计算方法 HW7

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1.

1. (1) 对应的 Jacobi 迭代格式分量形式：

$$\begin{cases} x_1^{(k+1)} = \frac{1}{2}x_2^{(k)} + 1 \\ x_2^{(k+1)} = \frac{1}{2}x_1^{(k)} + \frac{1}{2}x_3^{(k)} + 1 \\ x_3^{(k+1)} = \frac{1}{2}x_2^{(k)} + \frac{1}{2}x_4^{(k)} + 1 \\ x_4^{(k+1)} = \frac{1}{2}x_3^{(k)} + 1 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \\ x_4^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \\ x_4^{(k)} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

迭代格式如上，Jacobi迭代矩阵

$$G \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

(2) 对于 G，求 G 的特征值：

$$\det(\lambda E - G) = \begin{vmatrix} \lambda & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \lambda & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \lambda & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^4 - \frac{3}{4}\lambda^2 + \frac{1}{16} = 0$$

$\therefore \lambda_1 = -\frac{\sqrt{5}+1}{4}$ $\lambda_2 = -\frac{\sqrt{5}-1}{4}$ $\lambda_3 = \frac{\sqrt{5}-1}{4}$ $\lambda_4 = \frac{\sqrt{5}+1}{4}$，都是特征值

$\therefore \rho(G) = \max_{1 \le i \le 4} |\lambda_i| = \frac{\sqrt{5}+1}{4} < 1$

$\therefore$ 此时对于所有迭代都收敛，Jacobi迭代收敛.

2.

2.(1)

$$A=\begin{pmatrix}5 & -3 & 2\\ -3 & 5 & 2\\ 2 & 2 & 7\end{pmatrix}\qquad b=\begin{pmatrix}5\\5\\7\end{pmatrix}$$

先求Jacobi迭代格式：
$$\begin{cases}x_1^{(k+1)}= \frac{3}{5}x_2^{(k)}-\frac{2}{5}x_3^{(k)}+1\\ x_2^{(k+1)}=\frac{3}{5}x_1^{(k)}-\frac{2}{5}x_3^{(k)}+1\\ x_3^{(k+1)}=-\frac{2}{7}x_1^{(k)}-\frac{2}{7}x_2^{(k)}+1\end{cases}$$

∴ Gauss－Seidel迭代格式：
$$\begin{cases}x_1^{(k+1)}= \frac{3}{5}x_2^{(k)}-\frac{2}{5}x_3^{(k)}+1\\ x_2^{(k+1)}=\frac{3}{5}x_1^{(k+1)}-\frac{2}{5}x_3^{(k)}+1\\ x_3^{(k+1)}=-\frac{2}{7}x_1^{(k+1)}-\frac{2}{7}x_2^{(k+1)}+1\end{cases}$$

(2) ∵ $A=U+D+L$

∴
$$U=\begin{pmatrix}0 & -3 & 2\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}\quad D=\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7\end{pmatrix}\quad L=\begin{pmatrix}0 & 0 & 0\\ -3 & 0 & 0\\ 2 & 2 & 0\end{pmatrix}$$

∴ 分裂矩阵 $Q=D+L=\begin{pmatrix}5 & 0 & 0\\ -3 & 5 & 0\\ 2 & 2 & 7\end{pmatrix}$ 迭代矩阵 $G=-(D+L)^{-1}U$

$$G=-\begin{pmatrix}\frac{1}{5} & 0 & 0\\ \frac{3}{25} & \frac{1}{5} & 0\\ -\frac{16}{175} & -\frac{2}{35} & \frac{1}{7}\end{pmatrix}\cdot\begin{pmatrix}0 & -3 & 2\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & \frac{3}{5} & -\frac{2}{5}\\ 0 & \frac{9}{25} & -\frac{16}{25}\\ 0 & -\frac{48}{175} & \frac{52}{175}\end{pmatrix}$$

(3) 对于矩阵G，求特征值：

$$\det(\lambda E-G)=\begin{vmatrix}\lambda & -\frac{3}{5} & \frac{2}{5}\\ 0 & \lambda-\frac{9}{25} & \frac{16}{25}\\ 0 & \frac{48}{175} & \lambda-\frac{52}{175}\end{vmatrix}=\lambda\left[\left(\lambda-\frac{9}{25}\right)\left(\lambda-\frac{52}{175}\right)-\frac{48}{175}\times\frac{16}{25}\right]=0$$

∴ 三个单特征值 $\lambda_1=0$, $\lambda_2\approx0.748727$ $\lambda_3\approx-0.091584$

∴ $\rho(G)=\max\limits_{1\le i\le3}|\lambda_i|=0.748727$

$\rho(G)<1$

∴ Gauss－Seidel迭代收敛.