HW9

PB17111614

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1.

1、$h=0.02$.

向前差商：$f'(0.02)=\dfrac{f(0.02+h)-f(0.02)}{h}=-100.0$

$$f'(0.04)=\frac{f(0.06)-f(0.04)}{h}=\frac{8.0-2.0}{0.02}=300.0$$

向后差商：$f'(0.02)=\dfrac{f(0.02)-f(0)}{h}=\dfrac{4.0-6.0}{0.02}=-100.0$

$$f'(0.04)=\frac{f(0.04)-f(0.02)}{h}=\frac{2.0-4.0}{0.02}=-100.0$$

中心差商：$f'(0.02)=\dfrac{f(0.04)-f(0)}{2h}=\dfrac{2.0-6.0}{2\times0.02}=-100.0$

$$f'(0.04)=\frac{f(0.06)-f(0.02)}{2h}=\frac{8.0-4.0}{2\times0.02}=100.0$$

2.

2、由 Gram-Schmidt 正交化：

$$\begin{cases} P_0(x) = f_0(x) = 1 \\ P_1(x) = f_1(x) - \dfrac{(x, P_0(x))}{(P_0(x), P_0(x))} P_0(x) = x - \dfrac{3}{4}, \\ P_2(x) = x^2 - \dfrac{(x^2, P_0(x))}{(P_0(x), P_0(x))} P_0(x) - \dfrac{(x^2, P_1(x))}{(P_1(x), P_1(x))} P_1(x) = x^2 - \dfrac{28}{23}x + \dfrac{12}{115} \end{cases}$$

$P_2(x)$ 两个零点：$x_1 = 0.092786$，$x_2 = 1.124605$.

∴积分系数：$A_1 = \int_0^1 x^2 l_1(x) dx = \int_0^1 x^2 \dfrac{x-x_2}{x_1-x_2} dx = 0.121018$

$A_2 = \int_0^1 x^2 l_2(x) dx = \int_0^1 x^2 \dfrac{x-x_1}{x_2-x_1} dx = 0.212316$.

∴两点Gauss公式 $\int_0^1 x^2 f(x) dx \simeq G_2(f) = 0.121018 f(0.092786) + 0.212316 f(1.124605)$

3.

(1)

3、(1) $f(x,y) = -y(x)$　$x_0 = 0$，$y_0 = 1$.

向前：$y_{n+1} = y_n + h f(x_n, y_n)$

$= y_n + h(-y_n(x_n))$

$= (1-h) y_n$

向后Euler：$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$

$= y_n + h(y(x_{n+1}))$

$= y_n - h y_{n+1}$

∴ $y_{n+1} = \dfrac{1}{1+h} y_n$.

梯形格式：$y_{n+1} = y_n + \dfrac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$

$= y_n + \dfrac{h}{2}[-y(x_n) - y(x_{n+1})]$.

$= y_n - \dfrac{h}{2} y_n - \dfrac{h}{2} y_{n+1}$

∴ $y_{n+1} = \dfrac{2-h}{2+h} y_n$.

改进Euler：$\overline{y}_{n+1} = y_n + h f(x_n + y_n)$

$y_{n+1} = y_n + \dfrac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \overline{y}_{n+1})]$.

$\overline{y}_{n+1} = (1-h) y_n$

$y_{n+1} = y_n + \dfrac{h}{2}[-y_n + (h-1) y_{n+1}]$

$= \dfrac{2-h}{-h^2+h+2} y_n \doteq \dfrac{1}{1+h} y_n$.

(2) 向前Euler：∵ $y_{n+1} =$

∴ $y_n = ($

向后Euler：$y_n = ($

梯形格式：

(2)

(b) 向前Euler：$\because y_{n+1}=(1-h)y_n,\ y_0=1$

$$\therefore y_n=(1-h)^n=\left(\frac{n-1}{n}\right)^n$$

向后Euler：$y_n=\left(\frac{1}{1+h}\right)^n=\left(\frac{n}{n+1}\right)^n$

梯形格式：$y_n=\left(\frac{2-h}{2+h}\right)^n=\left(\frac{2n-1}{2n+1}\right)^n$

改进Euler：$\cancel{y_n=\left(\frac{2-h}{h^2+h+2}\right)^n=\left(\frac{2n-1}{2n^2+n+1}\right)^n}$

$$y_n=\left(\frac{1}{1+h}\right)^n=\left(\frac{n}{n+1}\right)^n.$$

(3)

(2) ∵ $\varepsilon_0 = y(x_0) - y_0 = 0$

∴ 向前Euler公式收敛

$$\lim_{h\to 0} y_n = \lim_{n\to+\infty} y_n = y(1).$$

同理向后Euler公式收敛.

原方程: $\begin{cases} y'(x) = -y(x) \\ y(0) = 1 \end{cases}$ 通解: $y(x) = e^{-x} \quad 0 \le x \le 1.$

∴ $y(1) = e^{-1}$

∵ 对向前Euler: $y_n = \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n$

$$\therefore \lim_{n\to+\infty} y_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n\to+\infty} \left[\left(1 + \left(-\frac{1}{n}\right)\right)^{-n}\right]^{-1} = e^{-1} = y(1).$$

∴ 向前Euler收敛到 $y(1)$.

向后Euler, $\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right)^n = \lim_{n\to\infty} \left[\left(1 - \frac{1}{n+1}\right)^{-(n+1)}\right]^{-\frac{n}{n+1}}$

$$= \lim_{n\to\infty} e^{-\frac{n}{n+1}} = e^{-1} = y(1).$$

∴ 向后Euler收敛到 $y(1)$.

梯形格式: $\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} \left(1 - \frac{2}{2n+1}\right)^n = \lim_{n\to+\infty} \left[\left(1 - \frac{2}{2n+1}\right)^{-\frac{2n+1}{2}}\right]^{-\frac{2n}{2n+1}}$

$$= \lim_{n\to+\infty} e^{-\frac{2n}{2n+1}} = e^{-1} = y(1).$$

∴ 梯形格式收敛到 $y(1)$.

~~改进Euler:~~ $\cancel{\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} \left(\frac{2n-1}{n^2+n-1}\right)^n = \lim_{n\to+\infty} \left(1 - \frac{n^2-n}{n^2+n-1}\right)^n}$

$\cancel{= \lim_{n\to+\infty} \left[\left(1 - \frac{n^2-n}{n^2+n-1}\right)^{-\frac{n^2+n-1}{n^2-n}}\right]^{-\frac{n^3-n^2}{n^2+n-1}}}$

改进Euler: $\lim_{n\to\infty} y_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right)^n = \lim_{n\to\infty} e^{-\frac{n}{n+1}} = e^{-1} = y(1)$

∴ 改进Euler收敛到 $y(1)$.