Scalar QED

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1 Hydrogen Wavefunction Divergence Near Origin in Dirac Equation and Schrödinger Equation

1.1 The Dirac part

The Klein-Gordon Hydrogen Equation is

$$((i\partial_0+\frac{Z\alpha}{r})^2+\nabla^2-m^2)\Psi=0 \tag{1}$$

For the bound state, the eigen value and the wave function are

$$E=m\frac{1}{\sqrt{1+\frac{\alpha^2Z^2}{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}}} \tag{2}$$

$$\Psi=\frac{c}{\sqrt{4\pi}}e^{-kr}r^{\lambda} \tag{3}$$

where

$$\lambda=-\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2} \qquad c=\sqrt{\frac{(2k)^{2(1+\sqrt{\frac{1}{4}-Z^2\alpha^2})}}{\Gamma(2+2\sqrt{\frac{1}{4}-Z^2\alpha^2}))}} \qquad k=\frac{m}{\sqrt{1+\frac{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}{\alpha^2Z^2}}} \tag{4}$$

c is the normaliztion factor for $\int d^3r|\Psi|^2=1$. For convenience, define

$$\Psi'=\frac{\Psi}{2(mZ\alpha)^{\frac{3}{2}}} \tag{5}$$

Now Ψ' is dimensionless and expand it in α, we get the origin divergence comes from a term

$$-(Z\alpha)^2\log(mr) \tag{6}$$

the m in log could be interpreted as a subtraction point μ.

1.2 The Schrödinger part

The Hamiltonian is

$$H = H_0 + H_{int} \tag{7}$$

$$H_0 = -\frac{\nabla^2}{2m} - \frac{Z\alpha}{r}, \quad H_{int} = \frac{\nabla^4}{8m^3} + \frac{1}{32m^4}[-\nabla^2, [-\nabla^2, -\frac{Z\alpha}{r}]] \tag{8}$$

The first term of H_{int} is the relativistic kinematic v^2 correction, the second one is the Darwin term. The H_0 gives the radial wave functions as follows

$$R_{n0} = \frac{2(mZ\alpha)^{\frac{3}{2}}}{n^{\frac{3}{2}}} e^{-\frac{mZ\alpha}{n}r} F(1-n, 2, \frac{2mZ\alpha r}{n}), \quad E_n = -\frac{Z^2\alpha^2 m}{2n^2} \tag{9}$$

$$R_{k0} = \sqrt{\frac{2}{\pi}}(mZ\alpha)^{\frac{3}{2}} k e^{\frac{\pi}{2k}} |\Gamma(1-\frac{i}{k})| e^{-imz\alpha kr} F(1+\frac{i}{k}, 2, 2imZ\alpha kr), \quad E_k = \frac{mZ^2\alpha^2 k^2}{2} \tag{10}$$

Within perturbation theory,$E_1^{(1)} =< \phi|H_{int}|\phi >$, in quantum mechanics, the NLO energy correction is

$$E_1^{(1)} = E_1 Z^2 \alpha^2 \tag{11}$$

The NLO corretion of the bound state wave function is

$$\sum_{n\neq 1} a_{n1}\phi_{n00} + \int dk a_{k1}\phi_{k00} \tag{12}$$

with

$$a_{n1} = \frac{< \phi_{n00}|H_{int}|\phi_{100} >}{E_1 - E_n} \tag{13}$$

the discrete part of (12) is not divergent at $r = 0$. We now focus on the integration part and seperate the relativstic kinematic term and the Darwin term. Since we are only interested in the divergent part, here we give a hard cutoff $\frac{\Lambda}{m}$ as the up-limit of the integration and a also a down-limit λ, with $\lambda >> 1$ (note that the following wave function have been multiplied by $2(mZ\alpha)^{\frac{3}{2}}$)

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk \frac{2Z^2\alpha^2 k^{\frac{3}{2}}}{2\pi(\sqrt{1-\exp(-\frac{2\pi}{k})})} (1 - \frac{2}{1+k^2} \exp\left(-\frac{2\arctan(k)}{k}\right)) e^{\frac{\pi}{2k}} |\Gamma(1-\frac{i}{k})| \tag{14}$$

with the integral region we defined($k >> 1$), it would be OK to expand the integrand in $\frac{1}{k}$ (I havn't prove it yet), then the UV divergent term is

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk (Z\alpha)^2 (\frac{1}{\pi} + \frac{1}{k}) \tag{15}$$

$$\sim (\alpha Z)^2 (\frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right)) \tag{16}$$

The UV divergent part of Darwin term is

$$\Phi^{(1)}(0)_D = -\frac{(Z\alpha)^4}{8\pi}\int_\lambda^{\frac{\Lambda}{m}} dk k^2 e^{\frac{\pi}{k}} |\Gamma(1-\frac{i}{k})|^2 \tag{17}$$

with the same trick as (15), the UV divergen part is

$$\Phi^{(1)}(0)_D = -(\alpha Z)^4 \int_\lambda^{\frac{\Lambda}{m}} dk \frac{k^2}{8\pi} + \frac{k}{8} + \frac{1}{24}\pi \tag{18}$$

$$\sim -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m}) \tag{19}$$

Now collect all the results we get as follow.
The K-G wave function's origin UV divergence is

$$K-G \;\; UV : -(Z\alpha)^2 \log(mr) \tag{20}$$

The purterbative Schrödinger wave function's origin UV divergence, with a k cutoff $\frac{\Lambda}{m}$, is

$$Kin \;\; UV : (\alpha Z)^2 (\frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right)) \tag{21}$$

$$Darwin \;\; UV : -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m}) \tag{22}$$

All the m, under Λ or in a log, can be interpreted as a subtraction point μ.

2 Non-relativistic QED (NRQED) Matching

2.1 Feynman Rules

2.1.1 QED

Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi + \Phi_v^* iv \cdot D\Phi_v \tag{23}$$

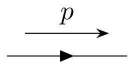
with

$$D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi$$

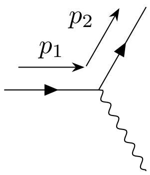
and

$$D_\mu \Phi_v = \partial_\mu \Phi - iZeA_\mu \Phi_v$$

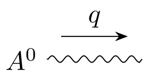
But note that no **A** can appear in actual calculation because here only static scalar potential exists. And the Feynman rules are standard QED and HQET Feynman rules except that photons only appear as zero component of Coulomb gauge photon to describe Coulomb potential exchange.



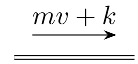
$$= \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$



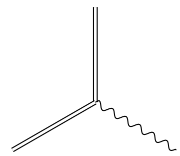
$$= -ie\gamma^\mu$$



$$= \frac{i}{\mathbf{q}^2}$$



$$= \frac{i}{v \cdot k}$$



$$= iZev^\mu$$

Here v satisfies $v^2 = 1$ and the k with it stands for the offshellness of the propagating momentum.

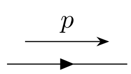
2.1.2 NRQED

Using the Foldy-Wouthuysen transformation, we can have the Lagrangian

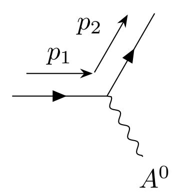
$$\mathcal{L}_{NRQED} = \bar{\psi}_e \left(iD_0 + \frac{\mathbf{D}^2}{2m} \right) \psi_e + \delta\mathcal{L} + \Phi_v^* iv \cdot D\Phi_v \tag{24}$$

with the same notation above. Here $\mathbf{D} = \nabla - ie\mathbf{A}$.

Feynman rules are also the same except for the scalar electron side which becomes



$$= \frac{i}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon}$$



$$= -ie$$

We can ignore all interacting terms involving $\mathbf{A}$.

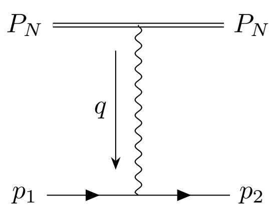
Another way to achieve it is to use the transform rules of heavy quark effective theory (HQET) and change the power counting.

2.2 LO Matching

2.2.1 QED

In tree level[1]

$$i\mathcal{M}^{(0)}_{QED} =$$

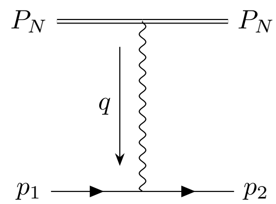


$$= -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \bar{u}_e(p_2)\gamma_0 u_e(p_1)$$

[1]Note that there's no Gamma matrice in the heavy particle side, they can only appear in the QED side.

2.2.2 NRQED

$$i\mathcal{M}^{(0)}_{NRQED} =$$



$$= -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \psi^\dagger(p_2)\psi(p_1)$$

Using Dirac representation, the Dirac spinor is

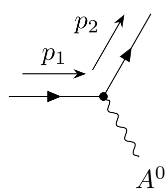
$$u_e(p) = \sqrt{\frac{p^0+m}{2p^0}} \begin{pmatrix} \psi(p) \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p^0+m}\psi(p) \end{pmatrix}$$

Expand to v^2 order, we have an extra vertex $\frac{\mathbf{p_1}^2+\mathbf{p_2}^2-2\mathbf{p_1}\cdot\mathbf{p_2}-2i(\mathbf{p_1}\times\mathbf{p_2})\cdot\boldsymbol{\sigma}}{8m^2}$ which is exactly those terms with denorminator $1/8m^2$ in BBL[2].

Rather than write down the effective electron-photon vertex up to $\mathcal{O}(v^2)$

$$\left[1 - \frac{(\mathbf{p}_1+\mathbf{p}_2)^2 - 2i(\mathbf{p}_1\times\mathbf{p}_2)\cdot\boldsymbol{\sigma}}{8m^2}\right]$$

we can add an additional vertex



$$= ie\frac{(\mathbf{p}_1+\mathbf{p}_2)^2 - 2i(\mathbf{p}_1\times\mathbf{p}_2)\cdot\boldsymbol{\sigma}}{8m^2}$$

3 Local Operator and Matrix Element of NRQED

The perterbative expansion looks like this:

$$\langle 0|\psi(0)N(0)|v_N\rangle \tag{25}$$

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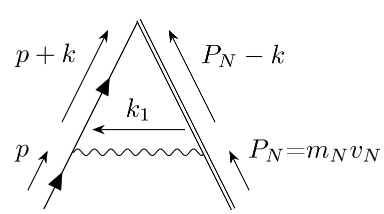
3.1 NLO

$$\langle 0|\psi_e(0)N(0)(-ie\mu^{-\epsilon})\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0(-ie\mu^{-\epsilon})\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle =$$



which doesn't have logarithm divergence. We can rigorously prove that so long as there's no dynamic photon, NRQED has no logarithmic divergence at one loop order (at least for this problem we're considering).

[2]Which contains both Darwin and spin-orbital terms

3.2 NNLO

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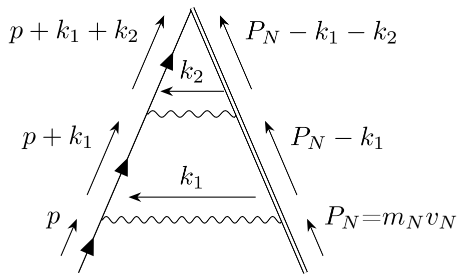
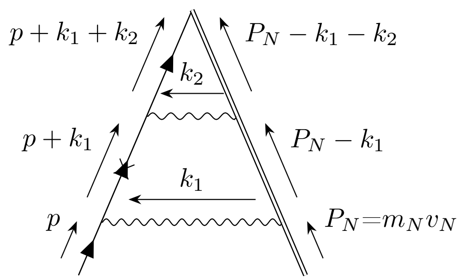
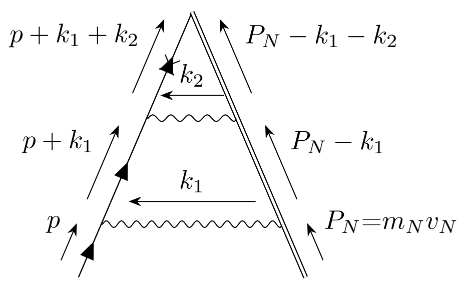


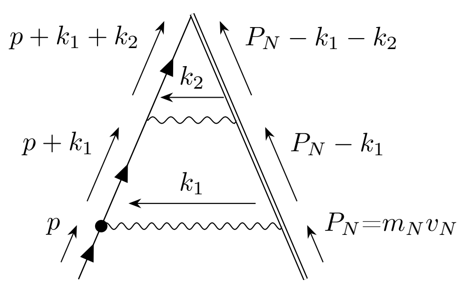
$$= 0 + \text{finite terms}$$



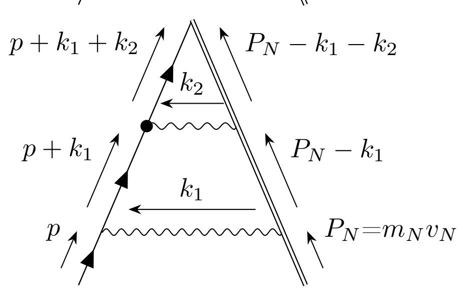
$$= Z^2\alpha^2\left(\frac{1}{2(3-d)} + \log\mu\right) + \text{finite terms}$$



$$= 0 + \text{finite terms}$$



$$= Z^2\alpha^2\left(\frac{1}{4(d-3)} - \frac{\log\mu}{2}\right) + \text{finite terms}$$



$$= Z^2\alpha^2\left(\frac{1}{2(d-3)} - \log\mu\right)\text{finite terms}$$