{(By Axiom 3)}$$

$$\Rightarrow \qquad P(A \cap \bar{B}) = P(A) - P(B)$$

(ii) $P(A \cap \overline{B}) \geq 0 \quad \Rightarrow \quad P(A) - P(B) \geq 0 \quad \Rightarrow \quad P(B) \leq P(A)$

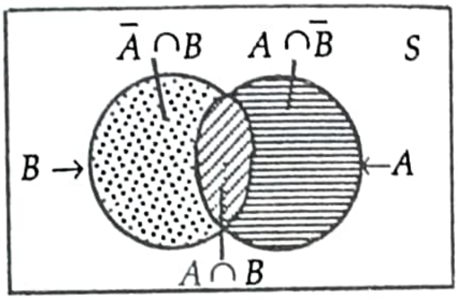
Hence $\quad B \subset A \quad \Rightarrow \quad P(B) \leq P(A)$ ... (3·12)

3·9·1. Addition Theorem of Probability

Theorem 3·6. *If A and B are any two events (subsets of sample space S) and are not disjoint, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \text{... (3·13)}$$

Proof. From the Venn diagram, we have



$$A \cup B = A \cup (\overline{A} \cap B),$$

where A and $\overline{A} \cap B$ are mutually disjoint.

$\therefore \quad P(A \cup B) = P[A \cup (\overline{A} \cap B)]$

$= P(A) + P(\overline{A} \cap B)$ [By Axiom 3] ... (*)

$= P(A) + P(B) - P(A \cap B)$ [From Theorem 3·3 (i)]

OR From (*) onwards.

$P(A \cup B) = P(A) + [P(\overline{A} \cap B) + P(A \cap B)] - P(A \cap B)$

$= P(A) + P[(\overline{A} \cap B) \cup (A \cap B)] - P(A \cap B)$

[$\because$ $(\overline{A} \cap B)$ and $(A \cap B)$ are disjoint]

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Aliter. $P(A \cup B) = \dfrac{n(A \cup B)}{n(S)} = \dfrac{n(A) + n(B) - n(A \cap B)}{n(S)}$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} = P(A) + P(B) - P(A \cap B)$$

Cor. 1. If the events A and B are mutually disjoint, then

$$A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = P(\phi) = 0$$

$\therefore \quad P(A \cup B) = P(A) + P(B)$, which is Axiom 3 of probability.

Cor. 2. For three non-mutually exclusive events A, B and C, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C). \qquad (3·13a)$$

Proof. $P(A \cup B \cup C) = P[A \cup (B \cup C)]$

$= P(A) + P(B \cup C) - P[A \cap (B \cup C)]$ [From (3·13)]

$= P(A) + [P(B) + P(C) - P(B \cap C)] - P[(A \cap B) \cup (A \cap C)]$

$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$

3·9·2. Extension of Addition Theorem of Probability to n Events.

Theorem 3·7. *For n events $A_1, A_2, \ldots, A_n$, we have*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum\sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k)$$
$$+ \ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n) \qquad \ldots (3\cdot 14)$$

Proof. For two events A_1 and A_2, we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \qquad (*)$$

Hence (3·14) is true for $n = 2$.

Let us now suppose that (3·14) is true for $n = r$, (say) so that

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r) \ldots (**)$$

Now

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left\{\left(\bigcup_{i=1}^{r} A_i\right) \cup A_{r+1}\right\}$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left\{\left(\left(\bigcup_{i=1}^{r} A_i\right) \cap A_{r+1}\right)\right\} \qquad \ldots [\text{Using } (*)]$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} \left(A_i \cap A_{r+1}\right)\right\} \qquad \text{(By Distributive Law)}$$

$$= \sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$
$$+ P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right\} \qquad [\text{From } (**)]$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$
$$- \left\{\sum_{i=1}^{r} P(A_i \cap A_{r+1}) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j \cap A_{r+1})\right.$$
$$\left. + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r \cap A_{r+1})\right\} \qquad \ldots [\text{From } (**)]$$

$$\Rightarrow P\left(\bigcup_{i=1}^{r+1} A_i\right) = \sum_{i=1}^{r+1} P(A_i) - \left\{\sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \sum_{i=1}^{r} P(A_i \cap A_{r+1})\right\}$$
$$+ \ldots + (-1)^{r} P\{(A_1 \cap A_2 \cap \ldots \cap A_{r+1}\}$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le (r+1)} P(A_i \cap A_j) + \ldots + (-1)^{r} P(A_1 \cap A_2 \cap \ldots \cap A_{r+1})$$

Hence if (3·14) is true for $n = r$, it is also true for $n = (r + 1)$. But we have proved in (*) that (3·14) is true for $n = 2$. Hence by the principle of mathematical induction, it follows that (3·14) is true for all positive integral values of n.

Remarks 1. If we write $P(A_i) = p_i$, $P(A_i \cap A_j) = p_{ij}$, $P(A_i \cap A_j \cap A_k) = p_{ijk}$ and so on and

$$S_1 = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} P(A_i)$$

$$S_2 = \sum\sum_{1 \le i < j \le n} p_{ij} = \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$S_3 = \sum\sum\sum_{1 \le i < j < k \le n} p_{ijk}$$ and so on, then from (3·14), we get :

$$P\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \ldots + (-1)^{n-1} S_n \qquad \ldots(3{\cdot}14a)$$

2. If all the events A_i, $(i = 1, 2, \ldots, n)$ are mutually disjoint then (3·14) gives

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

3. ***Practical Use of (3·14)*** From practical point of view the theorem can be restated in a slightly different form. Let us suppose that an event A can materialise in several mutually exclusive forms, *viz.*, $A_1, A_2, \ldots, A_n$ which may be regarded as that many mutually exclusive events. If A happens then any one of the events A_i, $(i = 1, 2, \ldots, n)$ must happen and conversely if any one of the events A_i, $(i = 1, 2, \ldots, n)$ happens, then A happens. Hence the probability of happening of A is same as the probability of happening of any one of its (unspecified) mutually exclusive forms. From this point of view, the total addition probability theorem can be restated as follows :

The probability of happening of an event A is the sum of the probabilities of happening of its mutually exclusive forms $A_1 A_2, \ldots, A_n$. Symbolically,

$$P(A) = P(A_1) + P(A_2) + \ldots + P(A_n) \qquad \ldots(3{\cdot}14b)$$

The probabilities $P(A_1), P(A_2), \ldots P(A_n)$ of the mutually exclusive forms of A are known as the *partial probabilities*. Since $P(A)$ is their sum, it may be called the *total probability* of A. Hence the name of the theorem.

3·9·3. Boole's Inequality

Statement. *For n events $A_1, A_2, \ldots, A_n$, we have*

$$(a)\ P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1) \qquad \ldots(3{\cdot}15)$$

$$(b)\ P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i) \qquad \ldots(3{\cdot}15a)$$

Proof. *(a)* $\quad P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \le 1$

$\Rightarrow \quad P(A_1 \cap A_2) \ge P(A_1) + P(A_2) - 1 \qquad (*)$

Hence (3·15) is true for $n = 2$.

Let us now suppose that (3·15) is true for $n = r$ (say), such that

$$P\left(\bigcap_{i=1}^{r} A_i\right) \ge \sum_{i=1}^{r} P(A_i) - (r-1) \qquad (**)$$

Then $\quad P\left(\bigcap_{i=1}^{r+1} A_i\right) = P\left(\bigcap_{i=1}^{r} A_i \cap A_{r+1}\right)$

$$\ge P\left(\bigcap_{i=1}^{r} A_i\right) + P(A_{r+1}) - 1 \qquad \text{[From (*)]}$$

$$\geq \sum_{i=1}^{r} P(A_i) - (r-1) + P(A_{r+1}) - 1 \qquad \text{[From (**)]}$$

$$\Rightarrow \quad P\left(\bigcap_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - r, \quad \Rightarrow \quad \text{(3·15) is true for } n = r+1 \text{ also.}$$

The result now follows by the principle of mathematical induction.

(*b*) Applying the inequality (3·15) to the event $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$, we get

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n) \geq \{P(\bar{A}_1) + P(\bar{A}_2) + \ldots + P(\bar{A}_n)\} - (n-1)$$

$$= \{1 - P(A_1)\} + \{1 - P(A_2)\} + \ldots + \{1 - P(A_n)\} - (n-1)$$

$$= 1 - P(A_1) - P(A_2) - \ldots - P(A_n)$$

$$\Rightarrow \quad P(A_1) + P(A_2) + \ldots + P(A_n) \geq 1 - P(\bar{A}_1) \cap \bar{A}_2 \cap \ldots \cap A_n.$$

$$= 1 - P\,\overline{(A_1 \cup A_2 \cup \ldots \cup A_n)}$$

$$= P(A_1 \cup A_2 \cup \ldots \cup A_n)$$

$$\Rightarrow \quad P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2) + \ldots + P(A_n)$$

as desired.

Aliter for (b), i.e., (3·15a). We have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2) \qquad [\because P(A_1 \cap A_2) \geq 0] \ \ldots (***)$$

Hence (3·15*a*) is true for $n = 2$.

Let us now suppose that (3·15*a*) is true for $n = r$ (say), so that

$$P\left(\bigcup_{i=1}^{r} A_i\right) \leq \sum_{i=1}^{r} P(A_i) \qquad \ldots (****)$$

Now

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left(\bigcup_{i=1}^{r} A_i \cup A_{r+1}\right)$$

$$\leq P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) \qquad \text{[Using (***)]}$$

$$\leq \sum_{i=1}^{r} P(A_i) + P(A_{r+1}) \qquad \text{[Using(****)]}$$

$$\therefore \quad P\left(\bigcup_{i=1}^{r+1} A_i\right) \leq \sum_{i=1}^{r+1} P(A_i)$$

Hence if (3·15*a*) is the true for $n = r$, then it is also true for $n = r + 1$. But we have proved in (***) that (3·15*a*) is true for $n = 2$. Hence by mathematical induction we conclude that (3·15*a*) is true for all positive integral values of n.

Theorem 3·8. *For n events $A_1, A_2, \ldots, A_n$,*

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) \qquad \ldots (3·16)$$

Proof. We shall prove this theorem by the principle of mathematical induction.

We know that

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - \{P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_3 \cap A_1)\} + P(A_1 \cap A_2 \cap A_3)$$

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{3} A_i\right) \geq \sum_{i=1}^{3} P(A_i) - \sum_{1 \leq i < j \leq 3} P(A_i \cap A_j) \qquad \ldots (*)$$

Thus the result is true for $n = 3$. Let us suppose that the result is true for $n = r$, (say),

so that : $$P\left(\bigcup_{i=1}^{r} A_i\right) \geq \sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \leq i < j \leq r} P(A_i \cap A_j) \qquad \ldots (**)$$

Now $$P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left(\bigcup_{i=1}^{r} A_i \cup A_{r+1}\right)$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left[\left(\bigcup_{i=1}^{r} A_i\right) \cap A_{r+1}\right]$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right\}$$

$$\geq \left\{\sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \leq i < j \leq r} P(A_i \cap A_j)\right\}$$

$$+ P(A_{r+1}) - P\left\{\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right\} \quad \text{[From (**)]} \qquad \ldots (***)$$

From Boole's inequality, we get

$$P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \leq \sum_{i=1}^{r} P(A_i \cap A_{r+1}) \Rightarrow -P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \geq -\sum_{i=1}^{r} P(A_i \cap A_{r+1})$$

$\therefore$ From (***), we get

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \leq i < j \leq r} P(A_i \cap A_j) - \sum_{i=1}^{r} P(A_i \cap A_{r+1})$$

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \leq i < j \leq r+1} P(A_i \cap A_j)$$

Hence, if the theorem is true for $n = r$, it is also true for $n = r + 1$. But we have seen in (*) that the result is true for $n = 3$. Hence by the principle of mathematical induction, the result is true for all positive integral values of n.

Example 3·24. *A letter of the English alphabet is chosen at random. Calculate the probability that the letter so chosen*

(i) is a vowel, (ii) precedes m and is a vowel, (iii) follows m and is a vowel.

Solution. The sample space of the experiment is :

$$S = (a, b, c, d, \ldots, x, y, z), \qquad n(S) = 26.$$

(*i*) Let E_1 be the event that the letter chosen is a vowel, Then

$$E_1 = \{a, e, i, o, u\}; \quad n(E_1) = 5 \qquad \therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{5}{26}$$

(*ii*) Let E_2 be the event that the letter precedes m and is a vowel, Then

$$E_2 = \{a, e, i\}; \quad n(E_2) = 3 \qquad \therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{26}$$

(*iii*) Let E_3 be the event that the letter follows m and is a vowel. Then,

$$E_3 = \{o, u\}; \quad n(E_3) = 2 \qquad \therefore \quad P(E_3) = \frac{n(E_3)}{n(S)} = \frac{2}{26} = \frac{1}{13}$$

Example 3·25. *Five salesmen of B, C, D and E of a company are considered for a three-member trade delegation to represent the company in an international trade conference. Construct the sample space and find the probability that :*

(i) A is selected. (ii) A is not selected, and (iii) Either A or B (not both) is selected. (Assume the natural assignment of probability.)

Solution. The sample space for selecting three salesmen out of 5 salesmen A, B, C, D and E for the trade delegation is given by :

$$S = \{ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE\} \Rightarrow n(S) = 10$$

Under the assumption of natural assignment of probabilities, each of these outcomes (elementary events) has an equal chance of being selected.

Let us define the following events :

E_1 : A is selected and E_2 : A or D (not both) is selected.

(*i*) $E_1 = \{ABC, ABD, ABE, ACD, ACE, ADE\} \Rightarrow n(E_1) = 6.$

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{10} = \frac{3}{5}$$

(*ii*) $\overline{E_1}$ = A is not selected = $\{BCD, BCE, BDE, CDE\} \Rightarrow n(E_2) = 4$

$$\therefore \quad P(\overline{E_1}) = \frac{n(\overline{E_1})}{n(S)} = \frac{4}{10} = \frac{2}{5} \quad \text{or} \quad P(\overline{E_1}) = 1 - P(E_1) = \frac{2}{5}.$$

(*iii*) $E_2 = \{ABC, ABE, ACE, BCD, BDE, CDE\} \Rightarrow n(E_2) = 6$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{6}{10} = \frac{3}{5}.$$

Important Remark. In all the problems that follow, we shall always assume natural assignment of probabilities to the elementary events, unless specified otherwise.

Example 3·26. *A, B and C are three mutually exclusive and exhaustive events associated with a random experiment. Find P(A) given that :*

$$P(B) = \frac{3}{2}P(A) \quad \text{and} \quad P(C) = \frac{1}{2}P(B)$$

Solution. Let $P(A) = p$, then $P(B) = \frac{3}{2}p$ and $P(C) = \frac{1}{2} \times \frac{3}{2}p = \frac{3}{4}p$.

Since A, B, C are mutually exclusive and exhaustive events,

$$P(A) + P(B) + P(C) = 1 \Rightarrow p + \frac{3}{2}p + \frac{3}{4}p = 1 \Rightarrow \frac{13}{4}p = 1 \Rightarrow p = \frac{4}{13}.$$

Example 3·27. *If* $p_1 = P(A)$, $p_2 = P(B)$, $p_3 = P(A \cap B)$, $(p_1, p_2, p_3 > 0)$, *express the following in terms of* p_1, p_2, p_3

(*a*) $P(\overline{A \cup B})$, (*b*) $P(\overline{A} \cup \overline{B})$, (*c*) $P(\overline{A} \cap B)$, (*d*) $P(\overline{A} \cup B)$, (*e*) $P(\overline{A} \cap \overline{B})$

(*f*) $P(A \cap \overline{B})$, (*g*) $P(A \mid B)$, (*h*) $P(B \mid \overline{A})$, (*i*) $P[\overline{A} \cap (A \cup B)]$.

Solution.

(a) $P\,\overline{(A \cup B)} = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - p_1 - p_2 + p_3.$

(b) $P(\overline{A} \cup \overline{B}) = P\,\overline{(A \cap B)} = 1 - P(A \cap B) = 1 - p_3$

(c) $P(\overline{A} \cap B) = P(B) - P(A \cap B) = p_2 - p_3$

(d) $P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B) = 1 - p_1 + p_2 - (p_2 - p_3) = 1 - p_1 + p_3$

(e) $P(\overline{A} \cap \overline{B}) = P\,\overline{(A \cup B)} = 1 - p_1 - p_2 + p_3.$ [Part (a)]

(f) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = p_1 - p_3$

(g) $P(A \mid B) = P(A \cap B)/P(B) = p_3/p_2$

(h) $P(B \mid \overline{A}) = P(\overline{A} \cap B)/P(\overline{A}) = (p_2 - p_3)/(1 - p_1)$ [Part (c)]

(i) $P(\overline{A} \cap (A \cup B)] = P[(\overline{A} \cap A) \cup (\overline{A} \cap B)] = P(\overline{A} \cap B) = p_2 - p_3$ $[\because A \cap \overline{A} = \phi]$

Example 3·28. *Let $P(A) = p$, $P(A \mid B) = q$, $P(B \mid A) = r$. Find relations between the number p, q, r for the following cases :*

(a) Events A and B are mutually exclusive.

(b) A and B are mutually exclusive and collectively exhaustive.

(c) A is sub-event of B ; B is a sub-event of A.

(d) $\overline{A}$ and $\overline{B}$ are mutually exclusive.

Solution. From the given data : $P(A) = p,\ P(A \cap B) = P(A)\,P(B \mid A) = rp$

$$\therefore \quad P(B) = \frac{P(A \cap B)}{P(A \mid B)} = \frac{rp}{q} \quad \text{and} \quad P(A) + P(B) = p + \frac{rp}{q} = \frac{p(q+r)}{q}$$

(a) Since A and B are mutually exclusive

$$A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = P(\phi) = 0 \quad \Rightarrow \quad rp = 0.$$

(b) Since A and B are mutually exclusive and collectively exhaustive,

$$P(A \cap B) = 0 \quad \text{and} \quad P(A) + P(B) = 1$$

$$\Rightarrow \quad p(q + r) = q\,;\ rp = 0 \quad \text{or} \quad pq = q \quad \Rightarrow \quad p = 1 \quad \text{or} \quad q = 0$$

(c) $A \subseteq B \Rightarrow A \cap B = A$ or $P(A \cap B) = P(A) \Rightarrow rp = p$, i.e., $r = 1$ or $p = 0$.

$B \subseteq A \Rightarrow A \cap B = B$ or $P(A \cap B) = P(B)$

$$\Rightarrow \quad rp = (rp/q) \quad \Rightarrow \quad rp(q - 1) = 0 \quad \Rightarrow \quad r = 0 \quad \text{or} \quad p = 0.$$

(d) Since $\overline{A}$ and $\overline{B}$ are mutually exclusive, $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 0$

$$\Rightarrow \quad 1 - [P(A) + P(B) - P(A \cap B)] = 0 \quad \Rightarrow \quad P(A) + P(B) = 1 + P(A \cap B)]$$

$$\Rightarrow \quad p[1 + (r/q)] = 1 + rp \quad \Rightarrow \quad p(q + r) = q(1 + pr).$$

Example 3·29. *A die is loaded in such a manner that for $n = 1, 2, 3, 4, 5, 6$, the probability of the face marked n, landing on top when the die is rolled is proportional to n. Find the probability that an odd number will appear on tossing the die.*

Solution. Here we are given :

$$P(n) \propto n \quad \Rightarrow \quad P(n) = kn \quad \dots (*), \quad \text{where } k \text{ is the constant of proportionality.}$$

Also $P(1) + P(2) + \dots + P(6) = 1 \quad \Rightarrow \quad k(1+2+3+4+5+6) = 1 \quad \Rightarrow \quad k = \frac{1}{21}$

$$\text{Required Probability} = P(1) + P(3) + P(5) = \frac{1+3+5}{21} = \frac{3}{7} \qquad [\text{Using } (*)]$$

Example 3·30. *If two dice are thrown, what is the probability that the sum is (a) greater than 8, and (b) neither 7 nor 11 ?*

Solution. (*a*) If S denotes the sum on the two dice, then we want $P(S > 8)$.

The required event can happen in the following mutually exclusive ways :

(*i*) $S = 9$ (*ii*) $S = 10$ (*iii*) $S = 11$ (*iv*) $S = 12$.

Hence by addition theorem of probability

$$P(S > 8) = P(S = 9) + P(S = 10) + P(S = 11) + P(S = 12) \qquad \dots (*)$$

In a throw of two dice, the sample space contains $6^2 = 36$ points. The number of favourable cases can be enumerated as follows :

$S = 9$: (3, 6), (6, 3), (4, 5), (5, 4), *i.e.*, 4 sample points $\quad \therefore P(S = 9) = \frac{4}{36}$

$S = 10$: (4, 6), (6, 4), (5, 5), *i.e.*, 3 sample points. $\quad \therefore P(S = 10) = \frac{3}{36}$

$S = 11$: (5, 6), (6, 5), *i.e.*, 2 sample points $\quad \therefore P(S = 11) = \frac{2}{36}$

$S = 12$: (6, 6), *i.e.*, 1 sample point. $\quad \therefore P(S = 12) = \frac{1}{36}$

$$\therefore \quad P(S > 8) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18} \qquad [\text{From } (*)]$$

(*b*) Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11 with a pair of dice.

$S = 7$: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), *i.e.*, 6 distinct sample points.

$$\therefore \quad P(A) = P(S = 7) = \frac{6}{36} = \frac{1}{6}$$

$S = 11$: (5, 6), (6, 5), *i.e.*, 2 distinct sample points.

$$\therefore \quad P(B) = P(S = 11) = \frac{2}{36} = \frac{1}{18}$$

$$\begin{aligned} \therefore \text{ Required probability} &= P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B)] \qquad (\because A \text{ and } B \text{ are disjoint events.}) \\ &= 1 - \frac{1}{6} - \frac{1}{18} = \frac{7}{9}. \end{aligned}$$

Example 3·31. *Two dice are tossed. Find the probability of getting 'an even number on the first die or a total of 8'.*

Solution. In a random toss of two dice, sample space S is given by :

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(S) = 6 \times 6 = 36$$

Let us define the events :

A : Getting an even number on the first dice

B : The sum of the points obtained on the two dice 8.

These events are represented by the following subsets of S.

$A = \{2, 4, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(A) = 3 \times 6 = 18$

$B = \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\} \quad \Rightarrow \quad n(B) = 5$

Also $A \cap B = \{(2, 6), (6, 2), (4, 4)\} \quad \Rightarrow \quad n(A \cap B) = 3.$

$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$, $P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}$, and $P(A \cap B) = \frac{3}{36} = \frac{1}{12}$

Hence, the required probability is given by :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{18}{36} + \frac{5}{36} - \frac{3}{36} = \frac{20}{36} = \frac{5}{9}.$$

Example 3·32. *An integer is chosen at random from two hundred digits. What is the probability that the integer is divisible by 6 or 8 ?*

Solution. The sample space of the random experiment is :

$$S = \{1, 2, 3, \ldots, 199, 200\} \Rightarrow n(S) = 200$$

The event A : 'integer chosen is divisible by 6' has the sample points given by :

$$A = \{6, 12, 18, \ldots, 198\} \Rightarrow n(A) = \frac{198}{6} = 33. \quad \therefore P(A) = \frac{n(A)}{n(S)} = \frac{33}{200}$$

Similarly the event B : 'integer chosen is divisible by 8' has the sample points given by :

$$B = \{8, 16, 24, \ldots, 200\} \Rightarrow n(B) = \frac{200}{8} = 25 \therefore P(B) = \frac{n(B)}{n(S)} = \frac{25}{200}$$

The LCM of 6 and 8 is 24. Hence, a number is divisible by both 6 and 8, if it is divisible by 24.

$$\therefore \quad A \cap B = \{24, 48, 72, \ldots, 192\} \Rightarrow n(A \cap B) = \frac{192}{24} = 8 \quad \Rightarrow \quad P(A \cap B) = \frac{8}{200}$$

Hence, the required probability is :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{1}{4}.$$

Example 3·33. *The probability that a student passes a Physics test is $\frac{2}{3}$ and the probability that he passes both a Physics test and an English test is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that he passes the English test ?*

Solution. Let us define the following events :

A : The student passes a Physics test ; B : The student passes an English test

In the usual notations, we are given :

$$P(A) = \frac{2}{3}, P(A \cap B) = \frac{14}{45}, \quad P(A \cup B) = \frac{4}{5} \text{ and we want, } P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow \frac{4}{5} = \frac{2}{3} + P(B) - \frac{14}{45}$$

$$\therefore \quad P(B) = \frac{4}{5} + \frac{14}{45} - \frac{2}{3} = \frac{36 + 14 - 30}{45} = \frac{4}{9}.$$

Example 3·34. *An investment consultant predicts that the odds against the price of a certain stock will go up during the next week are 2 : 1 and the odds in favour of the price* *remaining the same are 1 : 3. What is the probability that the price of the stock will go down during the next week ?*

Solution. Let A denote the event that 'stock price will go up', and B be the event 'stock price will remain same'. Then $P(A) = \frac{1}{2+1} = \frac{1}{3}$ and $P(B) = \frac{1}{1+3} = \frac{1}{4}$.

$\therefore$ P(stock price will either go up or remain same) is given by :

$$P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Hence, the probability that stock price will go down is given by :

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}.$$

Example 3·35. *An MBA applies for a job in two firms X and Y. The probability of his being selected in firm X is 0·7 and being rejected at Y is 0·5. The probability of at least one of his applications being rejected is 0·6. What is probability that he will be selected in one of the firms ?*

Solution. Let A and B denote the events that the person is selected in firms X and Y respectively. Then in the usual notations, we are given :

$$\left.\begin{aligned} P(A) &= 0{\cdot}7 \quad \Rightarrow \quad P(\bar{A}) = 1 - 0{\cdot}7 = 0{\cdot}3 \\ P(\bar{B}) &= 0{\cdot}5 \quad \Rightarrow \quad P(B) = 1 - 0{\cdot}5 = 0{\cdot}5 \end{aligned}\right\} \qquad \ldots (*)$$

and $$P(\bar{A} \cup \bar{B}) = 0{\cdot}6 = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B}) \qquad \ldots (**)$$

The probability that the persons will be selected in one of the two firms X or Y is given by :

$$\begin{aligned} P(A \cup B) &= 1 - P(\bar{A} \cap \bar{B}) = 1 - \{P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cup \bar{B})\} && [\text{From } (**)] \\ &= 1 - (0{\cdot}3 + 0{\cdot}5 - 0{\cdot}6) = 0{\cdot}8. && [\text{From } (*)] \end{aligned}$$

Example 3·36. *Three newspapers A, B and C are published in a certain city. It is estimated from a survey that of the adult population : 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read at least one of the papers ?*

Solution. Let E, F and G denote the events that the adult reads newspapers A, B and C respectively. Then we are given :

$$P(E) = \frac{20}{100}, \quad P(F) = \frac{16}{100}, \quad P(G) = \frac{14}{100}, \quad P(E \cap F) = \frac{8}{100}$$

$$P(E \cap G) = \frac{5}{100}, \quad P(F \cap G) = \frac{4}{100}, \quad \text{and} \quad P(E \cap F \cap G) = \frac{2}{100}$$

The required probability that an adult reads at least one of the newspapers (by addition theorem) is given by :

$$\begin{aligned} P(E \cup F \cup G) &= P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(E \cap G) + P(E \cap F \cap G) \\ &= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{4}{100} - \frac{5}{100} + \frac{2}{100} = \frac{35}{100} = 0{\cdot}35 \end{aligned}$$

Hence 35% of the adult population reads at least one of the newspapers.

Example 3·37. *A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.*

Solution. Let us define the following events :

A : the card drawn is a king ; $\qquad$ B : the card drawn is a heart ;

C : the card drawn is a red card.

Then A, B and C are not mutually exclusive.

$A \cap B$: the card drawn is the king of hearts $\Rightarrow n(A \cap B) = 1$

$B \cap C = B$: the card drawn is a heart $(\because B \subset C)$ $\Rightarrow n(B \cap C) = 13$

$C \cap A$: the card drawn is a red king $\Rightarrow n(C \cap A) = 2$

$A \cap B \cap C = A \cap B$: the card drawn is the king of hearts $\Rightarrow n(A \cap B \cap C) = 1$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}; \quad P(B) = \frac{13}{52}; \quad P(C) = \frac{26}{52}$$

$$P(A \cap B) = \frac{1}{52}; \quad P(B \cap C) = \frac{13}{52}; \quad P(C \cap A) = \frac{2}{52}; \quad P(A \cap B \cap C) = \frac{1}{52}$$

The required probability of getting a king or heart or a red card is given by :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}.$$

3·10. CONDITIONAL PROBABILITY

As discussed earlier, the probability $P(A)$ of an event A represents the likelihood that a random experiment will result in an outcome in the set A relative to the sample space S of the random experiment. However, quite often, while evaluating some event probability, we already have some information stemming from the experiment. For example, if we have prior information that the outcome of the random experiment must be in a set B of S, then this information must be used to re-appraise the likelihood that the outcome will also be in B. This re-appraised probability is denoted by $P(A \mid B)$ and is read as the conditional probability of the event A, given that the event B has already happened.

We give below some illustrations to explain this concept.

***Illustrations* 1.** Let us consider a random experiment of drawing a card from a pack of cards. Then the probability of happening of the event A : "The card drawn is a king", is given by :

$$P(A) = \frac{4}{52} = \frac{1}{13}.$$

Now suppose that a card is drawn and we are informed that the drawn card is red. How does this information affect the likelihood of the event A ?

Obviously, if the event B : 'The card drawn is red', has happened, the event 'Black card' is not possible. Hence the probability of the event A must be computed relative to the new sample space 'B' which consists of 26 sample points (red cards only), *i.e.*, $n(B) = 26$. Among these 26 red cards, there are two (red) kings so that $n(A \cap B) = 2$. Hence, the required probability is given by :

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{26} = \frac{1}{13}.$$

2. Consider a random experiment of tossing three fair coins. Then, as explained earlier, the sample space S is :

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$
$$= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \text{ (say)},$$

so that : $$P(\{\omega_i\}) = \frac{1}{8}; \; i = 1, 2, \ldots 8.$$

Now suppose that the same experiment is performed by another person and nothing is known about its outcome. However, we have the information that he obtained 'at least two heads'. We are interested to find how this additional information affects the probabilities of the elementary outcomes.

This means that if the event A : 'At least two heads are obtained', has happened then the elementary outcomes ω_4, ω_6, ω_7 and ω_8 could not have happened. However, the remaining four outcomes ω_1, ω_2, ω_3, and ω_5 are still possible and we assign the probability $\frac{1}{4}$ to each one of them.

$$\therefore \quad P(\omega_1 \mid A) = P(\omega_2 \mid A) = P(\omega_3 \mid A) = P(\omega_5 \mid A) = \frac{1}{4}\cdot$$

From the above illustrations we observe that some additional information may change the probability of the happening of some event. We now proceed to develop procedure to calculate the probabilities of events when we know some additional information.

Remark. When we know that a particular event B has occurred, instead of S, we concentrate our attention on B only and the conditional probability of A given B will be analogously the ratio of the probability of that part of A which is included in B (*i.e.*, $A \cap B$) to the probability of B. It, therefore, reflects the change of viewpoint only, namely, instead of S we have to concentrate on B only.

3·11. MULTIPLICATION THEOREM OF PROBABILITY

Theorem 3·9. *For two events A and B,*

$$\left.\begin{aligned} P(A \cap B) &= P(A) . P(B \mid A), \quad P(A) > 0 \\ &= P(B) . P(A \mid B), \quad P(B) > 0 \end{aligned}\right\} \qquad \dots (3\cdot17)$$

where $P(B \mid A)$ represents conditional probability of occurrence of B when the event A has already happened and $P(A \mid B)$ is the conditional probability of happening of A, given that B has already happened.

Proof. In the usual notations, we have

$$P(A) = \frac{n(A)}{n(S)}; \qquad P(B) = \frac{n(B)}{n(S)} \quad \text{and} \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad \dots (*)$$

For the conditional event $A \mid B$, the favourable outcomes must be one of the sample points of B, *i.e.*, for the event $A \mid B$, the sample space is B and out of the $n(B)$ sample points, $n(A \cap B)$ pertain to the occurrence of the event A. Hence

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (*), we get

$$P(A \cap B) = \frac{n(B)}{n(S)} \times \frac{n(A \cap B)}{n(B)} = P(B) . P(A \mid B) \qquad \dots (**)$$

Similarly, we get from (*) :

$$P(A \cap B) = \frac{n(A)}{n(S)} \times \frac{n(A \cap B)}{n(A)} = P(A) . P(B \mid A) \qquad \dots (***)$$

From (**) and (***), we get the result (3·17).

Thus, we have proved that "*the probability of the simultaneous occurrence of two events*

A and B is equal to the product of the probability of one of these events and the conditional probability of the other, given that the first one has occurred". Any of the events may be called the first event.

Remarks 1. $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ and $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ ... (3·18)

Thus the conditional probabilities $P(B \mid A)$ and $P(A \mid B)$ are defined if and only if $P(A) \neq 0$ and $P(B) \neq 0$, respectively.

2. (*i*) For $P(B) > 0$, $P(A \mid B) \leq P(A)$

Proof. $n(A \cap B) \leq n(A)$ and $n(B) \leq n(S)$. Dividing, we get

$$\frac{n(A \cap B)}{n(B)} \leq \frac{n(A)}{n(S)} \quad \Rightarrow \quad P(A \mid B) \leq P(A).$$

(*ii*) The conditional probability $P(A \mid B)$ is not defined if $P(B) = 0$.

(*iii*) $P(B \mid B) = 1$.

3·12. INDEPENDENT EVENTS

Two or more events are said to be *independent* if the happening or non-happening of any one of them, does not, in any way, affect the happening of others.

Consider the experiment of throwing two dice, say die 1 and die 2. It is obvious that the occurrence of a certain number of dots on the die 1 has nothing to do with a similar event for the die 2. The two are quite independent of each other, so to say. But suppose, the two dice were connected with a piece of thread before being thrown. The situation changes. This time the two events are not independent in as much as that the uppermost face of one die will have something to do in causing a particular face of the other die to be uppermost ; and the shorter the thread the more is this influence or dependence.

Similarly, if we draw two cards from a pack of cards in succession, then the results of the two draws are independent if the cards are drawn with replacement (*i.e.*, if the first card drawn is placed back in the pack before drawing the second card) and the results of the two draws are not independent if the cards are drawn without replacement.

Definition. *An event A is said to be independent (or statistically independent) of another event B, if the conditional probability of A given B, i.e., $P(A \mid B)$ is equal to the unconditional probability of B, i.e., if* $P(A \mid B) = P(A)$. ... (3·19)

It may be noted that the above definition is meaningful only when $P(A \mid B)$ is defined, *i.e.*, if $P(B) \neq 0$.

Similarly, an event B is said to be independent (or statistically independent) of event A, if

$$P(B \mid A) = P(B)\,; \quad P(A) \neq 0. \qquad \text{... (3·20)}$$

Theorem 3·10. *If the events A and B are such that $P(A) \neq 0$, $P(B) \neq 0$ and A is independent of B, then B is independent of A.*

Proof. Since the event A is independent of B, we have

$$P(A \mid B) = P(A) \quad \Rightarrow \quad \frac{P(A \cap B)}{P(B)} = P(A) \quad \Rightarrow \quad P(A \cap B) = P(A)\,P(B)$$

$$\therefore \quad \frac{P(B \cap A)}{P(A)} = P(B) \qquad [\because \; P(A) \neq 0 \text{ and } A \cap B = B \cap A]$$

$$\Rightarrow \quad P(B \mid A) = P(B) \quad \Rightarrow \quad B \text{ is independent of } A.$$

Remarks 1. Thus, we see that if A is independent of B, then B is independent of A. Hence, instead of saying that 'A is independent of B' or 'B is independent of A', we may say that A and B are independent events.

2. For any event A in S,

(*a*) A and the null event ϕ are independent

(*b*) A and S are independent.

Proof. (*a*) $P(A \cap \phi) = P(\phi) = 0 = P(A) \cdot P(\phi) \quad \Rightarrow \quad A$ and ϕ are independent.

(*b*) $P(A \cap S) = P(A) = P(A) \,.\, 1 = P(A)\, P(S) \qquad [\because A \subset S \text{ and } P(S) = 1]$

$\Rightarrow \quad A$ and S are independent.

3·13. MULTIPLICATION THEOREM OF PROBABILITY FOR INDEPENDENT EVENTS

Theorem 3·11. *If A and B are two events with positive probabilities* $\{P(A) \neq 0, P(B) \neq 0\}$, *then A and B are independent if and only if* $P(A \cap B) = P(A).\, P(B)$ ... (3·21)

Proof. We have :

$$P(A \cap B) = P(A)\, P(B \mid A) = P(B)\, P(A \mid B)\,;\, P(A) \neq 0, \;\; P(B) \neq 0 \qquad \text{... (*)}$$

If A and B are independent, *i.e.*, A is independent of B and B is independent of A, then, we have

$$P(A \mid B) = P(A) \qquad \text{and} \qquad P(B \mid A) = P(B) \qquad \text{... (**)}$$

From (*) and (**), we get $\qquad P(A \cap B) = P(A)\, P(B)$, as required.

Conversely, if (3·21) holds, then we get

$$\left.\begin{aligned} \frac{P(A \cap B)}{P(B)} &= P(A) \quad \Rightarrow \quad P(A \mid B) = P(A) \\ \text{and} \quad \frac{P(A \cap B)}{P(A)} &= P(B) \quad \Rightarrow \quad P(B \mid A) = P(B) \end{aligned}\right\} \qquad \text{...(***)}$$

(***) implies that A and B are independent events.

Hence, for independent events A and B, the probability that both of these occur simultaneously is the product of their respective probabilities.

This rule is known as the *Multiplication Rule of Probability* .

3·14. EXTENSION OF MULTIPLICATION THEOREM OF PROBABILITY TO *n* EVENTS

Theorem 3·12. *For n events* $A_1, A_2, \ldots, A_n$, *we have*

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \ldots \times P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \qquad \text{... (3·22)}$$

where $P(A_i \mid A_j \cap A_k \cap \ldots \cap A_l)$ *represents the conditional probability of the event* A_i *given that the events* $A_j, A_k, \ldots, A_l$ *have already happened.*

Proof. For two events A_1 and A_2, $P(A_1 \cap A_2) = P(A_1)\, P(A_2 \mid A_1)$

We have for three events A_1, A_2, and A_3

$$P(A_1 \cap A_2 \cap A_3) = P\{A_1 \cap (A_2 \cap A_3)\}$$

$$= P(A_1)\, P\{(A_2 \cap A_3) \mid A_1\} \qquad \text{[Using (3·17)]}$$
$$= P(A_1)\, P(A_2 \mid A_1)\, P\{(A_3 \mid (A_1 \cap A_2)\} \qquad \text{[Using (3·17)]}$$

Thus we find that (3·22) is true for $n = 2$ and $n = 3$. Let us suppose that (3·22) is true for $n = k$, so that

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \ldots P(A_k \mid A_1 \cap A_2 \cap \ldots \cap A_{k-1}) \qquad \ldots (3·23)$$

Now

$$P[(A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}] = P(A_1 \cap A_2 \cap \ldots \cap A_k) \times P(A_{k+1} \mid A_1 \cap A_2 \cap \ldots \cap A_k) \qquad \text{[Using (3·17)]}$$
$$= P(A_1)\, P(A_2 \mid A_1) \ldots P(A_k \mid A_1 \cap A_2 \cap \ldots \cap A_{k-1}) \times P(A_{k+1} \mid A_1 \cap A_2 \cap \ldots \cap A_k) \qquad \text{[From (3·23)]}$$

Thus (3·22) is true for $n = k + 1$ also. Since (3·22) is true for $n = 2$ and $n = 3$, by the principle of mathematical induction, it follows that (3·22) is true for all positive integral values of n.

3·14·1. Extension of Multiplication Theorem of Probability for *n*-Independent Events.

Theorem 3·13. *Necessary and sufficient condition for independence of n events A_1, A_2, ..., A_n is that the probability of their simultaneous happening is equal to the product of their respective probabilities, i.e.,*

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2)\, P(A_3) \ldots P(A_n) \qquad \ldots (3·24)$$

Proof. If $A_1, A_2, \ldots, A_n$ are independent events then

$$P(A_2 \mid A_1) = P(A_2),\ P(A_3 \mid A_1 \cap A_2) = P(A_3),\ \ldots,\ P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}) = P(A_n)$$

Hence, from (3·22), we get

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$$

Conversely if (3·24) holds, then from (3·22) and (3·24), we get

$$P(A_2) = P(A_2 \mid A_1);\ P(A_3) = P(A_3 \mid A_1 \cap A_2),\ \ldots,\ P(A_n) = P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

$\Rightarrow$ $A_1, A_2, A_3, \ldots, A_n$ are independent events.

Hence the theorem.

Remark. **Mutually Exclusive (Disjoint) Events and Independent Events.** Let A and B be mutually exclusive (disjoint) events with positive probabilities $[P(A) > 0, P(B) > 0]$,

$$A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = P(\phi) = 0 \qquad \ldots (i)$$

Further, by compound probability theorem, we have

$$P(A \cap B) = P(A).\, P(B \mid A) = P(B).\, P(A \mid B) \qquad \ldots (ii)$$

Since $P(A) \neq 0$; $P(B) \neq 0$, from (*i*) and (*ii*), we get

$$P(A \mid B) = 0 \neq P(A),\quad P(B \mid A) = 0 \neq P(B) \qquad \ldots (iii)$$

$\Rightarrow$ A and B are dependent events.

Hence, two mutually disjoint events with positive probabilities are always dependent (not independent) events.

However, if A and B are independent events with $P(A) > 0$ and $P(B) > 0$, then

$$P(A \cap B) = P(A)\, P(B) \neq 0$$

$\Rightarrow$ A and B cannot be mutually exclusive.

Hence two independent events (both of which are possible events), cannot be mutually disjoint.

Theorem 3·14. *For a fixed B with $P(B) > 0$, $P(A \mid B)$ is probability function.*

Proof. (i) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

(ii) $$P(S \mid B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(iii) If $\{A_n\}$ is any finite or infinite sequences of disjoint events, then

$$P[\bigcup_n A_n \mid B] = \frac{P[(\bigcup_n A_n) \cap B]}{P(B)} = \frac{P[\bigcup_n (A_n \cap B)]}{P(B)}$$

$$= \frac{\sum_n P(A_n \cap B)}{P(B)} = \sum_n \left[\frac{P(A_n \cap B)}{P(B)}\right] = \sum_n P(A_n \mid B)$$

Hence the theorem.

Remark. For given B satisfying $P(B) > 0$, the conditional probability $P[\cdot \mid B]$ also enjoys the same properties as the unconditional probability.

For example, in the usual notations, we have

(i) $P(\phi \mid B) = 0$

(ii) $P(\overline{A} \mid B) = 1 - P(A \mid B)$,

(iii) $P\left[\bigcup_{i=1}^{n} A_i \mid B\right] = \sum_{i=1}^{n} P(A_i \mid B)$,

where $A_1, A_2, \ldots, A_n$ are mutually disjoint events.

(iv) $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 \cap A_2 \mid B)$

(v) $P(A \cap \overline{B} \mid C) + P(A \cap B \mid C) = P(A \mid C)$

(vi) If $E \subset F$, then $P(E \mid B) \leq P(F \mid B)$

and so on.

The proofs of results (*iv*), (*v*) and (*vi*) are given in Theorems 3·15, 3·16 and 3·17 respectively. Others are left as exercises to the reader.

Theorem 3·15. *For any three events A, B and C,*

$$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Proof. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P[(A \cap C) \cup (B \cap C)] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Dividing both sides by $P(C)$, we get

$$\frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}, \; P(C) > 0$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad \frac{P[(A \cup B) \cap C]}{P(C)} = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

$$\Rightarrow \quad P[(A \cup B) \mid C] = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Theorem 3·16. *For any three events A, B and C,*

$$P(A \cap \overline{B} \mid C) + P(A \cap B \mid C) = P(A \mid C)$$

Proof. $P(A \cap \overline{B} \mid C) + P(A \cap B \mid C)$

$$= \frac{P(A \cap \overline{B} \cap C)}{P(C)} + \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap \overline{B} \cap C) + P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} = P(A \mid C).$$

Theorem 3·17. *For any three events A, B and C defined on the sample space S such that $B \subset C$ and $P(A) > 0$, $P(B \mid A) \le P(C \mid A)$.*

Proof. $P(C \mid A) = \dfrac{P(C \cap A)}{P(A)} = \dfrac{P[(B \cap C \cap A) \cup (\overline{B} \cap C \cap A)]}{P(A)}$

$$= \frac{P[(B \cap C \cap A)]}{P(A)} + \frac{P(\overline{B} \cap C \cap A)}{P(A)} \qquad \text{(Using Axiom 3)}$$

$$= P(B \cap C \mid A) + P(\overline{B} \cap C \mid A)$$

$$= P(B \mid A) + P(\overline{B} \cap C \mid A) \qquad [\because B \subset C \Rightarrow B \cap C = B]$$

$\Rightarrow \qquad P(C \mid A) \ge P(B \mid A) \qquad [\because P(\overline{B} \cap C \mid A) \ge 0]$

Theorem 3·18. *If A and B are independent events, then*

(i) A and $\overline{B}$ (ii) $\overline{A}$ and B (iii) $\overline{A}$ and $\overline{B}$, are also independent

Proof. Since A and B are independent, $P(A \cap B) = P(A)\,P(B)$ ... (*)

(*a*) $P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)\,P(B)$ [From (*)

$= P(A)\,[1 - P(B)] = P(A)\,P(\overline{B})$

$\Rightarrow$ A and $\overline{B}$ are independent events.

Aliter. $P(A \cap B) = P(A)\,P(B) = P(A)\,P(B \mid A) = P(B)\,P(A \mid B)$

i.e., $P(B \mid A) = P(B) \Rightarrow B$ is independent of A.

and $P(A \mid B) = P(A) \Rightarrow A$ is independent of B.

Also $P(B \mid A) + P(\overline{B} \mid A) = 1 \Rightarrow P(B) + P(\overline{B} \mid A) = 1$

$\therefore \quad P(\overline{B} \mid A) = 1 - P(B) = P(\overline{B})$

$\therefore$ $\overline{B}$ is independent of A and by symmetry we say that A is independent of $\overline{B}$ Hence, A and $\overline{B}$ are independent events.

(*ii*) $P(\overline{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\,P(B)$ [From (*)]

$= P(B)\,[1 - P(A)] = P(\overline{A})\,P(B)$

$\Rightarrow$ $\overline{A}$ and B are independent events.

(*iii*) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A)\,P(B)$ [From (*)]

$= [1 - P(B)] - P(A)\,[1 - P(B)]$

$= [1 - P(A)]\,[1 - P(B)] = P(\overline{A})\,P(\overline{B})$

$\therefore$ $\overline{A}$ and $\overline{B}$ are independent events.

Aliter. We know that: $P(\overline{A} \mid \overline{B}) + P(A \mid \overline{B}) = 1$ ... (**)

Since A and B are independent; by Part (*i*) of the above theorem A and $\overline{B}$ are also independent.

$\therefore \quad P(A \mid \overline{B}) = P(A)$. Hence from (**), we get

$$P(\overline{A} \mid \overline{B}) + P(A) = 1 \quad \Rightarrow \quad P(\overline{A} \mid \overline{B}) = 1 - P(A) = P(\overline{A})$$

Hence $\overline{A}$ and $\overline{B}$ are independent events.

3·15. PAIRWISE INDEPENDENT EVENTS

Consider n events $A_1, A_2, \ldots, A_n$ defined on the same sample space so that $P(A_i) > 0$; $i = 1, 2, \ldots, n$. These *events are said to be pairwise independent if every pair of two events is independent in the sense of the definition given in § 3·13.*

> **Definition**. *The events $A_1, A_2, \ldots, A_n$ are said to be pairwise independent if and only if:*
>
> $$P(A_i \cap A_j) = P(A_i)\, P(A_j),\ i \neq j = 1, 2, \ldots, n \qquad \ldots (3·25)$$

In particular, three events A_1, A_2, A_3 are pairwise independent if and only if :

$$\left.\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\, P(A_2) \\ P(A_1 \cap A_3) &= P(A_1)\, P(A_3) \\ P(A_2 \cap A_3) &= P(A_2)\, P(A_3) \end{aligned}\right\} \qquad \ldots (3·26)$$

3·15·1. Mutually Independent Events. Let S denote the sample space for a number of events. *The events in S are said to be mutually independent if the probability of the simultaneous occurrence of (any) finite number of them is equal to the product of their separate probabilities.*

> **Definition.** *The n events $A_1, A_2, \ldots, A_n$ in a sample space S are said to be mutually independent if*
>
> $$P(A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}) = P(A_{i1})\, P(A_{i2}) \ldots P(A_{ik})\ ;\ k = 2, 3, \ldots, n \qquad \ldots (3·27)$$

Hence, the events are mutually independent if they are independent by pairs, and by triplets, and by quadruples, and so on.

Conditions for mutual independence of n events. Mathematically, n events $A_1, A_2, \ldots A_n$ are mutually independent if and only if the following conditions hold.

$$\left.\begin{aligned} &(i) \quad P(A_i \cap A_j) = P(A_i)\, P(A_j),\ (i \neq j\ ;\ i, j = 1, 2, \ldots, n) \\ &(ii) \quad P(A_i \cap A_j \cap A_k) = P(A_i)\, P(A_j)\, P(A_k),\ (i \neq j \neq k\ ;\ i\ ;\ j\ ;\ k = 1, 2, \ldots, n) \\ &\qquad \vdots \\ &(n-1) : P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n) \end{aligned}\right\} \ldots (3·28)$$

It is interesting to note that the above equations in (3·28) give respectively ${}^nC_2, {}^nC_3, {}^nC_n$ conditions to be satisfied by $A_1, A_2, \ldots A_n$.

Hence the total number of conditions for mutual independence of $A_1, A_2, \ldots, A_n$ is :

$${}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n - ({}^nC_0 + {}^nC_1) = 2^n - 1 - n$$

In particular for three events A_1, A_2 and A_3, $(n = 3)$, we have the following $2^3 - 1 - 3 = 4$, conditions for their mutual independence.

$$\left.\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\, P(A_2) \\ P(A_2 \cap A_3) &= P(A_2)\, P(A_3) \\ P(A_1 \cap A_3) &= P(A_1)\, P(A_3) \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)\, P(A_2)\, P(A_3) \end{aligned}\right\} \quad \ldots (3{\cdot}29)$$

Remarks **1.** It may be observed that pairwise or mutual independence of events : $A_1, A_2, \ldots, A_n$, is defined only when $P(A_i) \neq 0$, for $i = 1, 2, \ldots, n$.

2. From (3·26) and (3·29), it is obvious that mutual independence of events implies that they are pairwise independent. However, the converse is not true, *i.e.*, the events may be pairwise independent but not mutually independent. For illustrations, see Examples 3·54 and 3·55.

Theorem 3·9. *If A, B, C are mutually independent events then $A \cup B$ and C are also independent.*

Proof. We are required to prove :

$$P[(A \cup B) \cap C] = P(A \cup B)\, P(C)$$

$$\text{L.H.S.} = P[(A \cap C) \cup P(B \cap C)] \quad \text{[By Distributive Law]}$$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

$$= P(A)\, P(C) + P(B)\, P(C) - P(A)\, P(B)\, P(C)$$

[$\because$ *A*, *B* and *C* are mutually independent]

$$= P(C)\, [P(A) + P(B) - P(A \cap B)] = P(C)\, P(A \cup B) = \text{R.H.S.}$$

Hence $(A \cup B)$ and *C* are independent.

Theorem 3·20. *If A, B and C are random events in a sample space and if A, B and C are pairwise independent and A is independent of $(B \cup C)$, then A, B and C are mutually independent.*

Proof. We are given

$$\left.\begin{aligned} P(A \cap B) &= P(A)\, P(B) \\ P(B \cap C) &= P(B)\, P(C) \\ P(A \cap C) &= P(A)\, P(C) \\ P[A \cap (B \cup C)] &= P(A)\, P(B \cup C) \end{aligned}\right\} \quad \ldots (*)$$

Now $\quad P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A)\,.\,P(B) + P(A)\,.\,P(C) - P(A \cap B \cap C)$$

[From (*)] ... (**)

and $\quad P(A)\, P(B \cup C) = P(A)\, [P(B) + P(C) - P(B \cap C)]$

$$= P(A)\,.\,P(B) + P(A)\, P(C) - P(A)\, P(B \cap C) \quad \ldots (***)$$

From (**) and (***), on using (*), we get

$$P(A \cap B \cap C) = P(A)\, P(B \cap C) = P(A)\, P(B)\, P(C) \quad \text{[From (*)]}$$

Hence *A*, *B*, *C* are mutually independent.

3·15·2. Given *n* independent events A_i, $(i = 1, 2, \ldots, n)$ with respective probabilities of occurrence p_i, to find the probability of occurrence of at least one of them.

We have $P(A_i) = p_i \quad \Rightarrow \quad P(\bar{A}_i) = 1 - p_i\,;\ i = 1, 2, \ldots, n \quad \ldots (*)$

Hence the probability '*p*' of happening of at least one of the events is given by :

$$\left.\begin{aligned} p = P(A_1 \cup A_2 \cup \ldots \cup A_n) &= 1 - P(\overline{A_1 \cup A_2 \cup \ldots \cup A_n}) \\ &= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n) = 1 - P(\bar{A}_1)\, P(\bar{A}_2) \ldots P(\bar{A}_n) \end{aligned}\right\} \quad \ldots (**)$$

($\because A_1, A_2, \ldots, A_n$ are independent $\Rightarrow \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ are also independent)

$$= 1 - [(1 - p_1)(1 - p_2) \ldots (1 - p_n)]$$

$$= \left[\sum_{i=1}^{n} p_i - \underset{\substack{i, j = 1 \\ i < j}}{\sum\sum^{n}} (p_i p_j) + \underset{\substack{i, j, k = 1 \\ i < j < k}}{\sum\sum\sum^{n}} (p_i p_j p_k) - \ldots + (-1)^{n-1} (p_1 p_2 \ldots p_n) \right]$$

Remark. The results in (**) are very important and are used quite often in numerical problems. Result (**) stated in words gives :

P [happening of at least one of the events $A_1, A_2, \ldots, A_n$]

$= 1 - P$ (none of the events $A_1, A_2, \ldots, A_n$ happens) ... (3·30)

or equivalently,

P {none of the given events happens} $= 1 - P$ {at least one of them happens}. ... (3·30a)

Example 3.38. *If $A \cap B = \phi$, then show that $P(A) \le P(\bar{B})$.*

Solution. We have

$$A = (A \cap B) \cup (A \cap \bar{B}) = \phi \cup (A \cap \bar{B}) = A \cap \bar{B} \qquad [\because A \cap B = \phi \text{ (Given)}]$$

$$\therefore \qquad A \subseteq \bar{B} \quad \Rightarrow \quad P(A) \le P(\bar{B}), \text{ as desired.}$$

Aliter. Since $A \cap B = \phi$, we have $A \subset \bar{B}$, which implies that $P(A) \le P(\bar{B})$.

Example 3·39. *Let A and B be two events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$, show that*

$$(a)\ P(A \cup B) \ge \frac{3}{4}, \quad \text{and} \quad (b)\ \frac{3}{8} \le P(A \cap B) \le \frac{5}{8}.$$

Solution. (*a*) we have

$$A \subset (A \cup B) \Rightarrow P(A) \le P(A \cup B) \Rightarrow \frac{3}{4} \le P(A \cup B) \Rightarrow P(A \cup B) \ge \frac{3}{4}$$

(*b*) $A \cap B \subseteq B \Rightarrow P(A \cap B) \le P(B) = \frac{5}{8}$ ... (*i*)

Also $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1 \Rightarrow \frac{3}{4} + \frac{5}{8} - 1 \le P(A \cap B)$

$\therefore \quad \frac{6 + 5 - 8}{8} \le P(A \cap B) \Rightarrow \frac{3}{8} \le P(A \cap B)$ ... (*ii*)

From (*i*) and (*ii*), we get $\frac{3}{8} \le P(A \cap B) \le \frac{5}{8}$.

Example 3·40. *For any two events A and B,*

$$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$$

Proof. We have

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

Using axiom 3, we have

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)] = P(A \cap \bar{B}) + P(A \cap B)$$

(From axiom 1)

Now $P[(A \cap \bar{B}) \ge 0$

$\therefore \qquad P(A) \ge P(A \cap B)$ ... (*)

Similarly $\qquad P(B) \ge P(A \cap B)$

$\Rightarrow \qquad P(B) - P(A \cap B) \ge 0$

Now $\quad P(A \cup B) = P(A) + [P(B) - P(A \cap B)] \quad \ldots (**)$

$\therefore \quad P(A \cup B) \geq P(A) \quad \Rightarrow \quad P(A) \leq P(A \cup B) \quad \ldots (***)$

Also $\quad P(A \cup B) \leq P(A) + P(B) \quad$ [From (**)] $\quad \ldots (****)$

Hence from (*), (***) and (****), we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

Aliter. Since $A \cap B \subset A$, by *Theorem 4·6 (ii)*, we get

$$P(A \cap B) \leq P(A).$$

Also $A \subset (A \cup B) \quad \Rightarrow \quad P(A) \leq P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\leq P(A) + P(B) \qquad [\because P(A \cap B) \geq 0]$$

Combining the above results, we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

EXAMPLES ON ADDITION AND MULTIPLICATION THEOREMS OF PROBABILITY

Example 3·41. *The odds against Manager X settling the wage dispute with the workers are 8 : 6 and odds in favour of manager Y settling the same dispute are 14 : 16.*

(i) What is the chance that neither settles the dispute, if they both try, independently of each other ?

(ii) What is the probability that the dispute will be settled ?

Solution. Let A be the event that the manager X will settle the dispute and B be the event that the Manager Y will settle the dispute. Then clearly

$$P(\bar{A}) = \frac{8}{8+6} = \frac{4}{7} \quad \Rightarrow \quad P(A) = 1 - P(\bar{A}) = \frac{6}{14} = \frac{3}{7}$$

$$P(B) = \frac{14}{14+16} = \frac{7}{15} \quad \Rightarrow \quad P(\bar{B}) = 1 - P(B) = \frac{16}{14+16} = \frac{8}{15}$$

The required probability that neither settles the dispute is given by :

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B}) = \frac{4}{7} \times \frac{8}{15} = \frac{32}{105}$$

[Since A and B independent $\Rightarrow$ $\bar{A}$ and $\bar{B}$ are also independent]

(*ii*) The dispute will be settled if at least one of the managers X and Y settles the dispute. Hence the required probability is given by :

$$P(A \cup B) = \text{Prob. [At least one of } X \text{ and } Y \text{ settles the dispute.]}$$
$$= 1 - \text{Prob. [None settles the dispute.]}$$
$$= 1 - P(\bar{A} \cap \bar{B}) = 1 - \frac{32}{105} = \frac{73}{105}.$$

[From Part (*i*)]

Example 3·42. *The odds that person X speaks the truth are 3 : 2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point.*

Solution. Let us define the events :

A : X speaks the truth, $\qquad B$: Y speaks the truth

Then $\bar{A}$ and $\bar{B}$ represent the complementary events the X and Y tell a lie respectively. We are given :

$$P(A) = \frac{3}{3+2} = \frac{3}{5} \quad \Rightarrow \quad P(\bar{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

and
$$P(B) = \frac{5}{5+3} = \frac{5}{8} \quad \Rightarrow \quad P(\bar{B}) = 1 - \frac{5}{8} = \frac{3}{8}$$

The event E that X and Y contradict each other on an identical point can happen in the following mutually exclusive ways :

(*i*) X speaks the truth and Y tells a lie, *i.e.*, the event $A \cap \bar{B}$ happens,

(*ii*) X tells a lie and Y speaks the truth, *i.e.*, the event $\bar{A} \cap B$ happens.

Hence by addition theorem of probability, the required probability is given by :

$$P(E) = P(i) + P(ii) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A) \times P(\bar{B}) + P(\bar{A}) \times P(B) \qquad \text{[Since } A \text{ and } B \text{ are independent]}$$

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40} = 0{\cdot}475$$

Hence A and B are likely to contradict each other on an identical point in 47·5% of the cases.

Example 3·43. *One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third guns respectively. If $P(E_1) = 0{\cdot}5$ $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$ and E_1, E_2, E_3, are independent events, find the probability that (a) exactly one hit is registered, and (b) at least two hits are registered.*

Solution. We are given : $P(E_1) = 0{\cdot}5$, $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$

$\Rightarrow \quad P(\bar{E}_1) = 0{\cdot}5, \quad P(\bar{E}_2) = 0{\cdot}4$ and $P(\bar{E}_3) = 0{\cdot}2$

(*a*) Exactly one hit can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap \bar{E}_2 \cap \bar{E}_3$ happens, (*ii*) $\bar{E}_1 \cap E_2 \cap \bar{E}_3$ happens, (*iii*) $\bar{E}_1 \cap \bar{E}_2 \cap E_3$ happens.

Hence by addition probability theorem, the required probability 'p' is given by :

$$p = P(E_1 \cap \bar{E}_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap E_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap \bar{E}_2 \cap E_3)$$

$$= P(E_1)\,P(\bar{E}_2)\,P(\bar{E}_3) + P(\bar{E}_1)\,P(E_2)\,P(\bar{E}_3) + P(\bar{E}_1)\,P(\bar{E}_2)\,P(E_3)$$

[Since E_1, E_2 and E_3 are independent.]

$$= 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}26.$$

(*b*) At least two hits can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap E_2 \cap \bar{E}_3$ happens (*ii*) $E_1 \cap \bar{E}_2 \cap E_3$ happens

(*iii*) $\bar{E}_1 \cap E_2 \cap E_3$ happens, (*iv*) $E_1 \cap E_2 \cap E_3$ happens.

$\therefore$ Required probability

$$= P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8$$

$$= 0{\cdot}06 + 0{\cdot}16 + 0{\cdot}24 + 0{\cdot}24 = 0{\cdot}70.$$

Example 3·44. *An urn contains 4 tickets numbered 1, 2, 3, 4 and another contains 6 tickets numbered 2, 4, 6, 7, 8, 9. If one of the two urns is chosen at random and a ticket is*

Now $\quad P(A \cup B) = P(A) + [P(B) - P(A \cap B)]$ ... (**)

$\therefore \quad P(A \cup B) \geq P(A) \quad \Rightarrow \quad P(A) \leq P(A \cup B)$ ... (***)

Also $\quad P(A \cup B) \leq P(A) + P(B) \quad$ [From (**)] ...(****)

Hence from (*), (***) and (****), we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$$

Aliter. Since $A \cap B \subset A$, by *Theorem 4·6 (ii)*, we get

$$P(A \cap B) \leq P(A).$$

Also $A \subset (A \cup B) \quad \Rightarrow \quad P(A) \leq P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\leq P(A) + P(B) \qquad [\because P(A \cap B) \geq 0]$$

Combining the above results, we get

$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

EXAMPLES ON ADDITION AND MULTIPLICATION THEOREMS OF PROBABILITY

Example 3·41. *The odds against Manager X settling the wage dispute with the workers are 8 : 6 and odds in favour of manager Y settling the same dispute are 14 : 16.*

(i) What is the chance that neither settles the dispute, if they both try, independently of each other ?

(ii) What is the probability that the dispute will be settled ?

Solution. Let A be the event that the manager X will settle the dispute and B be the event that the Manager Y will settle the dispute. Then clearly

$$P(\bar{A}) = \frac{8}{8+6} = \frac{4}{7} \quad \Rightarrow \quad P(A) = 1 - P(\bar{A}) = \frac{6}{14} = \frac{3}{7}$$

$$P(B) = \frac{14}{14+16} = \frac{7}{15} \quad \Rightarrow \quad P(\bar{B}) = 1 - P(B) = \frac{16}{14+16} = \frac{8}{15}$$

The required probability that neither settles the dispute is given by :

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B}) = \frac{4}{7} \times \frac{8}{15} = \frac{32}{105}$$

[Since A and B independent $\Rightarrow$ $\bar{A}$ and $\bar{B}$ are also independent]

(ii) The dispute will be settled if at least one of the managers X and Y settles the dispute. Hence the required probability is given by :

$P(A \cup B)$ = Prob. [At least one of X and Y settles the dispute.]

= 1 – Prob. [None settles the dispute.]

$$= 1 - P(\bar{A} \cap \bar{B}) = 1 - \frac{32}{105} = \frac{73}{105}.$$ [From Part (*i*)]

Example 3·42. *The odds that person X speaks the truth are 3 : 2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point.*

Solution. Let us define the events :

A : X speaks the truth, $\qquad B$: Y speaks the truth

Then $\bar{A}$ and $\bar{B}$ represent the complementary events the X and Y tell a lie respectively. We are given :

$$P(A) = \frac{3}{3+2} = \frac{3}{5} \quad \Rightarrow \quad P(\bar{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

and $$P(B) = \frac{5}{5+3} = \frac{5}{8} \quad \Rightarrow \quad P(\bar{B}) = 1 - \frac{5}{8} = \frac{3}{8}$$

The event E that X and Y contradict each other on an identical point can happen in the following mutually exclusive ways :

(*i*) X speaks the truth and Y tells a lie, *i.e.*, the event $A \cap \bar{B}$ happens,

(*ii*) X tells a lie and Y speaks the truth, *i.e.*, the event $\bar{A} \cap B$ happens.

Hence by addition theorem of probability, the required probability is given by :

$$P(E) = P(i) + P(ii) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A) \times P(\bar{B}) + P(\bar{A}) \times P(B) \qquad \text{[Since } A \text{ and } B \text{ are independent]}$$

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40} = 0{\cdot}475$$

Hence A and B are likely to contradict each other on an identical point in 47·5% of the cases.

Example 3·43. *One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third guns respectively. If $P(E_1) = 0{\cdot}5$ $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$ and E_1, E_2, E_3, are independent events, find the probability that (a) exactly one hit is registered, and (b) at least two hits are registered.*

Solution. We are given : $P(E_1) = 0{\cdot}5$, $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$

$\Rightarrow$ $P(\bar{E}_1) = 0{\cdot}5$, $P(\bar{E}_2) = 0{\cdot}4$ and $P(\bar{E}_3) = 0{\cdot}2$

(*a*) Exactly one hit can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap \bar{E}_2 \cap \bar{E}_3$ happens, (*ii*) $\bar{E}_1 \cap E_2 \cap \bar{E}_3$ happens, (*iii*) $\bar{E}_1 \cap \bar{E}_2 \cap E_3$ happens.

Hence by addition probability theorem, the required probability 'p' is given by :

$$p = P(E_1 \cap \bar{E}_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap E_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap \bar{E}_2 \cap E_3)$$

$$= P(E_1)\, P(\bar{E}_2)\, P(\bar{E}_3) + P(\bar{E}_1)\, P(E_2)\, P(\bar{E}_3) + P(\bar{E}_1)\, P(\bar{E}_2)\, P(E_3)$$

[Since E_1, E_2 and E_3 are independent.]

$$= 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}26.$$

(*b*) At least two hits can be registered in the following mutually exclusive ways :

(*i*) $E_1 \cap E_2 \cap \bar{E}_3$ happens (*ii*) $E_1 \cap \bar{E}_2 \cap E_3$ happens

(*iii*) $\bar{E}_1 \cap E_2 \cap E_3$ happens, (*iv*) $E_1 \cap E_2 \cap E_3$ happens.

$\therefore$ Required probability

$$= P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8$$

$$= 0{\cdot}06 + 0{\cdot}16 + 0{\cdot}24 + 0{\cdot}24 = 0{\cdot}70.$$

Example 3·44. *An urn contains 4 tickets numbered 1, 2, 3, 4 and another contains 6 tickets numbered 2, 4, 6, 7, 8, 9. If one of the two urns is chosen at random and a ticket is* *drawn at random from the chosen urn, find the probabilities that the ticket drawn bears the number* *(i) 2 or 4,* *(ii) 3,* *(iii) 1 or 9.*

Solution. (*i*) Required event can happen in the following mutually exclusive ways :

(*I*) First urn is chosen and then a ticket is drawn.

(*II*) Second urn is chosen and then a ticket is drawn.

Since the probability of choosing any urn is $\frac{1}{2}$, required probability '*p*' is given by :

$$p = P(I) + P(II) = \frac{1}{2}\times\frac{2}{4} + \frac{1}{2}\times\frac{2}{6} = \frac{5}{12}$$

(*ii*) Required probability $= \frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times 0 = \frac{1}{8}$

($\because$ In the 2nd urn there is no ticket with number 3.)

(*iii*) Required probability $= \frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{6} = \frac{5}{24}$.

Example 3·45. *A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.*

Solution. The required event *E* that 'in a draw of 4 balls from the box at random there is at least one ball of each colour', can materialise in the following mutually disjoint ways :

(*i*) 1 Red, 1 White, 2 Black balls ; (*ii*) 2 Red, 1 White, 1 Black balls; (*iii*) 1 Red, 2 White, 1 Black balls.

Hence by addition theorem of probability, the required probability is given by :

$$\begin{aligned} P(E) &= P(i) + P(ii) + P(iii) \\ &= \frac{{}^6C_1\times{}^4C_1\times{}^5C_2}{{}^{15}C_4} + \frac{{}^6C_2\times{}^4C_1\times{}^5C_1}{{}^{15}C_4} + \frac{{}^6C_1\times{}^4C_2\times{}^5C_1}{{}^{15}C_4} \\ &= \frac{1}{{}^{15}C_4}[6\times4\times10 + 15\times4\times5 + 6\times6\times5] = \frac{4!}{15\times14\times13\times12}(240+300+180) \\ &= \frac{24\times720}{15\times14\times13\times12} = 0{\cdot}5275. \end{aligned}$$

Example 3·46. *Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, and 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is 13/32.*

Solution. The required event of getting 1 girl and two boys among the three selected children can materialise in the following three mutually disjoint cases :

Group No. →	I	II	III
(*i*)	Girl	Boy	Boy
(*ii*)	Boy	Girl	Boy
(*iii*)	Boy	Boy	Girl

Hence by addition theorem of probability,

Required probability = $P(i) + P(ii) + P(iii)$ ...(*)

Since the probability of selecting a girl from the first group is 3/4, of selecting a boy from the second group is 2/4, and of selecting a boy from the third group is 3/4, and since these three events of selecting children from three groups are independent of each other, by compound probability theorem, we have

$$P(i) = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}; \quad P(ii) = \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}; \quad \text{and} \quad P(iii) = \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}$$

Substituting in (*), we get

$$\text{Required probability} = \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}.$$

Example 3·47. *It is 8 : 5 against the wife who is 40 years old living till she is 70 and 4 : 3 against her husband now 50 living till he is 80. Find the probability that*

(i) Both will be alive, *(ii) None will be alive,*

(iii) Only wife will be alive, *(iv) Only husband will be alive,*

(v) Only one will be alive, *(vi) At least one will be alive,*

30 years hence.

Solution. Let us define the events :

A : Wife will be alive, and B : Husband will be alive; 30 years hence.

Then, we are given :

$$P(A) = \frac{5}{8+5} = \frac{5}{13} \quad \Rightarrow \quad P(\bar{A}) = 1 - P(A) = \frac{8}{13}$$

$$P(B) = \frac{3}{4+3} = \frac{3}{7} \quad \Rightarrow \quad P(\bar{B}) = 1 - P(B) = \frac{4}{7}$$

If we assume that A and B are independent so that A and $\bar{B}$, $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$ are also independent, then the required probabilities are given by :

(i) $P(A \cap B) = P(A)\,P(B) = \frac{5}{13} \times \frac{3}{7} = \frac{15}{91}$ ($\because$ A and B are independent.)

(ii) $P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B}) = \frac{8}{13} \times \frac{4}{7} = \frac{32}{91}$ ($\because$ $\bar{A}$, $\bar{B}$ are independent.)

(iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{5}{13} - \frac{15}{91} = \frac{20}{91}$ [From Part (*i*)]

(iv) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{3}{7} - \frac{15}{91} = \frac{24}{91}$ [From Part (*i*)]

(v) $P(A \cap \bar{B}) + P(\bar{A} \cap B) = \frac{20}{91} + \frac{24}{91} = \frac{44}{91}$ [From Parts (*iii*) and (*iv*)]

(vi) $P(A \cup B) = 1 - (\bar{A} \cap \bar{B}) = 1 - \frac{32}{91} = \frac{59}{91}.$ [From Part (*ii*)]

Example 3·48. *A problem in Statistics is given to three students A, B and C whose chances of solving it are 1/2, 3/4 and 1/4 respectively.*

What is the probability that the problem will be solved if all of them try independently ?

Solution. Let A, B, C denote the events that the problem is solved by the students A, B, C respectively. Then

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{3}{4}, \quad \text{and} \quad P(C) = \frac{1}{4}$$

The problem will be solved if at least one of them solves the problem. Thus we have to calculate the probability of occurrence of at least one of the three events A, B, C, *i.e.*, $P(A \cup B \cup C)$.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B)P(C) + P(A)P(B)P(C)$$

($\because$ A, B, C are mutually independent events.)

$$= \frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \frac{1}{2}\cdot\frac{3}{4} - \frac{3}{4}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{29}{32}.$$

Aliter. $P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C}) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$$= 1 - P(\bar{A})\,P(\bar{B})\,P(\bar{C})$$

[$\because$ A, B, C are mutually independent $\Rightarrow \bar{A}, \bar{B}$ and $\bar{C}$ are mutually independent.]

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right) = 1 - \frac{3}{32} = \frac{29}{32}.$$

Example 3·49. *A manager has two assistants and he bases his decision on information supplied independently by each one of them. The probability that he makes a mistake in his thinking is 0·005. The probability that an assistant gives wrong information is 0·3. Assuming that the mistakes made by the manager are independent of the information given by the assistants, find the probability that he reaches a wrong decision.*

Solution. Let us define the following events :

A : The manager makes a mistake in his thinking.

B : The 1st assistant gives him wrong information.

C : The 2nd assistant gives him wrong information.

In usual notations, we are given :

$$P(A) = 0{\cdot}005,\; P(B) = 0{\cdot}3 = P(C) \quad \Rightarrow \quad P(\bar{A}) = 0{\cdot}995,\; P(\bar{B}) = P(\bar{C}) = 0{\cdot}7$$

Assuming that the mistakes made by the manager are independent of the information supplied independently by each of the two assistants, we conclude that A, B and C, and consequently $\bar{A}, \bar{B}$ and $\bar{C}$ are mutually independent.

$\therefore$ $p = P$ [Manager reaches a wrong decision]

$= 1 - P$ [Manager reaches a correct decision] $= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$; because manager will reach a correct decision if he does not make a mistake in his thinking (*i.e.*, $\bar{A}$ happens) *and* both the assistants supply him correct information (*i.e.*, $\bar{B} \cap \bar{C}$ happens).

$\therefore$ $p = 1 - P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$

[Since the events $\bar{A}, \bar{B}$ and $\bar{C}$ are independent.]

$$= 1 - 0{\cdot}995 \times 0{\cdot}7 \times 0{\cdot}7 = 1 - 0{\cdot}48755 = 0{\cdot}51245.$$

Example 3·50. *The odds that a book on Statistics will be favourably reviewed by 3 independent critics are 3 to 2, 4 to 3 and 2 to 3 respectively. What is the probability that of the three reviews :*

(*i*) *All will be favourable,*

(*ii*) *Majority of the reviews will be favourable,*

(*iii*) *Exactly one review will be favourable,*

(*iv*) *Exactly two reviews will be favourable, and*

(*v*) *At least one of the reviews will be favourable.*

Solution. Let A, B and C denote respectively the events that the book is favourably reviewed by first, second and third critic respectively. Then we are given :

$$P(A) = \frac{3}{5}, P(B) = \frac{4}{7} \text{ and } P(C) = \frac{2}{5} \quad \Rightarrow \quad P(\bar{A}) = \frac{2}{5}, P(\bar{B}) = \frac{3}{7} \text{ and } P(\bar{C}) = \frac{3}{5}$$

(*i*) The probability that all the three reviews will be favourable is :

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C) = \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{24}{175}$$

($\because$ A, B and C are mutually independent events.)

(*ii*) The event that majority, *i.e.*, at least 2 reviews are favourable can materialise in the following mutually exclusive ways :

(*a*) $A \cap B \cap \bar{C}$ happens, (*b*) $A \cap \bar{B} \cap C$ happens, (*c*) $\bar{A} \cap B \cap C$ happens, and (*d*) $A \cap B \cap C$ happens.

Hence, the required probability is :

$$P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= P(A)\, P(B)\, P(\bar{C}) + P(A)\, P(\bar{B})\, P(C) + P(\bar{A})\, P(B)\, P(C) + P(A)\, P(B)\, P(C)$$

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{2}{5} + \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{94}{175}$$

(*iii*) Arguing as in case (*ii*), the probability that exactly one review will be favourable is

$$P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C)$$

$$= \frac{3}{5} \times \frac{3}{7} \times \frac{3}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{7} \times \frac{2}{5} = \frac{63}{175}$$

(*iv*) Similarly, the probability that exactly two reviews will be favourable is :

$$P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$$

$$= P(A) \times P(B) \times P(\bar{C}) + P(A) \times P(\bar{B}) \times P(C) + P(\bar{A}) \times P(B) \times P(C)$$

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{70}{175}$$

(*v*) The probability that at least one of the reviews will be favourable is :

$$P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap (\bar{C}) = 1 - P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$$

$$= 1 - \frac{2}{5} \times \frac{3}{7} \times \frac{3}{5} = \frac{157}{175}$$

In (*ii*) to (*v*) we have used that A, B and C are mutually independent.

Example 3·51. *A and B alternately cut a pack of cards and the pack is shuffled after each cut. If A starts and the game is continued until one cuts a diamond, what are the respective chances of A and B first cutting a diamond ?*

Solution. Let A_i and B_i denote the events of A and B cutting a diamond respectively in the ith trial. Then, we are given :

$$P(A_i) = P(B_i) = \frac{13}{52} = \frac{1}{4} \quad \Rightarrow \quad P(\bar{A}_i) = P(\bar{B}_i) = \frac{3}{4}\,;\ i = 1, 2, 3, \ldots$$

If A starts the game, he can first cut the diamond in the following mutually exclusive ways :

(*i*) A_1 happens, (*ii*) $\bar{A}_1 \cap \bar{B}_2 \cap A_3$ happens, (*iii*) $\bar{A}_1 \cap \bar{B}_2 \cap \bar{A}_3 \cap \bar{B}_4 \cap A_5$ happens, and so on. Hence by addition theorem of probability, the probability 'p' that A first cuts a diamond is given by :

$$p = P(i) + P(ii) + P(iii) + \dots\dots$$
$$= P(\bar{A}_1) + P(\bar{A}_1 \cap \bar{B}_2 \cap A_3) + P(\bar{A}_1 \cap \bar{B}_2 \cap \bar{A}_3 \cap \bar{B}_4 \cap A_5) + \dots$$
$$= P(\bar{A}_1) + P(\bar{A}_1)\, P(\bar{B}_2)\, P(A_3) + P(\bar{A}_1)\, P(\bar{B}_2)\, P(\bar{A}_3)\, P(\bar{B}_4)\, P(A_5) + \dots$$

(By multiplication Law of Probability for independent events.)

$$= \frac{1}{4} + \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} + \dots\dots = \frac{\frac{1}{4}}{1 - \frac{9}{16}} = \frac{4}{7}$$

$\therefore$ The probability that B first cuts a diamond $= 1 - p = 1 - \frac{4}{7} = \frac{3}{7}$.

Example 3·52. *(Huyghen's Problem) A and B throw alternatively with a pair of balanced dice. A wins if he throws a sum of six points before B throws a sum of seven points, while B wins if he throws a sum of seven points before A throws a sum of six points. If A begins the game, show that his probability of winning is 30/61.*

Solution. Let A_i denote the event of A's throwing '6' in the ith throw, $i = 1, 2, \dots$, and B_i denote the event of B's throwing '7' in the ith throw, $i = 1, 2, \dots$; with a pair of dice. Then $\bar{A}_i$ and $\bar{B}_i$ are the complementary events.

'6' can be obtained with two dice in the following ways :

(1, 5), (5, 1), (2, 4), (4, 2), (3, 3), *i.e.*, in 5 distinct ways.

$$\therefore \quad P(A_i) = \frac{5}{36} \quad \Rightarrow \quad P(\bar{A}_i) = 1 - \frac{5}{36} = \frac{31}{36};\ i = 1, 2, 3, \dots \qquad \dots(*)$$

'7' can be obtained with two dice as follows :

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), *i.e.*, in 6 distinct ways.

$$\therefore \quad P(B_i) = \frac{6}{36} = \frac{1}{6} \quad \Rightarrow \quad P(\bar{B}_i) = 1 - \frac{1}{6} = \frac{5}{6}; \qquad i = 1, 2, 3, \qquad \dots(**)$$

If A starts the game, he will win in the following mutually exclusive ways :

(*i*) A_1 happens, (*ii*) $\bar{A}_1 \cap \bar{B}_2 \cap A_3$ happens

(*iii*) $\bar{A}_1 \cap \bar{B}_2 \cap \bar{A}_3 \cap \bar{B}_4 \cap A_5$ happens, and so on.

Hence by addition theorem of probability, the required probability of A's winning, (say) $P(A)$ is given by :

$$P(A) = P(i) + P(ii) + P(iii) + \dots$$
$$= P(A_1) + P(\bar{A}_1 \cap \bar{B}_2 \cap A_3) + P(\bar{A}_1 \cap \bar{B}_2 \cap \bar{A}_3 \cap \bar{B}_4 \cap A_5) + \dots$$
$$= P(A_1) + P(\bar{A}_1)\, P(\bar{B}_2)\, P(A_3) + P(\bar{A}_1)\, P(\bar{B}_2)\, P(\bar{A}_3)\, P(\bar{B}_4)\, P(A_5) + \dots$$

[$\because$ The events A_i, B_i; $i = 1, 2, 3, \dots$ are mutually independent.]

$$= \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \dots \qquad \text{[Using (*) and (**)]}$$

$$= \frac{\frac{5}{36}}{1 - \frac{31}{36} \times \frac{5}{6}} = \frac{30}{61}.$$

Example 3·53. *A, B and C play a game and the chances of their winning it in an attempt are $\frac{2}{3}, \frac{1}{2}$ and $\frac{1}{4}$ respectively. A has the first chance, followed by B and then by C. This cycle is repeated till one of them wins the game. Find their respective chances of winning the game.*

Solution. Let A_i, B_i and C_i denote the events that A, B and C respectively win the game in their ith ($i = 1, 2, 3, \ldots$) attempt. Then we have :

$$P(A_i) = \frac{2}{3}, P(B_i) = \frac{1}{2}, P(C_i) = \frac{1}{4} \quad \Rightarrow \quad P(\bar{A}_i) = \frac{1}{3}, P(\bar{B}_i) = \frac{1}{2}, P(\bar{C}_i) = \frac{3}{4}$$

Since A starts the game, he can win in the following mutually exclusive ways :

(*i*) He wins in the first attempt, *i.e.*, the event A_1 happens.

(*ii*) A does not win in the first attempt, B does not win in the first attempt, C does not win in the first attempt and then A wins in the second attempt, *i.e.*, the event $\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap A_2$, happens.

(*iii*) Arguing as in (*ii*), the event :

$$\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap \bar{A}_2 \cap \bar{B}_2 \cap \bar{C}_2 \cap A_3 \text{ happens, and so on.}$$

The probability of A's winning (by addition theorem) is given by :

$P(A) = P(i) + P(ii) + P(iii) + \ldots$

$$= P(A_1) + P(\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap A_2) + P(\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap \bar{A}_2 \cap \bar{B}_2 \cap \bar{C}_2 \cap A_3) + \ldots$$

$$= P(A_1) + P(\bar{A}_1)\, P(\bar{B}_1)\, P(\bar{C}_1)\, P(A_2) + P(\bar{A}_1)\, P(\bar{B}_1) P(\bar{C}_1)\, P(\bar{A}_2)\, P(\bar{B}_2)\, P(\bar{C}_2) P(A_3) + \ldots$$

[$\because$ The events A_i, B_i, C_i ; $i = 1, 2, 3, \ldots$ are mutually independent.]

$$= \frac{2}{3} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{2}{3} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{2}{3} + \ldots = \frac{2}{3} + \frac{1}{12} + \frac{1}{96} + \ldots$$

$$= \frac{\frac{2}{3}}{1 - \frac{1}{8}} = \frac{2}{3}\times\frac{8}{7} = \frac{16}{21}.$$

If A starts the game, then B can win in the following mutually exclusive ways :

(*i*) $\bar{A}_1 \cap B_1$ happens

(*ii*) $\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap \bar{A}_2 \cap B_2$ happens

(*iii*) $\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap \bar{A}_2 \cap \bar{B}_2 \cap \bar{C}_2 \cap \bar{A}_3 \cap B_3$ happens, and so on.

The chance of B's winning is :

$$P(B) = P(\bar{A}_1 \cap B_1) + P(\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap \bar{A}_2 \cap B_2)$$
$$+ P(\bar{A}_1 \cap \bar{B}_1 \cap \bar{C}_1 \cap \bar{A}_2 \cap \bar{B}_2 \cap \bar{C}_2 \cap \bar{A}_3 \cap B_3) + \ldots$$

$$= P(\bar{A}_1) \times P(B_1) + P(\bar{A}_1) \times P(\bar{B}_1) \times P(\bar{C}_1) \times P(\bar{A}_2) \times P(B_2)$$
$$+ P(\bar{A}_1) \times P(\bar{B}_1) \times P(\bar{C}_1) \times P(\bar{A}_2) \times P(\bar{B}_2) \times P(\bar{C}_2) \times P(\bar{A}_3) \times P(B_3) + \ldots$$

[$\because$ The events A_i, B_i, C_i ; $i = 1, 2, 3, \ldots$ are mutually independent.]

$$= \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}\times\frac{1}{2} + \ldots = \frac{\frac{1}{6}}{1 - \frac{1}{8}} = \frac{4}{21}.$$

Hence the chance of C's winning is :

$$P(C) = 1 - P(A) - P(B) = 1 - \frac{16}{21} - \frac{4}{21} = \frac{1}{21}.$$

Example 3·54. *An urn contains four tickets marked with numbers 112, 121, 211, 222 and one ticket is drawn at random. Let A_i, ($i = 1, 2, 3$) be the event that ith digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.*

Solution. A_1 is the event that the first digit of the number of the ticket drawn is 1 and the favourable cases for this are 112 and 121, *i.e.*, two cases.

$\therefore \quad P(A_1) = \frac{2}{4} = \frac{1}{2}$. Similarly, we get $\quad P(A_2) = P(A_3) = \frac{2}{4} = \frac{1}{2}$

$A_1 \cap A_2$ is the event that the first two digits in the number which the selected ticket bears are each equal to unity and the only favourable case is ticket with number 112.

$$\therefore \quad P(A_1 \cap A_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A_1)\, P(A_2)$$

Similarly, $P(A_2 \cap A_3) = \frac{1}{4} = P(A_2)\, P(A_3)$ and $P(A_3 \cap A_1) = \frac{1}{4} = P(A_3)\, P(A_1)$

Thus we conclude that the events A_1, A_2 and A_3 are pairwise independent. Now

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P\,\{\text{all the three digits in the number selected are 1's}\} \\ &= P(\phi) = 0 \\ &\neq P(A_1)\, P(A_2)\, P(A_3) \end{aligned}$$

Hence A_1, A_2 and A_3, though pairwise independent are not mutually independent.

Example 3·55. *Two fair dice are thrown independently. Three events A, B and C are defined as follows :*

A : Odd face with first dice

B : Odd face with second dice

C : Sum of points on two dice is odd.

Are the events A, B and C (i) Pairwise independent, (ii) Mutually independent ?

Solution. In a random toss of two dice

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \quad \Rightarrow \quad n(S) = 6 \times 6 = 36$$

Event	*Favourable cases*	*No. of favourable cases*
A	$\{1, 3, 5\} \times \{1, 2, 3, 4, 5, 6\}$	$3 \times 6 = 18$
B	$\{1, 2, 3, 4, 5, 6\} \times (1, 3, 5\}$	$6 \times 3 = 18$
*C	$\{1, 3, 5\} \times \{2, 4, 6\} \cup \{2, 4, 6\} \times (1, 3, 5\}$	$3 \times 3 + 3 \times 3 = 18$
$A \cap B$	$\{1, 3, 5\} \times \{1, 3, 5\}$	$3 \times 3 = 9$
** $A \cap C$	$\{1, 3, 5\} \times \{2, 4, 6\}$	$3 \times 3 = 9$
** $B \cap C$	$\{2, 4, 6\} \times \{1, 3, 5\}$	$3 \times 3 = 9$
$A \cap B \cap C$	ϕ	0

* The sum of points on two dice be odd if one shows odd number and the other shows even number.

** If one die shows odd number and the sum is also odd, then the other die must show even number.

$$\therefore \quad P(A) = \frac{18}{36} = \frac{1}{2} = P(B) = P(C)$$

$$\left.\begin{aligned} P(A \cap B) &= \frac{9}{36} = \frac{1}{4} = P(A)\,P(B) \\ P(A \cap C) &= \frac{9}{36} = \frac{1}{4} = P(A)\,P(C) \\ P(B \cap C) &= \frac{9}{36} = \frac{1}{4} = P(B)\,P(C) \end{aligned}\right\} \quad \ldots (*)$$

and $\quad P(A \cap B \cap C) = P(\phi) = 0 \neq P(A)\,P(B)\,P(C) \quad \ldots (**)$

Hence (*) implies that the events A, B and C are pairwise independent but (**) implies that they not mutually independent.

Example 3·56. *Why does it pay to bet consistently on seeing 6 at least once in 4 throws of a die, but not on seeing a double six at least once in 24 throws with two dice ? (de Mere's Problem).*

Solution. The probability of getting a '6' in throw of die $= \frac{1}{6}$

$\therefore$ The probability of not getting a '6' in a throw of die $= 1 - \frac{1}{6} = \frac{5}{6}$

By compound probability theorem, the probability that in 4 throws of die no '6' is obtained $= \left(\frac{5}{6}\right)^4$

Hence, the probability of obtaining '6' at least once in 4 throws of a die

$$= 1 - \left(\frac{5}{6}\right)^4 = 0{\cdot}516$$

Now, if a trial consists of throwing two dice at a time, then the probability of getting a 'double' of '6' in a trial $= \frac{1}{36}$

Thus the probability of not getting a 'double of 6' in a trial $= \frac{35}{36}$

The probability that in 24 throws, with two dice each, no 'double of 6' is obtained

$$= \left(\frac{35}{36}\right)^{24}$$

Hence the probability of getting a 'double of 6' at least once in 24 throws

$$= 1 - \left(\frac{35}{36}\right)^{24} = 0{\cdot}491$$

Since the probability in the first case is greater than the probability in the second case, the result follows.

Example 3·57. *Let $A_1, A_2, \ldots A_n$ be independent events and $P(A_k) = p_k$. Further, let p be the probability that none of the events occurs ; then show that $p \leq \exp\left(-\sum_k p_k\right)$.*

Solution. We have $\quad p_i = P(A_i)\,;\ i = 1, 2, \ldots, n$

$$p = P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n)$$

$$= \prod_{i=1}^{n} P(\bar{A}_i) = \prod_{i=1}^{n} [1 - P(A_i)] = \prod_{i=1}^{n} (1 - p_i) \qquad \text{(Since } A_i\text{'s are independent.)}$$

$$\leq \prod_{i=1}^{n} \exp(-p_i) \qquad [\because 1 - x \leq e^{-x} \text{ for } 0 \leq x \leq 1 \text{ and } 0 \leq p_i \leq 1]$$

$$\Rightarrow p \leq \exp\left[-\sum_{i=1}^{n} p_i\right], \text{ as desired.}$$

Remark. We have $1 - x \leq \exp(-x)$ for $0 \leq x \leq 1$. ... (*)

Proof. The inequality (*) is obvious for $x = 0$ and $x = 1$. Consider $0 < x < 1$. Then

$$\log(1-x)^{-1} = -\log(1-x) = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right) \qquad \ldots (**)$$

the expansion being valid since $0 < x < 1$. Further since $x > 0$, we get from (**)

$$\log(1-x)^{-1} > x \quad \Rightarrow \quad -\log(1-x) > x$$

$$\Rightarrow \quad \log(1-x) < -x, \quad i.e., \quad 1 - x < e^{-x}, \text{ as desired.}$$

Example 3·58. *p is the probability that a man aged x years will die in a year. Find the probability that out of n men $A_1, A_2, \ldots, A_n$ each aged x, A_1 will die in a year and will be the first to die.*

Solution. Let E_i, $(i = 1, 2, \ldots, n)$ denote the event that A_i dies in a year. Then

$$P(E_i) = p, \ (i = 1, 2, \ldots, n) \quad \text{and} \quad P(\bar{E}_i) = 1 - p.$$

The probability that none of n men $A_1, A_2, \ldots, A_n$ dies in a year

$$= P(\bar{E}_1 \cap \bar{E}_2 \cap \ldots \cap \bar{E}_n) = P(\bar{E}_1)\, P(\bar{E}_2) \ldots P(\bar{E}_n)$$

(By compound probability theorem)

$$= (1-p)^n$$

$\therefore$ The probability that at least one of $A_1, A_2, \ldots, A_n$ dies in a year

$$= 1 - P(\bar{E}_1 \cap \bar{E}_2 \cap \ldots \cap \bar{E}_n) = 1 - (1-p)^n$$

The probability that among n men, A_1 is the first to die is $1/n$ and since this event is independent of the event that at least one man dies in a year, the required probability is : $\frac{1}{n}\left[1 - (1-p)^n\right]$.

EXAMPLES ON CONDITIONAL PROBABILITY

Example 3·59. *Data on the readership of a certain magazine show that the proportion of 'male readers under 35 is 0·40 and over 35 is 0·20. If the proportion of readers under 35 is 0·70, find the proportion of subscribers that are 'females over 35 years'. Also calculate the probability that a randomly selected male subscriber is under 35 years of age.*

Solution. Let us define the following events :

A : Reader of the magazine is a male.

B : Reader of the magazine is over 35 years of age.

Then in usual notations, we are given :

$$P(A \cap B) = 0{\cdot}20, \quad P(A \cap \bar{B}) = 0{\cdot}40 \quad \text{and} \quad P(\bar{B}) = 0{\cdot}70 \ \Rightarrow \ P(B) = 0{\cdot}30$$

(*i*) The proportion of subscribers that are 'females over 35 years' is :

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = 0{\cdot}30 - 0{\cdot}20 = 0{\cdot}10$$

(*ii*) The probability that a randomly selected male subscriber is under 35 years is :

$$P(\bar{B} \mid A) = \frac{P(A \cap \bar{B})}{P(A)} = \frac{0{\cdot}40}{0{\cdot}60} = \frac{2}{3}.$$

$$[\because \; P(A) = P(A \cap B) + P(A \cap \bar{B}) = 0{\cdot}20 + 0{\cdot}40 = 0{\cdot}60]$$

Example 3·60. *From a city population, the probability of selecting (i) a male or a smoker is 7/10, (ii) a male smoker is 2/5, and (iii) a male, if a smoker is already selected is 2/3. Find the probability of selecting (a) a non-smoker, (b) a male, and (c) a smoker, if a male is first selected.*

Solution. Define the following events :

A : a male is selected, $\quad B$: a smoker is selected

We are given :

$$P(A \cup B) = \frac{7}{10}, \qquad P(A \cap B) = \frac{2}{5}, \qquad P(A \mid B) = \frac{2}{3}$$

(*a*) The probability of selecting a non-smoker is :

$$P(\bar{B}) = 1 - P(B) = 1 - \frac{P(A \cap B)}{P(A \mid B)} \qquad \left[\because \; P(A \mid B) = \frac{P(A \cap B)}{P(B)}\right]$$

$$= 1 - \frac{2/5}{2/3} = 1 - \frac{3}{5} = \frac{2}{5}$$

$$\Rightarrow \qquad P(B) = 1 - \frac{2}{5} = \frac{3}{5}$$

(*b*) The probability of selecting a male (by Addition theorem) is :

$$P(A) = P(A \cup B) + P(A \cap B) - P(B) = \frac{7}{10} + \frac{2}{5} - \frac{3}{5} = \frac{1}{2}$$

(*c*) The probability of selecting a smoker if a male is first selected is :

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{2/5}{1/2} = \frac{4}{5}.$$

Example 3·61. *Sixty per cent of the employees of the XYZ Corporation are college graduates. Of these, ten per cent are in sales. Of the employees who did not graduate from college, eighty per cent are in sales. What is the probability that*

(i) an employee selected at random is in sales ?

(ii) an employee selected at random is neither in sales nor a college graduate ?

Solution. Let us define the following events :

A : An employee is a college graduate ; $\quad B$: An employee is in sales

Then we are given : $P(A) = 0{\cdot}60, \qquad P(B \mid A) = 0{\cdot}10, \qquad P(B \mid \bar{A}) = 0{\cdot}80$

(*i*) The probability that an employee is in sales is :

$$P(B) = P(A \cap B) + P(\bar{A} \cap B) = P(A) \times P(B \mid A) + P(\bar{A}) \times P(B \mid \bar{A})$$

$$= 0{\cdot}60 \times 0{\cdot}10 + (1 - 0{\cdot}60) \times 0{\cdot}80 = 0{\cdot}38$$

(*ii*) The probability that an employee is neither in sales nor a college graduate is :

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - \{P(A) + P(B) - P(A \cap B)\}$$

$$= 1 - \{P(A) + P(B) - P(A)\,P(B \mid A)\} = 1 - (0{\cdot}60 + 0{\cdot}38 - 0{\cdot}60 \times 0{\cdot}10) = 0{\cdot}08.$$

Example 3·62. *Two computers A and B are to be marketed. A salesman who is assigned the job of finding customers for them has 60% and 40% chances respectively of succeeding in case of computer A and B. The two computers can be sold independently. Given that he was able to sell at least one computer, what is the probability that computer A has been sold ?*

Solution. Let E denote the event that computer A is marketed and F denote the event that computer B is marketed. We are given :

$$P(E) = \frac{60}{100} = 0{\cdot}60 \quad \Rightarrow \quad P(\bar{E}) = 0{\cdot}40 \qquad \text{and} \qquad P(F) = \frac{40}{100} = 0{\cdot}40 \Rightarrow P(\bar{F}) = 0{\cdot}60$$

$$\text{Required probability} = P[E \mid (E \cup F)] = \frac{P\{E \cap (E \cup F)\}}{P(E \cup F)}$$

$$= \frac{P(E)}{1 - P(\bar{E} \cap \bar{F})} = \frac{P(E)}{1 - P(\bar{E})\,P(\bar{F})} = \frac{0{\cdot}6}{1 - 0{\cdot}4 \times 0{\cdot}6} = \frac{0{\cdot}60}{0{\cdot}76} = 0{\cdot}79.$$

Example 3·63. *A certain drug manufactured by a company is tested chemically for its toxic nature. Let the event 'the drug is toxic' be denoted by E and the event 'the chemical test reveals that the drug is toxic' be denoted by F. Let P (E) = θ, P (F | E) = P ($\bar{F} \mid \bar{E}$) = 1– θ. Then show that probability that the drug is not toxic given that the chemical test reveals that it is toxic is free from θ.*

Solution. We are given $P(E) = \theta \Rightarrow P(\bar{E}) = 1 - \theta$ and $P(F \mid E) = P(\bar{F} \mid \bar{E}) = 1 - \theta$. ...(*)

We want $P(\bar{E} \mid F)$ and we have to show that it is independent of θ.

$$P(\bar{E} \mid F) = 1 - P(E \mid F) = 1 - \frac{P(E \cap F)}{P(F)} = 1 - \frac{P(E) \cdot P(F \mid E)}{P(F)} = 1 - \frac{\theta(1-\theta)}{P(F)} \quad \text{[From (*)] ... (**)}$$

The event F can materialise in the following mutually disjoint and exhaustive cases : $E \cap F$ and $\bar{E} \cap F$. Hence by addition theorem of probability, we have

$$\begin{aligned} P(F) &= P(E \cap F) + P(\bar{E} \cap F) = P(E)\,P(F \mid E) + P(\bar{E}) \cdot P(F \mid \bar{E}) \\ &= \theta(1-\theta) + (1-\theta)\left[1 - P(\bar{F} \mid \bar{E})\right] && \text{[Using (*)]} \\ &= \theta(1-\theta) + (1-\theta)\left[1 - (1-\theta)\right] && \text{[From (*)]} \\ &= \theta(1-\theta) + (1-\theta)\,\theta = 2\theta(1-\theta). \end{aligned}$$

Substituting in (**), we get $\quad P(\bar{E} \mid F) = 1 - \dfrac{\theta(1-\theta)}{2\theta(1-\theta)} = 1 - \dfrac{1}{2} = \dfrac{1}{2}$,

which is independent of θ as desired.

Example 3·64. *A bag contains 17 counters marked with the numbers 1 to 17. A counter is drawn and replaced ; a second drawing is then made. What is the probability that :*

(i) the first number drawn is even and the second odd ?

(ii) the first number is odd and the second even ?

How will your results in (i) and (ii) be effected if the first counter drawn is not replaced ?

Solution. (*i*) Let A denote the event of getting even numbered counter on the first draw and B denote the event of getting odd numbered counter on the second draw. Since the counter drawn is replaced, events A and B are independent.

Now from 1 to 17, the even numbers are 2, 4, 6, 8, 10, 12, 14, and 16, *i.e.*, 8 and odd numbers are 9.

$$\therefore \qquad P(A) = \frac{8}{17} \qquad \text{and} \qquad P(B) = \frac{9}{17}$$

Using multiplication theorem of probability, the probability of getting even number on the first draw and odd number on the second draw is given by :

$$P(A \cap B) = P(A) \times P(B) = \frac{8}{17} \times \frac{9}{17} = \frac{72}{289}$$

However, if the first counter drawn is not replaced before the second counter is drawn, the events A and B are not independent. In this case,

$$P(A \cap B) = P(A) \times P(B \mid A) = \frac{8}{17} \times \frac{9}{16} = \frac{9}{34}$$

(*ii*) The probabilities of the first counter drawn being odd and the second counter drawn being even are :

$$\frac{9}{17} \times \frac{8}{17} = \frac{72}{289}, \quad \text{if replacement is made, and}$$

$$\frac{9}{17} \times \frac{8}{16} = \frac{9}{34}, \quad \text{if the replacement is not made.}$$

Example 3·65. *A bag contains 10 gold and 8 silver coins. Two successive drawings of 4 coins are made such that : (i) coins are replaced before the second trial, (ii) the coins are not replaced before the second trial. Find the probability that the first drawing will give 4 gold and the second 4 silver coins.*

Solution. Let A denote the event of drawing 4 gold coins in the first draw and B denote the event of drawing 4 silver coins in the second draw. Then we have to find the probability of $P(A \cap B)$.

(*i*) *Draws with replacement.* If the coins drawn in the first draw are replaced back in the bag before the second draw then the events A and B are independent and the required probability is given (using the multiplication rule of probability) by the expression

$$P(A \cap B) = P(A) \,.\, P(B) \qquad \dots (*)$$

1st draw. Four coins can be drawn out of $10 + 8 = 18$ coins in ${}^{18}C_4$ ways, which gives the exhaustive number of cases. In order that all these coins are of gold, they must be drawn out of the 10 gold coins and this can be done in ${}^{10}C_4$ ways. Hence

$$P(A) = \frac{{}^{10}C_4}{{}^{18}C_4}$$

2nd draw. When the coins drawn in the first draw are replaced before the 2nd draw, the bag contains 18 coins. The probability of drawing 4 silver coins in the 2nd draw is given by : $P(B) = \frac{{}^{8}C_4}{{}^{18}C_4}$. Substituting in (*), $P(A \cap B) = \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{18}C_4}$.

(*ii*) *Draws without replacement.* If the coins drawn are not replaced back before the second draw, then the events A and B are not independent and the required probability is given by :

$$P(A \cap B) = P(A) \,.\, P(B \mid A) \qquad \dots (**)$$

As discussed in part (*i*), $P(A) = \frac{{}^{10}C_4}{{}^{18}C_4}$

Now, if the 4 gold coins which were drawn in the first draw are not replaced back, there are $18 - 4 = 14$ coins left in the bag and $P(B \mid A)$ is the probability of drawing 4 silver coins from the bag containing 14 coins out of which 6 are gold coins and 8 are silver coins.

Hence $P(B \mid A) = \frac{{}^{8}C_4}{{}^{14}C_4}$. Substituting in (**), we get $P(A \cap B) = \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{14}C_4}$.

Example 3·66. *A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one, and examined. Those examined are not put back. What is the probability that the 9th one examined is the last defective ?*

Solution. Let A be the event of getting exactly 3 defectives in examination of 8 record players and let B denote the event that the 9th piece examined is a defective one.

Since it is a problem of sampling without replacement and since there are 4 defectives (and 11 non-defectives) out of 15 record players, $P(A) = {}^4C_3 \times {}^{11}C_5 / {}^{15}C_8$.

$P(B \mid A)$ = Probability that the 9th examined record player is defective given that there were 3 defectives in the first 8 pieces examined $= \dfrac{4-3}{15-8} = \dfrac{1}{7}$,

since there is only one defective piece left among the remaining $15 - 8 = 7$ record players.

Hence, required probability $= P(A \cap B) = P(A)\, P(B \mid A)$

$$= \frac{{}^4C_3 \times {}^{11}C_5}{{}^{15}C_8} \times \frac{1}{7} = \frac{8}{195}.$$

Example 3·67. *Two dice, one green and the other red, are thrown. Let A be the event that the sum of the points on the faces is odd, and B be the event of at least one ace (number '1').*

(a) Describe the (i) complete sample space, (ii) events A, B, $\bar{B}$, $A \cap B$, $A \cup B$, and $A \cap \bar{B}$ and find their probabilities assuming that all the 36 sample points have equal probabilities.

(b) Find the probabilities of the events :

(i) $(\bar{A} \cup \bar{B})$ (ii) $(\bar{A} \cap \bar{B})$ (iii) $(A \cap \bar{B})$ (iv) $(\bar{A} \cap B)$

(v) $\overline{(A \cap B)}$ (vi) $(\bar{A} \cup B)$ (vii) $\overline{(A \cup B)}$ (viii) $\bar{A} \cap (A \cup B)$

(ix) $A \cup (\bar{A} \cap B)$ (x) $(A \mid B)$ and $(B \mid A)$, and (xi) $(\bar{A} \mid \bar{B})$ and $(\bar{B} \mid \bar{A})$.

Solution. (*a*) (*i*) The sample space S, consisting of the 36 elementary events is given by :

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

$$= \left\{ \begin{array}{llllll} (1,1); & (1,2); & (1,3); & (1,4); & (1,5); & (1,6), \\ (2,1); & (2,2); & (2,3); & (2,4); & (2,5); & (2,6), \\ (3,1); & (3,2); & (3,3); & (3,4); & (3,5); & (3,6), \\ (4,1); & (4,2); & (4,3); & (4,4); & (4,5); & (4,6), \\ (5,1); & (5,2); & (5,3); & (5,4); & (5,5); & (5,6), \\ (6,1); & (6,2); & (6,3); & (6,4); & (6,5); & (6,6) \end{array} \right\}$$

where, for example, the ordered pair (4, 5) refers to the elementary event that the green die shows 4 and the red die shows 5.

A = The event that the sum of the numbers shown by the two dice is odd.

$= \{1, 3, 5\} \times \{2, 4, 6\} \cup \{2, 4, 6\} \times \{1, 3, 5\}$

$= \{ (1, 2); (2, 1); (1, 4); (2, 3); (3, 2); (4, 1); (1, 6); (2, 5); (3, 4),$
$(4, 3); (5, 2); (6, 1); (3, 6); (4, 5); (5, 4); (6, 3); (5, 6); (6, 5) \}$

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{18}{36}$

B = The event that at least one face is 1,

$= \{(1, 1) ; (1, 2) ; (1, 3) ; (1, 4) ; (1, 5) ; (1, 6) ; (2, 1) ; (3, 1), (4, 1) ; (5, 1) ; (6, 1)\}$

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{11}{36}$$

$\bar{B}$ = The event that none of the faces obtained is an ace.

$= \{2, 3, 4, 5, 6\} \times \{2, 3, 4, 5, 6\}$

$= \{(2, 2) ; (2, 3) ; (2, 4) ; (2, 5) ; (2, 6) ; (3, 2) ; (3, 3) ; (3, 4) ;$
$(3, 5) ; (3, 6) ; (4, 2) ; (4, 3) ; (4, 4) ; (4, 5) (4, 6) ; (5, 2) ;$
$(5, 3) ; (5, 4) ; (5, 5) ; (5, 6) ; (6, 2); (6, 3) ; (6, 4) ; (6, 5) (6, 6)\}$

$$\therefore \quad P(\bar{B}) = \frac{n(\bar{B})}{n(S)} = \frac{25}{36}$$

$A \cap B$ = The event that sum is odd and at least one face is an ace.

$= \{(1, 2) ; (2, 1) ; (1, 4) ; (4, 1) ; (1, 6) ; (6, 1)\}$

$$\therefore \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$A \cup B = \{(1, 2) ; (2, 1) ; (1, 4) ; (2, 3) ; (3, 2) ; (4, 1) ; (1, 6) ; (2, 5) ;$
$(3, 4) ; (4, 3) ; (5, 2) ; (6, 1) ; (3, 6) ; (4, 5) ; (5, 4) ; (6, 3) ;$
$(5, 6) ; (6, 5) ; (1, 1) ; (1, 3) ; (1, 5) ; (3, 1) ; (5, 1)\}$

$$\therefore \quad P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{23}{36}$$

$A \cap \bar{B} = \{(2, 3); (2, 5) ; (3, 2); (3, 4) ; (3, 6) ; (4, 3) ; (4, 5) ; (5, 2); (5, 4); (5, 6) ; (6, 3) ; (6, 5)\}$

$$P(A \cap \bar{B}) = \frac{n(A \cap \bar{B})}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

(b) (i) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(ii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

(iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{18}{36} - \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$

(iv) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{11}{36} - \frac{6}{36} = \frac{5}{36}$

(v) $P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(vi) $P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B) = \left(1 - \frac{18}{36}\right) + \frac{11}{36} - \frac{5}{36} = \frac{2}{3}$

(vii) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

(viii) $P[\bar{A} \cap (A \cup B)] = P[(A \cap \bar{A}) \cup (\bar{A} \cap B)] = P(\bar{A} \cap B) = \frac{5}{36}$ $[\because A \cap \bar{A} = \phi]$

(ix) $P[A \cup (\bar{A} \cap B)] = P(A) + P(\bar{A} \cap B) - P(A \cap \bar{A} \cap B)$

$= P(A) + P(\bar{A} \cap B) = \frac{18}{36} + \frac{5}{36} = \frac{23}{36}$

$[\because P(A \cap \bar{A} \cap B) = P(\phi) = 0]$

(x) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{6/36}{11/36} = \frac{6}{11}$; $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{6/36}{18/36} = \frac{6}{18} = \frac{1}{3}$

(xi) $P(\bar{A} \mid \bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \dfrac{13/36}{25/36} = \dfrac{13}{25}$; $P(\bar{B} \mid \bar{A}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{A})} = \dfrac{13/36}{18/36} = \dfrac{13}{18}$.

Example 3·68. *If $p_1 = P(A), p_2 = P(B), p_3 = P(A \cap B), (p_1, p_2, p_3 > 0)$; express the following in terms of p_1, p_2, p_3.*

(i) $P(\overline{A \cup B})$, *(ii)* $P(\bar{A} \cup \bar{B})$, *(iii)* $P(\bar{A} \cap B)$, *(iv)* $P(\bar{A} \cup B)$, *(v)* $P(\bar{A} \cap \bar{B})$,

(vi) $P(A \cap \bar{B})$, *(vii)* $P(A \mid B)$, *(viii)* $P(B \mid \bar{A})$, *(ix)* $P[\bar{A} \cap (A \cup B)]$.

Solution. (*i*) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$$= 1 - p_1 - p_2 + p_3$$

(*ii*) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - p_3$

(*iii*) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = p_2 - p_3$

(*iv*) $P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B) = 1 - p_1 + p_2 - (p_2 - p_3) = 1 - p_1 + p_3$

(*v*) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - p_1 - p_2 + p_3$ [From Part (*i*)]

(*vi*) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = p_1 - p_3$

(*vii*) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{p_3}{p_2}$ (*viii*) $P(B \mid \bar{A}) = \dfrac{P(\bar{A} \cap B)}{P(\bar{A})} = \dfrac{p_2 - p_3}{(1 - p_1)}$

(*ix*) $P[\bar{A} \cap (A \cup B)] = P[(\bar{A} \cap A) \cup (\bar{A} \cap B)] = P[\phi \cup (\bar{A} \cap B)] = P(\bar{A} \cap B) = p_2 - p_3$.

Example 3·69. *Plant I of XYZ manufacturing organisation employs 5 production and 3 maintenance engineers, another plant II of same organisation employs 4 production and 5 maintenance engineers. From any one of these plants, a single selection of two engineers is made. Find the probability that one of them would be production engineer and the other maintenance engineer.*

Solution. Let us define the following events :

A_1 : Plant I is selected ; A_2 : Plant II is selected

B : In a selection of 2 persons, one is production engineer and the other is maintenance engineer.

The required event of selecting one production and one maintenance engineer in a selection of two persons can materialise in the following mutually exclusive ways :

(*i*) $A_1 \cap B$ happens, (*ii*) $A_2 \cap B$ happens

Hence, the required probability p (by addition theorem) is given by :

$$p = P(i) + P(ii) = P(A_1 \cap B) + P(A_2 \cap B)$$
$$= P(A_1) \times P(B \mid A_1) + P(A_2) \times (B \mid A_2) \qquad \ldots (*)$$

Since there are two plants, the selection of each being equally likely, we have

$$P(A_1) = P(A_2) = \frac{1}{2}$$

$\therefore$ $P(B \mid A_1)$ = Probability of selecting one production and one maintenance engineer in a selection of two engineers from the first plant.

$$= \frac{{}^5C_1 \times {}^3C_1}{{}^8C_2} = \frac{5 \times 3 \times 2\,!}{8 \times 7} = \frac{15}{28}$$

Similarly, $$P(B \mid A_2) = \frac{{}^4C_1 \times {}^5C_1}{{}^9C_2} = \frac{4 \times 5 \times 2\,!}{9 \times 8} = \frac{5}{9}$$

Substituting in (*), we obtain $$p = \frac{1}{2} \times \frac{15}{28} + \frac{1}{2} \times \frac{5}{9} = \frac{275}{504}.$$

Matching Problem. Let us have n letters corresponding to which there exist n envelopes bearing different addresses. Considering various letters being put in various envelopes at random, a *match is said to occur* if a letter goes into the right envelope. (Alternatively, if in a party there are n persons with n different hats, *a match is said to occur* if in the process of selecting hats *at random*, the ith person righrly gets the ith hat.)

A match at the kth position for k = 1, 2, ..., n. Let us first consider the event A_k when a match occurs at the kth place. For better understanding let us put the envelopes bearing numbers 1, 2, ..., n in ascending order. When A_k occurs, k th letter goes to the kth envelope but $(n-1)$ letters can go to the remaining $(n-1)$ envelopes in $(n-1)\,!$ ways. Hence

$$P(A_k) = \frac{(n-1)\,!}{n\,!} = \frac{1}{n},$$

where $P(A_k)$ denotes the probability of the kth match. It is interesting to see that $P(A_k)$ *does not depend on k.*

Example 3·70. *(a) 'n' different objects 1, 2, ..., n are distributed at random in n places marked 1, 2, ... n, Find the probability that none of the objects occupies the place corresponding to its number.*

(b) If n letters are randomly placed in correctly addressed envelopes, prove that the probability that exactly r letters are placed in correct envelopes is given by :

$$\frac{1}{r\,!} \sum_{k=0}^{n-r} (-1)^k \frac{1}{k\,!}\,; \quad r = 1, 2, \ldots, n$$

Solution. (Probability of no match) Let E_i, $(i = 1, 2, \ldots, n)$ denote the event that the ith object occupies the place corresponding to its number so that $\bar{E}_i$, is the complementary event. Then the probability 'p' that none of the objects occupies the place corresponding to its number is given by

$$\begin{aligned} p &= P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap \ldots \cap \bar{E}_n) \\ &= 1 - P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n) \\ &= 1 - \Bigg[\sum_{i=1}^{n} P(E_i) - \sum\sum_{\substack{i,j=1 \\ i<j}}^{n} P(E_i \cap E_j) + \sum\sum\sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} P(E_i \cap E_j \cap E_k) - \ldots \\ &\qquad + (-1)^{n-1} P(E_1 \cap E_2 \cap \ldots \cap E_n) \Bigg] \quad \ldots (*) \end{aligned}$$

Now $$P(E_i) = \frac{1}{n}, \forall\, i$$

$$P(E_i \cap E_j) = P(E_i)\, P(E_j \mid E_i) = \frac{1}{n} \times \frac{1}{n-1}, \forall\, i, j\ (i < j)$$

From (4·6), we get

$$P(A) = P\left[\left(\bigcup_{i=1}^{n} A_i\right) \cup B_{n+1}\right] = \sum_{i=1}^{n} P(A_i) + P(B_{n+1})$$

(By axiom of Additivity)

$$\Rightarrow \qquad P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \sum_{i=1}^{n} P(A_i) + \lim_{n \to \infty} (B_{n+1}) = \sum_{i=1}^{\infty} P(A_i), \qquad \text{[From (4·8)]}$$

which is the extended axiom of addition.

4·2. BAYES' THEOREM

In the discussion of conditional probability we indicated that revising probability when new information is obtained is an important phase of probability analysis. Often, we begin our analysis with initial or *prior* probability estimates for specific events of interest. Then, from sources such as a sample, a special report, a product test, and so on we obtain some additional information about the events. Given this new information, we update the prior probability values by calculating revised probabilities, referred to as *posterior probabilities*. Bayes' theorem *which was given by Thomas Bayes, a British Mathematician,* in 1763, provides a means for making these probability calculations. The steps in this probability revision process are shown in the following diagram :

Prior Probabilities	→	New Information	→	Application of Bayes' Theorem	→	Posterior Probabilities

Theorem 4·2. *Bayes' Theorem. If $E_1, E_2, E_2 \ldots, E_n$ are mutually disjoint events with $P(E_i) \neq 0$, $(i = 1, 2, \ldots, n)$, then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have*

$$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} = \frac{P(E_i)\, P(A \mid E_i)}{P(A)}\,; \; i = 1, 2, \ldots, n \qquad \ldots (4·9)$$

Proof. Since $A \subset \bigcup_{i=1}^{n} E_i$, we have, $A = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (A \cap E_i)$

[By distributive law]

Since $(A \cap E_i) \subset E_i$, $(i = 1, 2, \ldots, n)$ are mutually disjoint events, we have by addition theorem of probability :

$$P(A) = P\left\{\bigcup_{i=1}^{n} (A \cap E_i)\right\} = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i), \qquad \ldots (*)$$

by multiplication theorem of probability.

Also we have $\quad P(A \cap E_i) = P(A)\, P(E_i \mid A)$

$$\Rightarrow \qquad P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \text{[From (*)]}$$

***Remarks* 1.** The probabilities $P(E_1), P(E_2), \ldots, P(E_n)$ are termed as the '*a prior probabilities*' because they exist before we gain any information from the experiment itself.

2. The probabilities $P(A \mid E_i), i = 1, 2, \ldots, n$ are called '*likelihoods*' because they indicate how likely the event A under consideration is to occur, given each and every *a prior* probability.

3. The probability $P(E_i \mid A), i = 1, 2, \ldots, n$ are called '*posterior probabilities*' because they are determined after the results of the experiment are known.

4. From (*) we get the following important result :

"If the events $E_1, E_2, \ldots, E_n$ constitute a disjoint partition of the sample space S and $P(E_i) \neq 0; i = 1, 2, \ldots, n$, then for any event A is S, we have

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i) \qquad (4\cdot10)$$

4·2·1. Bayes' Theorem for Future Events. *The probability of the materialisation of another event C, given $P(C \mid A \cap E_1), P(C \mid A \cap E_2), \ldots, P(C \mid A \cap E_n)$ is given by :*

$$P(C \mid A) = \frac{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)\, P(C \mid E_i \cap A)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \ldots (4\cdot11)$$

Proof. Since the occurrence of event A implies the occurrence of one and only one of the events $E_1, E_2, \ldots, E_n$, the event C (granted that A has occurred) can occur in the following mutually exclusive ways :

$$C \cap E_1, C \cap E_2, \ldots, C \cap E_n$$

i.e.,
$$C = (C \cap E_1) \cup (C \cap E_2) \cup \ldots \cup (C \cap E_n)$$
$$\Rightarrow \quad C \mid A = \{(C \cap E_1) \mid A\} \cup \{(C \cap E_2) \mid A\} \cup \ldots \cup \{(C \cap E_n) \mid A\}$$
$$\therefore \quad P(C \mid A) = P\{(C \cap E_1) \mid A\} + P\{(C \cap E_2) \mid A\} + \ldots + P\{(C \cap E_n) \mid A\}$$
$$= \sum_{i=1}^{n} P\{(C \cap E_i) \mid A\} = \sum_{i=1}^{n} P(E_i \mid A)\, P\{C \mid (E_i \cap A)\}$$

Substituting the value of $P(E_i \mid A)$ from (4·9), we get

$$P(C \mid A) = \frac{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)\, P(C \mid E_i \cap A)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)}$$

***Remarks* 1.** It may happen that the materialisation of the event E_i makes C independent of A, then, we have
$$P(C \mid E_i \cap A) = P(C \mid E_i),$$

and the above formula reduced to
$$P(C \mid A) = \frac{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)\, P(C \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \ldots (4\cdot12)$$

The event C can be considered in regard to A as Future Event.

2. Bayes theorem is extensively used in statistical inference, and by business and management executives in arriving at valid decisions in the face of uncertainty.

The Tabular Approach. The posterior probabilities, using Bayes' Theorem can be obtained conveniently in a tabular form, which involves the following steps.

Step s1 : Prepare the following three columns :

Column 1 — The list of all mutually exclusive events E_i, $i = 1, 2, \ldots, n$; occur in the problem.

Column 2 — The prior probabilities $P(E_i)$; $i = 1, 2, \ldots, n$ for the events.

Column 3 — The conditional probabilities of the new information (A) given each event, *viz.*, $P(A \mid E_i)$, $i = 1, 2, \ldots, n$

2 : In column 4, compute joint probabilities for each event using the formula :

$$P(E_i \cap A) = P(E_i)\, P(A \mid E_i).$$

3 : Sum the joint probability column to find the probability of the new information *viz.*, $P(A)$.

4 : In column 5, compute the posterior probabilities using the basic relationship of conditional probability, *viz.*,

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} \; ; i = 1, 2, \ldots, n.$$

Example 4·1. *Suppose that a product is produced in three factories X, Y and Z. It is known that factory X produces thrice as many items as factory Y, and that factories Y and Z produce the same number of items. Assume that it is known that 3 per cent of the items produced by each of the factories X and Z are defective while 5 per cent of those manufactured by factory Y are defective. All the items produced in the three factories are stocked, and an item of product is selected at random.*

(i) What is the probability that this item is defective ?

(ii) If an item selected at random is found to be defective, what is the probability that it was produced by factory X, Y and Z respectively ?

Solution. Let the number of items produced by each of the factories Y and Z be n. Then the number of items produced by the factory X is $3n$. Let E_1, E_2 and E_3 denote the events that the items are produced by factory X, Y and Z respectively and let A be the event of the item being defective. Then we have

$$P(E_1) = \frac{3n}{3n + n + n} = 0{\cdot}6 \; ; P(E_2) = \frac{n}{5n} = 0{\cdot}2 \; ; \quad \text{and} \quad P(E_3) = \frac{n}{5n} = 0{\cdot}2$$

Also, $P(A \mid E_1) = P(A \mid E_3) = 0{\cdot}03$ and $P(A \mid E_2) = 0{\cdot}05$ (Given)

(*i*) The probability that an item selected at random from the stock is defective is given by :

$$\begin{aligned} P(A) &= \sum_{i=1}^{3} P(A \cap E_i) = \sum_{i=1}^{3} P(E_i)\, P(A \mid E_i) \\ &= P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) + P(E_3)\, P(A \mid E_3) \\ &= 0{\cdot}6 \times 0{\cdot}03 + 0{\cdot}2 \times 0{\cdot}05 + 0{\cdot}2 \times 0{\cdot}03 = 0{\cdot}034 \end{aligned}$$

(*ii*) By Bayes' Rule, the required probabilities are given by :

$$P(E_1 \mid A) = \frac{P(E_1)\, P(A \mid E_1)}{P(A)} = \frac{0{\cdot}6 \times 0{\cdot}03}{0{\cdot}034} = \frac{0{\cdot}018}{0{\cdot}034} = \frac{9}{17}$$

$$P(E_2 \mid A) = \frac{P(E_2)\, P(A \mid E_2)}{P(A)} = \frac{0{\cdot}2 \times 0{\cdot}05}{0{\cdot}034} = \frac{0{\cdot}010}{0{\cdot}034} = \frac{5}{17}.$$

$$P(E_3 \mid A) = \frac{P(E_3)\, P(A \mid E_3)}{P(A)} = \frac{0{\cdot}006}{0{\cdot}034} = \frac{3}{17}$$

$$\Rightarrow \quad P(E_3 \mid A) = 1 - [P(E_1 \mid A) + P(E_2 \mid A)] = 1 - \left(\frac{9}{17} + \frac{5}{17} \right) = \frac{3}{17}.$$

Example 4·2. *In 2002 there will be three candidates for the position of principal – Mr. Chatterji, Mr. Ayangar and Dr. Singh –whose chances of getting the appointment are in the proportion 4 : 2 : 3 respectively. The probability that Mr. Chatterji if selected would introduce co-education in the college is 0·3. The probabilities of Mr. Ayangar and Dr. Singh doing the same are respectively 0·5 and 0·8.*

(i) What is probability that there will be co-education in the college in 2003 ?

(ii) If there is coeducation in the college in 2003, what is the probability that Dr. Singh is the principal.

Solution. Let us define the following events :

A : Introduction of co-education ; E_1 : Mr. Chatterji is selected as principal ;

E_2 : Mr. Ayangar is selected as principal ; E_3 : Dr. Singh is selected as principal

Then we are given :

$$P(E_1) = \frac{4}{9}, \qquad P(E_2) = \frac{2}{9} \qquad \text{and} \qquad P(E_3) = \frac{3}{9}$$

$$P(A \mid E_1) = \frac{3}{10}, \qquad P(A \mid E_2) = \frac{5}{10} \qquad \text{and} \qquad P(A \mid E_3) = \frac{8}{10}$$

(*i*) The required probability that there will be coeducation in the college in 2003 is given by :

$$\begin{aligned} P(A) &= P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3)] \\ &= P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) \\ &= P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) + P(E_3)\, P(A \mid E_3) \\ &= \frac{4}{9} \cdot \frac{3}{10} + \frac{2}{9} \cdot \frac{5}{10} + \frac{3}{9} \cdot \frac{8}{10} = \frac{46}{90} = \frac{23}{45} \end{aligned}$$

(*ii*) The required probability is given by Bayes' Rule, by :

$$\begin{aligned} P(E_3 \mid A) &= \frac{P(E_3)\, P(A \mid E_3)}{P(A)} \\ &= \frac{\frac{3}{9} \times \frac{8}{10}}{\frac{46}{90}} = \frac{24}{46} = \frac{12}{23}. \end{aligned}$$

Example 4·3. *The probabilities of X, Y and Z becoming managers are* $\frac{4}{9}$, $\frac{2}{9}$ *and* $\frac{1}{3}$ *respectively. The probabilities that the Bonus Scheme will be introduced if X, Y and Z becomes managers are* $\frac{3}{10}$, $\frac{1}{2}$ *and* $\frac{4}{5}$ *respectively.*

(i) What is the probability that Bonus Scheme will be introduced, and (ii) if the Bonus Scheme has been introduced, what is the probability that the manager appointed was X ?

Solution. Let E_1, E_2, E_3 denote the events that X, Y and Z become managers respectively and A denote event that 'Bonus Scheme' is introduced. We are given:

$$P(E_1) = \frac{4}{9}, \; P(E_2) = \frac{2}{9}, \quad P(E_3) = \frac{1}{3}; \quad P(A \mid E_1) = \frac{3}{10}, \quad P(A \mid E_2) = \frac{1}{2}, \quad P(A \mid E_3) = \frac{4}{5}$$

The event A can materialise in the following mutually exclusive ways :

(*i*) Mr. X becomes manager and bonus scheme is introduced, *i.e.*, $E_1 \cap E$ happens,

(*ii*) $E_2 \cap A$ happens, (*iii*) $E_3 \cap A$ happens. Thus

$$A = (E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A),$$

where $E_1 \cap A$, $E_2 \cap A$ and $E_3 \cap A$ are disjoint.

(*i*) Using Addition theorem of probability, the required probability that the bonus scheme is introduced is given by :

$$\begin{aligned} P(A) &= P(E_1 \cap A) + P(E_2 \cap A) + P(E_3 \cap A) \\ &= P(E_1) \times P(A \mid E_1) + P(E_2) \times P(A \mid E_2) + P(E_3) \times P(A \mid E_3) \\ &= \frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{1}{2} + \frac{1}{3} \times \frac{4}{5} = \frac{23}{45} \end{aligned}$$

(*ii*) Using Baye's Theorem, the required probability is :

$$P(E_1 \mid A) = \frac{P(E_1) \times P(A \mid E_1)}{P(A)} = \frac{12/90}{23/45} = \frac{6}{23}$$

Aliter. The above probabilities can be elegantly obtained in a tabular form as given below.

Events E_i (1)	*Prior Probabilities* $P(E_i)$ (2)	*Conditional Probabilities* $P(A \mid E_i)$ (3)	*Joint Probabilities* $P(E_i \cap A)$ (4) = (2) × (3)	*Posterior Probabilities* $P(E_i \mid A)$ (5) = (4) ÷ $P(A)$
E_1	$\frac{4}{9}$	$\frac{3}{10}$	$\frac{12}{90}$	$\frac{6}{23}$
E_2	$\frac{2}{9}$	$\frac{1}{2}$	$\frac{1}{9} = \frac{10}{90}$	$\frac{5}{23}$
E_3	$\frac{1}{3}$	$\frac{4}{5}$	$\frac{4}{15} = \frac{24}{90}$	$\frac{12}{23}$
Total			$P(A) = \frac{46}{90}$	

Example 4·4. *A factory produces a certain type of outputs by three types of machine. The respective daily production figures are :*

Machine I : 3,000 Units; Machine II : 2,500 Units; Machine III : 4,500 Units

Past experience shows that 1 per cent of the output produced by Machine I is defective. The corresponding fraction of defectives for the other two machines are 1·2 per-cent and 2 per-cent respectively. An item is drawn at random from the day's production run and is found to be defective. What is probability that it comes from the output of

(i) Machine I, *(ii) Machine II,* *(iii) Machine (iii) ?*

Solution. Let E_1, E_2 and E_3 denote the events that the output is produced by machines I, II and III respectively and let A denote the event that the output is defective. Then we have :

$$P(E_1) = \frac{3000}{10{,}000} = 0{\cdot}30, \quad P(E_2) = \frac{2500}{10{,}000} = 0{\cdot}25, \quad P(E_3) = \frac{4500}{10{,}000} = 0{\cdot}45$$

$$P(A \mid E_1) = 1\% = 0{\cdot}01, \quad P(A \mid E_2) = 1{\cdot}2\% = 0{\cdot}012, \quad P(A \mid E_3) = 2\% = 0{\cdot}02$$

The probability that an item selected at random from day's production is defective is given by :

$$\begin{aligned} P(A) &= \sum_{i=1}^{3} P(E_i \cap A) = \sum_{i=1}^{3} P(E_i) \cdot P(A \mid E_i) \\ &= 0{\cdot}30 \times 0{\cdot}01 + 0{\cdot}25 \times 0{\cdot}012 + 0{\cdot}45 \times 0{\cdot}02 = 0{\cdot}015 \end{aligned}$$

By Baye's rule, the required probabilities are given by :

(i) $$P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{P(A)} = \frac{0.003}{0.015} = \frac{1}{5}$$

(ii) $$P(E_2 \mid A) = \frac{P(E_2) \cdot P(A \mid E_3)}{P(A)} = \frac{0.003}{0.015} = \frac{1}{5}$$

(iii) $$P(E_3 \mid E) = \frac{P(E_3) \cdot P(A \mid E_3)}{P(A)} = \frac{0.009}{0.015} = \frac{3}{5}$$

The probabilities in (i), (ii) and (iii) are known as posterior probabilities of events E_1, E_2 and E_3 respectively.

Aliter. The posterior probabilities can be obtained elegantly in a tabular form as given below.

Events E_i	Prior Probabilities $P(E_i)$	Conditional Probabilities $P(A \mid E_i)$	Joint Probabilities $P(E_i \cap A)$	Posterior Probabilities $P(E_i \mid A)$
(1)	(2)	(3)	(4) = (2) × (3)	(5) = (4) ÷ P (A)
E_1	0·30	0·010	0·003	$\frac{0.003}{0.015} = \frac{1}{5}$
E_2	0·25	0·012	0·003	$\frac{0.003}{0.015} = \frac{1}{5}$
E_3	0·45	0·020	0·009	$\frac{0.009}{0.015} = \frac{3}{5}$
Total	1·00		$P(A) = 0.015$	1

Example 4·5. *There are two bags A and B. A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is $\frac{6}{7}$, find the value of n.*

Solution. Let E_1 denote the event that bag A is selected and E_2 denote the event that bag B is selected. Let E be the event that two balls drawn are white. We have

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(E \mid E_1) = \frac{{}^nC_2}{{}^{n+2}C_2} = \frac{n(n-1)}{(n+2)(n+1)}$$

and $$P(E \mid E_2) = \frac{{}^2C_2}{{}^{n+2}C_2} = \frac{2}{(n+2)(n+1)}$$

Using Baye's Theorem, the probability that the two white balls drawn are from the bag A, is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{P(E_1)\,P(E \mid E_1) + P(E_2)\,P(E \mid E_2)} = \frac{6}{7} \qquad \text{(Given)}$$

$$\Rightarrow \quad \frac{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)}}{\frac{1}{2} \cdot \frac{n(n-1)}{(n+2)(n+1)} + \frac{1}{2} \cdot \frac{2}{(n+2)(n+1)}} = \frac{6}{7} \quad \Rightarrow \quad \frac{n(n-1)}{n(n-1)+2} = \frac{6}{7}$$

$$\therefore \quad 7n(n-1) = 6n(n-1) + 12 \quad \Rightarrow \quad n^2 - n - 12 = 0 \quad \Rightarrow \quad n = 4, -3.$$

Since n cannot be negative, we get $n = 4$.

Example 4·6. *A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter came from CALCUTTA ?*

Solution. Let E_1 and E_2 denote the events that the letter came from TATANAGAR and CALCUTTA respectively. Let A denote the event that two consecutive visible letters on the envelop are TA. We have

$$P(E_1) = P(E_2) = \frac{1}{2}, \; P(A \mid E_1) = \frac{2}{8} \quad \text{and} \quad P(A \mid E_2) = \frac{1}{7}.$$

Using the Bayes' theorem, we get

$$P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2)} = \frac{\frac{1}{2}\cdot\frac{1}{7}}{\frac{1}{2}\cdot\frac{2}{8} + \frac{1}{2}\cdot\frac{1}{7}} = \frac{4}{11}.$$

Example 4·7. *The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease X, died. What is the chance that his disease was diagnosed correctly ?*

Solution. Let us define the following events :

E_1 : Disease X is diagnosed correctly by doctor A.
E_2 : Disease X is not diagnosed correctly by doctor A.
E : A patient (of Dr A) who had disease X dies.

Then, we are given :

$$P(E_1) = 0{\cdot}6 \quad ; \quad P(E \mid E_1) = 0{\cdot}4$$

$$P(E_2) = P(\bar{E}_1) = 1 - P(E_1) = 0{\cdot}4 \quad ; \quad P(E \mid E_2) = 0{\cdot}7$$

$$\therefore \quad P(E) = \sum_{i=1}^{2} P(E_i)\,P(E \mid E_i) = 0{\cdot}6 \times 0{\cdot}4 + 0{\cdot}4 \times 0{\cdot}7 = 0{\cdot}52$$

Using the Bayes' thereon, the required probability is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{P(E)} = \frac{0{\cdot}6 \times 0{\cdot}4}{0{\cdot}52} = \frac{0{\cdot}24}{0{\cdot}52} = \frac{6}{13}.$$

Example 4·8. *The contents of urns I, II and III are as follows :*

1 white, 2 black and 3 red balls,
2 white, 1 black and 1 red balls, and
4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn from it. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Solution. Let E_1, E_2, and E_3 denote the events that the urn, I, II and III is chosen, respectively, and let A be the event that the two balls taken from the selected urn are white and red.

Then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A \mid E_1) = \frac{1 \times 3}{^6C_2} = \frac{1}{5}, \; P(A \mid E_2) = \frac{2 \times 1}{^4C_2} = \frac{1}{3}, \text{ and } P(A \mid E_3) = \frac{4 \times 3}{^{12}C_2} = \frac{2}{11}$$

$$\therefore \quad P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{\sum_{i=1}^{3} P(E_i)\,P(A \mid E_i)} = \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{11}} = \frac{55}{118}$$

Similarly, $$P(E_3 \mid A) = \frac{\frac{1}{3}\times\frac{2}{11}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}\times+\frac{1}{3}\times\frac{2}{11}} = \frac{30}{118}$$

$\therefore$ $$P(E_1 \mid A) = 1 - \frac{55}{118} - \frac{30}{118} = \frac{33}{118}.$$

Example 4·9. *From a vessel containing 3 white and 5 black balls, 4 balls are transferred into an empty vessel. From this vessel a ball is drawn and is found to be white. What is the probability that out of four balls transferred 3 are white and 1 is black ?*

Solution. Let us define the following events :

E_1 : Transfer of 0 white and 4 black balls

E_2 : Transfer of 1 white and 3 black balls

E_3 : Transfer of 2 white and 2 black balls

E_4 : Transfer of 3 white and 1 black balls

(Since the urn contains 3 white balls, more than 3 white balls cannot be transferred from the vessel)

E : Drawing of a white ball from the second vessel.

Then $P(E_1) = \frac{^5C_4}{^8C_4} = \frac{1}{14}$, $P(E_2) = \frac{^3C_1 \times {^5C_3}}{^8C_4} = \frac{3}{7}$

$P(E_3) = \frac{^3C_2 \times {^5C_2}}{^8C_4} = \frac{3}{7}$, $P(E_4) = \frac{^3C_3 \times {^5C_1}}{^8C_4} = \frac{1}{14}$

Also $P(E \mid E_1) = 0$, $P(E \mid E_2) = \frac{1}{4}$, $P(E \mid E_3) = \frac{2}{4}$ and $P(E \mid E_4) = \frac{3}{4}$

Hence, by Bayes Theorem, the probability that out of four balls transferred, 3 are white and is black is :

$$P(E_4 \mid E) = \frac{\frac{1}{14}\times\frac{3}{4}}{\frac{1}{14}\times 0 + \frac{3}{7}\times\frac{1}{4} + \frac{3}{7}\times\frac{1}{2} + \frac{1}{14}\times\frac{3}{4}} = \frac{3}{6+12+3} = \frac{1}{7} = 0{\cdot}14.$$

Example 4·10. *A and B are two weak students of statistics and their chances of solving a problem in statistics correctly are $\frac{1}{6}$ and $\frac{1}{8}$ respectively. If the probability of their making a common error is $\frac{1}{525}$ and they obtain the same answer, find the probability that their answer is correct.*

Solution. Let us define the following events :

E_1 : Both A and B solve the problem correctly.

E_2 : Exactly one of them solves the problem correctly.

E_3 : Neither of them solves the problem correctly.

E : They get the same answer.

Then, according to the data given in the problem, assuming that A and B try the problem independently, we have.

$P(E_1) = \frac{1}{6}\times\frac{1}{8} = \frac{1}{48}$; $P(E \mid E_1) = 1$

$P(E_2) = \frac{1}{6}\times\frac{7}{8} + \frac{5}{6}\times\frac{1}{8} = \frac{12}{48}$; $P(E \mid E_2) = 0$

$P(E_3) = \frac{5}{6} \times \frac{7}{8} = \frac{35}{48}$; $P(E \mid E_3) = \frac{1}{525}$ (Given)

Hence, by Bayes' Rule, the required probability is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{\sum_{i=1}^{3} P(E_i)\,P(E \mid E_i)}$$

$$= \frac{\frac{1}{48} \times 1}{\frac{1}{48} \times 1 + \frac{12}{48} \times 0 + \frac{35}{48} \times \frac{1}{525}} = \frac{\frac{1}{48}}{\frac{1}{48} + \frac{1}{48 \times 15}} = \frac{1}{48} \times \frac{48 \times 15}{(15+1)} = \frac{15}{16}.$$

Aliter. Let us define the events :

E_1 : A and B get the same correct answer

E_2 : A and B get the same wrong answer

E : A and B get the same answer.

Then we have :

$$P(E_1) = \frac{1}{6} \times \frac{1}{8} = \frac{1}{48}; \quad P(E_2) = \frac{5}{6} \times \frac{7}{8} \times \frac{1}{525} = \frac{1}{720}$$

$$P(E) = P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{48} + \frac{1}{720} = \frac{15+1}{720} = \frac{1}{45}$$

The required probability that the answer is correct is given by :

$$P(E_1 \mid E) = \frac{P(E_1 \cap E)}{P(E)} = \frac{P(E_1)}{P(E)} = \frac{\frac{1}{48}}{\frac{1}{45}} = \frac{45}{48} = \frac{15}{16}. \qquad (\because E_1 \subset E)$$

Example 4·11. *A speaks truth 4 out of 5 times. A die is tossed. He reports that there is a six. What is the chance that actually there was six ?*

Solution. Let us define the following events

E_1 : A speaks truth ; E_2 : A tells a lie ; E : A reports a six

From the data given in the problem, we have

$$P(E_1) = \frac{4}{5}, \quad P(E_2) = \frac{1}{5}; \quad P(A \mid E_1) = \frac{1}{6}, \quad P(A \mid E_2) = \frac{5}{6}$$

The required probability that actually there was six (by Bayes Theorem) is :

$$P(E_1 \mid E) = \frac{P(E_1) \times P(E \mid E_1)}{P(E_1) \times P(E \mid E_1) + P(E_2) \times P(E \mid E_2)} = \frac{\frac{4}{5} \cdot \frac{1}{6}}{\frac{4}{5} \cdot \frac{1}{6} + \frac{1}{5} \cdot \frac{5}{6}} = \frac{4}{9}.$$

Example 4·12. *In answering a question on a multiple choice test a student either knows the answer or he guesses. Let p be the probability that he knows the answer and 1 – p the probability that he guesses. Assume that a student who guesses at answer will be correct with probability 1/5, where 5 is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he answered it correctly ?*

Solution. Let us define the following events :

E_1 : The student knows the right answer.

E_2 : The student guesses the right answer.

A : The student gets the right answer.

Then we are given : $P(E_1) = p, \quad P(E_2) = 1-p, \quad P(A \mid E_2) = \frac{1}{5}$

$P(A \mid E_1) = P$ (student gets right answer given that he knew the right answer) = 1

We want $P(E_1 \mid A)$. Using Bayes' rule, we get

$$P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2)} = \frac{p \times 1}{p \times 1 + (1-p) \times \frac{1}{5}} = \frac{5p}{4p+1}$$

Example 4·13. *(Baye's Theorem for repeated trials). In n independent trials with constant probability p of success in each trial, m successes have been observed. p is not known, but it is known that it takes one of the values $p_1, p_2, \ldots, p_k$ and the probability with which p takes the value p_i is α_i $(i = 1, 2, \ldots, k)$, What is the probability that*

(i) $p = p_i$,

(ii) *p lies between the two given limits α and β* $(0 \le \alpha < \beta \le 1)$ *or what is the probability of the inequality* $\alpha \le p \le \beta$?

Solution. Let us define the events as follows :

A = m successes in n trials

E_i = p takes the value p_i, $(i = 1, 2, \ldots, k)$.

Then $\quad P(E_i) = \alpha_i \quad$ (Given)

and $\quad P(A \mid E_i) = {}^nC_m\, p_i^m (1-p_i)^{n-m}$. (By Binomial Probability Law)

Using Baye's Theorem, we get :

$$P(E_i \mid A) = \frac{\alpha_i \cdot {}^nC_m\, p_i^m (1-p_i)^{n-m}}{\sum\limits_{i=1}^{k} \alpha_i \cdot {}^nC_m\, p_i^m (1-p_i)^{n-m}} = \frac{\alpha_i\, p_i^m (1-p_i)^{n-m}}{\sum\limits_{i=1}^{k} \alpha_i\, p_i^m (1-p_i)^{n-m}}$$

Now applying the theorem of total probability, the probability of inequality $\alpha \le p \le \beta$ is given by :

$$P(\alpha \le p \le \beta \mid A) = \frac{\sum\limits_{p_i=\alpha}^{\beta} \alpha_i\, p_i^m (1-p_i)^{n-m}}{\sum\limits_{i=1}^{k} \alpha_i\, p_i^m (1-p_i)^{n-m}}$$

Example 4·14. *A printing machine can print n 'letters', say $\alpha_1\ \alpha_2 \ldots, \alpha_n$. It is operated by electrical impulses, each letter being produced by a different impulse. Assume that p is the constant probability of printing the correct letter and the impulses are independent. One of the n impulses, chosen at random, was fed into the machine twice and both times the letter α_1 was printed. Compute the probability that the impulse chosen was meant to print α_1.*

Solution. Let us define the following events :

A : Letter α_1 is printed both the times.

B_i : The impulse chosen is meant to print α_i, $(i = 1, \ldots, n)$.

E_i : Letter α_i is printed

Then we have : $\quad P(B_i) = \frac{1}{n}, \ (i = 1, 2, \ldots, n)$.

$P(E_i \mid B_i) = P$ [A letter is printed correctly] $= p, \ (i = 1, 2, \ldots, n)$

and $\quad P(E_i \mid B_j) = P$ [A letter is printed wrongly] $= 1 - p, \ (i \ne j = 1, 2, \ldots, n)$