Homework: Gauge Field Theory

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1. Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda^2}{4}\phi^4$$

which satisfies

$$\phi \to -\phi$$

For such symmetry to break, we perform the following presedure:

First, the minimum of $V(\phi)$ can be found in $\phi = \pm\frac{\mu^2}{\lambda^2}$, and we can define $v^2 = |\langle 0|\phi|0\rangle|^2 = \frac{\mu^2}{\lambda}$, which yields the broken symmetry of vacuum.

By redefining the field $\phi(x) = \rho(x) + v$ such that $\rho(x)$ has the right vacuum, the Lagrangian is now

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 - \mu^2\rho^2 - \lambda^2\rho^3 v - \frac{\lambda^2}{4}\rho^4 + \frac{\mu^4}{4\lambda^2}$$

and we can see that there is no massless Goldstone particle. That's because that although the symmetry $\phi \to -\phi$ has broken, but it's discrete symmetry, therefore can't produce Goldstone particles.

2. R$_\xi$ Gauge. The Lagrangian is

$$\mathcal{L}(\phi, A^\mu) = (D^\mu\phi)^\dagger(D_\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where $D^\mu = \partial^\mu + igA^\mu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Also we have $|\langle 0|\phi|0\rangle| = v$, $v^2 = \frac{\mu^2}{\lambda}$.

R_ξ gauge

$$\mathcal{L} \to \mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu - \xi g v b)^2$$

Choose ϕ to be $\phi = \frac{1}{\sqrt{2}}(v + h(x) + ib(x))$,

$$D^\mu\phi = \frac{1}{\sqrt{2}}[\partial^\mu h + i\partial^\mu b + igA^\mu(v+h) - gbA^\mu] = \frac{1}{\sqrt{2}}[(\partial^\mu h - gbA^\mu) + i(\partial^\mu b + g(v+h)A^\mu)]$$

so the kinetic term

$$(D^\mu\phi)^\dagger(D_\mu\phi) = \frac{1}{2}[(\partial^\mu h - gbA^\mu)^2 + (\partial^\mu b + g(v+h)A^\mu)^2]$$

this gives

$$\begin{aligned}(D^\mu\phi)^\dagger(D_\mu\phi) &= \frac{1}{2}\partial^\mu h\partial_\mu h - gb\partial^\mu hA_\mu + \frac{1}{2}g^2b^2A^\mu A_\mu + \frac{1}{2}\partial^\mu b\partial_\mu b + g(v+h)\partial^\mu bA_\mu + \frac{1}{2}g^2(v+h)^2A^\mu A_\mu \\ &= \frac{1}{2}\partial^\mu h\partial_\mu h + \frac{1}{2}\partial^\mu b\partial_\mu b + \frac{1}{2}g^2v^2A^\mu A_\mu + gv\partial^\mu bA_\mu + g^2vhA^\mu A_\mu + \frac{1}{2}g^2(b^2+h^2)A^\mu A_\mu + g(h\partial^\mu b - b\partial^\mu h)A_\mu\end{aligned}$$

now we got the kinetic terms of scalar fields $h(x)$ and $b(x)$, mass term for gauge field A^μ, crossing term of b and A^μ, and some interacting terms in the end.

The mass term of original scala field gives

$$\mu^2\phi^\dagger\phi = \frac{1}{2}\mu^2(v+h)^2 - \frac{1}{2}\mu^2b^2$$

so the rest part of scalar field is

$$-\frac{b^4\lambda}{4}-\frac{1}{2}b^2h^2\lambda-b^2h\lambda v-\frac{h^4\lambda}{4}-h^3\lambda v-h^2\mu^2+\frac{\mu^4}{4\lambda}$$

Now the gauge fixing term is

$$-\frac{1}{2\xi}\partial^\mu A_\mu\partial^\nu A_\nu+gvb\partial_\mu A^\mu-\frac{\xi g^2v^2}{2}b^2$$

we know that $F^{\mu\nu}F_{\mu\nu}$ can always be written in two terms, so

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}=-\frac{1}{2}(\partial^\mu A_\nu)^2+\frac{1}{2}(1-\xi^{-1})(\partial^\mu A_\mu)^2+gvb\partial^\mu A_\mu-\frac{\xi g^2v^2}{2}b^2$$

and

$$gvb\partial^\mu A_\mu=-gvA_\mu\partial^\mu b$$

the crossing term is cancelled. The last term also gives b field mass $\frac{\xi g^2v^2}{2}$.

The Lagrangian is now

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}\partial^\mu h\partial_\mu h+\frac{1}{2}\partial^\mu b\partial_\mu b-\frac{1}{2}(\partial^\mu A_\nu)^2+\frac{1}{2}(1-\xi^{-1})(\partial^\mu A_\mu)^2+\frac{1}{2}g^2v^2A^\mu A_\mu-\mu^2h^2-\frac{\xi g^2v^2}{2}b^2(+\frac{\mu^4}{4\lambda})\\&+g^2vhA^\mu A_\mu+\frac{1}{2}g^2(b^2+h^2)A^\mu A_\mu+g(h\partial^\mu b-b\partial^\mu h)A_\mu-\frac{b^4\lambda}{4}-b^2h\lambda v-\frac{h^4\lambda}{4}-h^3\lambda v\end{aligned}$$

Then we have some standard 3 and 4 particle vertexs. Now we just need to deal with the propagators and the vertex with derivative.

The propagators of both scalar fields are trival, with $m_h=\sqrt{2}\mu$, $m_b=\sqrt{\xi}gv$. The propagator of the vector field is, however, a bit more complicated.

$$\Delta_A^{\mu\nu}(x-y)=\frac{g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}}{k^2-m^2+i\epsilon}+\frac{\xi\frac{k^\mu k^\nu}{k^2}}{k^2-\xi m^2+i\epsilon}$$

where the mass of vector field $m=gv$.

Now we'll show how to derive the propagator: Define $\mathcal{L}_0$

$$\mathcal{L}_0=-\frac{1}{2}\partial_\mu A^\nu\partial^\mu A_\nu+\frac{1}{2}(1-\xi^{-1})\partial^\nu A_\mu\partial^\mu A_\nu+\frac{1}{2}m^2A^\nu A_\nu$$

and

$$S_0=\int\mathrm{d}^4x\mathcal{L}_0$$

Transform to momentum space

$$S_0=-\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Big\{\tilde{A}_\mu(k)\big(g^{\mu\nu}k^2-(1-\xi^{-1})k^\mu k^\nu-m^2g^{\mu\nu}\big)\tilde{A}_\nu(-k)-\tilde{J}^\mu(k)\tilde{A}_\mu(-k)-\tilde{J}^\mu(-k)\tilde{A}_\mu(k)\Big\}$$

Define $\tilde{D}^{\mu\nu}(k)=g^{\mu\nu}k^2-(1-\xi^{-1})k^\mu k^\nu-m^2g^{\mu\nu}$

$$\begin{aligned}\tilde{D}^{\mu\nu}(k)&=g^{\mu\nu}k^2-(1-\xi^{-1})k^\mu k^\nu-m^2g^{\mu\nu}\\&=(k^2-m^2)g^{\mu\nu}-(1-\xi^{-1})k^\mu k^\nu\\&=(k^2-m^2)(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2})+(k^2-m^2)\frac{k^\mu k^\nu}{k^2}-(1-\xi^{-1})k^\mu k^\nu\\&=(k^2-m^2)(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2})+\xi^{-1}(k^2-\xi m^2)\frac{k^\mu k^\nu}{k^2}\end{aligned}$$

then to have the result

$$S_0=-\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{J}_\mu(k)\tilde{\Delta}_F^{\mu\nu}(k)\tilde{J}_\nu(-k)$$

we must have

$$\tilde{D}_{\mu\nu}\tilde{\Delta}_F^{\nu\rho}=\delta_\mu^\rho$$

that is

$$
\begin{aligned}
&\tilde{D}_{\mu\nu}(k)\tilde{\Delta}_F^{\nu\rho}(k) = \delta_\mu^\rho \\
=&\left\{(k^2 - m^2)(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + \xi^{-1}(k^2 - \xi m^2)\frac{k_\mu k_\nu}{k^2}\right\}\{A g^{\nu\rho} + B k^\nu k^\rho\} \\
=&A(k^2 - m^2)\delta_\mu^\rho - A(k^2 - m^2)\frac{k_\mu k^\rho}{k^2} + \xi^{-1}(k^2 - \xi m^2)A k_\mu k^\rho + \xi^{-1}(k^2 - \xi m^2)B k_\mu k^\rho
\end{aligned}
$$

such that $A = \frac{1}{k^2 - m^2 + i\epsilon}$ and $B = \frac{\xi}{(k^2 - \xi m^2 + i\epsilon)k^2} - \frac{1}{k^2(k^2 - m^2 + i\epsilon)}$ (with the Feynman prescription). The propagator is now

$$\tilde{\Delta}_F^{\mu\nu}(k) = \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - m^2 + i\epsilon} + \frac{\xi k^\mu k^\nu / k^2}{k^2 - \xi m^2 + i\epsilon}$$

3. $Z^0 \to l\bar{l}$.

Write down the Lagrangian

$$\mathcal{L} = \mathcal{L}_W - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \left(\bar{\nu}_L, \bar{e}_L, \bar{e}_R\right) i\not{\partial} \begin{pmatrix} \nu_L \\ e_L \\ e_R \end{pmatrix} + \mathcal{L}_N + \mathcal{L}_W$$

$$\mathcal{L}_N = (\frac{gg'}{\sqrt{g^2 + g'^2}}\bar{e}_L\gamma^\mu e_L - \frac{gg'}{\sqrt{g^2 + g'^2}}Y_R\bar{e}_R\gamma^\mu e_R)A_\mu + (-\frac{g'^2}{\sqrt{g^2 + g'^2}}Y_R\bar{e}_R\gamma^\mu e_R + \frac{g'^2 - g^2}{2\sqrt{g^2 + g'^2}}\bar{e}_L\gamma^\mu e_L)Z_\mu - \frac{\sqrt{g^2 + g'^2}}{2}\bar{\nu}_L\gamma^\mu\nu_L Z_\mu$$

The interaction term of Z boson and leptons is

$$\mathcal{L}_Z = (-\frac{g'^2}{\sqrt{g^2 + g'^2}}Y_R\bar{e}_R\gamma^\mu e_R + \frac{g'^2 - g^2}{2\sqrt{g^2 + g'^2}}\bar{e}_L\gamma^\mu e_L)Z_\mu$$

Note that the outstate don't have any explicit handness, so we can rewrite it as

$$\mathcal{L}_Z = \bar{e}\gamma^\mu(\alpha + \beta\gamma^5)eZ_\mu$$

where $\alpha = \frac{g'^2(1 - 2Y_R) - g^2}{4\sqrt{g^2 + g'^2}}$, $\beta = \frac{-g'^2(2Y_R + 1) + g^2}{4\sqrt{g^2 + g'^2}}$. Take $Y_R = -1$ and $\frac{gg'}{\sqrt{g^2 + g'^2}} = e$, then

$$\alpha = -\frac{e}{c_W s_W}(\frac{1}{4} - s_W^2), \quad \beta = \frac{e}{4c_W s_W}$$

where $c_W = \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}$, $s_W = \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}$, so that

$$\mathcal{L}_Z = \bar{e}\gamma^\mu(\alpha + \beta\gamma^5)eZ_\mu$$

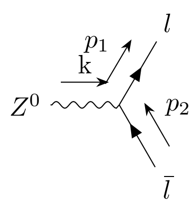
Now the Lagrangian in the full form is

$$\mathcal{L}_Z = -\frac{e}{c_W s_W}\bar{e}\gamma^\mu(\frac{1 - \gamma^5}{4} - s_W^2)eZ_\mu$$

And $m_Z = \frac{ev}{2s_W c_W}$.

Now the amplitude is

$i\mathcal{M} =$



$$= i\bar{u}^s\gamma^\mu(\alpha + \beta\gamma^5)v^r\epsilon_\mu^\lambda$$

And do the spin & polarization sum

$$\begin{aligned}
\frac{1}{3}\sum_{r,s,\lambda}|\mathcal{M}|^2 &= \frac{1}{3}\sum_{r,s,\lambda}[\bar{u}\gamma^\mu(\alpha+\beta\gamma^5)v\epsilon_\mu][\bar{v}\gamma^\nu(\alpha+\beta\gamma^5)u\epsilon_\nu^*] \\
&= \frac{1}{3}\sum_{r,s,\lambda}\epsilon_\mu\epsilon_\nu^*\,\mathrm{tr}\big\{u\bar{u}\gamma^\mu(\alpha+\beta\gamma^5)v\bar{v}\gamma^\nu(\alpha+\beta\gamma^5)\big\} \\
&= \frac{1}{3}(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_Z^2})\,\mathrm{tr}\Big\{(\not{p}_1+m)\gamma^\mu(\alpha+\beta\gamma^5)(\not{p}_2-m)\gamma^\nu(\alpha+\beta\gamma^5)\Big\}
\end{aligned}$$

Note that

$$\begin{aligned}
(\alpha+\beta\gamma^5)(\not{p}-m)\gamma^\nu(\alpha+\beta\gamma^5) &= [\alpha(\not{p}-m)-\beta(\not{p}+m)\gamma^5]\gamma^\nu(\alpha+\beta\gamma^5) \\
&= \alpha(\not{p}-m)\gamma^\nu(\alpha+\beta\gamma^5)+\beta(\not{p}+m)\gamma^\nu\gamma^5(\alpha+\beta\gamma^5) \\
&= \alpha(\not{p}-m)\gamma^\nu(\alpha+\beta\gamma^5)+\beta(\not{p}+m)\gamma^\nu(\beta+\alpha\gamma^5) \\
&= (\alpha^2+\beta^2)\not{p}\gamma^\nu+2\alpha\beta\not{p}\gamma^\nu\gamma^5-(\alpha^2-\beta^2)m\gamma^\nu \\
&= (\alpha'\not{p}-\beta' m)\gamma^\nu+2\alpha\beta\not{p}\gamma^\nu\gamma^5
\end{aligned}$$

where $\alpha'=\alpha^2+\beta^2$, $\beta'=\alpha^2-\beta^2$.

So the trace part becomes

$$\mathrm{tr}\Big\{(\not{p}_1+m)\gamma^\mu[(\alpha'\not{p}_2-\beta' m)\gamma^\nu+2\alpha\beta\not{p}_2\gamma^\nu\gamma^5]\Big\} = \mathrm{tr}\Big\{(\not{p}_1+m)\gamma^\mu(\alpha'\not{p}_2-\beta' m)\gamma^\nu+2\alpha\beta(\not{p}_1+m)\gamma^\mu\not{p}_2\gamma^\nu\gamma^5\Big\}$$

$$\mathrm{tr}\Big\{(\not{p}_1+m)\gamma^\mu(\alpha'\not{p}_2-\beta' m)\gamma^\nu\Big\} = 4[\alpha' p_1^\mu p_2^\nu+\alpha' p_1^\nu p_2^\mu-g^{\mu\nu}(\alpha' p_1\cdot p_2+\beta' m^2)]$$

$$\begin{aligned}
\mathrm{tr}\Big\{(\not{p}_1+m)\gamma^\mu\not{p}_2\gamma^\nu\gamma^5\Big\} &= \mathrm{tr}\Big\{\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\gamma^5\Big\} \\
&= -4i\epsilon^{\rho\mu\sigma\nu}p_{1\rho}p_{2\sigma}
\end{aligned}$$

and the latter term will vanish (anti symmetry multiplies symmetry).

$$\begin{aligned}
\frac{1}{3}\sum_{r,s,\lambda}|\mathcal{M}|^2 &= \frac{4}{3}(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_Z^2})[\alpha' p_1^\mu p_2^\nu+\alpha' p_1^\nu p_2^\mu-g^{\mu\nu}(\alpha' p_1\cdot p_2+\beta' m^2)] \\
&= -\frac{4}{3}[2\alpha' p_1\cdot p_2-4(\alpha' p_1\cdot p_2+\beta' m^2)]+\frac{4}{3}[\frac{2}{m_Z^2}\alpha'(p_1\cdot k)(p_2\cdot k)-(\alpha' p_1\cdot p_2+\beta' m^2)] \\
&= \frac{4}{3}[\frac{2}{m_Z^2}\alpha'(p_1\cdot k)(p_2\cdot k)-(\alpha' p_1\cdot p_2+\beta' m^2)+2\alpha' p_1\cdot p_2+4\beta' m^2] \\
&= \frac{4}{3}[\frac{2}{m_Z^2}\alpha'(p_1\cdot k)(p_2\cdot k)+\alpha' p_1\cdot p_2+3\beta' m^2] \\
&= \frac{4}{3}[\frac{2}{m_Z^2}\alpha'(m^2+p_1\cdot p_2)^2+\alpha' p_1\cdot p_2+3\beta' m^2]
\end{aligned}$$

Knowing that in centre-of-mass frame

$$p_1\cdot p_2 = E_1^2+\mathbf{p}_1^2 = 2E_1^2-m^2,\quad E_1=\frac{m_Z}{2}$$

$$\frac{1}{3}\sum_{r,s,\lambda}|\mathcal{M}|^2 = \frac{4}{3}[\alpha' m_Z^2-(\alpha'-3\beta')m^2]$$

The decay width is

$$
\begin{aligned}
\Gamma &= \frac{1}{2m_Z}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{|\mathcal{M}|^2}{4E_1E_2}(2\pi)^4\delta^4(k-p_1-p_2) \\
&= \frac{1}{2\pi m_Z}\int \mathrm{d}|\mathbf{p_1}||\mathbf{p_1}|^2\frac{|\mathcal{M}|^2}{4E_1^2}\delta(m_Z-2E_1) \\
&= \frac{1}{2\pi m_Z}\int \mathrm{d}E_1|\mathbf{p_1}|\frac{|\mathcal{M}|^2}{8E_1}\delta(m_Z-2E_1) \\
&= \frac{|\mathcal{M}|^2}{16\pi m_Z^2}\sqrt{m_Z^2-4m^2} \\
&= \frac{\frac{4}{3}[\alpha' m_Z^2-(\alpha'-3\beta')m^2]}{16\pi m_Z^2}\sqrt{m_Z^2-4m^2} \\
&= \frac{\alpha' m_Z^2-(\alpha'-3\beta')m^2}{12\pi m_Z^2}\sqrt{m_Z^2-4m^2}
\end{aligned}
$$

Use $m_Z = 91.187GeV$, $s_W^2 = 0.231$, $e^2 = \frac{4\pi}{128}$, we have $\Gamma = 84.032MeV$.