Homework: Quantum Field Theory #2

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Prove that $S^{\mu\nu}$ satisfy the Lie algebra of Lorentz group, that's to say, satisfy the commutation relation the same as $[J^{\mu\nu}, J^{\rho\sigma}]$.

First we know that $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. We can write down the commutation relation of $S^{\mu\nu}$

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}[[\gamma^\mu, \gamma^\nu], [\gamma^\rho, \gamma^\sigma]] \\
&= -\frac{1}{16}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho] \\
&= -\frac{1}{16}\{[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] - [\gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma] - [\gamma^\mu\gamma^\nu, \gamma^\sigma\gamma^\rho] + [\gamma^\nu\gamma^\mu, \gamma^\sigma\gamma^\rho]\}
\end{aligned}$$

We evaluate the commutation relation $[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma]$ separately

$$\begin{aligned}
[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] &= \gamma^\mu[\gamma^\nu, \gamma^\rho]\gamma^\sigma + [\gamma^\mu, \gamma^\rho]\gamma^\sigma\gamma^\nu + \gamma^\rho[\gamma^\mu, \gamma^\sigma]\gamma^\nu + \gamma^\mu\gamma^\rho[\gamma^\nu, \gamma^\sigma] \\
&= \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma + \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu + \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu \\
&= 2\gamma^\mu g^{\nu\rho}\gamma^\sigma + 2g^{\mu\rho}\gamma^\sigma\gamma^\nu - 2\gamma^\mu\gamma^\rho g^{\nu\sigma} - 2\gamma^\rho g^{\mu\sigma}\gamma^\nu
\end{aligned}$$

So

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] = -\frac{1}{8}\{&\gamma^\mu g^{\nu\rho}\gamma^\sigma + g^{\mu\rho}\gamma^\sigma\gamma^\nu - \gamma^\mu\gamma^\rho g^{\nu\sigma} - \gamma^\rho g^{\mu\sigma}\gamma^\nu - \gamma^\nu g^{\mu\rho}\gamma^\sigma - g^{\nu\rho}\gamma^\sigma\gamma^\mu + \gamma^\nu\gamma^\rho g^{\mu\sigma} + \gamma^\rho g^{\nu\sigma}\gamma^\mu \\
&- \gamma^\mu g^{\nu\sigma}\gamma^\rho - g^{\mu\sigma}\gamma^\rho\gamma^\nu + \gamma^\mu\gamma^\sigma g^{\nu\rho} + \gamma^\sigma g^{\mu\rho}\gamma^\nu\gamma^\nu g^{\mu\sigma}\gamma^\rho + g^{\nu\sigma}\gamma^\rho\gamma^\mu - \gamma^\nu\gamma^\sigma g^{\mu\rho} - \gamma^\sigma g^{\nu\rho}\gamma^\mu\} \\
= -\frac{1}{4}\{&g^{\nu\rho}[\gamma^\mu, \gamma^\sigma] + g^{\mu\rho}[\gamma^\sigma, \gamma^\nu] - g^{\nu\sigma}[\gamma^\mu, \gamma^\rho] - g^{\mu\sigma}[\gamma^\rho, \gamma^\nu]\} \\
= i\{&g^{\nu\rho}S^{\mu\sigma} + g^{\mu\rho}S^{\sigma\nu} - g^{\nu\sigma}S^{\mu\rho} - g^{\mu\sigma}S^{\rho\nu}\}
\end{aligned}$$

which is the same as the commutation relation of $[J^{\mu\nu}, J^{\rho\sigma}]$.