Homework: Quantum Field Theory #9

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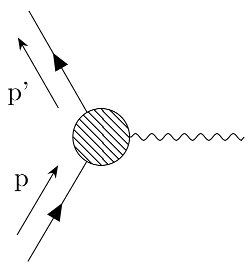
February 19, 2017

6.1. Rosenbluth Formula.

The QED vertex:



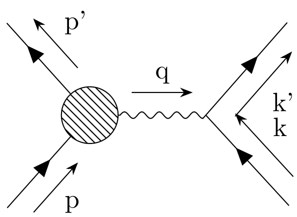
$$= \bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p)$$

where $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Consider the fermions strongly interacting ones (such as protons), the form factor reflects the structure of strong interaction and therefore can only be determined by experiment. Consider a electron ($E \ll m_e$) scattering from a proton initially at rest. Show that this leads to the Rosenbluth formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2\left[(F_1^2 - \frac{q^2}{4m^2}F_2^2)cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right]}{2E^2\left[1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}}$$

for elastic scattering cross section, computed to leading order in α.

The whole diagram



$$= \bar{u}_p(p')(-ie\Gamma^\mu)u_p(p)\frac{-i}{q^2}\bar{u}_e(k')(-ie\gamma_\mu)u_e(k)$$

Use spin sum

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\sum_{spins}\left\{\bar{u}_p(p')\Gamma^\mu u_p(p)\bar{u}_e(k')\gamma_\mu u_e(k)[\bar{u}_p(p')\Gamma^\nu u_p(p)\bar{u}_e(k')\gamma_\nu u_e(k)]^\dagger\right\}$$

while

$$\begin{aligned}\sum_{spins}\bar{u}_e(k')\gamma_\mu u_e(k)[\bar{u}_e(k')\gamma_\nu u_e(k)]^\dagger &= \sum_{spins}\bar{u}_e(k')\gamma_\mu\bar{u}_e(k)[u_e(k)\gamma_\nu u_e(k')] \\ &= \sum_{spins}\mathrm{tr}\{u_e(k')\bar{u}_e(k')\gamma_\mu u_e(k)\bar{u}_e(k)\gamma_\nu\} \\ &= \mathrm{tr}\Big\{(\not{k}' + m_e)\gamma_\mu(\not{k} + m_e)\gamma_\nu\Big\}\end{aligned}$$

and

$$\sum_{spins} \bar{u}_p(p')\Gamma^\mu u_p(p)\bar{u}_p(p')\Gamma^\nu u_p(p)$$

$$= \sum_{spins} \bar{u}_p(p')\left\{\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\rho}q_\rho}{2m}F_2(q^2)\right\}u_p(p)\bar{u}_p(p)\left\{\gamma^\nu F_1(q^2) - \frac{i\sigma^{\nu\sigma}q_\sigma}{2m}F_2(q^2)\right\}u_p(p')$$

$$= \mathrm{tr}\big\{(\not p' + m)\gamma^\mu(\not p + m)\gamma^\nu\big\}F_1^2 + (\mathrm{tr}\big\{(\not p' + m)\sigma^{\mu\rho}(\not p + m)\gamma^\nu\big\} - \mathrm{tr}\big\{(\not p' + m)\gamma^\mu(\not p + m)\sigma^{\nu\rho}\big\})\frac{iq_\rho}{2m}F_1F_2$$

$$+ \mathrm{tr}\left\{(\not p' + m)\frac{\sigma^{\mu\rho}q_\rho}{2m}(\not p + m)\frac{\sigma^{\nu\sigma}q_\sigma}{2m}\right\}F_2^2$$

(Still need to prove $[\bar{u}(p')\sigma^{\mu\nu}u(p)]^\dagger = \bar{u}(p)\sigma^{\mu\nu}u(p')$, which only requires $[\bar{u}(p')\gamma^\mu\gamma^\nu u(p)]^\dagger = \bar{u}(p)\gamma^\mu\gamma^\nu u(p')$, so

$$[\bar{u}(p')\gamma^\mu\gamma^\nu u(p)]^\dagger = (u(p))^\dagger\gamma^{\nu\dagger}\gamma^{\mu\dagger}\gamma^{0\dagger}u(p') = (u(p))^\dagger\gamma^{\nu\dagger}\gamma^0\gamma^\mu u(p') = (u(p))^\dagger\gamma^0\gamma^\nu\gamma^\mu u(p') = \bar{u}(p)\gamma^\nu\gamma^\mu u(p')$$

note that there's a extra minus sign.)

Now we calculate the traces term by term.

$$\mathrm{tr}\left\{(\not k' + m_e)\gamma_\mu(\not k + m_e)\gamma_\nu\right\} = 4(k'_\mu k_\nu + k_\mu k'_\nu - k'\cdot k g_{\mu\nu}) + 4m_e^2 g_{\mu\nu}$$

m_e is small comparing to m so we ignore it: $= 4(k'_\mu k_\nu + k_\mu k'_\nu - k'\cdot k g_{\mu\nu})$

$$\mathrm{tr}\big\{(\not p' + m)\gamma^\mu(\not p + m)\gamma^\nu\big\} = 4(p'^\mu p^\nu + p^\mu p'^\nu - p'\cdot p g^{\mu\nu}) + 4m^2 g^{\mu\nu}$$

$$\mathrm{tr}\big\{(\not p' + m)\gamma^\mu\gamma^\rho(\not p + m)\gamma^\nu\big\} = 4m(p'^\mu g^{\rho\nu} + p'^\nu g^{\rho\mu} - p'^\rho g^{\mu\nu}) + 4m(p^\nu g^{\mu\rho} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu})$$

$$\begin{aligned}\mathrm{tr}\big\{(\not p' + m)\sigma^{\mu\rho}(\not p + m)\gamma^\nu\big\} &= \frac{i}{2}4m\{[p'^\mu g^{\rho\nu} + p'^\nu g^{\rho\mu} - p'^\rho g^{\mu\nu} + p^\nu g^{\mu\rho} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu}] \\ &\quad - [p'^\rho g^{\mu\nu} + p'^\nu g^{\mu\rho} - p'^\mu g^{\rho\nu} + p^\nu g^{\rho\mu} + p^\mu g^{\rho\nu} - p^\rho g^{\mu\nu}]\} \\ &= 4m\frac{i}{2}\{2p'^\mu g^{\rho\nu} - 2p'^\rho g^{\mu\nu} + 2p^\rho g^{\mu\nu} - 2p^\mu g^{\rho\nu}\} \\ &= 4mi\{p'^\mu g^{\rho\nu} - p'^\rho g^{\mu\nu} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu}\}\end{aligned}$$

$$\begin{aligned}&\mathrm{tr}\big\{(\not p' + m)\sigma^{\mu\rho}(\not p + m)\gamma^\nu\big\} - \mathrm{tr}\big\{(\not p' + m)\gamma^\mu(\not p + m)\sigma^{\nu\rho}\big\} \\ =&\,\mathrm{tr}\big\{(\not p' + m)\sigma^{\mu\rho}(\not p + m)\gamma^\nu\big\} - \mathrm{tr}\big\{(\not p + m)\sigma^{\nu\rho}(\not p' + m)\gamma^\mu\big\} \\ =&\,4mi\{[p'^\mu g^{\rho\nu} - p'^\rho g^{\mu\nu} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu}] - [p^\nu g^{\rho\mu} - p^\rho g^{\mu\nu} + p'^\rho g^{\mu\nu} - p'^\nu g^{\rho\mu}]\} \\ =&\,4mi\{p'^\mu g^{\rho\nu} - 2p'^\rho g^{\mu\nu} + 2p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu} - p^\nu g^{\rho\mu} + p'^\nu g^{\rho\mu}\}\end{aligned}$$

$$\begin{aligned}\mathrm{tr}\left\{(\not p' + m)\frac{\sigma^{\mu\rho}q_\rho}{2m}(\not p + m)\frac{\sigma^{\nu\sigma}q_\sigma}{2m}\right\} &= -4(p'^\mu p^\nu g^{\rho\sigma} - p'^\mu p^\sigma g^{\rho\nu} + p'^\rho p^\sigma g^{\mu\nu} - p'^\rho p^\nu g^{\mu\sigma} + p'\cdot p g^{\mu\sigma}g^{\rho\nu} - p'\cdot p g^{\mu\nu}g^{\rho\sigma} + p'^\nu p^\mu g^{\rho\sigma} \\ &\quad - p'^\nu p^\rho g^{\mu\sigma} + p'^\sigma p^\rho g^{\mu\nu} - p'^\sigma p^\mu g^{\rho\nu})\frac{q_\rho}{2m}\frac{q_\sigma}{2m} + 4m^2(g^{\mu\nu}g^{\rho\sigma} - g^{\rho\nu}g^{\mu\sigma})\frac{q_\rho}{2m}\frac{q_\sigma}{2m} \\ &= -\frac{1}{m^2}(p'^\mu p^\nu q^2 - p'^\mu(p\cdot q)q^\nu + (p'\cdot q)(p\cdot q)g^{\mu\nu} - (p'\cdot q)p^\nu q^\mu + (p'\cdot p)q^\mu q^\nu - (p'\cdot p)g^{\mu\nu}q^2 \\ &\quad + p'^\nu p^\mu q^2 - p'^\nu(p\cdot q)q^\mu + (p'\cdot q)(p\cdot q)g^{\mu\nu} - (p'\cdot q)p^\mu q^\nu) + (g^{\mu\nu}q^2 - q^\nu q^\mu) \\ &= -\frac{1}{m^2}(p'^\mu p^\nu q^2 - p'^\mu(p\cdot q)q^\nu + 2(p'\cdot q)(p\cdot q)g^{\mu\nu} - (p'\cdot q)p^\nu q^\mu + (p'\cdot p)q^\mu q^\nu - (p'\cdot p)g^{\mu\nu}q^2 \\ &\quad + p'^\nu p^\mu q^2 - p'^\nu(p\cdot q)q^\mu - (p'\cdot q)p^\mu q^\nu) + (g^{\mu\nu}q^2 - q^\nu q^\mu)\end{aligned}$$

(Trace of six gamma matrices: (the Latin letters are just for convienece)

$$\begin{aligned}\mathrm{tr}\big\{\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f\big\} &= 2g^{ab}\,\mathrm{tr}\big\{\gamma^c\gamma^d\gamma^e\gamma^f\big\} - 2g^{ac}\,\mathrm{tr}\big\{\gamma^b\gamma^d\gamma^e\gamma^f\big\} + 2g^{ad}\,\mathrm{tr}\big\{\gamma^b\gamma^c\gamma^e\gamma^f\big\} \\ &\quad - 2g^{ae}\,\mathrm{tr}\big\{\gamma^b\gamma^c\gamma^d\gamma^f\big\} + 2g^{af}\,\mathrm{tr}\big\{\gamma^b\gamma^c\gamma^d\gamma^e\big\} - \mathrm{tr}\big\{\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f\big\} \\ &= 4\{g^{ab}(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}) - g^{ac}(g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}) + g^{ad}(g^{bc}g^{ef} - g^{be}g^{cf} + g^{bf}g^{ce}) \\ &\quad - g^{ae}(g^{bc}g^{df} - g^{bd}g^{cf} + g^{bf}g^{cd}) + g^{af}(g^{bc}g^{de} - g^{bd}g^{ce} + g^{be}g^{cd})\}\end{aligned}$$

and what we need is a and d to contract with momentum and b, c & e, f reversed and cancelled, so

$$\begin{aligned}
\frac{1}{4}\operatorname{tr}\{\not p'[\gamma^b,\gamma^c]\not p[\gamma^e,\gamma^f]\} &= p'_a p_d\{[g^{ab}(g^{cd}g^{ef}-g^{ce}g^{df}+g^{cf}g^{de})-g^{ac}(g^{bd}g^{ef}-g^{be}g^{df}+g^{bf}g^{de})+g^{ad}(g^{bc}g^{ef}-g^{be}g^{cf}+g^{bf}g^{ce}) \\
&-g^{ae}(g^{bc}g^{df}-g^{bd}g^{cf}+g^{bf}g^{cd})+g^{af}(g^{bc}g^{de}-g^{bd}g^{ce}+g^{be}g^{cd})] \\
&+[-g^{ac}(g^{bd}g^{ef}-g^{be}g^{df}+g^{bf}g^{de})+g^{ab}(g^{cd}g^{ef}-g^{ce}g^{df}+g^{cf}g^{de})-g^{ad}(g^{cb}g^{ef}-g^{ce}g^{bf}+g^{cf}g^{be}) \\
&+g^{ae}(g^{cb}g^{df}-g^{cd}g^{bf}+g^{cf}g^{bd})-g^{af}(g^{cb}g^{de}-g^{cd}g^{be}+g^{ce}g^{bd})] \\
&+[-g^{ab}(g^{cd}g^{fe}-g^{cf}g^{de}+g^{ce}g^{df})+g^{ac}(g^{bd}g^{fe}-g^{bf}g^{de}+g^{be}g^{df})-g^{ad}(g^{bc}g^{fe}-g^{bf}g^{ce}+g^{be}g^{cf}) \\
&+g^{af}(g^{bc}g^{de}-g^{bd}g^{ce}+g^{be}g^{cd})-g^{ae}(g^{bc}g^{df}-g^{bd}g^{cf}+g^{bf}g^{cd})] \\
&+[g^{ac}(g^{bd}g^{fe}-g^{bf}g^{de}+g^{be}g^{df})-g^{ab}(g^{cd}g^{fe}-g^{cf}g^{de}+g^{ce}g^{df})+g^{ad}(g^{cb}g^{fe}-g^{cf}g^{be}+g^{ce}g^{bf}) \\
&-g^{af}(g^{cb}g^{de}-g^{cd}g^{be}+g^{ce}g^{bd})+g^{ae}(g^{cb}g^{df}-g^{cd}g^{bf}+g^{cf}g^{bd})]\} \\
&= p'_a p_d\{4g^{ab}g^{cf}g^{de}-4g^{ab}g^{ce}g^{df}+4g^{ac}g^{be}g^{df}-4g^{ac}g^{bf}g^{de}+4g^{ad}g^{bf}g^{ce}-4g^{ad}g^{be}g^{cf} \\
&+4g^{ae}g^{bd}g^{cf}-4g^{ae}g^{bf}g^{cd}+4g^{af}g^{be}g^{cd}-4g^{af}g^{bd}g^{ce}\} \\
&= 4(p'^b p^e g^{cf}-p'^b p^f g^{ce}+p'^c p^f g^{be}-p'^c p^e g^{bf}+p'\cdot p g^{bf}g^{ce}-p'\cdot p g^{be}g^{cf}+p'^e p^b g^{cf} \\
&-p'^e p^c g^{bf}+p'^f p^c g^{be}-p'^f p^b g^{ce})
\end{aligned}$$

Trace of two sigma bilinear:

$$\begin{aligned}
\operatorname{tr}\{\sigma^{\mu\rho}\sigma^{\nu\sigma}\} &= -\frac{1}{4}\{\operatorname{tr}\{\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma\}-\operatorname{tr}\{\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma\}-\operatorname{tr}\{\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu\}+\operatorname{tr}\{\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\}\} \\
&= -\{(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\nu})-(g^{\mu\rho}g^{\nu\sigma}-g^{\rho\nu}g^{\mu\sigma}+g^{\rho\sigma}g^{\mu\nu})-(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\rho\nu}+g^{\mu\nu}g^{\rho\sigma}) \\
&+(g^{\mu\rho}g^{\nu\sigma}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma})\} \\
&= 4(g^{\mu\nu}g^{\rho\sigma}-g^{\rho\nu}g^{\mu\sigma})
\end{aligned}$$

)

The invariant amplitude

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{e^4}{q^4}(k'_\mu k_\nu+k_\mu k'_\nu-k'\cdot k g_{\mu\nu})\Big\{(p'^\mu p^\nu+p^\mu p'^\nu-p'\cdot p g^{\mu\nu}+m^2 g^{\mu\nu})F_1^2 \\
&-m(p'^\mu g^{\rho\nu}-2p'^\rho g^{\mu\nu}+2p^\rho g^{\mu\nu}-p^\mu g^{\rho\nu}-p^\nu g^{\rho\mu}+p'^\nu g^{\rho\mu})\frac{q_\rho}{2m}F_1F_2 \\
&+[-\frac{1}{m^2}(p'^\mu p^\nu q^2-p'^\mu(p\cdot q)q^\nu+2(p'\cdot q)(p\cdot q)g^{\mu\nu}-(p'\cdot q)p^\nu q^\mu+(p'\cdot p)q^\mu q^\nu-(p'\cdot p)g^{\mu\nu}q^2 \\
&+p'^\nu p^\mu q^2-p'^\nu(p\cdot q)q^\mu-(p'\cdot q)p^\mu q^\nu)+(g^{\mu\nu}q^2-q^\nu q^\mu)]F_2^2\Big\} \\
&= \frac{e^4}{q^4}\Big\{[(k'\cdot p')(k\cdot p)+(k'\cdot p)(k\cdot p')-(p'\cdot p)(k'\cdot k)+m^2(k'\cdot k)+(k\cdot p')(k'\cdot p)+(k\cdot p)(k'\cdot p')-(p\cdot p')(k\cdot k') \\
&+m^2(k\cdot k')-2(k\cdot k')(p\cdot p')+4(k\cdot k')(p\cdot p')-4m^2(k\cdot k')]F_1^2-[(k'\cdot p')k^\rho-2p'^\rho(k'\cdot k)+2p^\rho(k'\cdot k)-(p\cdot k')k^\rho \\
&-k'^\rho(p\cdot k)+k'^\rho(k\cdot p')+(p'\cdot k)k'^\rho-2p'^\rho(k\cdot k')+2p^\rho(k\cdot k')-(p\cdot k)k'^\rho-(p\cdot k')k^\rho+(p'\cdot k')k^\rho-(k'\cdot k)p'^\rho \\
&+8(k'\cdot k)p'^\rho-8(k'\cdot k)p^\rho+(k'\cdot k)p^\rho+(k'\cdot k)p^\rho-(k'\cdot k)p'^\rho]\frac{q_\rho}{2}F_1F_2-\frac{1}{m^2}\Big\} \\
&= \frac{e^4}{q^4}\Big\{[2(k'\cdot p')(k\cdot p)+2(k'\cdot p)(k\cdot p')-2m^2(k'\cdot k)]F_1^2-[2(k'\cdot p')k^\rho+2(k\cdot k')p'^\rho-2(k\cdot k')p^\rho-2(p\cdot k')k^\rho-(p\cdot k)k'^\rho \\
&-k'^\rho(p\cdot k)+k'^\rho(k\cdot p')+(p'\cdot k)k'^\rho]\frac{q_\rho}{2}F_1F_2\Big\}
\end{aligned}$$

7.3 Consider

$$H_{int} = \int \mathrm{d}^3x\frac{\lambda}{\sqrt{2}}\phi\bar{\psi}\psi+\int \mathrm{d}^3x e A_\mu\bar{\psi}\gamma^\mu\psi$$