Homework: Gauge Field Theory #1

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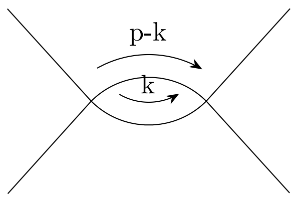
1. ϕ^4 theory ($\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$). Verify optical theorem in the lowest order.



$$= \frac{(-i\lambda)^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p-k)^2-m^2+i\epsilon}$$

For simplicity, we ignore the mass term.

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2(p-k)^2}$$

Apply feynman parameterization

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{[x(p-k)^2+(1-x)k^2]^2}$$

$k \to k + xp$

$$= \frac{(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{[k^2+x(1-x)p^2+i\epsilon]^2}$$

Set $\Delta \equiv -x(1-x)p^2 + i\epsilon$, and apply wick rotation

$$i\mathcal{M}_2 = \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2}$$

Dimensional regularization

$$\begin{aligned}
i\mathcal{M}_2 &= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{1}{[k_E^2+\Delta]^2} \\
&= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\mathrm{d}k_E\frac{k_E^{d-1}}{[k_E^2+\Delta]^2} \\
&= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \frac{\pi^{d/2}\Gamma(2-d/2)}{\Gamma(2)(2\pi)^d}\Delta^{d/2-2} \\
&\xrightarrow{d\to 4} -i\lambda^2\frac{\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)}{32\pi^2}\int_0^1 \mathrm{d}x(\frac{\Delta}{4\pi})^{-\epsilon/2} \\
&= -i\lambda^2\frac{\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)}{32\pi^2}\int_0^1 \mathrm{d}x(1-\frac{\epsilon}{2}\ln\frac{\Delta}{4\pi}) \\
&= \frac{-i\lambda^2}{32\pi^2}(\frac{2}{\epsilon}-\gamma+2-\ln\left(-p^2\right)+\ln(4\pi)+\mathcal{O}(\epsilon))
\end{aligned}$$

where $\epsilon = 4-d$.

So

$$i\mathcal{M}(s) = -i\lambda + \frac{-i\lambda^2}{32\pi^2}(\frac{2}{\epsilon}-\gamma+2-\ln(-s)+\ln(4\pi))$$

$$\mathcal{M}(s) = -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \gamma + 2 - \ln(-s) + \ln(4\pi)) = -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s) + finite\ terms)$$

where $finite\ terms = \ln(4\pi) + 2 - \gamma$.

$$\lambda_R = \lambda + \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms)$$

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms)$$

$$\begin{aligned}\mathcal{M}(s) &= -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s) + finite\ terms) \\ &= -\lambda_R + \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms) - \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s) + finite\ terms) \\ &= -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{s}\end{aligned}$$

As the lowest order, the results are always $-\lambda$.

Optical theorem concludes that

$$\frac{\lambda^2}{16\pi} = \int \mathrm{d}\Pi\lambda^2$$

where

$$\begin{aligned}\int \mathrm{d}\Pi\lambda^2 &= \int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 4E_1E_2}(2\pi)^4\delta^4(p - p_1 - p_2)\lambda^2 \\ &= \frac{1}{16\pi}\lambda^2\end{aligned}$$

2. Proca field, QED with massive photon. Calculate the leading order of $e^-e^- \to e^-e^-$.

The propagator

$$\langle 0|T\{A^\mu_{in}(x)A^\nu_{in}(y)\}|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i(-g^{\mu\nu} + \frac{k^\mu k^\nu}{\mu^2})}{k^2 - \mu^2 + i\epsilon} + \frac{i}{\mu^2}\delta^4(x-y)\delta^{\mu 0}\delta^{\nu 0}$$

The Lagarangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu + \bar{\psi}(i\not{D} - m)\psi$$

and the interaction part

$$\mathcal{L}_I = e\bar{\psi}\gamma^\mu\psi A_\mu$$

($\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu + \bar{\psi}(i\not{\partial} - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$). The corresponding Hamiltonian is

$$\mathcal{H}_I = A^\mu J_\mu + \frac{1}{2\mu^2}J_0^2 = -e\bar{\psi}\gamma^\mu\psi A_\mu + \frac{e^2}{2\mu^2}\bar{\psi}\gamma^0\psi\bar{\psi}\gamma_0\psi$$

and we have the propagator

$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i(-g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2})}{k^2 - \mu^2 + i\epsilon} + \frac{i}{\mu^2}\delta^4(x-y)\delta^0_\mu\delta^0_\nu$$

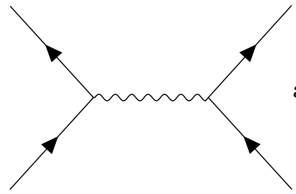
and

$$\langle k_1k_2|T\{-i\mathcal{H}_I\}|p_1p_2\rangle = i\,\langle k_1k_2|T\{e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{e^2}{2\mu^2}\bar{\psi}\gamma^0\psi\bar{\psi}\gamma_0\psi\}|p_1p_2\rangle$$

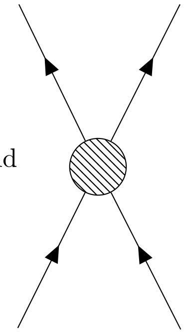
At tree level(to e^2 order), the first part must be

$$-e^2\,\langle k_1k_2|T\{\bar{\psi}\gamma^\mu\psi A_\mu\bar{\psi}\gamma^\nu\psi A_\nu\}|p_1p_2\rangle$$

so generally we have two diagrams



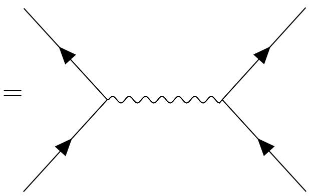
and



with some exchange in external legs.

The contribution of the first one is

$$-\frac{1}{e^2}i\mathcal{M}_1 =$$



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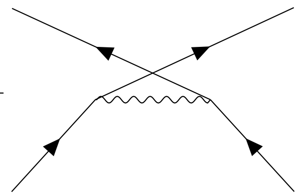

$$\begin{aligned}
=&\bar{u}(k_1)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}+\frac{i}{\mu^2}\delta^0_\mu\delta^0_\nu]\bar{u}(k_2)\gamma^\nu u(p_2)-\bar{u}(k_2)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}+\frac{i}{\mu^2}\delta^0_\mu\delta^0_\nu]\bar{u}(k_1)\gamma^\nu u(p_2)\\
=&\bar{u}(k_1)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}]\bar{u}(k_2)\gamma^\nu u(p_2)-\bar{u}(k_2)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}]\bar{u}(k_1)\gamma^\nu u(p_2)\\
&+\frac{i}{\mu^2}\bar{u}(k_1)\gamma^0 u(p_1)\bar{u}(k_2)\gamma^0 u(p_2)-\frac{i}{\mu^2}\bar{u}(k_2)\gamma^0 u(p_1)\bar{u}(k_1)\gamma^0 u(p_2)
\end{aligned}$$

and the second one is

$$i\mathcal{M}_2 =\frac{ie^2}{\mu^2}(\bar{u}(k_1)\gamma^0 u(p_1)\bar{u}(k_2)\gamma^0 u(p_2)-\bar{u}(k_2)\gamma^0 u(p_1)\bar{u}(k_1)\gamma^0 u(p_2))$$

Combine these two and the incovariant terms are automatically canceled.

3. Vacuum polarization of massive photon.