

Hydrogen

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1 Matching

QED Lagrangian is

$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \quad (1)$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \quad (2)$$

In tree level¹

$$\begin{aligned}
 i\mathcal{M}_{QED}^{(0)} &= \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \text{---} \text{---} p_2 \end{array} = -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \bar{u}_e(p_2) \gamma_0 u_e(p_1) \\
 i\mathcal{M}_{NRQED}^{(0)} &= \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \text{---} \text{---} p_2 \end{array} = -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \psi^\dagger(p_2) \psi(p_1)
 \end{aligned}$$

The box diagram for NRQED process is

$$\begin{aligned}
 i\mathcal{M}_{NRQED}^{(1)} &= \begin{array}{c} P_N \text{---} \text{---} P_N \\ \begin{array}{c} \xrightarrow{P_N - k} \\ \downarrow k \\ \uparrow k - q \end{array} \\ p_1 \text{---} \text{---} p_2 \\ \xrightarrow{p_1 + k} \end{array} \\
 &= e^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int [dk] \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (-k^0 + i\epsilon) (p_1^0 + k^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\
 &= -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (E_1 - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m})} \psi(p_1) \\
 &= -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 (E_1 - \frac{\mathbf{k}^2}{2m})} \psi(p_1)
 \end{aligned}$$

¹Note that there's no Gamma matrix in the heavy particle side, they can only appear in the QED side.

The box and crossed box diagram for QED process is

$$\begin{aligned}
i\mathcal{M}_1^{(1)} &= \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 - p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.

$$\begin{aligned}
i\mathcal{M}_2^{(1)} &= \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 - \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 + p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 - \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 + p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

$$\begin{aligned}
i\mathcal{M}_1^{(1)} + i\mathcal{M}_2^{(1)} &= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^{0^2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 - p_1^{0^2}]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^{0^2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 - \mathbf{p}_1^2]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1)
\end{aligned}$$

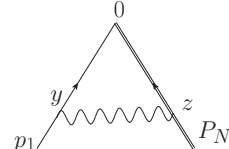
Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order.

Now consider operator product expansion.

Tree level matching: 

$$\begin{aligned}\langle 0|T\psi(x)N(0)|pP_N\rangle &= \text{diagram} = u_e(p)u_N(P_N)e^{-ip\cdot x} \\ \langle 0|T\psi_e(x)N(0)|pP_N\rangle &= \text{diagram} = \psi_e(p)u_N(P_N)\end{aligned}$$

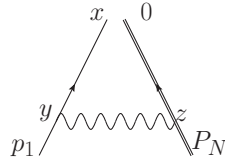
At leading order $u_e(p) = \begin{pmatrix} \psi_e(p) \\ 0 \end{pmatrix}$. (If we're only interested in the hard region contribution, which is independent of states, the leading order is independent of any on-shell momentums.)

One loop scenario for NRQED case: 

$$\begin{aligned}\langle 0|\psi_e(0)N(0)e \int d^4y \bar{\psi}_e \psi_e A^0 e \int d^4z \bar{N} N A^0 |eN\rangle &= e^2 u_N(v_N) \int [dk] \frac{1}{\mathbf{k}^2(-k^0 + i\epsilon)(p_1^0 + k^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\ &= -ie^2 u_N(v_N) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(E_1 - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\ &= -ie^2 u_N(v_N) \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p}_1)^2(E_1 - \frac{\mathbf{k}^2}{2m} + i\epsilon)} \psi(p_1)\end{aligned}$$

drop p_1

$$= -ie^2 u_N(v_N) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(E_1 - \frac{\mathbf{k}^2}{2m} + i\epsilon)} \psi(p_1) = \pi i e^2 \sqrt{\frac{2m}{E_1}} u_N(v_N) \psi(p_1)$$

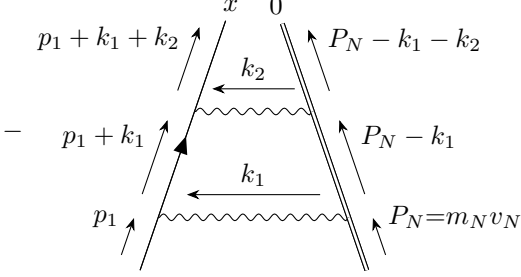
For QED case: 

$$\begin{aligned}\langle 0|\psi(x)N(0)e \int d^4y \bar{\psi} \gamma^0 \psi A^0 e \int d^4z \bar{N} N A^0 |eN\rangle &\stackrel{2}{=} e^2 u_N(v_N) \int [dk] e^{-i(\mathbf{k} + \mathbf{p}_1) \cdot \mathbf{x}} \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2[(p_1 + k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\ &= e^2 u_N(v_N) \int [dk] e^{-i(\mathbf{k} + \mathbf{p}_1) \cdot \mathbf{x}} \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2[(p_1 + k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\ &= ie^2 u_N(v_N) \int \frac{d^3k}{(2\pi)^3} e^{-i(\mathbf{k} + \mathbf{p}_1) \cdot \mathbf{x}} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 - p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\ &= ie^2 u_N(v_N) \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)\end{aligned}$$

drop \mathbf{p}_1

$$= ie^2 u_N(v_N) \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{p_1^0 + \sqrt{\mathbf{k}^2 + m^2}}{2\mathbf{k}^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)$$

Two loop scenario for QED case $\langle 0|T\psi(x)N(0)e \int d^4 y_1 \bar{\psi}\gamma^0 \psi A^0 e \int d^4 z_1 \bar{N} N A^0 e \int d^4 y_2 \bar{\psi}\gamma^0 \psi A^0 e \int d^4 z_2 \bar{N} N A^0 |eN\rangle$:



$$\begin{aligned} &= e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon} \frac{\not{p}_1 + \not{k}_1 + \not{k}_2 + m}{(p_1 + k_1)^2 - m^2 + i\epsilon} \gamma^0 \frac{\not{p}_1 + \not{k}_1 + m}{(p_1 + k_1)^2 - m^2 + i\epsilon} \gamma^0 u_N(v_N) u_e(p_1) \\ &= e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} \frac{1}{|\mathbf{k}_1|^2 |\mathbf{k}_2|^2} \frac{4p_1^0{}^2 + 2p_1^0 k_1^0 + 2\mathbf{p}_1 \cdot \mathbf{k}_1 + 2(p_1^0 + k_1^0)(\not{k}_1 + \not{k}_2)\gamma^0 - \not{k}_2 \not{k}_1}{[(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon][(p_1 + k_1)^2 - m^2 + i\epsilon][-k_1^0 - k_2^0 + i\epsilon][-k_1^0 + i\epsilon]} u_N(v_N) u_e(p_1)^3 \\ &= ie^4 \int [dk_1] \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{2(k_1^0 + p_1^0)[(\sqrt{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2} - k_1^0 - p_1^0) + (k_2^i \gamma_i + \not{k}_1)\gamma^0] - [\gamma^0(\sqrt{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2} - k_1^0 - p_1^0) - k_2^i \gamma_i] \not{k}_1}{2\sqrt{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2}(\sqrt{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2} - p_1^0 + \frac{2((\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2) + 2\sqrt{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2} - p_1^0}{2((\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2)} i\epsilon)} \\ &\quad \frac{1}{-k_1^0 + i\epsilon} \frac{1}{(p_1 + k_1)^2 - m^2 + i\epsilon} \frac{1}{|\mathbf{k}_1|^2 |\mathbf{k}_2|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} \end{aligned}$$

define $a = (\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2$ and $b = \sqrt{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2} - k_1^0 - p_1^0 + k_2^i \gamma_i \gamma^0 = \sqrt{a} - k_1^0 - p_1^0 + k_2^i \gamma_i \gamma^0$ (pole location is $\sqrt{a} - k_1^0 - p_1^0 - \frac{i\epsilon}{2\sqrt{a}}$), and note that the long coefficient of the first ϵ above is positive

$$= ie^4 \int [dk_1] \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{4p_1^0{}^2 + 2p_1^0 k_1^0 + 2\mathbf{p}_1 \cdot \mathbf{k}_1 + 2(k_1^0 + p_1^0)[b + \not{k}_1 \gamma^0] - \gamma^0 b \not{k}_1}{2\sqrt{a}(\sqrt{a} - p_1^0)[-k_1^0 + i\epsilon][(p_1 + k_1)^2 - m^2 + i\epsilon]} \frac{1}{|\mathbf{k}_1|^2 |\mathbf{k}_2|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}$$

also define b' so that $b = b' - k_1^0$ ($b' = \sqrt{a} - p_1^0 + k_2^i \gamma_i \gamma^0$) and $a' = (\mathbf{p}_1 + \mathbf{k}_1)^2 + m^2$

$$= ie^4 \int [dk_1] \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{4p_1^0{}^2 + 2p_1^0 k_1^0 + 2\mathbf{p}_1 \cdot \mathbf{k}_1 + 2(k_1^0 + p_1^0)[b' + k_1^i \gamma_i \gamma^0] - \gamma^0(b' - k_1^0)(k_1^0 \gamma^0 + k_1^i \gamma_i)}{2\sqrt{a}(\sqrt{a} - p_1^0)[-k_1^0 + i\epsilon][(p_1 + k_1)^2 - m^2 + i\epsilon]|\mathbf{k}_1|^2 |\mathbf{k}_2|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}$$

the pole location is $\sqrt{a'} - p_1^0 - \frac{i\epsilon}{2\sqrt{a'}}$

$$\begin{aligned} &= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} - p_1^0 + (k_1 + k_2)^i \gamma_i \gamma^0) - (\sqrt{a'} - \sqrt{a} + k_2^i \gamma_i \gamma^0)(\sqrt{a'} - p_1^0 - k_1^i \gamma_i \gamma^0) + 2p_1^0{}^2 + 2\mathbf{k}_1 \cdot \mathbf{p}_1 + 2\sqrt{a'} p_1^0}{4\sqrt{a}\sqrt{a'}(\sqrt{a'} - p_1^0)(\sqrt{a} - p_1^0)|\mathbf{k}_1|^2 |\mathbf{k}_2|^2} \\ &\quad u_N(v_N) u_e(p_1) e^{-i(\mathbf{p}_1+\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} \end{aligned}$$

shift both loop momentum⁴ so that $a = |\mathbf{k}_2|^2 + m^2$ and $a' = |\mathbf{k}_1|^2 + m^2$, now $b = \sqrt{a} - k_1^0 + (k_2 - k_1)^i \gamma_i \gamma^0$ and $b' = \sqrt{a} + (k_2 - k_1)^i \gamma_i \gamma^0$

$$\begin{aligned} &= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} u_N(v_N) u_e(p_1) e^{-i\mathbf{k}_2\cdot\mathbf{x}} \\ &\quad \frac{2\sqrt{a'}(\sqrt{a} - p_1^0 + (k_2 - k_1)^i \gamma_i \gamma^0) - (\sqrt{a'} - \sqrt{a} + (k_2 - k_1)^i \gamma_i \gamma^0)(\sqrt{a'} - p_1^0 - (k_1 - p_1)^i \gamma_i \gamma^0) + 2m^2 + 2\mathbf{k}_1 \cdot \mathbf{p}_1 + 2\sqrt{a'} p_1^0}{4\sqrt{a}\sqrt{a'}(\sqrt{a'} - p_1^0)(\sqrt{a} - p_1^0)|\mathbf{k}_1 - \mathbf{p}_1|^2 |\mathbf{k}_2 - \mathbf{k}_1|^2} \end{aligned}$$

drop \mathbf{p}_1

$$\begin{aligned} &= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} - m + k_2^i \gamma_i \gamma^0) - (\sqrt{a'} - \sqrt{a} + (k_2 - k_1)^i \gamma_i \gamma^0)(\sqrt{a'} - m - k_1^i \gamma_i \gamma^0) + 2m^2 + 2\sqrt{a'} m}{4\sqrt{a}\sqrt{a'}(\sqrt{a'} - m)(\sqrt{a} - m)|\mathbf{k}_1|^2 |\mathbf{k}_2 - \mathbf{k}_1|^2} \\ &\quad u_N(v_N) u_e(p_1) e^{-i\mathbf{k}_2\cdot\mathbf{x}} \end{aligned}$$

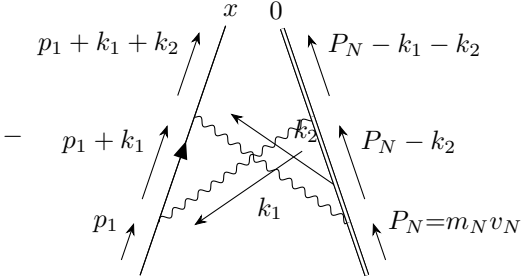
² $\langle 0|\psi(x)N(0)e \int d^4 y \bar{\psi}\gamma^0 \psi A^0 e \int d^4 z \bar{N} N A^0 |eN\rangle = e^2 \int d^4 y \int d^4 z \int \frac{d^4 k}{(2\pi)^4} \frac{i}{\mathbf{k}^2} e^{-i\mathbf{k}\cdot(\mathbf{z}-\mathbf{y})} \int \frac{d^4 k_1}{(2\pi)^4} \tilde{S}_e(k_1) e^{-i\mathbf{k}_1\cdot(\mathbf{y}-\mathbf{x})} \int \frac{d^4 k_2}{(2\pi)^4} \tilde{S}_N(k_2) u_N(v_N) u_e(p) e^{-i\mathbf{p}_1\cdot\mathbf{y}}.$

rewrite it with $a_1 = a'$ and $a_2 = a$

$$= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{2\sqrt{a_1}(\sqrt{a_2} - m + k_2^i \gamma_i \gamma^0) - (\sqrt{a_1} - \sqrt{a_2} + (k_2 - k_1)^i \gamma_i \gamma^0)(\sqrt{a_1} - m - k_1^i \gamma_i \gamma^0) + 2m^2 + 2\sqrt{a_1}m}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1} - m)(\sqrt{a_2} - m)|\mathbf{k}_1|^2|\mathbf{k}_2|^2} u_N(v_N)u_e(p_1)e^{-i\mathbf{k}_2 \cdot \mathbf{x}}$$

to investigate the divergent behaviour of the integral, rewrite the integral before the shift ($a_1 = (\mathbf{p}_1 + \mathbf{k}_1)^2 + m^2$ and $a_2 = (\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2 + m^2$)

$$= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{2\sqrt{a_1}(\sqrt{a_2} - p_1^0 + (k_1 + k_2)^i \gamma_i \gamma^0) - (\sqrt{a_1} - \sqrt{a_2} + k_2^i \gamma_i \gamma^0)(\sqrt{a_1} - p_1^0 - k_1^i \gamma_i \gamma^0) + 2p_1^{0^2} + 2\mathbf{k}_1 \cdot \mathbf{p}_1 + 2\sqrt{a_1}p_1^0}{4\sqrt{a_2}\sqrt{a_1}(\sqrt{a_1} - p_1^0)(\sqrt{a_2} - p_1^0)|\mathbf{k}_1|^2|\mathbf{k}_2|^2} u_N(v_N)u_e(p_1)e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}}$$



$$= e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_2^0 + i\epsilon} \frac{1}{(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon} \frac{\not{p}_1 + \not{k}_1 + \not{k}_2 + m}{(p_1 + k_1)^2 - m^2 + i\epsilon} \gamma^0 u_N(v_N)u_e(p_1)$$

$$= e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \frac{1}{|\mathbf{k}_1|^2 |\mathbf{k}_2|^2} \frac{4p_1^{0^2} + 2p_1^0 k_1^0 + 2\mathbf{p}_1 \cdot \mathbf{k}_1 + 2(p_1^0 + k_1^0)(\not{k}_1 + \not{k}_2)\gamma^0 - \not{k}_2 \not{k}_1}{[(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon][(p_1 + k_1)^2 - m^2 + i\epsilon][-k_1^0 - k_2^0 + i\epsilon][-k_2^0 + i\epsilon]} u_N(v_N)u_e(p_1)$$

$$= ie^4 \int [dk_1] \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{4p_1^{0^2} + 2p_1^0 k_1^0 + 2\mathbf{p}_1 \cdot \mathbf{k}_1 + 2(k_1^0 + p_1^0)[b + \not{k}_1 \gamma^0] - \gamma^0 \not{k}_1}{2\sqrt{a_2}(\sqrt{a_2} - k_1^0 + i\epsilon) \left[k_1^0 + p_1^0 - \sqrt{a_2} + \frac{i\epsilon}{2\sqrt{a_2}} + i\epsilon \right] [(p_1 + k_1)^2 - m^2 + i\epsilon]} \frac{1}{|\mathbf{k}_1|^2 |\mathbf{k}_2|^2} u_N(v_N)u_e(p_1) e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}}$$

$$= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} - p_1^0 + (k_1 + k_2)^i \gamma_i \gamma^0) + (\sqrt{a'} + \sqrt{a} + k_2^i \gamma_i \gamma^0)(\sqrt{a'} + p_1^0 + k_1^i \gamma_i \gamma^0) - 2p_1^{0^2} - 2\mathbf{k}_1 \cdot \mathbf{p}_1 + 2\sqrt{a'}p_1^0}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1} + \sqrt{a_2})(\sqrt{a_2} - p_1^0)|\mathbf{k}_1|^2|\mathbf{k}_2|^2} u_N(v_N)u_e(p_1)e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}}$$

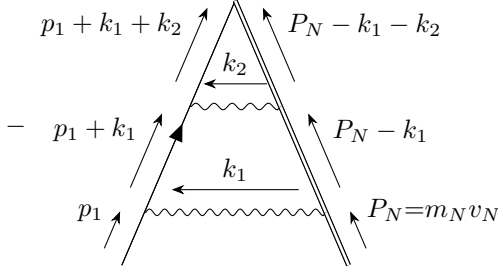
The sum of NRQED diagram at NNLO is

$$= e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} u_N(v_N)u_e(p_1)e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right]$$

³With a $u_e(p_1)$ on the right hand side, $(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0(\not{p}_1 + \not{k}_1 + m)\gamma^0 = (\not{p}_1 + \not{k}_1 + \not{k}_2 + m)[2(p_1^0 + k_1^0)\gamma^0 - (\not{p}_1 + \not{k}_1 - m)] = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (\not{p}_1 + \not{k}_1 + \not{k}_2 + m)(\not{p}_1 + \not{k}_1 - m) = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\not{k}_1 = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (2p_1 \cdot k_1 - \not{k}_1 \not{p}_1 + \not{k}_2 \not{k}_1 + \not{k}_1 m) = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (2p_1 \cdot k_1 + \not{k}_2 \not{k}_1) = 2(p_1^0 + k_1^0)[2(p_1^0 + k_1^0 + k_2^0) - \gamma^0(\not{p}_1 + \not{k}_1 + \not{k}_2 - m)] - (2p_1 \cdot k_1 + \not{k}_2 \not{k}_1) = 2(p_1^0 + k_1^0)[2p_1^0 + (\not{k}_1 + \not{k}_2)\gamma^0] - (2p_1 \cdot k_1 + \not{k}_2 \not{k}_1) = 4p_1^{0^2} + 2p_1^0 k_1^0 + 2\mathbf{p}_1 \cdot \mathbf{k}_1 + 2(p_1^0 + k_1^0)(\not{k}_1 + \not{k}_2)\gamma^0 - \not{k}_2 \not{k}_1.$

⁴ $k_1 \rightarrow k'_1 = k_1 + p_1$ and $k_2 \rightarrow k'_2 = k_1 + k_2 + p_1 = k'_1 + k_2$.

For NRQED case ($\langle 0 | \psi_e(0) N(0) e \int d^4 y_1 \bar{\psi}_e \psi_e A^0 e \int d^4 z_1 \bar{N} N A^0 e \int d^4 y_2 \bar{\psi}_e \psi_e A^0 e \int d^4 z_2 \bar{N} N A^0 | e N \rangle$)



$$\begin{aligned}
&= e^4 \int [dk_1] [dk_2] \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{p_1^0 + k_1^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k}_1)^2}{2m} + i\epsilon} \frac{1}{p_1^0 + k_1^0 + k_2^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2}{2m} + i\epsilon} \psi_e(p_1) u_N(v_N) \\
&= i e^4 \int [dk_1] \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{p_1^0 + k_1^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k}_1)^2}{2m} + i\epsilon} \frac{1}{p_1^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2}{2m} + 2i\epsilon} \psi_e(p_1) u_N(v_N) \\
&= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{p_1^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k}_1)^2}{2m} + 2i\epsilon} \frac{1}{p_1^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2)^2}{2m} + 2i\epsilon} \psi_e(p_1) u_N(v_N)
\end{aligned}$$

do the shift as above

$$= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{p_1^0 - m - \frac{|\mathbf{k}_1|^2}{2m} + 2i\epsilon} \frac{1}{p_1^0 - m - \frac{|\mathbf{k}_2|^2}{2m} + 2i\epsilon} \psi_e(p_1) u_N(v_N)$$

drop \mathbf{p}_1

$$= -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{-\frac{|\mathbf{k}_1|^2}{2m} + 2i\epsilon} \frac{1}{-\frac{|\mathbf{k}_2|^2}{2m} + 2i\epsilon} \psi_e(p_1) u_N(v_N)$$

2 HSET

2.1 Lagrangian

For scalar QED

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi$$

where

$$D_\mu = \partial_\mu + ie A_\mu$$

In Schwartz's QFT (Chap. 35) he mentioned a choice of χ_v and $\tilde{\chi}_v$:

$$\phi(x) = e^{-imv \cdot x} \frac{1}{\sqrt{2m}} (\chi_v(x) + \tilde{\chi}_v(x)) \quad (3)$$

$$\chi_v(x) = e^{imv \cdot x} \frac{1}{\sqrt{2m}} (iv \cdot D + m) \phi(x), \quad \tilde{\chi}_v(x) = e^{imv \cdot x} \frac{1}{\sqrt{2m}} (-iv \cdot D + m) \phi(x) \quad (4)$$

Put (3) into (4), a simple relation is derived:

$$(-iv \cdot D) \chi_v(x) = (2m + iv \cdot D) \tilde{\chi}_v(x)$$

It can also be written as

$$2m \tilde{\chi}_v = (-iv \cdot D) (\chi_v + \tilde{\chi}_v)$$

⁵Clearly in this line, if this NRQCD diagram is crossed, the second pole would become $-k_2^0 + i\epsilon$ and the whole formula is zero (since both poles of k_1^0 would be in the same side).

Use this result

$$\begin{aligned}\mathcal{L} &= \frac{1}{2m} \left\{ [D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger + imv^\mu(\chi_v + \tilde{\chi}_v)^\dagger \right\} \{ [D_\mu(\chi_v + \tilde{\chi}_v)] - imv_\mu(\chi_v + \tilde{\chi}_v) \} - m^2(\chi_v + \tilde{\chi}_v)^\dagger(\chi_v + \tilde{\chi}_v) \\ &= (\chi_v + \tilde{\chi}_v)^\dagger (iv \cdot D)(\chi_v + \tilde{\chi}_v) + \frac{1}{2m} [D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger D_\mu(\chi_v + \tilde{\chi}_v) \end{aligned} \quad (5)$$

$$= (\chi_v(x) + \tilde{\chi}_v(x))^\dagger (iv \cdot D)(\chi_v(x) + \tilde{\chi}_v(x)) + \mathcal{O}\left(\frac{1}{m}\right) \quad (6)$$

(note that $D_\mu \phi = e^{-imv \cdot x} [D_\mu(\chi_v + \tilde{\chi}_v) - imv_\mu(\chi_v + \tilde{\chi}_v)]$ and $-imv^\mu [D_\mu(\chi_v + \tilde{\chi}_v)]^\dagger (\chi_v + \tilde{\chi}_v) = imv^\mu (\chi_v + \tilde{\chi}_v)^\dagger D_\mu(\chi_v + \tilde{\chi}_v) - \text{total derivative term}$)

Use the leading order of (5)

$$\begin{aligned}\mathcal{L}^{(0)} &= (\chi_v + \tilde{\chi}_v)^\dagger (iv \cdot D)(\chi_v + \tilde{\chi}_v) \\ &= \chi_v^\dagger iv \cdot D \chi_v + \tilde{\chi}_v^\dagger iv \cdot D(\chi_v + \tilde{\chi}_v) + \chi_v^\dagger iv \cdot D \tilde{\chi}_v \\ &= \chi_v^\dagger iv \cdot D \chi_v - 2m \tilde{\chi}_v^\dagger \tilde{\chi}_v + (iv \cdot D \chi_v)^\dagger \tilde{\chi}_v \\ &= \chi_v^\dagger iv \cdot D \chi_v - 2m \tilde{\chi}_v^\dagger \tilde{\chi}_v + [(-2m - iv \cdot D) \tilde{\chi}_v]^\dagger \tilde{\chi}_v \\ &= \chi_v^\dagger iv \cdot D \chi_v - \tilde{\chi}_v^\dagger (iv \cdot D + 4m) \tilde{\chi}_v\end{aligned}$$

We can have the final form⁶

$$\mathcal{L} = \chi_v^\dagger iv \cdot D \chi_v - \tilde{\chi}_v^\dagger (iv \cdot D + 4m) \tilde{\chi}_v + \mathcal{O}\left(\frac{1}{m}\right)$$

2.2 Quantization

2.2.1 HQET as an example

The leading term of HQET Lagrangian is

$$\mathcal{L} = \bar{Q}_v (iv \cdot D) Q_v$$

$$Q_v(x) = e^{imv \cdot x} \frac{1 + \not{v}}{2} \psi(x)$$

In free Dirac fermion theory we know that

$$\begin{aligned}\left\{ \psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y}) \right\} &= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_{ab} \\ \{a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger\} &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')\end{aligned}$$

also the plane wave expansion of ψ is

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}} u(p) e^{-ip \cdot x} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2mv^0}} a_v \sqrt{m} u(v) e^{-imv \cdot x - ik \cdot x}\end{aligned}$$

using normalization of states $u(k) = \sqrt{m} u(v)$ ⁷, $\langle p' | p \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\mathbf{p}' - \mathbf{p})$ and $\langle v', k' | v, k \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta^{(3)}(\mathbf{k}' - \mathbf{k})$ we have $|p\rangle = \sqrt{m} |v\rangle$ ($|p\rangle = \sqrt{2E_p} a_{\mathbf{p}}^\dagger |0\rangle$ while $|v, k\rangle = \sqrt{2v^0} a_{v, \mathbf{k}}^\dagger |0\rangle$)

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v u(v) e^{-imv \cdot x - ik \cdot x}$$

⁶With one problem: if we can tolerate coupled particle-anti particle pair, we can trade $iv \cdot D$ for mass term, so the leading part is the same but the anti-particle part could be different with the mixing?

Using the definition of $Q_v(x)$

$$\begin{aligned} Q_v(x) &= e^{imv \cdot x} \frac{1 + \not{v}}{2} \psi(x) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v \frac{1 + \not{v}}{2} u(v) e^{-ik \cdot x} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v u(v) e^{-ik \cdot x} \end{aligned}$$

The commutation relation should be

$$\{Q_{va}(\mathbf{x}), Q_{v'b}(\mathbf{x}')\} = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{4v^0 v'^0}} \{a_v, a_{v'}^\dagger\} u_a(v) u_b^\dagger(v') e^{-ik \cdot x + ik' \cdot x'}$$

using $\sum_s u_a(v) u_b^\dagger(v) = \frac{1}{m} \sum_s u_a(p) u_b^\dagger(p) = [(\not{v} + 1)\gamma^0]_{ab}$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{4v^0 v'^0}} \{a_v, a_{v'}^\dagger\} [(\not{v} + 1)\gamma^0]_{ab} e^{-ik \cdot x + ik' \cdot x'}$$

assuming $\{a_v, a_{v'}\} = (2\pi)^3 \delta_{vv'} \delta^{(3)}(\mathbf{k} - \mathbf{k}')$

$$\begin{aligned} &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2v^0} [(\not{v} + 1)\gamma^0]_{ab} e^{-ik \cdot (x - x')} \delta_{vv'} \\ &= [\frac{(\not{v} + 1)\gamma^0}{2v^0}]_{ab} \delta_{vv'} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \end{aligned}$$

2.2.2 HSET

The anti-particle field is decoupled so we don't have to consider that for now. The equation-of-motion is

$$\begin{cases} v \cdot D \chi_v^\dagger = 0 \\ v \cdot D \chi_v = 0 \end{cases}$$

By definition

$$\begin{aligned} \chi_v(x) &= \frac{e^{imv \cdot x}}{\sqrt{2m}} (iv \cdot D + m) \phi(x) \\ &= \frac{1}{\sqrt{2m}} (iv \cdot D + 2m) e^{imv \cdot x} \phi(x) \end{aligned}$$

Obviously the plane wave expansion should be irrelevant of the heavy particle mass, which means the exponential part is $e^{-ik \cdot x}$ where k marks the offshellness.

⁷The relation $\bar{u}^s(p) \gamma^\mu u^s(p) = 2p^\mu$ can be derived using Gordon identity, same for $\bar{u}^s(v) \gamma^\mu u^s(v) = 2v^\mu$, but it's actually $\bar{u}u$.