Capital Growth: Theory and Practice*

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Abstract

In capital accumulation under uncertainty, a decision-maker must determine how much capital to invest in riskless and risky investment opportunities over time. The investment strategy yields a stream of capital, with investment decisions made so that the dynamic distribution of wealth has desirable properties. The distribution of accumulated capital to a fixed point in time and the distribution of the first passage time to a fixed level of accumulated capital are variables controlled by the investment decisions. An investment strategy which has many attractive and some not attractive properties is the growth optimal strategy, where the expected logarithm of wealth is maximized. This strategy is also referred to as the Kelly strategy It maximizes the rate of growth of accumulated capital.. With the Kelly strategy, the first passage time to arbitrary large wealth targets is minimized, and the probability of reaching those targets is maximized. However, the strategy is very aggressive since the Arrow-Pratt risk aversion index is essentially zero. Hence, the chances of losing a substantial portion of wealth are very high, particularly if the estimates of the returns distribution are in error. In the time domain, the chances are high

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that the first passage to subsistence wealth occurs before achieving the established wealth goals.

This chapter is a survey of the theoretical results and practical uses of the capital growth approach. It is a companion to the chapter in this volume on the Kelly criterion by E. O. Thorp. Alternative formulations for capital growth models in discrete and continuous time are presented. Various criteria for performance and requirements for feasibility are related in an expected utility framework. Typically, there is a trade-off between growth and security with a fraction invested in an optimal growth portfolio determined by the risk aversion criteria. Models for calculating the optimal fractional Kelly investment with alternative performance criteria are formulated. The effect of estimation and modeling error on strategies and performance is discussed.

Various applications of the capital growth approach are made to futures trading, lotto games, horseracing, and the fundamental problem of asset allocation between stocks, bonds and cash. The chapter concludes with a discussion of some of the great investors and speculators, and how they used Kelly and fractional Kelly strategies in their investment programs.

Keywords: Kelly criterion, betting, long run investing, portfolio allocation, logarithmic utility, capital growth

JEL Classifications: C61, D81, G1

Contents

1 Introduction

The study of growth is a significant topic in economic theory. The capital raised through securities and bonds supports the economic growth of industry. At the same time, personal wealth is accumulated through the buying and selling of financial instruments. In the capital marketplace the investor trades assets, with the rates of return generating a trajectory of wealth over time.

In capital growth under uncertainty, an investor must determine how much capital to invest in riskless and risky instruments at each point in time, with a focus on the trajectory of accumulated capital to a planning horizon. Assuming prices are not affected by individual investments but rather aggregrate investments, individual decisions are made based on the projected price processes given the history of prices to date. An investment strategy which has generated considerable interest is the growth optimal or Kelly strategy, where the expected logarithm of wealth is maximized. (Kelly, 1956.) Researchers such as Thorp (1971, 1975), Hausch, Ziemba and Rubinstein (1981), Grauer and Hakansson (1986, 1987), and Mulvey and Vladimirow (1992) have used the optimal growth strategy to compute optimal portfolio weights in multi-asset and worldwide asset allocation problems. The wealth distribution of this strategy has many attractive characteristics (see, e.g. Hakansson, 1970, 1971; and Markowitz, 1976).

The Kelly or capital growth criteria maximizes the expected logarithm as its utility function period by period. (Hakansson, 1971.) It has desirable properties such as being myopic - todays optimal decision does not depend upon yesterdays or tomorrows data, asymptotically maximizing long run wealth almost surely, and attaining arbitrarily large wealth goals faster than any other strategy. Also in an economy with one log bettor and all other investors with essentially different strategies, the log bettor will eventually get all the economy's wealth. (Hens and Schenk-Hoppe, 2005.) The drawback of log, with its essentially zero Arrow-Pratt absolute risk aversion, is that in the short run it is the most risky utility function one would ever consider. Since there is essentially no risk aversion, the wagers it suggests are very large and typically undiversified. Simulations show that log investors have much more final wealth most of the time than those using other strategies, but those investors can essentially go bankrupt a small percentage of the time, even facing a large number of very favorable investment choices. (Ziemba and Hausch, 1986.) One way to modify the growth-security profile is to use either ad hoc or scientifically computed (MacLean et al, 2004) fractional Kelly strategies that blend the log optimal portfolio with cash. For instance, a fractional Kelly strategy will keep accumulated capital above a specified wealth path with high probability given log normally distributed assets. This is equivalent to using a negative power utility function whose coefficient (equivalent to a risk aversion index) is determined by the fraction and vice versa. Thus one moves the risk aversion away from zero to a higher level. This results in a smoother wealth path but has less growth. For non-lognormal asset returns distributions, the fractional Kelly is an approximate solution to the optimal risk-return trade-off.

In this paper the traditional capital growth model and modifications to control the risk of capital loss are developed. Parameter estimation and risk control are considered in a Bayesian dynamic model where the filtration and control processes are separate. In continuous time, the Bayesian model is a generalization to the multiasset case of the random coefficients model of Browne and Whitt (1996). Given the estimated price dynamics, an investment decision is made to control the path of future wealth. Approaches to risk control of wealth at a planning horizon based on expected utility can be put in the context of stochastic dominance (Hanoch and Levy, 1969; De Giorgio, 2005). This leads naturally to bi-criteria problems of risk and return, which can be solved explicitly in continuous time (MacLean, Zhao, and Ziemba, 2005) or as a stochastic program in discrete time (Orgyczak and Ruszczynski, 1998; MacLean et. al., 2005). If wealth is to be controlled at all points along the trajectory, then the setting of wealth goals and first passage times is applicable. In continuous time, growth strategies minimize the time to wealth goals subject to a probability constraint requiring goal attainment before falling to a subsistence level (MacLean, Ziemba, and Li, 2005).

In practice an investment portfolio cannot be continuously rebalanced and a realistic approach is to reconsider the investment decision at discrete points in time (Rogers 2001). The time points can be at fixed intervals or at random times determined by the deviation of the wealth trajectory from expectations. At each rebalance time, with additional data and a change in wealth, price parameters are re-estimated and a new investment strategy is developed. This dynamic process is illustrated in Figure 1.

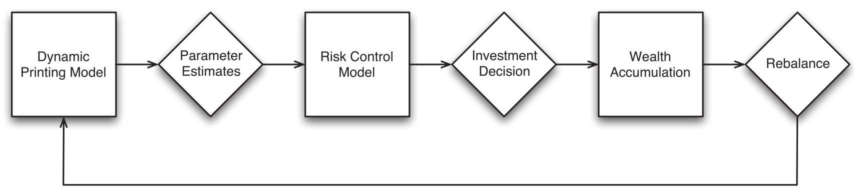

Figure 1: Dynamic Investment Process

In the random rebalancing times approach, the excessive deviation from expectations is determined from wealth goals (MacLean, Zhao and Ziemba, 2005). Upper and lower wealth targets serve as control limits to determine random rebalance times or decision points. The control limits are significant when estimates for future price changes are substantially in error and the wealth trajectory is not proceeding as expected. The investment risk-return strategy controls risk as defined by the estimated distribution for returns on assets. The wealth limits control the risk resulting from estimation error.

Before considering the components of the dynamic system in Figure 1, the process of capital accumulation is defined.

2 Capital Accumulation

Consider a competitive financial market with K assets whose prices are stochastic, dynamic processes, and a single asset whose price is non-stochastic. Let the vector of prices at time t be

$$P(t) = (P_0(t), P_1(t), ..., P_K(t))', \tag{1}$$

where $P_0(t)$ is the price of the non-stochastic or risk free asset. If the prices are given at points in time t_1 and t_2, with $t_1 < t_2$, then the rate of return over that time on a unit of capital invested in asset i is

$$R_i(t_1, t_2) = \frac{P_i(t_2)}{P_i(t_1)} = 1 + r_i(t_1, t_2), i = 0, ..., K. \tag{2}$$

In the financial market, assets are traded at points in time and the return on assets leads to the accumulation of capital for an investor. In the analysis of trading strategies, the following structure is assumed:

(a) All assets have limited liability.

(b) There are no transactions costs, taxes, or problems with indivisibility of assets.

(c) Capital can be borrowed or lent at the risk free interest rate.

(d) Short sales of all assets is allowed.

Suppose an investor has w_t units of capital at time t, and that capital is fully invested in the assets, with the proportions invested in each asset given by $x_i(t), i = 0, ...K$. Then an investment or trading strategy at time t is the vector process

$$X(t) = (x_0(t), x_1(t), ..., x_K(t))'. \tag{3}$$

Given the investments $w_{t_1}X(t_1)$ at time t_1, the accumulated capital at time t_2 is

$$W(t_2) = w_{t_1}R'(t_1, t_2)X(t_1) = w_{t_1}\sum_{i=0}^{K} R_i(t_1, t_2)x_i(t_1). \tag{4}$$

The trajectory of returns between time t_1 and time t_2 depends on the asset, and is typically non-linear. So changing the investment strategy at points in time between t_1 and t_2 will possibly improve capital accumulation. If trades could be timed to correspond to highs and lows in prices, then greater capital would be accumulated. The history of asset prices provides information that is useful in predicting future prices, so monitoring prices and revising the investment portfolio would seem appropriate.

To consider the effect of changes in strategy, partition the time interval into n segments, with $d = \frac{t_2 - t_1}{n}$, so that the accumulated capital is monitored, and the investment strategy is possibly revised at times $t_1, t_1 + d, ..., t_1 + nd = t_2$. Then wealth at time t_2 is

$$W_n(t_2) = w_{t_1} \prod_{i=0}^{n-1} R'(t_1 + id, t_1 + (i+1)d)X(t_1 + id). \tag{5}$$

Wealth is

$$W_n(t_2) = w_{t_1} \left(exp[\frac{1}{n} \sum_{i=0}^{n-1} ln(R'(t_1 + id, t_1 + (i+1)d)X(t_1 + id))] \right)^n. \tag{6}$$

The exponential form highlights the *growth rate* with the strategy $X = (X(t_1), ..., X(t_1 + (n-1)d))$,

$$G_n(X) = \frac{1}{n} \sum_{i=0}^{n-1} ln(R'(t_1 + id, t_1 + (i+1)d)X(t_1 + id)). \tag{7}$$

As the partitioning of the interval gets finer, so that $d \to 0$, then monitoring and trading are continuous. Furthermore, $n \to \infty$, and when the process $V_i = ln(R(t_1 + id, t_1 + (i+1)d)X'(t_1 + id), i = 0, ..., n-1$, is ergodic in the mean, $G_n(X) \to 1 + g$. If the random variables $V_i, i = 0, ...,$ are independent and identically distributed, then $W_n(t_2) = w_{t_1} exp(\sum_{i=0}^{n-1} V_i)$ converges to a lognormal random variable.

The capital accumulation process is straight forward, but there are details in the setup which have significant implications for the trajectory of wealth.

[1] ASSET PRICES

The return on investment results from changes in asset prices over time. The investment decisions are made based on expectations about unknown future prices. The standard approach to forcasting is to propose a model for price dynamics. In practice the model is a mechanism for transforming the record of past prices into projections for future prices. There are many models that have been proposed. Probabilistic models which focus on the dynamics of the price distributions have been successful, particularily if they contain a combination of Gaussian and Poisson processes. The models approximate the transition between rates of the return with a linear combination of random processes. This format is very suited to the analysis of capital accumulation, but it is important to understand that expectations generated by the approximating model can be seriously in error. Some model flexibility is introduced by defining the parameters in the combination to be random variables and working with a Bayesian dynamic linear model. Bayes Theorem provides a natural mechanism for model revision, which would be called for when there is evidence of a significant deviation between actual and expected prices.

[2] DECISION CRITERIA

The trajectory of accumulated capital over time depends on the returns on a unit of capital invested in assets, and the decisions on the amount of capital to invest in assets at each point in time. If the distributions for future returns are known, decisions can be based on the desired distributions of capital accumulation over time or at a planning horizon. Concepts of stochastic dominance can be used to order distributions. However, the future returns distributions are estimated with error and that risk needs to be reflected in decisions on investment. With the volatility of accumulated capital exceeding the conditional volatility for given returns distributions, higher levels of risk aversion are appropriate. Decision models based on measures of risk and return can be used to shape the distribution of accumulated capital.

[3] Timing of decisions

As the capital accumulation process evolves over time, information on asset returns is available. The information is important for evaluating the investment decision as well as estimating future returns distributions. There is an issue over when to react to information. It is standard practice to reconsider decisions at regular intervals in time. An alternative approach, based on concepts from statistical process control, is to set upper and lower control limits on accumulated capital, and to reconsider decisions at random times when the trajectory of capital is out of control.

These aspects of capital accumulation are developed in subsequent sections.

3 Asset Prices

The returns on assets are generated by price changes. Future trading prices and therefore returns are uncertain, and characterizing the distributions for returns is the foundation for investment planning and capital growth. Over the interval (t_1, t_2), the return on asset i, $R_i(t_1, t_2)$, is the product of incremental returns:

$$R_i(t_1, t_2) = \frac{P_i(t_2)}{P_i(t_1)} = \prod_{i=1}^{n} \left(\frac{P_i(t_1 + d)}{P_i(t_1)} \right) \cdots \left(\frac{P_i(t_2)}{P_i(t_1 + (n-1)d} \right),$$

and with the geometric form of returns, it is natural to consider the logarithm of returns when characterizing the distribution. The distribution of log returns can be defined by its moments and approximated using a small number of moments. It is known that the significance of moments diminishes with order. Chopra and Ziemba (1993) show that equal size errors in estimators for the means, variances, and covariances affect portfolio performance in the order of 20:2:1, respectively. (See also, the earlier studies of Kallberg and Ziemba, 1981, 1984.) The performance is also a function of the investors preference for risk as reflected in the investment strategy. Low risk aversion magnifies the effect of average estimation error, and high risk aversion diminishes the effect, so for log or near log investors the 20 times error can be 80 - 100 times. The performance measure in the Chopra-Ziemba study was the certainty equivalent of wealth, defined as

$$CE = u^{-1}(E(W)), \tag{8}$$

where W is accumulated capital and u is the power utility function.

The comparison of the CE with the correct moment and moment contaminated by estimation error is the certainly equivalent loss

$$CEL = \left(\frac{CE_{true} - CE_{cont}}{CE_{cont}} \right) 100. \tag{9}$$

The contamination was determined by the random factor $(1+kZ)$, where $Z \propto N(0,1)$, $k = 0.05, 0.10.0.15, 0.20$. So $Moment_{cont} = (1 + kZ)Moment$. A Monte Carlo study of the effects of error (contamination) was carried out. The optimal mean-variance one period wealth was calculated for the true and contaminated moments.

Table 1 compares the ratio of the CEL for errors in the mean versus errors in variances/covariances, with various risk tolerances. Risk tolerance is the reciprocal of the Arrow-Pratt risk aversion index.

Risk Tolerance	Errors in Mean vs Variance	Errors in Mean vs Variances
25	5.38	3.22
50	22.50	10.98
75	56.84	21.42

Table 1: Average Ratio for Errors in Means, Variances and Covariances - source: Chopra and Ziemba (1993)

Figure 2 shows the relative effects of errors in moment estimation as the size of the error increases.

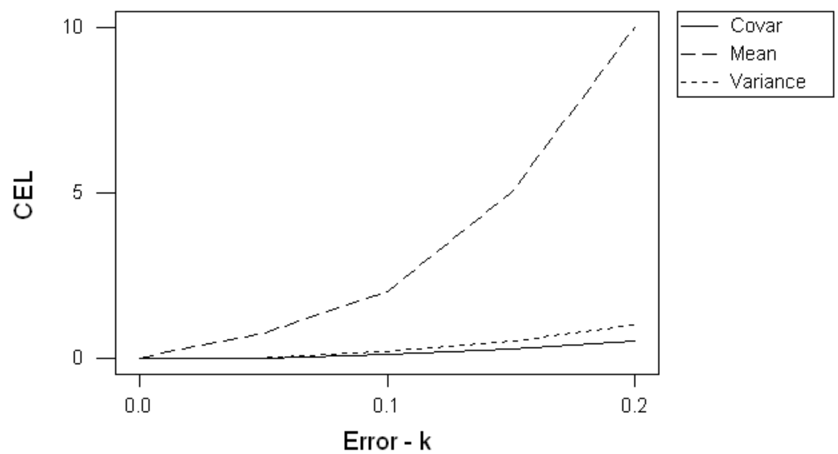

Figure 2: Mean Percent CEL for Various Errors -

source: Chopra and Ziemba (1993)

This one period picture of moments for the returns distributions is informative, but it is important to remember the dynamics of investing. At points in time information on past returns is used to forecast future returns. Accurate estimates for means are crucial. If investment decisions are based on expected returns, conditional on the history of returns, then capital growth is most likely. Methods of estimation for the mean rate of return in a dynamic model are considered next.

3.1 Pricing Model

It is clear that getting the mean right in forcasting the returns distributions is most important. For the incremental rate of return on asset i, $ln\left(\frac{P_i(t_1+jd)}{P_i(t_1+(j-1)d)}\right)$, consider $Y_{ij}(t_1) = \ln P_i(t_1{+}jd)$, and $Y_{ij}(t_1) - Y_{i(j-1)}(t_1) = \Delta Y_{ij}(t_1)$, for $i = 1, ...K$. So $ln(R_i(t_1, t_2) = \sum_{j=1}^{n} \Delta Y_{ij}(t_1)$. In this setup, the log-returns over a time interval are approximately normal. If the mean of $\Delta Y_{ij}(t_1)$ is λ_{ij} and $\mu_i = \frac{1}{n}\sum_{j=1}^{n}\lambda_{ij}$, then the increment means can be considered as sampled from a family of means for which μ_i is the overall mean. To formalize this concept, consider the infinitesmal increments process defined by the stochastic linear equations

$$dY_0(t) = rdt \tag{10}$$

$$dY_i(t) = \lambda_i dt + \delta_i dV_i, i = 1, ..., K, \tag{11}$$

where $dV_i, i = 1, ..., K,$ are independent Brownian motions and the rate parameters λ_i are random variables, with

$$\lambda_i = \mu_i + \beta_i Z_i, i = 1, ..., K. \tag{12}$$

To allow for the relationship between the rates of return on assets, the $Z_i, i = 1, ..., K$ are correlated Gaussian variables with ρ_{ij} the instantaneous correlation between variables $Z_i$and Z_j. So $E(\lambda_i) = \mu_i$, and $cov(\lambda_i, \lambda_j) = \sigma_{ij} = \beta_i\beta_j\rho_{ij}$.

The hierarchial linear model in (10) - (12) is a generalization of a single stock model in Browne and Whitt (1996), where the rate of return is a random variable. This model is also used in Rogers (2001) to study parameter estimation error.

The correlated Gaussian variables can be represented in terms of independent gaussian variables. If $U_j, j = 1, ..., m, m \leq K$, are i.i.d. standard Gaussian variables, then $Z_i = \sum_{j=1}^{m}\alpha_{ij}U_j$ and $\rho_{ij} = \sum_{k=1}^{m}\alpha_{ik}\alpha_{kj}$. In terms of the independent Gaussian variables, the model for increments is

$$dY_i(t) = [\mu_i + \sum_{i=1}^{m}\beta_i\alpha_{ij}U_j]dt + \delta_i dV_i, i = 1, ...K. \tag{13}$$

Typically the number of independent Gaussian variables (factors) needed to define the correlation between asset returns would be small and to have the parameters identifiable from the covariance, it is required that $m \leq$

$(K-1)/2$.

Although the equation in (13) has a familiar linear form, it is important to distinguish between terms defining the rates and the volatility terms. The volatilities $\delta_i, i = 1, ...K$, represent the specific variance of each asset return. The asset returns are correlated, but the correlation is generated by the factors in the expected rates of return.

There is evidence that in addition to factors in the mean rates of return, stochastic volatility factors are important for capturing certain aspects of returns distributions such as heavy tails (Chernov et. al. 2002). An alternative approach to extreme returns involves adding independent shock terms to capture dramatic price changes. The dynamic equations become

$$dY_i = \lambda_i dt + \delta_i dV_i + \vartheta_i dN_i(\pi_i),$$

where $dN_i(\pi_i)$ is a poisson process with intensity π_i, and shock size $\vartheta_i, i = 1, ..., K$. Between shocks, the financial market is in the same regime, and is described by the equations in (10) - (12). Within a regime, the random rate model is sufficient to explain the mean, variance and covariance for returns. An approach to the model with shocks is to define a conditional model given the shocks. The conditional dynamics are in (13).

The rates of return within a regime may be dynamic, with defining equations $d\lambda_i(t) = \mu_i dt + \beta_i dq_i$, where dq_i,$i = 1, ..., K$, are correlated Brownian motions. So $\lambda_i(t) = \lambda_i(0) + \mu_i t + \beta_i \sqrt{t} Z_i, i = 1, ..., K$. The estimation methods discussed later are easily adapted to the estimation of parameters with dynamic stochastic rates of return.

The distribution of asset prices, and in particular the mean vector and covariance matrix are the objects approximated by the linear model. Returning to the pricing equations, let

$$
\begin{aligned}
Y(t) &= (Y_1(t), ..., Y_K(t))', \\
\lambda &= (\lambda_1, ..., \lambda_K)', \\
\Delta &= diag(\delta_1^2, ..., \delta_K^2), \\
\mu &= (\mu_1, ..., \mu_K)', \\
\Gamma &= (\gamma_{ij}) = (\beta_i \beta_j \rho_{ij}).
\end{aligned}
$$

Without loss of generality assume $Y(0) = 0$. Given (λ, Δ), the conditional distribution of log-prices at time t is

$$(Y(t)|\lambda, \Delta) \propto N(\lambda t, t\Delta). \tag{14}$$

From (12) the rate has a prior distribution

$$\lambda \propto N(\mu, \Gamma). \tag{15}$$

The marginal distribution of log-prices is

$$Y(t) \propto N(\mu t, \Sigma_t), \tag{16}$$

with

$$\Sigma_t = t^2\Gamma + t\Delta = \Gamma_t + \Delta_t. \tag{17}$$

The hierarchial Bayes model for returns characterizes asset prices in terms of means and variances/covariances. It is an important property of the model that the covariance for log-prices is partitioned into a component determined by the random drift and a component determined by the diffusion.

The model defines an evolution of asset prices, but the practical context for a model is defining future price distributions given information on past prices. At any point in time assume that information is available on the history of prices. Consider the data available at time t, $\{Y(s), 0 \leq s \leq t\}$, and the corresponding filtration $\Im_t^Y = \sigma\{Y(s), 0 \leq s \leq t\}$. The observed mean rate of return to time t is

$$\bar{Y}_t = \frac{1}{t} Y(t). \tag{18}$$

With the prior distribution for λ in (12) and the conditional distribution for $Y(t)$ in (11), the posterior distribution for the mean rate of return λ, given $\Im_t^Y$ is

$$(\lambda|\Im_t^Y) \propto N(\hat{\lambda}_t, \Gamma_t^0), \tag{19}$$

where $\hat{\lambda}_t = \mu + (I - \Delta_t \Sigma_t^{-1})(\bar{Y}_t - \mu)$ and $\Gamma_t^0 = \frac{1}{t^2}(I - \Delta_t \Sigma_t^{-1})\Delta_t$.

So the hierarchial model provides a natural filter for information on prices. The conditional distribution for incremental rates of return is modified as a trajectory of asset prices unfolds. It follows that the Bayes estimate for the mean rate of return at time t is the conditional expectation

$$\hat{\lambda}_t = E(\lambda|\Im_t^Y) = \mu + (I - \Lambda_t \Sigma_t^{-1})(\overline{Y_t} - \mu) \tag{20}$$

.

The conditional expected return is the key to wealth accumulation. An investor allocating wealth across assets according to their conditional expectated return eventually accumulates total market wealth. (Amir, et. al., 2005.)

The Bayes estimate in (20) and the posterior distribution depend upon unknown parameters $(\lambda, \Gamma, \Delta)$. If the parameters can be estimated from the data on past returns, then replacing $(\lambda, \Gamma, \Delta)$ with estimates $(\hat{\lambda}, \hat{\Gamma}, \hat{\Delta})$ will provide an empirical Bayes estimate for the conditional mean rate of return.

3.2 Estimation

Assume that securities have been observed at regular intervals of width $\frac{t}{n}$ in the time period $(0, t)$. The log prices at times $\frac{(s+1)t}{n}$, given the log prices at times $\frac{st}{n}, s = 0, ..., n$, are

$$Y(s) = y(s-1) + \frac{t}{n}\lambda + \sqrt{\frac{t}{n}} \Delta^{\frac{1}{2}} Z. \tag{21}$$

The first order increments process generates sample rates

$$e(s) = (Y(s) - y(s-1)) \div \frac{t}{n} = \lambda + \sqrt{\frac{n}{t}}\Delta^{\frac{1}{2}}Z, \tag{22}$$

which are stationary with covariance

$$\Sigma_{nt} = \Gamma + \frac{n}{t}\Delta = \Gamma_{nt} + \Delta_{nt}, \tag{23}$$

and mean $E(e) = \lambda$.

From the realized trajectory of prices, the observations on log-prices at times $\frac{st}{n}, s = 0, ..., n$, are $\{Y_{is}, i = 1, ..., K; s = 1, ..., n\}$. The corresponding sample rates are $\{e_{is}, i = 1, ..., K; s = 1, ...n\}$. With $e'_s = (e_{1s}, ..., e_{Ks})$, it follows that

$$\bar{Y}_t = \frac{1}{n}\sum_{s=1}^{n} e_s \tag{24}$$

So $\bar{Y}_t$ is the maximum liklihood estimate of λ, given the sample rates $e_s, s = 1, ..., n$. Let the covariance matrix computed from the observed rates be S_{nt}, the usual estimate of Σ_{nt}. The theoretical covariance is partitioned as $\Sigma_{nt} = \Gamma_{nt} + \Delta_{nt}$, and the objective is to reproduce that partition with the sample covariance matrix. If the eigenvalues of Γ_{nt} are $\gamma_1, ..., \gamma_K$, and $\Delta_{nt} = diag(\delta_1^2, ..., \delta_K^2)$, then the eigenvalues of Σ_{nt} are $\gamma_1 + \delta_1^2, ..., \gamma_K + \delta_K^2$. When the rank of Γ_{nt} is $m < K$, then $\lambda_{m+1} = ... = \lambda_K = 0$. Consider the spectral decomposition of S_{nt}, with the ordered eigenvalues $g_1, ..., g_K$ and the corresponding eigenvectors $l_1, ..., l_K$. To generate the desired sample sample covariance structure, choose a *truncation* value $m < K$, and define the matrices

$$D = diag(g_1, ..., g_m) \tag{25}$$

$$L = (l_1, ..., l_m) \tag{26}$$

$$G_{nt} = L_{nt} D L'_{nt} \tag{27}$$

$$D_{nt} = diag(S_{nt} - G_{nt}) = diag(d_1, ...d_K) \tag{28}$$

$$S^*_{nt} = G_{nt} + D_{nt}. \tag{29}$$

In the theoretical covariance, it is possible that the eigenvalues $\lambda_j, j = 1, ...K$, are all positive. However, it is expected that the covariance between securities prices is generated by a small number of underlying portfolio's (factors), and therefore the number of positive eigenvalues (m) is small relative to K. In any case m is indeterminate and an arbitrary choice introduces error. Since many of the eignvalues and eigenvectors in the above construction are discarded, the method is referred to as *truncation.*

The matrices G_{nt}, D_{nt}, S^*_{nt} are estimates of $\Gamma_{nt}, \Delta_{nt}, \Sigma_{nt},$ respectively. Therefore, $\hat{\Gamma} = G_{nt}$ and $\hat{\Delta} = \frac{t}{n} D_{nt}$ are estimates of model parameters Γ and Δ. The estimate of Σ_t is

$$\hat{\Sigma}_t = t^2 \hat{\Gamma} + t\hat{\Delta} = \hat{\Gamma}_t + \hat{\Delta}_t. \tag{30}$$

For the parameter μ, the prior mean, assumptions about the financial market can guide estimation. If it is assumed that the there is a long term equilibrium value for returns on equities, it is reasonable to say $\lambda_i, i = 1, ..., K$, have a common mean. So $\mu' = (\mu, ..., \mu)$ and the prior mean is estimated by $\hat{\mu_t} 1$, where 1 is a vector of ones and

$$\hat{\mu_t} = \frac{1}{nK} \sum_i \sum_s e_{is}. \tag{31}$$

The *truncation estimator* for the conditional mean rate of return at time t is

$$\hat{\lambda}_{Tr} = \hat{\mu_t} 1 + (I - \hat{\Delta}_t \hat{\Sigma}_t^{-1})(\overline{Y_t} - \hat{\mu}_t 1). \tag{32}$$

The truncation estimator is an empirical Bayes estimator since it is in the form of the Bayes estimator, with estimates for the prior parameters. The assumption of a common prior mean could be relaxed to a common mean within asset classes, or some other grouping of securities.

An alternative empirical Bayes estimator has been developed by Jorion (1986). The prior mean is estimated by a weighted grand mean

$$\tilde{\mu}_t = 1' S_{nt}^{-1} \overline{Y_t} / (1' S_{nt}^{-1} 1). \tag{33}$$

The Bayes-Stein estimate of λ is

$$\hat{\lambda}_{BS} = \tilde{\mu}_t 1 + (\frac{n}{\varphi + K})(\overline{Y_t} - \tilde{\mu}_t 1), \tag{34}$$

with

$$\varphi = \frac{K+2}{(\overline{Y_t} - \tilde{\mu} 1)' S_{nt}^{-1} (\overline{Y_t} - \tilde{\mu} 1)}. \tag{35}$$

Although they have similar forms, the concept behind the Bayes-Stein estimator is quite different from the truncation estimator. The truncation estimator adjusts the maximum likelihood estimate $\bar{Y}_t$ based on the correlation between securities prices, or equivalently the scores on the latent market factors. The prior distribution is multivariate normal and the conditional covariance (specific variance) is diagonal. The Bayes-Stein estimator shrinks all the $\bar{Y}_{it}$ toward the grand mean, based on variance reduction. In this case, the prior is univariate normal, and the conditional covariance is not diagonal.

3.3 Comparison

A comparison of the various estimators for the rate of return is provided in MacLean, Foster, and Ziemba (2002) (See also MacLean, Foster and Ziemba, 2005). Two data sets for the years 1990-2002 of asset returns were used: (i) end of month prices for 24 leading stocks from the Toronto Stock Exchange (TSE); (ii) end of month prices for 24 leading stocks from the New York Stock Exchange (NYSE). The correlation structure for prices is different for the exchanges, so a performance comparison of estimators will show the significance of structure. The percent of variance accounted for by the top 5 eigenvalues for each correlation matrix is shown in Table 2

Table 2: Leading Eigenvalues (% of variance)

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	Total
NYSE	21.9	12.4	7.9	6.6	5.5	54.3
TSE	28.3	18.7	14.8	8.9	6.7	77.4

Source: MacLean, Foster and Ziemba (2005)

From the data on monthly closing prices for the set of 24 stocks on the Toronto Stock Exchange and the separate set of stocks on the New York Stock Exchange, the price increments (natural log of gross monthly rates of return) were computed. The mean vector and covariance matrix of the increments for each exchange were calculated, and these values were used as parameters in the dynamic model. Then trajectories of prices for 50 months were simulated and the expected rates of return were estimated by the various methods: average, truncation estimator, and Bayes-Stein estimator. For the truncation estimator the number of factors was preset at $m^* = 5$. Since the TSE data has a more compact market structure, it is expected that the truncation estimator with $m^* = 5$ will perform better in that case. This expiriment was repeated 1000 times.

The standard criterion for comparing estimators $\hat{\lambda}$ of a true mean rate of return λ is based on the mean squared error matrix:

$$E(\hat{\lambda} - \lambda)(\hat{\lambda} - \lambda)'. \tag{36}$$

The risk of an estimator is defined as the trace of the MSE matrix:

$$R(\lambda, \hat{\lambda}) = trE(\hat{\lambda} - \lambda)(\hat{\lambda} - \lambda)'. \tag{37}$$

In the simulation experiment, the value $\left[\frac{1}{K}R(\lambda, \hat{\lambda})\right]^{\frac{1}{2}}$ - root mean squared error, as well as the error for each estimator was computed. The results are shown in Table 2.

Table 3: %Root Mean Squared Error - source: MacLean, Foster and Ziemba (2005)

NYSE Stock	Estimator			TSE Stock	Estimator		
	Tr	Avg	BS		Tr	Avg	BS
$S_{NYSE,1}$	1.9725	2.8432	2.1462	$S_{TSE,1}$	1.7668	4.5063	3.1027
$S_{NYSE,2}$	4.1581	4.8305	4.2945	$S_{TSE,2}$	2.7083	4.4457	3.2007
$S_{NYSE,3}$	1.8415	2.9378	2.2138	$S_{TSE,3}$	1.9782	4.3989	3.0300
$S_{NYSE,4}$	2.2514	3.0818	2.3202	$S_{TSE,4}$	1.7466	4.3747	3.0131
$S_{NYSE,5}$	2.5917	2.8565	2.2977	$S_{TSE,5}$	1.6950	4.5301	3.1046
$S_{NYSE,6}$	2.8090	2.9188	2.2270	$S_{TSE,6}$	1.8265	4.6527	3.2517
$S_{NYSE,7}$	4.9898	5.3249	5.8516	$S_{TSE,7}$	1.6682	4.2899	2.9744
$S_{NYSE,8}$	2.1711	2.9338	2.2597	$S_{TSE,8}$	1.8485	4.3125	2.9897
$S_{NYSE,9}$	2.3748	3.1987	2.5019	$S_{TSE,9}$	1.8717	4.8132	3.3046
$S_{NYSE,10}$	2.0109	3.0979	2.4649	$S_{TSE,10}$	2.0050	4.4784	3.1089
$S_{NYSE,11}$	2.5031	3.0759	2.4323	$S_{TSE,11}$	2.3007	4.5624	3.2394
$S_{NYSE,12}$	1.9821	2.9752	2.2563	$S_{TSE,12}$	2.2495	5.0386	3.4564
$S_{NYSE,13}$	2.0238	2.9266	2.2242	$S_{TSE,13}$	2.6598	4.3629	3.1579
$S_{NYSE,14}$	1.9424	3.1798	2.5272	$S_{TSE,14}$	1.9208	4.5340	3.1757
$S_{NYSE,15}$	1.9684	3.0972	2.3393	$S_{TSE,15}$	1.7682	4.4389	3.0406
$S_{NYSE,16}$	2.2664	2.6545	2.0528	$S_{TSE,16}$	1.9065	4.3880	3.0011
$S_{NYSE,17}$	2.2813	2.7824	2.1058	$S_{TSE,17}$	2.1732	4.8737	3.2946
$S_{NYSE,18}$	4.5069	5.5240	5.7066	$S_{TSE,18}$	2.1746	4.7862	3.3036
$S_{NYSE,19}$	1.8315	2.8630	2.2293	$S_{TSE,19}$	2.2255	4.4604	3.1621
$S_{NYSE,20}$	5.3000	5.6623	6.3081	$S_{TSE,20}$	1.8554	4.3601	3.0257
$S_{NYSE,21}$	2.3198	2.7973	2.1474	$S_{TSE,21}$	1.6668	4.2875	2.9623
$S_{NYSE,22}$	2.2613	3.2026	2.3916	$S_{TSE,22}$	2.0788	4.5036	3.1099
$S_{NYSE,23}$	2.7386	2.8809	2.3177	$S_{TSE,23}$	2.0384	4.4373	3.1094
$S_{NYSE,24}$	1.8536	2.9950	2.2968	$S_{TSE,24}$	1.9518	4.1811	2.8882
AVG	2.6229	3.3600	2.8297	AVG	2.0026	4.5007	3.1253

The truncation estimator has smaller mean squared error for most stocks on the NYSE and for all stocks on the TSE. As predicted, the performance is better for the stocks on the TSE.

The parameter estimation presented above has focused on the mean return, since that value has such significance in investment decisions and capital accumulation. Studies of the long term effects of improved estimates of the mean justify that emphasis. (Grauer and Hakansson, 1986; Amir, et. al., 2005). The hierarchial model has volatility terms, and the empirical Bayes (truncation) approach also provides estimates of asset specific volatilities. The hierarchial model is in the class of stochastic volatility models. There are other approaches to forcasting returns. Most notable are CAPM and GARCH models. The model most consistent with the capital growth formulation is the hierarchial linear model, which characterizes the posterior mean rates of return given the data on past returns.

4 Growth Strategies

Accurate forcasts for future returns are the foundation for investment strategies. The accumulated wealth process from time t, $W(\tau), \tau \geq t$, is defined by the price dynamics and the investment strategy . The decision at time t controls the stochastic dynamic wealth process until a decision is made to revise the strategy. A standard approach in portfolio problems is to set a planning horizon at time T, and specify performance criteria for $W(T)$ as the basis of decision. The decision may be reconsidered between time t and T, either at regular intervals or at random times determined by the trajectory of wealth, but the emphasis is on the horizon. However, the path to the horizon is important, and paths with the same terminal wealth can be very different from a risk/survival perspective. Conditions which define acceptable paths can be imposed. For example, drawdown constraints are used to avoid paths with a substantial fall-off in accumulated capital (MacLean, Sanegre, Zhao, and Ziemba 2004, Grossman and Zhou 1993).

4.1 The Kelly Strategy

If the distribution of accumulated capital (wealth) at the horizon is the criterion for deciding on an investment strategy, then the rate of growth of capital becomes the determining factor when the horizon is distant. In fact, the optimal growth strategy is the unique *evolutionary stable* strategy, in the sense that it overwhelms other portfolio rules. Strategies that survive in the long run must converge to the optimal growth strategy. (Hens and Schenk-Hoppe, 2005.) Consider then the average growth rate between t_1 and t_2, for strategy $X = (X(t_1), ..., X(t_1 + (n-1)d))$,

$$EG_n(X) = \frac{1}{n}\sum_{i=0}^{n-1} Eln(R'(t_1 + id, t_1 + (i+1)d)X(t_1 + id). \tag{38}$$

The case usually discussed is where the incremental returns are serially independent. So the maximization of $EG_n(X)$ is

$$max\left\{Eln(R'(t_1 + id, t_1 + (i+1)d)X(t_1 + id)\right\}, \tag{39}$$

separately for each i. If the returns distribution is the same for each i, a fixed strategy holds over time. The strategy which solves (39) is called the KELLY or optimal growth strategy.

In the continuous time case with infinitesmal increments which are defined by the hierarchial Bayes model, the decision is determined from

$$max\left\{(\phi - re)'\tilde{X} + r - \frac{1}{2}\tilde{X}'\Delta\tilde{X}\right\}, \tag{40}$$

where $\phi_i = \lambda_i + \frac{1}{2}\delta_i^2, i = 1, ..., K$. Also, $\tilde{X}' = (x_1, ..., x_K)$ is the investment in risky assets. For this continuous time problem, the Kelly strategy is

$$\tilde{X}^* = \Delta^{-1}(\phi - re). \tag{41}$$

The Kelly or log optimal portfolio is $X^{*\prime} = (x_0^*, \tilde{X}^{*\prime})$, where $x_0^* = 1 - \sum_{i=1}^{K} x_i^*$. The continuous time formula can be viewed as an approximate solution to the discrete time investment problem. The Kelly strategy is a *fixed mix.* That is, the fraction of wealth invested in assets is determined by X^*, but rebalancing as wealth varies is required to maintain the fractions.

Table 4: Good and Bad Properties of the Optimal Growth Strategy

Feature	Property	Reference
Good	Maximizes rate of growth	Breiman (1960, 1961); Algeot and Cover (1988).
	Maximizes median log wealth	Ethier (1987).
	Minimizes expected time to asymptotically large goals	Breiman (1961); Algeot and Cover (1988); Browne (1997).
	Never risks ruin	Hakansson and Miller (1975).
	Absolute amount invested is monotone in wealth	MacLean, Ziemba and Li (2005).
	On average never behind any other investor	Finkelstein and Whitley (1981).
	The chance of being ahead of any other investor at least 0.5	Bell and Cover (1980).
	Wealth pulls way ahead of wealth for other strategies	Ziemba and Hausch (1986); MacLean, Ziemba and Blazenko (1992); Amir, et. el. (2005).
	Kelly is the unique evolutionary strategy	Hens and Schenk-Hoppe (2005).
	Growth optimal policy is myopic	Hakansson (1971).
	Can trade growth for security with negative power utility or fractional Kelly	MacLean and Ziemba (1999); MacLean, Ziemba and Li (2005).
	Fractional Kelly strategy to stay above a growth path with given probability	MacLean, et. al. (2004); Stutzer (2003).
Bad	It takes a long time to outperform with high probability	Aucamp (1993); Browne (1997b); Thorp (2005).
	The investments are extremely large if risk is low	Ziemba and Hausch (1986)
	The total amount invested swamps the gains	Ethier and Tavare (1983); Griffin (1985)
	There is overinvestment when returns are estimated with error	MacLean and Ziemba (1999); Rogers (2001); MacLean, Foster and Ziemba (2002).
	The average return converges to half the return from optimal expected wealth	Ethier and Tavare (1983); Griffin (1985).
	Investing double the Kelly fraction has a zero growth rate	Janacek (1998); Markowitz in Ziemba (2003).
	The chances of losing big in the short term (drawdown) can be high	Ziemba and Hausch (1986).
	Kelly strategy does not optimize expected utility of wealth. Example: Bernoulli trials with $1/2 < p < 1$,and $u(w) = w$. Then $x = 1$ maximizes $u(w)$, but $x = 2p - 1$ maximizes $Eln(w)$.	Samuelson (1971); Thorp (1971, 1975).

A variation on the Kelly strategy is the fractional Kelly strategy defined as $\tilde{X}_f = f\tilde{X}^*, f \geq 0$. The fractional Kelly strategy has the same distribution of wealth across risky assets, but varies the fraction of wealth invested in those risky assets. The optimal growth/Kelly strategy has been studied extensively. A summary of its properties is given in Table 3.

The Kelly portfolio is connected to the fundamental concepts in portfolio theory. (Luenberger, 1998.) If the instantaneous expected return on the Kelly portfolio is μ^*, and the instantaneous expected return on an asset is μ_i, then $\mu_i - r = \beta_i^*(\mu^* - r)$, where $\beta_i^* = \frac{\sigma_{i,*}}{\sigma^{*2}}$. In that expression, $\sigma_{i,*}$is the covariance between the i^{th} asset and the Kelly portfolio, and ${\sigma^*}^2$is the variance of the Kelly portfolio. This is exactly the CAPM equation, with the Kelly in the role of the market portfolio. If the asset is a derivative with price $y_i = F(P_i, t)$, with P_i the price of asset i, then the CAPM equation implies $\mu_{y_i} - r = \beta_{y_i}^*(\mu^* - r)$. With some rearrangement, this equation becomes $\frac{\partial F}{\partial t} + \frac{\partial F}{\partial P_i} r P_i + \frac{1}{2}\frac{\partial^2 F}{\partial^2 P_i}{\sigma^*}^2 P_i^2 = rF$, which is the Black-Scholes pricing formula for derivatives.

The Kelly strategy is central to the theory of capital growth, particularly in the long run. However, it can result in wealth trajectories which are unfavorable. As a measure of *underperformance*, consider the probability that wealth does not exceed a level defined by a fixed rate of return ρ. Whenever the growth rate $\frac{1}{T}\sum_{t=1}^{T} lnE\left[R'(t)X(t)\right]$ exceeds the target growth rate $ln\rho$, this underperformance eventually decays to zero exponentially fast as $T \to \infty$. The exponential decay rate of this probability is

$$D(X, \rho) = max_{\theta > 0}\left\{lim_{T\to\infty}\left\{-\frac{1}{T}lnE(\phi_T(\theta))\right\}\right\}, \tag{42}$$

where

$$\phi_T(\theta) = exp\left(-\theta\sum_{t=1}^{T}(lnR(t)'X(t) - ln\rho)\right). \tag{43}$$

The strategy X_{max} that maximizes $D(X, \rho)$ will result in the lowest large-T probabilities of failing to exceed the constantly growing wealth target $W(0)\rho^T$. When the $R(t)$ are independent and identically distributed, the

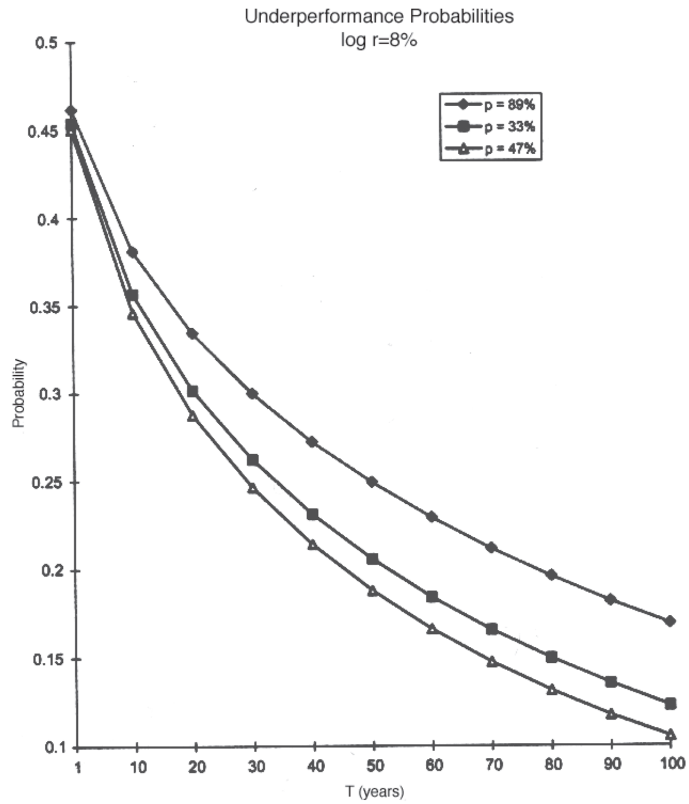

Figure 3: Kelly Underperformance Probability - source: Stutzer (2003)

solution X_{max} can be found from the problem

$$max_{\theta>0}\left\{-\frac{1+\theta}{\rho}max_X E\left[\frac{1}{1+\theta}\left(R'X\right)^{-\theta}\right]\right\}. \tag{44}$$

The $\theta_{max}$is an endogenously determined parameter in a power utility function.

In terms of the probability of not exceeding a target growth rate ρ, the Kelly strategy might not be preferable. In an example with $(\mu = .15, \sigma = .3)$ for a lognormal asset and a risk free asset with $r = .07$, Stutzer compared the Kelly strategy which invests 89% of capital in stock with portfolios with 47% of capital in stock and with 33% in stock. The target growth rate was set at $ln(\rho) = .08$. That comparison is provided in Figure 3.

In this example the Kelly is dominated by a fractional Kelly strategy with 47% in stock ($f = 0.528$) at each planning horizon from a risk perspective. The investment fraction is linked to risk or the underperformance probability.

The concept of concentrating on wealth above a target path is also referred to as *the discretionary wealth hypothesis* (Wilcox, 2005). The investment objective is to maximize the utility of wealth in excess of the amount that would trigger a shortfall. The discretionary wealth model would have a log wealth objective and a constraint requiring wealth to exceed the shortfall level. Variations on this model, and the connection to fractional Kelly strategies are explored later in the paper.

A general framework for simultaneously considering preferences for shortfall risk and return is stochastic dominance, which is discussed in the next section. That material is presented for the case where the distribution of returns is known. In many situations, the risks associated with the Kelly strategy are too high for known distributions. If the estimation of returns contains significant error, the motivation for a more conservative strategy such as fractional Kelly is even stronger. To see this, recall that the risk of an estimator is given as $R(\lambda, \hat{\lambda}) = trE(\hat{\lambda} - \lambda)(\hat{\lambda} - \lambda)'$. If the optimal estimator is λ^*, then $R(\lambda, \hat{\lambda}) = R(\lambda, \lambda^*) + R(\hat{\lambda})$, with $R(\hat{\lambda})$ being the additional risk for $\hat{\lambda}$. Consider a contamination of the estimator $\hat{\lambda}$ resulting from adding error. The additional risk that is incurred by using a contaminated estimate $\hat{\lambda}_{cont}$can be represented by the relative savings loss for the estimator $\hat{\lambda}_{cont}$

$$RSL(\hat{\lambda}_{cont}) = \frac{R(\hat{\lambda})}{R(\hat{\lambda}_{cont})}. \tag{45}$$

Consider an investor with wealth w_t at time t and as an investment strategy using the fractional optimal growth: $\tilde{X}_f = f\tilde{X}^*$. Let the $(t+1)^{th}$ period wealth be $W_{\hat{\lambda}}(t+1)$ and $W_{\lambda^*}(t+1)$ for the growth strategy with the estimated rate of return $\hat{\lambda}$ and the optimal estimateλ^*, respectively. The ratio

$$WL(X_{\hat{\lambda}_{cont}}) = E \log \frac{W_{\hat{\lambda}}(t+1)}{W_{\lambda^*}(t+1)} \tag{46}$$

indicates the wealth loss from the estimate $\hat{\lambda}$ for the rates of return.

With $\tilde{X}_{\hat{\lambda}}(t), \tilde{X}_{\hat{\lambda}_{cont}}(t)$ being the fractional Kelly strategies from using standard and contaminated estimates for the rate of return, the one period ahead relative wealth loss from a contaminated estimate is defined as

$$RWL(X_{\hat{\lambda}_{cont}}) = \frac{WL(X_{\hat{\lambda}})}{WL(X_{\hat{\lambda}_{cont}})}. \tag{47}$$

With greater contamination (poorer estimate of mean rate of return), the relative savings loss and the relative wealth loss both increase. However, for the growth strategies the wealth loss is modulated by the risk aversion index and thus the fraction in the Kelly strategy. That is, for a fixed contamination

$$RWL(X_{\hat{\lambda}_{cont}}) < RSL(\hat{\lambda}_{cont}), f < 1$$

$$RWL(X_{\hat{\lambda}_{cont}}) = RSL(\hat{\lambda}_{cont}), f = 1$$

$$RWL(X_{\hat{\lambda}_{cont}}) > RSL(\hat{\lambda}_{cont}), 1 > f > 1.$$

For proof of this result see MacLean, Foster, and Ziemba (2005). The wealth loss depends upon the risk aversion at the time of decision. In the decision rule, the risk aversion parameter β defines a fraction of capital invested in the optimal growth portfolio. When $f < 1$, the control of decision risk also reduces the impact of estimation error. Correspondingly, when $f > 1$, the overinvestment increases the effect of estimation error. The risk reduction effects of fractional Kelly strategies are separate from the reduction in estimation error through improved estimation methodologies.

4.2 Stochastic Dominance

Usually an investors' preferences for wealth over time are the basis for the investment decision. Those preferences involve growth of capital, but also an aversion to serious decay in wealth. The traditional method for defining preferences for wealth at the planning horizon is expected utility. The uncertain accumulated capital at the horizon with various strategies can be compared using that criterion. Wealth $W_1(T)$ dominates $W_2(T)$ *iff* $Eu(W_1(T))$ is greater than or equal to $Eu(W_2(T))$ for every $u \in U$, with strict inequality for at least one u (Hanoch and Levy, 1969).

There are alternative orders of dominance based on the class of utility function. The classes are defined by the sign of the k^{th}order derivative of the utility function. Consider

$$U_k = \{u|(-1)^{j-1}u^{(j)} \geq 0, j = 1, ...k\}. \tag{48}$$

So U_1 is the class of monotone nondecreasing functions, U_2 is the class of concave,monotone nondecreasing functions,and so on, with $U_1 \supseteq U_2 \supseteq ....$ If $U_\infty = lim_{j\to\infty}U_j$, then U_∞ contains the set of power utility functions, $U_* = \{u|u(w) = c_1 w^{c_2}, c_1 > 0, c_2 < 1\}$. This special class is important since it captures features such as risk aversion, and positive skewness, and is amenable to analytic results. The utility $u(w) = ln(w)$ is a member of the class of power utilities, and is the utility generating the Kelly investment strategy. From a utility perspective it is quite specialized, although it generates the most systematic and dominant portfolio.

Increasingly restrictive orders of dominance are defined by the classes of utilities. Consider the dominance criteria:

Wealth $W_1(T)$ k^{th}order dominates wealth $W_2(T)$, denoted $W_1(T) \succeq_k W_2(T)$, if $Eu(W_1(T)) \geq Eu(W_2(T))$for all $u \in U_k$,with strict inequality for at least one u.

Another formulation for stochastic dominance follows from the distribution functions for wealth. Let $W_1(T)$ and $W_2(T)$ have densities f_1 and f_2, respectively. For a density f, consider for $k = 1, 2, ...$, the nested integrations

$$I_k^f(w) = \int_{-\infty}^{w} I_{k-1}^f(z)dz, \tag{49}$$

where $I_0 = f$. Then

Wealth W_1 k^{th} order dominates wealth W_2 iff $I_k^{f_1}(w) \leq I^{f_2}(w)$, for all w, with strict inequality for at least one w.

It is convenient to write the nested integral in terms of loss functions. Again consider wealth W, with distribution F. Let $\rho_1(\alpha) = inf\{\eta | F(\eta) \geq \alpha\}$, the α^{th} percentile of the wealth distribution. For $k \geq 2$, define

$$\rho_k^W(\alpha) = \int_{-\infty}^{\rho_{k-1}(\alpha)} (\rho_{k-1}(\alpha) - w)^{k-1} dF(w), \tag{50}$$

which is a normed measure of the loss along the distribution. This loss function is a rearrangement of the nested integration. Then

Wealth W_1 k^{th} order dominates wealth W_2 iff For $k \geq 2$,$\rho_k^{W_1}(\alpha) \leq \rho_k^{W_2}(\alpha)$, for all α,with strict inequality for at least one α.For $k = 1$,$\rho_1^{W_1}(\alpha) \geq \rho_1^{W_2}(\alpha)$,for all α,with strict inequality for at least one α.

The variations on the dominance relation provide insight into the conditions for the wealth process. The loss function definition of dominance is particularly useful since all the measures have the same argument α, and values of α are easy to comprehend. It is clear that $\rho_1^W(\alpha), 0 \leq \alpha \leq 1$, is the inverse cumulative distribution, and ordering based on this measure expresses a preference for wealth. Since $\int_{-\infty}^{\rho_1(\alpha)}(\rho_1(\alpha) - w)dF(w) = \alpha\rho_1(\alpha) - \int_{-\infty}^{\rho_1(\alpha)} wdF(w)$, the second order measure, $\rho_2^W(\alpha), 0 \leq \alpha \leq 1$, is equivalent to the Lorenz curve (Lorenz, 1905). It is defined as $L(\alpha) = \int_{-\infty}^{\rho_1(\alpha)} wdF(w)$, and reflects risk aversion in that it is sensitive to by the lower tail of the distribution. The measure $\rho_3^W(\alpha), 0 \leq \alpha \leq 1$, captures the aversion to variance. These characteristics are components of the preferences which investors express with a utility function.

4.3 Bi-criteria Problems: Fractional Kelly Strategies

In general, the ordering of wealth distributions at the horizon using stochastic dominance is not practical. Rather than use the full range of α values with the moments definition, it is more realistic to identify specific values and work with a vector order. The vector ordering $(x_1, ..., x_p) < (y_1, ..., y_p)$ holds iff $x_i \leq y_i, i = 1, ..., p$, with at least one strict inequality. Selecting two values of α to set up bi-criteria problems $(\rho_k(\alpha_1), \rho_k(\alpha_2))$, has some appeal. Consider terminal wealth variables $W_1(T)$ and $W_2(T)$, and values $\alpha_1 < \alpha_2$.

For $k \geq 2$, $W_1(T)$ is k^{th} order $(\alpha_1, \alpha_2)-$preferred to $W_2(T)$, denoted $W_1(T) \gg_k W_2(T)$, iff $(\rho_k^{W_1}(\alpha_1), \rho_k^{W_1}(\alpha_2)) < (\rho_k^{W_2}(\alpha_1), \rho_k^{W_2}(\alpha_2))$. $W_1(T)$ is 1^{st} order $(\alpha_1, \alpha_2)-$preferred to $W_2(T)$ iff $(\rho_1^{W_1}(\alpha_1), \rho_1^{W_1}(\alpha_2)) > (\rho_1^{W_2}(\alpha_1), \rho_1^{W_2}(\alpha_2))$.

The preference ordering is *isotonic* with respect to the dominance ordering: $W_1(T) \succeq_k W_2(T) \Rightarrow W_1(T) \gg_k W_2(T)$.

The preference ordering on wealth leads to analogous optimization problems for determining a preferred investment strategy. The definition of a problem requires the specification of lower and upper $\alpha-$values, $\alpha_L$and α_U, capturing risk and return. There are natural values based on conventional use. So $\rho_1(\alpha_U) = \rho_1(0.5) =$median wealth, which is equivalent to the mean log-wealth $= E(ln(W(T))$. If the lower value is α_L, then $\rho_1(\alpha_L) =$ the $100\alpha_L^{th}$percentile. A bi-criteria problem defined by the first order preference ordering is

P_{VaR} :

$$Max \left\{ E(ln(W(T))|Pr[W(T) \leq w_{VaR}] \leq \alpha \right\}. \tag{51}$$

This is a variation on the unconstrained optimal growth problem, with a value-at-risk constraint imposed. Since $\{W(T) \leq w_{VaR}\} \Leftrightarrow \{ln(W(T)) \leq ln(w_{VaR})\}$, the constraint is anologous to a condition on the growth rate of trajectories as in the underperformance measure (Stutzer, 2003).

For second order preference, let $\rho_2(\alpha_U) = \rho_2(1.0) = E(W(T)) =$ mean wealth. With $\rho_2(\alpha_L) =$ the lower $100\alpha^{th}$ percent incomplete mean, the second order preference problem is

$$P_{CVaR}: \quad Max\left\{E(W(T))|E(W(T)I_{\alpha_L}) \geq w_{CVaR}\right\}. \tag{52}$$

In this notation $I_{\alpha_L}$is an indicator variable for the lower $100\alpha_L\%$ of the wealth distribution. This problem is discussed in DiGiorgio (2005), where the consistency of the objective and constraint combination for second order dominance is established.

The preference problems are defined by wealth at a horizon time. The properties of wealth at the horizon are significant, but the path of wealth to the horizon is also important as was indicated earlier with the discretionary wealth hypothesis. There are unsustainable wealth levels and losses, so measures which reflect the chance of survival should be included in the decision process. There are two approaches considered: (i) acceptable paths based on the *rate of change* or growth in wealth; (ii) acceptable paths based on wealth *levels*. The chance that a path is acceptable defines a path measure. Table 5 gives a statement of such measures. The notation τ_w refers to the first passage time to the wealth level w.

Table 5: Path Measures

Criterion	Specification	Path Measure
Change	$b<0$	$\delta_1(b) = Pr[\frac{1}{W(\tau)}dW(t+\xi) > b\xi, \xi > 0]$
Levels	(w_L, w_U)	$\delta_2(w_L, w_U) = Pr[\tau_{w_U} < \tau_{w_L}]$

The change measure puts a lower bound on the fallback in any period. The levels measure requires trajectories to reach an upper level before falling to a lower level, eg, doubling before halving. These measures could be used in place of a moment measure or even in addition to those measures to control the path of wealth. A variety of decision models for determining an investment strategy, which have a form of optimality referred to as growth-security efficiency, are displayed in Table 6.

Table 6: Alternative Decision Models

Model	criterion	Problem
M_1	Expected utility:power utility with risk aversion index $\frac{1}{1-\beta}, \beta < 1$.	$MaxE\left[\frac{1}{1-\beta}W(T)^{\beta}\right]$
M_2	First Order Dominance: optimal median wealth subject to a VaR constraint.	$Max\left\{E(ln(W(T))\|Pr[W(T) \leq w_{VaR}] \leq \alpha_L\right\}$
M_3	Second Order Dominance: optimal mean wealth subject to a CVaR constraint.	$Min\left\{E(W(T))\|E(W(T)I_{\alpha_L}) \geq w_{CVaR}\right\}$
M_4	Drawdown: optimal median wealth subject to a drawdown constraint.	$Max\left\{E(ln(W(T))\|\delta_1(b) \geq 1-\alpha_L\right\}$
M_5	Wealth Goals: optimal median wealth subject to control limits.	$Max\left\{E(ln(W(T))\|\delta_2(w_L, w_U) \geq 1-\alpha_L\right\}$

The usual emphasis in the bi-criteria problems is on median wealth. That is important since the unconstrained solution to a median wealth problem is the Kelly strategy. The problems in Table 6 are defined for continuous or discrete time models. Information on prices is recorded at discrete points in time, and decisions on allocation of capital to investment instruments are taken at discrete points in time. However, wealth accumulates continuously and it reasonable to analyze capital growth in continuous time. So discrete price data is used to estimate parameters in a continuous time pricing model, but the forecast growth in capital between decision points is continuous. There are some limitations to this continuous time parametric approach. It is possible that there is not a good model for prices, and discrete time and state scenarios are better able to capture future prospects. Also, certain psuedo-investment instruments such as lotteries and games of chance are discrete by nature. In the analysis of investment strategies in this section, it is assumed that the continuous time Bayesian pricing model is correct, although knowledge of the model parameters is uncertain. At the end of this section an example of a discrete scenario, discrete time problem is considered.

For continuous time investment with asset prices defined by geometric Brownian motion, the class of feasible investment strategies for the single risk-free and m risky opportunities is

$$\chi_t = \left\{ X(t) | \left[X(t)'(\hat{\phi}(t) - re) + r - \frac{1}{2} X(t)' \hat{\Delta} X(t) \right] \geq 0 \right\}. \tag{53}$$

The condition in (53) is for positive growth. The expected growth rate for wealth, given an investment strategy $X(t) \in \chi_t$, is defined as $G(X) = ElnW(T)^{\frac{1}{T}}$. The Kelly or optimal growth strategy is defined by $X^*(t) = argmaxG(X)$. In the continuous time problem, the Kelly strategy, defining the investments in risky assets, has the closed form $X^*(t) = \Delta^{-1}(\phi(t) \hat{-} re)$, where $e' = (1, ..., 1)$. The subclass of fractional Kelly strategies is defined as

$$\chi_t^* = \{ X(t) | X(t) = fX^*(t), f \geq 0 \}. \tag{54}$$

Since $X^*(t) \in \chi_t$, then $\chi_t^* \subseteq \chi_t$. The significance of the fractional Kelly strategies lies in their optimality for the problems in Table 6, assuming the Bayesian geometric Brownian motion model for prices is correct.

Let $X_{M_j}(t)$be the optimal solution to growth problem $M_j, j = 1, 2, 3, 4, 5$ defined in Table 6. Then $X_{M_j}(t) \in \chi_t^*$,that is, the solution is fractional Kelly.

For proof see MacLean, Zhao and Ziemba (2005).

In the continuous time formulation, the optimal investment strategies for the various problems have the same form. However, the actual fraction in each problem, which controls the allocation of capital to risky and risk-free instruments, depends on the decision model and parameters. The formulas for the fractions for different models are displayed in Table 7. The notation $\tilde{\mu} = (\hat{\phi} - re)'X^*(t) + r$, and $\tilde{\sigma}^2 = X^{*'}(t)\Delta X^*(t)$ is used for the mean and variance of the rate of return on the Kelly strategy. Also y^* is the minimum positive root of the equation $\gamma y^{c+1} - y + (1-\gamma) = 0$, for $c = \frac{ln(w_U) - ln(w_t)}{ln(w_t) - ln(w_L)}$.

Table 7: Investment Fractions

Model	Parameters	Fraction
M_1	β	$f_1 = \frac{1}{1-\beta}$
M_2	(ρ_1^*, α)	$f_2 = \frac{B_1 + \sqrt{B_1^2 + 2A_1C_1}}{A_1}$ $A_1 = (\hat{\phi} - re)'\Delta^{-1}(\hat{\phi} - re)$ $B_1 = A_1 + z_\alpha\sqrt{\frac{A_1}{T-t}}$ $C_1 = r - (T-t)^{-1} ln \frac{\rho_1^*}{w_t}$
M_3	(ρ_2^*, α)	$f_3 = \frac{B_2 + \sqrt{B_2^2 + 2A_2C_2}}{A_2}$ $A_2 = \left[z_\alpha \frac{\Phi'(z_\alpha)}{\alpha} + \left(\frac{\Phi'(z_\alpha)}{\alpha} \right)^2 \right] A_1$ $B_2 = A_1 - \frac{\Phi'(z_\alpha)}{\alpha}\sqrt{\frac{A_1}{T-t}}$ $C_2 = r - (T-t)^{-1} ln \frac{\rho_2^{**}}{w_t}$
M_4	(b, α)	$f_4 = \frac{b}{\tilde{\mu} + z_\alpha \tilde{\sigma}}$
M_5	(w_L, w_U)	$f_5 = h_t \cdot \tilde{H} + \sqrt{\left[h_t \cdot \tilde{H} \right]^2 + \frac{2rh_t}{\tilde{\sigma}^2}}$ $\tilde{H} = \frac{\tilde{\mu} - r}{\tilde{\sigma}^2}$ $h_t = \frac{ln(w_t) - ln(w_L)}{ln(w_t) - ln(y^* w_L)}$

Although it is not obvious from the formulas, the fractions in Table 7 reflect the orderings from the sequence of stochastic dominance relations. So the fraction $f_1 = \frac{1}{1-\beta}$ in M_1 is the most specific, but has fewer degrees of freedom in defining risk. There is a set of specifications for (ρ_1^*, α) in M_2, for example, which yield the same fraction, that is $f_2 = f_1$. In general, if the settings for model M_i are represented by π_i, there are equivalence classes of settings

$$\Pi_i(\beta) = \left\{ \pi_i | argmax\left(M_i(\pi_i)\right) = \frac{1}{1-\beta} X^* \right\}, i = 2, ..., 5. \tag{55}$$

The fractional Kelly strategy arises from the aversion to risk, which can be characterized by a power utility or by explicit risk constraints. In Figure

3 it was shown that the Kelly strategy was dominated from an underperformance probability perspective. The definition of the underperformance problem with iid returns in equation (44) leads to the result that the strategy which minimizes the underperformance probability for a target growth rate ρ less than the optimal growth rate, is fractional Kelly. Alternatively, the strategy which maximizes the probability of outperforming a target growth rate is fractional Kelly. The wealth beyond the target growth path has been termed *discretionary wealth*. (Wilcox, 2003.) So the fractional Kelly strategy maximizes the probability of discretionary wealth, a result referred to as the *discretionary wealth hypothesis*.

Example: Investment in stocks and bonds in continuous time

To illustrate the connection between the investment fraction in the optimal growth strategy and and the specifications for ρ_1 and α_U, the allocation of investment capital to stocks, bonds, and cash over time is considered. Daily prices were generated for 260 trading days, based on the statistics in Table 8. They were determined from statistics on total returns from the S&P500, Solomon Brothers bond index, and US treasury bills.

Table 8: Daily Rates of return

	Stocks	Bonds	Cash
Mean	0.00050	0.00031	0.00019
Variance	0.00062	0.00035	0.000
Covariance	0.000046		

Initial wealth was set at $1. For a trajectory of simulated prices, an investment strategy was determined every 10 days. At a rebalance time, the data on past prices was used to update estimates for model parameters based on the empirical Bayes methodology. With revised estimates, the continuous time formulas were used to calculate the investment strategy. The fractions of wealth invested in the optimal portfolio of stocks and bonds in the initial period is shown in Table 9. The fractions are particularly sensitive to the

VaR level ρ_1. The strategies are fractional Kelly, since the distribution of investment capital in (stocks, bonds) is (.6, .4) for each combination of (ρ_1, α_L).

Table 9: VaR Strategies (Stocks, Bonds) - source: MacLean, Zhao and Ziemba (2005).

			α_L		
ρ_1^*	0.01	0.02	0.03	0.04	0.05
0.99	(.30,.20)	(.34,.23)	(.38,.25)	(.40,.27)	(.43,.29)
0.98	(.56,.37)	(.64,.42)	(.69,.46)	(.74,.50)	(.79,.53)
0.97	(.82,.54)	(.93,.62)	(1.01,.68)	(1.09,.72)	(1.16,.77)
0.96	(1.08,..72)	(1.22,.81)	(1.33,.89)	(1.43,.95)	(1.52,.1.01)
0.95	(1.34,.89)	(1.52,1.01)	(1.65,1.10)	(1.78,1.18)	(1.89,1.26)

With 1000 trajectories simulated, the average wealth at the end of the planning horizon was calculated; see Table 10.

Table 10: Expected Wealth - source: MacLean, Zhao and Ziemba (2005).

			α_L		
ρ_1^*	0.01	0.02	0.03	0.04	0.05
0.99	1.0629	1.0649	1.0666	1.0681	1.0696
0.98	1.0764	1.0797	1.0826	1.0854	1.0884
0.97	1.0881	1.0953	1.1019	1.1084	1.1154
0.96	1.1053	1.1189	1.1310	1.1432	1.1557
0.95	1.1292	1.1509	1.1703	1.1895	1.2091

This example is intended for illustration of the effect of risk constraints such as VaR on the fractional Kelly strategy and the accumulated capital. The impact of the VaR level, ρ_1^* is most pronounced. Strict requirements in the VaR level lead to very conservative fractional Kelly strategies.

Example: Investment in Stocks and Bonds in discrete time

For comparison with the continuous time problem, M_2 was implemented in discrete time, with years as time units. The statistics on annual returns from the data are in Table 11.

Table 11: Annual Return Statistics

Parameter	Stocks	Bonds	Cash
Mean	0.08750	0.0375	0
Variance	0.1236	0.0597	0
Correlation	0.32		

A corresponding set of scenarios was created (sampling from a lognormal distribution for stocks and bonds), and they are displayed in Table 12. The sampling process was structured so that sample statistics were as close as possible to the statistics in Table 10. (MacLean, Sanegre, Zhao, and Ziemba, 2004.)

Table 12: Return Scenarios

Stocks	Bonds	Cash	Probability
0.95	1.015	1	0.25
1.065	1.100	1	0.25
1.085	0.965	1	0.25
1.250	1.070	1	0.25

The planning horizon was set at 3 years and the same scenarios were used each year. So there were 64 scenarios, each with a probability of 1/64. With this discrete time and discrete scenario formulation, problem M_2 was solved with $\alpha = 0.01$ and a variety of values for the VaR level ρ_1^*. Starting wealth was \$1. The results from solving the problems are shown in Table 13. (Details on this problem are in MacLean, Sanegre, Zhao and Ziemba, 2004.) If the annual returns are compared to the results for the continuous time problem with the same α, the returns in this case are slightly lower. However, the continuous time application involved rebalancing every 10 days. In Table 13, it can be observed that the very strict VaR condition almost eliminates the possibility of growth. As well, the optimal strategy is not fractional, with the investment mix changing as the horizon approaches.

Table 13: Investment Strategy and Rate of Return - source: MacLean, et. al. (2004).

	Year 1			Year 2			Year 3			Yearly	
ρ_1^*	stocks	bonds	cash	stocks	bonds	cash	stocks	bonds	cash	Return	
0.950	1	0	0	0.492	0.508	0	0.492	0.508	0	1.061	
0.970	1	0	0	0.333	0.667	0	0.333	0.667	0	1.057	
0.990	0.456	0.544	0	0.270	0.730	0	0.270	0.730	0	1.041	
0.995	0.270	0.730	0	0.219	0.590	0.191	0.218	0.590	0.192	1.041	
0.999	0.270	0.730	0	0.008	0.020	0.972	0.008	0.020	0.972	1.017	

4.4 Growth-Security Trade-off

The solutions displayed in the Table 7 are derived from the continuous time wealth equation, although the strategies are calculated at discrete decision points in time. The alternative problems in Table 6 can be based on the discrete time wealth equation, but the optimal solution is not necessarily fractional Kelly. That point was demonstrated in the discrete time example. However, the fractional Kelly solution may be near-optimal. If the feasible strategies for the discrete time problem are restricted to the class of fractional strategies, the solutions are *effective* (MacLean, Ziemba and Blazenko, 1992). That is, as the fraction changes, the growth (objective) and security (constraint) move in opposite directions, so that growth is monotone non-increasing in security. Specifically, for $0 \leq f \leq 1$, $\alpha_L < .5, \alpha_U \geq .5$,

$$\frac{d}{df}\rho_i(\alpha_L) \geq 0, \frac{d}{df}\delta_i(\alpha_L) \geq 0, \frac{d}{df}\rho_i(\alpha_U) \leq 0, i = 1, 2. \tag{56}$$

The implication of this monotonicity is that growth can be traded for security using the fraction allocated to the optimal growth portfolio. With the fractional Kelly strategy fX^*, the computation of the growth and security measures as a function of f is accomplished with known formulas (MacLean et. al. 1992; Dohi et. al. 1995). The approximating continuous time formulas are in Table 14.

Table 14: Computation of Measures

Measure	Specification	Formula
$\rho_1(\alpha_U)$	$\alpha_{U=0.5}$	$w_t exp\left\{(\mu(X)-r)\lambda(X)+r-\frac{1}{2}X'\Delta X)T)\right\}$
$\rho_2(\alpha_U)$	$\alpha_{U=1.0}$	$w_t exp\left\{(\mu(X)-r)\lambda(X)+r)T\right\}$
$\rho_1(\alpha_L)$	w_{VaR}	$\Phi(\frac{D(X)T-ln\frac{w_{VaR}}{w_t}}{\sigma(X)\lambda(X)T^{\frac{1}{2}}})$
$\rho_2(\alpha_L)$	w_{CVaR}	$w_t exp\left\{(\mu(X)-r)\lambda(X)+r)T\right\}[\Phi(z_{\alpha_L}-\sigma(X)]$
$\delta_1(\alpha_L)$	β	$\left[1-\Phi(\frac{ln\beta-\mu(X)}{\sigma(X)/T^{\frac{1}{2}}})\right]^{nT}$
$\delta_2(\alpha_L)$	(w_L, w_U)	$\frac{1-(\frac{w_U}{w_t})^{2D(X)/\sigma^2(X)\lambda^2(X)}}{1-(\frac{w_U}{w_L})^{2D(X)/\sigma^2(X)\lambda^2(X)}}$

So the growth - security trade-off can be observed for various fractional Kelly straregies and suitable fractions (meeting investor preferences) can be determined. This will be demonstrated with some examples.

Example: Investing in turn-of-the-year effect

The trade-off between growth and security was used by Clark and Ziemba (1987) in the analysis of investment in the turn of the year effect. The excess return of small cap stocks minus large cap stocks is most pronounced in January. The distribution of gains at the turn of the year from holding long positions in the Value Line Index of small stocks, and short positions in the Standard and Poors Index of large cap stocks is given in Table 15, where each point is worth $500. The data covers the period 1976-77 to 1986-87.

Table 15: Returns distribution for VL/S&P Spread

Points Spread	7	6	5	4	3	2	1	0	-1
Probability	.007	.024	.07	.146	.217	.229	.171	.091	.045

The Kelly strategy calculated from this distribution invests 74% of ones fortune in the trade. This is very aggressive considering the possible estimation errors and the market volatility.

In Figure 4 is a graph for fractional Kelly strategies with the turn of the year trade, showing the chance of reaching a wealth of 10 million before ruin, starting from various initial wealth levels. The graph for .25 Kelly is much more secure. Similarly in Figure 5, the trade-off between relative growth and probability of achieving a desired wealth level is displayed. Going from Kelly to quarter Kelly realizes a probability gain (security) of about .25 and an almost equivalent loss in relative growth. An approximate 0.25 Kelly strategy was used with consistent success in actual trades on this commodity by W. T. Ziemba in the 14 years from 1982/83 to 1996/97, winning each year. Because the declining Value Line volume makes the trade risky, he has not done the trade since 1997.

The current markets have become much more dangerous than during the period of the study of the turn of the year effect: see Rendon and Ziemba (2005) for an update to 2004/5. The January effect still exists, however, in the futures markets, but now is confined to the second half of December.

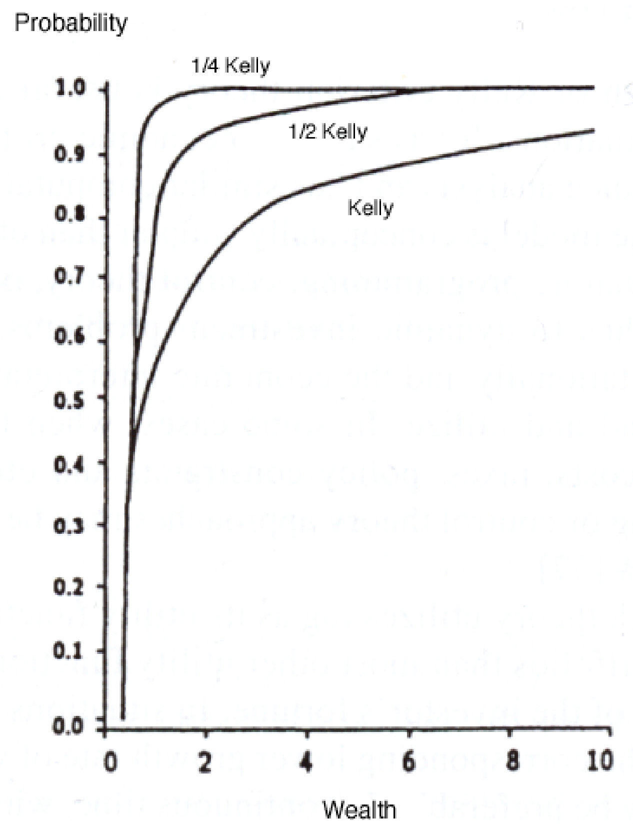

Figure 4: Turn-of-the-year effect: probability of reaching \$10 million before ruin for Kelly, half Kelly and quarter Kelly strategies - source: MacLean, Ziemba and Blazenko (1992).

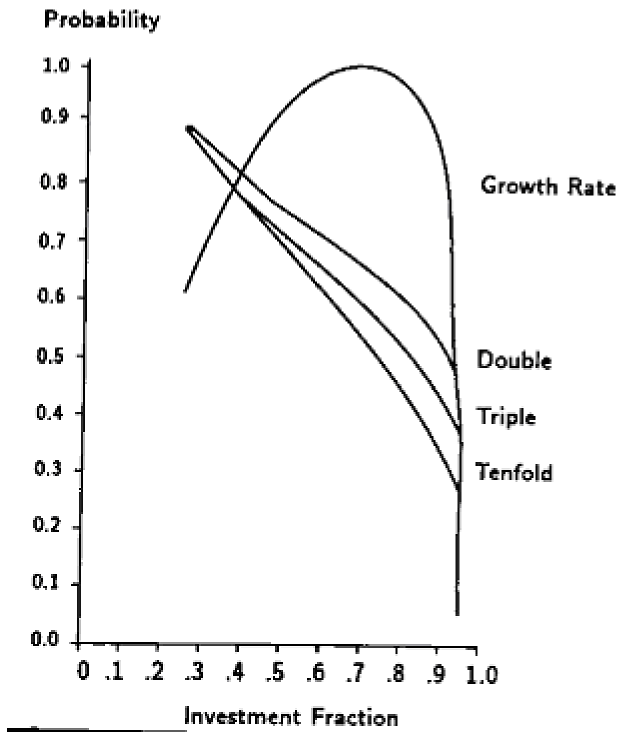

Figure 5: Relative growth versus probability of wealth - source: MacLean, Ziemba and Blazenko (1992).

Example: Kentucky Derby

Don Hausch and William Ziemba have written extensively on the advantages of the capital growth approach to wagering on horseraces. They have developed a betting approach, the Dr. Z system, based on the Kelly strategy. (Ziemba and Hausch, 1984, 1986, 1987). In Figure 6 is shown wealth level histories from place and show betting on the Kentucky Derby from 1934 to 1994 using the Dr. Z system (Bain, Hausch and Ziemba, 2005). The system uses probabilities from the simpler win market to determine bets in the more complex place/show market, where inefficiencies are more likely to occur. Starting with initial wealth of \$2500, and a 4.00 dosage index filter rule, Kelly and half Kelly strategies are compared with \$200 flat bets on the favorite. The full Kelly yields a final wealth of \$16,861, while half kelly has a final wealth of $6,945, but with a much smoother wealth path.

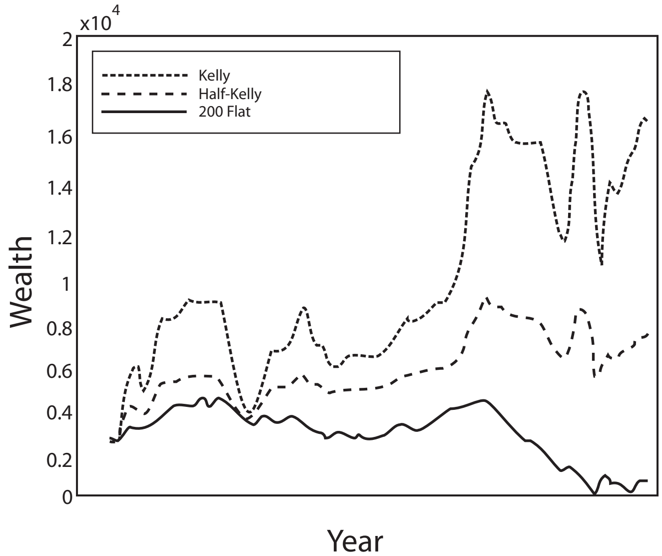

Figure 6: Kentucky Derby Capital Growth - source: Bain, et. al. (2005).

Example:Lotto Games

In lotto games players select a small set of numbers from a given list. The prizes are shared by those with the same numbers as those selected in the random drawing. The lottery organization bears no risk in the pari-mutuel system and takes its profits before the prizes are shared. Hausch and Ziemba (1995) survey these games. Ziemba et al (1986) studied the 6/49 game played in Canada and several other countries. Numbers ending in eight and especially nine and zero tend to be unpopular. Six tuples of unpopular numbers have an edge with expected returns exceeding their cost. The expected value approaches $2.25 per dollar wagered when there are carryovers (that is when the Jackpot is accumulating because it has not been won.) However, investors may still lose because of mean reversion (the unpopular numbers tend to become less unpopular over time) and gamblers' ruin (the investor has used up his resources before winning). MacLean, Ziemba and Blazenko (1992) investigated how an investor might do playing sets of unpopular numbers with a combined advantage using the data in Table 16.

Table 16: Lotto 6/49 Data - source: MacLean, et. al. (1992).

Prizes	Prob.	Value	Contribution %
Jackpot	1/13983816	$6M	42.9
Bonus	1/2330636	$0.8M	34.3
5/6	1/55492	M	9.0
4/6	1/1032	$5,000	14.5
3/6	1/57	$150	17.6
Edge			18.1%
Kelly bet			0.00000011
Number of Tickets with 10M bankroll			11

The optimal Kelly wagers are extremely small. The reason for this is that the bulk of the expected value is from prizes that occur with less than one in a million probability. A wealth level of $1 million is needed to justify even one $1 ticket. Figure 7 provides the chance that the investor will double, quadruple or tenfold this fortune before it is halved using Kelly and fractional Kelly strategies. These chances are in the 40-60%. With fractional Kelly strategies in the range of 0.00000004 and 0.00000025 or less of the investor's initial wealth, the chance of increasing one's initial fortune tenfold before halving it is 95% or more. However, it takes an average of 294 billion years to achieve this goal, assuming there are 100 draws per year as there are in the Canadian Lotto 6/49.

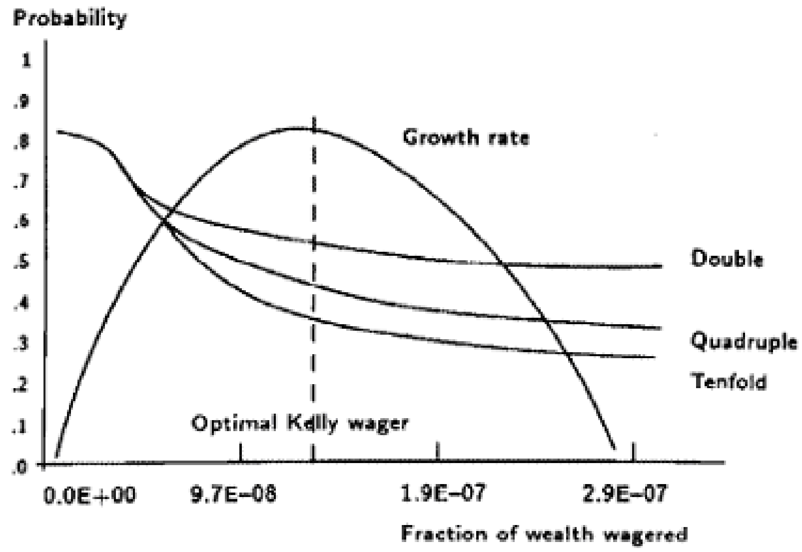

Figure 7: Lotto 6/49 - Probability of multiplying before losing half of ones fortune vs bet size - source: Maclean et. al. (1992)

The conclusion is that except for millionaires and pooled syndicates, it is not possible to use the unpopular numbers in a scientific way to beat the lotto and have high confidence of becoming rich; these aspiring millionaires are also most likely going to be residing in a cemetery when their distant heir finally reaches the goal.

5 Timing of decisions

After the investment decision is taken, it is necessary to monitor the performance of the portfolio in terms of accumulated capital. This is particularly true in the case where the dynamics of prices have changed since the time of decision. The use of control limits to detect unacceptable or out of control paths is an additional component in the management of the capital accumulation process (MacLean, Zhao, and Ziemba 2005). If the trajectory of capital accumulation is unacceptable, then corrective action is required.

When control limits identify significant deviations from expectations, a time for adjustment is identified. The adjustment requires an evaluation of the pricing model and a revision of investment decisions. Using information on prices collected since the last decision, a new price model is developed, an updated investment strategy is calculated (based on the new forecasts and possible financial constraints). Then new control limits are determined to monitor the trajectory of wealth over the upcoming period.

5.1 Control Limits

The investment strategy to accumulate wealth is based on estimates of the parameters which drive the asset prices and preferences for accumulated capital. The strategy and the resulting wealth trajectory are very sensitive to estimation errors for the parameters in returns distributions (Kallberg and Ziemba, 1981, 1984, and Chopra and Ziemba, 1993). The estimation errors are far more important than the frequency of the rebalancing of decisions (Rogers 2001). A natural way to deal with the uncertain direction of a trajectory of the stochastic dynamic wealth process is to set process control limits and to adjust (update estimates for returns and re-solve the growth-security problem) when a limit is reached. This is standard practice in the control of production processes.

In statistical process control, the control limits are based on the mean and standard deviation, with the convention being six-sigma limits. In the capital accumulation problem, the symmetry of deviations from expectations does not hold. Risk aversion makes the investor more sensitive to large losses. The limits can be selected so that they are consistent with risk aversion and the axioms for risk measures. To develop the control limits consider

$\tau_w(X(t)|w_t)$ = the first passage time to wealth w, starting from wealth w_t at time t and following strategy $X(t)$.

Then the upper control limit (UCL) will be set from the expected return at the planning horizon, the level expected if the estimates for model parameters are correct. It is expected that the UCL is reached exactly as the time horizon arrives. The lower control limit (LCL) provides downside risk control. To match the security provided by the VaR problem for example, the LCL is selected so that the wealth process with an optimal strategy will reach the LCL before the UCL at most 100α % of the time.

Suppose the optimal strategy is $X(t)$. The control limits are determined for the wealth trajectory resulting from the decision $X(t)$. Define the UCL as

$$w_U = E[W(T)] \tag{57}$$

Then the measure $\delta_2(w_L, w_U)$ can be used to determine the LCL, with

$$w_L = sup\left\{w|\delta_2(w, w_U) \geq 1 - \alpha\right\}. \tag{58}$$

The computational formulas for ρ_2 and δ_2 make the calculation of limits straightforward, when the investment decision is a fixed mix such as the fractional Kelly. For the continuous time model the limits have a closed form solution (MacLean, Zhao and Ziemba 2005) as displayed in Table 17.

Table 17: Control Limits

$w_U = w_t exp\left\{(X'(\phi - re) + r)(T - t)\right\}$
$w_L = w_t \left(\frac{\alpha_L}{1-(1-\alpha_L)\left(\frac{w_t}{w_U}\right)^{\theta(X)}} \right)^{\frac{1}{\theta(X)}}$

To summarize the approach, the growth-security problem is solved at time t for an optimal strategy based on forecast returns and the planning horizon of $(T - t)$. Then control limits are computed, which are consistent with the growth-security specifications, and serve as stopping boundaries for the wealth process. The intention is that the portfolio rebalancing would only take place when a boundary is reached and the trajectory is not proceeding as anticipated.

EXAMPLE:INVESTMENT IN STOCKS AND BONDS

The information basis for investment will be daily trading prices for stocks, bonds and cash. These prices will be generated from the random rates of return model. The baseline values for the true price process were presented in Table 8 previously.

The approach to analyzing investment decisions will be to take the VaR model as the standard and to add upper and lower wealth limits calculated from the VaR strategy . That is, at each rebalance time, first the strategy is computed with risk specifications. Then the upper and lower wealth limits will be calculated from the formulas in Table 17. The upper and lower control limits are calculated to determine the next rebalance time. The control limits, corresponding to the VaR strategies are shown in Table 18. The initial wealth is $w_0 = 1$, and the VaR horizon is $T = 10$ days. The values in these tables are updated as the portfolio is rebalanced.

Table 18: Control Limits (w_L, w_U) for VaR Strategies -

	α_L				
ρ_1^*	0.01	0.02	0.03	0.04	0.05
0.99	(0.9995,1.0031)	(0.9996,1.0033)	(1,1.0034)	(1,1.0035)	(1,1.0037
0.98	(0.9986,1.0042)	(0.9992,1.0045)	(0.9999,1.0047)	(0.9999,1.0047)	(1,1.0047)
0.97	(0.9976,1.0047)	(0.9989,1.0047)	(0.9999,1.0047)	(0.9999,1.0047)	(1,1.0047)
0.96	(0.9976,1.0047)	(0.9989,1.0047)	(0.9999,1.0047)	(0.9999,1.0047)	(1,1.0047)
0.95	(0.9976,1.0047)	(0.9989,1.0047)	(0.9999,1.0047)	(0.9999,1.0047)	(1,1.0047)

Source: MacLean, Zhao and Ziemba (2005)

The control limits are tight in this example. In all cases the lower limit is above the VaR value ρ_1^*. This is important because the VaR strategy is computed with estimated returns distributions. Although the constraint requires falling below the VaR value at most $100\alpha_L$ percent of the time, the actual trajectories drop below VaR much more frequently - around 4 times as often. The control limits will keep the failures close to the planned $100\alpha_L$ percent. The upper limit hits a ceiling at the optimal growth value of 1.0047.

The values in these tables are updated as the portfolio is rebalanced. The computational experiment which tests the risk control methodology consists of generating daily prices for stocks and bonds for one year (260 trading days), using the lognormal model with parameter values in Table 8. The results for VaR strategies with fixed time rebalancing set at every 10 days were presented in Table 10. In contrast, the results for VaR strategies with random time rebalancing determined by the control limits, where the upper limit is the expected VaR wealth after 10 days, are presented in Table 19.

Table 19: Expected Wealth with Control Limits. Source: MacLean, Zhao and Ziemba (2005)

			α_L		
ρ_1	0.01	0.02	0.03	0.04	0.05
0.99	1.0905	1.0799	1.0730	1.0714	1.0726
0.98	1.1050	1.1024	1.1187	1.1227	1.1270
0.97	1.1395	1.1315	1.1547	1.1708	1.1696
0.96	1.1737	1.1700	1.2059	1.2045	1.2085
0.95	1.2156	1.2227	1.2573	1.2802	1.2853

The average terminal wealth when control limits are included is higher in all the scenarios considered. The advantage grows as the value at risk (fallback) decreases. This improved performance is attributable to rebalancing at the right time, that is when the forecast for returns on assets is clearly in error.

6 Legends of Capital Growth

The powerful results for the capital growth approach to investing might imply that it is the obvious methodology to use when planning a strategy. However, it is not common in investment planning and the terminology "growth portfolio" does not usually refer to the Kelly strategy. There are, however, notable practitioners of the optimal growth methodology that are centi-millionaires. Foremost is Edward O.Thorp, who has championed the Kelly strategy in gambling and investment for decades; see his companion chapter that details his experiences and successes and extensive mathematical results on capital growth theory. In this section, some very successful applications of the capital growth strategy are described.

Princeton Newport Partners

In 1969 an investment partnership specializing in convertible hedging was established, with the plan to use the Kelly strategy to allocate assets. Princeton Newport Partners was managed by Edward Thorp, who used the Kelly strategy in the gambling game Blackjack.(Thorp, 1962.) To say PNP was successful is an understatement. PNP found risk-arbitrage opportunities in convertible securities that allowed it to get remarkable returns. (Thorp and Kassouf, 1967; Thorp, 1969). Figure 8 shows the cumulative earnings results between 1968 and 1988.

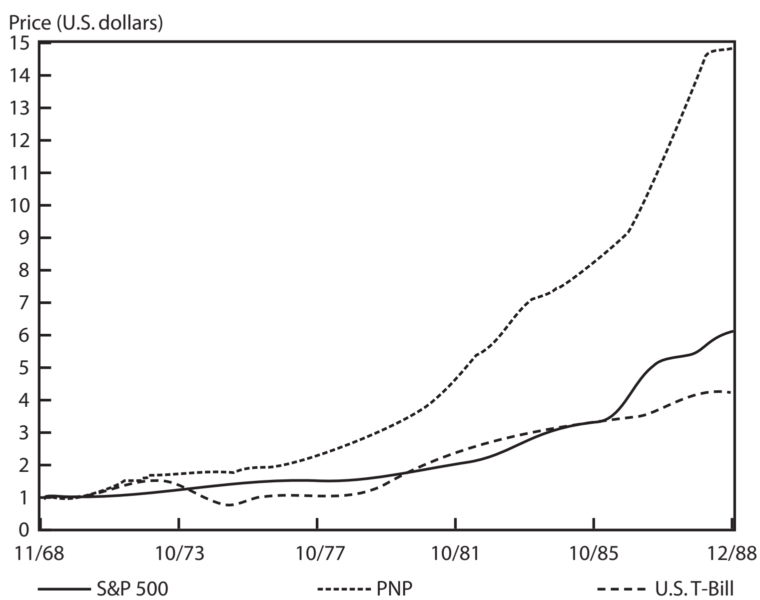

Figure 8: PNP Performance - source: Ziemba (2005)

Thorp's fund had a net mean return of 15.1% and a yearly standad deviation of 4%. The fund had no losing quarters and only 3 losing months.

PNP was closed in 1988. Later Thorp founded two other equally successful funds - Ridgeline Partners (1994 - 2002), and XYZ (1992 - 2002).

Kings College Chest Fund

Another exceptional investment record was achieved by the economist John Maynard Keynes. He ran the Kings College Chest Fund from 1927 until his death in 1945. The fund performance is shown in Figure 9.

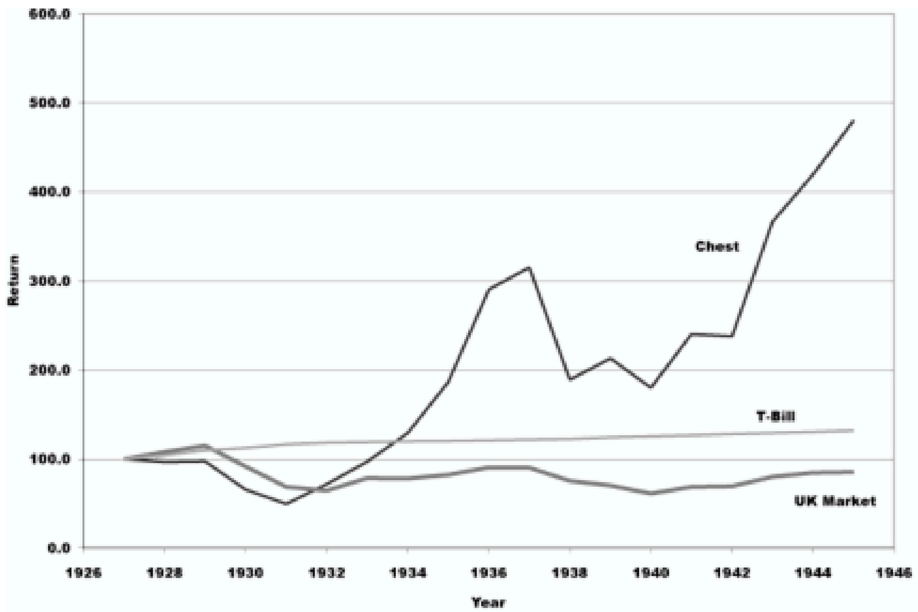

Figure 9: Chest Fund Performance - source: Ziemba (2005)

Keynes lost more than 50% of the fund during the difficult years of the depression. Otherwise the performance was very good but volatile. By 1945, the geometric mean return was 9.12% versus the UK market rate of -0.89%. So Keynes outperformed the market by more than 10% per year. Ziemba (2003) found that The Kings College Chest Fund Performance is well approximated by a fractional Kelly investor, with 80% Kelly, which is equivalent to the negative power utility function $-w^{-0.25}$.

Berkshire -Hathaway

The world's most famous investor is Warren Buffet, who runs the Berkshire - Hathaway Fund. Buffet's investment style is aggressive, with emphasis on value, large holdings and patience. The strong performance of that fund is displayed in Figure 10. The geometric mean of BKH for the 40 years from 1965 to 2004 was 22.02% versus 10% for the S&P500, and $15 invested in BKH in 1965 was worth nearly $90,000 in May 2005. The fund outperformed other well known funds, although the wealth trajectory is more variable. Thorp (2005) indicates that Buffet closely follows a Kelly strategy and this could explain the volatility. Using as a risk measure a modified Sharpe ratio which only considers losses, Ziemba (2005) shows that BH is the only fund in the set:{Ford Foundation, Harvard University Endowment, Quantum, Windsor, Tiger, BH}, whose risk measure improves. However, based on the return over the risk free asset per unit of standard deviation, BH was unable to beat the Ford Foundatioin or the Harvard University Endowment. This confirms that the Kelly capital growth approach must be measured based on long run wealth, not intermediate levels of wealth or their volatility.

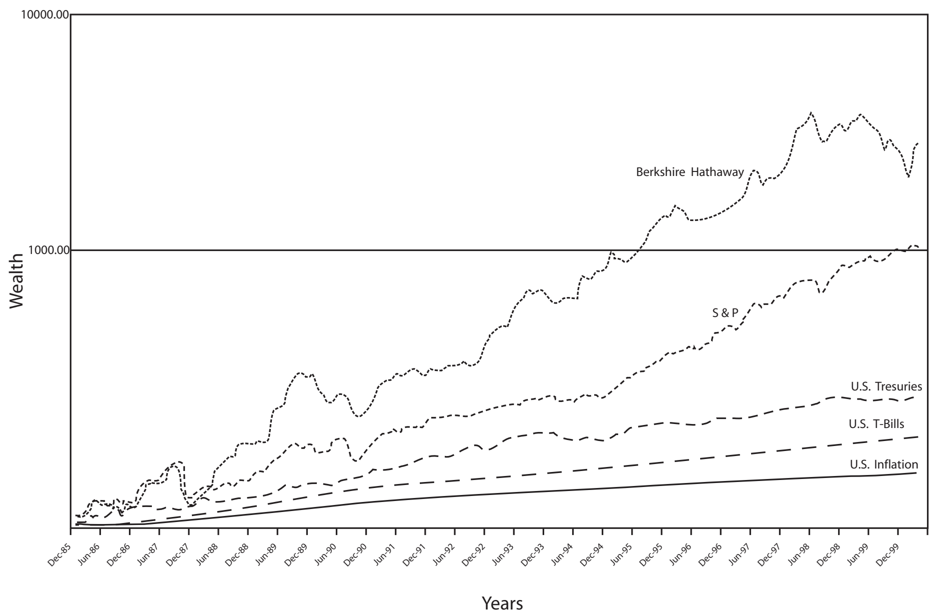

Figure 10: Berkshire - Hathway Performance - source Ziemba (2005)

Hong Kong Betting Syndicate

The world's most successful racetrack bettor is William Benter of the Hong Kong Betting Syndicate (Benter, 1994, 2001). He used a conservative fractional Kelly betting system to wager on horse races in Hong Kong over a 12 year period. The performance, as shown in Figure 11, is similar to the pattern of the other growth investors. The growth rate is variable, but averages around 50% per year. One aspect of this application is the importance of accurate estimates of race odds and returns. The strong returns after 1000 races correspond to improvements in the handicapping system used by Bender.

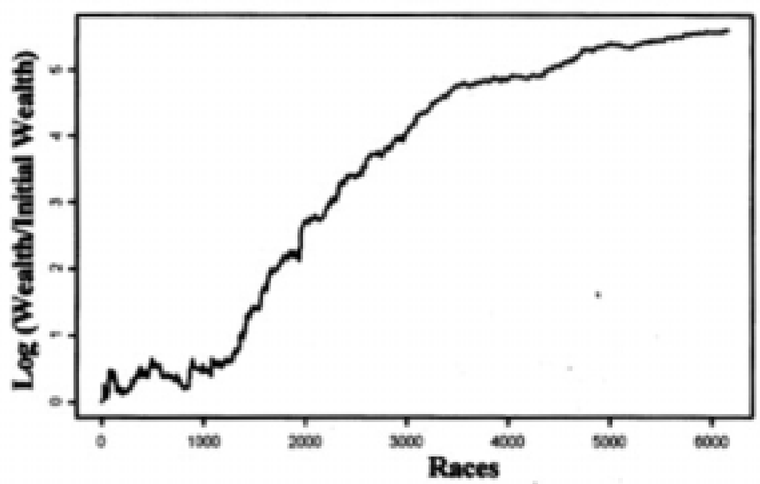

Figure 11: Hong Kong Betting - source Benter (2001)

One significant feature of the applications described in this section is the long planning horizon. In each case, the early performance was not dominant. But the patience of a long run investor paid off with the capital growth approach. The experience gained over time also led to better estimates of conditional expected returns. As the powerful results of Hens and Schenk-Hoppe (2005) have demonstrated, the capital growth strategy overpowers others when the conditional expected returns are correct; see also Breiman (1961) whose results also suggested this property.

References

[1] Algeot, P. and T. Cover (1988). Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. Annals of probability 16, 876-898.

[2] Amir, R., I. Evstigneev, T. Hens and K. Schenk-Hoppe (2005) Market selection and survival of investment strategies. J. of Mathematical Economics 41, 105-122.

[3] Artzner, P., F. Delbaen, J. Eber, and D. Heath (1999). Coherent measures of risk. Mathematical Finance, 9, 203-228.

[4] Aucamp, D. (1993) On the extensive number of plays to achieve superior performance with the geometric mean strategy. Management Science 39, 1163-1172.

[5] Bain, R., D. B. Hausch, and W. T. Ziemba (2005). An application of expert information to win betting on the Kentucky derby, 1981-2005. Working paper, University of British Columbia.

[6] Basak, S. and A. Shapiro (2001). Value at Risk based risk management: optimal policies and asset prices. Review of Financial Studies.

[7] Bell, R. M. and T. M. Cover (1980). Competitive optimality of logarithmic investment. Math. of Operations research 5, 161-166.

[8] Benter, W. (1994). Computer based horse race handicapping and wagering systems: a report. Efficiency of Racetrack Betting Markets, Hausch, Lo and Ziemba, eds. Academic Press, San Diego, 183-198.

[9] Benter, W. (2001) Development of a mathematical model for successful horse race wagering, presentation at the University of Nevada, Las Vegas.

[10] Blake,D., B.N. Lehmann, and A. Timmermann (1999). Asset allocation dynamics and pension fund performance. Journal of Business 72,429-461.

[11] Bordley, R. and LiCalzi, M. (2000). Decision analysis using targets instead of utility functions. Decisions in Economics and Finance, 23, 53-74.

[12] Brennan, M.J. (1998). The role of learning in dynamic portfolio decisions. European Finance Review I, 295-306.

[13] Brieman, L. (1960). Investment policies for expanding business optimal in a long-run sense. Naval Research Logistics Quarterly 7, 647-651.

[14] Brieman, L. (1961) Optimal gambling system for favorable games, in Proc 4th Berkeley Symposium on Mathematics, Statistics and Probability 1: 63-68

[15] Browne, S. (1997). The return on investment from proportional investment strategies. Advances in Applied Probability 29.

[16] Browne, S. and W. Whitt (1996). Portfolio choice and the Bayesian Kelly criterion. Adv. Appl. Prob., 28, 1145-1176.

[17] Burkhardt, T. (1998). A mean-variance of first passage time approach to portfolio selection in a lognormal world. Presentation at The VIII International Conference on Stochastic Programming, Vancouver, August.

[18] Campbell, J., Lo, A. and C. MacKinlay (1997) The Econometrics of Financial Markets. Princeton University Press, Princeton, NJ.

[19] Chopra,V. and W. T. Ziemba (1993). The effect of errors in the mean, variance, and covariance estimates on optimal portfolio choice. Journal of Portfolio Management, Winter,6-11.

[20] Chernov, M., Gallant, A.R., Ghysels, E., and G. Tauchen (2002). Alternative Models for Stock Price Dynamics. Working paper, Columbia Business School, Columbia University.

[21] Clark, R. and W. T. Ziemba (1987). Playing the turn of the year effect with index futures. Operations Research, 35, 799-813.

[22] Consigli, G., MacLean, L. C., Zhao, Y., and W. T. Ziemba (2005). Modelling market returns with a speculative bubble. submitted to J. of Banking and Finance.

[23] De Giorgio, E. (2005). Reward-risk portfolio selection and stochastic dominance. Journal of Banking and Finance 29(4), 895-926.

[24] Dohi, T., H. Tanaka, N. Kaio, and S. Osaki (1995). Alternative growth versus security in continuous dynamic trading. European Journal of Operations Research 84, 430 - 443.

[25] Ethier, S. N. and S. Tavare (1983). The proportional bettor's return on investment. Journal of Applied Probability 20, 563-573.

[26] Fleming, W.H. and R. W. Rishel (1975), Deterministic and stochastic optimal control. Springer-Verlag, New York.

[27] Grauer, R.R., and N. H. Hakansson (1986). A half century of returns on levered and unlevered portfolios of stocks, bonds and bills, with and without small stocks. Journal of Business, 59.2, 287-318.

[28] Grauer, R.R., and N. H. Hakansson (1987). Gains from international diversification: 1968-85 returns on portfolios of stocks and bonds. Journal of Finance, 42, 721-739.

[29] Griffin, P. (1985). Different measures of win rates for optimal proportional betting. Management Science 20, 1540-1547.

[30] Grossman, S.J. and Z. Zhou, (1993) Optimal investment strategies for controlling drawdowns. Mathematical Finance 3, 241-276.

[31] Hausch, D. B. and W. T. Ziemba (1995). Efficiency of sports and lottery betting markets in Finance, Jarrow, R., Maksimovic, V and Ziemba, W. eds. North Holland Handbooks in Operations Research and Management Science, 545 - 580.

[32] Hausch, D. B. and Ziemba, W. T. and M. Rubinstein (1981). Efficiency of the markets for racetrack betting. Management Science 27, 1435 - 1452.

[33] Hakansson, N.H. (1970). Optimal investment and consumption strategies under risk for a class of utility functions. Econometrica 38, 587 - 607.

[34] Hakansson, N.H. (1971). On optimal myopic portfolio policies, with and without serial correlation of yields. Journal of Business 44, 324 - 334.

[35] Hakansson, N.H., and W. T. Ziemba (1995). Capital growth theory, in Finance, R.A. Jarrow, V. Maksimovic and W.T. Ziemba, Eds, North Holland, 65-86.

[36] Hanoch, G. and H. Levy (1969). The efficiency analysis of choices involving risk. Review of Economic Studies 36, 335-346.

[37] Hens, T. and K. Schenk-Hoppe (2005). Evolutionary stability of portfolio rules in incomplete markets. J. of Mathematical Economics 41, 43-66.

[38] Hensel, C.R., D.D. Ezra and J.H. Ilkiw (1991) The importance of the asset allocation decision. The Financial Analysts Journal (July-August), 65-72.

[39] Hensel, C. R. and W. T. Ziemba (2000). Anticipation of the January small firm effect in the US futures markets. in Security Market Imperfections in Worldwide Equity Markets, Kiem, D. and W. T. Ziemba (Eds). Cambridge University Press, 179-202.

[40] Janacek, K. (1998). Optimal Growth in Gambling and Investing. M. Sc. Thesis, Charles University, Prague.

[41] Jorion, J. (1986). Bayes-Stein estimation for portfolio analysis. Journal of Financial and Quantitative Analysis 21, 279 - 329.

[42] Jorion,J.(1997) Value-at-Risk: The Benchmark for Controlling Market Risk. Irwin, Chicago.

[43] Kallberg, J. G., and W. T. Ziemba (1981)Remarks on optimal portfolio selection. Methods of Operations Research 44, 507-520.

[44] Kallberg, J.G., and W. T. Ziemba (1983). Comparison of alternative utility functions in portfolio selection problems, Management Science 29, 1257-1276.

[45] Kallberg, J.G., and W. T. Ziemba (1984). Mis-specification in portfolio selection problems. in Risk and capital, Bamberg and Spreemann, Eds, Springer-Verlag, 74-87.

[46] Karatzas, I. and S. Shreve (1988). Brownian motion and stochastic calculus, Springer-Verlag, New York.

[47] Kelly, J. (1956). A new interpretation of information rate. Bell System Technology Journal 35, 917-926.

[48] Lawley, D.N. and A. E. Maxwell (1971). Factor analysis as a statistical method. Butterworths, London.

[49] Levy, H. (1998). Stochastic Dominance. Kluwer Academic Publishers. Norwell, MA.

[50] Lorenz, M. (1905). Methods for Measuring the Concentration of Wealth. J. Amer. Stat. Assoc. 9, 209-219.

[51] Luenberger, D.G. (1998). Investment Science, Oxford University Press, New York.

[52] Luenberger, D.G. (1993). A Preference Foundation for Log Mean-Variance Criteria in Portfolio choice Problems. Journal of Economic Dynamics and Control, 17, 887-906.

[53] MacLean, L. C., M.E. Foster and W. T. Ziemba (2002). Empirical Bayes estimation of securities price parameters. in Financial Engineering, E-Commerce and Supply Chain, P. Paradolos, Ed., Kluwer, Boston, MA., pp 47-57.

[54] MacLean, L. C., M. E. Foster and W. T. Ziemba (2005) Empirical Bayes estimation with dynamic portfolio models, Technical Report. Sauder School of Business, UBC, Vancouver.

[55] MacLean, L., C., R. Sanegre, Y. Zhao and W. T. Ziemba (2004). Capital Growth with Security. J. of Economic Dynamics and Control 28, 937-954.

[56] MacLean, L.C. and K. L. Weldon (1996). Estimating multivariate random effects without replication. Commun. Statist. Theory Meth. 24, 1447-1469.

[57] MacLean, L. C., Y. Zhao, and W. T. Ziemba (2005). Dynamic portfolio selection with process control. J. of Banking and Finance, in press.

[58] MacLean, L.C. and W. T. Ziemba (1991). Growth -Security profiles in capital accumulation under risk. Annals of Operations Research 31, 501 - 510.

[59] MacLean, L.C. and W. T. Ziemba (1999). Growth versus security tradeoffs in dynamic investment analysis. Annals of Operations Research 85, 193-225.

[60] MacLean, L.C. and W. T. Ziemba (2000). Efficiency concepts in capital accumulation models. Philosophical Transactions of the Royal Society A, v357, 2029 - 2037.

[61] MacLean, L.C., W. T. Ziemba, and G. Blazenko (1992). Growth versus security in dynamic investment analysis. Management Science 38, 1562-1585.

[62] MacLean. L., Ziemba, W., and Y. Li (2005). Time to wealth goals in capital accumulation. Quantitative Finance, in press.

[63] Markowitz, H.M. (1952). Portfolio Selection. Journal of Finance, 7, 77-91.

[64] Markowitz, H.M. (1976). Investment for the long run: New evidence for an old rule. Journal of Finance, 31.5, 1273-1286.

[65] Markowitz, H.M. (1987). Mean-variance analysis in portfolio choice and capital markets. Basil Blackwell, New York.

[66] Merton, Robert C. (1990). Continuous Time Finance. Malden, MA Blackwell Publishers Inc.

[67] Mulvey, J. and H. Vladimirow (1992). Stochastic network programming for financial planning problems. Management Science 38:11, 1642 - 1664.

[68] Ogryczak, W. and A. Ruszczynski (2002). Dual stochastic dominance and related mean-risk models. Siam J. Optimization 13, 60 - 78.

[69] Pratt, J.W. (1964). Risk Aversion in the small and in the large, Econometrica.

[70] Rendon, J. and W. T. Ziemba (2005). Is the January effect still alive in the futures market? working paper, Sauder School of Business, UBC, Vancouver, Canada.

[71] Rockafellar, R.T. and S. Uryasev. (2000). Optimization of Conditional value-at-risk. Journal of Risk 2, 21-41.

[72] Rogers, L.C.G. (2001) The relaxed investor and parameter uncertainty. Finance and Stochastics, 5, 131-154.

[73] Rotando, L. M. and E. O. Thorp (1992). The Kelly criterion and the stock market. American Mathematical Monthly, Dec,. 922-931.

[74] Rubinstein, M. (1976). The strong case for the generalized logarithmic utility model as the premier model of financial markets. Journal of Finance 31, 551-571.

[75] Samuelson, P.A. (1971) The fallacy of maximizing the geometric mean in long sequences of investing or gambling. Proceedings National Academy of Science 68: 2493-2496

[76] Stutzer, M. (2003). Portfolio choice with endogenous utility: a large deviations approach. Journal of Econometrics 116, 365 - 386.

[77] Stutzer, M. (2004). Asset allocation without unobservable parameters. Financial Analysts Journal 60:5, 38 - 51.

[78] Thorp, E. O. (1962). Beat The dealer. Random House, New york.

[79] Thorp, E.O. (1975). Portfolio choice and the Kelly criterion. In: Stochastic Optimization Models in Finance (eds. W.T. Ziemba and R.G. Vickson). Academic Press, New York, 599-619. reprinted from (1971) paper in Proceedings of the American Statistical Association, Business and Economic Statistics Section, 215 -224.

[80] Thorp, E. O. and S. T. Kassouf (1967). Beat The Market. Random House, New york.

[81] Wilcox, J. (2003). Harry Markowitz and the discretionary wealth hypothesis. Journal of Portfolio Management, 29(3), 58-65.

[82] Wilcox, J. (2005). A Better Paradigm for Finance. Finance Letters 3(1), 5-11.

[83] Xia, Y. (2001) Learning about predictability: the effects of parameter uncertainty on dynamic asset allocation. Journal of Finance, 56, 205246.

[84] Ziemba, W. T. (1994). Investing in the Turn-of-the-Year Effect in the US Futures market. Interfaces 24(3), 46-61.

[85] Ziemba, W. T. (2003) The Stochastic Programming Approach to Asset, Liability, and Wealth Management. AMIR, Virginia, USA.

[86] Ziemba, W. T. (2005) The symmetric downside-risk Sharpe ratio and the evaluation of great investors and speculators. Journal of Portfolio Management, Fall, in press.

[87] Ziemba, W. T., S. L. Brumelle, A. Gautier and S. L. Schwartz(1986). Dr. Z's 6/49 Guidebook. Dr. Z Investments, Inc., San Luis Obispo, Ca.

[88] Ziemba, W. T. and D. B. Hausch (1986). Betting at the Racetrack. Dr. Z Investments, Inc., San Luis Obispo, Ca.

[89] Ziemba, W. T. and D. B. Hausch (1987) Dr. Z's Beat The Racetrack. William Morrow, New York. Update of (1984).Beat the Racetrack. Harcourt, Brace and Jovanovich, San Deigo.