Lattice Quantum Field Theory Lectures

Lecture 2: 1+1 dimensional quantum field theory

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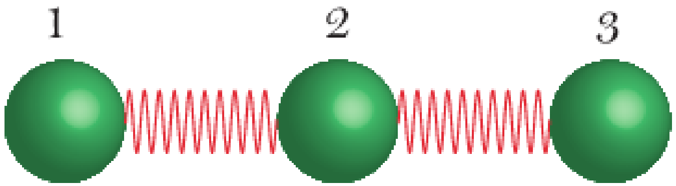
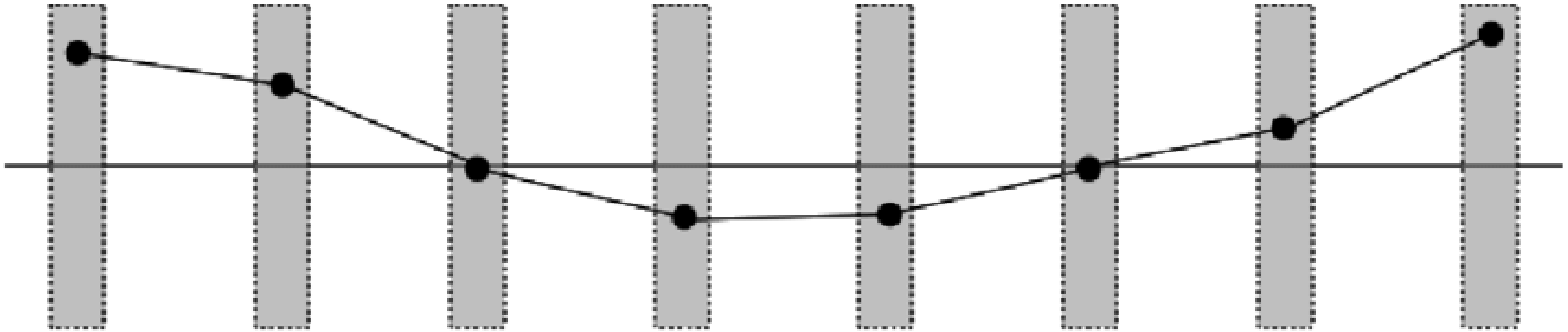
outline

- N-coupled oscillators
- Normal mode expansion and solutions
- Continuum limit
- 1+1 dimensional quantum field theory
- Mass and dispersion relation
- Soliton solution of phi-4 theory

N coupled oscillators





1D chain (ring)

- We label oscillators by *i* = 1, 2,...., N, with periodic condition such that *i*=0 and N are identical.
- Each oscillator has 1D coordinate $x_i = ia$, where a can be viewed as the basic length unit.
- The total kinetic energy,

$$T = \frac{1}{2} m \sum_{i=1,N} \dot{q}^2(ia)$$ where dot is the t-derivative

- The total potential energy ([N+1]=1)

$$V = \frac{1}{2}\kappa \sum_{n=1}^{N_a} \left(q(na) - q([n+1]a)\right)^2 ,$$

Equations of motion (E.O.M)

- The EOM are coupled linear differential equations

$$
\begin{aligned}
m\ddot{q}(na) &= -\frac{\partial V}{\partial q(na)} \\
&= -\kappa\left(2q(na) - q([n-1]a) - q([n+1]a)\right)
\end{aligned}
$$

- We can diagonalize these Eqs by introducing the normal coordinates,

$$
q(na) = \frac{1}{\sqrt{N_{\rm a}}}\sum_{k_l} e^{ik_l na} u_{k_l},
$$

$$
k_l = \frac{2\pi}{N_{\rm a}a} l \text{ with } l = 0, \pm 1, \pm 2, \cdots, \frac{N_{\rm a}}{2}.
$$

$\boldsymbol{\ell}$ **must be integer**
$\boldsymbol{\ell = 0}$ **is zero − mode**

Zero mode etc

- The periodic boundary condition is satisfied.
- There is always one zero mode. Zero-mode $l=0$ corresponds all coordinates move together. The potential energy is zero. It is a free motion.
- For $N=3$, there are two additional modes corresponds to $l=\pm 1$.
- For $N=4$, there are three additional modes, correspond to $l=\pm 1, 2$. The mode $l=-2$ is the same as $l=2$.
- Positive and negative l's are complex conjugate of each other, with opposite chirality.

Normal mode dynamics

- The lagrangian of the normal modes are

$$L = \frac{m}{2}\sum_{k_l} \dot{u}_{k_l}\dot{u}_{-k_l} - \frac{\kappa}{2}\sum_{k_l} 2\left(1-\cos(k_l a)\right) u_{k_l} u_{-k_l}$$

- Introduce the canonical coordinates,

$$p_{k_l} = \frac{\partial L}{\partial \dot{u}_{k_l}} = m\dot{u}_{-k_l}$$
$$p_{-k_l} = \frac{\partial L}{\partial \dot{u}_{-k_l}} = m\dot{u}_{k_l}.$$

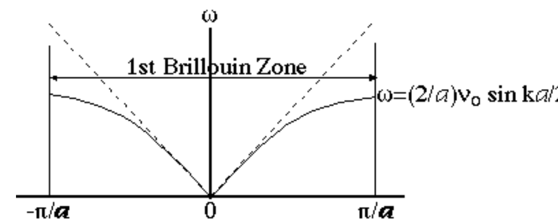
- New Hamiltonian is a sum of non-interacting normal modes

$$\mathrm{H} = \sum_{k_l}\left(\frac{1}{2m}p_{k_l}p_{-k_l} + \frac{1}{2}m\omega_{k_l}^2 u_{k_l}u_{-k_l}\right),$$

Dispersion relation and quantization

- Dispersion relation: Frequency related to different k

$$\omega_{k_l} = \sqrt{\frac{2\kappa\,(1-\cos(k_l a))}{m}} = 2\sqrt{\frac{\kappa}{m}}\left|\sin(\frac{k_l a}{2})\right|$$



- Introduce creation and annihilation operators

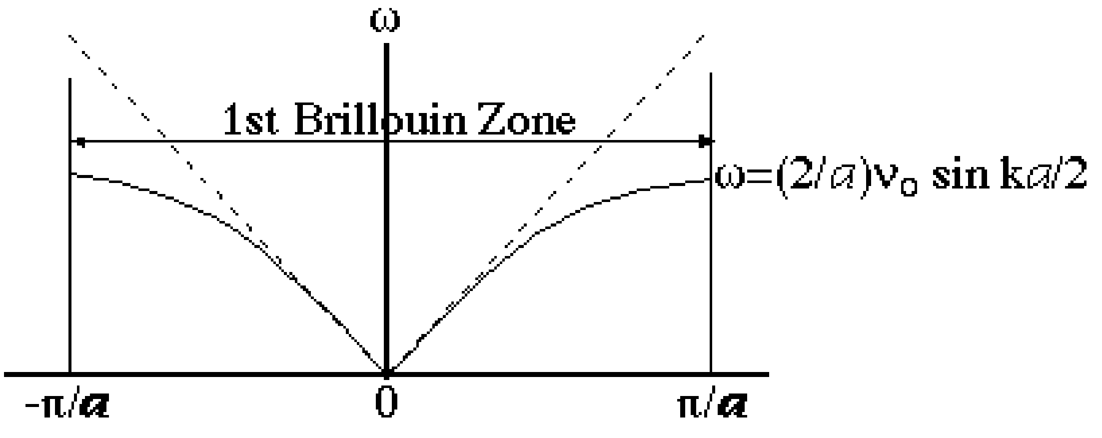
$$\hat{a}_{k_l} = \sqrt{\frac{m\omega_{k_l}}{2\hbar}}\left(\hat{u}_{-k_l} + \frac{i}{m\omega_{k_l}}\hat{p}_{k_l}\right)$$

$$\hat{a}^{\dagger}_{k_l} = \sqrt{\frac{m\omega_{k_l}}{2\hbar}}\left(\hat{u}_{k_l} - \frac{i}{m\omega_{k_l}}\hat{p}_{-k_l}\right).$$

- Now we have N-non-interacting harmonic oscillators,

$$H = \sum_{k_l} \mathcal{H}_{k_l} \qquad \mathcal{H}_{k_l} = \hbar\omega_{k_l}\left(\hat{a}^{\dagger}_{k_l}\hat{a}_{k_l} + \frac{1}{2}\right)$$

- It is interesting to note that even though every term of pot. energy seems to support an oscillator with angular frequency ω, the normal modes can have a range of angular frequency, going from 0 to 2ω.



Quantum states

- The ground state of the system is when all oscillators are the ground state

$$|0,0,....,0\rangle \quad \text{with } E_0 = \frac{\hbar}{2}\sum \omega_{k_l} \quad \text{(vacuum energy)}$$

The w. f. is $\Pi_{kl}\,\varphi_0(u_{kl})$ which is a complicated function of the original coordinates.

- The first excited state is a set of states with one quantum in one of the oscillators (kl)

$$|0,1,....,0\rangle \quad \text{with energy } E(kl)=E_0+\hbar\omega_{k_l}$$

which has the excitation energy $\Delta E(kl) = \hbar\omega_{k_l}$.

Only the excitation energy is measurable experimentally!

Taking continuum limit

- Let a→0 and N→∞, Na=L finite, we have infinite number of quantum mechanical degrees of freedom (field theory!)

we define a field through

$$q(x,t) = \lim_{\substack{a\to 0\\ N_a\to\infty}} \frac{q_n(t)}{\sqrt{a}} = \lim_{\substack{a\to 0\\ N_a\to\infty}} \frac{1}{\sqrt{N_a a}} \sum_k u_k(t) e^{ikx} = \frac{1}{\sqrt{L}} \sum_k u_k(t) e^{ikx}$$

$$p(x,t) = \lim_{\substack{a\to 0\\ N_a\to\infty}} \frac{p_n(t)}{\sqrt{a}} = \lim_{\substack{a\to 0\\ N_a\to\infty}} \frac{1}{\sqrt{N_a a}} \sum_k p_k(t) e^{-ikx} = \frac{1}{\sqrt{L}} \sum_k p_k(t) e^{-ikx}$$

More on the limit

- In the $a \rightarrow 0$, we pack ∞ number of dof in the finite line segment L.
- Correspondingly, there are infinite number of non-interacting normal modes corresponding to

$$k = \frac{2\pi}{L} l \quad \text{with } l = 0, \pm 1, \pm 2, \ldots, \infty$$

Now $\omega = \omega_0 a\, k$ (k is still discrete)

now $\omega_0 a$ has a unit of velocity, v_s it is the sound speed in this one dimensional medium.

Thus $\omega = v_s k,$

Wave equation

- The classical e.o.m now becomes the wave equation

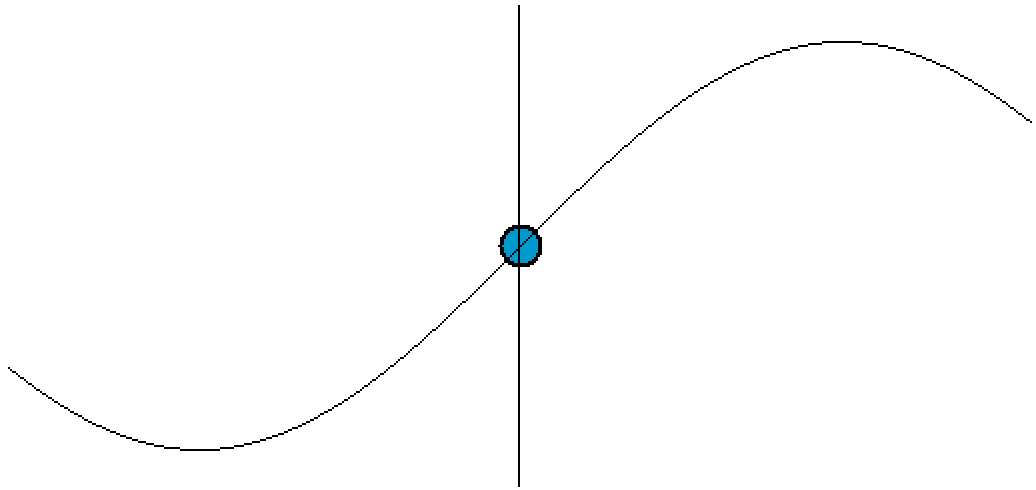
$$\left(\frac{1}{v_s^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) q(x,t) = 0,$$

whereas $k = \frac{\omega}{v_s} = 2\pi/\lambda$ where λ is the wavelength.

- Thus in this finite-length L, 1D system (a string), with ∞ number of h.o., one equivalently can represent the system by infinite number of waves with variable k.

q is the wave field. Large k means small w.l. (UV mode), small k means large w.l. (IR mode), smallest are $\pm\frac{2\pi}{L}$ and $0.$

Single oscillator and continuous wave (classical)



1D field theory

- 1D field theory deals with this 1D systems of waves.
- In the above example, we have free waves, i.e., the waves do not interact.
- However, more meaningful examples deals with waves that interact.
- We can easily add interactions when using Lagrangian dynamics for the field theory.

Relativistic waves

- When we are dealing with fundamental theories, we know that relativity is important. Therefore, the wave equation must be invariant under relativity transformation.

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$
$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

- This is the case if $v_s = c$,

$$\left[\frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2}\right]\phi(x,t) = \left[\frac{\partial^2}{c^2 \partial t'^2} - \frac{\partial^2}{\partial x'^2}\right]\phi(x',t') = 0$$

Quantum mechanical wave

- In QM, particles are described by QM waves, just like that the electron is described by electron wave. For non-relativistic particles, they are described by waves satisfying Schrodinger eq. which corresponds to $E = p^2/2m$
- For a relativistic QM particle, it shall satisfy the relativistic wave equation.
- For a free particle, relativistic w.e. shall be derived from $E^2 = p^2c^2 + c^4m^2$, where m is the rest mass.

Klein-Gordon equation

For the relativistic energy-momentum relation, one can derive the following wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0.$$

This is famous Klein-Gordon equation. Comparing to our earlier example, one has an extra mass term

$$\frac{m^2c^2}{\hbar^2}$$

which has the Planck constant $\hbar$, indicating it is a Quantum w.e.

It reduced to the Schrodinger eq. in small velocity limit.

Natural unit in relativistic theory

- In dealing with relativistic problem. It is quite common to use the so-called natural unit.

$$\hbar = c = 1$$

- Usually, there are three different units in SI systems, mass in kg, time in sec and length in m.
- After choosing this natural unit, [second] and [kg] can be expressed in term of [m].
- c = 1, $3\text{x}10^{23}$ fm/sec=1, 1 sec = 3 x 10^{23} fm, 1fm = 3.3 $\text{x}10^{-24}$s

Natural unit and mass dimension

- And also,

$$\hbar c = 1 = 197 \text{ fm MeV}$$

thus, 197 MeV = 1 fm^{-1}

- That means [E]=[p]=[1/x]=[1/t]= MeV
- All physical quantity can have eV or keV or MeV as dimension, or mass dimension.
- The mass dimension of the action S is 0.
- Lagrangian has mass dimension 1, [L]=1.

Quantum field theory

- In relativistic theories, the mass and energy can convert into each other.
- Thus, particles can disappear into energy, and reversely energy can create particles.
- Thus the single particle quantum mechanics as described by Klein-Gordon eq. is useless. One needs a theory which can create and annihilate particles.
- For this, one needs to discuss the quantized wave systems (coupled h.o.) or quantum ∞ dof systems or quantum field theory.

Quantization of 1+1 wave system

- One needs to quantize 1+1 dimensional wave system, which is in a sense already quantum mechanical (it contains Planck const).
- One can quantize by assuming the field $\varphi(x,t)$ is an operator and find the conjugate field operator $\pi(x,t)$

  and postulate commutation relations among quantum field
- However, for a numerical approach, the above strategy is of little use. One can again, however, use Feynman's path integral approach. To do this, we need to start with a lagrangian.

Lagrangian for a field

- The lagrangian is a sum over all modes, thus

$$L = \int L dx$$

where the lagrangian density can be written as

$$L = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - \frac{1}{2}m^2\phi^2.$$

One can verify that EL eq. reproduces KG eq.

When quantized, the first excited state of the system with a set of h.o. angular frequency,

$$\omega^2 = k^2 + m^2$$

describes a particle of mass m and momentum k.

Introducing interactions

- 1D interaction-free field theory is very simple and not interesting.
- To make a non-trivial field theory, we can introduce an interaction term

$$L = -\frac{\lambda}{4!}\phi^4$$

with $\lambda > 0$, so that the total energy has a lower bound.

- We will try to focus on solving this so-called φ^4 theory next.

Dimensional analysis

- In 1+1 D field theory, the mass dimension of the field is zero. $[\phi] = 0$
- The lagrangian density has mass-dim 2.
- The coupling constant λ>0 also has mass dimension 2.
- It can be shown that the system still supports a free propagating wave as the first excited state of the system, corresponding to a "physical particle" with non-trivial internal structure.

Calculations to do

1. Calculate the physical mass M of the free propagating wave as a function of bare mass m_0 and λ, which also will depend on the momentum cut-off Λ or lattice spacing a. Show it logarithmically depends a.

   $$M(m_0, \lambda, a) \sim \ln a$$

2. Calculate the dispersion relation of the particle satisfying the relativistic relation, i.c.

   $$E^2 = k^2 + M^2$$

which is the relationship between momentum and energy of a relativistic particle!

Euclidean time

- Again to make numerical calculation possible, one has to use Euclidean time
- One needs to consider evolution in imaginary time.

Ground state and filtering

- Again label the exact ground state of 1+1 field theory as

$$|0\rangle$$

- A quantum wave with momentum k=0 can be generated by

$$\hat{\phi}_{k=0}(\tau = 0) \quad |0\rangle$$

which can be expanded into a set of exact eigenstates. After long "time" T,

$$e^{-TH}\hat{\phi}_{k=0}(\tau = 0)|0\rangle \sim e^{-TM}|k = 0\rangle$$

Only the first excited with k=0 remains.

Two-point correlation function

- Now define the two-point correlation function

$$\langle 0 \middle| \hat{\phi}(x,T)\hat{\phi}_{k=0}(\tau = 0) \middle| 0 \rangle$$

which reduces to at large T,

$$C_2(T,M) \sim ce^{-TM}$$

Thus by studying the large-T behavior of the of the two-point correlation function, one can get the physical mass M.

Calculating "dispersion" relation

- To find the dispersion relation, E(k), one can calculate the two-point correlation function

$$\mathrm{C}_2(k,T) = \langle 0 \big| \hat{\phi}(x,\tau = T)\hat{\phi}_k(\tau = 0) \big| 0 \rangle$$

- At large T, the first excited state with momentum k dominates, which produces the following exponential

$$\mathrm{C}_2(k,T,E) \sim e^{-E(k)T}$$

one can get the E(k) by checking the leading large-T behavior

Lattice implementation

- Two-point function as a functional integral

$$C_2\ (k,T) = \int [D\phi(x,\tau)]\phi(x,T)\int dy\phi(y,0)e^{-S_E}$$

where the action is

$$S_E = \int dxd\tau[\frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4]$$

where again λ is positive and dimension-2.

Lattice calculation

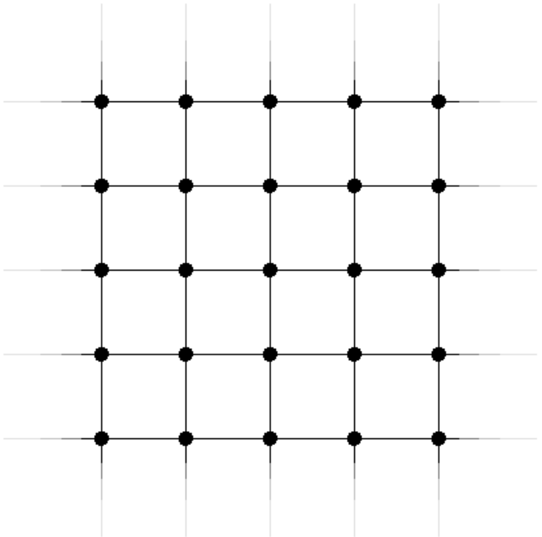
- We consider field configurations in 2-D lattice, with N points in “time” as well as space directions, N^2.
- Assume the lattice spacing is a in both directions.

  Thus, the size of the box is L=Na.
- To simulate the theory well, one needs to have

$$\frac{1}{L} \ll m, \quad \sqrt{\lambda} \ll \frac{1}{a}$$

where 1/a is the UV cut-off and 1/L is IR cutoff.

Lattice implementation



- On the lattice, one has ϕ_{ij} degrees

 of freedom with I, j = 1, …., N

 with periodic boundary condition

$$\phi_{i+N,j+N} = \phi_{ij}$$

- One generate configuration $\{\phi_{ij}\}$ using Monte Carlo method

$$C_2(k,m,T) = \sum \phi(x\,,T)\sum_y e^{iky}\,\phi(y,0)$$

Actual consideration

- For 2D simulation, a reasonable choice is N=100.

  If we one choose, m=1, λ=1, a=0.1, L=10.
- Finite-volume effect

one can do the same simulation, but with N=500, L=50 with the same a, m, λ.

- Finite-a effect: one can do the same simulation with a=0.05, N=200, or a=0.02, N=500.

Thus mass M will have lna-dependence, which can be computed in pert. theory.

- The continuum limit exists when all physical observables are expressed in terms of M and λ.

Calculating the mass as a function of a.

- Calculate the C_2 for several different T.
- Plot $\ln C_2$ as a function of T.
- Find the mass.
- For several different a, plot the relation between M^2 and $\ln a$.
- This is the famous UV divergence in the field theory. However, this divergence does not affect the physical observables in terms of physical mass and coupling.

Sine-Gordon eq.

- Sin-Gordon equation has a solition solution.
- One can calculate the mass of a quantum soliton.