SET-1

MATHEMATICS

Series ONS

Paper & Solution

Code: 65/1/C

Time: 3 Hrs.

Max. Marks: 100

General Instructions:

(i) ***All*** *questions are* ***compulsory****.*

(ii) Please check that this question paper contains ***26*** *questions.*

(ii) Questions ***1- 6*** *in Section A are very short-answer type questions carrying* ***1*** *mark each.*

(iv) Questions ***7 - 19*** *in Section B are long-answer* ***I*** *type questions carrying* ***4*** *marks each.*

(v) Questions ***20 - 26*** *in Section C are long-answer* ***II*** *type questions carrying* ***6*** *marks each.*

(vi) Please write down the serial number of the question before attempting it.

SECTION - A

1. If x $\in$ N and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x.

Solution:

$$(x+3)2x-(-2)(-3x)=8$$
$$x=2$$

2. Use elementary column operation $C_2 \to C_2 + 2C_1$ in the following matrix equation:

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2 & 5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

3. Write the number of all possible matrices of order 2 x 2 with each entry 1, 2 or 3.

Solution:

$$\left.\begin{aligned} \text{No. of possible matrices} &= 3^4 \\ &\text{or } 81 \end{aligned}\right\}$$

4. Write the position vector of the point which divides the join of points with position vectors $3\vec{a}-2\vec{b}$ and $2\vec{a}+3\vec{b}$ in the ratio 2 : 1.

Solution:

$$\frac{2\left(2\vec{a}+3\vec{b}\right)+1\left(3\vec{a}-2\vec{b}\right)}{2+1}$$

$=\frac{7}{3}\vec{a}+\frac{4}{3}\vec{b}$ (or external division may also be considered)

5. Write the number of vectors of unit length perpendicular to both the vectors $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $b=\hat{j}+\hat{k}$

Solution:

2

6. Find the vector equation of the plane with intercepts 3, -4 and 2 on x, y and z-axis respectively.

Solution:

$$\frac{x}{3}+\frac{y}{-4}+\frac{z}{2}=1$$

$$\Rightarrow \vec{r}.\left(4\hat{i}-3\hat{j}+6\hat{k}\right)=12 \text{ or } \vec{r}.\left(\frac{\hat{i}}{3}-\frac{\hat{j}}{4}+\frac{\hat{k}}{2}\right)=1$$

SECTION - B

7. Find the coordinates of the point where the line through the points A(3, 4, 1) and B(5, 1, 6) crosses the XZ plane. Also find the angle which this line makes with the XZ plane.

Solution:

Equation of line through A(3, 4, 1) and B(5, 1, 6)

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5}=K\left(\text{say}\right)$$

General point on the line:

$x=2k+3,\ y=-3k=4,\ z=5k+1$

line crosses xz plane i.e. y = 0 if –3k + 4 = 0

$$\because k=\frac{4}{3}$$

Co-ordinate of required point $\left(\frac{17}{3},0,\frac{23}{3}\right)$

Angle, which line makes with xz plane:

$$\sin\theta = \left|\frac{2(0)+(-3)(1)+5(0)}{\sqrt{4+9+25}\sqrt{1}}\right| = \frac{3}{\sqrt{38}} \Rightarrow \theta = \sin^{-1}\left(\frac{3}{\sqrt{38}}\right)$$

8. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.

Solution:

$\vec{d}_1$ & $\vec{d}_2$ be the two diagonal vectors:

$\therefore \vec{d}_1 = 4\hat{i}-2\hat{j}-2\hat{k}, \vec{d}_2 = -6\hat{j}-8\hat{k}$

or $\vec{d}_2 = 6\hat{j}+8\hat{k}$

Unit vectors parallel to the diagonals are:

$$\hat{d}_1 = \frac{2}{\sqrt{6}}\hat{i}-\frac{1}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$$

$$\hat{d}_2 = -\frac{3}{5}\hat{j}-\frac{4}{5}\hat{k} \quad \left(\text{or } \hat{d}_2 = \frac{3}{5}\hat{j}+\frac{4}{5}\hat{k}\right)$$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & -6 & -8 \end{vmatrix} = 4\hat{i}+32\hat{j}-24\hat{k}$$

Area of parallelogram $= \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right| = \sqrt{404}$ or $2\sqrt{101}$ sq.units

9. In a game, a man wins ₹ 5 for getting a number greater than 4 and loses ₹ 1 otherwise, when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a number greater than 4. Find the expected value of the amount he wins/loses.

OR

A bag contains 4 balls. Two balls are drawn at random (without replacement) and are found to be white. What is the probability that all balls in the bag are white?

Solution:

let X = Amount he wins then x = ₹ 5, 4, 3, – 3

P = Probability of getting a no. $>4 = \frac{1}{3}, q = 1-p = \frac{2}{3}$

X:	5	4	3	-3
P(x)	$\frac{1}{2}$	$\frac{2}{3}.\frac{1}{3}=\frac{2}{9}$	$\left(\frac{2}{3}\right)2.\frac{1}{3}=\frac{4}{27}$	$\left(\frac{2}{3}\right)^3=\frac{8}{27}$

Expected amount he wins = $\sum XP(X) = \frac{5}{3} + \frac{8}{9} + \frac{12}{27} - \frac{24}{27}$

= ₹ $\frac{19}{9}$ or ₹ $2\frac{1}{9}$

OR

$$\left.\begin{array}{l} E_1\text{=Event that all balls are white,} \\ E_2\text{=Event that 3 balls are white and 1 ball is non white} \\ E_3\text{=Event that 2 balls are white and 2 balls are non white} \\ \text{A=Event that 2 balls drawn without replacement are white} \end{array}\right\}$$

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A/E_1) = 1,\ P(A/E_2) = \frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2},\ P(A/E_3) = \frac{2}{4}\cdot\frac{1}{3} = \frac{1}{6}$$

$$P(E_1/A) = \frac{1.\ {}^1\!/_3}{1\cdot {}^1\!/_3 + {}^1\!/_3 \cdot {}^1\!/_2 + {}^1\!/_3 \cdot {}^1\!/_6} = \frac{3}{5}$$

10. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.

OR

If y = 2 cos (logx) + 3 sin (logx), prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$

Solution:

let $y = u + v, u = x^{\sin x}, v = (\sin x)^{\cos x}$

$$\log u = \sin x.\log x \Rightarrow \frac{du}{dx} = x^{\sin x}.\left\{\cos x.\log x + \frac{\sin x}{x}\right\}$$

$$\log v = \cos x.\log(\sin x) \Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\cdot\{\cos x\cdot\cot x - \sin x\cdot\log(\sin x)\}$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\sin x}\cdot\left\{\cos x.\log x + \frac{\sin x}{x}\right\} + (\sin x)^{\cos x}\{\cos x\cdot\cot x - \sin x\cdot\log(\sin x)\}$$

OR

$$\frac{dy}{dx} = \frac{-2\sin(\log x)}{x} + \frac{3\cos(\log x)}{x}$$

$\Rightarrow x\frac{dy}{dx} = -2\sin(\log x) + 3\cos(\log x)$, Differentiate w.r.t 'x'

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-2\cos(\log x)}{x} - \frac{3\sin(\log x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

11. If x = a sin 2t(1 + cos 2t) and y = b cos 2t(1 – cos 2t), find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

Solution:

$$\frac{dx}{dt} = 2a\cos 2t\,(1+\cos 2t) - 2a\sin 2t \cdot \sin 2t$$

$$\frac{dy}{dt} = -2b\sin 2t\,(1-\cos 2t) + 2b\cos 2t \cdot \sin 2t$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \left.\frac{2b\cos 2t \cdot \sin 2t - 2b\sin 2t(1-\cos 2t)}{2a\cos 2t\,(1+\cos 2t) - 2a\sin 2t \cdot \sin 2t}\right]_{t=\frac{\pi}{4}} = \frac{b}{a}$$

12. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is y = 4x – 5. Find the values of a and b.

Solution:

$$y^2 = ax^3 + b \Rightarrow 2y\frac{dy}{dx} = 3ax^2 \ \therefore \frac{dy}{dx} = \frac{3a}{2}\frac{x^2}{y}$$

Slope of tangent at (2, 3) = $= \left.\frac{dy}{dx}\right]_{(2,3)} = \frac{3a}{2}.\frac{4}{3} = 2a$

Comparing with slope of tangent y = 4x – 5, we get, 2a = 4 $\therefore \boxed{a = 2}$

Also (2, 3) lies on the curve $\therefore$ 9 = 8a + b, put a = 2, we get b = – 7

13. Find: $\int \frac{x^2}{x^4 + x^2 - 2}$

Solution:

Let $x^2 = t$ $\therefore \frac{x^2}{x^4 + x^2 - 2} = \frac{x^2}{(x^2-1)(x^2+2)} = \frac{t}{(t-1)(t+2)} = \frac{A}{t-1} + \frac{B}{t+2}$

Solving for A and B to get, $A = \frac{1}{3}, B = \frac{2}{3}$

$$\int \frac{x^2}{x^4 + x^2 - 2}\,dx = \frac{1}{3}\int \frac{1}{x^2-1}\,dx + \frac{2}{3}\int \frac{1}{x^2+2}\,dx = \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

14. $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin x + \cos x}$

OR

Evaluate: $\int_0^{\frac{3}{2}} |x\cos \pi x|\,dx$

Solution:

Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx$, Also $I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx = \int_0^{\pi/2}\frac{\cos^2 x}{\cos x + \sin x}dx$

Adding to get, $2I = \int_0^{\pi/2}\frac{1}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{1}{\cos\left(x-\frac{\pi}{4}\right)}dx$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\sec\left(x-\pi/4\right)dx = \frac{1}{\sqrt{2}}\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|_0^{\pi/2}$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}}\left\{\log|\sqrt{2}+1|-\log|\sqrt{2}-1|\right\}$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}}\left\{\log|\sqrt{2}+1|-\log|\sqrt{2}-1| \text{ or } \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right\}$$

OR

$$\int_0^{3/2}|x\cos\pi x|\,dx = \int_0^{1/2}x\cos\pi x\,dx - \int_{1/2}^{3/2}x\cos\pi x\,dx$$

$$= \left\{\frac{x\sin\pi x}{\pi}+\frac{\cos\pi x}{\pi^2}\right\}_0^{1/2} - \left\{\frac{x\sin\pi x}{\pi}+\frac{\cos\pi x}{\pi^2}\right\}_{1/2}^{3/2}$$

$$= \frac{1}{2\pi}-\frac{1}{\pi^2}-\left(-\frac{3}{2\pi}-\frac{1}{2\pi}\right) = \frac{5}{2\pi}-\frac{1}{\pi^2}$$

15. Find: $\int(3x+1)\sqrt{4-3x-2x^2}\,dx$

Solution:

$$\int(3x+1)\sqrt{4-3x-2x^2}\,dx = -\frac{3}{4}\int(-4x-3)\sqrt{4-3x-2x^2}dx - \frac{5}{4}\int\sqrt{4-3x-2x^2}\,dx$$

$$= -\frac{1}{2}\left(4-3x-2x^2\right)^{3/2} - \frac{5}{4}\sqrt{2}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$= -\frac{1}{2}(4-3-2x^2)^{3/2} - \frac{5}{4}\sqrt{2}\left\{\frac{4x+3}{8}\sqrt{\frac{41}{16}-\left(x+\frac{3}{4}\right)^2}+\frac{41}{32}\cdot\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right\}+C$$

$$= -\frac{1}{2}(4-3x-2x^2)^{3/2} - \frac{5}{4}\left\{\frac{4x+3}{8}\sqrt{4-3x-2x^2}+\frac{41\sqrt{2}}{32}\cdot\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right\}+C$$

16. Solve the differential equation:

$$y + x\frac{dy}{dx} = x - y\frac{dy}{dx}$$

Solution:

The differential equation can be re-written as:

$$\frac{dy}{dx} = \frac{x-y}{x+y}, \text{ put } y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1-v}{1+v} \Rightarrow \frac{1+v}{1-2v-v^2}dv = \frac{1}{x}dx$$

Integrating we get

$$\Rightarrow \frac{1}{2}\int \frac{2V+2}{V^2+2V-1}dv = -\int \frac{1}{x}dx = \frac{1}{2}\log|V^2+2V-1| = -\log x + \log C$$

$\therefore$ Solution of the differential equation is:

$$\frac{1}{2}\log\left|\frac{y^2}{x^2} + \frac{2y}{x} - 1\right| = \log C - \log x \text{ or, } y^2 + 2xy - x^2 = C^2$$

17. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Solution:

Let radius of any of the circle touching co-ordinate axes in the second quadrant be "a" then centre is (–a, a)

$\therefore$ Equation of the family of circles is:

$$(x+a)^2 + (y-a)^2 = a^2, a \in R$$

$$\Rightarrow x^2 + y^2 + 2ax - 2ay + a^2 = 0$$

Differentiate w.r.t. "x", 2x + 2yy′ + 2a – 2ay′ = 0 $\Rightarrow a = \dfrac{x+yy'}{y'-1}$

$\therefore$ the differential equation is:

$$\left.\begin{array}{l}\left(x + \dfrac{x+yy'}{y'-1}\right)^2 + \left(y - \dfrac{x+yy'}{y'-1}\right)^2 = \left(\dfrac{x+yy'}{y'-1}\right)^2 \\ \Rightarrow \left(\dfrac{xy'+yy'}{y'-1}\right)^2 + \left(\dfrac{x+y}{y'-1}\right)^2 = \left(\dfrac{x+yy'}{y'-1}\right)^2\end{array}\right\}$$

18. Solve the equation for x : $\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$

OR

If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, prove that $\frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

Solution:

$$\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x\Rightarrow\sin^{-1}(1-x)=\frac{\pi}{2}-2\sin^{-1}x$$

$$\Rightarrow 1-x=\sin\left(\frac{\pi}{2}-2\sin^{-1}x\right)\Rightarrow 1-x=\cos(\sin^{-1}x)\Rightarrow 1-x=1-2\sin^2(\sin^{-1}x)$$

$$\Rightarrow 1-x=1-2x^2$$

Solving we get, x = 0 or $x=\frac{1}{2}$

OR

From the equation: $\cos^{-1}\frac{x}{a}=\alpha-\cos^{-1}\frac{y}{b}$

$$\frac{x}{a}=\cos\left(\alpha-\cos^{-1}\frac{y}{b}\right)\Rightarrow\frac{x}{a}=\cos\alpha\cdot\cos\left(\cos^{-1}\frac{y}{b}\right)+\sin\alpha\cdot\sin\left(\cos^{-1}\frac{y}{b}\right)$$

$$\Rightarrow\frac{x}{a}=\frac{y.\cos\alpha}{b}+\sin\alpha\sqrt{1-\frac{y^2}{b^2}}\Rightarrow\frac{x}{a}-\frac{y}{b}\cos\alpha=\sin\alpha\sqrt{1-\frac{y^2}{b^2}}$$

Squaring both sides,

$$\Rightarrow\left(\frac{x}{a}-y\frac{\cos\alpha}{b}\right)^2=\left(\sin\alpha\sqrt{1-\frac{y^2}{b^2}}\right)^2$$

$$\Rightarrow\frac{x^2}{a^2}-\frac{2xy}{ab}\cdot\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

19. A trust invested some money in two type of bonds. The first bond pays 10% interest and second bond pays 12% interest. The trust received ₹ 2,800 as interest. However, if trust had interchanged money in bonds, they would have got ₹ 100 less as interest. Using matrix method, find the amount invested by the trust. Interest received on this amount will be given to Help age India as donation. Which value is reflected in this question ?

Solution:

let ₹ x be invested in first bond

and ₹ y be invested in second bond

then the system of equations is:

$$\left.\begin{array}{l}\dfrac{10x}{100}+\dfrac{12y}{100}=2800\\[2mm] \dfrac{12x}{100}+\dfrac{10y}{100}=2700\end{array}\right\}\Rightarrow \left.\begin{array}{l}5x+6y=140000\\ 6x+5y=135000\end{array}\right\}$$

Let $A=\begin{bmatrix}5 & 6\\ 6 & 5\end{bmatrix};\ X=\begin{bmatrix}x\\ y\end{bmatrix};\ B=\begin{bmatrix}140000\\ 135000\end{bmatrix}$

$\therefore A\cdot X=B$

$|A|=-11;\ A^{-1}=\dfrac{1}{-11}\begin{bmatrix}5 & -6\\ -6 & 5\end{bmatrix}$

$\therefore$ Solution is $\left.\begin{array}{l}X=A^{-1}B\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\dfrac{1}{-11}\begin{bmatrix}5 & -6\\ -6 & 5\end{bmatrix}\begin{bmatrix}140000\\ 135000\end{bmatrix}=\begin{bmatrix}10000\\ 15000\end{bmatrix}\\ \therefore x=10000,\ y=15000,\ \therefore \text{Amount invested}=25000\text{ rupees}\end{array}\right\}$

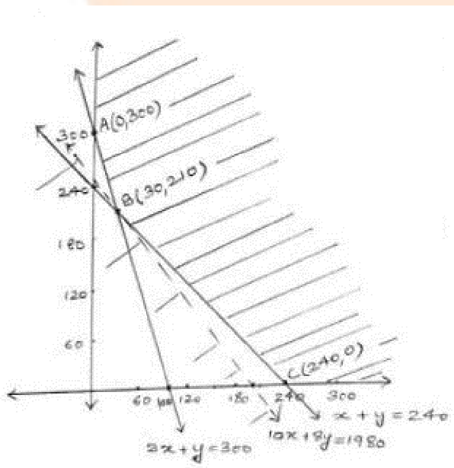
Value: caring elders

SECTION – C

20. There are two types of fertilisers 'A' and 'B'. 'A' consists of 12% nitrogen and 5% phosphoric acid whereas 'B' consists of 4% nitrogen and 5% phosphoric acid. After testing the soil conditions, farmer finds that he needs at least 12 kg of nitrogen and 12 kg of phosphoric acid for his crops. If 'A' costs? ₹ 10 per kg and 'B' cost ₹ 8 per kg, then graphically determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost.

Solution:

Let x kg of fertilizer A be used and y kg of fertilizer B be used then the linear programming problem is:
Minimize cost: z = 10x + 8y



$$\left.\begin{array}{l}\text{Subject to } \dfrac{12x}{100}+\dfrac{4y}{100}\geq 12 \Rightarrow 3x+y\geq 300\\ \qquad\qquad\ \dfrac{5x}{100}+\dfrac{5y}{100}\geq 12 \Rightarrow x+y\geq 240\\ \qquad\qquad\ x,\ y\geq 0\end{array}\right\}$$

Correct Graph

Value of Z at corners of the unbounded region ABC:

Corner	Value of Z
A(0,300)	2400 rupees
B(30, 210)	1980 rupees (Minimum)
C(240, 0)	2400 rupees

The region of 10x + 8y < 1980 or 5x + 4y < 990 has no point in common to the feasible region. Hence, minimum cost = ₹ 1980 at x = 30 and y = 210

21. Five bad oranges are accidently mixed with 20 good ones. If four oranges are drawn one by one successively with replacement, then find the probability distribution of number of bad oranges drawn. Hence find the mean and variance of the distribution.

Solution:

Let X = Number of bad oranges out of 4 drawn = 0, 1, 2, 3, 4

P = Probability of a bad orange = $\dfrac{1}{5}, q=1-p=\dfrac{4}{5}$

$\therefore$ Probability distribution is:

X:	0	1	2	3	4
P(X):	${}^4C_0\left(\frac{4}{5}\right)^4=\frac{256}{625}$	${}^4C_1\,\frac{1}{5}\left(\frac{4}{5}\right)^3$	${}^4C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2$	${}^4C_3\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)$	${}^4C_4\left(\frac{1}{5}\right)^4$
		$=\frac{256}{625}$	$=\frac{96}{625}$	$=\frac{16}{625}$	$=\frac{1}{625}$

$$\text{Mean } (\mu)=\Sigma X.P(X)=0\times\frac{256}{625}+1\times\frac{256}{625}+2\times\frac{96}{625}+3\times\frac{16}{625}+4\times\frac{1}{625}=\frac{4}{5}$$

$$\text{Variance } (\sigma^2)=\Sigma x^2.P(x)-[\Sigma x.P(x)]^2$$

$$=0\times\frac{256}{625}+\frac{1\times 256}{625}+\frac{4\times 96}{625}+\frac{9\times 16}{625}+\frac{16}{625}-\left(\frac{4}{5}\right)^2=\frac{16}{25}$$

22. Find the position vector of the foot of perpendicular and the perpendicular distance from the point P with position vector $2\hat{i}+3\hat{j}+4\hat{k}$ to the plane. $\vec{r}.(2\hat{i}+\hat{j}+3\hat{k})-26=0$. Also find image of P in the plane.

Solution:

Line through 'P' and perpendicular to plane is:

$\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+\lambda(2\hat{i}+\hat{j}+3\hat{k})$

General point on line is: $\vec{r}=(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}$

For some $\lambda \in$ R, $\vec{r}$ is the foot of perpendicular, say Q, from P to the plane, since it lies on plane

$\therefore\ [(2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}]\cdot(2\hat{i}+\hat{j}+3\hat{k})-26=0$

$\Rightarrow 4+4\lambda+3+\lambda+12+9\lambda-26=0 \Rightarrow \lambda=\frac{1}{2}$

$\therefore$ Foot of perpendicular is $Q\left(3\hat{i}+\frac{7}{2}\hat{j}+\frac{11}{2}\hat{k}\right)$

let $P'(a\hat{i}+b\hat{j}+c\hat{k})$ be the image of P in the plane then Q is midpoint of PP$'$

$\therefore\ \ Q\left(\frac{a+2}{2}\hat{i}+\frac{b+3}{2}\hat{j}+\frac{c+4}{2}\hat{k}\right)=Q\left(3\hat{i}+\frac{7}{2}\hat{j}+\frac{11}{2}\hat{k}\right)$

$\Rightarrow\ \ \frac{a+2}{2}=3, \frac{b+3}{2}=\frac{7}{2}, \frac{c+4}{2}=\frac{11}{2} \Rightarrow a=4, b=4, c=7 \quad P'(4\hat{i}+4\hat{j}+7\hat{k})$

Perpendicular distance of P from plane $=PQ=\sqrt{(2-3)^2+\left(3-\frac{7}{2}\right)^2+\left(4-\frac{11}{2}\right)^2}=\sqrt{\frac{7}{2}}$

23. Show that the binary operation * on A= **R** — {—1} defined as a*b = a + b + ab for all a, b $\in$ A is commutative and associative on A. Also find the identity element of * in A and prove that every element of A is invertible.

Solution:

Commutative: For any elements a, b $\in$ A

a * b = a + b + ab = b + a + ba = b * a. Hence * is commutative

Associative: For any three elements a, b, c, $\in$ A

a * (b * c) = a * (b + c + bc) = a + b + c + bc + ab + ac + abc

(a * b) * c = (a + b + ab) * c = a + b + ab + c + ac + bc + abc

$\therefore$ a * (b * c) = (a * b) * c, Hence * is Associative.

Identity element: let e $\in$ A be the identity element them a * e = e * a = a

$\Rightarrow$ a + e + ae = e + a + ea = a $\Rightarrow$ e (1 + a) = 0, as a $\neq$ – 1

e = 0 is the identity element

Invertible: let a, b $\in$ A so that 'b' is inverse of a

$\therefore$ a * b = b * a = e

$\Rightarrow$ a + b + ab = b + a + ba = 0

As a $\neq$ –1, b = $\dfrac{-a}{1+a} \in A.$ Hence every element of A is invertible.

24. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$.

OR

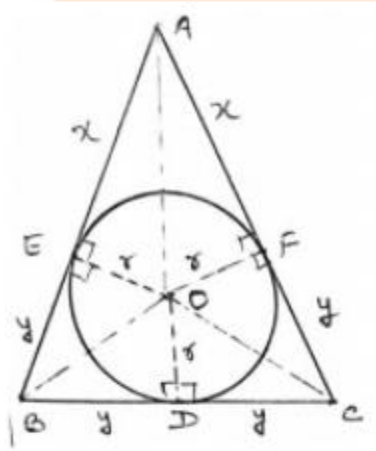
If the sum of lengths of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

Solution:

Let ΔABC be isosceles with inscribed circle of radius 'r' touching sides AB, AC and BC at E, F and D respectively.

let AE = AF = x, BE = BD = y, CF = CD = y then area (ΔABC) = ar(ΔAOB) + ar(ΔAOC) + ar(ΔBOC)

$$\Rightarrow \frac{1}{2}.2y\left(r+\sqrt{r^2+x^2}\right)=\frac{1}{2}\{2yr+2(x+y)r\} \Rightarrow x=\frac{2r^2y}{y^2-r^2}$$



Then,

P(Perimeter of Δ ABC) = $2x+4y=\frac{4r^2y}{y^2-r^2}+4y$

$$\frac{dP}{dy}=\frac{-4r^2(r^2+y^2)}{(y^2-r^2)^2}+4 \text{ and } \frac{dP}{dy}=0 \Rightarrow y=\sqrt{3}r$$

$$\left.\frac{d2P}{dy^2}\right]_{y=\sqrt{3}r}=\frac{4r^2y(2y^2+6r^2)}{(y^2-r^2)^3}=\frac{6\sqrt{3}}{r}>0$$

$\therefore$ Perimeter is least iff y = $\sqrt{3}r$ and least perimeter is

$$P=4y+\frac{4r^2y}{y^2-r^2}=4\sqrt{3}r+\frac{4r^2\sqrt{3}r}{2r^2}=6\sqrt{3}r$$

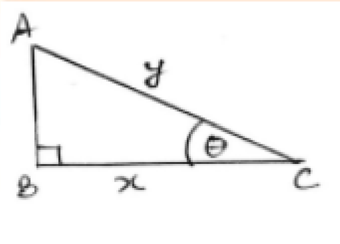
OR

let ABC be the right triangle with $\angle$B = 90°

$\angle$ACB = θ, AC = y, BC = x, x + y = k (constant)

A (Area of triangle) $=\frac{1}{2}.BC.AB=\frac{1}{2}.x\sqrt{y^2-x^2}$

let $z=A^2=\frac{1}{4}x^2(y^2-x^2)=\frac{1}{4}x^2\left\{(k-x)^2-x^2\right\}=\frac{1}{4}(x^2k^2-2kx^3)$



$$\frac{dz}{dx}=\frac{1}{4}(2xk^2-6kx^2) \text{ and } \frac{dz}{dx}=0 \Rightarrow x=\frac{k}{3},\ y=k-x=\frac{2k}{3}$$

$$\left.\frac{d^2z}{dx^2}\right]_{x=\frac{k}{3}}=\left.\frac{1}{4}(2k^2-12kx)\right]_{x=\frac{k}{3}}=-\frac{k^2}{2}<0$$

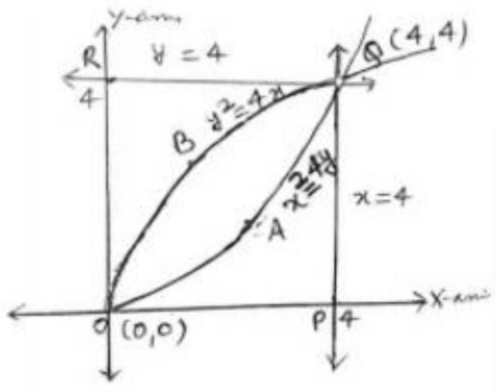
$\therefore$ z and area of ΔABC is max at $x=\frac{k}{3}$ and, $\cos\theta=\frac{x}{y}=\frac{k}{3}.\frac{3}{2k}=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$

25. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of square bounded by x = 0, x = 4, y = 4 and y = 0 into three equal parts.

Solution:

Point of intersection of $y^2 = 4x$ and $x^2 = 4y$ are (0, 0) and (4, 4);

Correct Graph



are (OAQBO) $= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$$= \left\{\frac{4}{3} x^{3/2} - \frac{x^3}{12}\right\}\Bigg]_0^4$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

Area (OPQAO) $= \int_0^4 \frac{x^2}{4} dx = \frac{1}{12} x^3 \Big]_0^4 = \frac{16}{3}$

Area (OBQRO) $= \int_0^4 \frac{y^2}{4} dy = \frac{1}{12} y^3 \Big]_0^4 = \frac{16}{3}$

Hence the areas of the three regions are equal.

26. Using properties of determinants show that Δ ABC is isosceles if:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

OR

A shopkeeper has 3 varieties of pens 'A', 'B' and 'C'. Meenu purchased 1 pen of each variety for a total of ₹ 21. Jeevan purchased 4 pens of 'A' variety, 3 pens of variety and 2 pens of 'C' variety for ₹ 60. While Shikha purchased 6 pens of 'A' variety, 2 pens of 'B' variety and 3 pens of 'C' variety for ₹ 70. Using matrix method. Find cost of each variety of pen.

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix}=0$$

Apply $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\Leftrightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B-\cos A & \cos C-\cos A \\ \cos^2 A+\cos A & (\cos B-\cos A)(\cos B+\cos A+1) & (\cos C-\cos A)(\cos C+\cos A+1) \end{vmatrix}=0$$

Taking (cos B – cos A), (cos C – cos A) common from C_2 & C_3

$$\Leftrightarrow (\cos B-\cos A)(\cos C-\cos A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 1 \\ \cos^2 A+\cos A & \cos B+\cos A+1 & \cos C+\cos A+1 \end{vmatrix}=0$$

Expand along R_1

$$\Leftrightarrow (\cos B-\cos A)(\cos C-\cos A)(\cos C-\cos B)=0$$

$$\Leftrightarrow \begin{array}{c} \cos A=\cos B \\ \text{or} \\ \cos B=\cos C \\ \text{or} \\ \cos C=\cos A \end{array} \quad \Leftrightarrow \begin{array}{c} A=B \\ \text{or} \\ B=C \\ \text{or} \\ C=A \end{array} \quad \Leftrightarrow \Delta ABC \text{ is an isosceles triangle}$$

OR

let the cost of one pen of variety ‘A’, ‘B’ and ‘C’ be ₹ x. `₹ y and ₹ z respectively then the system of equations is:

$$\left.\begin{aligned} x+y+z&=21 \\ 4x+3y+2z&=60 \\ 6x+2y+3z&=70 \end{aligned}\right\}$$

Matrix form of the system is:

A×X = B, where A = $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}; X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}; B=\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$

|A| = (5) –1 (0) + 1 (–10) = – 5

Co-factors of the matrix A are:

$$\left.\begin{matrix} C_{11}=5; & C_{21}=-1; & C_{31}=-1 \\ C_{12}=0; & C_{22}=-3 & C_{32}=2 \\ C_{13}=-10; & C_{23}=4; & C_{33}=-1 \end{matrix}\right\}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} Adj\ A \frac{1}{-5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

Solution of the matrix equation is X = A^{-1} B

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix} \quad \therefore x = 5,\ y = 8,\ z = 8$$