3.2 Properties of z-Transform

Linearity: If $x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z)$

$x_2(n) \longleftrightarrow X_2(z)$

Then, $x(n) = a_1 x_1(n) + a_2 x_2(n) \stackrel{z}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z)$

Ex. 3.2.1

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

$$= \underline{3(2^n)} u(n) - \underline{4(3^n)} u(n)$$

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

$$\boxed{a^n u(n) \xrightarrow{z} \frac{1}{1-az^{-1}}}$$

Time shifting

If $x(n) \xrightarrow{z} X(z)$

$x(n-k) \xrightarrow{z} z^{-k} X(z)$

$\delta(n) \xrightarrow{z} 1$

$\delta(n-k) \rightarrow z^{-k}$

Ex. 3.2.3 (H.W)

Ex. 3.2.4 $x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

$\boxed{a = z^{-1}}$

$$X(z) = \sum_{n=0}^{N-1} 1 \cdot z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots z^{N-1}$$

$$= \frac{1 - z^{-N}}{1 - z^{-1}}$$

$\boxed{\dfrac{1 - a^{N}}{1 - a}}$ $\boxed{\dfrac{1}{1 - a}}$

Find the z-Transform

$$x(n) = u(n) - u(n-N)$$

$$X(z) = Z\left[u(n) - u(n-N)\right]$$

$$= Z\left[u(n)\right] - Z\left[u(n-N)\right]$$

$$= \frac{1}{1-z^{-1}} - z^{-N} \cdot \frac{1}{1-z^{-1}}$$

$$= \left(1 - z^{-N}\right) \cdot \frac{1}{1-z^{-1}}$$

Differentiation in the z-domain

if $x(n) \overset{z}{\longleftrightarrow} X(z)$

then $n\,x(n) \longleftrightarrow -z \dfrac{dX(z)}{dz} \longrightarrow$ Proof: H.W.

Ex. 3.2.7

$$x(n) = n\,a^n u(n)$$

$$a^n u(n) \longrightarrow \frac{1}{1-az^{-1}} \longrightarrow (1-az^{-1})^{-1}$$

$$z\left[n\,a^n u(n)\right] = -z \frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] = -z \cdot \frac{-az^{-2}}{(1-az^{-1})^2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}.$$

Convolution of two sequences

if $\quad x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

$$x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z) \cdot X_2(z)$$

$$X(z)$$

$$\downarrow$$

$$x(n)$$

Inverse z $\rightarrow$ $x_1(n) * x_2(n)$

Ex. 3.2.9 $\boxed{X(z) = \sum x(n) z^{-n}}$

$x_1(n) = \{\underline{1}, -2, 1\}$, $x_2(n) = \{1, 1, 1, 1, 1, 1\}$

Compute the convolution $x(n)$ of the signals.

$X_1(z) = 1 - 2z^{-1} + z^{-2}$, $X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$X(z) = X_1(z) \cdot X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$

$x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$

Steps for finding convolution using z-Tranform

1. Compute the z-Tran of the signals to be convolved

$$\checkmark X_1(z) = Z\{x_1(n)\}$$

$$\checkmark X_2(z) = Z\{x_2(n)\}$$

2. Multiply the two z-transforms

$$X(z) = X_1(z)X_2(z)$$

3. Find the inverse z-transform of $X(z)$

$$x(n) = Z^{-1}\{X(z)\}$$