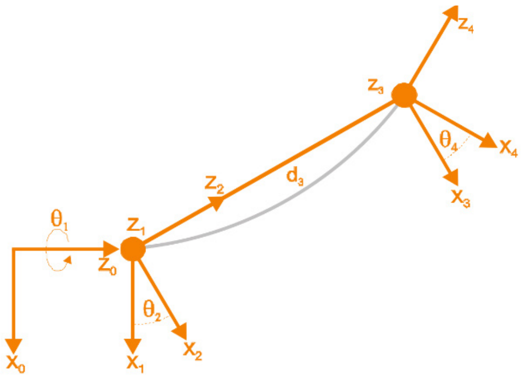
Continuum Robot Virtual RRPR Robot Dynamics Derivation

Step 1: Forward Kinematics to form the $H_1^0 \;\; H_2^0 \; H_3^0 \; H_4^0$

From ECE893 lecture notes, the DH table is:



Link	θ	d	a	α
1	*	0	0	90
2	*	0	0	-90
3	0	*	0	90
4	*	0	0	-90

Thus, from Robotica calculator (developed by Dr. Spong) we obtain:

$$H_1^0 = \begin{bmatrix} \cos[q1] & 0 & \sin[q1] & 0 \\ \sin[q1] & 0 & -\cos[q1] & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_2^0 = \begin{bmatrix} \cos[q1]\cos[q2] & -\sin[q1] & -\cos[q1]\sin[q2] & 0 \\ \cos[q2]\sin[q1] & \cos[q1] & -\sin[q1]\sin[q2] & 0 \\ \sin[q2] & 0 & \cos[q2] & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = \begin{bmatrix} \cos[q1]\cos[q2] & -\cos[q1]\sin[q2] & \sin[q1] & -\cos[q1]\sin[q2]d_3 \\ \cos[q2]\sin[q1] & -\sin[q1]\sin[q2] & -\cos[q1] & -\sin[q1]\sin[q2]d_3 \\ \sin[q2] & \cos[q2] & 0 & \cos[q2]d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_4^0 = \begin{bmatrix} \cos[q1]\cos[q2+q4] & -\sin[q1] & -\cos[q1]\sin[q2+q4] & -\cos[q1]\sin[q2]d_3 \\ \cos[q2+q4]\sin[q1] & \cos[q1] & -\sin[q1]\sin[q2+q4] & -\sin[q1]\sin[q2]d_3 \\ \sin[q2+q4] & 0 & \cos[q2+q4] & \cos[q2]d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: Form the Jacobian Matrix for the center of mass of each link.

For Revolute Joint: $J_i = \begin{bmatrix} J_{vi} \\ J_{\omega i} \end{bmatrix} = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}$ and for Prismatic Joint: $J_i = \begin{bmatrix} J_{vi} \\ J_{\omega i} \end{bmatrix} = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}$

Thus, we obtain:

Jacobian for mass center of link 1: $J_1 = \begin{bmatrix} J_{v1} \\ J_{\omega 1} \end{bmatrix} = \begin{bmatrix} z_0 \times (o_{c1} - o_0) & 0 & 0 & 0 \\ z_0 & 0 & 0 & 0 \end{bmatrix}$

Jacobian for mass center of link 2: $J_2 = \begin{bmatrix} J_{v2} \\ J_{\omega 2} \end{bmatrix} = \begin{bmatrix} z_0 \times (o_{c2} - o_0) & z_1 \times (o_{c2} - o_1) & 0 & 0 \\ z_0 & z_1 & 0 & 0 \end{bmatrix}$

Jacobian for mass center of link 3: $J_3 = \begin{bmatrix} J_{v3} \\ J_{\omega 3} \end{bmatrix} = \begin{bmatrix} z_0 \times (o_{c3} - o_0) & z_1 \times (o_{c3} - o_1) & z_2 & 0 \\ z_0 & z_1 & 0 & 0 \end{bmatrix}$

Jacobian for mass center of link 4: $J_4 = \begin{bmatrix} J_{v4} \\ J_{\omega 4} \end{bmatrix} = \begin{bmatrix} z_0 \times (o_{c4} - o_0) & z_1 \times (o_{c4} - o_1) & z_2 & z_3 \times (o_{c4} - o_3) \\ z_0 & z_1 & 0 & z_3 \end{bmatrix}$

where $z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and z_1, z_2, z_3 can be found from the third column of the R_1^0, R_2^0, R_3^0 which is located in H_1^0, H_2^0, H_3^0.

and $o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which represent base frame coordinate, and o_1, o_2, o_3 can be found from the fourth column of H_1^0, H_2^0, H_3^0.

Also, the center of mass coordinate of the 4 links $o_{c1}, o_{c2}, o_{c3}, o_{c4}$ is represented as

$o_{c1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $o_{c2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $o_{c3} = \begin{bmatrix} -\frac{d_3}{2}\sin(q_2)\cos(q_1) \\ -\frac{d_3}{2}\sin(q_2)\sin(q_1) \\ \frac{d_3}{2}\cos(q_2) \end{bmatrix}$, $o_{c4} = \begin{bmatrix} -d_3\sin(q_2)\cos(q_1) \\ -d_3\sin(q_2)\sin(q_1) \\ d_3\cos(q_2) \end{bmatrix}$ considering that all links are zero except for the third link, which is equal to d_3.

So, after matrix calculation:

$$J_1 = \begin{bmatrix} J_{v1} \\ J_{\omega 1} \end{bmatrix} = \begin{bmatrix} z_0 \times (o_{c1} - o_0) & 0 & 0 & 0 \\ z_0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) & 0 & 0 & 0 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & 0 & 0 & 0 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & 0 & 0 & 0 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} J_{v2} \\ J_{\omega 2} \end{bmatrix} = \begin{bmatrix} z_0 \times \left(o_{c2} - o_0 \right) & z_1 \times \left(o_{c2} - o_1 \right) & 0 & 0 \\ z_0 & z_1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) & \begin{bmatrix} \sin[q1] \\ -\cos[q1] \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) & 0 & 0 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \sin[q1] \\ -\cos[q1] \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & 0 & 0 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \mathrm{s}1 \\ -\mathrm{c}1 \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} J_{v3} \\ J_{\omega 3} \end{bmatrix} = \begin{bmatrix} z_0 \times \left(o_{c3} - o_0 \right) & z_1 \times \left(o_{c3} - o_1 \right) & z_2 & 0 \\ z_0 & z_1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} -\frac{d_3}{2} \sin(q_2) \cos(q_1) \\ -\frac{d_3}{2} \sin(q_2) \sin(q_1) \\ \frac{d_3}{2} \cos(q_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) & \begin{bmatrix} \sin[q1] \\ -\cos[q1] \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} -\frac{d_3}{2} \sin(q_2) \cos(q_1) \\ -\frac{d_3}{2} \sin(q_2) \sin(q_1) \\ \frac{d_3}{2} \cos(q_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) & \begin{bmatrix} -\cos[q_1] \sin[q_2] \\ -\sin[q_1] \sin[q_2] \\ \cos[q2] \end{bmatrix} & 0 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \sin[q1] \\ -\cos[q1] \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \frac{d_3}{2} s2s1 \\ -\frac{d_3}{2} s2c1 \\ 0 \end{bmatrix} & \begin{bmatrix} -\frac{d_3}{2} \mathrm{c}1\mathrm{c}2 \\ -\frac{d_3}{2} s1c2 \\ -\frac{d_3}{2} s1s1s2 - \frac{d_3}{2} c1c1s2 \end{bmatrix} & \begin{bmatrix} -\mathrm{c}1s2 \\ -s1s2 \\ c2 \end{bmatrix} & 0 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} s1 \\ -c1 \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

$$J_4=\begin{bmatrix} J_{v4} \\ J_{\omega 4}\end{bmatrix}=\begin{bmatrix} z_0\times(o_{c4}-o_0) & z_1\times(o_{c4}-o_1) & z_2 & z_3\times(o_{c4}-o_3) \\ z_0 & z_1 & 0 & z_3\end{bmatrix}$$

$$=\begin{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}\times\left(\begin{bmatrix}-d_3\,\mathrm{s}2\,\mathrm{c}1\\-d_3\,\mathrm{s}2\,\mathrm{s}1\\d_3\,\mathrm{c}2\end{bmatrix}-\begin{bmatrix}0\\0\\0\end{bmatrix}\right) & \begin{bmatrix}\mathrm{s}1\\-\mathrm{c}1\\0\end{bmatrix}\times\left(\begin{bmatrix}-d_3\,\mathrm{s}2\,\mathrm{c}1\\-d_3\,\mathrm{s}2\,\mathrm{s}1\\d_3\,\mathrm{c}2\end{bmatrix}-\begin{bmatrix}0\\0\\0\end{bmatrix}\right) & \begin{bmatrix}-c1\,\mathrm{s}2\\-s1\,\mathrm{s}2\\c2\end{bmatrix} & \begin{bmatrix}s1\\-c1\\0\end{bmatrix}\times\left(\begin{bmatrix}-d_3\,\mathrm{s}2\,\mathrm{c}1\\-d_3\,\mathrm{s}2\,\mathrm{s}1\\d_3\,\mathrm{c}2\end{bmatrix}-\begin{bmatrix}-d_3c1\,\mathrm{s}2\\-d_3s1\,\mathrm{s}2\\c2d_3\end{bmatrix}\right) \\ \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}\mathrm{s}1\\-\mathrm{c}1\\0\end{bmatrix} & 0 & \begin{bmatrix}s1\\-c1\\0\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix} \begin{bmatrix}d_3\,\mathrm{s}2\,\mathrm{s}1\\-d_3\,\mathrm{s}2\,\mathrm{c}1\\0\end{bmatrix} & \begin{bmatrix}-d_3\,\mathrm{c}1\,\mathrm{c}2\\-d_3s1c2\\-d_3s1s1s2-d_3c1c1s2\end{bmatrix} & \begin{bmatrix}-c1\,\mathrm{s}2\\-s1\,\mathrm{s}2\\c2\end{bmatrix} & \begin{bmatrix}0\\0\\0\end{bmatrix} \\ \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}\mathrm{s}1\\-\mathrm{c}1\\0\end{bmatrix} & 0 & \begin{bmatrix}s1\\-c1\\0\end{bmatrix}\end{bmatrix}$$

Step 3: Derive the D matrix

D matrix is equal to: $D(q)=\sum_{i=1}^{n} m_i J_{vi}^{\ T} J_{vi} + J_{\omega i}^{\ T} R_i^0 I_i \left(R_i^0\right)^T J_{\omega i}$

where m_i denotes the mass of i^{th} link, I_i denotes the inertia matrix in the link-fixed frame with its origin at the center of mass of the link, J_{vi} denotes the velocity Jacobian for the center of mass of link i, and $J_{\omega i}$ denotes the angular velocity Jacobian for link i.

From last step we know that the angular velocity Jacobian matrices for the four links are:

$$J_{\omega 1}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix};\ J_{\omega 2}=\begin{bmatrix}0&s1&0&0\\0&-c1&0&0\\1&0&0&0\end{bmatrix};\ J_{\omega 3}=\begin{bmatrix}0&s1&0&0\\0&-c1&0&0\\1&0&0&0\end{bmatrix};\ J_{\omega 4}=\begin{bmatrix}0&s1&0&s1\\0&-c1&0&-c1\\1&0&0&0\end{bmatrix}$$

The linear velocity Jacobian matrices for the four links are:

$$J_{v1}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix};\ J_{v2}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix};\ J_{v3}=\begin{bmatrix}\frac{d_3}{2}s2s1 & -\frac{d_3}{2}\mathrm{c}1\,\mathrm{c}2 & -\mathrm{c}1s2 & 0\\ -\frac{d_3}{2}s2c1 & -\frac{d_3}{2}s1c2 & -s1s2 & 0\\ 0 & -\frac{d_3}{2}s1s1s2-\frac{d_3}{2}c1c1s2 & c2 & 0\end{bmatrix};$$

$$J_{v4}=\begin{bmatrix}d_3\,\mathrm{s}2\,\mathrm{s}1 & -d_3\,\mathrm{c}1\,\mathrm{c}2 & -c1\,\mathrm{s}2 & 0\\ -d_3\,\mathrm{s}2\,\mathrm{c}1 & -d_3s1c2 & -s1\,\mathrm{s}2 & 0\\ 0 & -d_3s1s1s2-d_3c1c1s2 & c2 & 0\end{bmatrix}$$

The inertia matrix for the four links are (If we model 1st, 2nd, and 4th link as point mass and 3rd link as cylinder)

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{1z} \end{bmatrix},\ I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{2z} \end{bmatrix},\ I_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{4z} \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} I_{3x} & 0 & 0 \\ 0 & I_{3y} & 0 \\ 0 & 0 & I_{3z} \end{bmatrix} \text{ where}$$

$I_{1z} = m_1 r_1, I_{2z} = m_2 r_2, I_{4z} = m_4 r_4$, $I_{3x} = I_{3y} \frac{1}{12} m_3 \left(3r_3^2 + d_3^2\right)$, $I_{3z} = \frac{1}{2} m_3 r_3^2$, r_1, r_2, r_4 are the distance from point mass to the axis of rotation and r_3 is the cylinder radius.

Hence, the matrix $D(q)$ is given by

$$\begin{aligned} D(q) &= m_1 J_{v1}{}^T J_{v1} + m_2 J_{v2}{}^T J_{v2} + m_3 J_{v3}{}^T J_{v3} + m_4 J_{v4}{}^T J_{v4} \\ &+ J_{\omega 1}{}^T R_1^0 I_1 \left(R_1^0\right)^T J_{\omega 1} + J_{\omega 2}{}^T R_2^0 I_2 \left(R_2^0\right)^T J_{\omega 2} + J_{\omega 3}{}^T R_3^0 I_3 \left(R_3^0\right)^T J_{\omega 3} + J_{\omega 4}{}^T R_4^0 I_4 \left(R_4^0\right)^T J_{\omega 4} \\ &= \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} \end{aligned}$$

where (computed from Mathematica)

$$d_{11} = c2^2 I_{2z} + c2^2 I_{3y} + I_{3x} s2^2 + m_3 \left(\frac{1}{4} c1^2 d3^2 s2^2 + \frac{1}{4} d3^2 s1^2 s2^2 \right) + m_4 \left(c1^2 d3^2 s2^2 + d3^2 s1^2 s2^2 \right) + I_{4z} \left(c2c4 - s2s4 \right)^2$$

$$d_{12} = I_{4z} s1 \left(-c1c4s2 - c1c2s4\right)\left(c2c4 - s2s4\right) - c1 I_{4z} \left(-c4s1s2 - c2s1s4\right)\left(c2c4 - s2s4\right)$$

$$d_{13} = 0$$

$$d_{14} = I_{4z} s1 \left(-c1c4s2 - c1c2s4\right)\left(c2c4 - s2s4\right) - c1 I_{4z} \left(-c4s1s2 - c2s1s4\right)\left(c2c4 - s2s4\right)$$

$$d_{21} = d_{12} = I_{4z} \left(c2c4 - s2s4\right)\left(s1\left(-c1c4s2 - c1c2s4\right) - c1\left(-c4s1s2 - c2s1s4\right)\right)$$

$$\begin{aligned} d_{22} &= c1^2 I_{3z} \left(c1^2 + s1^2\right) + I_{3z} s1^2 \left(c1^2 + s1^2\right) + m_4 (c1^2 c2^2 d3^2 + c2^2 d3^2 s1^2 + \left(-c1d3c1s2 - d3s1^2 s2\right)^2) \\ &+ m_3 \left(\frac{1}{4} c1^2 c2^2 d3^2 + \frac{1}{4} c2^2 d3^2 s1^2 + \left(-\frac{1}{2} c1^2 d3s2 - \frac{1}{2} d3s1^2 s2 \right)^2 \right) \\ &+ I_{4z} s1 \left(-c1c4s2 - c1c2s4\right)\left(s1\left(-c1c4s2 - c1c2s4\right) - c1\left(-c4s1s2 - c2s1s4\right)\right) \\ &- c1 I_{4z} \left(-c4s1s2 - c2s1s4\right)\left(s1\left(-c1c4s2 - c1c2s4\right) - c1\left(-c4s1s2 - c2s1s4\right)\right) \end{aligned}$$

$$\begin{aligned} d_{23} &= m_4 \left(c1^2 c2d3s2 + c2d3s1^2 s2 + c2\left(-c1d3c1s2 - d3s1^2 s2\right) \right) \\ &+ m_3 \left(\frac{1}{2} c1^2 c2d3s2 + \frac{1}{2} c2d3s1^2 s2 + c2 \left(-\frac{1}{2} c1^2 d3s2 - \frac{1}{2} d3s1^2 s2 \right) \right) \end{aligned}$$

$$d_{24} = I_{4z} s1(-c1c4s2 - c1c2s4)(s1(-c1c4s2 - c1c2s4) - c1(-c4s1s2 - c2s1s4))$$
$$-c1 I_{4z}(-c4s1s2 - c2s1s4)(s1(-c1c4s2 - c1c2s4) - c1(-c4s1s2 - c2s1s4))$$

$$d_{31} = 0$$

$$d_{32} = m_4\left(c1^2 c2d3s2 + c2d3s1^2 s2 + c2\left(-c1d3c1s2 - d3s1^2 s2\right)\right)$$
$$+m_3\left(\frac{1}{2}c1^2 c2d3s2 + \frac{1}{2}c2d3s1^2 s2 + c2\left(-\frac{1}{2}c1^2 d3s2 - \frac{1}{2}d3s1^2 s2\right)\right)$$

$$d_{33} = m_3\left(c2^2 + c1^2 s2^2 + s1^2 s2^2\right) + m_4\left(c2^2 + c1^2 s2^2 + s1^2 s2^2\right)$$

$$d_{34} = 0$$

$$d_{41} = I_{4z}(c2c4 - s2s4)(s1(-c1c4s2 - c1c2s4) - c1(-c4s1s2 - c2s1s4))$$

$$d_{42} = I_{4z} s1(-c1c4s2 - c1c2s4)(s1(-c1c4s2 - c1c2s4) - c1(-c4s1s2 - c2s1s4))$$
$$-c1 I_{4z}(-c4s1s2 - c2s1s4)(s1(-c1c4s2 - c1c2s4) - c1(-c4s1s2 - c2s1s4))$$

$$d_{43} = 0$$

$$d_{44} = I_{4z} s1(-c1c4s2 - c1c2s4)(s1(-c1c4s2 - c1c2s4) - c1(-c4s1s2 - c2s1s4))$$
$$-c1 I_{4z}(-c4s1s2 - c2s1s4)(s1(-c1c4s2 - c1c2s4) - c1(-c4s1s2 - c2s1s4))$$

Step 4: Derive C Matrix

From the Matrix $D(q)$ that was found, the Christoffel symbols c_{ijk} are found as

$$c_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$$

For $i = 1,2,3,4$; $j = 1,2,3,4$; $k = 1,2,3,4$ and writing the matrix $C(q,\dot{q})$ with its $(k,j)^{th}$ element being

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$$

We get

$$C(q,\dot{q}) = \begin{bmatrix} C_{11}(q,\dot{q}) & C_{12}(q,\dot{q}) & C_{13}(q,\dot{q}) & C_{14}(q,\dot{q}) \\ C_{21}(q,\dot{q}) & C_{22}(q,\dot{q}) & C_{23}(q,\dot{q}) & C_{24}(q,\dot{q}) \\ C_{31}(q,\dot{q}) & C_{32}(q,\dot{q}) & C_{33}(q,\dot{q}) & C_{34}(q,\dot{q}) \\ C_{41}(q,\dot{q}) & C_{42}(q,\dot{q}) & C_{43}(q,\dot{q}) & C_{44}(q,\dot{q}) \end{bmatrix}$$

Using Maple 2018 software to calculate each term we obtained:

$$C_{11}(q,\dot{q})=\sum_{i=1}^{n} c_{i11}(q)\dot{q}_i = c_{111}(q)\dot{q}_1 + c_{211}(q)\dot{q}_2 + c_{311}(q)\dot{q}_3 + c_{411}(q)\dot{q}_4$$

$$=\frac{\left(-8\sin(q_4) I_{4z}(q_2 d + q_4 d)\cos(q_4) - \dot{q}_3 q_3 (m_3+4m_4)\right)\cos(q_2)^2}{4}$$

$$+\frac{\sin(q_2)\left(-8I_{4z}(\dot{q}_2+\dot{q}_4)\cos(q_4)^2 + \left(4I_{4z}+(m_3+4m_4)q_3^{\,2} - 4I_{2z} + 4I_{3x} - 4I_{3y}\right)\dot{q}_2 + 4I_{4z}\dot{q}_4\right)\cos(q_2)}{4}$$

$$+\sin(q_4) I_{4z}(\dot{q}_2+\dot{q}_4)\cos(q_4) + \frac{\dot{q}_3 q_3 (m_3+4m_4)}{4}$$

$$C_{12}(q,\dot{q})=\sum_{i=1}^{n} c_{i21}(q)\dot{q}_i = c_{121}(q)\dot{q}_1 + c_{221}(q)\dot{q}_2 + c_{321}(q)\dot{q}_3 + c_{421}(q)\dot{q}_4$$

$$=\frac{1}{4}\dot{q}_1\left(\begin{array}{l} -8I_{4z}\cos(q_2)^2\sin(q_4)\cos(q_4) \\ +\left(-8I_{4z}\cos(q_4)^2 + 4I_{4z} + (m_3+4m_4)q_3^{\,2} - 4I_{2z} + 4I_{3x} - 4I_{3y}\right)\sin(q_2)\cos(q_2) \\ +4I_{4z}\sin(q_4)\cos(q_4) \end{array}\right)$$

$$C_{13}(q,\dot{q})=\sum_{i=1}^{n} c_{i31}(q)\dot{q}_i = c_{131}(q)\dot{q}_1 + c_{231}(q)\dot{q}_2 + c_{331}(q)\dot{q}_3 + c_{431}(q)\dot{q}_4$$

$$=\frac{\dot{q}_1\sin(q_2)^2(m_3+4m_4)q_3}{4}$$

$$C_{14}(q,\dot{q})=\sum_{i=1}^{n} c_{i41}(q)\dot{q}_i = c_{141}(q)\dot{q}_1 + c_{241}(q)\dot{q}_2 + c_{341}(q)\dot{q}_3 + c_{441}(q)\dot{q}_4$$

$$=-2I_{4z}\dot{q}_1\left(\cos(q_2)^2\sin(q_4)\cos(q_4) + \sin(q_2)\left(\cos(q_4)^2 - \frac{1}{2}\right)\cos(q_2) - \frac{\sin(q_4)\cos(q_4)}{2}\right)$$

$$C_{21}(q,\dot{q})=\sum_{i=1}^{n} c_{i11}(q)\dot{q}_i = c_{112}(q)\dot{q}_1 + c_{212}(q)\dot{q}_2 + c_{312}(q)\dot{q}_3 + c_{412}(q)\dot{q}_4$$

$$=-\frac{1}{4}\dot{q}_1\left(\begin{array}{l} -8I_{4z}\cos(q_2)^2\sin(q_4)\cos(q_4) \\ +\left(-8I_{4z}\cos(q_4)^2 + 4I_{4z} + (m_3+4m_4)q_3^{\,2} - 4I_{2z} + 4I_{3x} - 4I_{3y}\right)\sin(q_2)\cos(q_2) \\ +4I_{4z}\sin(q_4)\cos(q_4) \end{array}\right)$$

$$C_{22}(q,\dot{q})=\sum_{i=1}^{n} c_{i22}(q)\dot{q}_i = c_{122}(q)\dot{q}_1 + c_{222}(q)\dot{q}_2 + c_{322}(q)\dot{q}_3 + c_{422}(q)\dot{q}_4$$

$$=\frac{\dot{q}_3 q_3 (m_3+4m_4)}{4}$$

$$C_{23}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i32}\left(q\right)\dot{q}_i = c_{132}\left(q\right)\dot{q}_1+c_{232}\left(q\right)\dot{q}_2+c_{332}\left(q\right)\dot{q}_3+c_{432}\left(q\right)\dot{q}_4$$
$$=\frac{\dot{q}_2 q_3\left(m_3+4m_4\right)}{4}$$

$$C_{24}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i42}\left(q\right)\dot{q}_i = c_{142}\left(q\right)\dot{q}_1+c_{242}\left(q\right)\dot{q}_2+c_{342}\left(q\right)\dot{q}_3+c_{442}\left(q\right)\dot{q}_4$$
$$=0$$

$$C_{31}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i13}\left(q\right)\dot{q}_i = c_{113}\left(q\right)\dot{q}_1+c_{213}\left(q\right)\dot{q}_2+c_{313}\left(q\right)\dot{q}_3+c_{413}\left(q\right)\dot{q}_4$$
$$=-\frac{\dot{q}_1\sin\left(q_2\right)^2\left(m_3+4m_4\right)q_3}{4}$$

$$C_{32}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i23}\left(q\right)\dot{q}_i = c_{123}\left(q\right)\dot{q}_1+c_{223}\left(q\right)\dot{q}_2+c_{323}\left(q\right)\dot{q}_3+c_{423}\left(q\right)\dot{q}_4$$
$$=-\frac{\dot{q}_2 q_3\left(m_3+4m_4\right)}{4}$$

$$C_{33}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i33}\left(q\right)\dot{q}_i = c_{133}\left(q\right)\dot{q}_1+c_{233}\left(q\right)\dot{q}_2+c_{333}\left(q\right)\dot{q}_3+c_{433}\left(q\right)\dot{q}_4$$
$$=0$$

$$C_{34}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i43}\left(q\right)\dot{q}_i = c_{143}\left(q\right)\dot{q}_1+c_{243}\left(q\right)\dot{q}_2+c_{343}\left(q\right)\dot{q}_3+c_{443}\left(q\right)\dot{q}_4$$
$$=0$$

$$C_{41}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i14}\left(q\right)\dot{q}_i = c_{114}\left(q\right)\dot{q}_1+c_{214}\left(q\right)\dot{q}_2+c_{314}\left(q\right)\dot{q}_3+c_{414}\left(q\right)\dot{q}_4 =$$
$$2I_{4z}\dot{q}_1\left(\cos\left(q_2\right)^2\sin\left(q_4\right)\cos\left(q_4\right)+\sin\left(q_2\right)\left(\cos\left(q_4\right)^2-\frac{1}{2}\right)\cos\left(q_2\right)-\frac{\sin\left(q_4\right)\cos\left(q_4\right)}{2}\right)$$

$$C_{42}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i24}\left(q\right)\dot{q}_i = c_{124}\left(q\right)\dot{q}_1+c_{224}\left(q\right)\dot{q}_2+c_{324}\left(q\right)\dot{q}_3+c_{424}\left(q\right)\dot{q}_4$$
$$=0$$

$$C_{43}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i34}\left(q\right)\dot{q}_i = c_{134}\left(q\right)\dot{q}_1+c_{234}\left(q\right)\dot{q}_2+c_{334}\left(q\right)\dot{q}_3+c_{434}\left(q\right)\dot{q}_4$$
$$=0$$

$$C_{44}\left(q,\dot{q}\right)=\sum_{i=1}^{n}c_{i44}\left(q\right)\dot{q}_i = c_{144}\left(q\right)\dot{q}_1+c_{244}\left(q\right)\dot{q}_2+c_{344}\left(q\right)\dot{q}_3+c_{444}\left(q\right)\dot{q}_4$$
$$=0$$

Step 5. Derive the g matrix

Finally, the potential energy of the manipulator is given by:

$P = m_1 g \cdot 0 + m_2 g \cdot 0 + m_3 g D_{Oc3} + m_4 g D_{Oc4}$ where D_{Oc3} and D_{Oc4} are distance from the mass center of link 3 and 4 to the $y_0 z_0$ plane and are equal to

$$D_{Oc3} = -\frac{d_3}{2}\sin(q_2)\cos(q_1)$$

$$D_{Oc3} = -d_3 \sin(q_2)\cos(q_1)$$

Hence, the $g(q)$ matrix becomes

$$g(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} \\ \frac{\partial P}{\partial q_2} \\ \frac{\partial P}{\partial q_3} \\ \frac{\partial P}{\partial q_4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} m_3 g \sin(q_1)\sin(q_2) d_3 + m_4 g \sin(q_1)\sin(q_2) d_3 \\ -\frac{1}{2} m_3 g \cos(q_1)\cos(q_2) d_3 - m_4 g \cos(q_1)\cos(q_2) d_3 \\ -\frac{1}{2} m_3 g \sin(q_2)\cos(q_1) - m_4 g \sin(q_2)\cos(q_1) \\ 0 \end{bmatrix}$$

The dynamical equations of the manipulator are given by

$$D(q)\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{bmatrix} + C(q,\dot{q})\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} + g(q) = \begin{bmatrix} \tau_1 \\ \tau_2 \\ f_3 \\ \tau_4 \end{bmatrix}$$