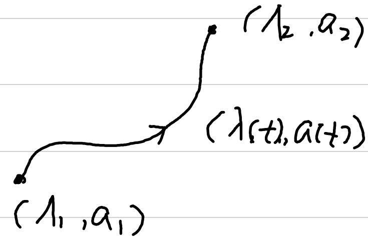
# 分立时空对称性与正反粒子对称性（4维）

## Poincaré 群的四个连通分支

$(\Lambda_1, a_1) \to (\Lambda_2, a_2)$




是否存在连续函数 $\lambda(t)$ 和 $a(t)$, s.t.

$$\begin{cases} \lambda(0) = \Lambda_1 \\ a(0) = a_1 \end{cases} \qquad \begin{cases} \lambda(1) = \Lambda_2 \\ a(1) = a_2 \end{cases}$$

如果存在，则 $(\Lambda_1, a_1)$ 与 $(\Lambda_2, a_2)$ 处在同一连通分支中，

考虑

$$\Lambda = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{30} & \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} \qquad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\because \ \Lambda^T G \Lambda = G$

$\therefore \ (\det \Lambda)^2 = 1 \qquad \therefore \ \det \Lambda = \pm 1$

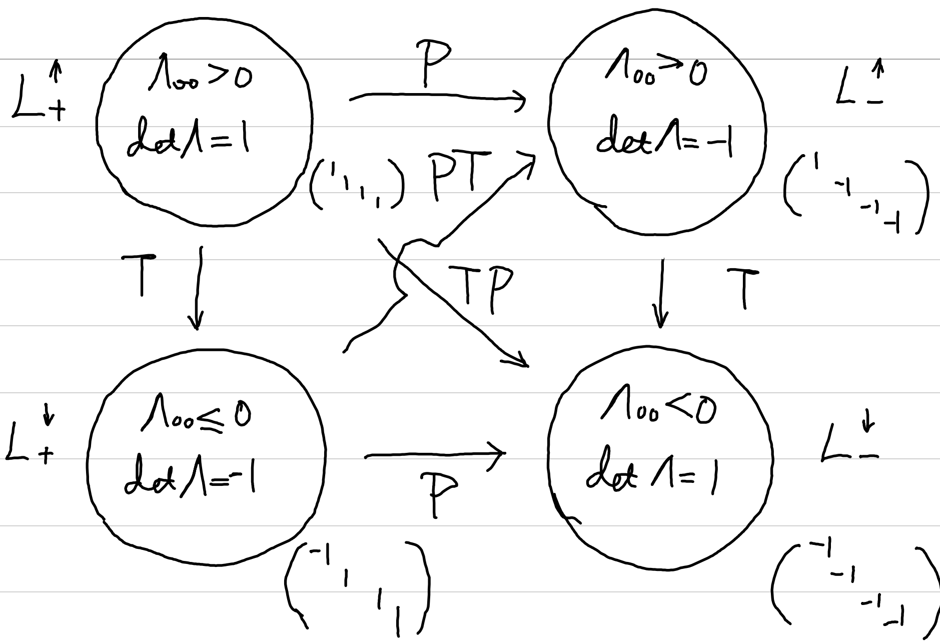
显然，$\det \lambda(t)$ 是连续函数 $\therefore$ 同一连通分支中的 $(\Lambda, a)$ $\det \Lambda$ 的符号一定相同（因为连续函数不能跳过 $(-1, 1)$ 从 $+1$ 直接变成 $-1$）

再考虑 $\Lambda^T G \Lambda = G$ 的第一行第一列：

$$\Lambda_{00}^2 - \Lambda_{11}^2 - \Lambda_{22}^2 - \Lambda_{33}^2 = 1$$

显然，若 $\Lambda_{\mu\nu} \in \mathbb{R}$，$|\Lambda_{00}| \geq 1 \quad \therefore \Lambda_{00} = 1$ 和 $\Lambda_{00} = -1$ 在不同的连通分支！

于是，Poincaré 群的元素被 $\det \Lambda$ 和 $\mathrm{sgn}\, \Lambda_{00}$ 分为 4 个连通分支。




| | | | |
|---|---|---|---|
| $\Lambda^0{}_0 \geq 1$ , | orthochronous | 正时的 | $L^\uparrow = L_+^\uparrow \cup L_-^\uparrow$ |
| $\det\Lambda = +1$ , | proper | 正常的 | $L^+ = L_+^\uparrow \cup L_+^\downarrow$ |
| $\mathrm{sgn}\Lambda^0{}_0 \det\Lambda = +1$ | orthochorous | 正统的(?) | $L_0 = L_+^\uparrow \cup L_-^\downarrow$ |
| $L_+^\uparrow$ | restricted | 限制的(?) | |

联系不同连通分支的(标准)Lorentz变换，称为分立Lorentz对称变换：

空间反射 $$P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

时间反演 $$T \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

量子物理中的对称性 ⇒ 态矢量空间中的变换.

Wigner定理: 保证两个态之间跃迁概率不变(不需要概率幅不变)的变换, 一定可以表示为态矢量空间中的幺正或反幺正变换!

$P: \ |\psi\rangle \to P|\psi\rangle$ , $T: |\psi\rangle \to T|\psi\rangle$

满足 $(P\varphi, P\psi) = (\varphi, \psi)$ 或 $(P\varphi, P\psi) = (\psi, \varphi)$

$(T\varphi, T\psi) = (\varphi, \psi)$ 或 $(T\varphi, T\psi) = (\psi, \varphi)$

待定!

(1) 空间反射变换.

$P$ 改变 3 动量方向, 但不改变自旋方向.

$$\therefore \begin{cases} P a^s_{\vec{p}} P^{-1} = \eta_a a^s_{-\vec{p}} & |\eta_a|^2 = 1 \\ P b^s_{\vec{p}} P^{-1} = \eta_b b^s_{-\vec{p}} & |\eta_b|^2 = 1 \end{cases}$$

而 $PP a^s_{\vec{p}} P^{-1} P^{-1} = \eta_a^2 a^s_{\vec{p}}$ $\quad \therefore \quad \eta_a^2, \eta_b^2 = \pm 1$ ($\because$ 费米子算符)

$\therefore$

$$\psi(x) \to P\psi(x)P^{-1} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( \eta_a a^s_{-\vec{p}} u^s(p) e^{-ipx} + \eta_b^* b^{s\dagger}_{-\vec{p}} v^s(p) e^{ipx} \right)$$

积分变换 $p \to \tilde{p} = (p^0, -\vec{p})$ $\qquad p\cdot x = \tilde{p}\cdot(t, -\vec{x})$

$\tilde{p}\cdot\sigma = p\cdot\bar{\sigma}$ $\qquad \tilde{p}\cdot\bar{\sigma} = p\cdot\sigma$

$$\therefore u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi \\ \sqrt{\tilde{p}\cdot\sigma}\,\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} u(\tilde{p}) = \gamma^0 u(\tilde{p})$$

$$v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi \\ -\sqrt{\tilde{p}\cdot\sigma}\,\xi \end{pmatrix} = -\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} v(\tilde{p}) = -\gamma^0 v(\tilde{p})$$

$$\therefore P\psi(x)P^{-1}=\int\frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s\Big[\eta_a a^s_{\tilde{p}}\gamma^0 u(\tilde{p})e^{-i\tilde{p}(t,-\vec{x})}$$
$$-\eta_b^* b^{s+}_{\tilde{p}}\gamma^0 v(\tilde{p})e^{i\tilde{p}(t,-\vec{x})}\Big]$$

$\psi(x)$ 在 $P$ 变换下应变为

$$\psi(x)\to\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)=\Lambda_{\frac{1}{2}}(P)\cdot\psi(t,-\vec{x})$$

显然我们已经将 $P\psi(x)P^{-1}$ 写为对 $(t,-\vec{x})$ 的依赖形式，
如果选取 $\eta_b^*=-\eta_a$ （即 $\eta_a\eta_b=\eta_a(-\eta_a)^*=-1$ ）
则

$$P\psi(x)P^{-1}=\eta^a\gamma^0\psi(t,-\vec{x})$$

也就是说，$\Lambda_{\frac{1}{2}}(P)=\eta^a\gamma^0$.

对于双线性型，$\bar{\psi}(x)\psi(x)$，$\bar{\psi}(x)\gamma^\mu\psi(x)$，$i\bar{\psi}(x)[\gamma^\mu,\gamma^\nu]\psi(x)$，
$\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$，$i\bar{\psi}\gamma_5\psi(x)$，它们有各自的 $P$ 变换性质.

例1：$$P\bar{\psi}(x)P^{-1}=P\psi^+(x)\gamma^0P^{-1}=P\psi^+(x)P^{-1}\gamma^0$$
$$=(P\psi(x)P^{-1})^+\gamma^0=\eta_a^*\bar{\psi}(t,-\vec{x})\gamma^0$$

例2：$$P\bar{\psi}(x)\psi(x)P^{-1}=P\bar{\psi}(x)P^{-1}P\psi(x)P^{-1}$$
$$=\eta_a^*\bar{\psi}(t,-\vec{x})\gamma^0\gamma^0\eta_a\psi(t,-\vec{x})$$
$$=\bar{\psi}(t,-\vec{x})\psi(t,-\vec{x})$$

例3：$$iP\bar{\psi}(x)\gamma_5\psi(x)P^{-1}=iP\bar{\psi}(x)P^{-1}\gamma_5P\psi(x)P^{-1}$$
$$=i\eta_a^*\bar{\psi}(t,-\vec{x})\gamma^0\gamma_5\gamma^0\psi(t,-\vec{x})\eta_a$$
$$=-i\bar{\psi}(t,-\vec{x})\gamma_5\psi(t,-\vec{x})$$

例4：复标量场.

$$\phi(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ipx}+b_p^+e^{ipx}\right)$$

$$P a_p P^{-1}=\eta_a a_{-p} \qquad P b_b P^{-1}=\eta_b b_{-p}.$$

$\therefore\ |\eta_a|=|\eta_b|=\eta_a^2=\eta_b^2=1$

$(\because P^2 a_p P^{-2}=a_p$，而 $P^2 a_p P^{-2}=\eta_a^2 a_p\ \therefore \eta_a^2=+1)$

$$\therefore P\phi(x)P^{-1}=\int\frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\left(\eta_a a_{\tilde{p}}e^{-i\tilde{p}\cdot(t,-\vec{x})}+\eta_b^* b_p^+ e^{i\tilde{p}(t,-\vec{x})}\right)$$

$$=\eta_a\phi(t,-\vec{x}) \qquad (\text{选取 } \eta_b^*=\eta_a)$$

例5：$P\bar{\psi}(x)\gamma^\mu\psi(x)P^{-1}=P\bar{\psi}(x)P^{-1}\gamma^\mu P\psi(x)P^{-1}$

$$=\eta_a^*\bar{\psi}(t,-\vec{x})\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x})\eta_a$$

$$=\bar{\psi}(t,-\vec{x})\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x})$$

对 $\mu=0$，$\gamma^0\gamma^\mu\gamma^0=\gamma^0=+\gamma^\mu$

对 $\mu\neq0$，$\gamma^0\gamma^\mu\gamma^0=-\gamma^\mu\gamma^0\gamma^0=-\gamma^\mu$

$$\therefore P\bar{\psi}(x)\gamma^\mu\psi(x)P^{-1}=\begin{cases}\bar{\psi}(t,-\vec{x})\gamma^\mu\psi(t,-\vec{x}) & \mu=0\\ -\bar{\psi}(t,-\vec{x})\gamma^\mu\psi(t,-\vec{x}) & \mu=1,2,3\end{cases}$$

$\therefore$ 如果记 $J^\mu(x)=\bar{\psi}(x)\gamma^\mu\psi(x)$，

则 $PJ^\mu(x)P^{-1}=P^\mu{}_\nu J^\nu(t,-\vec{x}) \qquad P^\mu{}_\nu=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$

$\therefore J^\mu(x)$ 是 Lorentz 矢量.

练习：证明 $P\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)P^{-1}=-P^\mu{}_\nu\bar{\psi}(t,-\vec{x})\gamma^\nu\gamma_5\psi(t,-\vec{x})$

即 $J_5^\mu(x)\equiv\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$ 为 Lorentz 赝矢量.

可以证明，在双线性型中，$\eta_a$，$\eta_b$ 总以 $|\eta_a|^2$ 的形式出现

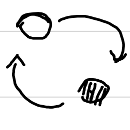
$\therefore\ \eta_a$ 本身的幅角没有物理可观测效应，不妨取为 1.

然而 $\eta_a \eta_b = -1$ 是重要的.

⇒ 费米子与其反粒子的P宇称差负号.

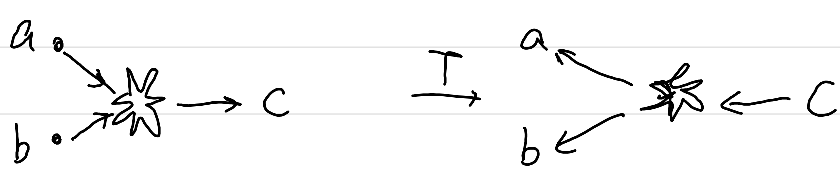
$$a_p^{s+} b_q^{r+}|0\rangle \to P(a_p^{s+} b_q^{r+}|0\rangle) = P a_p^{s+} P^{-1} P b_q^{r+} P^{-1} P|0\rangle$$
$$= \eta_a^* \eta_b^* a_{-p}^{s+} b_{-p}^{r+}|0\rangle$$

∴ 由 $f\bar{f}$ 组成的复合系统, 在P变换下有额外的(−1)因子.

∴  系统, $L=0$ (对称波函数) → 奇宇称 (赝标粒子)

$L=1$ (反对称波函数) → 偶宇称 (矢量粒子)

(2) 时间反演变换 T.




$(c, ab) = \langle c|ab\rangle$ $(ab, c) = \langle ab|c\rangle$

! 交换了顺序 反幺正变换?

$[T, H] = 0$

若 $T\psi(x)T^{-1} = \Lambda_{\pm}(T)\cdot\psi(-t,\vec{x})$, 则 $\psi(-t,\vec{x})$ 中全为负频解.

$T|\psi\rangle$ 在$(-t)$时刻看起来和 $t$时刻的$|\psi\rangle$一样的态!

即.

$\mathcal{O}(x)$: $e^{iHt}\,\mathcal{O}(\vec{x})\,e^{-iHt}$

$T\mathcal{O}(x)T^{-1}$: $e^{-iHt}\,T\mathcal{O}(\vec{x})T^{-1}\,e^{iHt}$

$\because [T, H] = 0 \quad \therefore \quad T i = -iT \longrightarrow$ 反幺正算符.

$T$ 改变动量方向，也改变自旋方向

$\left( \vec{p}' = \frac{d\vec{x}}{d(-t)} = -\vec{p} \,,\quad \vec{L}' = \vec{x}' \times \vec{p}' = \vec{x} \times (-\vec{p}) = -\vec{L} \right)$

$\therefore \quad T a^{s}_{\vec{p}} T^{-1} = \eta_a \, a^{-s}_{-\vec{p}} \qquad T b^{s}_{\vec{p}} T^{-1} = \eta_b \, b^{-s}_{-\vec{p}}$

此时我们记 $\xi^{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi^{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

对于沿 $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ 方向的自旋本征态

$\because \vec{n} \cdot \vec{\sigma} \, \xi_{\pm} = \pm \xi_{\pm} \,, \Rightarrow$

$$\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \xi_{\pm 1} \\ \xi_{\pm 2} \end{pmatrix} = \pm \begin{pmatrix} \xi_{\pm 1} \\ \xi_{\pm 2} \end{pmatrix}$$

$\therefore$ "+" $\begin{pmatrix} \cos\theta - 1 & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -1-\cos\theta \end{pmatrix} = 2 \begin{pmatrix} -\sin^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -\cos^2\frac{\theta}{2} \end{pmatrix}$

$\therefore \xi_{+} \propto \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$

同理： $\xi_{-} \propto \begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$

定义 $\xi_{+} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \qquad \xi_{-} = \begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$

则 $\xi_{+} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi_{-}^{*} \,,\quad \xi_{-} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi_{+}^{*}$

$\therefore \xi_{\pm} = (-i\sigma^{2}) \, \xi_{\mp}^{*}$

更一般地

$(\vec{n}\cdot\vec{\sigma})(-i\sigma^2\xi^*) = -i(\vec{n}\cdot\vec{\sigma})\cdot\sigma^2\xi^*$

而 $\sigma^1\cdot\sigma^2 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}i&0\\0&-i\end{pmatrix} = -\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = -\sigma^2\cdot(\sigma^1)^*$

$\sigma^2\cdot\sigma^2 = -\sigma^2\cdot\sigma^{2*}$

$\sigma^3\cdot\sigma^2 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&-i\\-i&0\end{pmatrix} = -\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = -\sigma^2\cdot(\sigma^3)^*$

$\therefore -i(\vec{n}\cdot\vec{\sigma})\cdot\sigma^2 = i\sigma^2(\vec{n}\cdot\vec{\sigma}^*)$

$\therefore (\vec{n}\cdot\vec{\sigma})(-i\sigma^2\xi^*) = -i(\vec{n}\cdot\vec{\sigma})\cdot\sigma^2\xi^* = i\sigma^2(\vec{n}\cdot\vec{\sigma}^*)\xi^*$

$= -i\sigma^2(-\vec{n}\cdot\vec{\sigma}^*)\xi^*$

$\therefore$ if $\vec{n}\cdot\vec{\sigma}\,\xi = \lambda\xi$

则 $(\vec{n}\cdot\vec{\sigma})(-i\sigma^2\xi^*) = -i\sigma^2(-\vec{n}\cdot\vec{\sigma}^*)\xi^* = -i\sigma^2(-(\vec{n}\cdot\vec{\sigma}\xi)^*)$

$= (-\lambda)\cdot(-i\sigma^2\xi^*)$

$\therefore T\psi(x)T^{-1} = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s T\left(a_p^s u^s(p)e^{-ipx} + b_p^s v^s(p)e^{ipx}\right)T^{-1}$

由于 T 为反幺正算符, $T(\alpha a_p^s)T^{-1} = \alpha^* T a_p^s T^{-1}$

$\therefore T\psi(x)T^{-1} = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s u^s(p)^* e^{ipx}\, T a_p^s T^{-1} + v^s(p)^* e^{-ipx}\, T b_p^s T^{-1}$

而 $(-t,\vec{x})\cdot\tilde{p} = -p\cdot x$

$\therefore T\psi(x)T^{-1} = \int\frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s\Big(u^s(p^*)e^{-i\tilde{p}(t,-\vec{x})}\, T a_p^s T^{-1}$

$+ v^s(p^*)e^{i\tilde{p}(t,-\vec{x})}\, T b_p^s T^{-1}\Big)$

$\therefore$ 需要将 $u^s(p)^*$ 和 $v^s(p)^*$ 写为 $\tilde{p}$ 的形式.

$$u^s(p)^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}^* = \begin{pmatrix} \sqrt{p\cdot\sigma^*}\,\xi^{s*} \\ \sqrt{p\cdot\bar\sigma^*}\,\xi^{s*} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma^*}\; i\sigma^2 \cdot (-i\sigma^2 \xi^{s*}) \\ \sqrt{p\cdot\bar\sigma^*}\; i\sigma^2\, (-i\sigma^2 \xi^{s*}) \end{pmatrix}$$

$$= \begin{pmatrix} (i\sigma^2)\sqrt{\tilde p\cdot\sigma}\,(-i\sigma^2\xi^{s*}) \\ (i\sigma^2)\sqrt{\tilde p\cdot\bar\sigma}\,(-i\sigma^2\xi^{s*}) \end{pmatrix} \qquad p\to\tilde p \;\because\; \sigma^0\sigma^2 = +\sigma^2\sigma^{\vec 0}$$

$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\tilde p\cdot\sigma}\,\xi^{-s} \\ \sqrt{\tilde p\cdot\bar\sigma}\,\xi^{-s} \end{pmatrix}$$

$$= i \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \cdot u^{-s}(\tilde p)$$

$$\therefore\; u^{-s}(\tilde p) = -\gamma^1\gamma^3 [u^s(p)]^*$$

类似地, $v^{-s}(\tilde p) = +\gamma^1\gamma^3 [v^s(p)]^*$

而 $T a_p^s T^{-1} = \eta_a a_{-p}^{-s} \qquad T b_p^s T^{-1} = \eta_b b_{-p}^{-s}$

$$\therefore\; T\psi(x)T^{-1} = \int \frac{d^3\tilde p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde p}}}\sum_s \Big[(-\gamma^1\gamma^3)\eta_a\, a_{\tilde p}^{-s} u^{-s}(\tilde p) e^{-i\tilde p(-t,\vec x)}$$
$$- (-\gamma^1\gamma^3)\eta_b\, b_{\tilde p}^{-s} v^{-s}(\tilde p) e^{i\tilde p(-t,\vec x)}\Big]$$
$$= (-\gamma^1\gamma^3)\,\psi(-t,\vec x) \qquad \eta_b = -\eta_a$$

类似的例子, ……

(3) 正反粒子变换. C

C 不是时空对称变换! 将粒子变为反粒子.

$$C a_p^s C^{-1} = b_p^s \qquad C b_p^s C^{-1} = a_p^s$$

$$\therefore C\psi(x)C^{-1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left( u^s(p)e^{-ipx} b_p^s + v^s(p)e^{ipx} a_p^{s\dagger} \right)$$

∴ 需要将 $v^s(p)$ 以 $u^s(p)^*$ 表示

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \qquad u^s(p)^* = \begin{pmatrix} \sqrt{p\cdot\sigma^*}\,\xi^{s*} \\ \sqrt{p\cdot\bar\sigma^*}\,\xi^{s*} \end{pmatrix}$$

$$\therefore u^s(p)^* = \begin{pmatrix} \sqrt{p\cdot\sigma^*}\,(i\sigma^2)(-i\sigma^2)\,\xi^{s*} \\ \sqrt{p\cdot\bar\sigma^*}\,(i\sigma^2)(-i\sigma^2)\,\xi^{s*} \end{pmatrix}$$

$$= \begin{pmatrix} i\sigma^2\sqrt{p\cdot\bar\sigma}\,\xi^{-s} \\ i\sigma^2\sqrt{p\cdot\sigma}\,\xi^{-s} \end{pmatrix}$$

!!! $$v^s(p) \equiv \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\,\xi^{-s} \end{pmatrix}$$

$$\therefore u^s(p)^* = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} v^s(p) = -i\gamma^2 v^s(p)$$

$$v^s(p)^* = -i\gamma^2 u^s(p)$$

$$\therefore C\psi(x)C^{-1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s (-i\gamma^2)\left[ v^s(p)^* b_p^s e^{-ipx} + u^s(p)^* a_p^{s\dagger} e^{ipx} \right]$$

$$= -i\gamma^2\psi(x)^* = -i\gamma^2(\psi^\dagger)^T = -i\gamma^2(\bar\psi\gamma^0)^T$$

$$= -i(\bar\psi\gamma^0\gamma^2)^T$$

CPT定理:

$\Theta \equiv PCT.$

$\forall \psi, \varphi,\qquad (\Theta\psi, \Theta\varphi) \equiv (\varphi, \psi)$

$$\begin{aligned}\Theta \psi(x) \Theta^{-1} &= CPT\ \psi(x) T^{-1} P^{-1} C^{-1} \\ &= CP(-\gamma^1\gamma^3)\psi(-t, \vec{x}) P^{-1} C^{-1} \\ &= C(-\gamma^1\gamma^3)\gamma^0 \psi(-t, -\vec{x}) C^{-1} \\ &= (-\gamma^1\gamma^3\gamma^0)\ C\psi(-x) C^{-1} \\ &= i\gamma^1\gamma^3\gamma^0\gamma^2 \cdot \psi(-x)^* \\ &= -i\gamma^0\gamma^1\gamma^2\gamma^3\ \psi(-x)^* \quad = -\gamma_5 \psi(-x)^*\end{aligned}$$

对于带任意 Lorentz 指标的场(们)

$\varphi_\mu, \cdots \psi_\nu$

$$\langle 0 | \varphi_\mu(x_1) \cdots \psi_\nu(x_n) | 0 \rangle \equiv i^F (-1)^J \langle 0 | \psi_\nu(-x_n) \cdots \varphi_\mu(-x_1) | 0 \rangle$$

(只要满足弱微观因果性条件)

其中, F 为费米子算符的数目, J 为"左手指标"的数目

为了得到标量, $2 | J + F/2$