Meson-meson scattering in 1+1 Dimension

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On-going Work

- Meson-meson scattering amplitude in 1+1 dimension, Guo-ying Chen, Yingsheng Huang, Yu Jia and Rui Yu.
- Divergence of Klein-Gordon hydrogen atom wave-function near origin, Yingsheng Huang, Yu Jia and Rui Yu.

Content

1. 1+1-d QCD and 't Hooft model
2. Quark-Antiquark Amplitude and Normalized Bound State Wave-function
3. Meson-meson scattering amplitude
4. Numerical Calculation
5. Conclusion
6. Divergence in Relativistic Quantum Mechanics

1+1-d QCD and 't Hooft model ('t Hooft, 1974)

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}\,{}_i{}^j G^{\mu\nu}\,{}_j{}^i + \bar{q}^{ai}(i\gamma^\mu D_\mu - m_a) q_i^a, \tag{1}$$

where

$$\begin{aligned} G_{\mu\nu}\,{}_i{}^j &= \partial_\mu A_i{}^j{}_\nu - \partial_\nu A_i{}^j{}_\mu + ig[A_\mu, A_\nu]_i{}^j, \\ D_\mu q_i^a &= \partial_\mu q_i^a + ig A_i{}^j{}_\mu q_j^a, \\ i, j &= 1, 2, ..., N_c, \qquad a = 1, 2, ..., N_f. \end{aligned} \tag{2}$$

Choose light-cone gauge condition

$$A_- = A^+ = 0, \tag{3}$$

where $A_- = \frac{1}{\sqrt{2}}(A^0 + A^1) = \frac{1}{\sqrt{2}}(A_0 - A_1)$. With this condition gauge field can be solved and transformed into instantaneous potential.

In the light-cone gauge, the nonvanishing components of the field strength tensor reads

$$G_{+-} = -G_{-+} = -\partial_- A_+, \tag{4}$$

and the Lagrangian can then be written as

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}(\partial_- A_+)^2 + \bar{q}^a(i\partial_+\gamma_- + i\partial_-\gamma_+ - g\gamma_- A_+ - m_a)q^a. \tag{5}$$

The definition and the algebra for the γ matrices read

$$\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^1), \quad (\gamma^+)^2 = (\gamma^-)^2 = 0, \quad \{\gamma^+, \gamma^-\} = 2. \tag{6}$$

The Feynman rules in the light-cone gauge

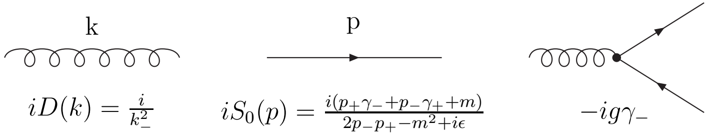

Figure: Feynman rules in the light-cone gauge.

Dyson-Schwinger equation in the large N_c limit, no crossed gluons

$$S(p) = S_0(p) + iN_c g^2 S(p) \left[\int \frac{d^2k}{(2\pi)^2} D(p-k)\gamma_- S(k)\gamma_- \right] S_0(p), \tag{7}$$

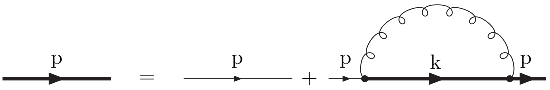

Figure: The thin line denotes the free quark propagator and the solid line denotes the dressed quark propagator.

Solution to the above equation is found to be

$$S(p) = \frac{p_-\gamma_+}{2p_+p_- - M^2 - \frac{N_c g^2}{\pi}\frac{|p_-|}{\lambda} + i\epsilon}, \quad M^2 = m^2 - \frac{N_c g^2}{\pi}, \tag{8}$$

The Bethe-Salpeter equation can be written as

$$\begin{aligned}\psi(p,r) = {} & 4iN_c g^2 p_-(p_- - r_-)[2p_+p_- - M_1^2 - \frac{N_c g^2}{\pi}\frac{|p_-|}{\lambda} + i\epsilon]^{-1} \\ & \times [2(p_+ - r_+)(p_- - r_-) - M_2^2 - \frac{N_c g^2}{\pi}\frac{|p_- - r_-|}{\lambda} + i\epsilon]^{-1} \\ & \times \int \frac{d^2k}{(2\pi)^2}\frac{1}{k_-^2}\psi(p+k,r).\end{aligned} \tag{9}$$

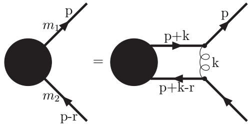

Figure: The Bethe-Salpeter equation of the bound state. Arrow lines are dressed quark propagators.

After defining $\varphi(p_-,r)=\int dp_+\psi(p,r)$, one can have

$$\begin{aligned}\varphi(p_-,r) &= i\frac{N_c g^2}{(2\pi)^2}\int dp_+[p_+-\frac{M_1^2}{2p_-}-\frac{N_c g^2}{2\pi}\frac{sgn(p_-)}{\lambda}+i\epsilon\cdot sgn(p_-)]^{-1}\\ &\times [p_+-r_+-\frac{M_2^2}{2(p_- - r_-)}-\frac{N_c g^2}{2\pi}\frac{sgn(p_- - r_-)}{\lambda}+i\epsilon\cdot sgn(p_- - r_-)]^{-1}\\ &\times \int dk_-\frac{\varphi(p_- + k_-,r)}{k_-^2}. \end{aligned}\tag{10}$$

By taking the p_+ integral and using

$$\int dk_-\frac{\varphi(p_- + k_-,r)}{k_-^2}=\frac{2}{\lambda}\varphi(p_-,r)+P\int dk_-\frac{\varphi(p_- + k_-,r)}{k_-^2},\tag{11}$$

where $P\frac{1}{k_-^2}=\frac{1}{2}(\frac{1}{(k_-+i\epsilon)^2}+\frac{1}{(k_- -i\epsilon)^2})$, one can have the following

$$
\begin{aligned}
&[r_+ - \frac{M_2^2}{2(r_- - p_-)} - \frac{M_1^2}{2p_-} - \frac{N_c g^2}{\pi\lambda} + i\epsilon]\varphi(p_-, r) \\
= \quad & -\frac{N_c g^2}{2\pi}\theta(p_-)\theta(r_- - p_-) \times [\frac{2}{\lambda}\varphi(p_-, r) + P\int dk_- \frac{\varphi(p_- + k_-, r)}{k_-^2}]. \qquad (12)
\end{aligned}
$$

Clearly, the infra-red singularities in both sides cancel with each other. After timing the factor $\frac{2\pi}{N_c g^2} r_-$ in both sides of the above equation and defining the following dimensionless quantities

$$
\frac{2\pi r_+ r_-}{N_c g^2} = \mu^2, \quad \frac{\pi M_{1,2}^2}{N_c g^2} = \alpha_{1,2}, \quad \frac{p_-}{r_-} = x, \qquad (13)
$$

we obtain the famous 't Hooft equation

$$
\mu^2 \varphi(x) = (\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x})\varphi(x) - P\int_0^1 dy \frac{\varphi(y)}{(x-y)^2}. \qquad (14)
$$

The solutions to the 't Hooft equation have discrete enginevalues μ_n^2, $n = 0, 1, 2, \cdots$. The corresponding enginefunctions φ_n satisfy the complete and orthogonal relations

$$
\sum_n \varphi_n(x)\varphi_n^*(x') = \delta(x - x'), \qquad \int_0^1 \varphi_n^*(x)\varphi_m(x)dx = \delta_{nm}. \qquad (15)
$$

Quark-Antiquark Amplitude and Normalized Bound State Wave-function(Callen, Coote and Gross, 1976)

The Bethe-Salpeter equation for the quark-antiquark scattering amplitude can be written as

$$\mathcal{T}(p,p';r) = -\frac{ig^2}{(p_- - p'_-)^2} + i4N_c g^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k_- - p_-)^2} \tilde{S}(k)\tilde{S}(k-r)\mathcal{T}(k,p';r), \tag{16}$$

where $\tilde{S}(p)\gamma_+ = S(p)$. This equation has been solved (Callan, Coote and Gross, 1975) and the result is

$$\begin{aligned}\mathcal{T}(x,x';r) \quad = \quad & -\frac{ig^2}{r_-^2(x-x')^2} + \sum_n \frac{i}{r^2 - r_n^2} \left\{ \varphi_n(x) \frac{g^2}{|r_-|} \sqrt{\frac{N_c}{\pi}} \left[\theta(x(1-x)) \frac{2|r_-|}{\lambda} + \frac{\alpha_1}{x} + \frac{\alpha_2}{1-x} - \mu_n^2 \right] \right\} \\ & \times \left\{ \varphi_n^*(x') \frac{g^2}{|r_-|} \sqrt{\frac{N_c}{\pi}} \left[\theta(x'(1-x')) \frac{2|r_-|}{\lambda} + \frac{\alpha_1}{x'} + \frac{\alpha_2}{1-x'} - \mu_n^2 \right] \right\}, \end{aligned} \tag{17}$$

where $x = \frac{p_-}{r_-}, \quad x' = \frac{p'_-}{r_-}$.

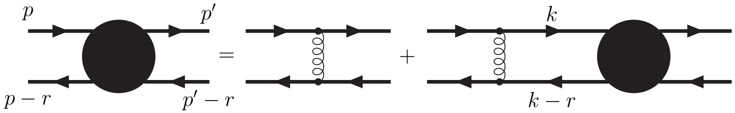

Figure: The Bethe-Salpeter equation for quark-antiquark scattering amplitude.

The amplitude has infinite poles at $r^2 = r_n^2,\ n = 0, 1, 2\cdots$. Physical interpretation of the above solution is clear, the summation of the t-channel multi-gluon exchange is equivalent to the summation of the s-channel quark-antiquark bound state exchange. The residue of the pole gives the normalized bound-state wave function

$$\Phi_n^{1,2}(x) = \varphi_n(x)\frac{g^2}{|r_-|}\sqrt{\frac{N_c}{\pi}}\left[\theta(x(1-x))\frac{2|r_-|}{\lambda} + \frac{\alpha_1}{x} + \frac{\alpha_2}{1-x} - \mu_n^2\right]. \tag{18}$$

Function $\Phi_n^{1,2}(x)$ can also be interpreted as the transition amplitude between the bound state and the quark pair.

This so-called form factor serves as an external state and link between bound state and quarks in our amplitude calculation.

Meson-meson scattering amplitude(Gou-ying Chen and Rui Yu)

For process $A(q^a\bar{q}^b)+B(q^c\bar{q}^a)\to C(q^a\bar{q}^b)+D(q^c\bar{q}^a)$ (where a,b,c are different flavor indexes), the amplitude reads

$$i\mathcal{M}=(1+\mathcal{C})i\mathcal{M}_0,$$

$$i\mathcal{M}_0=\theta(\omega_2-\omega_1)i4g^2\omega_1\int_0^1 dx\int_0^1 dy\frac{1}{(y\omega_1-\omega_2-x)^2}\varphi_A(\frac{\omega_2-\omega_1+x}{\omega_2-\omega_1+1})\varphi_B(y)\varphi_C(x)\varphi_D(\frac{y\omega_1}{\omega_2}),$$

where

$$\omega_1=\frac{r_{B-}}{r_{C-}},\quad \omega_2=\frac{r_{D-}}{r_{C-}}. \tag{19}$$

Here and in the following, we define the operation $(A\leftrightarrow C,\ \ B\leftrightarrow D,\ \ \omega_1\to\frac{\omega_2}{1+\omega_2-\omega_1},\ \ \omega_2\to\frac{\omega_1}{1+\omega_2-\omega_1})$ as $\mathcal{C}$. One can find that the final expression is infra-red safe, thus we postpone $\lambda\to 0$ in our final expression.

$A(q^a\bar{q}^b)+B(q^b\bar{q}^a)\to C(q^a\bar{q}^b)+D(q^b\bar{q}^a)$ reads

$$i\mathcal{M}=(1+\mathcal{P})(1+\mathcal{C})i\mathcal{M}_0. \tag{20}$$

where the operation $\mathcal{P}$ is defined as $\mathcal{P}=(A\leftrightarrow B,\ \ C\leftrightarrow D,\ \ \omega_1\to\frac{1+\omega_2-\omega_1}{\omega_2},\ \ \omega_2\to\frac{1}{\omega_2})$.

Diagrams for the three-flavor processes

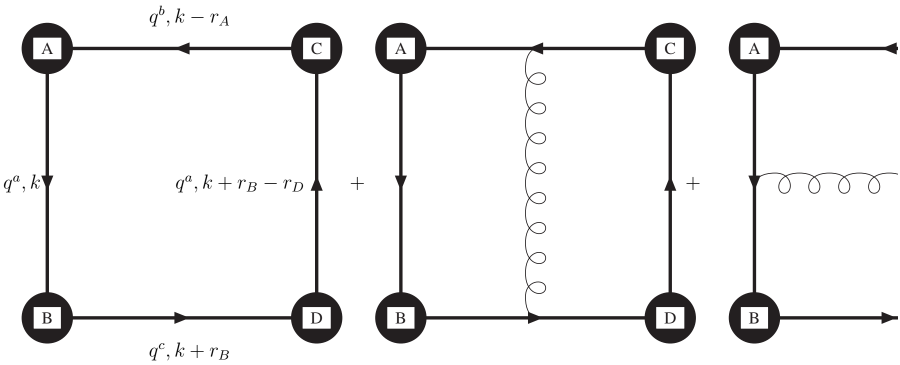

Figure: Four-body contact interaction part for $A(q^a\bar{q}^b) + B(q^c\bar{q}^a) \to C(q^a\bar{q}^b) + D(q^c\bar{q}^a)$. r_A, r_B are the incoming momenta of A and B respectively, and r_C, r_D are the outgoing momenta of C and D respectively.

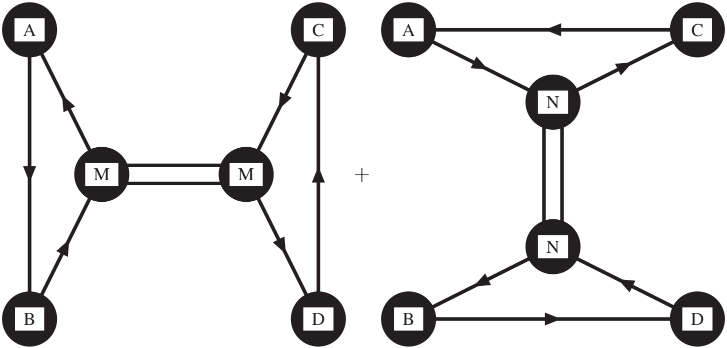

Figure: The meson exchange part for $A(q^a\bar{q}^b) + B(q^c\bar{q}^a) \to C(q^a\bar{q}^b) + D(q^c\bar{q}^a)$

Diagrams for the two-flavor processes

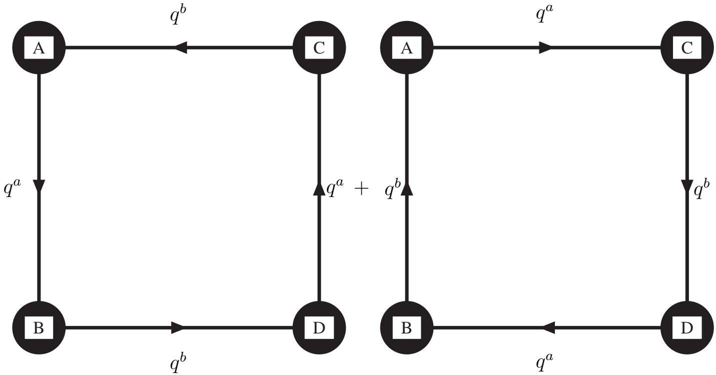

Figure: Basic diagrams for $A(q^a\bar{q}^b) + B(q^b\bar{q}^a) \to C(q^a\bar{q}^b) + D(q^b\bar{q}^a)$.

$A(q^a\bar{q}^a)+B(q^a\bar{q}^a)\to C(q^a\bar{q}^a)+D(q^a\bar{q}^a)$ reads

$$i\mathcal{M}=(1+\mathcal{R})(1+\mathcal{P})(1+\mathcal{C})i\mathcal{M}_0+(1+\mathcal{R})i\mathcal{M}_1, \tag{21}$$

where

$$\begin{aligned}
&i\mathcal{M}_1\\
&=-(1+\mathcal{Q})\theta(1-\omega_1)i4g^2\int_0^1 dxP\int_0^1 dy\frac{\omega_1\omega_2}{[(y-1)\omega_1+(1-x)\omega_2]^2}\varphi_A(\frac{x\omega_2}{1+\omega_2-\omega_1})\varphi_B(y)\varphi_C(y\omega_1)\varphi_D(x)\\
&-(1+\mathcal{C})\theta(\omega_2-\omega_1)i4g^2\int_0^1 dxP\int_0^1 dy\frac{\omega_1}{(y\omega_1-x)^2}\varphi_A(\frac{x+\omega_2-\omega_1}{1+\omega_2-\omega_1})\varphi_B(y)\varphi_C(x)\varphi_D(\frac{(y-1)\omega_1+\omega_2}{\omega_2})\\
&-(1+\mathcal{Q}+\mathcal{P}+\mathcal{C})\theta(\omega_2-\omega_1)\theta(\omega_1-1)i\frac{4\pi}{N_c}\int_0^1 dx\left[2r_{C+}r_{C-}+2r_{D+}r_{C-}+\frac{M_a^2}{x-\omega_1}+\frac{M_a^2}{x-1}\right.\\
&\left.-\frac{M_a^2}{x-\omega_1+\omega_2}-\frac{M_a^2}{x}\right]\times\varphi_A(\frac{x-\omega_1+\omega_2}{1+\omega_2-\omega_1})\varphi_B(x/\omega_1)\varphi_C(x)\varphi_D(\frac{x-\omega_1+\omega_2}{\omega_2}),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}&=(C\leftrightarrow D,\ \ \omega_1\to\frac{\omega_1}{\omega_2},\ \ \omega_2\to 1/\omega_2),\\
\mathcal{Q}&=(B\leftrightarrow C,\ \ A\leftrightarrow D,\ \ \omega_1\to 1/\omega_1,\ \ \omega_2\to\frac{1+\omega_2-\omega_1}{\omega_1}).
\end{aligned} \tag{22}$$

The amplitude can be different when the flavour composition is different, which leads to 23 scenarios in total.

Diagrams for the one-flavor processes

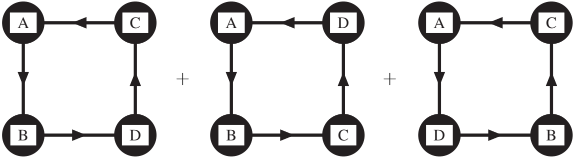

Figure: Basic diagrams for $A(q^a\bar{q}^a) + B(q^a\bar{q}^a) \to C(q^a\bar{q}^a) + D(q^a\bar{q}^a)$. There are also diagrams with clock-wised fermion loop.

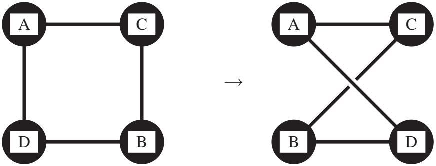

Figure: Replotting the third diagram in Fig. 8.

Numerical Calculation(Yingsheng Huang)

The whole process is simple:

1. Solve 't Hooft equation and obtain meson masses (the eigen value) and the corresponding eigen states;
   (a) BSW method

   Best for heavy quark mass, easy to solve, minimum time consumption, large matrix size cause difficulty in calculating amplitude.

   (b) 't Hooft's original method

   Only for light quark, time consuming, not reliable in high exciting states, small matrix size makes it ideal for amplitude calculation.
2. Put the solution into the corresponding scattering amplitude formula.

Numerical difficulties:

- Cauchy principal value integration.

  The main problem in the process. In other words, it's singularity problem in numerical calculation. This cause problems with light (u, s) quark
- Integration accuracy and instability.

Dimensions

The unit of mass is $\beta = 340MeV$. In this case, the mass of charm quark is $m_c = 4.19\beta = 1.425GeV$, the mass of strange quark is $m_s = 0.749\beta = 0.2547GeV$ and the mass of bottom quark is $m_b = 13.5565\beta = GeV$.

CHARMONIUM $c\bar{c}$.

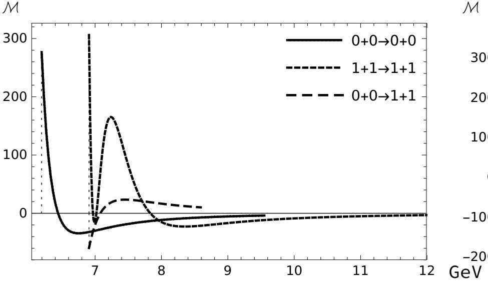

Figure: Quark mass: 1.425 GeV

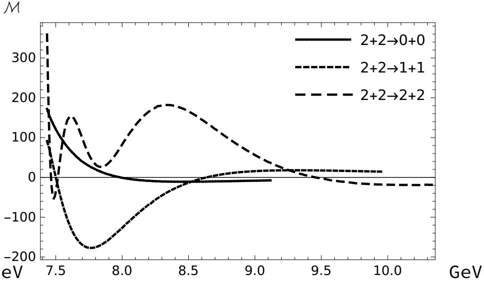

Figure: Quark mass: 1.425 GeV

CHARMED, STRANGE MESONS $c\bar{s}$. It's $12 + 21 \to 12 + 21$.

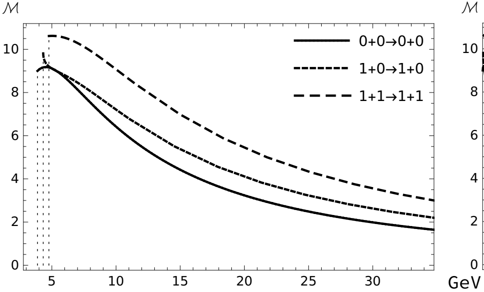

Figure: Quark mass: 1.425 GeV & 0.2547 GeV

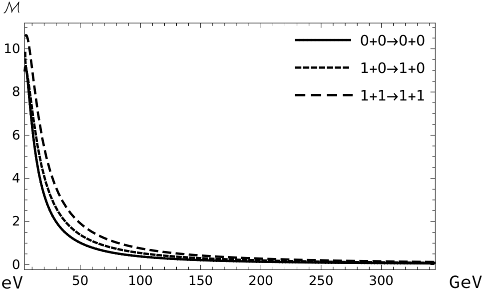

Figure: Quark mass: 1.425 GeV & 0.2547 GeV (HE)

BOTTOM MESONS Both $B^-(b\bar{u})$ and $\bar{B}^0(b\bar{d})$. $B^- + \bar{B}^0 \to B^- + \bar{B}^0$ (TBD).

Conclusion

- Derive 't Hooft equation (Large N_c limit)
- Derive normalized bound state wave-function
- Calculate amplitudes
- Numerical study
- We were looking for four-quark state in 1+1-d QCD, and we thought the bump might have something to do with resonance (similar fitting with Breit-Wigner formula was done by Batiz, Peña and Stabler in 2013), and that could mean exotic state.

  But a cut through the quark line can't produce four quarks, thus four quark intermediate state seems not possible.

  No concrete conclusion for now.

Divergence in Relativistic Quantum Mechanics

Ground state Klein-Gordon Wave-function with Coulomb potential:

$$\psi = \frac{c}{\sqrt{4\pi}} e^{-kr} r^{\lambda} \tag{23}$$

where

$$\lambda = -\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2\alpha^2},$$

$$c = \sqrt{\frac{(2k)^{2(1+\sqrt{\frac{1}{4}-Z^2\alpha^2})}}{\Gamma(2+2\sqrt{\frac{1}{4}-Z^2\alpha^2})}},$$

$$k = \frac{m}{\sqrt{1+\frac{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}{Z^2\alpha^2}}}$$

expand over $Z\alpha$ we got logarithmic divergence.

$$R(r) \sim -(Z\alpha)^2 \log(2mZ\alpha r)$$

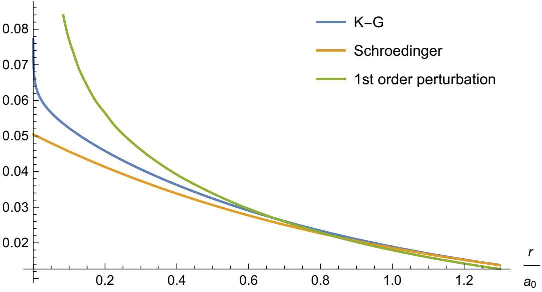

Figure: Comparison between Klein-Gordon wavefunction and Schrödinger wavefunction, with parameters set at $Z = 1$, $\alpha = 0.2$, $m = 1$.

This behavior can be reproduced by simple perturbation theory, with extra power divergence.

$$H = H_0 + H_{int}, \quad H_{int} = H_{kin} + H_{Darwin} + \mathcal{O}(v^6)$$
$$H_0 = -\frac{\nabla^2}{2m} - \frac{Z\alpha}{r}, \quad H_{kin} = \frac{\nabla^4}{8m^3}, \quad H_{Darwin} = \frac{1}{32m^4}[-\nabla^2, [-\nabla^2, -\frac{Z\alpha}{r}]] \tag{24}$$

The NLO correrction to wave-function is

$$\phi^{(1)} = \sum_{n\neq 1} a_{n1}\phi^{(0)}_{n00} + \int d\kappa a_{\kappa 1}\phi^{(0)}_{\kappa 00} \tag{25}$$

where $\kappa = \frac{|\mathbf{k}|}{mZ\alpha}$ and $\phi^{(0)}_{nlm}$, $\phi^{(0)}_{\kappa lm}$ are Schrödinger wave-functions in bound state and scattering state.

The scattering part would cause a divergence when integrating over very-high momentum states, by introducing a hard cutoff Λ we can regularize it (Darwin term won't contribute)

$$R^{(1)}(0)_{kin} = \int^{\frac{\Lambda}{m}} d\kappa (Z\alpha)^2 (\frac{1}{\pi} + \frac{1}{\kappa}) \tag{26}$$
$$\sim (Z\alpha)^2 (\frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right)) \tag{27}$$

- This problem can be dealt with in the framework of QFT!
- NRQED, HQET, OPE, Composite Operator, RGE

Thanks for your attention!