Homework: General Relativity #2

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1. Assuming

$$K^\mu(x) \to \tilde{K}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} K^\nu(x)$$
$$\xi^\mu(x) \to \tilde{\xi}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \xi^\nu(x)$$

and the Lie derivative of counter-variant vector $K^\mu(x)$ is

$$L_\xi K^\mu = \lim_{\epsilon \to 0} \frac{K^\mu(P) - K^\mu(P \Rightarrow Q)}{\epsilon} = K^\mu_{,\nu}\xi^\nu - \xi^\mu_{,\nu}K^\nu$$

so

$$L_{\tilde{\xi}}\tilde{K}^\mu = \tilde{K}^\mu_{,\nu}\tilde{\xi}^\nu - \tilde{\xi}^\mu_{,\nu}\tilde{K}^\nu = \frac{\partial \tilde{K}^\mu}{\partial \tilde{x}^\nu}\tilde{\xi}^\nu - \frac{\partial \tilde{\xi}^\mu}{\partial \tilde{x}^\nu}\tilde{K}^\nu$$

Then we have

$$\begin{aligned}
L_{\tilde{\xi}}\tilde{K}^\mu &= (\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}K^\alpha(x))_{,\nu}\tilde{\xi}^\nu - \tilde{\xi}^\mu_{,\nu}\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}K^\alpha(x) \\
&= \frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial \tilde{x}^\nu}K^\alpha(x)\tilde{\xi}^\nu + \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial K^\alpha}{\partial \tilde{x}^\nu}\tilde{\xi}^\nu - \frac{\partial \tilde{\xi}^\mu}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}K^\alpha(x) \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial K^\alpha}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\nu}{\partial x^\sigma}\xi^\sigma - \frac{\partial \tilde{x}^\mu}{\partial x^\sigma}\frac{\partial \xi^\sigma}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\nu}{\partial x^\alpha}K^\alpha \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial K^\alpha}{\partial x^\sigma}\xi^\sigma - \frac{\partial \tilde{x}^\mu}{\partial x^\sigma}\frac{\partial \xi^\sigma}{\partial x^\alpha}K^\alpha \\
&= \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}L_\xi K^\mu
\end{aligned}$$

which satisfy the trnasformation law of counter-variant vector.

2. Prove for a complete antisymmetric tensor $H^{\mu\nu\cdots\sigma}(x)$ of any order, $\nabla_\rho H^{\rho\nu\cdots\sigma} = \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}H^{\rho\nu\cdots\sigma})$.

The covariant derivative for H is

$$\nabla_\rho H^{\rho\nu\cdots\sigma} = \partial_\rho H^{\rho\nu\cdots\sigma} + \Gamma^\rho_{\rho\alpha}H^{\alpha\nu\cdots\sigma} + \Gamma^\nu_{\rho\alpha}H^{\rho\alpha\cdots\sigma}\cdots$$

and note that H is antisymmetric, so the rest terms vanishes. And also $\Gamma^\rho_{\rho\alpha} = \frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g}$

$$\begin{aligned}
\nabla_\rho H^{\rho\nu\cdots\sigma} &= \partial_\rho H^{\rho\nu\cdots\sigma} + \Gamma^\rho_{\rho\alpha}H^{\alpha\nu\cdots\sigma} \\
&= \partial_\rho H^{\rho\nu\cdots\sigma} + \frac{1}{\sqrt{-g}}(\partial_\rho\sqrt{-g})H^{\rho\nu\cdots\sigma} \\
&= \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}H^{\rho\nu\cdots\sigma})
\end{aligned}$$

3. Calculate the Riemann curvature tensor on a sphere of radius R.

The metric is

$$\mathrm{d}s^2 = R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\varphi^2$$

The definition of Riemann curvature tensor is

$$R^\rho_{\sigma\mu\nu} = \Gamma^\rho_{\sigma\nu,\mu} - \Gamma^\rho_{\sigma\mu,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

and definition of Christoffel symbol

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$$

Note that

$$g^{22} = R^{-2}, g^{33} = \frac{1}{R^2\sin^2\theta}$$

we have

$$\Gamma^2_{33} = -\frac{1}{2}R^{-2}R^2 2\sin\theta\cos\theta = -\sin\theta\cos\theta, \Gamma^3_{23} = \frac{1}{2}R^{-2}\sin^{-2}\theta R^2 2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta}$$

and the rest are zero.

So

$$\begin{aligned}
R^2_{323} &= \Gamma^2_{33,2} - \Gamma^2_{32,3} + \Gamma^2_{2\sigma}\Gamma^\sigma_{33} - \Gamma^2_{3\sigma}\Gamma^\sigma_{23} = \sin^2\theta - \cos^2\theta + \cos^2\theta = \sin^2\theta \\
R^2_{332} &= -R^2_{323} = -\sin^2\theta \\
R^3_{223} &= \Gamma^3_{23,2} - \Gamma^3_{22,3} + \Gamma^3_{2\sigma}\Gamma^\sigma_{23} - \Gamma^3_{3\sigma}\Gamma^\sigma_{22} = -\csc\theta + \frac{\cos^2\theta}{\sin^2\theta} = -1 = -R^3_{232}
\end{aligned}$$

and the rest are zero.

4. Radial equation

$$(\frac{\mathrm{d}r}{\mathrm{d}\tau})^2 = E^2 - (1 - \frac{2GM}{r})(1 + \frac{L^2}{r^2})$$

when $r \to \infty$, it becomes

$$(\frac{\mathrm{d}r}{\mathrm{d}\tau})^2 = E^2 - 1$$

In SR

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = p_r$$

and here we don't have any angular quantity, so the former one becomes

$$\mathbf{p}^2 = E^2 - 1$$

which is exactly the mass-energy equation in SR.

For a mass point, if it can travel to infinity, we can assume it's at rest there. It's easy to know that if $\frac{L}{GM} > 4$ and $E > 1$ it can travel there.

5. To calculate the minimum circular orbit radius of bound state, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}U^2 = 0 \text{ and } \frac{\mathrm{d}^2}{\mathrm{d}r^2}U^2 = 0$$

we know that

$$U^2 = (1 - \frac{2GM}{r})(1 + \frac{L^2}{r^2}) = 1 - \frac{2}{\tilde{r}} + \frac{\tilde{L}^2}{\tilde{r}^2} - \frac{2\tilde{L}^2}{\tilde{r}^3}$$

where $\tilde{L} = \frac{L}{GM}$ and $\tilde{r} = \frac{r}{GM}$.

Applying the conditions given the first line, we have

$$\frac{1}{\tilde{r}^2} - \frac{\tilde{L}^2}{\tilde{r}^3} + \frac{3\tilde{L}^2}{\tilde{r}^4} = 0$$

and

$$-\frac{2}{\tilde{r}^3} + \frac{3\tilde{L}^2}{\tilde{r}^4} - \frac{12\tilde{L}^2}{\tilde{r}^5} = 0$$

so we have

$$\tilde{r} = 6$$

6. Prove that $-\frac{1}{2}(h_{\mu\nu,\alpha}^{;\alpha} + \eta_{\mu\nu} h_{\alpha\beta}^{;\alpha\beta} - h_{\mu\alpha,\nu}^{;\alpha} - h_{\nu\alpha,\mu}^{;\alpha}) = 8\pi G T_{\mu\nu}$.

The Ricci tensor (weak field condition applied) is

$$R_{\mu\nu} = -\frac{1}{2}(h_{\mu\nu,\alpha}^{;\alpha} + h_{,\mu\nu} - h_{\mu,\alpha,\nu}^{\alpha} - h_{\nu,\alpha,\mu}^{\alpha})$$

and its trace

$$R = -(h_{,\alpha}^{;\alpha} - h_{\alpha\beta}^{;\alpha\beta})$$

The trace-reverse tensor is

$$\bar{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = -\frac{1}{2}(h_{\mu\nu,\alpha}^{;\alpha} + h_{,\mu\nu} - h_{\mu,\alpha,\nu}^{\alpha} - h_{\nu,\alpha,\mu}^{\alpha} - \eta_{\mu\nu} h_{,\alpha}^{;\alpha} + \eta_{\mu\nu} h_{\alpha\beta}^{;\alpha\beta})$$

which is exactly the Einstein tensor G.

Now we know

$$h_{\mu\nu,\alpha}^{;\alpha} - \frac{1}{2}\eta_{\mu\nu} h_{,\alpha}^{;\alpha} = \bar{h}_{\mu\nu,\alpha}^{;\alpha}$$

$$\bar{h}_{\nu\alpha,\mu}^{;\alpha} = h_{\nu\alpha,\mu}^{;\alpha} - \frac{1}{2}\eta_{\nu\alpha} h_{,\mu}^{;\alpha} = h_{\nu\alpha,\mu}^{;\alpha} - \frac{1}{2} h_{,\mu\nu}$$

$$\bar{h}_{\mu\alpha,\nu}^{;\alpha} = h_{\mu\alpha,\nu}^{;\alpha} - \frac{1}{2}\eta_{\mu\alpha} h_{,\nu}^{;\alpha} = h_{\mu\alpha,\nu}^{;\alpha} - \frac{1}{2} h_{,\mu\nu}$$

and the rest terms

$$\eta_{\mu\nu} h_{\alpha\beta}^{;\alpha\beta} - \frac{1}{2}\eta_{\mu\nu} h_{,\alpha}^{;\alpha} = \eta_{\mu\nu}(h_{\alpha\beta}^{;\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h^{;\alpha\beta} = \eta_{\mu\nu}\bar{h}_{\alpha\beta}^{;\alpha\beta})$$

From the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

we have

$$-\frac{1}{2}(\bar{h}_{\mu\nu,\alpha}^{;\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}^{;\alpha\beta} - \bar{h}_{\mu\alpha,\nu}^{;\alpha} - \bar{h}_{\nu\alpha,\mu}^{;\alpha}) = 8\pi G T_{\mu\nu}$$

7. Define $A_\pm = A_{11} \mp i A_{12}$. Under the rotation

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

we have

$$\tilde{A}_\pm = \tilde{A}_{11} \mp i\tilde{A}_{12}$$

Now evaluate it term by term:

For a counter-variant tensor of order 2

$$\tilde{A}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}A_{\alpha\beta}$$

so

$$\begin{aligned}\tilde{A}_{11} &= \frac{\partial x}{\partial \tilde{x}}\frac{\partial x}{\partial \tilde{x}}A_{11} + 2\frac{\partial x}{\partial \tilde{x}}\frac{\partial y}{\partial \tilde{x}}A_{12} + \frac{\partial y}{\partial \tilde{x}}\frac{\partial y}{\partial \tilde{x}}A_{22}\\ &= \cos^2\theta A_{11} + 2\cos\theta\sin\theta A_{12} + \sin^2\theta A_{22}\\ &= \cos 2\theta A_{11} + \sin 2\theta A_{12}\end{aligned}$$

and

$$\begin{aligned}\tilde{A}_{12} &= \frac{\partial x}{\partial \tilde{x}}\frac{\partial x}{\partial \tilde{y}}A_{11} + \frac{\partial x}{\partial \tilde{x}}\frac{\partial y}{\partial \tilde{y}}A_{12} + \frac{\partial y}{\partial \tilde{x}}\frac{\partial x}{\partial \tilde{y}}A_{21} + \frac{\partial y}{\partial \tilde{x}}\frac{\partial y}{\partial \tilde{y}}A_{22}\\ &= -\cos\theta\sin\theta A_{11} + \cos^2\theta A_{12} - \sin^2\theta A_{21} + \sin\theta\cos\theta A_{22}\\ &= -\sin 2\theta A_{11} + \cos 2\theta A_{12}\end{aligned}$$

Thus

$$\begin{aligned}\tilde{A}_{\pm} &= (\cos 2\theta \pm i\sin 2\theta)A_{11} + (\sin 2\theta \mp i\cos 2\theta)A_{12}\\ &= (\cos 2\theta \pm i\sin 2\theta)A_{11} \mp i(\cos 2\theta \pm i\sin 2\theta)A_{12}\\ &= e^{\pm 2i\theta}A_{\pm}\end{aligned}$$

8. Prove $T^G_{\mu\nu}$ is invariant under gauge transformation.

First

$$T^G_{\mu\nu} = -\frac{1}{8\pi G}(G_{\mu\nu} - G^{(1)}_{\mu\nu})$$

so we are to prove $G_{\mu\nu}$ and $G^{(1)}_{\mu\nu}$ are gauge-invariant.

For $G_{\mu\nu}$, it's easy to know that with Lorentz gauge

$$G_{\mu\nu} = \bar{h}^{,\alpha}_{\mu\nu,\alpha}$$

which is Lorentz-invariant under a given transformation $x^\mu \to x^\mu + \xi^\mu$ and $\xi^{,\alpha}_{\mu,\alpha} = 0$. Similarly we can prove that $G^{(1)}_{\mu\nu}$ is gauge-invariant.

To prove $\langle T^G_{\mu\nu}\rangle$ is gauge-invariant, first we have

$$\langle T^G_{\mu\nu}\rangle = \frac{1}{16\pi G}(A^{\rho\sigma}A^*_{\rho\sigma} - \frac{1}{2}\left|A^\lambda_\lambda\right|^2)$$

and the gauge transformation

$$\begin{aligned}A_{\mu\nu} &\to A_{\mu\nu} + k_\mu X_\nu + k_\nu X_\mu\\ A^*_{\mu\nu} &\to A^*_{\mu\nu} + k_\mu X^*_\nu + k_\nu X^*_\mu\end{aligned}$$

so

$$\begin{aligned}\langle T^G_{\mu\nu}\rangle \to{}& \frac{1}{16\pi G}\Big\{(A^{\rho\sigma} + k^\rho X^\sigma + k^\sigma X^\rho)(A^*_{\rho\sigma} + k_\rho X^*_\sigma + k_\sigma X^*_\rho) - \frac{1}{2}(A^\lambda_\lambda + k^\lambda X_\lambda + k^\lambda X_\lambda)(A^{\tau *}_\tau + k^\tau X^*_\tau + k^\tau X^*_\tau)\Big\}\\ ={}& \frac{1}{16\pi G}\Big\{A^{\rho\sigma}A^*_{\rho\sigma} + 2k^\rho X^\sigma A^*_{\rho\sigma} + 2A^{\rho\sigma}k_\rho X^*_\sigma + 2k^\rho X^\sigma k_\rho X^*_\sigma + 2k^\rho X^\sigma k_\sigma X^*_\rho\\ &\qquad - \frac{1}{2}(AA^* + 2Ak^\tau X^*_\tau + 2A^*k^\lambda X_\lambda + 4k^\lambda X_\lambda k^\tau X^*_\tau)\Big\}\end{aligned}$$

Now we only need to prove

$$2k^\rho X^\sigma A^*_{\rho\sigma} + 2A^{\rho\sigma}k_\rho X^*_\sigma + 2k^\rho X^\sigma k_\rho X^*_\sigma + 2k^\rho X^\sigma k_\sigma X^*_\rho - Ak^\tau X^*_\tau - A^* k^\lambda X_\lambda - 2k^\lambda X_\lambda k^\tau X^*_\tau = 0$$

From gauge condition we have $k^2 = 0$, so it becomes

$$2k^\rho X^\sigma A^*_{\rho\sigma} + 2k_\rho X^*_\sigma A^{\rho\sigma} - Ak^\tau X^*_\tau - A^* k^\lambda X_\lambda = 0$$

Combining $X^\sigma A^*_{\rho\sigma}$ and $X^{\sigma *} A_{\rho\sigma}$, we can prove the result is the real part of $2X^\sigma A^*_{\rho\sigma}$, so it becomes

$$k^\rho \operatorname{Re}[X^\sigma A^*_{\rho\sigma}] - \frac{1}{2}k^\lambda \operatorname{Re}[X^*_\lambda A] = k^\rho \operatorname{Re}[X^\sigma \bar{A}^*_{\rho\sigma}] = 0$$

We already know that $k \cdot A = 0$ so this equation stands.

9. Derive the Newtonian TOV equation (see Chandrasekhar 1939).

First we have

$$m = \int_0^r 4\pi r^2 \rho \mathrm{d}r, \ \mathrm{d}m(r) = 4\pi r^2 \rho \mathrm{d}r$$

and use Chandrasekhar's cylinder model (an infinitesimal cylinder at distance r from the center and height dr), we have the force represented by the difference of pressure which acts on a

$$-\mathrm{d}p = \frac{Gm(r)\rho \mathrm{d}r}{r^2}$$

so

$$p' = -\frac{Gm\rho}{r^2}$$

The TOV equation is

$$p' = -(p+\rho)\frac{Gm + 4\pi G r^3 p}{r(r-2Gm)}$$

We require $p \ll \rho$ and $m \ll r$ for non-relativistic limit, the former one also means $4\pi r^3 p \ll m$, and the latter one is the requirement of flat metric. So the TOV equation becomes

$$p' = -\rho\frac{Gm}{r^2}$$

which meets the one from Newtonian mechanics.

Einstein equation:

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}$$

Define

$$t_{\mu\nu} \equiv \frac{1}{8\pi G}[G_{\mu\nu} - G^{(1)}_{\mu\nu}]$$

and

$$G^{(1)}_{\mu\nu} = -8\pi G(T_{\mu\nu} + t_{\mu\nu})$$
$$T_{\mu\nu} = 0 \& G^{(1)}_{\mu\nu} = 0 \Longrightarrow t_{\mu\nu} = 0$$

which can't be right because $t_{\mu\nu}$ is the energy-momentum tensor of gravitational field.