Stephen Kim 1947806

MAT 220—Homework 5 Part A:

10.9 – Multiplication by a diagonal matrix. Suppose that A is an m x n matrix, D is a diagonal matrix, and B = DA. Describe B in terms of A and the entries of D. You can refer to the rows or columns or entries of A.

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \quad D = \begin{bmatrix} y_{11} & 0 & \cdots & 0 \\ 0 & y_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{mm} \end{bmatrix}$$

$$B = AD = \begin{bmatrix} x_{11}y_{11} & x_{12}y_{11} & \cdots & x_{1n}y_{11} \\ x_{21}y_{22} & x_{22}y_{22} & \cdots & x_{2n}y_{22} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1}y_{mm} & x_{m2}y_{mm} & \cdots & x_{mn}y_{mm} \end{bmatrix}$$

10.13 – Laplacian matrix of a graph. Let A be the incidence matrix of a directed graph with n nodes and m edges. The Laplacian matrix associated with the graph is defined as L = AA^T, which is the Gram matrix of A^T. It is named after the mathematician Pierre-Simon Laplace.

10.13 (a) Show that D(v) = v^{T}Lv, where D(v) is the Dirichlet energy defined on page 135.

$D(v) = v^T L v = v^T A A^T v = (A^T v)^2$

$D(v) = ||A^T v||^2$

10.13 (b) Describe the entries of L. Hint. The following two quantities might be useful: The degree of a node, which is the number of edges that connect to the node (in either direction), and the number of edges that connect a pair of distinct nodes (in either direction).

The Laplacian matrix is a combination of the degree matrix, which has the number of edges connecting to the node in the diagonal spots of the matrix, and the adjacency matrix, which represents each edge connecting two nodes together as -1.

10.17 Patients and symptoms. Each of a set of N patients can exhibit any number of a set of n symptoms. We express this as an N x n matrix S, with

$$S_{ij} = \begin{cases} 1 & \text{patient } i \text{ exhibits symptom } j \\ 0 & \text{patient } i \text{ does not exhibit symptom } j \end{cases}$$

Give simple English descriptions of the following expressions. Include the dimensions and describe the entries.

symptoms

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad N \times n$$

10.17 (a) S1

$$S1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ N x 1} \rightarrow \text{Shows whether or not each patient has symptom 1.}$$

10.17 (b) S^T1

$S^T1 = [1 \quad 0]$ 1 x n → Shows all the symptoms for the first patient

10.17 (c) S^TS

$$S^TS = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 0+0+0 \\ 0+0+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ n x n}$$

The matrix shows the number of kinds of symptoms, 2.

10.17 (d) SS^T

$$SS^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 & 1+0 \\ 0+0 & 0+1 & 0+0 \\ 1+0 & 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ N x N}$$

The matrix shows the number of patients, 3.

11.6 Inverse of a block upper triangular matrix. Let B and D be invertible matrices of sizes m x m and n x n, respectively, and let C be any m x n matrix. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix}$$

In terms of B^{-1}, C, and D^{-1}. (The matrix A is called block upper triangular.) Hints. First get an idea of what the solution should look like by considering the case when B, C, D are scalars. For the matrix case, your goal is to find matrices W, X, Y, Z (in terms of B^{-1}, C, and D^{-1}) that satisfy

$$\mathbf{A}\begin{bmatrix} \boldsymbol{W} & \boldsymbol{X} \\ \boldsymbol{Y} & \boldsymbol{Z} \end{bmatrix} = \mathbf{I}.$$

Use block matrix multiplication to express this as a set of four matrix equations that you can then solve. The method you will find is sometime called block back substitution.

A = $\begin{bmatrix} B_{mxm} & C_{mxn} \\ 0_{nxm} & D_{nxn} \end{bmatrix} \rightarrow \begin{bmatrix} B_{mxm} & C_{mxn} \\ 0_{nxm} & D_{nxn} \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ = I

$\begin{bmatrix} BW + CY_{mxm} & BX + CZ_{mxn} \\ 0W + DY_{nxm} & 0X + DZ_{nxn} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

DZ = I → Z = D^{-1}I = D^{-1}

DY = 0 → Y = D^{-1}0 = 0

BW + CY = I → BW + C0 = I → BW = I → W = B^{-1}I = B^{-1}

BX + CZ = 0 → BX + CD^{-1} = I → BX = - CD^{-1}1 → X = - $B^{-1}CD^{-1}$

$A^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}$

Part B:

1. **Find an example of a 2 × 2 matrix A such that A^2 is the 2 × 2 zero matrix. Check your answer by calculating the square.**

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → A^2 = $\begin{bmatrix} a^2 + bc & ba + bd \\ ca + cd & bc + d^2 \end{bmatrix}$

$a^2 + bc = ba + bd = ca + cd = bc + d^2 = 0$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ = $\begin{bmatrix} 1^2 + (1)(-1) & (1)(1) + (1)(-1) \\ (-1)(1) + (-1)(-1) & (1)(-1) + (-1)^2 \end{bmatrix}$

= $\begin{bmatrix} 1 + (-1) & 1 + (-1) \\ -1 + 1 & (-1) + 1 \end{bmatrix}$ = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. **A student says that for any square matrix *X*,**

$$\boldsymbol{(X - I)(X + I) = X^2 - I.}$$

Is the student right? If so, explain why. If not, give a specific counterexample, *i.e.*, a square matrix *X* for which the identity does not hold.

$(X – I)(X + I) = X^2 + XI – XI – I^2$ //expand the product

$= X^2 – I^2$

$= X^2 – I$ //$I^2 = I$

I^2 = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 + 0 & 0 + 0 \\ 0 + 0 & 0 + 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = I

Yes, the student is correct.

3. ***(Circular shift matrices)*** **Let *C* be the 4 × 4 matrix**

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

(a) How is *Cx* related to *x*? Your answer should be in English. *Hint.* See exercise title.

$$\mathrm{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix} \qquad \mathrm{Cx} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$$\mathrm{Cx} = \begin{bmatrix} 0+2+0+0 & 0+2+0+0 & 0+2+0+0 & 0+2+0+0 \\ 0+0+3+0 & 0+0+3+0 & 0+0+3+0 & 0+0+3+0 \\ 0+0+0+4 & 0+0+0+4 & 0+0+0+4 & 0+0+0+4 \\ 1+0+0+0 & 1+0+0+0 & 1+0+0+0 & 1+0+0+0 \end{bmatrix}$$

$$\mathrm{Cx} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Multiplying by the matrix C will shift all the elements up one row, with the top row becoming the new bottom row.

(b) What is C^4? Calculate it explicitly by calculating C^2 and then $C^4 = C^2C^2$. Then explain why your answer makes sense, given your answer to part (a).

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+0 & 0+0+0+0 & 0+1+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+1+0 \\ 0+0+0+1 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 1+0+0+0 & 0+0+0+0 & 0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+1 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 1+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 1+0+0+0 & 0+0+0+0 & 0+1+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

C shifts the rows up by 1. C^2 shifts the rows up by 2. C^4 shifts the rows up by 4, but since there are only 4 rows, the matrix stays the same. C^4 is also the identity matrix.

4. **Find examples of 2 × 2 matrices *A, X,* and *Y* such that *AX* = *AY* but *X* ≠ *Y* .**

Show/explain why your examples satisfy the required properties.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $Y = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$AX = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $AY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

AX = AY; X ≠ Y

5. **(a) Use 11.6 and the formula for the inverse of 2 × 2 matrices from the videos to find the inverse of**

$$A = \begin{bmatrix} 4 & 3 & 1 & -3 \\ 2 & 1 & 2 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

A = $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ → $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ = I

A^{-1} = $\begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}$

B = $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ B^{-1} = 1/((1)(4) – (2)(3))$\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ = -1/2$\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ = $\begin{bmatrix} -1/2 & 3/2 \\ 1 & -2 \end{bmatrix}$

C = $\begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}$ C^{-1} = 1/((1)(-2) – (2)(-3))$\begin{bmatrix} -2 & 3 \\ -2 & 1 \end{bmatrix}$ = 1/4$\begin{bmatrix} -2 & 3 \\ -2 & 1 \end{bmatrix}$ = $\begin{bmatrix} -1/2 & 3/4 \\ -1/2 & 1/4 \end{bmatrix}$

D = $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ D^{-1} = 1/((2)(0) – (1)(-1))$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ = 1/1$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ = $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

B^{-1}CD^{-1} = $\begin{bmatrix} -1/2 & 3/2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} \left(-\frac{1}{2}\right)(1) + (\frac{3}{2})(2) & \left(-\frac{1}{2}\right)(-3) + (\frac{3}{2})(-2) \\ (1)(1) + (-2)(2) & (1)(-3) + (-2)(-2) \end{bmatrix}$ = $\begin{bmatrix} 5/2 & -3/2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\begin{bmatrix} \left(\frac{5}{2}\right)(0) + (-\frac{3}{2})(1) & \left(\frac{5}{2}\right)(-1) + (-\frac{3}{2})(2) \\ (-3)(0) + (1)(1) & (-3)(-1) + (1)(2) \end{bmatrix}$ = $\begin{bmatrix} -3/2 & -11/2 \\ 1 & 5 \end{bmatrix}$

A^{-1} = $\begin{bmatrix} -1/2 & 3/2 & 3/2 & 11/2 \\ 1 & 2 & -1 & -5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(b) If *B* is the inverse that you found, calculate *AB* and *BA* to check that your inverse is correct.

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{11}{2} \\ 1 & -2 & -1 & -5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 & -3 \\ 2 & 1 & 2 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} (-2+3+0+0) & (-\frac{3}{2}+\frac{3}{2}+0+0) & (-\frac{1}{2}+3+3-\frac{11}{2}) & (\frac{3}{2}-3+\frac{3}{2}+0) \\ (4-4+0+0) & (3-2+0+0) & (1-4-2+5) & (-3+4-1+0) \\ (0+0+0+0) & (0+0+0+0) & (0+0+0+1) & (0+0+0+0) \\ (0+0+0+0) & (0+0+0+0) & (0+0+2-2) & (0+0+1+0) \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 1 & -3 \\ 2 & 1 & 2 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{11}{2} \\ 1 & -2 & -1 & -5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} (-2+3+0+0) & (6-6+0+0) & (6-3+0-3) & (22-15-1-6) \\ (-1+1+0+0) & (3-2+0+0) & (3-1+0-2) & (11-5-2-4) \\ (0+0+0+0) & (0+0+0+0) & (0+0+0+1) & (0+0-2+2) \\ (0+0+0+0) & (0+0+0+0) & (0+0+0+0) & (0+0+1+0) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$