Real-Time Control of an Inverted Pendulum: A Comparative Study

Conference Paper · December 2011

DOI: 10.1109/FIT.2011.41 · Source: DBLP

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Real-Time Control of an Inverted Pendulum: A Comparative Study

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***Abstract*—In this paper, we consider the problem of real-time control of an Inverted Pendulum. We design and implement four different control algorithms, namely PID, Pole placement, LQR and Fuzzy Logic. These controllers are applied to the Inverted Pendulum in real-time and their performance is compared on the basis of Pendulum regulation, disturbance rejection and control energy specifications. We also provide a performance comparison based on the ISE index.**

I. INRTODUCTION

The balancing of an Inverted Pendulum through cart movement is a classical problem in the area of control theory and engineering [1],[2]. It provides a benchmark for engineers to test and verify the performance of their Control algorithms [3]. Inverted Pendulum system finds its applications in various real-life processes including Crane stabilization, Space Rocket lift-offs, Earth-Quake proof building designs, and Robot maneuvers [4],[5].

Inverted Pendulum system has been widely studied from a control perspective [6]. In some of the existing literature, the inverted pendulum is considered to be a one degree of freedom system (see e.g. [7]). However, in practice, the incorporation of the cart position regulation (second degree of freedom) provides a more realistic approach to the problem [6]. Furthermore, some existing work in the literature ignores disturbance rejection properties (see e.g. [8]).

The aim of this work is to design, test and compare various controllers for the real-time stabilization of the Pendulum (in an upright position) along with the regulation of the cart toward its origin. Disturbance rejection properties as well as Control energy requirements for the designed control schemes are also examined.

In this paper, we provide a comprehensive real-time comparison of classical and modern control techniques, including PID, Pole Placement, Linear Quadratic Regulator (LQR), and Fuzzy Logic controller, on the basis of regulation performance as well as their respective control energy requirements. The regulation performance for all techniques is compared using performance index Integral Square Error (ISE). Furthermore, the disturbance rejection properties are examined through the application of a step disturbance to the system.

This paper is organized as follows. In section II, we present the mathematical model of the Inverted Pendulum system. In section III, Controllers are designed and their MATLAB simulation results presented and discussed. In section IV, Real-Time results are presented and analyzed along with a discussion of the implementation issues. Finally, we conclude in section V.

II. MATHEMATICAL MODELING

We first derive the mathematical model of Inverted Pendulum System using Lagrangian Analysis [9].

The input to the considered system is force F (t) and the two outputs are the position of the cart (denoted 'x') and the pendulum angle from the vertical (denoted 'θ').

The free body diagram of Inverted Pendulum system is shown by Fig 1.

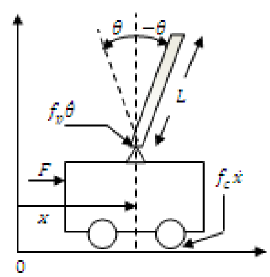

Fig 1. Free Body Diagram of Inverted Pendulum system.

A. Conventions

Force applied towards the right is considered Positive. Cart displacement is taken as positive in the right half Plane, and negative in the left half plane. Angular displacement of Pendulum is considered Positive when measured anticlockwise from the vertical.

B. Lagrangian Analysis

Inverted Pendulum has two degree of freedoms: the pendulum rotation and the cart movement. Hence we require two 2nd order differential equations (i.e. for $\ddot{\theta}$ and $\ddot{x}$) to completely model the system.

The Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varepsilon}_{pi}}\right) - \frac{\partial L}{\partial \varepsilon_{pi}} = \Xi_{pi} \tag{1}$$

where L is the Lagrange function given as:

L = Kinetic Energy – Potential Energy = T – V

and Ξ_{pi} is the net non-conservative force given as:

$$\Xi_{p1} = F(t) - f_c\dot{x}, \;\; \Xi_{p2} = f_p\dot{\theta} \tag{2}$$

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DOI 10.1109/FIT.2011.41

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The Kinetic Energy (T) for the complete system can be written as:

$$T = T_c + T_p$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}\dot{\theta}L\cos\theta + L^2\dot{\theta}^2) \qquad (3)$$

where T_p and T_c denote the Kinetic Energies for the Pendulum and the cart respectively.

Note that there is no change in potential energy of the cart since the surface is flat (non-inclined). The pendulum, however, loses height and, therefore, same potential energy which is given as:

$$V = ml\cos\theta \qquad (4)$$

Lagrangian function therefore becomes:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}\dot{\theta}L\cos\theta + L^2\dot{\theta}^2) - mgL\cos\theta$$

Using (1), with i = 1, we obtain the following nonlinear equation for the cart position x:

$$(M+m)\ddot{x} - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta + f_c\dot{x} = F\,(t) \qquad \mathbf{(5)}$$

Similarly, computing (1) with i = 2, provides the following equation for the pendulum angle θ:

$$(I + mL^2)\ddot{\theta} - mL\ddot{x}\cos\theta - mgL\sin\theta + f_p\dot{\theta} = 0 \qquad \mathbf{(6)}$$

Now Linearizing equations (5) and (6) by making use of the following small angle approximations:

$$\sin\theta \approx \theta,\ \cos\theta \approx 1,\ \dot{\theta}^2 \approx 0$$

yields:

$$\mathbf{(M+m)\ddot{x} - mL\ddot{\theta} + f_c\dot{x} = F\,(t)} \qquad \mathbf{(7)}$$

$$\mathbf{(I + mL^2)\ddot{\theta} - mL\ddot{x} - mgL\theta + f_p\dot{\theta} = 0} \qquad \mathbf{(8)}$$

Therefore, the state space model is given by

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{x}\\ \ddot{x}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ \frac{(M+m)mgL}{I(M+m)+MmL^2} & \frac{-(M+m)f_p}{I(M+m)+MmL^2} & 0 & \frac{-mLf_c}{I(M+m)+MmL^2}\\ 0 & 0 & 0 & 1\\ \frac{m^2L^2g}{I(M+m)+MmL^2} & \frac{-f_p mL}{I(M+m)+MmL^2} & 0 & \frac{-f_c(I+mL^2)}{I(M+m)+MmL^2}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ x\\ \dot{x}\end{bmatrix} + \begin{bmatrix}0\\ \frac{mL}{I(M+m)+MmL^2}\\ 0\\ \frac{(I+mL^2)}{I(M+m)+MmL^2}\end{bmatrix} F\,(t) \qquad (9)$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ x\\ \dot{x}\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix} F(t) \qquad (10)$$

C. *System Parameters*

The Parameters for the Inverted Pendulum System by Feedback [10] are provided in Table I below:

TABLE I
INVERTED PENDULUM PARAMETERS

Symbol	Quantity	Value
M	Mass of cart.	2.4Kg
m	Mass of Pendulum.	0.23Kg
L	Length of the pendulum.	0.36m
I	Moment of inertia.	$0.099Kg.m^2$
f_c	Coefficient of frictional force between ground and the cart.	0.05Kg/s
f_p	Coefficient of frictional force between pendulum and the pivot.	0.005 kgm/s rad
G	Gravitational force	$9.81m/s^2$

D. *System Analysis*

Inverted Pendulum is a 4th order Single Input Multiple Output (SIMO) system. Furthermore, it is an under-actuated and non-minimum phase system. It is a highly non-linear and unstable system in nature [1], which is why it is one of the more difficult problems in Control Engineering. For the considered system, the four open-loop poles are located at:
{0, -5.3928, 5.1603, -0.0830}.

III. CONTROLLER DESIGN

A. *Proportional Integral Derivative (PID) Controller*

PID controller is the most extensively used control scheme in industry [11],[12]. The conventional PID scheme is restricted to SISO systems. Therefore, for our SIMO Inverted Pendulum system, we modify the algorithm by considering a combination of two PID controllers - one for the stabilization of pendulum and the other for Cart regulation – as shown in Fig. 2.

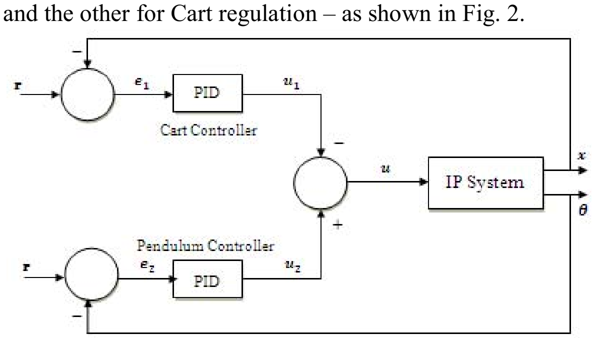


Fig 2. Considered PID Controller Structure [6],[13]

Tuning the PID controllers for improved performance yielded the following Gains:

TABLE II
PID GAINS

Controller	P	I	D
Cart	0.4	0.001	0.64
Pendulum	40	5	9

B. Pole Placement

In pole placement technique the closed-loop poles are determined using the closed-loop response requirements e.g. overshoot, settling time and steady state specifications [14]. In this scheme, the closed-loop poles, of a controllable system, can be placed at any desired location by choosing an appropriate feedback gain matrix K [11],[14].

For pole placement algorithm, the state feedback control law is given by:

$$u = -Kx$$

The feedback gain matrix K is formulated using Ackermann formula which states that [14]:

$$K = [0 \quad 0 \quad \cdots \quad 0 \quad 1][H \vdots GH \vdots \cdots \vdots G^{n-1}H]^{-1}\varphi(G)$$

with

$$\varphi(G) = G^n + \alpha_1 G^{n-1} + \alpha_2 G^{n-2} + \cdots + \alpha_n I$$

where n is the order of the system, and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ are the coefficients of the characteristics equation for the desired closed-loop poles.

For the case of the Inverted Pendulum (4th order system), we compute the two dominant poles, according to our design requirements, using the well-established formulas for 2nd order systems. Furthermore, we make the rest of the two poles non-dominant by placing them 5 times away from the real part of the designed dominant poles. This helps to ensure that their contribution towards the systems response is negligible [14], [15].

With the settling time of 4 seconds and overshoot of 5 percent, the closed poles computed are as {0.9501 + 0.0464i, 0.9501 − 0.0464i, 0.7788, 0.7788}.

The pole placement controller structure is shown in Fig 3. [14], [15]

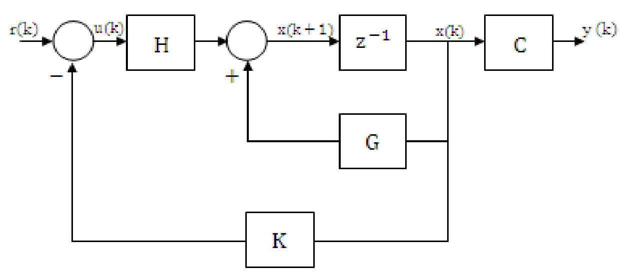


Fig 3. Pole Placement Controller Structure

C. Linear Quadratic Regulator (LQR) Controller

LQR refers to the optimal control problem for linear systems based on quadratic performance indexes [14].

The quadratic optimal control problem can be stated as follows [14], [16]. Given the controllable, linear discrete-time system

$$x(k+1) = Gx(k) + Hu(k) \tag{11}$$

where $x(k)$ is the n-dimensional State vector, $u(k)$ is the r-dimensional control vector, G is $(n \times n)$ system matrix and H is $(n \times r)$ input distribution Matrix, the optimal control problem is concerned with the computation of the control law which minimizes the following quadratic performance index.

$$J = \frac{1}{2}\sum_{k=0}^{N-1} x^*(k)Qx(k) + u^*(k)Ru(k)$$

The weighting matrices Q, R are selected based upon the desired level of performance and permissible control energy levels.

To solve the quadratic optimal control problem, consider the following discrete Ricatti equation [14].

$$P(k) = Q + G^*P(k+1)G\ldots$$
$$-G^*P(k+1)H[R + H^*P(k+1)H]^{-1}H^*P(k+1)G \tag{12}$$

It follows that optimal control vector $u(k)$can be given as:

$$u(k) = -[R + H^*P(k+1)H]^{-1}H^*P(K+1)Gx(k) \tag{13}$$

For the infinite horizon LQR scheme, used in this work, the steady-state Ricatti equation is obtained by iterating the Ricatti equation till P (k) becomes constant. After a certain trial and error the weighting matrices Q and R are chosen to be Q = diag(30, 30, 30, 30) and R = 1 respectively, for a reasonable performance. It is noted that it is the Q/R ratio that affects the overall response. The LQR controller structure is the same as that shown in Fig 3.

D. Fuzzy Logic

The Fuzzy Logic Controller [17] is different from the other controllers due to the fact that it does not require a mathematical model for the system. The controller only needs the output values from sensors along with the bounds on the control signal which can be applied to the system. The controller rules are designed through the analysis of system behavior.

In designing a fuzzy controller for an inverted pendulum, the Mamdani system [1] of two-outputs and one-input is employed. The structure of the Fuzzy logic controller is shown in Fig. 4. [18].

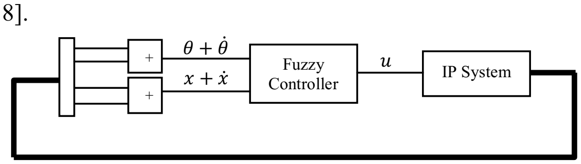


Fig. 4. Fuzzy Logic Controller Structure

Triangular memberships functions are used in the design of the Fuzzy controller as illustrated below:

First input to the controller is the sum of pendulum angle and pendulum velocity with the domain [-0.6 0.6] and membership functions [NL, NS, Z, PS, PL].

The second input to the controller is sum of cart position and cart velocity with the domain [-1 1] and membership functions [NL, NS, Z, PS, PL].

The output of the controller has domain [-60 60] and consists of seven membership functions which are [NL, NM, NS, Z, PS, PM, PL]. The rules table for the controller of inverted pendulum is given below:

θ/x	NL	NS	Z	PS	PL
NL	NL	NL	NL	NM	Z
NS	NL	NL	NM	Z	PM
Z	NS	NM	Z	PM	PS
PS	NM	Z	PM	PL	PL
PL	Z	PM	PL	PL	PL

E. MATLAB Simulation Results

All simulations are done in discrete time domain with a sampling time of 0.05 seconds. The MATLAB simulation results for Pendulum angle, Cart Position and the corresponding Control energy are given in Figs. 5, 6 and 7 respectively for all the designed controllers.

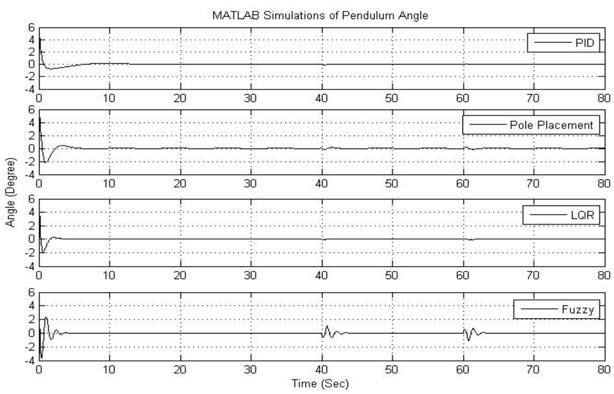


Fig. 5. MATLAB Simulation of Pendulum Angle

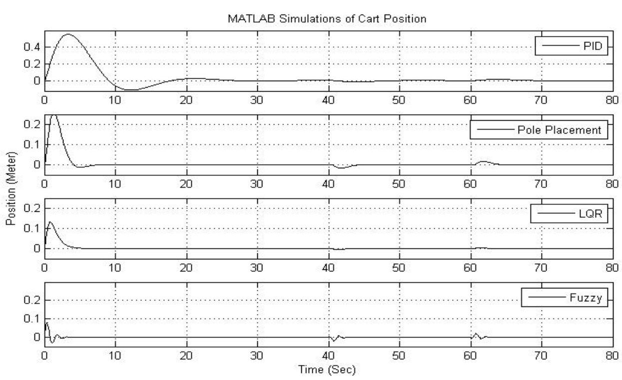


Fig. 6. MATLAB Simulation of Cart Position

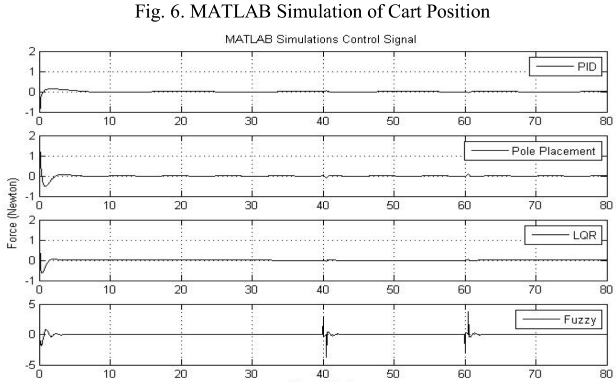


Fig. 7. MATLAB Simulation Control Signal

F. Simulation Results Analysis

Simulation Results shown in Figs. 5, 6, and 7 demonstrates that settling time and overshoot of pendulum angle is minimal in case of LQR and settling time and overshoot of cart position is minimum in case of Fuzzy logic. Inspecting the above figures, we see that LQR algorithm delivers the best overall performance whereas the PID scheme exhibits the worst response.

IV. Real-Time Implementation

For the real-time control implementation, we use the **Feedback Digital Pendulum Mechanical Unit 33-200.**

A. Implementation Issues

In real-time application, 20 samples/sec gave a very slow, unstable response. We found that 100 samples/sec was adequate for the controller implementation.

The pendulum system provides two outputs namely θ and x. Therefore, we estimated the remaining states ($\dot{\theta}$ and $\dot{x}$) for the successful implementation of the pole placement, LQR and fuzzy controllers.

The PID gains also needed some fine tuning for real-time implementation as given in the Table below

TABLE III
PID Gains

Controller	P	I	D
Cart	5	1.5	1.3
Pendulum	40	5	9

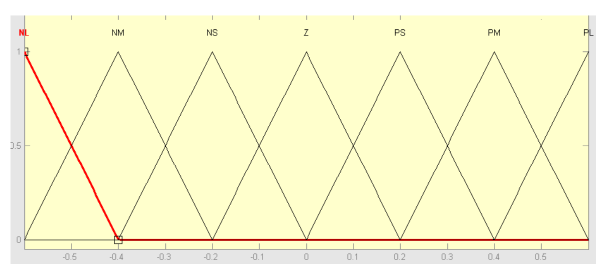
In real-time implementation of pole placement, the poles used in MATLAB simulation did not provide a reasonable response. Therefore, we recomputed the closed-loop poles at {0.9867+0.0112i, 0.9867−0.0112i, 0.9355, 0.9355} which lead to an improved response.

For LQR scheme, we redefined the weighting matrices to be Q = diag (1, 1, 300, 300) and R = 1 to ensure that the cart position remains within the physical system limits.

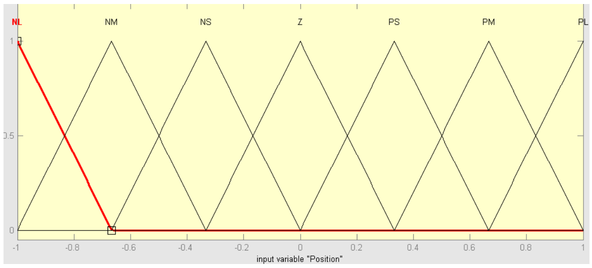
In real-time implementation of fuzzy logic controller, the 25 rules were extended to 49 rules which provided us with a more precise control.

The range of Angle input variable is [-0.6 0.6] radians as it is the stability region for the pendulum.

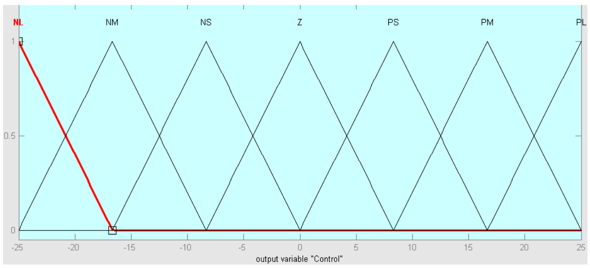
Triangular membership functions used in design of fuzzy controller are given below:



The Position input variable has the range of [-1 1] meters as it is the length of the base on which cart will slide.



The range of Control variable is [-25 25] Newton - the force applied by the controller.



The rules are prepared according to the physical behavior of the system as outlined in the figure below.

θ/x	NL	NM	NS	Z	PS	PM	PL
NL	NL	NL	NL	NL	NM	NM	Z
NM	NL	NL	NL	NM	NM	Z	PM
NS	NL	NL	NM	NM	Z	PM	PM
Z	NL	NM	NM	Z	PM	PM	PL
PS	NM	NM	Z	PM	PM	PL	PL
PM	NM	Z	PM	PM	PL	PL	PL
PL	Z	PM	PM	PL	PL	PL	PL

B. Real-Time Implementation Results

Real-time test runs for approx. 70 seconds, for the first 30 seconds the system is allowed to regulate and is left un-disturbed. Subsequently, at the 31st and 51st second, two identical step disturbances are applied to the system for half a second each as shown in the disturbance profile below.

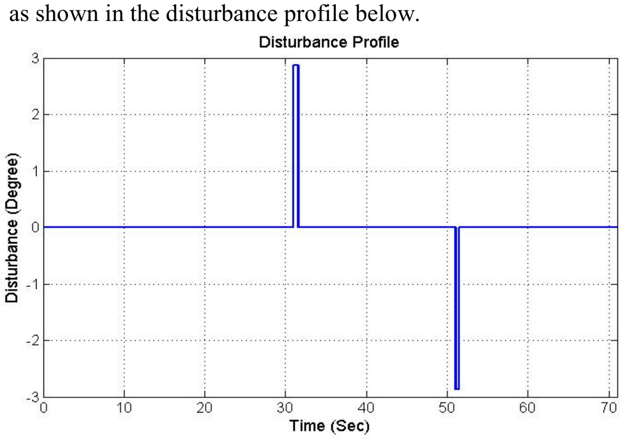


Fig. 8. Disturbance Profile (In Degree)

C. Result Analysis and Performance Comparison

In Fig. 10. for the first 30 seconds, fuzzy controller provides the best response giving smooth pendulum regulation. At the 1st application of disturbance, the PID-controlled system perturbs to around 2 degrees and the cart stabilizes the pendulum within approximately 2.5 seconds, whereas for the pole placement scheme, the perturbation is around 3.16 degrees, yielding damped oscillations which continue until

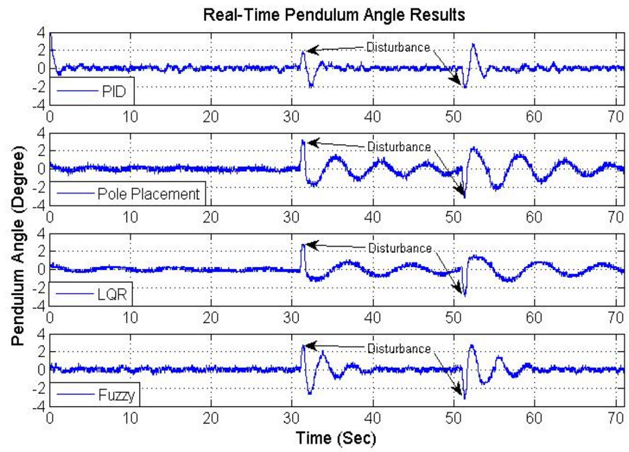


Fig. 9. Real-Time Pendulum Angle Results.

after the application of the 2nd disturbance (at t = 51s). For the 1st application of disturbance, LQR-controller system perturbs to 2.9 degrees and same damped oscillation are observed. However, these oscillations decay more quickly in comparison to pole placement scheme. With the fuzzy controller, the pendulum angle perturbs to 2.8 degrees and the controller stabilizes the pendulum within approximately 7 seconds.

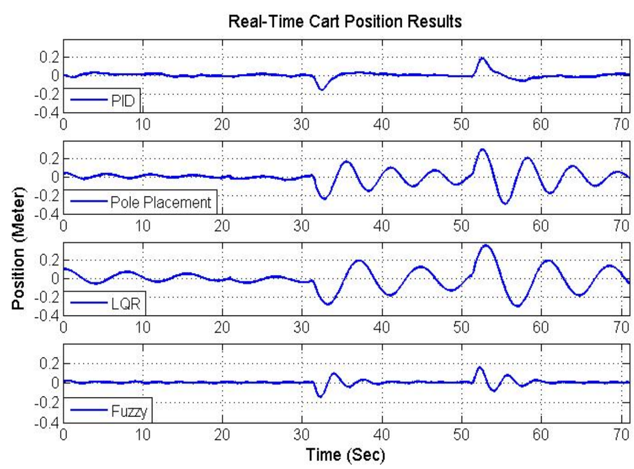


Fig. 10. Real-Time Cart Position Results.

From Fig. 11, it can be seen that the best cart regulation is achieved by the fuzzy controller. It provides the minimum perturbation even in the face of disturbance; however, the PID controller regulates the cart position more quickly in comparison to other schemes. The pole placement and LQR controllers struggle with disturbance rejection as can be seen in Figs. 10 and 11.

Control energy is the one of the most important considerations in any control scheme. It is generally undesirable to achieve good performance at the cost of high control energy. It can be seen in Fig. 12. that Fuzzy controller is delivering a good response using minimum energy i.e. between -6.5 to 6.8 Newton, while PID is consuming an extensive energy -28.40 to 28.73 Newton (approx).

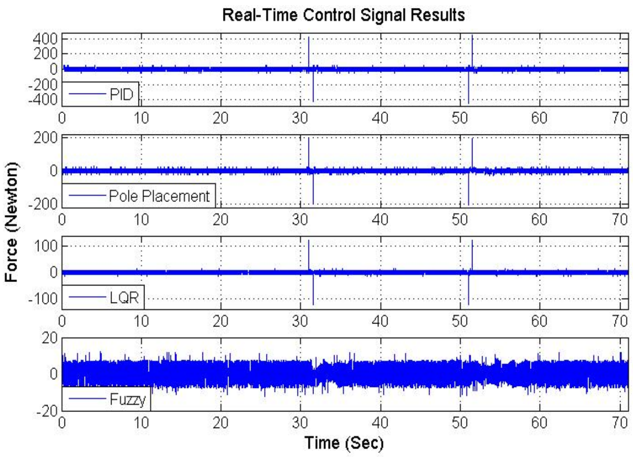


Fig. 11. Real-Time Control Signal Results

Performance indices are calculated and used to evaluate the regulation performance of the system. It is desirable for a system to have a low value of performance index [19]. We consider ISE as a performance index in this paper.

The ISE as well as the Net control energy formulae are given by:

$$\mathrm{ISE} = \int e(t)^2\, dt$$

$$\text{Net Control Energy} = \int F(t)^2\, dt$$

TABLE IV
PERFORMANCE COMPARISON

Controller	PID	Pole Placement	LQR	Fuzzy
Output	0.9203	6.0696	0.8334	0.7163
Net Control Energy	3.2e+04	9.17e+03	2.71e+03	1.77e+03

Table. IV shows that the fuzzy controller has the minimum value of ISE and also expends the smallest net control energy.

V. CONCLUSION

We have designed and implemented four control algorithms for the Inverted Pendulum system. Their performances have been compared on the basis of pendulum regulation, disturbance rejection and control energy requirements. The designed Fuzzy Logic controller delivers the best regulation and disturbance rejection performance and consumes relatively little control energy.

We are currently in the process of applying other schemes, including Model Predictive Control, to the Inverted Pendulum system. We are also focusing on improving the robustness of the designed control schemes.

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