Scalar QED

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1 Hydrogen Wavefunction Divergence in Klein-Gordon Equation and Schrödinger Equation

1.1 The Klein-Gordon part

The Klein-Gordon Hydrogen Equation is

$$((i\partial_0 + \frac{Z\alpha}{r})^2 + \nabla^2 - m^2)\Psi = 0 \tag{1}$$

For the bound state, the eigen value and the wave function are

$$E = m\frac{1}{\sqrt{1 + \frac{\alpha^2 Z^2}{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}}} \tag{2}$$

$$\Psi = \frac{c}{\sqrt{4\pi}} e^{-kr} r^{\lambda} \tag{3}$$

where

$$\lambda = -\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2\alpha^2} \qquad c = \sqrt{\frac{(2k)^{2(1+\sqrt{\frac{1}{4}-Z^2\alpha^2})}}{\Gamma(2+2\sqrt{\frac{1}{4}-Z^2\alpha^2}))}} \qquad k = \frac{m}{\sqrt{1 + \frac{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}{\alpha^2 Z^2}}} \tag{4}$$

c is the normaliztion factor for $\int d^3r|\Psi|^2 = 1$. For convenience, define

$$\Psi' = \frac{\Psi}{2(mZ\alpha)^{\frac{3}{2}}} \tag{5}$$

Now Ψ' is dimensionless and expand it in α, we get the origin divergence comes from a term

$$-(Z\alpha)^2 log(mr) \tag{6}$$

the m in log could be interpreted as a subtraction point μ.

1.2 The schrodinger part

The Hamiltonian is

$$H = H_0 + H_{int} \tag{7}$$
$$H_0 = -\frac{\nabla^2}{2m} - \frac{Z\alpha}{r}, \quad H_{int} = \frac{\nabla^4}{8m^3} + \frac{1}{32m^4}[-\nabla^2, [-\nabla^2, -\frac{Z\alpha}{r}]] \tag{8}$$

The first term of H_{int} is the relativistic kinematic v^2 correction, the second one is the Darwin term.
The H_0 gives the radial wave functions as follows

$$R_{n0} = \frac{2(mZ\alpha)^{\frac{3}{2}}}{n^{\frac{3}{2}}} e^{-\frac{mZ\alpha}{n}r} F(1-n, 2, \frac{2mZ\alpha r}{n}), \quad E_n = -\frac{Z^2\alpha^2 m}{2n^2} \tag{9}$$
$$R_{k0} = \sqrt{\frac{2}{\pi}}(mZ\alpha)^{\frac{3}{2}} k e^{\frac{\pi}{2k}} |\Gamma(1-\frac{i}{k})| e^{-imz\alpha kr} F(1+\frac{i}{k}, 2, 2imZ\alpha kr), \quad E_k = \frac{mZ^2\alpha^2 k^2}{2} \tag{10}$$

Within perturbation theory,$E_1^{(1)} =< \phi|H_{int}|\phi >$, in quantum mechanics, the NLO energy correction is

$$E_1^{(1)} = E_1 Z^2 \alpha^2 \tag{11}$$

The NLO corretion of the bound state wave function is

$$\sum_{n\neq 1} a_{n1}\phi_{n00} + \int dk a_{k1}\phi_{k00} \tag{12}$$

with

$$a_{n1} = \frac{< \phi_{n00}|H_{int}|\phi_{100} >}{E_1 - E_n} \tag{13}$$

the discrete part of (12) is not divergent at $r = 0$. We now focus on the integration part and seperate the relativstic kinematic term and the Darwin term. Since we are only interested in the divergent part, here we give a hard cutoff $\frac{\Lambda}{m}$ as the up-limit of the integration and a also a down-limit λ, with $\lambda >> 1$ (note that the following wave function have been multiplied by $2(mZ\alpha)^{\frac{3}{2}}$)

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk \frac{2Z^2\alpha^2 k^{\frac{3}{2}}}{2\pi(\sqrt{1-exp(-\frac{2\pi}{k})})} (1 - \frac{2}{1+k^2} exp(-\frac{2arctan(k)}{k})) e^{\frac{\pi}{2k}} |\Gamma(1-\frac{i}{k})| \tag{14}$$

with the integral region we defined($k >> 1$), it would be OK to expand the integrand in $\frac{1}{k}$ (I havn't prove it yet), then the UV divergent term is

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk (Z\alpha)^2 (\frac{1}{\pi} + \frac{1}{k}) \tag{15}$$
$$\sim (\alpha Z)^2 (\frac{\Lambda}{\pi m} + log(\frac{\Lambda}{m})) \tag{16}$$

The UV divergent part of Darwin term is

$$\Phi^{(1)}(0)_D = -\frac{(Z\alpha)^4}{8\pi}\int_{\lambda}^{\frac{\Lambda}{m}} dkk^2 e^{\frac{\pi}{k}}|\Gamma(1-\frac{i}{k})|^2 \tag{17}$$

with the same trick as (15), the UV divergen part is

$$\Phi^{(1)}(0)_D = -(\alpha Z)^4\int_{\lambda}^{\frac{\Lambda}{m}} dk\frac{k^2}{8\pi}+\frac{k}{8}+\frac{1}{24}\pi \tag{18}$$

$$\sim -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3}+\frac{\pi\Lambda^2}{2m^2}+\frac{\pi^2\Lambda}{3m}) \tag{19}$$

Now collect all the results we get as follow.
The K-G wave function's origin UV divergence is

$$K-G \;\; UV: -(Z\alpha)^2 log(mr) \tag{20}$$

The purterbative Schrodinger wave function's origin UV divergence, with a k cutoff $\frac{\Lambda}{m}$, is

$$Kin \;\; UV: (\alpha Z)^2(\frac{\Lambda}{\pi m}+log(\frac{\Lambda}{m})) \tag{21}$$

$$Darwin \;\; UV: -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3}+\frac{\pi\Lambda^2}{2m^2}+\frac{\pi^2\Lambda}{3m}) \tag{22}$$

All the m, under Λ or in a log, can be interpreted as a subtraction point μ.

2 Non-relativistic Scalar QED (NRSQED) Matching

2.1 Feynman Rules

2.1.1 Scalar QED (SQED)

Lagrangian

$$\mathcal{L}_{SQED} = |D_\mu\phi|^2 - m^2|\phi|^2 + \Phi_v^* iv\cdot D\Phi_v \tag{23}$$

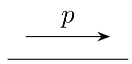
with

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$$

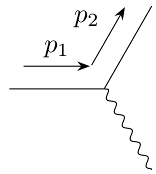
and

$$D_\mu\Phi_v = \partial_\mu\Phi - iZeA_\mu\Phi_v$$

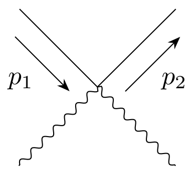
But note that no **A** can appear in actual calculation because here only static scalar potential exists. And the Feynman rules



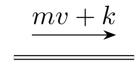
$= \dfrac{i}{p^2 - m^2 + i\epsilon}$



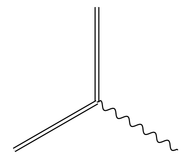
$= -ie(p_1^\mu + p_2^\mu)$



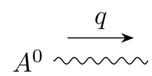
$= 2ie^2 g^{\mu\nu}$



$= \dfrac{i}{v \cdot k}$



$= iZev^\mu$



$= \dfrac{i}{\mathbf{q}^2}$

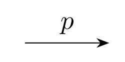
2.1.2 NRSQED

Using the transformation $\phi \to \dfrac{e^{-imt}}{\sqrt{2m}}\varphi$, we can have the Lagrangian

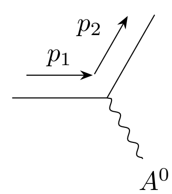
$$\mathcal{L}_{NRSQED} = \varphi^* \left(iD_0 + \frac{\mathbf{D}^2}{2m} \right) \varphi + \delta\mathcal{L} + \Phi_v^* iv \cdot D\Phi_v \tag{24}$$

with the same notation above. Here $\mathbf{D} = \nabla - ie\mathbf{A}$.

Feynman rules are also the same except for the scalar electron side which becomes



$= \dfrac{i}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon}$



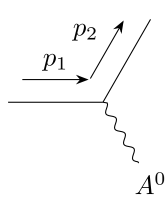
$= -ie$

We can ignore all interacting terms involving $\mathbf{A}$.

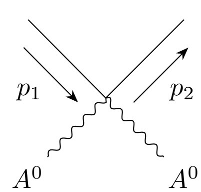
Since we need to match it to $\mathcal{O}(v^2)$ order

$$\delta\mathcal{L} = \frac{(D_0\varphi)^*(D_0\varphi)}{2m} = \frac{\dot{\varphi}^*\dot{\varphi}}{2m} + \frac{e^2\varphi^*\varphi A_0^2}{2m} - \frac{ie}{2m}A_0(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) \tag{25}$$

and it changes the Feynman rules to[1]



$= -ie(1 + \dfrac{E_1 + E_2}{2m})$



$= \dfrac{ie^2}{2m}$

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Since we rescaled ϕ by $\frac{1}{\sqrt{2m}}$ to get φ, the in/out states are also changed. We must multiply them by $\sqrt{2E}$ to compensate that change.

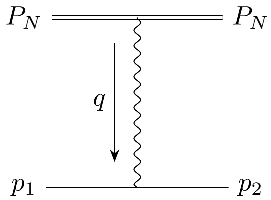
Another way to achieve it is to use the transform rules of heavy scalar effective theory (HSET).

[1] In this note, p^0 is the zero component of relativistic four momentum, and $E = p^0 - m$.

2.2 LO

2.2.1 SQED

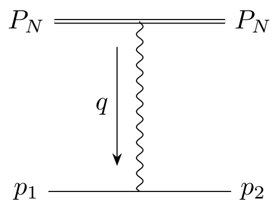
$$i\mathcal{M}^{(0)}_{SQED} =$$



$$= -e^2 v^0 \frac{i(p_1^0 + p_2^0)}{\mathbf{q}^2} = -e^2 v^0 \frac{i}{\mathbf{q}^2}(2m + E_1 + E_2)$$

2.2.2 NRSQED

$$i\mathcal{M}^{(0)}_{NRSQED} =$$

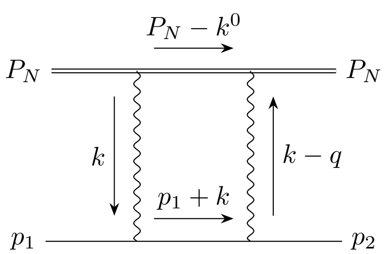


$$= -2\sqrt{E_1 E_2} e^2 v^0 \frac{i(1 + \frac{E_1+E_2}{2m})}{\mathbf{q}^2}$$

2.3 NLO

2.3.1 SQED

$$i\mathcal{M}^{(1)}_{SQED} =$$



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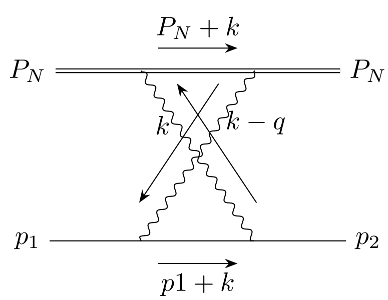

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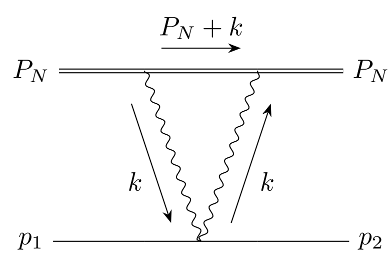

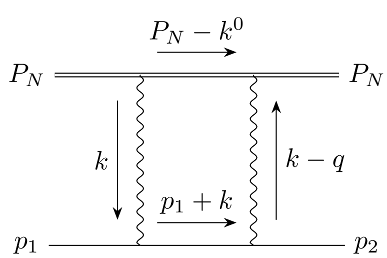

$$= -e^2 v^0 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{\mathbf{k}^2} \; < +content+ >$$

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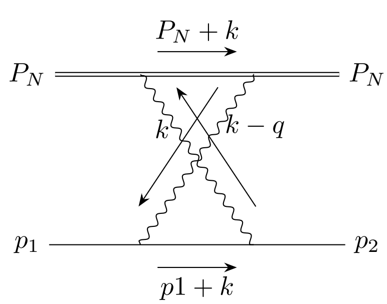

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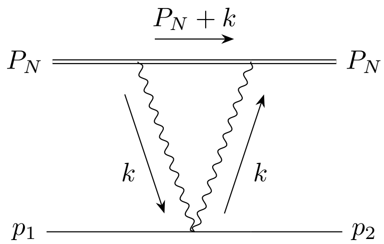

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2.3.2 NRSQED

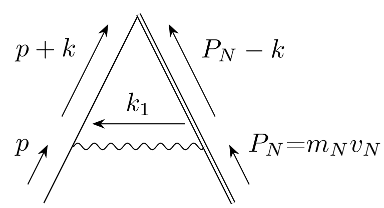
3 Local Operator and Matrix Element of NRSQED

To reproduce the singular behavior of "Klein-Gordon Hydrogen" wavefunction near origin, we can try OPE. But the dependence of x in OPE can be taken as a regularization scheme and thus the result should be the same as local one without renormalization. And the logarithmic terms of x in OPE can be reproduced by the logarithmic divergence of local operators. Since in the study of Klein-Gordon equation we know that the wavefunction only contains logarithmic divergence at the origin so that's the only type of divergence we're looking for.

3.1 LO

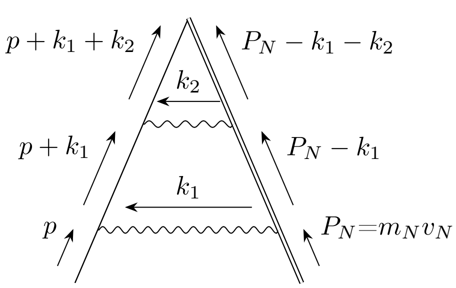
3.2 NLO

$$\langle 0|\psi_e(0)N(0)(-ie\mu^{-\epsilon})\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0(-ie\mu^{-\epsilon})\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle =$$



which doesn't have logarithm divergence[2].

3.3 NNLO

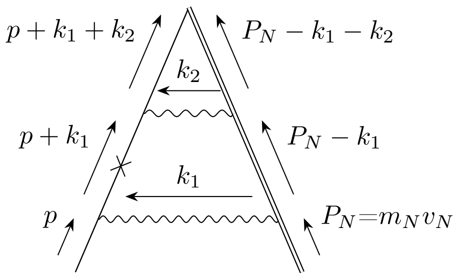


$$= -\mu^{-4\epsilon}e^4\left[\int [dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{2m+2E+k_1^0}{p^0+k_1^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{2m+2E+2k_1^0+k_2^0}{p^0+k_1^0+k_2^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\right]$$

[2] After dimensional regularization, the Gamma function in the numerator is something like $\Gamma(n-d/2)$ and Gamma function doesn't have pole at half integer. We can rigorously prove this kind of diagrams do not have logarithmic divergence at one loop.

do the shift as above

$$=e^4\left[\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{2m+2E}{E-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{2m+2E}{E-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]$$



$$= 16m^2(m+E)\mu^{-4\epsilon}e^4\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{1}{|\mathbf{k_2}|^2-2mE}$$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{\left(\frac{4\pi}{\Delta_2}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2\Gamma(2)}$$

where $\Delta_2=(1-x)\left(|\mathbf{k_1}|^2x-2Em\right)$

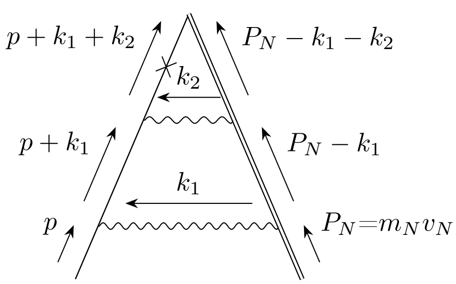
$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{(4\pi)^2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{1}{(|\mathbf{k_1}|^2-2mE/x)^{2-d/2}}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}\Gamma(2-d/2)$$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\mathrm{d}z\mathrm{d}t\delta(y+z+t-1)\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{zt^{1-d/2}|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2+\Delta_1]^{5-d/2}}\frac{\Gamma(5-d/2)}{\Gamma(2-d/2)}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}\Gamma(2-d/2)$$

where $\Delta_1=y(1-y)\mathbf{p}^2-2mE(z+t/x)$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2(4\pi)^2}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\mathrm{d}z\mathrm{d}t\delta(y+z+t-1)zt^{1-d/2}\frac{d(d+2)}{4}\frac{\Gamma\left(3-d\right)}{(4\pi)^{3-d/2}}\left(\frac{4\pi}{\Delta_1}\right)^{3-d}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}$$

$$=-16m^2(m+E)\frac{1}{128\pi^2m^2}(\frac{1}{d-3}+4\log\mu)+\text{finite terms}$$



$$= 16m^2(m+E)\mu^{-4\epsilon}e^4\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_1}|^2-2mE}\frac{|\mathbf{k_2}|^4/4m^2}{[|\mathbf{k_2}|^2-2mE]^2}$$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_1}|^2-2mE}\frac{(1-x)\Gamma(1-d/2)}{8\pi}\left(\frac{4\pi}{\Delta_2}\right)^{1-d/2}\frac{d(d+2)}{4}$$

where $\Delta_2=x(1-x)|\mathbf{k_1}|^2-2mE(1-x)$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\mathrm{d}z\mathrm{d}t\frac{t^{-d/2}}{[|\mathbf{k_1}|^2+\Delta_1]^{3-d/2}}\frac{\Gamma(3-d/2)}{\Gamma(1-d/2)}\delta(y+z+t-1)\frac{\Gamma(1-d/2)}{8\pi}\left(\frac{4\pi}{x(1-x)}\right)^{1-d/2}\frac{d(d+2)x}{4}$$

where $\Delta_1=y(1-y)\mathbf{p}^2-2mEz-2mE\frac{t}{x}$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\mathrm{d}z\mathrm{d}t\delta(y+z+t-1)\frac{1}{(4\pi)^{3-d/2}}\left(\frac{4\pi}{\Delta_1}\right)^{3-d}\frac{\Gamma\left(3-d\right)}{8\pi}\left(\frac{4\pi}{x(1-x)}\right)^{1-d/2}\frac{d(d+2)x}{4}t^{-d/2}$$

$$=16m^2(m+E)\frac{15}{8192\pi^2m^2}(\frac{1}{d-3}+4\log\mu)+\text{finite terms}$$

Appendices

Feynman parametrization used here is

$$\frac{1}{\prod A_i^{d_i}} = \int \prod \mathrm{d}x_i \delta(\sum x_i - 1) \frac{\prod x_i^{d_i-1}}{[\sum x_i A_i]^{\sum d_i}} \frac{\Gamma(\sum d_i)}{\prod \Gamma(d_i)} \tag{26}$$

Integral with the structure of the form

$$\int [dk_1][dk_2] \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{[p^0 + k_1^0 - m - \frac{(\mathbf{p}+\mathbf{k_1})^2}{2m} + i\epsilon]^m} \frac{1}{[p^0 + k_1^0 + k_2^0 - m - \frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m} + i\epsilon]^n}$$

will always produce

$$\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{[p^0 - m - \frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}]^m} \frac{1}{[p^0 - m - \frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m} + i\epsilon]^n}$$

with k_1^0 and k_2^0 goes to zero.

For arbitary one loop diagram of the following form, we have

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{2\beta}}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{n-\beta}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n - \beta - d/2)}{\Gamma(n)} \left(\frac{4\pi}{\Delta}\right)^{n-\beta-d/2} \tag{27a}$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n - \beta - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\beta-d/2} \tag{27b}$$

For two loop diagrams of this form ($\epsilon = 3 - d$)

$$\mu^{-4\epsilon} \int \frac{\mathrm{d}^d\mathbf{k}_1}{(2\pi)^d} \frac{\mathrm{d}^d\mathbf{k}_2}{(2\pi)^d} \frac{1}{(\mathbf{k_1} - \mathbf{a})^2} \frac{1}{(\mathbf{k_2} - \mathbf{k_1})^2} \frac{\mathbf{k}_1^{2\alpha}}{(\mathbf{k}_1^2 - c)^m} \frac{\mathbf{k}_2^{2\beta}}{(\mathbf{k}_2^2 - d)^n} \tag{28}$$

The integral is evaluated to

$$\begin{aligned}
&\mu^{-4\epsilon} \int_0^1 \prod_{i=1}^{2} \mathrm{d}x_i \delta(\sum x_i - 1) \prod x_i^{d_i-1} \frac{\Gamma(n+1)}{\Gamma(n)} \frac{1}{(4\pi)^{n+1-\beta}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n + 1 - \beta - d/2)}{\Gamma(n+1)} \left(\frac{4\pi}{\alpha(x_i)}\right)^{n+1-\beta-d/2} \\
&\int \frac{\mathrm{d}^d\mathbf{k}_1}{(2\pi)^d} \frac{1}{(\mathbf{k_1} - \mathbf{a})^2} \frac{\mathbf{k}_1^{2\alpha}}{(\mathbf{k}_1^2 - c)^m} \frac{1}{(\mathbf{k}_1 - \Delta_2)^{n+1-\beta-d/2}} \\
=&\mu^{-4\epsilon} \int_0^1 \prod_{i=1}^{2} \mathrm{d}x_i \delta(\sum x_i - 1) \prod x_i^{d_i-1} \frac{1}{(4\pi)^{n+1-\beta}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n + 1 - \beta - d/2)}{\Gamma(n)} \left(\frac{4\pi}{\alpha(x_i)}\right)^{n+1-\beta-d/2} \int_0^1 \prod_{i=1}^{3} \mathrm{d}y_j \delta(\sum y_j - 1) \\
&\prod y_j^{d_j-1} \frac{\Gamma(m + n + 2 - \beta - d/2)}{\Gamma(m)\Gamma(n + 1 - \beta - d/2)} \frac{1}{(4\pi)^{m+n+2-\alpha-\beta-d/2}} \frac{\Gamma(\alpha + d/2)}{\Gamma(d/2)} \frac{\Gamma(m + n + 2 - \alpha - \beta - d)}{\Gamma(m + n + 2 - \beta - d/2)} \left(\frac{4\pi}{\Delta_1}\right)^{m+n+2-\alpha-\beta-d} \\
=&\mu^{-4\epsilon} \int_0^1 \prod_{i=1}^{2} \mathrm{d}x_i \delta(\sum x_i - 1) \prod x_i^{d_i-1} \frac{1}{(4\pi)^d} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{1}{\Gamma(n)} \left(\frac{1}{\alpha(x_i)}\right)^{n+1-\beta-d/2} \\
&\int_0^1 \prod_{i=1}^{3} \mathrm{d}y_j \delta(\sum y_j - 1) \prod y_j^{d_j-1} \frac{\Gamma(\alpha + d/2)}{\Gamma(d/2)} \frac{\Gamma(m + n + 2 - \alpha - \beta - d)}{\Gamma(m)} \left(\frac{1}{\Delta_1}\right)^{m+n+2-\alpha-\beta-d}
\end{aligned}$$

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