Non-stochastic Low Rank Bandit

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Abstract

We study the problem of learning the maximum entry of a low-rank non-negative matrix, from sequential observations. In this setting, the learner chooses tuples of rows and columns at every round and observes the product of their values. The main challenge in this setting is that the learner does not observe the individual latent values of rows and columns as its feedback. Diverging from previous works we assume that the preference matrix is non-stochastic and hence our setting is more general in nature. Existing methods for solving similar problems rely on UCB-type algorithms based on constructing conservative confidence interval with the strong assumption that underlying distributions are stochastic and i.i.d. We depart from this standard approach and consider the case when the best row and column pair can be learned jointly with help of two separate bandit algorithms working individually on rows and columns. We propose a simple and computationally efficient algorithm that implements this procedure, which we call Low Rank Bandit, and prove a sub-linear bound on its n-step regret in the rank-1 special case. We evaluate the algorithm empirically on several synthetic and real-world datasets. In all experiments, we outperform existing state-of-the-art algorithms.

1 Introduction

In this work, we study the problem of learning the maximum entry of a low-rank matrix from sequential observations. The low-rank structure is observed in many real-world applications and is a standard assumption in recommender systems [Koren *et al.*, 2009; Ricci, 2011]. Our learning model is motivated by a real-world scenario, where a marketer wants to advertise a product and has K users and L marketing channels. The marketer can choose pairs of users and marketing channels to promote its product. Let the users and marketing channels be the rows and columns of a low rank matrix. Then the goal of the learner is to maximize its Click Through Rate (CTR) by finding the maximum entry of this matrix.

We formalize our learning problem as the following online learning problem. At round t, the learning agent chooses d-tuples of rows and columns where d is the rank of a non-negative and low-rank matrix M. We use the terminology row/user and column/item interchangeably, keeping in sync with our proposed application area. This matrix M is formed by the outer product of row and column latent preferences over d topics. Hence, M encodes the preference of row over items and is termed as row-column preference matrix. Note that the learner does not observe the individual latent vectors of row or column preferences but just their product. The row-column preference matrix M is low-rank at each round t, can vary substantially over time, and does not have to be stochastic. The goal of our learning agent is to minimize the cumulative regret with respect to a best solution in hindsight by finding the maximum entry in M as quickly as possible.

We make four major contributions. First, we formulate our online learning problem as a non-stochastic bandit problem on a class of non-negative low-rank matrices. We identify a family of non-negative low-rank matrices where our problem can be solved statistically efficiently, without actually observing the latent values of individual rows and columns. Second, we propose a computationally-efficient algorithm that implements this idea, which we call low rank bandit algorithm abbreviated as `LRB`. The algorithm has two components, column learning and row learning, which learn the pair of optimal columns and rows respectively. Since we are in the non-stochastic setting we use a variation of the `Exp3` [Auer *et al.*, 2002b] algorithm as our row and column learner. Note that we do not construct any confidence interval or eliminate rows and columns like the existing works. Infact, we use the well known fact that exponentially weighted algorithm like `Exp3` are robust and fast learner to design our algorithm. The Third, we analyze `LRB` and up to problem-specific factors, we prove a $O\left(\frac{(\sqrt{L}+\sqrt{K})\sqrt{n}}{\alpha}\right)$ upper bound on its n-step regret in the special case when rank is 1 and for some $\alpha > 0$. The regret of a naive solution is $O(\sqrt{KLn})$, and is much worse than that of `LRB` when $K \approx L$. Finally, we evaluate `LRB` empirically on several synthetic and real-world problems. Perhaps surprisingly, `LRB` performs well even when our modeling assumptions are violated.

The paper is organized as follows. In Section 2, we introduce the rank-1 setting. In Sections 3 and 4, we propose our rank-1 algorithm and derive a sublinear upper bound on its regret. In Sections 5 and 6, we introduce the rank-d setting and propose an algorithm for it. In Section 7, we evaluate our algorithms empirically. We discuss related works in Section 8 and conclude in Section 9.

2 Rank-1 Background and Settings

Notations: We denote $[n] = \{1, \ldots, n\}$ as the set of the first n positive integers. Let M denote any arbitrary matrix of size $m \times n$ matrix. Let the rank of the matrix be denoted by d. We denote by A^B the set of all vectors whose entries take values from set A and are indexed by set B, where A and B can be any arbitrary sets. For any d and $I \in [m]^d$, $M(I, :)$ denotes a $d \times n$ submatrix of M whose i-th row is $M(I(i), :)$. Similarly, for any d and $J \in [n]^d$, $M(:, J)$ denotes a $m \times d$ submatrix of M whose j-th column is $M(:, J(j))$. Finally, we denote Π_d as the set of all d-permutations. For an element $\pi \in \Pi_d$ and d-dimensional vector v, we denote by $\pi(v)$ the permutation of the entries of v according to π.

Rank-1 Setting: We study the online learning problem of finding the maximum entry of a family of non-stochastic, low-rank and non-negative matrices which we call as the *non-stochastic low-rank bandit problem*. We first analyze the simple rank-1 setup and propose our solution for this setting. Many of the key aspects of our design principle are captured in this rank-1 setting. Let $U_t \in [0, 1]^{K\times 1}$ be the row preference over a single topic and $V_t \in [0, 1]^{L\times 1}$ be the column preference over a single topic. The row-column *preference matrix* $M_t = U_t V_t^\mathsf{T}$ is non-negative, non-stochastic, and rank 1. We assume that,

$$i^* = \arg\max_{i\in[K], t\in[n]} U_t(i, 1), \qquad j^* = \arg\max_{j\in[L], t\in[n]} V_t(j, 1). \quad (1)$$

Thus, this assumption makes sure that the row and column preferences (U_t and V_t respectively) can change with time t, but the best row and column does not change over all time $t \in [n]$. At every round t, the learner chooses one pair of row and column indexed by i_t and j_t respectively and observes their product $U_t(i_t)V_t(i_t)$ as its feedback.

Regret Definition (Rank-1): The goal of the learner is to minimize the expected n-step regret with respect to the optimal solution in the hindsight as follows,

$$R(n) = \sum_{t=1}^{n} \mathbb{E}\left[M_t(i^*, j^*) - M_t(i_t, j_t)\right] \quad (2)$$

where, the expectation is over any random choice of rows and columns.

3 Rank 1 Algorithm

We present the algorithm LRB for the rank 1 setting. The LRB consist of two key components, a row learning algorithm and a column learning algorithm. At every round t the row algorithm suggest the row $i_t \in [K]$ and the column algorithm suggests the column $j_t \in [L]$. Note that in this non-stochastic scenario we use Exp3 as the row and column learning algorithm. The row Exp3 has K arms and the column Exp3 has L arms. The main idea is to use the row Exp3 to learn the best row on average while the column Exp3 learns the best column on average. The learner then observes the reward $M_t(i_t, j_t)$ and updates the row and column Exp3 simultaneously. A key insight to this design is that when both the row and column learner are run simultaneously, they will learn the most rewarding row and column on average and converge on the maximum entry of the the matrix M. From the definition of U_t, for any sequence of n columns, the maximum value is in row i^* (see eq (1)). This means that the row algorithm learns irrespective of what the column algorithm does. From the definition of V_t, for any sequence of n rows, the maximum value is in column j^* (see eq (1)). This means that the column algorithm learns irrespective of what the row algorithm does and even though other entries may change arbitrarily in M_t, i^* and j^* remain fixed for all $t \in [n]$.

Algorithm 1 Low Rank Bandit (LRB) (Rank-1)

```
1: Input: Time horizon n                        ▷ Initialization
2: Initialize RowAlg
3: Initialize ColAlg
4: for t = 1, ..., n do                         ▷ Generate response
5:     Row i_t suggested by RowAlg
6:     Column j_t suggested by ColAlg
7:     Observe M_t(i_t, j_t)                    ▷ Update statistics
8:     Update arm i_t of RowAlg with reward M_t(i_t, j_t).
9:     Update arm j_t of ColAlg with reward M_t(i_t, j_t).
```

4 Analysis of Rank 1 Setting

In this section we analyze the rank-1 LRB and show its regret for a horizon n.

Theorem 1. *Let ColAlg and RowAlg in LRB be Exp3 algorithm, respectively. Then the expected n-step regret of LRB is bounded as*

$$R(n) = O\left(\frac{\left(\sqrt{L} + \sqrt{K}\right)\sqrt{n}}{\alpha}\right)$$

for any $\alpha > 0$ such that $U_t(i, 1) \geq \alpha$ and $V_t(j, 1) \geq \alpha$ for all $i \in [K]$, $j \in [L]$, and $t \in [n]$.

Proof. Let, $(U_t V_t^\mathsf{T})_{t=1}^n$ be a sequence of n non-negative rank-1 matrices such that $U_t \in [0, 1]^{K\times 1}, V_t \in [0, 1]^{L\times 1}$, and the highest entry is $U_t(1)V_t(1)$. Let, $((i_t, j_t))_{t=1}^n$ be a sequence of n row-column pairs chosen by a learning agent. Then the expected n-step regret of the agent is,

$$R(n) = \sum_{t=1}^{n} \mathbb{E}\left[U_t(1)V_t(1) - U_t(i_t)V_t(j_t)\right]$$

where the expectation is over the randomness of the agent. Now note that for any U, V, i, and j in our problem we can show that,

$$\begin{aligned}
&2(U(1)V(1) - U(i)V(j)) \\
&= 2U(1)V(1) - U(i)V(1) - U(1)V(j) + \\
&U(i)V(1) + U(1)V(j) - 2U(i)V(j) \\
&= U(1)(V(1) - V(j)) + V(1)(U(1) - U(i)) + \\
&U(i)(V(1) - V(j)) + V(j)(U(1) - U(i)) \\
&= (U(1){+}U(i))\,(V(1){-}V(j)){+}(V(1){+}V(j))\,(U(1){-}U(i))
\end{aligned}$$

Therefore, the expected n-step regret can be decomposed as,

$$R(n) = \sum_{t=1}^{n} \mathbb{E}[(V_t(1) + V_t(j_t))(U_t(1) - U_t(i_t))] + \sum_{t=1}^{n} \mathbb{E}[(U_t(1)+U_t(i_t))(V_t(1) - V_t(j_t))]$$

Now suppose that all entries of U_t and V_t for all $t = 1, 2, \ldots, n$ are bounded from below by some $\alpha > 0$. Then we get that,

$$\begin{aligned}R(n) &= \sum_{t=1}^{n} \mathbb{E}[(1 + V_t(1)/V_t(j_t))V_t(j_t)(U_t(1) - U_t(i_t))] + \\ &\sum_{t=1}^{n} \mathbb{E}[(1 + U_t(1)/U_t(i_t))U_t(i_t)(V_t(1) - V_t(j_t))] \\ &\leq (1 + \frac{1}{\alpha})\Bigg[\sum_{t=1}^{n} \mathbb{E}[U_t(1)V_t(j_t) - U_t(i_t)V_t(j_t)] + \\ &\sum_{t=1}^{n} \mathbb{E}[U_t(i_t)V_t(1) - U_t(i_t)V_t(j_t)]\Bigg]\end{aligned}$$

B: The argument below is just bla bla bla and needs to be written properly. In particular, it needs to be clear what we condition on and what we take the expectation over.

Finally, we can show that the `ColAlg` using `Exp3` chooses the column j_t at time t and observe reward is $U_t(i_t)V_t(j_t)$. Therefore, the first sum above is bounded by $\sqrt{Ln}$ for any sequence of j_t, and thus also in expectation over the randomness in j_t. Similarly `RowAlg` using `Exp3` chooses the row i_t at time t, and observe reward is $U_t(i_t)V_t(j_t)$. Therefore, the second sum above is bounded by $\sqrt{Kn}$ for any sequence of i_t, and thus also in expectation over the randomness in i_t. Therefore we get the final regret as,

$$R(n) = O\left(\frac{\left(\sqrt{L} + \sqrt{K}\right)\sqrt{n}}{\alpha}\right)$$

□

Discussion: The main idea in Theorem 1 is to decompose the regret of `LRB` into two parts, where `ColAlg` does not suggest j^* and the `RowAlg` does not suggest i^*. The first part is analyzed as follows. `ColAlg` has a sub-linear regret, as we use `Exp3` as the `ColAlg`. Thus, the upper bound on the expected regret that `ColAlg` suggests sub-optimal columns for *any sequence* of rows is bounded. This regret scales as $O(\sqrt{Ln})$ based on the analysis of Auer *et al.* [2002b]. Similarly, we analyze the regret for the `RowAlg` as it also uses the exponentially weighted algorithm `Exp3`. The `RowAlg` suffers a regret of $O(\sqrt{Kn})$ for suggesting sub-optimal rows for *any sequence* of columns . Hence, combining both the pasts we get the regret of order $O\left(\frac{(\sqrt{L}+\sqrt{K})\sqrt{n}}{\alpha}\right)$ in Theorem 1. We use non-stochastic algorithm `Exp3` for `ColAlg` and `RowAlg` because our environment is non-stationary. Finally, we can consider any matrix M of dimension $K \times L$ as a K-bandit problem with each bandit having L arms and the optimal columns are learned separately for each bandit. Such a trivial setting gives rise to the regret bound of order $O(\sqrt{KLn})$. Our regret bound also improves upon this trivial approach.

5 Rank-d Background and Settings

In this section, we study the online learning problem of finding the maximum entry of a family of non-stochastic, low-rank and non-negative matrices for the general rank-d setting.

Hott-topics Assumption: We focus on a family of low-rank matrices, which are known as hott topics [Recht *et al.*, 2012]. We define a *hott-topics matrix* of rank d as $M = UV^{\mathsf{T}}$, where U is a $K \times d$ non-negative matrix and V is a $L \times d$ non-negative matrix that gives rise to the hott-topics structure. In particular, we assume that there exists d rows I^* in U such that each row in U can be represented as a convex combination of rows of I^* and the zero vector. Hence, for $A = \{a \in [0,1]^{d\times 1} : \|a\|_1 \leq 1\}$ each row of U can be expressed as,

$$\forall i \in [K]\ \exists \alpha \in A : U(I^*, :)\alpha = U(i, :)\,, \tag{3}$$

Similarly, we assume that there exist d rows J^* in V such that each row of V can be expressed as a convex combination of rows J^* and the zero vector,

$$\forall j \in [L]\ \exists \alpha \in A : V(J^*, :)\alpha = V(j, :)\,, \tag{4}$$

where $A = \{a \in [0,1]^{d\times 1} : \|a\|_1 \leq 1\}$. One way to conceptualize this very general hott-topics assumption in our setting is to assume that the matrix M represents preferences of rows for columns. $M(i, j)$ is the preference of row i for column j. The rank d of M is the number of latent topics. The matrix U are latent preferences of K rows over d topics, where $U(i, :)$ are the preferences of row $i \in [K]$. The matrix V are latent preferences of L columns in the space of d topics, where $V(j, :)$ are the coordinates of column $j \in [L]$. Without loss of generality, we assume that $U \in [0,1]^{K\times d}$ and $V \in [0,1]^{L\times d}$. We assume that the coordinates are points in a simplex, that is $\|U(i, :)\|_1 \leq 1$ for all $i \in [K]$ and $\|V(j, :)\|_1 \leq 1$ for all $j \in [L]$. Note that our assumptions imply that $M(i, j) \geq 0$ for any $i \in [K]$ and $j \in [L]$.

Rank-d Setting: Again, note that at time t, the preferences of rows over columns are encoded in a $K \times L$ *preference matrix* $M_t = U_t V_t^{\mathsf{T}}$, where U and V are defined as in (3) and (4). We assume that row and column preferences (U_t and V_t respectively) can change with time t. At every round t the learner chooses d-pairs of rows and columns from M_t denoted by $(I_t, J_t) \in \Pi_d([K]) \times \Pi_d([L])$. It then observes all the values from the matrix $M_t(I_t, J_t)$ for all $i_t \in I_t$ and $j_t \in J_t$. The *reward* for the agent for choosing arms (I_t, J_t) at time t is denoted by $r_t(i^*(I_t, J_t), j^*(I_t, J_t))$ such that,

$$(i^*(I, J), j^*(I, J)) = \operatorname*{arg\,max}_{(i,j)\in(I\times J)} M_t(i, j) \tag{5}$$

A remarkable property of our row-column preference matrices M_t is that for any row $i \in [K]$ and any column $j \in [L]$ at any time t,

$$\operatorname*{arg\,max}_{(i,j)\in([K]\times[L])} M_t(i, j) \in (I^*, J^*), \tag{6}$$

where I^* and J^* is defined in (3) and (4). Hence, the hott-topics assumption makes it possible to learn the maximum entry of M_t statistically efficiently as at any time $t \in [n]$ the maximum entry $M_t(i^*, j^*)$ will be in $M_t(I^*, J^*)$. Note that even though different entries of U_t and V_t can attain high rewards at different times but the I^* and J^* remain fixed for all time $t \in [n]$.

Regret Definition (Rank-d): Now we are ready to define our notion of optimality and regret for the general rank-d scenario. Our goal is to minimize the expected n-step regret,

$$R(n) = \sum_{t=1}^{n} \mathbb{E}\left[M_t(i_t^*, j_t^*) - M_t(i^*(I, J), j^*(I, J))\right] , \quad (7)$$

where the expectation is with respect to both randomly choosing rows (I_t) and columns (J_t) by the learning algorithm and potential randomness in the environment.

6 Algorithm for Rank d Setting

Now, we propose the general *Low Rank Bandit (*`LRB`*)* algorithm for solving the family of non-stochastic, non-negative and low-rank matrices of rank d. Again, the goal is to identify the maximum entry of the matrix by quickly identifying the d-best rows or columns. The pseudocode of `LRB` is in Algorithm 2. `LRB` has two main components, column learning and row learning algorithm.

At every round t, the row learning algorithm recommends a list of d rows. But we exploit an additional structure (eq (6)) in our problem to learn the optimal rows I^*. The row learning algorithm are d instances of multi-armed bandit algorithms, which we denote by $\texttt{RowAlg}(k)$ for algorithm $k \in [d]$. $\texttt{RowAlg}(1)$ learns the most rewarding row on average, $\texttt{RowAlg}(2)$ learns the second most rewarding row on average conditioned on the first learned column, and so on. Similarly, column learning algorithm recommends a list of d columns by exploiting the same structure in our rewards. Again the goal of the column learning algorithm `ColAlg` is to learn the optimal set of columns J^*. Hence, $\texttt{ColAlg}(1), \ldots, \texttt{ColAlg}(d)$ learns the most rewarding columns in $[J^*]$ on average. Note that this sequence of learning the rows or columns first does not matter because the *hott-topics* structure is defined on both U and V matrix (eq (3), (4)) generating $M_t = U_t V_t^{\mathsf{T}}$, and so we will be learning the d-best rows or columns in average. Another way of looking at this is to first realize that if we fix the column selection strategy, which is simply some distribution over d-tuples of chosen columns then for any such distribution, the d hott-topic rows are the optimal solution to the row selection problem. By symmetry, the same is true for the column selection problem. If we run both in parallel, and the distributions in the other dimensions do not change too fast (this is true by our design in eq (5)), then $i_1, \ldots, i_d$ and $j_1, \ldots, j_d$ would slowly converge to the d hott-topic rows and columns.

Finally, `LRB` observes the individual rewards of $M(i, j)$ for all $(i, j) \in (I_t, J_t)$. Then we update both column and row learning algorithms. The reward of the arm in $\texttt{RowAlg}(k)$, which selects the k-th row in I_t, is updated as follows. If the k-th arm was not previously suggested row its reward is updated d times such that, $\max_{k \le k_1} M_t(i_k, j_{k_2}) - \max_{k<k_1} M_t(i_k, j_{k_2})$ for all $k_1, k_2 \in [d]$. By the choice of our design and previous argument a similar update is performed on the k-th column learning algorithm `ColAlg` such that its reward is also updated d times such that, $\max_{k \le k_2} M_t(i_{k_1}, j_k) - \max_{k<k_2} M_t(i_{k_1}, j_k)$ for all $k_1, k_2 \in [d]$. Otherwise, if any of the row or column has been previously selected by the corresponding `RowAlg` or `ColAlg` algorithm then we update it with reward 0. This type of update makes sure that the $\texttt{RowAlg}(k)$ or $\texttt{ColAlg}(k)$ learns the k-th best row or column conditioned on the selection of previous $1, \ldots, k-1$ `RowAlg` or `ColAlg`. If there is conflict of suggestion, only one `RowAlg` or `ColAlg` is updated so that each can focus on a distinct row or column. A similar type of update is also done in a much simpler setting of Ranked Bandit Algorithm (RBA) in Radlinski *et al.* [2008].

Algorithm 2 Low Rank Bandit (`LRB`) (Rank-d)

Input: Time horizon n, Rank d
for $k = 1, \ldots, d$ **do** ▷ Initialization
  Initialize $\texttt{RowAlg}(k)$
  Initialize $\texttt{ColAlg}(k)$
for $t = 1, \ldots, n$ **do**
  for $k = 1, \ldots, d$ **do** ▷ Generate response
    $\hat{i}_k \leftarrow$ Suggested row i_t by $\texttt{RowAlg}(k)$
    if $\hat{i}_k \in \{i_1, \ldots, i_{k-1}\}$ **then**
      $i_k \leftarrow$ Random row not in $\{i_1, \ldots, i_{k-1}\}$
    else
      $i_k \leftarrow \hat{i}_k$
    $\hat{j}_k \leftarrow$ Suggested column j_t by $\texttt{ColAlg}(k)$
    if $\hat{j}_k \in \{j_1, \ldots, j_{k-1}\}$ **then**
      $j_k \leftarrow$ Random column not in $\{j_1, \ldots, j_{k-1}\}$
    else
      $j_k \leftarrow \hat{j}_k$
  $I_t \leftarrow (i_1, \ldots, i_d)$
  $J_t \leftarrow (j_1, \ldots, j_d)$
  Observe $M_t(I_t(k), J_t(k))$ for all $k \in [d]$
  for $k_1 = 1, \ldots, d$ **do**
    for $k_2 = 1, \ldots, d$ **do** ▷ Update statistics
      if $i_k = \hat{i}_k$ **then**
        Update arm i_k of $\texttt{RowAlg}(k)$ with reward
        $$\max_{k \le k_1} M_t(i_k, j_{k_2}) - \max_{k<k_1} M_t(i_k, j_{k_2})$$
      else
        Update $\hat{i}_k$ of $\texttt{RowAlg}(k)$ with reward 0
      if $j_k = \hat{j}_k$ **then**
        Update arm j_k of $\texttt{ColAlg}(k)$ with reward
        $$\max_{k \le k_2} M_t(i_{k_1}, j_k) - \max_{k<k_2} M_t(i_{k_1}, j_k)$$
      else
        Update $\hat{j}_k$ of $\texttt{ColAlg}(k)$ with reward 0

6.1 Practical Considerations

We leave the implementation of the `ColAlg` and `RowAlg` to the users. Motivated by rank 1 we use non-stochastic algorithm `Exp3` as `ColAlg` and `RowAlg` and showed how the regret scales for the rank-1 setting. For experimental purposes, stochastic algorithms like UCB1 or Thompson Sampling can also be used to improve the performance of LRB. This has also been explored in Radlinski *et al.* [2008] where RBA uses UCB1 for ranking items. The proposed LRB algorithm only has to update through $(K + L)d$ entries for the d ColAlg and the RowAlg respectively at every round t. This is in stark contrast to some of the existing matrix completion algorithms which has to reconstruct a $K \times L$ matrix [Sen *et al.*, 2016] or calculate second or third order tensors [Gopalan *et al.*, 2016].

7 Experiments

In this section, we compare LRB to several bandit algorithms in three experiments. The first two experiments are on synthetic problems where all modeling assumptions hold. The third experiment is on a real-world dataset where we evaluate LRB when our modeling assumptions fail. All results are averaged over 10 independent random runs. We test in both rank 1 and rank 2 settings to clearly illustrate the failures of the current Rank-1 algorithms and show the efficiency of our proposed method.

7.1 Experiments on Rank-1 Setting

Evaluated Algorithms for Rank-1: We compare against several state-of-the-art rank 1 algorithms. Note all the rank 1 algorithms suggest a single row and column at every round. The UCB1 algorithm from Auer *et al.* [2002a] builds a confidence set at every round t over all the entries of M_t as $c_{i,j}(t) = \sqrt{\frac{2\log t}{N_{i,j}(t)}}$ where $N_{i,j}(t)$ denotes the number of times the $M(i,j)$-th entry has been observed. It the suggests the best row-column pair based on the term $\hat{M}_t(i,j) + c_{i,j}(t)$ where $\hat{M}_t(i,j)$ denotes the empirical mean of all the observed rewards for $M(i,j)$. The UCB1-Elim [Auer and Ortner, 2010] is similar to UCB1 but it eliminates sub-optimal rows and columns based on a similar confidence set $c_{i,j}(t)$ till it finally converges on the best pair of row and column. The algorithm LinUCB was first proposed in Li *et al.* [2010] for the contextual bandit setting. Note that rank-1 bandit generalizes to the stochastic linear bandit setting and can be solved by LinUCB. Similarly, GLM-UCB from Filippi *et al.* [2010] can also be used solve the rank 1 bandit problem. Finally, we compare against the algorithm $\texttt{Rank1} - \texttt{Elim}$ from Katariya *et al.* [2016] which is an improved version of UCB1-Elim and employs row and column elimination and aggressive exploration to converge on the best row and column pair. For LRB we use the Algorithm 1 from Section 3. We use eq (2) definition to calculate regret in rank-1 setting.

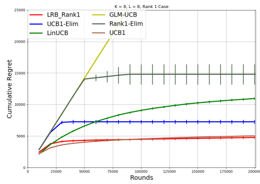
Synthetic Experiment 1 **for Rank-1:** This experiment is conducted to test the performance of LRB over a small number of rows and columns and to show how LRB scales with increasing number of rows and columns. Note that in this experiment all our modeling assumptions hold. This simulated testbed consist of two scenarios: (1) 8 rows and 8 columns and (2) 16 rows and 16 columns. In this setting, $U = \{0.7, 0.9\}^{K\times 1}$ and similarly $V = \{0.7, 0.9\}^{L\times 1}$ with only the entry $U(K/2, 1) = V(L/2, 1) = 0.9$. Hence, the matrix $M = UV^\intercal$ is rank 1 and the hott-topics structure is maintained. At every round t, we generate the matrix $M_t = UD_tV^\intercal$ where D_t is a randomly generated value from $[0, 1]$. Note for all time $t \in [n]$, M_t is rank-1 matrix with its maximum value always at $M_t(K/2, L/2)$, and other entries changing arbitrarily but always less than $M_t(K/2, L/2)$. The learner observes the entry $M_t(i, j)$ when it selects the i-th row and j-th column. A similar environment has been discussed as B_{spike} in Katariya *et al.* [2016]. From Figure 1(a) and 1(b) we can clearly see that LRB outperforms all the other algorithms. The regret curve of LRB flattens, indicating that it has learned the best row-column pair. As we scale the number of rows and columns we see that LRB performs even better than other algorithms.

7.2 Experiments on Rank-2 Setting

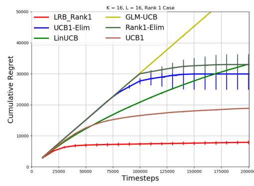
Evaluated Algorithms for Rank-2: We design the rank-2 algorithms by modifying the rank-1 algorithms. Again note that all the rank-2 algorithms suggest two pairs of rows and columns at every round t. For all of the algorithms UCB1, UCB1-Elim, LinUCB, $\texttt{GLM} - \texttt{UCB}$, and $\texttt{Rank1} - \texttt{Elim}$ we modify these algorithms so that they suggest 2 pairs of rows and columns based on their respective confidence interval set $c_{i,j}(t)$. The row and column pair with the highest and the second highest $\hat{M}_t(i,j) + c_{i,j}(t)$ are suggested for each round t and consequently after observing all the entries of $M_t(i,j)$ all of the algorithms update their estimates of $\hat{M}_t(i,j)$ for each $i, j \in [d]$. For LRB we use the Algorithm 2 from Section 6. Note that there are two RowAlg and ColAlg, each running an Exp3 algorithm with the exploration parameters as discussed before. For LRB-rank2 we use the Algorithm 2 from Section 6. We also modify the rank-1 LRB (Algorithm 1) so that the algorithm works in rank-2 setting. After LRB-rank1 has sampled one pair of rows and columns from $[K]$ and $[L]$, it then samples again another choice that does not clash with the first pair and then updates all the pairs with the feedback observed. We use eq (7) definition to calculate regret in rank-2 setting.

Synthetic Experiment 2**:** This experiment is conducted to test the performance of LRB over a large number of rows and columns. This simulated testbed consist of 64 rows, 64 columns, and rank$(M) = 2$. The vectors spanning U and V, generating the row-column preference matrix M, are shown Figure 1(c). The rows and columns are evenly distributed into a $50 : 50$ split such that 50% of rows prefer column 1 and 50% rows prefer column 2. The column hott-topics are $V(1,:) = (1, 0)$ and $V(2,:) = (0, 0.6)$ while 50% remaining columns has feature $V(j',:) = (0.45, 0.5)$ and the rest have $V(j,:) = (0.5, 0.45)$. Similarly, we create the row feature matrix U having a $50 : 50$ split such that $U(1,:) = (1, 0)$, $U(2,:) = (0, 0.6)$ and the remaining 50% rows having $U(i,:) = (0.5, 0.4)$ and the rest having $U(i',:) = (0.4, 0.5)$. At every timestep t the resulting matrix $M_t = UD_tV^\intercal$ is generated where D_t is a randomly-generated diagonal matrix. From Figure 1(b) we can clearly see that LRB-rank2 outperforms all the other algorithms. It's regret curve flattens, indicating that it has learned the best row-column pair. The key realization is that LRB takes advantage of the hott-topics structure and quickly identifies them. Note that for any rank d setting the best row-column pair must be one of the hott-topics in (I^*, J^*). Also note the failure of

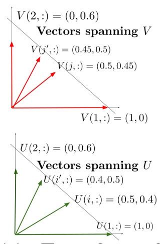
LRB-rank1 in this setting which clearly shows why a general rank-d algorithm with our specific type of update is required.



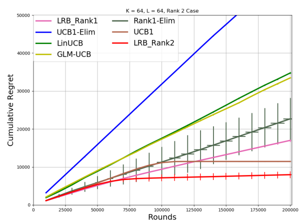
(a) Expt-1: 8 rows, 8 columns, Rank 1 Setting



(b) Expt-1: 16 rows, 16 columns, Rank 1 Setting



(c) Expt-2: 64 rows, 64 columns, row and column vectors



(d) Expt-2: 64 rows, 64 columns, Rank 2 Setting

Figure 1: A comparison of LRB vs state-of-the-art algorithms.

Real-world Experiment 3**:** We conduct the third experiment to test the performance of LRB when our modeling assumptions are violated. We use the Jester dataset [Goldberg *et al.*, 2001] which consist of over 4.1 million continuous ratings of 100 jokes from 73,421 users collected over 5 years. In this dataset there are many users who rated all jokes and we work with these users. We sample randomly 20 users (who have rated all jokes) from this dataset and use singular value decomposition (SVD) to obtain a rank 2 approximation of this user-joke rating matrix M. In the resultant matrix M, most of the rows belong to the two classes preferring jokes 98, and 28, while a very small percentage of users prefer some other jokes. Note that this condition results from the fact that this real-life dataset does not have the hott-topics structure. Furthermore, in this experiment we assume that the noise is independent Bernoulli over the entries of M and hence this experiment deviates from our modeling assumptions. In Figure 2 we see that LRB-rank2 outperform other algorithms. Finally, LRB-rank1 again fails to perform well in this setting.

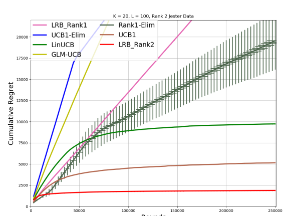


Figure 2: Expt-3: 20 rows, 100 columns, Jester Dataset

8 Related Work

Previous works that have studied this setting have focused either on the rank-1 setting or proposed solution where the underlying distributions are stochastic with additional structures and assumptions.

Rank-1 **Setting:** The work of Katariya *et al.* [2016] was proposed for a rank-1 bandit model with the assumption that the underlying distributions are stochastic. Similarly, Katariya *et al.* [2017] was proposed for the special case when the underlying distributions are Bernoulli. A more simpler setting has also been studied in Maillard and Mannor [2014]. All of these works used different variations of the Upper Confidence Bound (UCB) algorithm Auer *et al.* [2002a], [Auer and Ortner, 2010] algorithm to construct a confidence interval set over row-column pairs to identify and eliminate sub-optimal rows and columns. These naturally results in algorithms that explore conservatively (for the sake of row and column elimination) and cannot work beyond the stochastic distribution assumption. As opposed to these earlier works our method is focused on the more general and natural non-stochastic setting.

Rank-d **Setting:** The work of Kveton *et al.* [2017] can be viewed as a generalization of rank-1 bandits of Katariya *et al.* [2016] to a higher rank of d. However, this work proposes a phase-based algorithm that calculates the square of the determinant of a $d \times d$ sub-matrix to eliminate sub-optimal rows and columns at the end of phases which is impractical for very large non-negative low-rank matrices. The theoretical guarantees hold for only stochastic distributions. Some other approaches involving non-negative matrix factorization Sen *et al.* [2016] or tensor based methods [Gopalan *et al.*, 2016] to reconstruct the matrix have also been proposed. These works require strong assumptions on the structure of the matrix such as all the matrices satisfy a weak statistical Restricted Isometric Property (RIP) or calculate third order tensors as in Anandkumar *et al.* [2014]. On the contrary, our simple and statistically efficient algorithm is easily generalizable to rank-d and do not require any sort of costly matrix inversion or reconstruction operations or even row or column eliminations and hence are much easier to implement.

9 Conclusions

In this paper, we studied the problem of finding the highest entry of a non-stochastic, non-negative low-rank matrix. We formulated the above problem as an online-learning problem and proposed the LRB algorithm for this setting. We proved that an instance of algorithm has a regret bound in the special case of rank-1 setting that scales as $O\big(\frac{(\sqrt{L}+\sqrt{K})\sqrt{n}}{\alpha}\big)$ and has the correct order with respect to rows, columns and rank of the row-column preference matrix M. We also evaluated our proposed algorithm on several simulated and real-life datasets and show that it outperforms the existing state-of-the-art algorithms. There are several directions where this work can be extended. Note that we only proved our theoretical results for the rank 1 setting. Proving theoretical guarantees for LRB algorithm will require additional assumptions on the structure of rewards and the matrix M. Another interesting direction is to look at structures beyond hott-topics assumption on row and column matrix.

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