Note on Braaten's Paper

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1 Intro

Hamiltonian [1]:
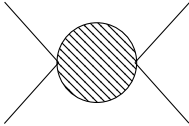
$$\mathcal{H} = \sum_\sigma \frac{1}{2m} \nabla \psi_\sigma^\dagger \cdot \nabla \psi_\sigma^{(\Lambda)} + \frac{g(\Lambda)}{m} \psi_1^\dagger \psi_2^\dagger \psi_3 \psi_4^{(\Lambda)} + \mathcal{V} \tag{1}$$

where the renormalized coupling

$$g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \tag{2}$$

2 Amplitude

Consider:

$$i\mathcal{A} = \langle 34 | \psi^\dagger \psi | 12 \rangle = \tag{3}$$

Define $P = p_1 + p_2 = (E, \mathbf{0})$, and $E = p^2/m$. The integral equation is

$$i\mathcal{A} = -\frac{ig(\Lambda)}{m} \left(1 + i\mathcal{A} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon} \frac{i}{k^0 - p^0 - \frac{|\mathbf{k}-\mathbf{p}|^2}{2m} + i\epsilon} \right) \tag{4}$$

The integral gives (redefine $\epsilon \to 2m\epsilon$)

$$\mathcal{I} = \frac{im}{2\pi^2} \left(-\Lambda + \sqrt{-mE - i\epsilon} \tan^{-1}\left(\frac{\Lambda}{\sqrt{-mE - i\epsilon}} \right) \right) = -\frac{i\Lambda m}{2\pi^2} + \frac{mp}{4\pi} \tag{5}$$

and

$$i\mathcal{A} = \frac{-1}{\mathcal{I} + \frac{m}{ig(\Lambda)}} = -\left[\frac{im\sqrt{-mE - i\epsilon} \tan^{-1}\left(\frac{\Lambda}{\sqrt{-mE - i\epsilon}} \right)}{2\pi^2} - \frac{im}{4\pi a} \right]^{-1} \tag{6}$$

$$\xrightarrow{\Lambda\to\infty} \frac{4i\pi/m}{-1/a + \sqrt{-mE - i\epsilon}} \tag{7}$$

Note that by definition, scattering length is the leading order momentum expansion of $1/\mathcal{A}$, which gives

$$\frac{1}{a} = \frac{4i\pi}{m}\left(\mathcal{I} + \frac{m}{ig(\Lambda)}\right)^{(0)} \tag{8}$$

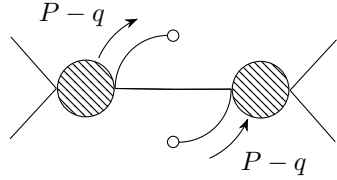$$= \frac{4\pi}{g(\Lambda)} + \frac{2\Lambda}{\pi} \tag{9}$$

$$\Rightarrow g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \tag{10}$$

and this is actually how we get the form of (2).

3 OPE

3.1 l.h.s.

Take what we got in the last section as a new non-perturbative vertex, we only need to deal with tree diagram this way. First we have Figure 2(a) in Braaten's paper:

$$= \langle 34|\psi^\dagger\left(-\frac{\mathbf{r}}{2}\right)\psi\left(\frac{\mathbf{r}}{2}\right)|12\rangle \tag{11}$$
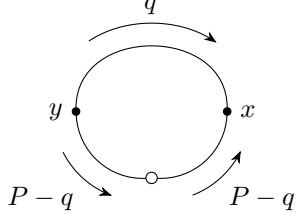
$$= (i\mathcal{A})^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{i}{q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon} \frac{i}{E - q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon}^2 e^{i\mathbf{q}\cdot\mathbf{r}} \tag{12}$$

$$= \mathcal{A}^2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{m^2 e^{i\mathbf{q}\cdot\mathbf{r}}}{(\mathbf{q}^2 - p^2 - i\epsilon)^2} \tag{13}$$

$$= \frac{im^2 \mathcal{A}^2 e^{ipr}}{8\pi p} \tag{14}$$

3.2 r.h.s.

For simplicity, we drop the external lines and focus on the internal subgraph. Consider Figure 2(b):

$$= \langle 34|\psi^\dagger \psi(0)|12\rangle_{amp} \tag{15}$$

$$= \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} e^{iP\cdot y} e^{-iP\cdot x} e^{-il_1\cdot y} e^{il_2\cdot x} e^{iq\cdot(x-y)} \tilde{D}(l_1)\tilde{D}(l_2)\tilde{D}(q) \tag{16}$$

$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \tilde{D}(P-q)\tilde{D}(P-q)\tilde{D}(q) \tag{17}$$

$$= -\int \frac{\mathrm{d}^3 \mathbf{q}}{(2\pi)^3} \frac{m^2}{(\mathbf{q}^2 - p^2 - i\epsilon)^2} \tag{18}$$
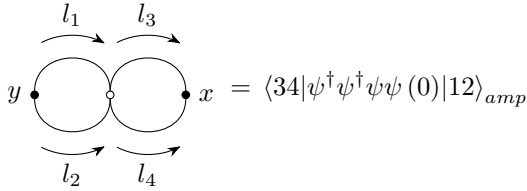
$$= -\frac{im^2}{8\pi p} \tag{19}$$

where $\tilde{D}$ marks momentum space propagator and two external vertexes give an $(i\mathcal{A})^2$ factor. The total contribution is

$$\frac{im^2 \mathcal{A}^2}{8\pi p}, \tag{20}$$

the first order Fourier expansion of the l.h.s.

3.3 Higher dimensional operators

Figure 2(c) gives

$$= \langle 34|\psi^\dagger \psi^\dagger \psi \psi(0)|12\rangle_{amp} \tag{21}$$

$$= \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{\mathrm{d}^4 l_4}{(2\pi)^4} e^{iP\cdot y} e^{-iP\cdot x} e^{-i(l_1+l_2)\cdot y} e^{i(l_3+l_4)\cdot x}$$
$$\tilde{D}(l_1)\tilde{D}(l_2)\tilde{D}(l_3)\tilde{D}(l_4) \tag{22}$$

3

which becomes

$$y \quad \bigcirc\bigcirc \quad x = \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \tilde{D}(l_1)\tilde{D}(P-l_1)\tilde{D}(l_2)\tilde{D}(P-l_2) \tag{23}$$

$$= -\int \frac{\mathrm{d}^3 \mathbf{l_1}}{(2\pi)^3} \frac{\mathrm{d}^3 \mathbf{l_2}}{(2\pi)^3} \frac{m^2}{\left(\mathbf{l_1}^2 - p^2 - i\epsilon\right)\left(\mathbf{l_2}^2 - p^2 - i\epsilon\right)} \tag{24}$$

$$= -\mathcal{I}^2 \tag{25}$$

There're four diagrams in total:

$$= \mathcal{A}^2 \mathcal{I}^2 \tag{26}$$

$$= \mathcal{A}\mathcal{I} \tag{27}$$

$$= \mathcal{A}\mathcal{I} \tag{28}$$

$$= 1 \tag{29}$$

We have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm\mathbf{p}} = (\mathcal{A}\mathcal{I} + 1)^2 \tag{30}$$

in total. Plug in

$$\mathcal{I} = -\frac{m}{ig(\Lambda)} - \frac{1}{\mathcal{A}} \tag{31}$$

we have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm\mathbf{p}} = m^2 g^{-2}(\Lambda) \mathcal{A}^2 \tag{32}$$

References

[1] Eric Braaten and Lucas Platter. Exact relations for a strongly interacting fermi gas from the operator product expansion. *Phys. Rev. Lett.*, 100(20), may 2008.