Stochastic Low-Rank Latent Bandits

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Abstract

To be written.

1 Introduction

STORY: We address a recommendation problem in the hard setting where no feature is available to the learner. Blah blah: recommendation and bandits, major problem, blah blah.

We rely on the assumption that the underlying click-through rate matrix has a latent sructure that we cannot directly observe but that we propose to leverage nonetheless. We formulate a rank-d bandit problem that generalizes previous works on rank-1 and on latent bandits (quote, quote). We propose a meta algorithm that uses two layers of bandit algorithms in order to learn 1/the best set of items overall and 2/ the individual preferences. This is a novel and efficient bandit startegy for the latent bandits and an elegant generalization of the rank-1 setting. We show a regret bound for our algorithm and run experiments on simulated and real data.

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Cla: I haven't changed this section yet, wanted to make sure the story is right before.

In this paper, we study the problem of recommending the best items to users who are coming sequentially. The learner has access to very less prior information about the users and it has to adapt quickly to the user preferences and suggest the best item to each user. Furthermore, we consider the setting where users are grouped into clusters and within each cluster the users have the same choice of the best item, even though their quality of preference may be different for the best item. These clusters along with the choice of the best item for each user are unknown to the learner. Also, we assume that each user has a single best item preference.

This complex problem can be conceptualized as a low rank stochastic bandit problem where there are K users and L items. The reward matrix, denoted by $\bar{M} \in [0,1]^{K\times L}$, generating the rewards for user, item pair has a low rank structure. The online learning game proceeds as follows, at every timestep t, nature reveals one user (or row) from $\bar{M}$ where user is denoted by i_t. The learner selects some items (or columns) from $\bar{M}$, where an item is denoted by $j_t \in [L]$. Then the learner receives one noisy feedback $r_t(i_t, j_t) \sim \mathcal{D}(\bar{M}(i_t, j_t))$, where $\mathcal{D}$ is a distribution over the entries in $\bar{M}$ and $\mathbb{E}[r_t(i_t, j_t)] = \bar{M}(i_t, j_t)$. Then the goal of the learner is to minimize the cumulative regret by quickly identifying the best item j^* for each $i \in [K]$ where $\bar{M}(i, j^*) = \arg\max_{j\in[L]}\{\bar{M}(i,j)\}$.

1.1 Notation and Learning Setting

Throughout the paper, we denote $[n] = \{1, 2, \ldots, n\}$. An instance of the *Low-Rank Bandit* problem is a matrix $R \in [0,1]^{K\times L}$ representing the expected click-through rates (CTRs) for each user $k \in [K]$ on each item $l \in [L]$. If, $J \subset [L]$ is a subset of columns, we denote $R(:, J) \in [0,1]^{K,|J|}$ the corresponding submatrix containing the $|J|$ columns of R.

We assume that there exists a latent structure, i.e that $R = UV^T$ where the rows of U and V contain the hidden users' and item's features. It is important to notice that none of those features are observable, meaning that we cannot build on a linear bandit model, and in particular our problem cannot be seen as a *clustering of bandits* problem Gentile et al. (2014). However, the rank of the CTR matrix is assumed to be low, that is $d << \min\{L, K\}$. This is the key assumption of our model. It implies, by definition, the following property.

Observation 1. *Let $M \in \mathbb{R}^{K\times L}$ be a rank-d matrix. Then,*

- *There exists a basis J^* of d column such that all the L columns' latent features are linear combinations of the vectors in J^*;*
- *There exists a basis I^* of d users such that all the K users' latent features are linear combinations of the vectors in I^*.*

Without loss of generality, the above mentioned bases can be chosen of maximal volume such that the corresponding transformation matrix is the least singular possible.

Proof. The existence of the basis on both dimensions comes directly by definition of the low rank assumption. The choice of the spanning vectors is arbitrary and maximising the volume means choosing vectors with larger norm and hence potentially larger payoff. ■

Cla: Here state the result on the existence of a best set of d items, I'm not sure how to state it. It is not an "assumption" though, it is a Lemma or a Fact but not an assumption. It is a consequence of the low rank assumption :)

The interaction at round $t \geq 1$ of the learner with the online recommender system characterized by R goes as follows:

- a user $i_t \in [K]$ shows up – it corresponds to the index of a row of the matrix. It can be seen as an unobserved context generated by the environment;
- the learner chooses a set $J_t \subset [L]$ such that $|J_t| = d$ to be sequentially presented to the user;
- the user browses those d options and send an individual feedback for each of them (semi-bandit setting): $\forall j \in J_t$, the learner observes $Y_{t,j} = R(i_t, j) + \eta_{t,j}$ where $(\eta_{t,j})_{t,j\geq 0}$ is a seqence of i.i.d centered random variables.

  Cla: fix your noise model here. Bernoulli ??

For each user $i \in [K]$, there exists one unique best item $j^*(i) \in [L]$

Cla: Define the best item, define the expected regret

The objective of the learning agent is to minimize the expected cumulative regret up to horizon n. We define the cumulative regret, denoted by $\mathcal{R}_n$ as,

1.2 Related Works

In Maillard and Mannor (2014) the authors propose the Latent Bandit model where there are two sets: 1) set of arms denoted by $\mathcal{A}$ and 2) set of types denoted by $\mathcal{B}$ which contains the latent information regarding the arms. The latent information for the arms are modeled such that the set $\mathcal{B}$ is assumed to be partitioned into $|C|$ clusters, indexed by $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_C \in \mathcal{C}$ such that the distribution $v_{a,b}, a \in \mathcal{A}, b \in \mathcal{B}_c$ across each cluster is same. Note, that the identity of the cluster is unknown to the learner. At every timestep t, nature selects a type $b_t \in \mathcal{B}_c$ and then the learner selects an arm $a_t \in \mathcal{A}$ and observes a reward $r_t(a, b)$ from the distribution $v_{a,b}$.

Another way to look at this problem is to imagine a matrix of dimension $|A| \times |B|$ where again the rows in $\mathcal{B}$ can be partitioned into $|C|$ clusters, such that the distribution across each of this clusters are same. Now, at every timestep t one of this row is revealed to the learner and it chooses one column such that the $v_{a,b}$ is one of the $\{v_{a,c}\}_{c\in\mathcal{C}}$ and the reward for that arm and the user is revealed to the learner.

This is actually a much simpler approach than the setting we considered because note that the distributions across each of the clusters $\{v_{a,c}\}_{c\in\mathcal{C}}$ are identical and estimating one cluster distribution will reveal all the information of the users in each cluster.

2 Algorithm

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for $t = 1, \ldots, n$ **do**
  User i_t comes to the system

  // Choose d arms in d column bandits
  Let $p_{c,t}(k, j)$ be the probability of playing arm $j \in [L]$ in c-bandit $k \in [d]$ at time t
  for $k = 1, \ldots, d$ **do**
    Sample $J_t[k] \sim \mathrm{Cat}(p_{c,t}(k, 1), \ldots, p_{c,t}(k, L))$

  // Choose an arm in row bandit i_t
  Let $p_{r,t}(i_t, k)$ be the probability of playing arm $k \in [d + 1]$ in r-bandit $i_t \in [K]$ at time t
  Sample $k_t \sim \mathrm{Cat}(p_{r,t}(i_t, 1), \ldots, p_{r,t}(i_t, d + 1))$

  // Update row bandit i_t
  for $k = 1, \ldots, d$ **do**
    if $k_t \leq d$ **then**
      $j_{t,k} \leftarrow J_t[k_t]$
    else
      $j_{t,k} \leftarrow J_t[k]$
  $s_{r,t} \leftarrow s_{r,t-1}$
  $s_{r,t}(i_t, k_t) \leftarrow s_{r,t}(i_t, k_t) + \sum_{k=1}^{d} \frac{R_t(i_t, j_{t,k})}{p_{r,t}(i_t, k_t)}$

  // Update d column bandits
  $s_{c,t} \leftarrow s_{c,t-1}$
  if $k_t > d$ **then**
    for $k = 1, \ldots, d$ **do**
      $s_{c,t}(k, J_t[k]) \leftarrow s_{c,t}(k, J_t[k]) + \frac{\max R_t(i_t, J_t[: k]) - \max R_t(i_t, J_t[: k-1])}{p_{c,t}(k, J_t[k])\, p_{r,t}(i_t, d + 1)}$

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3 Analysis

We assume that users come sequentially $i_1, \ldots, i_n \in [K]$. We denote by $j^*(i)$ the optimal arm of user i. When $J = (j_1, ..., j_k) \in [L]^k$ is a k-tuple, by $J[l]$ we will mean j_l, the l'th entry of J and $\max R(i, J) := \max_{l\in[k]} R(i, J[l])$.

Bra: "l" is a horrible letter because it looks like many other symbols. What about ℓ instead?

Let $U_t \in \mathbb{R}^{K\times d}_{\geq 0}$ and $V_t \in \mathbb{R}^{L\times d}_{\geq 0}$ to be time varying latent user and item factors. The reward matrix at time step $t \in [n]$ is $R_t = U_t V_t^T$.

Assumption 1 (Hott Topics)**.** *We will assume that there is a d-tuple $J^* \in [L]^d$ such that for every $j \in [L]$, there exists $\alpha^j_1, ..., \alpha^j_d \geq 0, \sum_k \alpha^j_k \leq 1$ and*

$$V_t[j, :] = \sum_{k\in J^*} \alpha^j_k V_t[k, :],$$

for every $t \in [n]$.

An important thing to note is that α^j_k's are *independent* of time t. With the above assumption, we have the following theorem.

Lemma 1. *For any set of columns J_t, we have*

$$\max_{j\in[L]} \sum_{t\in[n]} \max R_t(i_t,(J_t,j)) = \max_{l\in[d]} \sum_{t\in[n]} \max R_t(i_t,(J_t,J^*[l])).$$

Todo: What happens with the Bernoulli rounding trick? The above holds when the input is non-stochastic, but I haven't thought about the stochastic case. Without loss of generality, we will also assume that for every $1 \le k \le d$, $\max_{J\in[L]^k} \sum_t \max R_t(i_t,J) = \sum_t \max R_t(i_t, J^*[1:k])$

Anup: Need to say why this is possible: follows easily from hott topics

. Let $J_t = (\tilde{j}_{t,1}, \tilde{j}_{t,2}, ..., \tilde{j}_{t,d})$ be the tuple of d columns chosen by column-bandits at time t and $(j_{t,1},...,j_{t,d})$ be the d tuple of columns chosen by i_tth row EXP3 at time t. We want to bound the expected regret $R(n)$

$$R(n) = \mathbb{E}\left(d\sum_t R(i_t, j_t^*(i_t)) - \sum_t\sum_k R(i_t, j_{t,k})\right).$$

The row algorithm plays either one arm in J_t d times or plays every arm one time. We will use an indicator function $\mathbb{1}(j_{t,1} \neq j_{t,2})$ which takes value one only if the row algorithm

Anup: row algorithm? We need a better way to refer to the various EXP3s

plays every arm in J_t one time. Let $p_t = P(j_{t,1} \neq j_{t,2})$. We can write the expected regret as $R(n) = R_c(n) + R_r(n)$, where

$$R_c(n) = \mathbb{E}\left(d\sum_{t=1}^n R(i_t, j_t^*(i_t)) - d\sum_t \frac{\max R(i_t,J_t)}{p_t}\mathbb{1}(j_{t,1}\neq j_{t,2})\right)$$

and

$$R_r(n) = \mathbb{E}\left(d\sum_t \frac{\max R(i_t,J_t)}{p_t}\mathbb{1}(j_{t,1}\neq j_{t,2}) - \sum_t\sum_k R(i_t,j_{t,k})\right).$$

We will show that for every $\gamma > 0$, $R_c(n) = O\left(\frac{d^2}{\gamma}\sqrt{nL\log n}\right)$ and $R_r(n) = O\left(\frac{Kd\log d}{\gamma} + \gamma n\right)$.

Theorem 1. *By choosing γ appropriately, for all large enough n, we have*

$$R(n) = O\left(dL^{1/4}n^{3/4}\log^{1/4} n\right).$$

We now prove the bounds for $R_c(n)$ and $R_r(n)$ separately.

Todo: Make it clear what the randomness is when using $\mathbb{E}$ throughout.

Bra: Given the limited time, let's go with the current setting. This is most natural in the non-stochastic community and nobody will question it. Then the only randomness is with respect to random actions of the algorithm.

3.1 Bounding Column Regret

To bound $R_c(n)$, we first rewrite it as

$$\begin{aligned}
R_c(n) &= d\mathbb{E}\left(\sum_{t=1}^n \frac{R(i_t,j_t^*(i_t))}{p_t}\mathbb{1}(j_{t,1}\neq j_{t,2}) - \sum_t \frac{\max R(i_t,J_t)}{p_t}\mathbb{1}(j_{t,1}\neq j_{t,2})\right)\\
&= d\mathbb{E}\left(\sum_{t=1}^n \frac{\tilde{R}(i_t,j_t^*(i_t))}{p_t} - \sum_t \frac{\max \tilde{R}(i_t,J_t)}{p_t}\right)\\
&\le \frac{d}{\min_t p_t}\mathbb{E}\left(\sum_{t=1}^n \max\tilde{R}(i_t,J^*) - \sum_t \max\tilde{R}(i_t,J_t)\right).
\end{aligned}$$

Here, we define $\tilde{R}_t(i,j) = R_t(i,j)\mathbb{1}(j_{t,1}\neq j_{t,2})$. We are now ready to bound the regret. To avoid carrying tildes, we denote $\tilde{R}_t$ by R_t in the rest of the proof.

Lemma 2. *For any $k \in [d]$,*

$$\sum_t \mathbb{E} \max R_t(i_t, J_t[1:k]) \geq \mathbb{E} \sum_t \max R_t(i_t, J^*[1:k]) - O(k\sqrt{nL}).$$

Proof. We will show this by induction. Note that there are d column EXP3s in this case. The base case when $k = 1$ follows because of the guarantees of the first col-EXP3. Let $J^* = (j_1^*, j_2^*, ..., j_d^*)$. We will now assume that the result is true for $k-1$ for some $k > 1$. We have

$$\begin{aligned}
&\mathbb{E} \sum_t \max R_t(i_t, J_t[1:k]) && (1)\\
&\geq \max_{j_k} \mathbb{E} \sum_t \max R_t(i_t, (J_t[1:k-1], j_k)) - O\left(\sqrt{nL}\right) && (2)\\
&\geq \max_{j_k} \mathbb{E} \sum_t \max R_t(i_t, (J^*[1:k-1], j_k)) - O\left(\sqrt{nL}\right) - O\left((k-1)\sqrt{nL}\right) && (3)\\
&= \mathbb{E} \sum_t R_t(i_t, J^*[1:k]) - O\left(k\sqrt{nL}\right). && (4)
\end{aligned}$$

The last equality follows from Lemma 4. The first inequality is from the guarantees of kth col-EXP3.

Bra: State the regret bound of Exp3 in a lemma.

The crucial step is the second inequality. It says that we can replace $J_t[1:k-1]$ with $J^*[1:k-1]$ by just losing another additive $O\left((k-1)\sqrt{nL}\right)$ term. This follows from induction hypothesis and Lemma 3. We note that from Equation 4, we have

$$\max R_t(i_t, J_t[1:k]) \geq \mathbb{E} \sum_t \max R_t(i_t, J^*[1:k]) - O\left(k\sqrt{nL}\right),$$

which concludes the proof. ■

Lemma 3. *Suppose*

$$\mathbb{E} \sum_t (\max R_t(i_t, (J_t[1:k-1]) \geq \mathbb{E} \sum_t (\max R_t(i_t, (J^*[1:k-1]) - C$$

and let $j_k \in [L]$. Then,

$$\mathbb{E} \sum_t (\max R_t(i_t, (J_t[1:k-1], j_k) \geq \mathbb{E} \sum_t (\max R_t(i_t, (J^*[1:k-1], j_k) - O\left((k-1)\sqrt{nL}\right).$$

Proof. Let $T_1 = \{t \,|\, \max R_t(i_t, J^*[1:k-1]) < R_t(i_t, j_k)\}$ and $T_2 = [n] \backslash T_1$. We then have

$$\begin{aligned}
&\mathbb{E} \sum_t \max R_t(i_t, (J_t[1:k-1], j_k))\\
&= \mathbb{E} \sum_{t \in T_1} \max R_t(i_t, (J_t[1:k-1], j_k)) + \mathbb{E} \sum_{t \in T_2} \max R_t(i_t, (J_t[1:k-1], j_k))\\
&\geq \sum_{t \in T_1} \max R_t(i_t, (J^*[1:k-1], j_k)) + \mathbb{E} \sum_{t \in T_2} \max R_t(i_t, (J_t[1:k-1], j_k))\\
&\geq \sum_{t \in T_1} \max R_t(i_t, (J^*[1:k-1], j_k)) + \mathbb{E} \sum_{t \in T_2} \max R_t(i_t, J_t[1:k-1])\\
&\geq \sum_{t \in T_1} \max R_t(i_t, (J^*[1:k-1], j_k)) + \sum_{t \in T_2} \max R_t(i_t, J^*[1:k-1]) - C\\
&= \sum_{t \in T_1} \max R_t(i_t, (J^*[1:k-1], j_k)) + \sum_{t \in T_2} \max R_t(i_t, (J^*[1:k-1], j_2)) - C\\
&= \sum_{t \in [n]} \max R_t(i_t, (J^*[1:k-1], j_k)) - C.
\end{aligned}$$

The first inequality is easy because $\max R_t(i_t, (J^*[1:k-1], j_k) = R_t(i_t, j_k)$ for $t \in T_1$. Second inequality is trivial. Third inequality follows from the assumption. The next equality holds because of the definition of T_2. ■

Anup: Define a new slicing operator and a more compressed '-' operator so that the above expressions look a bit nicer?

3.2 Bounding Row Regret

To bound $R_r(n)$, we first note that

$$R_r(n) = E\left(\sum_t d \cdot \max R(i_t, J_t) - \sum_t \sum_k R(i_t, j_{t,k})\right).$$

We will decompose the regret as a sum of regret of row-EXP3s. There are K row-EXP3s and each one corresponds to a user. Let n_i be the number of times user i appears in the sequence $i_1, ..., i_n$. We then have

$$R_{r,i}(n) = \sum_{i \in [K]} R_{r,i}(n)$$

where $R_{r,i}(n) = E\left(\sum_t d \cdot \max R(i, J_t) - \sum_t \sum_k R(i, j_{t,k})\right)$. Since each user has a row-EXP3 is over $d+1$ arms, the regret is bounded by

$$R_{r,i}(n) = (e-1)\frac{(d+1)\log(d+1)}{\gamma} + \gamma n_i,$$

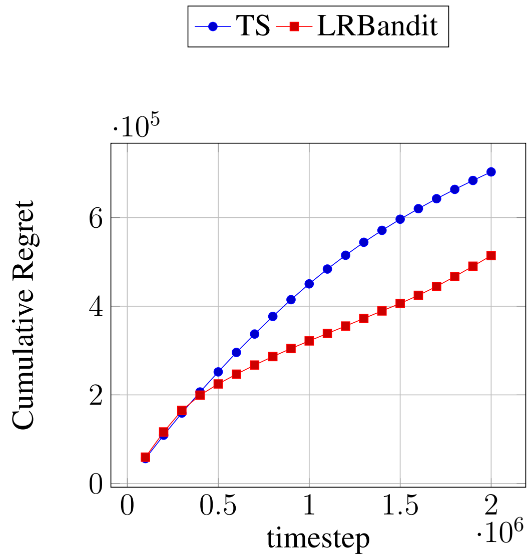
where $\gamma > 0$ is any positive number. Summing this over K users, we get

$$R(n) = (e-1)\frac{(d+1)\log(d+1)K}{\gamma} + \gamma n.$$

Anup: This proof needs to have a bit more details. Also, γ should appear in the algorithm and we should refer to that.

Bra: Please add more details. This needs to be done over all users.

4 Experiments



(a) Expt-1: 1024 Users, 128 arms, Round-Robin, Noisy Setting, Rank 2, equal sized clusters

Figure 1: A comparison of the cumulative regret by MRLG and MRLUCB.

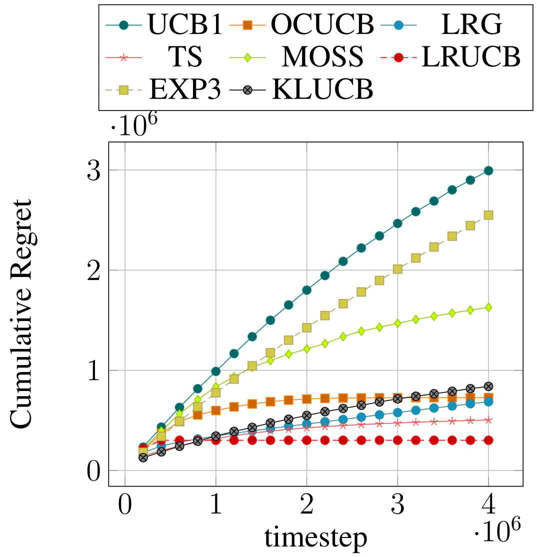
5 Conclusions and Future Direction

To be written.

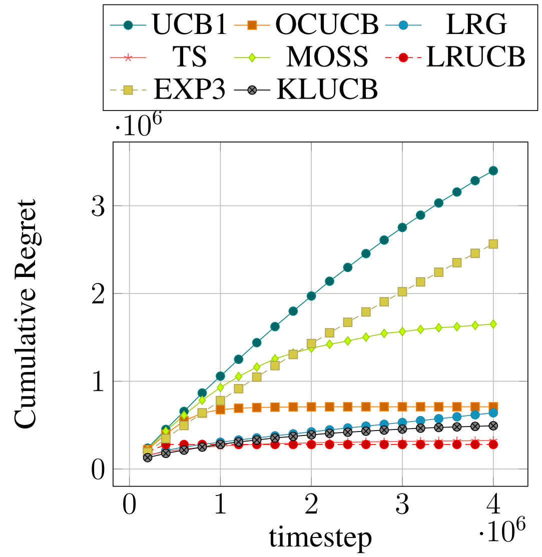
References

Gentile, C., Li, S., and Zappella, G. (2014). Online clustering of bandits. In *International Conference on Machine Learning*, pages 757–765.

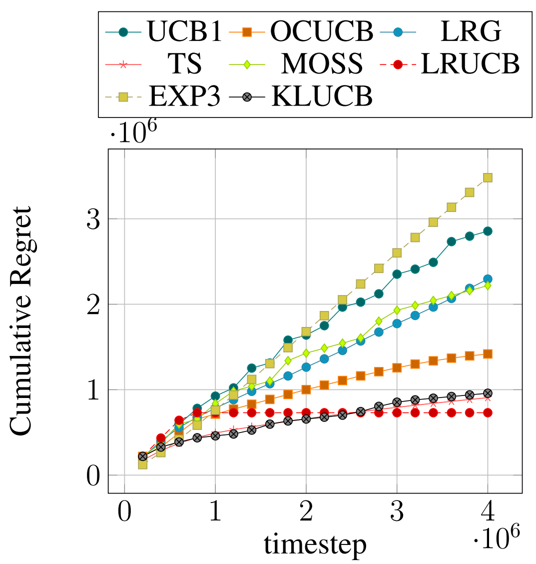
Maillard, O.-A. and Mannor, S. (2014). Latent bandits. In *International Conference on Machine Learning*, pages 136–144.



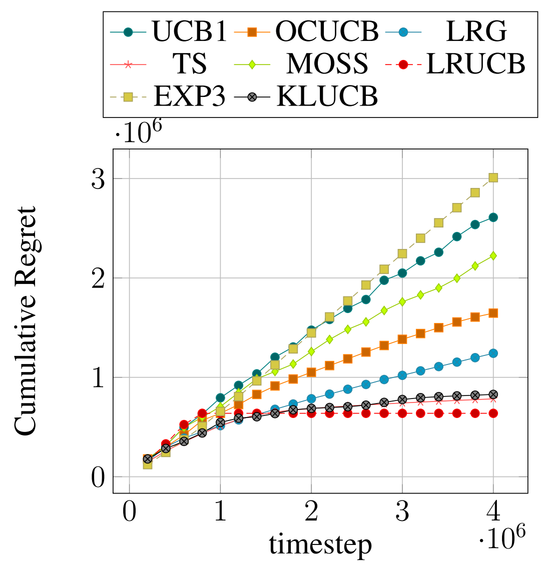
(a) Expt-1: 2048 Users, 64 Bernoulli-distributed arms, Round-Robin, Noisy Setting, Rank 2, equal sized clusters



(b) Expt-2: 2048 Users, 64 Bernoulli-distributed arms, Round-Robin, Noisy Setting, Rank 2, un-equal sized clusters, 70:30 split



(c) Expt-3: 4096 Users, 128 Bernoulli-distributed arms, Round-Robin, Noisy Setting, Rank 2, equal sized clusters



(d) Expt-4: 4096 Users, 128 Bernoulli-distributed arms, Round-Robin, Noisy Setting, Rank 2, un-equal sized clusters, 80:20 split

Figure 2: A comparison of the cumulative regret incurred by the various bandit algorithms.

A Claire's point of view

We assume that users come sequentially: the indices $i_1, \ldots, i_n \in [K]$ are *i.i.d.* random variables chosen by an unknown (and uncontrolled) distribution $\mathcal{D}_u$.

When $J = (j_1, ..., j_k) \in [L]^k$ is a k-tuple, by $J[\ell]$ we will mean j_ℓ, the l'th entry of J and $\max R(i, J) := \max_{l \in [k]} R(i, J[\ell])$. Let $U_t \in \mathbb{R}_{\geq 0}^{K \times d}$ and $V_t \in \mathbb{R}_{\geq 0}^{L \times d}$ be time varying latent user and item factors.

Assumption 2 (Hot Topics). *We will assume that there is a d-tuple $J^* \in [L]^d$ such that any user latent vector is a convex combination of the vectors in J^*. In other words, for every $j \in [L]$, there* *exists* $\alpha_1^j, ..., \alpha_d^j \in [0,1]^d$*,,* $\sum_k \alpha_k^j \leq 1$ *and*

$$V[j,:] = \sum_{k \in J^*} \alpha_k^j V[k,:].$$

Moreover, for any user $i \in [K]$*, we assume that there exists one unique best item that we denote* $B(i)$*. The mapping* B *is deternministic and defined my*

$$B(i) = \arg\max_{j \in [L]} u_i^T v_j$$

With the above assumption, we have the following cornerstone Lemma.

Lemma 4. *The mapping* B *has its image included in* J^**. This means that for any user* $i \in [K]$*,* $B(i) \in J^*$*.*

Proof. By definition,

$$B(i) = \arg\max_{j \in [L]} u_i^T v_j.$$

The function $v \mapsto u_i^T v$ is linear so its maximum on a convex is reached at one of its summits.

Cla: could be nice to write this down more properly.

■

This means that no matter which user i_t shows up at round t, the best item recommendation is one of the d elements of J^*.

At each round, the learner must choose d arms possibly with repetitions. Given a user i_t, the optimal action is $A^*(i_t) = (B(i_t), \dots, B(i_t))$. The instantaneous regret incurred by taking action $A_t = (j_{t,1}, \dots, j_{t,k})$ is

$$r_t = dR(i_t, B(i_t)) - \sum_{k \in A_t} R(i_t, k).$$

The goal of the learner is to minimize the expected regret

$$\begin{aligned}
R(T) &= \mathbb{E}_{D_u} \sum_{t=1}^{T} r_t \\
&= \mathbb{E}_{D_u} \sum_{t=1}^{T} dR(i_t, B(i_t)) - \sum_{k \in A_t} R(i_t, k) \\
&= \sum_{t=1}^{T} \sum_{k \in A_t} \mathbb{E}_{D_u} \left[R(i_t, B(i_t)) - R(i_t, k) \right] \\
&= \sum_{t=1}^{T} \sum_{j \in [L]} \mathbb{1}\{j \in A_t\} \mathbb{E}_{D_u} \left[R(i_t, B(i_t)) - R(i_t, j) \right] \\
&= \sum_{j \in [L]} [N_j(T)] \bar{\Delta}_j
\end{aligned}$$

where

$$N_j(t) := \sum_{t=1}^{T} \mathbb{1}\{j \in A_t\}; \quad \bar{\Delta}_j := \mathbb{E}_{D_u} \left[R(i_t, B(i_t)) - R(i_t, j) \right].$$

Cla: This decomposition is correct but it also hides a little bit too much information about the users. I'm not sure it will actually help bounding the regret of our algorithm but I wanted to write it down.

A.1 Lower bound discussion

Now that the problem is defined, we discuss its complexity through a problem-dependent lower bound on the expected regret. It appears that as soon as the probability of each user to show up is positive, each row bandit problem will be allocated a linear number of request. Moreover, under the assumption that the reward matrix R is rank d, the users' latent vectors are $d-$dimensional and they span the space (otherwise, the matrix rank would be lower). This implies in particular that each summit of the convex set defined by the master columns in J^* is the best arm for a linear number of rounds. Thus, intuitively, a *uniformly efficient* strategy should pull each arm in j^* a linear number of times. However, all the suboptimal arms $j \notin J^*$ are never $B(i_t)$ for any i_t so they should be pulled only for the sake of exploration.

In order to frame this exploration-exploitation problem in the usual finitely-armed stochastic bandit setting, we will rewrite the parameter of each arm $j \in [L]$. We introduce the parameters $p_i = \mathbb{P}(i_t = i)$ for each user i and we write

$$\theta_j = \mathbb{E}_{i\sim D_u}[R(i,j)] = \left(\sum_{i=1}^{K} p_i u_i\right)^T v_j,$$

which is the expected reward that the learner receives when he chooses action j in his set. On average over the columns, some rewards have a higher expectation than other due to the uneven representation of the users prefering them. We will denote j_d the $d-th$ best arm. We prove the following theorem

Theorem 2. *The distribution of the rewards associated with each arm $j \in [L]$ is a mixture of K distributions depending on the user. We denote $\mathcal{P}_j$ the probability distribution of the rewards when pulling arm j. For any $\theta^* \in [0,1]$,*

$$\mathcal{K}_{inf}(j;\theta^*) = \inf_{\mathcal{P}}\{KL(\mathcal{P}_j,\mathcal{P})|\mathbb{E}_P[reward] \geq \theta^*\}. \tag{5}$$

Cla: notation is needed here. It's a mess for now.

The expected regret of any uniformly efficient strategy is bounded from below by

$$\liminf_{T\to\infty} \frac{R(T)}{\log(T)} \geq \sum_{j\notin J^*} \frac{\bar{\Delta}_j}{\mathcal{K}_{inf}(j;\theta_{j_d})}.$$

Note that this lower bound takes into account the unknown probability distribution of the users both in the numerator (the gaps are defined in expectation wrt this distribution) and in the denominator (the information quantity $\mathcal{K}_{\text{inf}}$ also depends on it).

Proof. This proof relies on changes of measure (blah blah). Basically, $\mathcal{K}_{\text{inf}}(j;\theta_{j_d})$ is the expected log-likelihood ratio of the observations under two models : the original one and the one where arm j has a modified parameter v_j that gives it a higher expected reward than j_d. This is a quite standard result but the expectation over the users must be handled gently

Cla: I still need to think about it and fix the notations to make it right but I believe the result is true.

■

A.2 About the algorithm

Simple Multiple-Plays bandits as a baseline. One first idea is run a MPB algorithm that builds a list a d items that look better. Unfortunately, the expected regret of such strategy is linear ! Indeed, the optimal action of this kind of method is $A^*_{\text{MP}} = J^*$ that incurs a regret $\sum_{j\in J^*}\bar{\Delta}_j > 0$. This is because the bandit algorithm learns one fixed best action for all arms while the optimal strategy for our problem is to learn the best arm of each user *among the d best arms*.

Hierarchical bandits. To overcome this additional difficulty, we suggest the following general idea:

- We maintain a column bandit that takes the averaged rewards over the rows and learns the best d columns in expectaction over the users;
- At each round, a user i_t pops up and the corresponding *independent* row bandit will make his own recommendation decision after calling the column bandit for advice.
- The column bandit will send a possible set of d *different* arms S_t and the row bandit will choose the final action A_t by constructing a set out of these d suggestions. To simplify exposition, we will assume that two types of set can be constructed: either an *exploratory* one that simply pulls all the suggested arms and a *exploitation* action that decides which is the most promising item among the suggested ones and simply fills the whole list with d identical arms.

Cla: Note that it is really important to be consistent when using the terms action for the list and arms for each individual item of the lits, otherwise it's a mess. I'm doing my best...

In order to decompose the regret, we need to split the rounds when the column bandit recommended the best set J^*.

Fix a row $i \in [K]$ and consider the filtration $\mathcal{F}_i = \{t \leq T : i_i = i\}$. The regret of the corresponding bandit is equal to

$$R_i(T) = \mathbb{E}\left[\sum_{t\in\mathcal{F}_i} r_t \mathbb{S} = \mathbb{J}^*\right] + \mathbb{E}\left[\sum_{t\in\mathcal{F}_i} \mathbb{S} \neq \mathbb{J}^*\right]$$

The idea is now to say

- the first term is bounded by $O(d\log(p_i T))$ because the chosen row bandit algorithm is designed for that,
- the second term is bounded by $O(L\log(T))$ because the column bandit is designed for that.

Even if the intuition seems to go through, it is not completely immediate to prove. The recommendations of the column bandits will be base on the actions and observations gathered by the row bandits and simply averaged over the rows. This means for instance that if the column bandit is TS, its posteriot for arm j at round T is a gaussian (assuming Gaussian noise...) with mean $\sum_i S_{i,j}(T)/\sum_i N_{i,j}(T)$. In order to make sure that the column bandit does learn the best action J^*, we must make sure that the arms in J^* are pulled enough such that the expectation of the optimal action is not badly underestimated. Given that the column bandit cannot directly control the actions, it seems hard.