$$A_{jk}^{0\text{-MHV}} = A_n^0(1^{+}, 2^{+}, \ldots, j^{-}, j+1^{+}, \ldots, k^{-}, \ldots, n^{+})$$

$$= i\,\frac{\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots\cdots \langle n1\rangle}$$

$$A_4^0(1^{-}, 2^{+}, 3^{-}, 4^{-}) = i\,\frac{\langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$A_4^0(1^{+}, 2^{-}, 3^{+}, 4^{-}) = i\,\frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$A_4^0(1^{+}, 2^{+}, 3^{+}, 4^{+}) = A_4^0(1^{+}, 2^{+}, 3^{+}, 4^{-}) = 0$$

$$|A_4^0(1,2,3,4)|^2 = \frac{s_{34}^4 + s_{12}^4 + s_{14}^4}{s_{12}\, s_{23}\, s_{34}\, s_{41}}$$

$+ \; - \; - \; -$

$+$	$+$	$-$	$-$
$-$	$-$	$+$	$+$
$+$	$-$	$+$	$-$
$-$	$+$	$-$	$+$
$+$	$-$	$-$	$+$
$-$	$+$	$+$	$-$

6 MHV

2) IR Singularity in NLO Real part.

Tree QCD amplitude IR. divergence :

$g \to 0$ soft limit

$g // g \quad q // g, \; q // \bar{q}$ collinear limit

$$|A_4^0|^2 \xrightarrow[p_2 \to 0]{2 \to 0} \frac{s_{12}^4 + s_{13}^4 + s_{14}^4 + s_{23}^4 + s_{24}^4 + s_{34}^4}{s_{12}\, s_{23}\, s_{34}\, s_{41}} = \frac{s_{13}}{s_{12}\, s_{23}} \times \frac{s_{13}^4 + s_{14}^4 + s_{24}^4}{s_{13}\, s_{34}\, s_{41}}$$

$$s_{2i} = (p_2 + p_i)^2 = p_2^2 + p_i^2 + 2 p_2 \cdot p_i \to 0$$

$$= \frac{S_{13}}{S_{12} S_{23}} \left| A_3^0 \right|^2$$

$$|A_4^0|^2 \xrightarrow{2 \to 0} = \frac{S_{123}}{2} \left| A_3^0 \right|^2$$

$$S_{123} = \frac{2 S_{13}}{S_{12} S_{23}}$$ Eikonal factor $\sim \frac{1}{\rho^2}$

$$|A_4^0|^2 (1, 2, 3, 4) \xrightarrow{l \to 0} \underline{\underline{S_{iLk}}} \ \underline{\underline{|A_3^0|^2 (i, j, k)}}$$

$$S_{123}, \quad S_{234}, \quad S_{412}$$

$$\left[\, |A_4^0(1,2,3,4)|^2 + |A_4^0(1,4,3,2)|^2 \,\right] \xrightarrow{2 \to 0} \frac{2\, s_{13}}{s_{12}\, s_{23}} |A_3^0(1,3,4)|^2$$

Single collinear limit. $p_1 \parallel p_2$

$p_1 + p_2 = p$

$$\begin{cases} p_1 = z\, p \\ p_2 = (1-z)\, p \end{cases}$$

$$\sum_{\text{helicity}} |A_4^0(1,2,3,4)|^2 \quad 1/\!/2$$

$s_{12} \Rightarrow 0$

$s_{24} \to (1-z)\, s_{p4}$

$s_{13} \Rightarrow z\, s_{p3}$

$$= \frac{\left[z^4 + (1-z)^4\right](s_{p3}^4 + s_{p4}^4) + s_{34}^4}{z(1-z)\, s_{12}\, s_{p3}\, s_{34}\, s_{p4}}$$

$$= \left(\frac{P^0_{gg\to g}(z)}{s_{12}}\right) \left| A_5^0(P,3,4) \right|^2$$

$+$ Angular term.

$$P^0_{gg\to g}(z) = 2\left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right) = \frac{1+z^4+(1-z)^4}{z(1-z)}$$

$$\left|\frac{\langle ij\rangle^4}{\langle 12\rangle \cdots \langle a-1,a\rangle\langle ab\rangle\langle b,b+1\rangle \cdots \langle n1\rangle}\right|^2 \xrightarrow{a^{(+)}/b^{(+)}}$$

$$\underbrace{\frac{1}{\sqrt{(1-z)z}\,\langle ab\rangle}}_{} \times \frac{\langle ij\rangle^4}{\langle 12\rangle \cdots \langle a-1, P\rangle \langle P, b+1\rangle \cdots \langle n1\rangle}$$

$$\mathrm{Split}_-(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle ab\rangle}$$

$$\mathrm{Split}_+(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle ab\rangle}$$

$$\mathrm{Split}_+(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle ab\rangle}$$

$$|\mathrm{Split}_-(a^+, b^+)|^2 + |\mathrm{Split}_+(a^-, b^+)|^2 + |\mathrm{Split}_+(a^+, b^-)|^2$$

$$\approx \frac{1}{s_{ab}}\left[\frac{1}{z(1-z)} + \frac{z^4}{z(1-z)} + \frac{(1-z)^4}{z(1-z)}\right]$$

$$n = (1,0,0,-1)$$

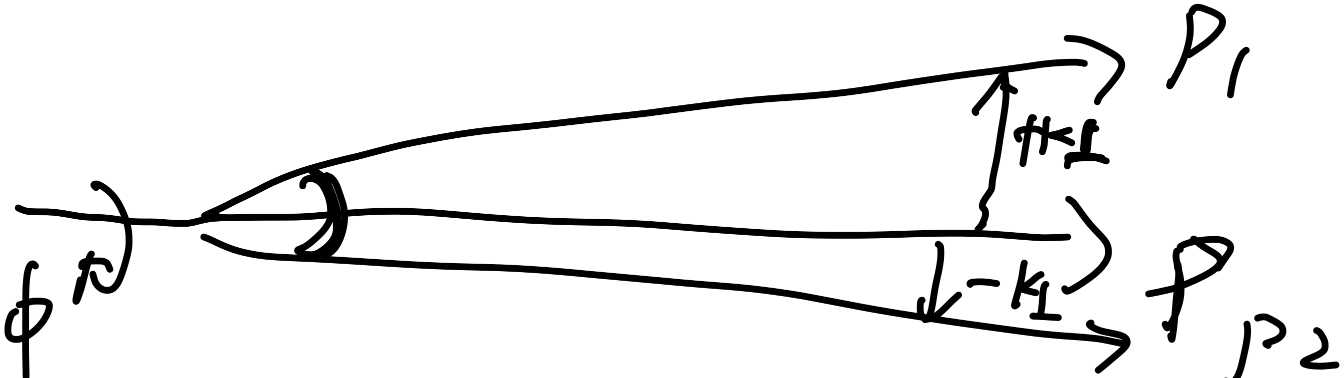
$$k_\perp = (0,\varepsilon,\varepsilon,0)$$

$$p_1 = z\,p + k_\perp - \frac{k_\perp^2}{z}\frac{n}{2p\cdot n}$$

$$p_2 = (1-z)\,p - k_\perp - \frac{k_\perp^2}{1-z}\frac{n}{2p\cdot n}$$

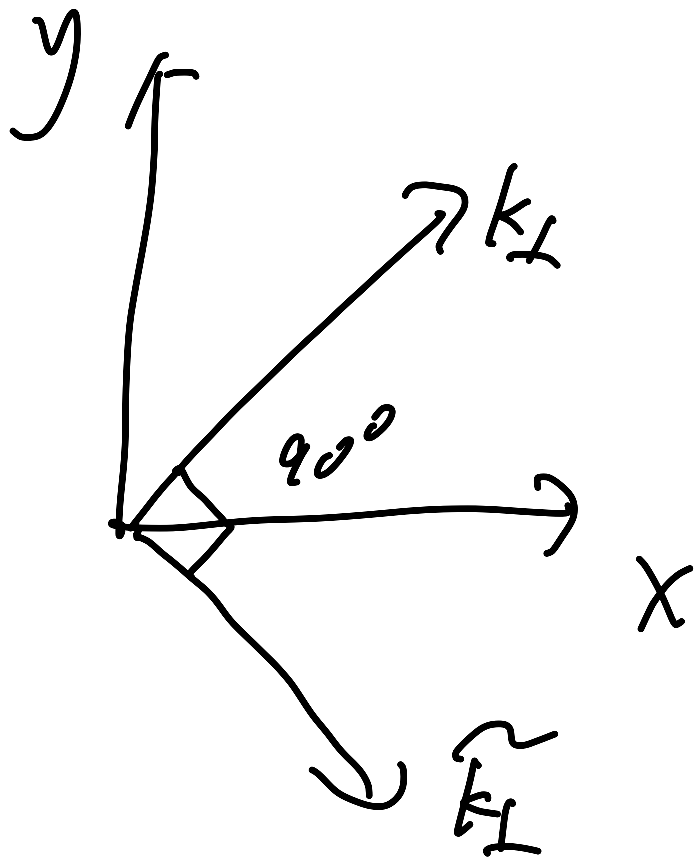
$$p_1 + p_2 \approx p \qquad p_1^2 = p_2^2 = p^2 = 0$$

$$p_1^2 = p_2^2 = 0$$



$$\begin{cases} P_1(p, k_\perp, z) \\ P_2(p, k_\perp, 1-z) \end{cases} \qquad \begin{cases} \tilde{P}_1(p, \tilde{k}_\perp, z) \\ \tilde{P}_2(p, \tilde{k}_\perp, 1-z) \end{cases}$$

$\tilde{k}_\perp \to k_\perp$



$$|A_4^0(1,2,3,4)^2|^2$$

$$+ |A_4^0(\tilde{1},\tilde{2},3,4)$$

$$\overset{1/\!/2}{\longrightarrow} \frac{2 P_{gg\to g}^0(z)}{s_{12}} \times |A_3^0(P,3,4)|^2$$

$$\left| A_4^0(1,2,3,4) \right|^2$$

$2 \to 0$

$2 // 1$ IR

$2 // 3$ divergene

$3 \to 0$ IR

$2 // 3$ $3 // 4$

$1 \to 0 \sim \frac{1}{0^2}$

$2 \to 0 \sim \frac{1}{0^2}$

$1 // 2 \longrightarrow \frac{1}{0} P^0_{gg\to g}(z)$

$2 // 4 \longrightarrow 1$

$1 // 3 \longrightarrow 1$

$3 // 4 \longrightarrow \frac{1}{0} P^0_{gg\to g}(z)$

$$P^0_{qg\to q}(z) = \frac{1+(1-z)^2}{z}$$

$$S_{ijk} = \frac{2S_{ik}}{S_{ij}S_{jk}} \sim \frac{1}{\epsilon^2}$$

$$P^0_{q\bar{q}\to g}(z) = z^2+(1-z)^2$$

$$\frac{P^0_{gg\to G}(z)}{S_{ij}} \sim \frac{1}{\epsilon} \quad \text{in} \quad \overbrace{z}^{55\%} \sim \overbrace{1-z}^{45\%}$$

$$\sim \frac{1}{\epsilon^2} \quad \text{in} \quad z\to 0 \quad 1-z\to 1$$

$$P_1 = zP \Rightarrow 0$$

$$\prod_j \frac{d^3 p_j}{(2\pi)^3\, 2E_j} (2\pi)^4 \delta(p_1+p_2-\sum_j p_j)\, \delta(p_j^2-m_j^2)$$

$$\frac{d^3 p_j}{2E_j} S_{ik} = \frac{\cancel{2}\, \cancel{E_j^2}\, dE_j\, d\cos\theta_j\, d\phi\, S_{ik}}{\cancel{2E_j}\; E_i \cancel{E_j} (1-\cos\theta_{ij}) E_j E_k (1-\cos\theta_{jk})}$$

2π

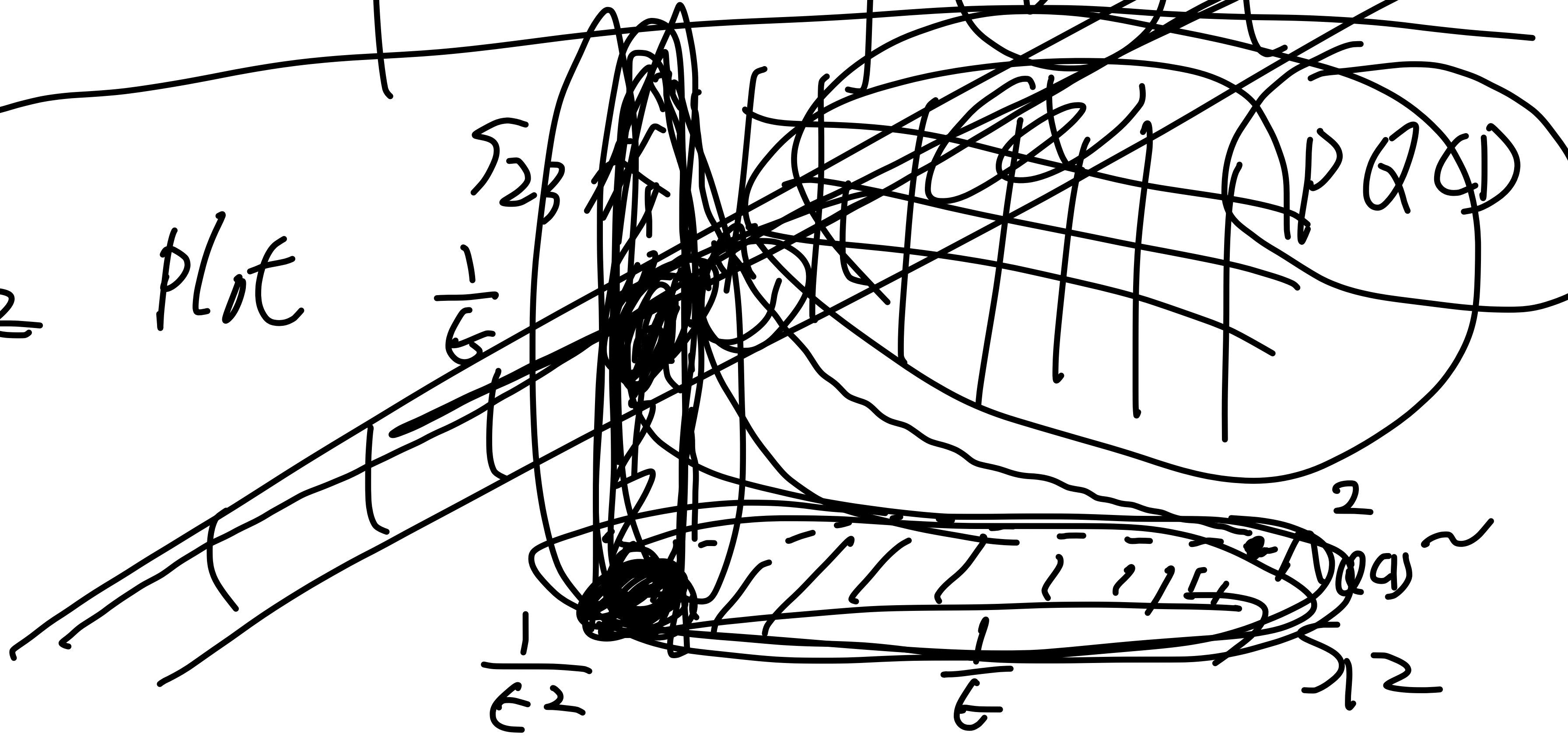
$p_j \to 0$

$$= \frac{2\pi S_{ik}\, dE_j\, d\cos\theta_j}{E_i E_k E_j (1-\cos\theta_{ij})(1-\cos\theta_{jk})} \sim$$

$\to 0$

$\frac{1}{\epsilon}$	$E_j \to 0$	$\theta_{ij}, \theta_{jk} \sim 1$
$\frac{1}{\epsilon^2}$,	$E_j \to 0$	$\theta_{ij} \sim 0$, or $\theta_{jk} \sim O(1)$

Dalitz plot



3) IR Singularity @ NLO Virtual.

$$A_n = A_n^0 + g A_{n+1}^0 + g^2 A_n^1 + g^3 A_{n+1}^1$$

$$+ g^4 A_n^2 \cdots\cdots$$

$$|A_n|^2 = |A_n^0|^2 \sim \text{LO / Born}$$

Real

Virtual

NLO $\quad + \; g^2 \left\{ |A_{n+1}^0|^2 + \left[A_n^0 A_n^{1+} + A_n^{1+} A_n^0 \right] \right\}$

NNLO $\quad + \; g^4 \left\{ |A_{n+2}^0|^2 + |A_n^1|^2 + \cdots \right\}$

N^3LO + ⋮

Real + Virtual = finite

KLN theorem

$$\text{Pole}\Big\{ A_4^{0}(g_1 g_2 g_3 g_4)\, A_4^{1\dagger}(g_1 g_2 g_3 g_4) + A_4^{0\dagger}(g_1 g_2 g_3 g_4)\, A_4^{1}(g_1 g_2 g_3 g_4) \Big\} \quad \text{Virtual}$$

$$= \Big[2 I_{gg}^{(1)}(\varepsilon, s_{12}) + 2 I_{gg}^{(1)}(\varepsilon, s_{23})$$

$$+ 2 I_{gg}^{(1)}(\epsilon, s_{34}) + 2 I_{gg}^{(1)}(\epsilon, s_{41}) \Big] \times \left| A_4^0(\overset{1234}{gggg}) \right|^2$$

$$I_{gg}^{(1)}(\epsilon, s_{ij}) = -\boxed{\frac{e^{\epsilon \gamma_E}}{2\Gamma(1-\epsilon)}} \left[\frac{1}{\epsilon^2} + \frac{11}{6\epsilon} \right] \boxed{\mathrm{Re}(-s_{ij})^{-\epsilon}}$$

$I_{qg}^{(1)}$, $I_{q\bar{q}}^{(1)}$ ...

Lecture 2 NLO Subtraction Method and application

1) structure of NLO subtraction :

Parton Model :

$$d\sigma(P_{H_1}, P_{H_2}) = \sum_{ij} \int \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} f_i^0(\xi_1) f_j^0(\xi_2) \, d\sigma_{ij}(\xi_1 P_{H_1}, \xi_2 P_{H_2})$$

$$f_i^{(0)}(z) = \int dx\,dy\, \delta(z - xy) \, f_{\hat{j}}(x, \mu_F) \, \Gamma_{\hat{j} i}(y, \mu_F)$$

$$f^0 = f \otimes \Gamma^+$$

$$\Gamma^+_{ij}(x, \mu_F) = \delta_{ij}\,\delta(1-x) - \frac{\alpha_s(\mu_F^2)}{2\pi}\Gamma^1_{ij}(x) + \frac{\alpha_s^2(\mu_F^2)}{(2\pi)^2}\Gamma^2_{ij}(x)$$

$$\Gamma^1_{ij}(x) = -\frac{1}{\epsilon}P^0_{ij}(x)$$

$\rightarrow$ DGLAP splitting functions

$$d\sigma(P_{H_1}, P_{H_2}) = f^0\, d\sigma\, f^{0'}$$

QCD improved Parton model

$$d\hat{\sigma}(p_{H_1}, p_{H_2}) = f \otimes \underbrace{\Gamma^{+} \cdot d\sigma \cdot \Gamma^{-1}}_{} \otimes f'$$

$$= f \otimes \underline{\underline{d\hat{\sigma}}} \otimes f'$$

$$d\hat{\sigma}^{LO}_{ij} = d\sigma^{LO}_{ij}$$

$$d\hat{\sigma}^{NLO}_{ij} = \underline{\underline{d\sigma^{NLO}_{ij}}} + \underline{\underline{d\hat{\sigma}^{MF,LO}_{ij}}}$$

$$
d\hat{\sigma}^{MF,LO}_{ij} = d\sigma^{R}_{ij} + d\sigma^{V}_{ij} + \int \Gamma'_{ki}(x_1)\, d\sigma^{LO}_{kj}\, \delta(1-x_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2} + \int \Gamma'_{lj}(x_2)\, d\sigma^{LO}_{il}\, \delta(1-x_1)\frac{dx_1}{x_1}\frac{dx_2}{x_2}
$$