

Hydrogen

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QED Lagrangian is

$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \quad (1)$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \quad (2)$$

The box diagram for NRQED process is

$$\begin{aligned}
 i\mathcal{M}_{NRQED} = & \quad \begin{array}{c} \text{Diagram: A box diagram with a top horizontal line labeled } P_N \text{ at both ends and } P_N - k \text{ above it with a right arrow. A bottom horizontal line labeled } p_1 \text{ at the left and } p_2 \text{ at the right, with } p_1 + k \text{ below it and a right arrow. Two vertical wavy lines connect the top and bottom lines. The left wavy line has a downward arrow labeled } k. The right wavy line has an upward arrow labeled } k - q. \end{array} \\
 & = e^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int [dk] \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (-k^0 + i\epsilon) (p_1^0 + k^0 - m - \frac{\mathbf{p}_1 + \mathbf{k}^2}{2m} + i\epsilon)} \psi(p_1) \\
 & = -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (E_1 - \frac{\mathbf{p}_1 + \mathbf{k}^2}{2m})} \psi(p_1) \\
 & = -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 (E_1 - \frac{\mathbf{k}^2}{2m})} \psi(p_1)
 \end{aligned}$$

The box and crossed box diagram for QED process is

$$\begin{aligned}
i\mathcal{M}_1 = & \text{Diagram: Box diagram with incoming electron } p_1 \text{ and outgoing electron } p_2. \text{ Incoming photon } P_N \text{ and outgoing photon } P_N. \text{ Internal photon lines } k \text{ and } k-q. \text{ Internal fermion lines } p_1+k \text{ and } P_N-k. \\
& = e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m) \gamma^0}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 [(p_1 + k)^2 - m^2 + i\epsilon] (-k^0 + i\epsilon)} u_e(p_1) \\
& = e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k} \gamma^0}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 [(p_1 + k)^2 - m^2 + i\epsilon] (-k^0 + i\epsilon)} u_e(p_1) \\
& = ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 [(\mathbf{k} + \mathbf{p}_1)^2 + m^2 - p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
& = ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i}) \gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 [\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_1$ has infrared log divergence and no ultraviolet divergence.

$$\begin{aligned}
i\mathcal{M}_2 = & \text{Diagram: Crossed box diagram with incoming electron } p_1 \text{ and outgoing electron } p_2. \text{ Incoming photon } P_N \text{ and outgoing photon } P_N. \text{ Internal photon lines } k \text{ and } k-q. \text{ Internal fermion lines } p_1+k \text{ and } P_N+k. \\
& = e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m) \gamma^0}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 [(p_1 + k)^2 - m^2 + i\epsilon] (k^0 + i\epsilon)} u_e(p_1) \\
& = e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k} \gamma^0}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 [(p_1 + k)^2 - m^2 + i\epsilon] (k^0 + i\epsilon)} u_e(p_1) \\
& = -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 - \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 [(\mathbf{k} + \mathbf{p}_1)^2 + m^2 + p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
& = -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i}) \gamma^i \gamma^0 - \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 [\mathbf{k}^2 + m^2 + p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_2$ has no infrared or ultraviolet divergence.

$$\begin{aligned}
i\mathcal{M}_1 + i\mathcal{M}_2 = & ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k^2 + m^2 + (k_i - p_{1i}) p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 [\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1) \\
& = ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k^2 + m^2 + (k_i - p_{1i}) p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 [\mathbf{k}^2 - \mathbf{p}_1^2] \sqrt{\mathbf{k}^2 + m^2}} u_e(p_1)
\end{aligned}$$