

# Appendix: A Plot is Worth a Thousand Tests: Assessing Residual Diagnostics with the Lineup Protocol

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## ARTICLE HISTORY

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## Appendix A. Additional details of testing procedures

### A.1. *Effect size derivation*

Effect size can be defined as the difference of a parameter for a particular model or distribution, or a statistic derived from a sample. Importantly, it needs to reflect the treatment we try to measure. Centred on a conventional statistical test, we usually can deduce the effect size from the test statistic by substituting the null parameter value. When considering the diagnostics of residual departures, there exist many possibilities of test statistics for a variety of model assumptions. Meanwhile, diagnostic plots such as the residual plot have no general agreement on measuring how strong a model violation pattern is. To build a bridge between various residual-based tests, and the visual test, we focus on the shared information embedded in the testing procedures, which is the distribution of residuals. When comes to comparison of distribution, Kullback-Leibler divergence (Kullback and Leibler 1951) is a classical way to represent the information loss or entropy increase caused by the approximation to the true distribution, which in our case, the inefficiency due to the use of false model assumptions.

Following the terminology introduced by Kullback and Leibler (1951),  $P$  represents the measured probability distribution, and  $Q$  represents the assumed probability distribution. The Kullback-Leibler divergence is defined as  $\int_{-\infty}^{\infty} \log(p(x)/q(x))p(x)dx$ , where  $p(\cdot)$  and  $q(\cdot)$  denote probability densities of  $P$  and  $Q$ .

Let  $\mathbf{X}$  denotes the  $p + 1$  regressors with  $n$  observations,  $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$  denotes the OLS solution,  $\mathbf{R} = \mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$  denotes the residual operator, and let  $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$  denotes the error. The residual vector

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$$\begin{aligned}
\mathbf{e} &= \mathbf{y} - \mathbf{X}\mathbf{b} \\
&= \mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\
&= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} \\
&= \mathbf{R}\mathbf{y} \\
&= \mathbf{R}(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) \\
&= \mathbf{R}\boldsymbol{\varepsilon}.
\end{aligned}$$

Because  $\text{rank}(\mathbf{R}) = n - p - 1 < n$ ,  $\mathbf{e}$  follows a degenerate multivariate normal distribution and does not have a density. Since the Kullback-Leibler divergence requires a proper density function, we need to simplify the covariance matrix of  $\mathbf{e}$  by setting all the off-diagonal elements to 0. Then, the residuals will assumed to follow  $N(\mathbf{0}, \text{diag}(\mathbf{R}\sigma^2))$  under the null hypothesis that the model is correctly specified. If the model is however misspecified due to omitted variables  $\mathbf{Z}$ , or a non-constant variance  $\mathbf{V}$ , the distribution of residuals can be derived as  $N(\mathbf{R}\mathbf{Z}\boldsymbol{\beta}_z, \text{diag}(\mathbf{R}\sigma^2))$  and  $N(\mathbf{0}, \text{diag}(\mathbf{R}\mathbf{V}\mathbf{R}'))$  respectively.

By assuming both  $P$  and  $Q$  are multivariate normal density functions, the Kullback-Leibler divergence can be rewritten as

$$KL = \frac{1}{2} \left( \log \frac{|\Sigma_p|}{|\Sigma_q|} - n + \text{tr}(\Sigma_p^{-1}\Sigma_q) + (\mu_p - \mu_q)' \Sigma_p^{-1} (\mu_p - \mu_q) \right).$$

Then, we can combine the two residual departures into one formula

$$KL = \frac{1}{2} \left( \log \frac{|\text{diag}(\mathbf{R}\mathbf{V}\mathbf{R}')|}{|\text{diag}(\mathbf{R}\sigma^2)|} - n + \text{tr}(\text{diag}(\mathbf{R}\mathbf{V}\mathbf{R}')^{-1}\text{diag}(\mathbf{R}\sigma^2)) + \boldsymbol{\mu}_z^T (\mathbf{R}\mathbf{V}\mathbf{R}')^{-1} \boldsymbol{\mu}_z \right).$$

When there are omitted variables but constant error variance, the formula can be reduced to

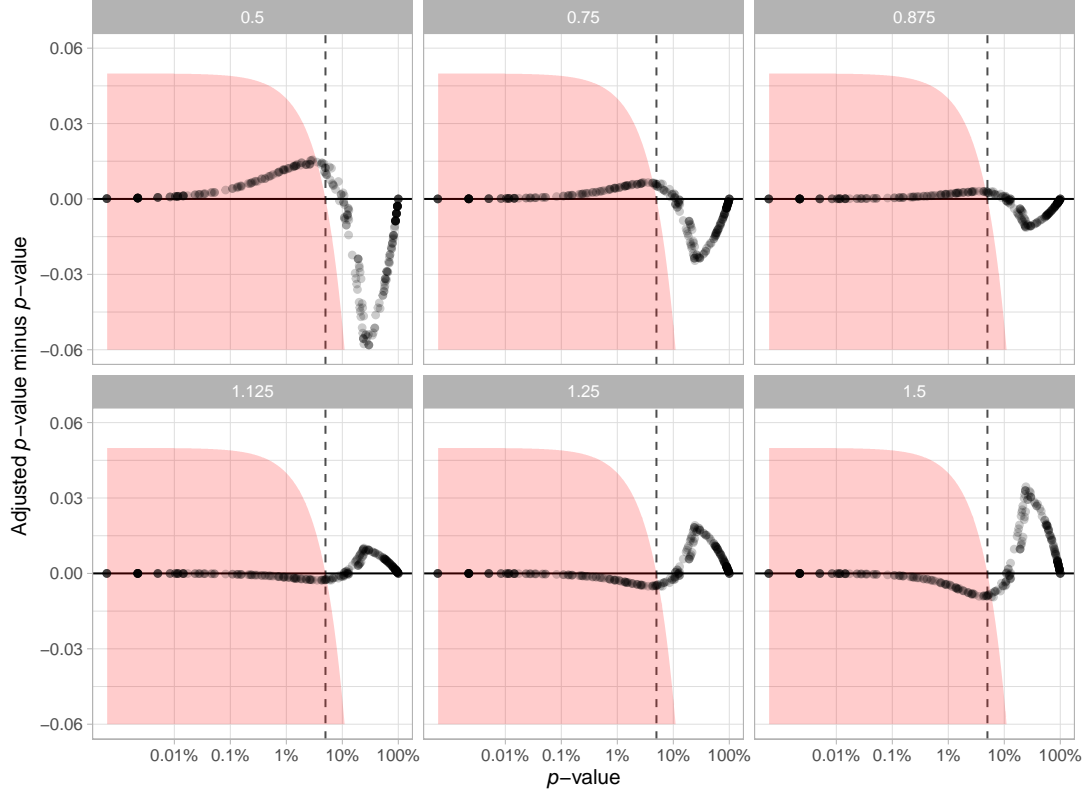
$$KL = \frac{1}{2} (\boldsymbol{\mu}_z^T (\text{diag}(\mathbf{R}\sigma^2))^{-1} \boldsymbol{\mu}_z).$$

And when the model equation is correctly specified but the error variance is non-constant, the formula can be reduced to

$$KL = \frac{1}{2} \left( \log \frac{|\text{diag}(\mathbf{R}\mathbf{V}\mathbf{R}')|}{|\text{diag}(\mathbf{R}\sigma^2)|} - n + \text{tr}(\text{diag}(\mathbf{R}\mathbf{V}\mathbf{R}')^{-1}\text{diag}(\mathbf{R}\sigma^2)) \right).$$

Since we assume  $\sigma = 1$  for the heteroskedasticity model, the final form of the formula is

$$KL = \frac{1}{2} \left( \log \frac{|\text{diag}(\mathbf{R}\mathbf{V}\mathbf{R}')|}{|\text{diag}(\mathbf{R})|} - n + \text{tr}(\text{diag}(\mathbf{R}\mathbf{V}\mathbf{R}')^{-1}\text{diag}(\mathbf{R})) \right).$$



**Figure A1.** Change of  $p$ -values with  $\hat{\alpha}$  multiplied by 0.5, 0.75, 0.875, 1.125, 1.25 and 1.5. Only lineups with uniform fitted value distribution is used. The vertical dashed line indicates  $p\text{-value} = 0.05$ . The area coloured in red indicates adjusted  $p\text{-value} < 0.05$ . The x-axis is drawn on logarithmic scale. For multipliers smaller than 1, the adjusted  $p$ -value will initially increase and decline when the  $p$ -value increases. The trend is opposite with multipliers greater than 1, but the difference eventually reaches 0.

## A.2. Sensitivity analysis for $\alpha$

The parameter  $\alpha$  used for the  $p$ -value calculation needs to be estimated from responses to null lineups. With a greater value of  $\hat{\alpha}$ , the  $p$ -value will be smaller, resulting in more lineups being rejected. However, The way we generate Rorschach lineup is not strictly the same as what suggested in VanderPlas et al. (2021) and Buja et al. (2009). Therefore, we conduct a sensitivity analysis in this section to examine the impact of the variation of the estimator  $\alpha$  on our primary findings.

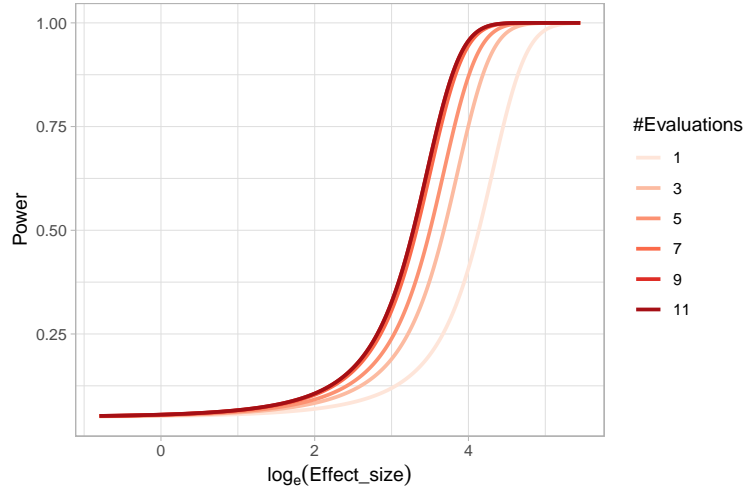
The analysis is conducted by setting up several scenarios, where the  $\alpha$  is under or overestimated by 12.5%, 25% and 50%. Using the adjusted  $\hat{\alpha}$ , we recalculate the  $p$ -value for every lineup and show the results in Figure A1. It can be observed that there are some changes to  $p$ -values, especially when the  $\hat{\alpha}$  is multiplied by 50%. However, Table A1 shows that adjusting  $\hat{\alpha}$  will not result in a huge difference in rejection decisions. There are only a small percentage of cases where the rejection decision change. It is very unlikely the downstream findings will be affected because of the estimate of  $\alpha$ .

## A.3. Effect of number of evaluations on the power of a visual test

When comparing power of visual tests across different fitted value distributions, we have discussed the number of evaluations on a lineup will affect the power of the visual

**Table A1.** Examining how decisions might change if  $\hat{\alpha}$  was different. Percentage of lineups that where the  $p$ -value would switch to above or below 0.05, when  $\hat{\alpha}$  is multiplied by a multiplier.

Multiplier	Reject to not reject	%	Not reject to reject	%
0.500	7	2.51	0	0.00
0.750	4	1.43	0	0.00
0.875	3	1.08	0	0.00
1.125	0	0.00	3	1.08
1.250	0	0.00	4	1.43
1.500	0	0.00	5	1.79

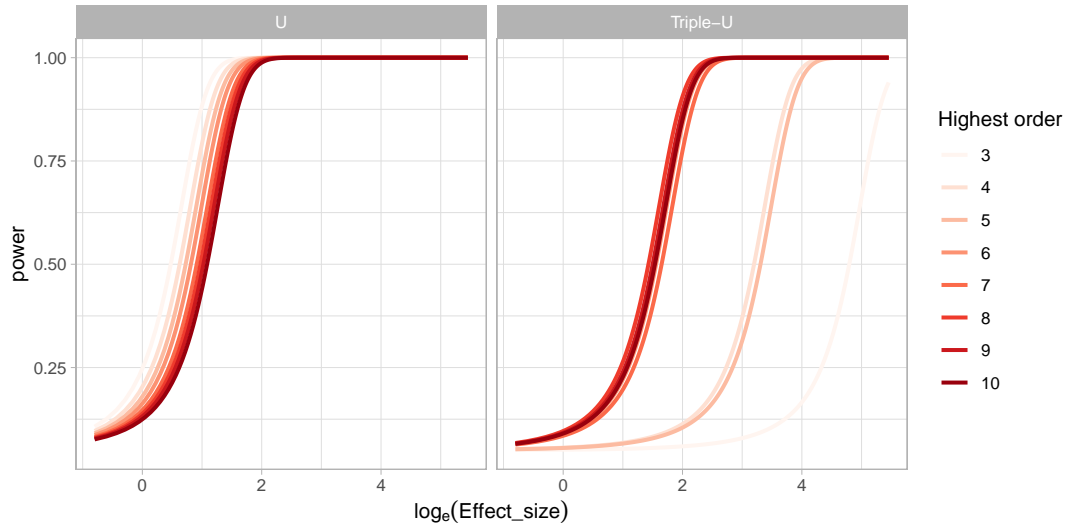


**Figure A2.** Change of power of visual tests for different number of evaluations on lineups with uniform fitted value distribution. The power will increase as the number of evaluations increases, but the margin will decrease.

test. Using the lineups with uniform fitted value distribution, we show in Figure A2 the change of power of visual tests due to different number of evaluations. It can be learned that as the number of evaluations increases, the power will increase but the margin will decrease. Considering we have eleven evaluations on lineups with uniform fitted value distribution, and five evaluations on other lineups, it is necessary to use the same number of evaluations for each lineup in comparison.

#### A.4. *Power of a RESET test under different auxiliary regression formulas*

It is found in the result that the power of a RESET test will be affected by the highest order of fitted values included in the auxiliary formula. And we suspect that the current recommendation of the highest order - four, is insufficient to test complex non-linear structures such as the “Triple-U” shape designed in this paper. Figure A3 illustrates the change of power of RESET test while testing the “U” shape and the “Triple-U” shape with different highest orders. Clearly, when testing a simple shape



**Figure A3.** Change of power of RESET tests for different orders of fitted values included in the auxiliary formula. The left panel is the power of testing the "U" shape and the right panel is the power of testing the "Triple-U" shape. The power will not be greatly affected by the highest order in the case of testing the "U" shape. In the case of testing the "Triple-U" shape, the highest order needs to be set to at least six to avoid the loss of power.

like the "U" shape, the highest order has very little impact on the power. But for testing the "Triple-U" shape, there will be a loss of power if the recommended order is used. To avoid the loss of power, the highest order needs to be set to at least six.

## Appendix B. Experiment setup

### B.1. Mapping of subjects to experimental factors

Mapping of subjects to experimental factors is an important part of experiment design. Essentially, we want to maximum the difference in factors exposed to a subject. For this purpose, we design an algorithm to conduct subject allocation. Let  $L$  be a set of available lineups and  $S$  be a set of available subjects. According to the experimental design, the availability of a lineup is associated with the number of subjects it can assign to. For lineups with uniform fitted value distribution, this value is 11. And other lineups can be allocated to at most five different subjects. The availability of a subject is associated with the number of lineups that being allocated to this subject. A subject can view at most 18 different lineups.

The algorithm starts from picking a random subject  $s \in S$  with the minimum number of allocated lineups. It then tries to find a lineup  $l \in L$  that can maximise the distance metric  $D$  and allocate it to subject  $s$ . Set  $L$  and  $S$  will be updated and the picking process will be repeated until there is no available lineups or subjects.

Let  $F_1, \dots, F_q$  be  $q$  experimental factors, and  $f_1, \dots, f_q$  be the corresponding factor values. We say  $f_i$  exists in  $L_s$  if any lineup in  $L_s$  has this factor value. Similarly,  $f_i f_j$  exists in  $L_s$  if any lineup in  $L_s$  has this pair of factor values. And  $f_i f_j f_k$  exists in  $L_s$  if any lineup in  $L_s$  has this trio of factor values. The distance metric  $D$  is defined between a lineup  $l$  and a set of lineups  $L_s$  allocated to a subject  $s$  if  $L_s$  is non-empty:

$$D = C - \sum_{1 \leq i \leq q} I(f_i \text{ exists in } L_s) - \sum_{\substack{1 \leq i \leq q-1 \\ i < j \leq q}} I(f_i f_j \text{ exists in } L_s) - \sum_{\substack{1 \leq i \leq q-2 \\ i < j \leq q-1 \\ j < k \leq q}} I(f_i f_j f_k \text{ exists in } L_s)$$

where  $C$  is a sufficiently large constant such that  $D > 0$ . If  $L_s$  is empty, we define  $D = 0$ .

The distance measures how different a lineup is from the set of lineups allocated to the subject in terms of factor values. Thus, the algorithm will try to allocate the most different lineup to a subject at each step.

### B.2. Data collection process

The survey data is collected via a self-hosted website designed by us. The complete architecture is provided in Figure B1. The website is built with the **Flask** (Grinberg 2018) web framework and hosted on **PythonAnywhere** (PythonAnywhere LLP 2023). It is configured to handle HTTP requests such that subjects can correctly receive web-pages and submit responses. Embedded in the resources sent to subjects, the **jsPsych** front-end framework (De Leeuw 2015) instructs subjects' browsers to render an environment for running behavioral experiments. During the experiment, this framework will automatically collect common behavioral data such as response time and clicks on buttons. Subjects' responses are first validated by a scheduled **Python** script run on the server, then push to a Github repository. Lineup images shown to users are saved in multiple Github repositories and hosted in corresponding Github pages. The URLs to these images are resolved by **Flask** and bundled in HTML files.

Once the participant is recruited from Prolific (Palan and Schitter 2018), it will be redirected to the entry page of our study website. An image of the entry page is provided in Figure B2. Then, the participant needs to submit the online consent form and fill in the demographic information as shown in B3 and B4 respectively. Before



Visual Inference

Completion Progress

Please do not refresh the page. If you have any question, please contact the research team via the Prolific messaging system.

Please read the **Explanatory Statement**. To continue, click the checkbox below and hit "Continue".

This study will take you **10-15 minutes**. You must complete the study in **1 hour**.

Please do not refresh the page during the study.

**EXPLANATORY STATEMENT**  
(Prolific users)

**Project ID:** 29199  
**Project title:** Advances in Artificial Intelligence for Data Visualization: Developing Computer Vision Models to Automate Reading of Data Plots, with Application to Predictive Model Diagnostics

<b>Chief Investigator's name:</b> Emi Tanaka Department of Econometrics and Business Statistics, Monash University email: emi.tanaka@monash.edu	<b>Student's name:</b> Weihao Li Department of Econometrics and Business Statistics, Monash University email: Patrick.li@monash.edu
<b>Other Investigator's name:</b> Dianne Cook Department of Econometrics and Business Statistics, Monash University email: dicook@monash.edu	

You are invited to take part in this study. Please read this Explanatory Statement in full before deciding whether or not to participate in this research. If you would like further information regarding any aspect of this project, you are encouraged to contact the researchers via the email addresses listed above.

☐ I have read and understood the Explanatory Statement.

**Figure B2.** The entry page of the study website.

Visual Inference

Completion Progress

Please do not refresh the page. If you have any question, please contact the research team via the Prolific messaging system.

To continue, click the checkbox below and hit "Continue".

### Consent Form

**Project ID:** 29199  
**Project title:** Advances in Artificial Intelligence for Data Visualization: Developing Computer Vision Models to Automate Reading of Data Plots, with Application to Predictive Model Diagnostics

**Chief Investigator:** Emi Tanaka  
**Other Investigator's name:** Dianne Cook  
**Student's name:** Weihao Li

**I consent to the following:**

☒ The data that I provide during this research may be used by the research team or other researchers in future research projects.

☒ I have been asked to take part in the Monash University research project specified above. I have read and understood the **Explanatory Statement** and I hereby consent to participate in this project.

Continue

**Figure B3.** The consent form provided in the study website.



*Visual Inference* Completion Progress  Please do not refresh the page. If you have any question, please contact the research team via the Prolific messaging system.

### Survey Questions

Please enter your Prolific ID:

Prolific ID

Please select your age category:

18-24 25-39 40-54 55-64 65 or above

Please select your highest level of education:

High school or below Diploma and Bachelor Degree Honours Degree Masters Degree Doctoral Degree

Please select your preferred pronoun:

☐ He  
☐ She  
☐ They  
☐ Other

Have you participated in any research that requires reading data graphs?

☐ Yes  
☐ No

**Figure B4.** The form to provide demographic information.

*Visual Inference* Completion Progress  Please do not refresh the page. If you have any question, please contact the research team via the Prolific messaging system.

Please read the **Training Page**. To continue, click the checkbox below and hit "Continue".

### Training Page (3 min read)

This document will provide you with the essential knowledge to finish the study.

**1 Webpage layout**

When you start the study, you will see a webpage like this:

*Visual Inference* Completion Progress  If you have any question, please contact the research team via the Prolific messaging system.

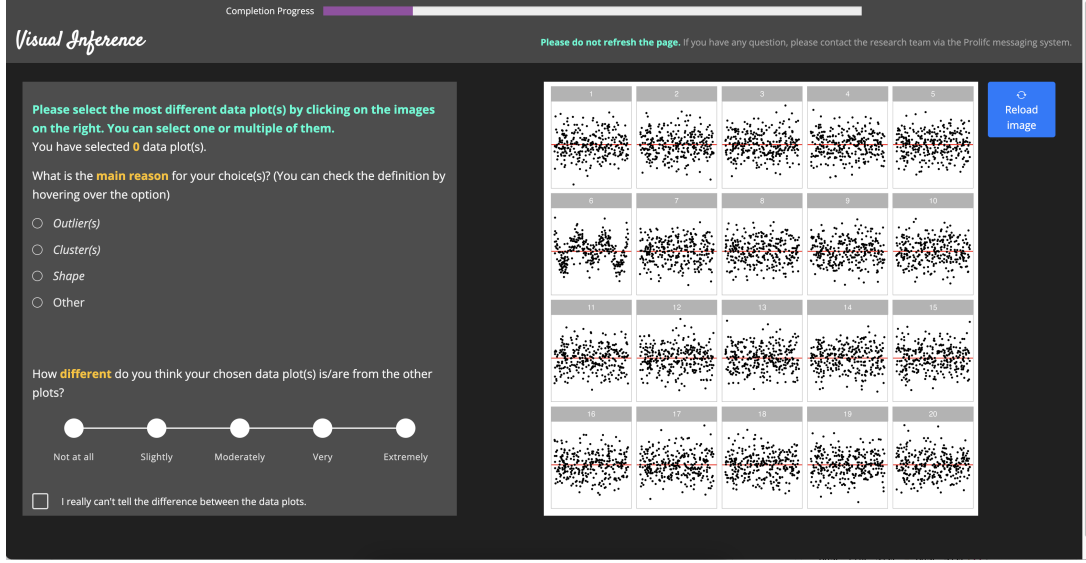
Please select the most different data plot(s) by clicking on the images on the right. You can select one or multiple of them. You have selected 0 data plot(s).

What is the **main reason** for your choice(s)? (You can check the definition by hovering over the option)

☐ Outlier(s)

☐ I have read and understood the Training Page.

**Figure B5.** The training page of the study website.



**Figure B6.** The lineup page of the study website.

**Table C1.** Count of lineups and evaluations over departure types and data collection periods.

Number	Non-linearity			Heteroskedasticity			Total
	I	II	III	I	II	II	
Lineups	576	0	144	0	540	135	1116
Evaluations	2880	0	864	0	2700	810	7254

## Appendix C. Analysis of results relative to data collection process

### C.1. Demographics

Throughout the study, we have collected 7254 evaluations on 1116 non-null lineups. Table C1 gives further details about the number of evaluations and lineups over departure types and data collection periods.

Along with the responses to lineups, we have collected a series of demographic information including age, pronoun, education background and previous experience in studies involved data visualization. Table C2, C3, C4 and C5 provide summary of the demographic data.

It can be observed from the tables that most participants have Diploma or Bachelor degrees, followed by High school or below and the survey data is gender balanced. Majority of participants are between 18 to 39 years old and there are slightly more participants who do not have previous experience than those who have.

### C.2. Data collection periods

We have the same type of model collected over different data collection periods, that may lead to unexpected batch effect. Figure C1 and C2 provide two lineups to examine whether there is an actual difference across data collection periods for non-linearity

**Table C2.** Summary of pronoun distribution of subjects recruited in this study.

Pronoun	Period I	%	Period II	%	Period III	%	Total	%
He	77	17.4	79	17.8	61	13.8	217	49.0
She	78	17.6	77	17.4	61	13.8	216	48.8
Other	5	1.1	4	0.9	1	0.2	10	2.3
	160	36.1	160	36.1	123	27.8	443	100.0

**Table C3.** Summary of age distribution of subjects recruited in this study.

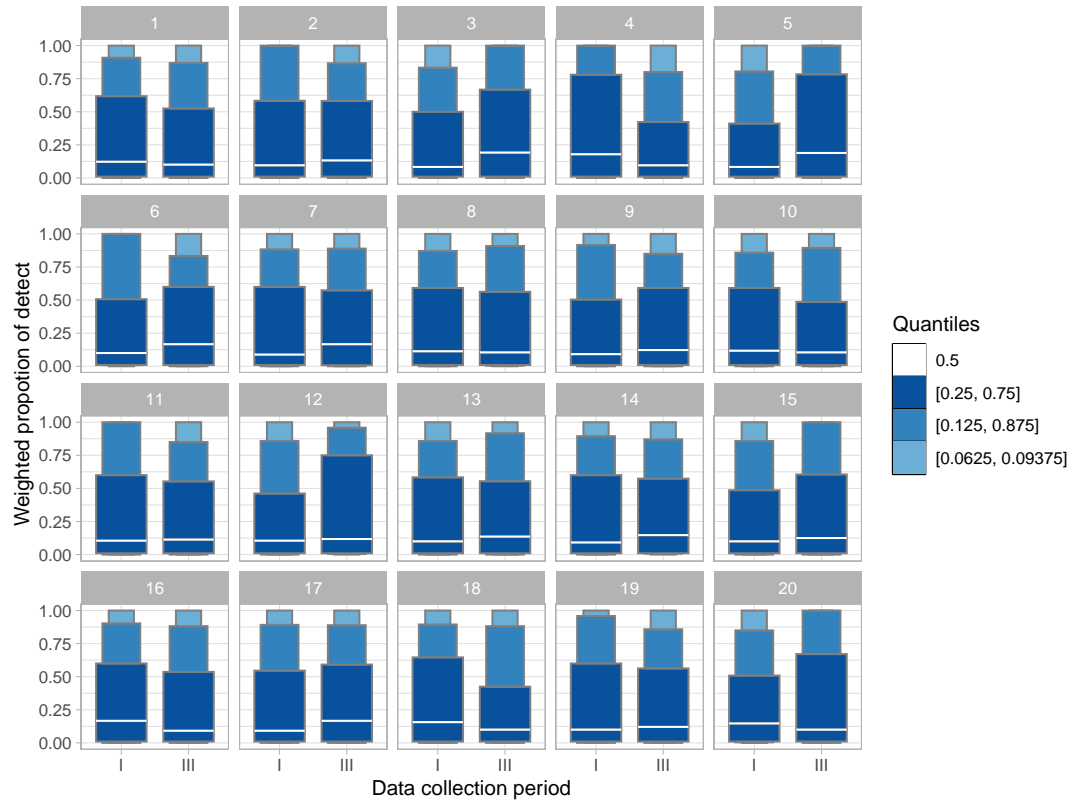
Age group	Period I	%	Period II	%	Period III	%	Total	%
18-24	83	18.7	86	19.4	51	11.5	220	49.7
25-39	69	15.6	63	14.2	63	14.2	195	44.0
40-54	6	1.4	8	1.8	6	1.4	20	4.5
55-64	2	0.5	3	0.7	3	0.7	8	1.8
	160	36.1	160	36.1	123	27.8	443	100.0

**Table C4.** Summary of education distribution of subjects recruited in this study.

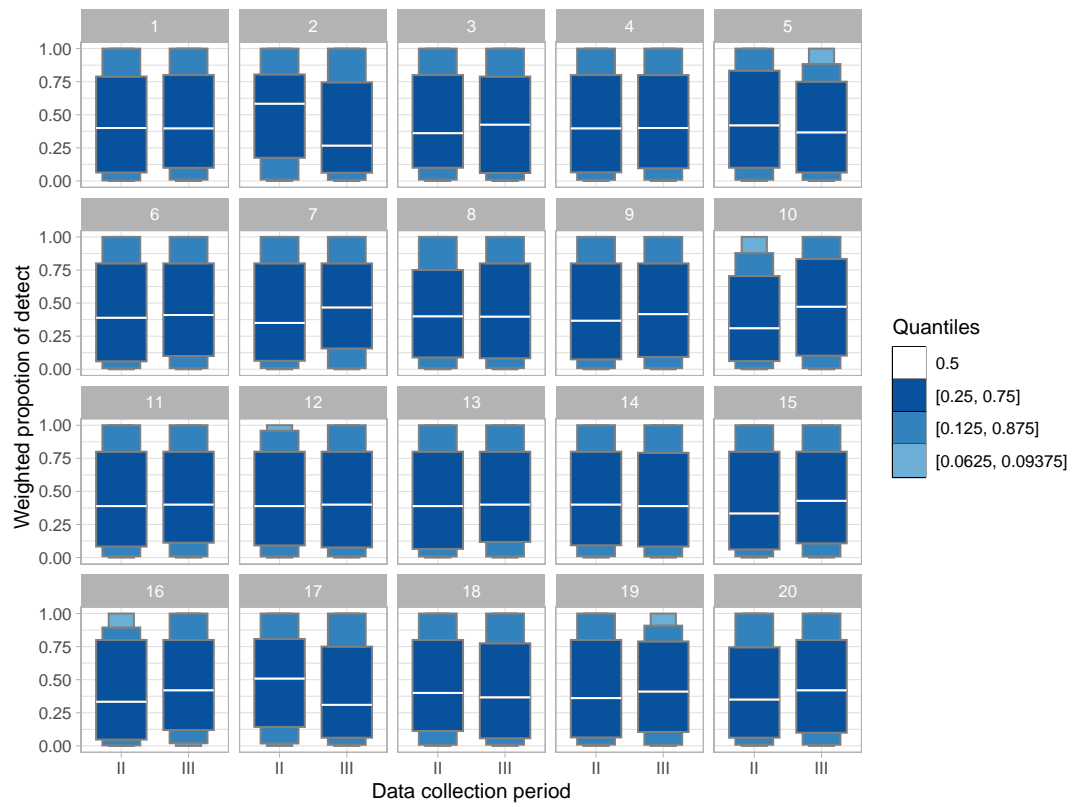
Education	Period I	%	Period II	%	Period III	%	Total	%
High School or below	41	9.3	53	12.0	33	7.4	127	28.7
Diploma and Bachelor Degree	92	20.8	79	17.8	66	14.9	237	53.5
Honours Degree	6	1.4	15	3.4	6	1.4	27	6.1
Masters Degree	21	4.7	13	2.9	16	3.6	50	11.3
Doctoral Degree	0	0.0	0	0.0	2	0.5	2	0.5
	160	36.1	160	36.1	123	27.8	443	100.0

**Table C5.** Summary of previous experience distribution of subjects recruited in this study.

Previous experience	Period I	%	Period II	%	Period III	%	Total	%
No	96	21.7	88	19.9	67	15.1	251	56.7
Yes	64	14.4	72	16.3	56	12.6	192	43.3
	160	36.1	160	36.1	123	27.8	443	100.0



**Figure C1.** A lineup of "letter-value" boxplots of weighted propotion of detect for lineups over different data collection periods for non-linearity model. Can you find the most different boxplot? The data plot is positioned in panel  $2^3 - 1$ .



**Figure C2.** A lineup of "letter-value" boxplots of weighted propotion of detect for lineups over different data collection periods for heteroskedasticity model. Can you find the most different boxplot? The data plot is positioned in panel  $2^4 - 2$ .

model and heteroskedasticity model respectively. To emphasize the tail behaviour and display fewer outliers, we use the “letter-value” boxplot (Hofmann, Wickham, and Kafadar 2017) which is an extension of the number of “letter value” statistics to check the weighed proportion of detect over different data collection period. The weighted proportion of detect is calculated by taking the average of  $c_i$  of a lineup over a data collection period. Within our research team, we can not identify the data plot from the null plots for these two lineups, result in  $p$ -values much greater than 5%. Thus, there is no clear evidence of batch effect.

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