One Loop Matching for Quasi PDF

Yingsheng Huang

October 25, 2019

1 Background

The definition of parton distribution function (PDF) is

$$q\left(x, \mu_{f}\right)=\frac{1}{2} \int \frac{d \eta^{-}}{2 \pi} e^{-i x P^{+} \eta^{-}}\left\langle P, S\left|\bar{\psi}\left(0, \eta^{-}, \mathbf{0}_{T}\right) \Gamma \mathcal{W}\left[\eta^{-} ; 0\right] \psi(0)\right| P, S\right\rangle \tag{1}$$

where with this unpolarized PDF case, $\Gamma=\gamma^{+}$. $\mathcal{W}$ is the gauge link defined as [Collins(2009)]

$$\mathcal{W}\left[w^{-}, 0\right]=P\left\{e^{-i g_{0} \int_{0}^{w^{-}} \mathrm{d} y^{-} A_{(0) \sigma}^{+}\left(0, y^{-}, \mathbf{0}_{\mathrm{T}}\right) t_{\sigma}}\right\} \tag{2}$$

The definition of quasi PDF is

$$\tilde{q}(x)=\frac{1}{2} \int \frac{\mathrm{d} z}{2 \pi} e^{i x P^{z} z}\langle P, S|\bar{\psi}(z) \tilde{\Gamma} \tilde{\mathcal{W}}[z, 0] \psi(0)| P, S\rangle \tag{3}$$

where

$$\tilde{\mathcal{W}}[z, 0]=\mathcal{P} \exp \left[i g \int_{0}^{z} \mathrm{d} z^{\prime} n \cdot A^{a}\left(z^{\prime}\right) \mathrm{t}^{a}\right], n^{\mu}=(0,0,0,-1) \tag{4}$$

and $\tilde{\Gamma}=\gamma^{z}$ in our case.

To make the gauge links equal to unity, we choose light cone gauge for PDF and axial gauge for quasi PDF.

2 Tree Level Matching

In axial gauge, the quasi PDF is

$$\tilde{q}(x)=\frac{1}{4 \pi} \int \mathrm{d} z e^{i x P^{z} z}\langle P|\bar{\psi}(z) \gamma^{z} \psi(0)| P\rangle \tag{5}$$

The frame is chosen such that $P^{\mu}=\left(P^{0}, \mathbf{0}, P^{z}\right)$.

$$P^{0}=\sqrt{m^{2}+P^{z 2}} \tag{6}$$

Up to one loop, we can use quark state as the external state to complete the matching process. The quark field ψ reads

$$\psi(x)=\int \frac{\mathrm{d}^{3} \vec{k}}{(2 \pi)^{3}} \frac{1}{2 E_{k}}\left[u(k) e^{-i k \cdot x} b_{k}+v(k) e^{i k \cdot x} d_{k}^{\dagger}\right] \tag{7}$$

Insert it to (5)

$$\tilde{q}^{(0)}(x)=\int \frac{\mathrm{d} z}{4 \pi} e^{i x P^{z} z}\langle 0| b_{P} \int \frac{\mathrm{d}^{3} \vec{p}}{(2 \pi)^{3}} \frac{1}{2 E_{p}}\left[\bar{u}(p) e^{i p \cdot x} b_{p}^{\dagger}+\bar{v}(p) e^{-i p \cdot x} d_{p}\right] \gamma^{z} \int \frac{\mathrm{d}^{3} \vec{k}}{(2 \pi)^{3}} \frac{1}{2 E_{k}}\left[u(k) e^{-i k \cdot x} b_{k}+v(k) e^{i k \cdot x} d_{k}^{\dagger}\right] b_{P}^{\dagger}|0\rangle \tag{8}$$

Look at the creation-annihilation operators, we have the following combinations:

$$b_P b_p^\dagger b_k b_P^\dagger,\ b_P d_p b_k b_P^\dagger,\ b_P b_p^\dagger d_k^\dagger b_P^\dagger,\ b_P d_p d_k^\dagger b_P^\dagger \tag{9}$$

Apparently the latter three all go to zero by moving the anti-quark operators to the side:

$$\begin{aligned}
\tilde{q}^{(0)}(x) &= \int \frac{\mathrm{d}z}{4\pi} e^{ixP^z z} \langle 0| \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\bar{u}(p)e^{ip\cdot z} b_P b_p^\dagger \gamma^z \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}u(k)e^{-ik\cdot 0} b_k b_P^\dagger |0\rangle \\
&= \int \frac{\mathrm{d}z}{4\pi} e^{ixP^z z} \langle 0| \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{e^{ip\cdot z}}{2E_p}\bar{u}(p)(2\pi)^3 2E_{\mathbf{P}}\delta^{(3)}(\mathbf{p}-\mathbf{P})\gamma^z \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{e^{-ik\cdot 0}}{2E_k}u(k)(2\pi)^3 2E_{\mathbf{P}}\delta^{(3)}(\mathbf{k}-\mathbf{P})|0\rangle \\
&= \int \frac{\mathrm{d}z}{4\pi} e^{ixP^z z + iP\cdot z}\bar{u}(P)\gamma^z u(P)
\end{aligned} \tag{10}$$

Using Gordon identity

$$\begin{aligned}
\tilde{q}^{(0)}(x) &= \int \frac{\mathrm{d}z}{4\pi} e^{ixP^z z - iP^z z}\bar{u}(P)\frac{P^z}{m}u(P) \\
&= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z - iP^z z} P^z \\
&= \delta(1-x)
\end{aligned} \tag{11}$$

3 One Loop Quasi PDF (Axial Gauge)

First we consider the matrix element in the definition of quasi PDF

$$\langle P|\bar{\psi}(z)\gamma^z\psi(0)|P\rangle \tag{12}$$

and in leading order this one gives

$$e^{-iP^z z}\bar{u}(P)\gamma^z u(P) \tag{13}$$

as mentioned above. This, in higher orders, is embedded via a Fourier transform. The full form of quasi PDF can be considered as a momentum space matrix element with an $1/4\pi$ factor.

Two diagrams are required with one loop corrections to quasi PDF. Detailed derivation with rigorous Wick contraction is to be found in Section B.

(a) (b)

Figure 1

The Feynman rule for the composite operator is

$$= e^{-ip_2^z z}\gamma^z \tag{14}$$

and two external lines give $\bar{u}(P)$ and $u(P)$ respectively.

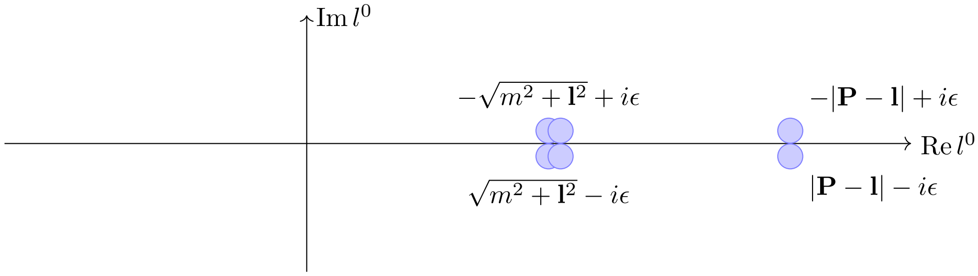
The first one is a quark self-energy correction

$$\bar{u}(P)e^{-iP^z z}\gamma^z \frac{i(\not{P}+m)}{P^2-m^2}(-i\Sigma_2(P))u(P) \tag{15}$$

The second one is

$$\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\,(-ig_st^a\gamma^\mu)\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}(-ig_st^a\gamma^\nu)\tilde{D}^A_{G\mu\nu}(P-l)u(P)\bigg|_{l^z=xP^z}$$
$$=-g_s^2C_F\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\,\gamma^\mu\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}\gamma^\nu\tilde{D}^A_{G\mu\nu}(P-l)u(P)\bigg|_{l^z=xP^z} \tag{16}$$

For the definition of $\tilde{D}^A_{G\mu\nu}$, see Section A. There're in total 6 poles:



For the result of numerator simplification, see Section D

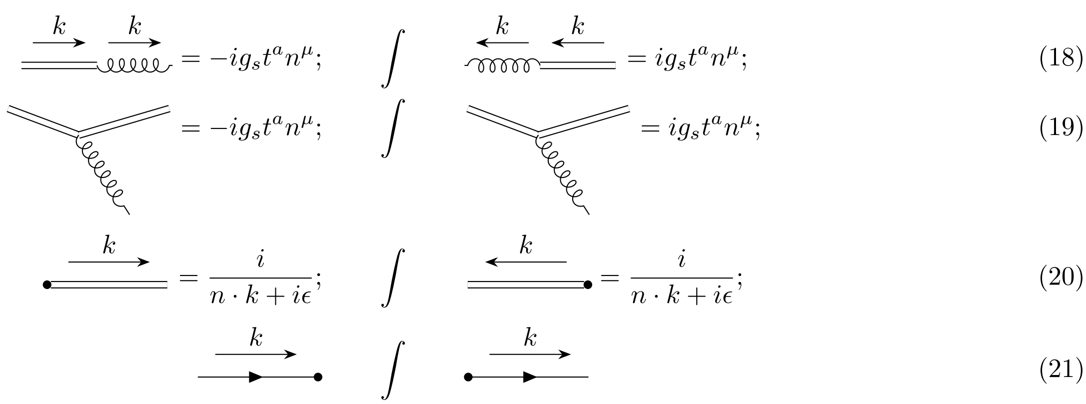
4 One Loop Quasi PDF (Feynman Gauge)

In Feynman gauge, we must have the full definition of quasi PDF. For unpolarized quasi PDF

$$\tilde{q}(x)=\frac{1}{2}\int\frac{\mathrm{d}z}{2\pi}e^{ixP^zz}\langle P,S|\bar{\psi}(z)\gamma^z\mathcal{P}\exp\left[ig\int_0^z\mathrm{d}z'A^{a,z}\left(z'\right)\mathrm{t}^a\right]\psi(0)|P,S\rangle \tag{17}$$

There're following 8 diagrams.

The corresponding Feynman rules are:



The last line stands for the momentum conservation between two dots. There're also an extra 1/2 factor on the outside and a Dirac delta function to eliminate all z-direction loop momenta. The delta function confines the sum of all momenta flow in z equals to xP^z.

Let's take the spin sum of external states.

4.1 Real corrections

First we must deal with those real diagrams.

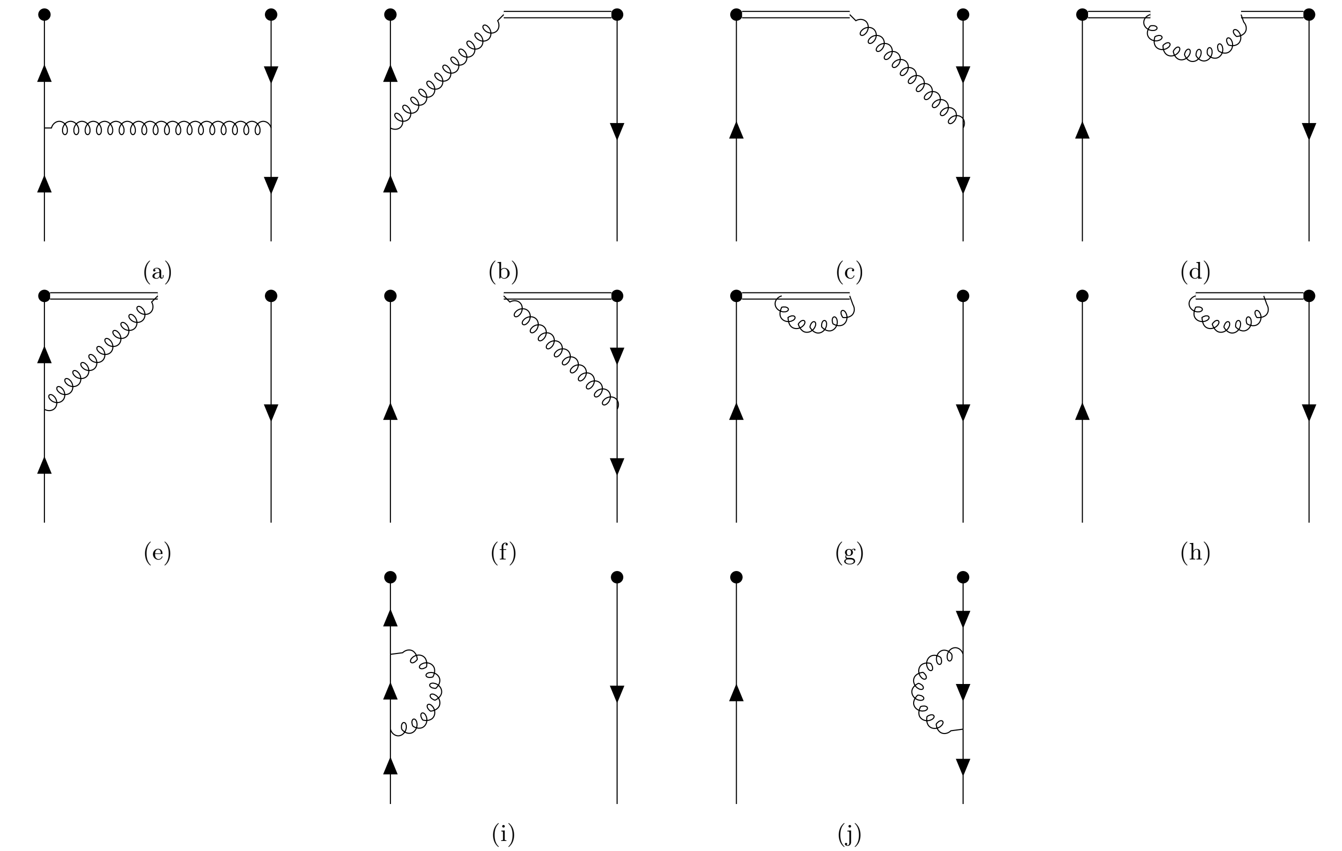
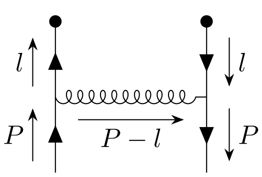


Figure 2: Diagrams of quasi PDF in Feynman gauge.

Diagram 2a gives

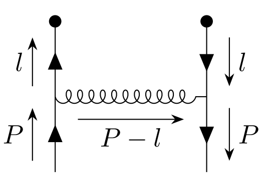


$$= \frac{1}{2}\bar{u}(P)(-ig_s t^a\gamma_\nu)\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{\not{l}-m+i\epsilon}\gamma^z\frac{-ig^{\mu\nu}}{(P-l)^2+i\epsilon}\frac{i}{\not{l}-m+i\epsilon}(-ig_s\gamma_\mu t^a)u(P)\delta(l^z-xP^z)$$

$$= -i\frac{g_s^2 C_F}{2}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^\mu\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma_\mu u(P)\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z) \tag{22}$$

After spin sum:

$$\frac{1}{2}\sum_s$$



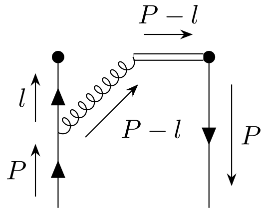
$$= \frac{-ig_s^2 C_F}{4}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\sum_s\bar{u}(P)\gamma^\mu\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma_\mu u(P)\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z)$$

$$= \frac{-ig_s^2 C_F}{4}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\mathrm{Tr}\big\{(\not{P}+m)\gamma^\mu(\not{l}+m)\gamma^z(\not{l}+m)\gamma_\mu\big\}}{(l^2-m^2+i\epsilon)^2(P-l)^2}\delta(l^z-xP^z)$$

the numerator and the denominator of the integrand is checked out in this step with Xiong's result

$$= \tag{23}$$

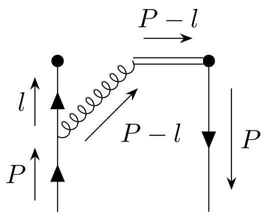
Diagram 2b gives



$$
= \frac{1}{2}\bar{u}(P)\gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{l^z - P^z + i\epsilon} (ig_s t^a) \frac{-ig^{\mu z}}{(P-l)^2 + i\epsilon} \frac{i}{\not{l} - m + i\epsilon} (-ig_s \gamma_\mu t^a) u(P) \delta(l^z - xP^z)
$$

$$
= \frac{ig_s^2 C_F}{2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P)\gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon} \gamma^z u(P) \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} \delta(l^z - xP^z) \tag{24}
$$

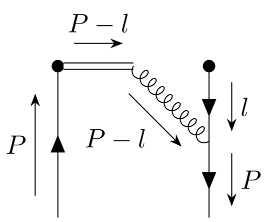
Take the spin sum

$\frac{1}{2}\sum_s$ 

$$
= \sum_s \frac{ig_s^2 C_F}{4} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P)\gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon} \gamma^z u(P) \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} \delta(l^z - xP^z)
$$

$$
= \frac{ig_s^2 C_F}{4} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not{P} + m)\gamma^z(\not{l} + m)\gamma^z\}}{l^2 - m^2 + i\epsilon} \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} \delta(l^z - xP^z) \tag{25}
$$

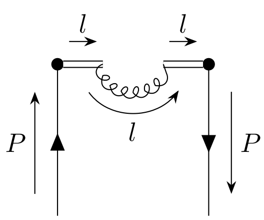
Diagram 2c should be the same with Diagram 2b.



$$
= \frac{1}{2}\bar{u}(P)(-ig_s\gamma_\mu t^a) \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{\not{l} - m + i\epsilon} \frac{-ig^{\mu z}}{(P-l)^2 + i\epsilon} \frac{i}{P^z - l^z + i\epsilon} (-ig_s t^a) \gamma^z u(P) \delta(l^z - xP^z)
$$

$$
= \frac{ig_s^2 C_F}{2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P)\gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon} \gamma^z u(P) \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} \delta(l^z - xP^z) \tag{26}
$$

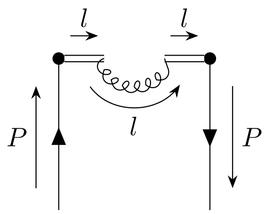
Diagram 2d is



$$
= \frac{1}{2}\bar{u}(P)\gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{l^z + i\epsilon} (ig_s t^a) \frac{-ig^{zz}}{l^2 + i\epsilon} (-ig_s t^a) \frac{i}{-l^z + i\epsilon} u(P) \delta(l^z - (1-x)P^z)
$$

$$
= \frac{-ig_s^2 C_F}{2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P)\gamma^z u(P) \frac{1}{l^z + i\epsilon} \frac{1}{l^2 + i\epsilon} \frac{1}{-l^z + i\epsilon} \delta(l^z - (1-x)P^z) \tag{27}
$$

Take the spin sum

$\frac{1}{2}\sum_s$ 

$$
= \sum_s \frac{-ig_s^2 C_F}{4} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P)\gamma^z u(P) \frac{1}{l^z + i\epsilon} \frac{1}{l^2 + i\epsilon} \frac{1}{-l^z + i\epsilon} \delta(l^z - (1-x)P^z)
$$

$$
= \frac{-ig_s^2 C_F}{4} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \mathrm{Tr}\{(\not{P} + m)\gamma^z\} \frac{1}{l^z + i\epsilon} \frac{1}{l^2 + i\epsilon} \frac{1}{-l^z + i\epsilon} \delta(l^z - (1-x)P^z) \tag{28}
$$

4.2 Virtual corrections

The quark self energy diagram gives

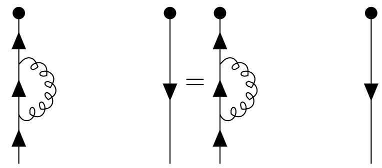
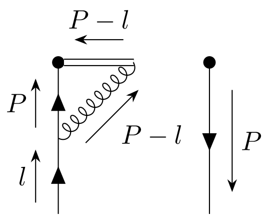
 (29)

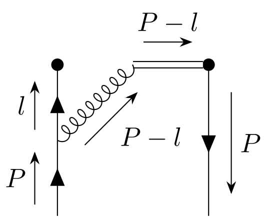
Diagram 2e



$$= \frac{1}{2P^z}\bar{u}(P)\gamma^z\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{l^z-P^z+i\epsilon}(-ig_st^a)\frac{-ig^{\mu z}}{(P-l)^2+i\epsilon}\frac{i}{\not{l}-m+i\epsilon}(-ig_s\gamma_\mu)u(P)\delta(1-x)$$

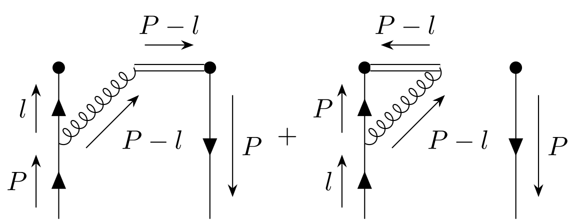
$$= \frac{-ig_s^2C_F}{2P^z}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\bar{u}(P)\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P)\frac{1}{l^z-P^z-i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(1-x) \tag{30}$$

The other diagram 2b is



$$= \frac{ig_s^2C_F}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\bar{u}(P)\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P)\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z) \tag{31}$$

The sum of both diagrams is



$$= \frac{ig_s^2C_F}{2}\int\mathrm{d}z\int\frac{\mathrm{d}^4l}{(2\pi)^4}\bar{u}(P)\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P)\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\left[e^{i(l^z-xP^z)z}-e^{i(P^z-xP^z)z}\right] \tag{32}$$

$$= \frac{ig_s^2C_F}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\bar{u}(P)\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P)\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}[\delta(l^z-xP^z)-\delta(P^z-xP^z)] \tag{33}$$

Here we're to take the spin sum. After integrated out l^0 and $\mathbf{l}_T$, the remaining integrand is [1]

$$\frac{g_s^2C_FP^{z3}}{8\pi^2(m^2+P^{z2})}\int\mathrm{d}l^z\frac{\delta(l^z-xP^z)-\delta(P^z-xP^z)}{|l^z-P^z|}+\frac{g_s^2C_FP^{z2}}{8\pi^2\sqrt{m^2+P^{z2}}}\int\mathrm{d}l^z\left(\frac{\log\frac{l^z-P^z}{\Lambda}}{l^z-P^z}\right)[\delta(l^z-xP^z)-\delta(P^z-xP^z)]+\text{Constant}$$

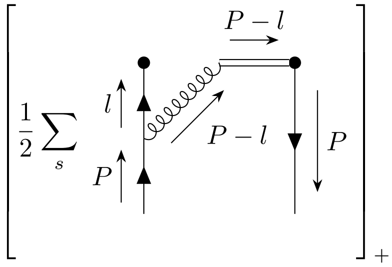
by setting $l^z=yP^z$, $\mathrm{d}l^z(\delta(l^z-xP^z)-\delta(P^z-xP^z))=\mathrm{d}y(\delta(y-x)-\delta(1-x))$

$$=\frac{g_s^2C_FP^{z3}/|P^z|}{8\pi^2(m^2+P^{z2})}\int\mathrm{d}y\frac{\delta(y-x)-\delta(1-x)}{|y-1|}+\frac{g_s^2C_FP^z}{8\pi^2\sqrt{m^2+P^{z2}}}\int\mathrm{d}y\left(\frac{\log\frac{y-1}{\Lambda/P^z}}{y-1}\right)[\delta(y-x)-\delta(1-x)]+\text{Constant}$$

Now we can transform the integration on y to plus functions[2]. By redefining the plus function to an extended version:

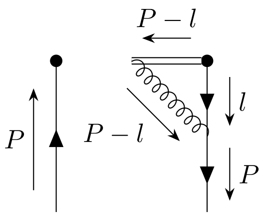
$$f_+(x)=f(x)-\delta(1-x)\int_{-\infty}^{\infty}\mathrm{d}yf(y) \tag{34}$$

The full result is



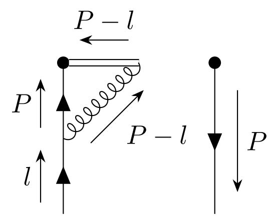
(35)

Diagram 2f is

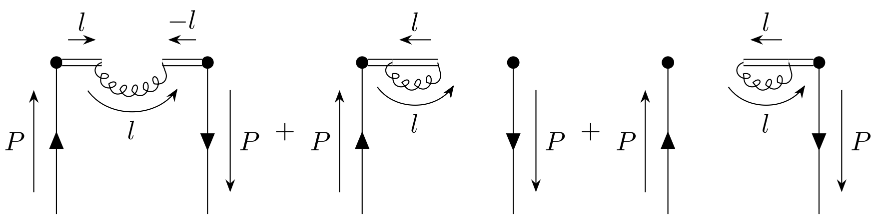


$$= \frac{1}{2P^z}\frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)(-ig_s\gamma_\mu t^a)\frac{i(\not{l}+m)}{l^2-m^2+i\epsilon}\gamma^z\frac{-ig^{\mu z}}{(P-l)^2+i\epsilon}\frac{i}{P^z-l^z}(ig_s t^a)u(P)\delta(1-x)$$

$$= \frac{-ig_s^2 C_F}{2P^z}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P)\frac{1}{l^z-P^z-i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(1-x)$$

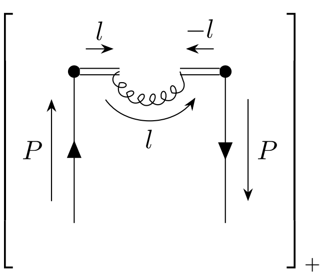
$=$ 

(36)



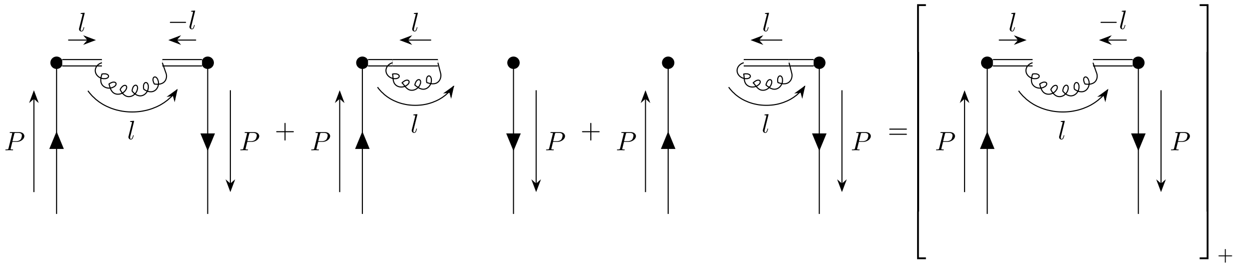
$$=\frac{g_s^2 C_F}{2}\bar{u}(P)\gamma^z\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}\delta(l^z-(1-x)P^z)u(P)-\frac{2g_s^2 C_F}{2}\delta(1-x)\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}$$

$$=g_s^2 C_F\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}[\delta(l^z/P^z-(1-x))-\delta(1-x)]$$

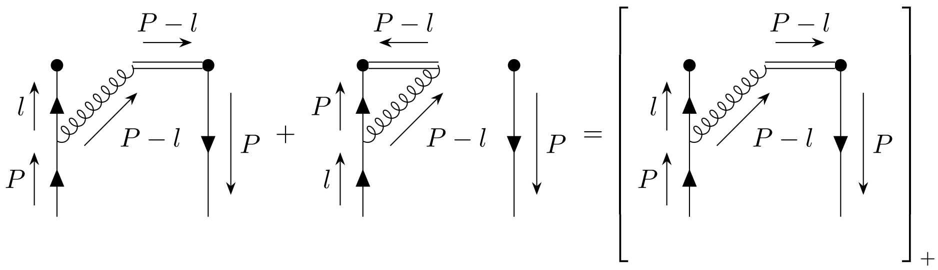
$=$ 

(37)

The final result is



(38)



(39)

5 Matching to PDF

The matching coefficient Z is defined as

$$\tilde{q}(x) = \int_0^1 \frac{dy}{y} Z\left(\frac{x}{y}, \frac{P^z}{\mu}\right) q(y) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{P_z^2}, \frac{M^2}{P_z^2}\right) \tag{40}$$

For leading order, it's

$$\delta(1-x) = \int_0^1 \frac{\mathrm{d}y}{y} Z\left(\frac{x}{y}, \frac{P^z}{\mu}\right)\delta(1-y) \tag{41}$$

It's straightforward to get

$$Z\left(\frac{x}{y}, \frac{P^z}{\mu}\right) = \delta\left(1 - \frac{x}{y}\right) \tag{42}$$

The one loop matching factor is

$$\left(1 + \delta\tilde{Z}_F\right)\delta(1-x) + \tilde{q}^{(1)}(x) = \int_0^1 \frac{dy}{y}\left[\delta\left(\frac{x}{y} - 1\right) + Z^{(1)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right)\right]\left[(1 + \delta Z_F)\,\delta(1-y) + q^{(1)}(y)\right] \tag{43}$$

A Conventions

The quark field ψ reads

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}\left[u(k)e^{-ik\cdot x}b_k + v(k)e^{ik\cdot x}d_k^\dagger\right] \tag{44}$$

and the projection of single particle state is

$$|p\rangle = b_p^\dagger\,|0\rangle \tag{45}$$

$$\left\{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\right\} = (2\pi)^3 2E\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs} \tag{46}$$

The Dirac spinor is normalized to

$$\bar{u}^s(p)u(p) = 2m\delta^{rs} \tag{47}$$

With Gordon identity, one can derive [Srednicki(2007)]

$$\bar{u}(P)\gamma^\mu u(P) = 2P^\mu \tag{48}$$

The axial gauge propagator is

$$\tilde{D}_G^{A\mu\nu}(p) = -i\delta_{ab}\left(g^{\mu\nu} - \frac{n^\mu p^\nu + n^\nu p^\mu}{n\cdot p} + n\cdot n\frac{p^\mu p^\nu}{(n\cdot p)^2}\right)\frac{1}{p^2} \tag{49}$$

The Feynman gauge propagator is

$$\tilde{D}_G^{F\mu\nu}(p) = \frac{-ig^{\mu\nu}\delta_{ab}}{p^2 + i\epsilon} \tag{50}$$

State contract with field:

$$\begin{aligned}\overbrace{\psi(x)|P}\rangle &= \int \frac{\mathrm{d}^3\mathbf{l}}{(2\pi)^3}\frac{1}{2E_\mathbf{l}}\left[b_\mathbf{l}u(l)e^{-il\cdot x} + d_\mathbf{l}^\dagger v(l)e^{il\cdot x}\right]b_\mathbf{P}^\dagger\,|0\rangle \\ &= \int \frac{\mathrm{d}^3\mathbf{l}}{(2\pi)^3}\frac{1}{2E_\mathbf{l}}u(l)e^{-il\cdot x}(2\pi)^3 2E\delta^{(3)}(\mathbf{l}-\mathbf{P})\,|0\rangle \\ &= u(P)e^{-iP\cdot x}\end{aligned} \tag{51}$$

and correspondingly

$$\langle\overbrace{P|\bar{\psi}}(x) = \bar{u}(P)e^{iP\cdot x} \tag{52}$$

Plus function is defined as [Collins(2009)]

$$\int_0^1 \mathrm{d}x\left(\frac{1}{1-x}\right)_+ T(x) \equiv \int_0^1 \mathrm{d}x\frac{T(x)-T(1)}{1-x} \tag{53}$$

$$\int_0^1 \mathrm{d}x\frac{A(x)}{(1-x)_+}T(x) \equiv \int_0^1 \mathrm{d}x\frac{A(x)T(x)-A(1)T(1)}{1-x} \tag{54}$$

The plus function could also be defined in a different fashion:

$$F_+(x) := \lim_{\beta\to 0}\left(F(x)\theta(1-\beta-x) - \delta(1-\beta-x)\int_0^{1-\beta}\mathrm{d}yF(y)\right) \tag{55}$$

and with a smooth test function it's defined the same way as above:

$$\int_0^1 \mathrm{d}xF_+(x)G(x) = \int_0^1 \mathrm{d}xF(x)[G(x)-G(1)] \tag{56}$$

with

$$\int_0^1 \mathrm{d}xF(x)_+ = 0 \tag{57}$$

For a flexible lower boundary:

$$\int_a^1 \mathrm{d}xF_+(x)G(x) = \int_a^1 \mathrm{d}xF(x)[G(x)-G(1)] + G(1)\int_0^a \mathrm{d}xF(x) \tag{58}$$

Also if rule out some smoothness problem

$$[f(x)]_+ = f(x) - \delta(1-x)\int_0^1 f(z)\mathrm{d}z \tag{59}$$

Here we use a modified version of plus function

$$f_+(x) = f(x) - \delta(1-x)\int_{-\infty}^{\infty} \mathrm{d}y f(y) \tag{60}$$

$$= f(x) - \delta(1-x)\int_{-\infty}^{\infty} \mathrm{d}l^z f(y)/P^z \tag{61}$$

The convention for path ordering is that the field with higher value of the integration variable s goes to the left. The definition of Heaviside theta function follows

$$\theta(z) = \int \frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz}}{w+i\epsilon} \tag{62}$$

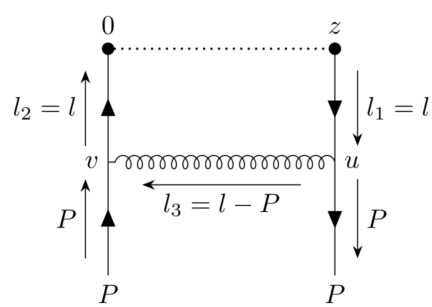
and here we choose

$$\theta(0) = 1 \tag{63}$$

B Wick Contraction

B.1 Axial Gauge

Take diagram 1b as an example



This corresponds to

$$\frac{1}{4\pi}\int \mathrm{d}z e^{ixP^z z}\langle P|\int \mathrm{d}^4u\bar{\psi}_u\psi_u A_u\bar{\psi}(z)\gamma^z\psi(0)\int \mathrm{d}^4v\bar{\psi}_v\psi_v A_v|P\rangle \tag{64}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z e^{ixP^z z}\int \mathrm{d}^4u\mathrm{d}^4v\bar{u}(P)e^{iP\cdot u}\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\tilde{D}_F(l_1)e^{-il_1\cdot(u-z)}\gamma^z\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\tilde{D}_F(l_2)e^{-il_2\cdot(-v)}$$
$$\int\frac{\mathrm{d}^4l_3}{(2\pi)^4}\tilde{D}_G(l_3)e^{-il_3\cdot(v-u)}u(P)e^{-iP\cdot v} \tag{65}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z\int \mathrm{d}^4u\mathrm{d}^4v\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\int\frac{\mathrm{d}^4l_3}{(2\pi)^4}e^{ixP^z z+il_1\cdot z}e^{i(P-l_1+l_3)\cdot u}e^{i(l_2-l_3-P)\cdot v}\bar{u}(P)\tilde{D}_F(l_1)\gamma^z\tilde{D}_F(l_2)$$
$$\tilde{D}_G(l_3)u(P) \tag{66}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z\int\frac{\mathrm{d}^4l}{(2\pi)^4}e^{ixP^z z+il\cdot z}\bar{u}(P)\tilde{D}_F(l)\gamma^z\tilde{D}_F(l)\tilde{D}_G(l-P)u(P) \tag{67}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z\int\frac{\mathrm{d}^4l}{(2\pi)^4}e^{i(xP^z-l^z)z}\bar{u}(P)\tilde{D}_F(l)\gamma^z\tilde{D}_F(l)\tilde{D}_G(l-P)u(P) \tag{68}$$

$$=\frac{1}{4\pi}\int\frac{\mathrm{d}l^0}{2\pi}\int\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\ \bar{u}(P)\tilde{D}_F(l)\gamma^z\tilde{D}_F(l)\tilde{D}_G(l-P)u(P)\Big|_{l^z=xP^z} \tag{69}$$

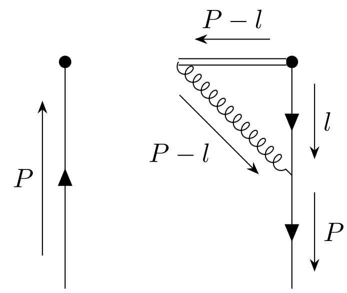
where

$$\int \mathrm{d}z e^{i(xP^z-l^z)z} = 2\pi\delta(l^z - xP^z) \tag{70}$$

This indicates what the Feynman diagram actually means: a normal Feynman diagram with only 3-momentum integration, where the axial momentum is fixed by $l^z = xP^z$, and with an extra $1/4\pi$ factor.

B.2 Feynman Gauge

Let's take diagram 2f as an example:



The one loop quasi PDF is

$$\tilde{q}_1^{(1)}(x) = \frac{1}{2}\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\int \mathrm{d}^4u(-ig_s t^a\gamma_\mu)\bar{\psi}_u\psi_u A_u^\mu \bar{\psi}(z)\gamma^z \tilde{\mathcal{W}}[z,0]\psi(0)|P,S\rangle \tag{71}$$

where

$$\tilde{\mathcal{W}}[z,0] = \mathcal{P}\exp\left[-ig_s\int_0^z \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right] \tag{72}$$

We should rewrite the gauge link to the product of two gauge links connect to infinity

$$\tilde{\mathcal{W}}[z,0] = \tilde{\mathcal{W}}[z,+\infty]\tilde{\mathcal{W}}[\infty,0] \tag{73}$$

and in one loop level it equals to

$$\mathcal{P}\exp\left[ig_s\int_\infty^z \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right]\mathcal{P}\exp\left[ig_s\int_0^\infty \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right] = \left[ig_s\mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z'+z)\,\mathrm{t}^a\right] - \left[ig_s\mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right]$$

The path ordering gives

$$\mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z') = \int \mathrm{d}z' A^{a,z}(z')\,\theta(z') = \int \mathrm{d}z' A^{a,z}(z')\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \tag{74}$$

and

$$\mathcal{P}\left[\int_0^\infty \mathrm{d}z' A^{a,z}(z')\right]^2 = \int \mathrm{d}z' A^{a,z}(z')\,\theta(z')\int \mathrm{d}z'' A^{a,z}(z'')\,\theta(z''-z') \tag{75}$$

with all momenta involved with z' must be in the z-direction (the exponent is actually $z'n\cdot w$ if a four-vector w actually exists). Consider the second gauge link first, the matrix element is then (discarding all couplings)

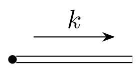
$$\begin{aligned}
&\langle P,S|\bar{\psi}_u\psi_u A_u^\mu\bar{\psi}(z)\gamma^z\int \mathrm{d}z' A^{a,z}(z'+z)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\psi(0)|P,S\rangle \\
=&\int \mathrm{d}^4u\langle P,S|\bar{\psi}_u\psi_u A_u\bar{\psi}(z)\gamma^z\int \mathrm{d}z' A^{a,z}(z'+z)\,\psi(0)|P,S\rangle\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \\
=&\int \mathrm{d}^4u\langle P,S|\bar{\psi}_u\psi_u A_u\bar{\psi}(z)\gamma^z\int \mathrm{d}z' A^{a,z}(z'+z)\,\psi(0)|P,S\rangle\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \\
=&\int \mathrm{d}z'\int \mathrm{d}^4u\,\bar{u}(P)e^{iP\cdot u}\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\tilde{D}_F(l_1)e^{-il_1\cdot(u-z)}\gamma^z\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\tilde{D}_G(l_2)e^{-il_2\cdot(u-z'-z)}u(P)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \\
=&\int \mathrm{d}z'\int \mathrm{d}^4u\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\int\frac{\mathrm{d}w}{2\pi}\bar{u}(P)e^{i(P-l_1-l_2)\cdot u}e^{i(l_1+l_2)\cdot z}e^{-iwz'-il_2\cdot z'}\tilde{D}_F(l_1)\gamma^z\tilde{D}_G(l_2)u(P)\frac{i}{w+i\epsilon}
\end{aligned}$$

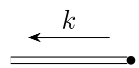
$$
\begin{aligned}
&= \int \mathrm{d}z' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \int \frac{\mathrm{d}w}{2\pi} \bar{u}(P) e^{iP\cdot z} e^{-iwz' + i(P-l)^z z'} \tilde{D}_F(l) \gamma^z \tilde{D}_G(P-l) u(P) \frac{i}{w+i\epsilon} \\
&= \bar{u}(P) e^{iP\cdot z} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_F(l) \gamma^z \tilde{D}_G^{\mu z}(P-l) \frac{i}{P^z - l^z + i\epsilon} u(P)
\end{aligned}
$$

The complete quasi PDF at one loop is

$$
\begin{aligned}
&\frac{g_s^2 C_F}{2} \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \bar{u}(P) \gamma_\mu e^{iP\cdot z} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_F(l) \gamma^z \tilde{D}_G^{\mu z}(P-l) \frac{i}{P^z - l^z + i\epsilon} u(P) \\
=& \frac{g_s^2 C_F}{2P^z} \bar{u}(P) \gamma_\mu \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_F(l) \gamma^z \tilde{D}_G^{\mu z}(P-l) \frac{i}{P^z - l^z + i\epsilon} u(P) \delta(1-x)
\end{aligned}
$$

multiplied by those couplings. This basically established that the momentum of a gluon equals to the momentum of the eikonal line it attaches to. We then have the Feynman rule:

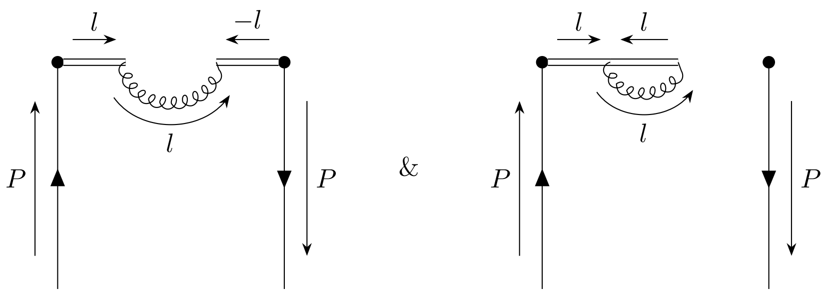


$= \frac{i}{n\cdot k + i\epsilon};$


$= \frac{i}{n\cdot k + i\epsilon}$ (76)

and for the gluon-eikonal vertex on the r.h.s., an extra minus sign is added for the normal $(-ig_s t^a)$.

The next job is to determine the Feynman rule for



The first one is

$$
\frac{1}{2} \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \left[ig_s \mathcal{P} \int_0^\infty \mathrm{d}z' A^{a,z}(z'+z)\, \mathrm{t}^a \right] \left[-ig_s \mathcal{P} \int_0^\infty \mathrm{d}z' A^{a,z}(z')\, \mathrm{t}^a \right] \psi(0) | P, S \rangle \tag{77}
$$

Let's look at the coupling-free form:

$$
\begin{aligned}
&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \left[\mathcal{P} \int_0^\infty \mathrm{d}z' A^{a,z}(z'+z) \right] \left[\mathcal{P} \int_0^\infty \mathrm{d}z'' A^{a,z}(z'') \right] \psi(0) | P, S \rangle \\
=& \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \int \mathrm{d}z' A^{a,z}(z'+z) \int \mathrm{d}z'' A^{a,z}(z'') \psi(0) | P, S \rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ihz''}}{h+i\epsilon} \\
=& \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \int \mathrm{d}z' A^{a,z}(z'+z) \int \mathrm{d}z'' A^{a,z}(z'') \psi(0) | P, S \rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ihz''}}{h+i\epsilon} \\
=& \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \bar{u}(P) e^{iP\cdot z} \gamma^z \int \mathrm{d}z' \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) e^{-il\cdot(z''-z'-z)} u(P) \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ihz''}}{h+i\epsilon} \\
=& \bar{u}(P) \int \frac{\mathrm{d}z}{2\pi} e^{i(x-1)P^z z + il^z z} \gamma^z \int \mathrm{d}z' \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \int \frac{\mathrm{d}w}{2\pi} \frac{i}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{i}{h+i\epsilon} e^{-i(w-l)\cdot z'} e^{-i(l+h)\cdot z''} u(P) \\
=& \bar{u}(P) \int \frac{\mathrm{d}z}{2\pi} e^{-i(1-x)P^z z + il^z z} \gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{l^z + i\epsilon} \frac{i}{-l^z + i\epsilon} u(P) \\
=& \bar{u}(P) \gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{l^z + i\epsilon} \frac{i}{-l^z + i\epsilon} \delta(l^z - (1-x)P^z) u(P)
\end{aligned}
$$

One can start with a different route:

$$
\bar{u}(P) \int \frac{\mathrm{d}z}{2\pi} e^{-i(1-x)P^z z + il^z z} \gamma^z \int \mathrm{d}z' \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \int \frac{\mathrm{d}w}{2\pi} \frac{i}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{i}{h+i\epsilon} e^{-i(w-l)\cdot z'} e^{-i(l+h)\cdot z''} u(P) \tag{78}
$$

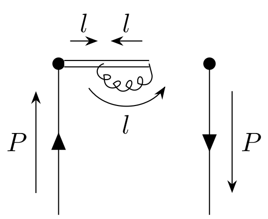
The second one is

$$\frac{1}{2}\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \frac{\mathcal{P}\left[-ig_s\int_0^\infty \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right]\left[-ig_s\int_0^\infty \mathrm{d}z'' A^{a,z}(z'')\,\mathrm{t}^a\right]}{2}\psi(0)|P,S\rangle \tag{79}$$

The coupling-free form is

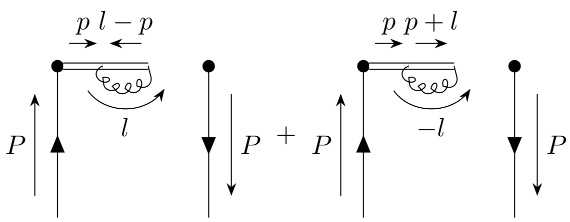
$$\begin{aligned}
&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \mathcal{P}\left[\int_0^\infty \mathrm{d}z' A^{a,z}(z')\int_0^\infty \mathrm{d}z'' A^{a,z}(z'')\right]\psi(0)|P,S\rangle\\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \int_0^\infty \mathrm{d}z' A^{a,z}(z')\int_0^\infty \mathrm{d}z'' A^{a,z}(z'')\left[\theta(z'-z'')+\theta(z''-z')\right]\psi(0)|P,S\rangle\\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \int_0^\infty \mathrm{d}z' A^{a,z}(z')\int_0^\infty \mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle\\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \int \mathrm{d}z' A^{a,z}(z')\int \mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle \int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \int \mathrm{d}z' A^{a,z}(z')\int \mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle \int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\bar{u}(P)e^{iP\cdot z}\gamma^z\int\mathrm{d}z'\int\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)e^{-il\cdot(z''-z')}u(P)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
=&\bar{u}(P)\int\frac{\mathrm{d}z}{2\pi}e^{-i(1-x)P^z z}\gamma^z\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\int\mathrm{d}z'\int\mathrm{d}z''\int\frac{\mathrm{d}w}{2\pi}\frac{i}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{i}{h+i\epsilon}e^{-i(w-l)\cdot z'}e^{-i(h+l)\cdot z''}u(P)\\
=&2\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}
\end{aligned}$$

This gives



$$= -\frac{g_s^2 C_F}{2}\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon} \tag{80}$$

We can reverse the loop momentum of one of the path and add a small inflowing momentum p (in the following diagrams each diagram only represents one specific path, that means the sum of both diagram is the value of the original diagram):



$$\begin{aligned}
&= -\frac{g_s^2 C_F}{2}\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\left[\frac{i}{p+i\epsilon}\frac{i}{p-l^z+i\epsilon}+\frac{i}{p+i\epsilon}\frac{i}{p+l^z+i\epsilon}\right]\\
&= -\frac{g_s^2 C_F}{2}\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{p+i\epsilon}\frac{2ip}{(p-l^z+i\epsilon)(p+l^z+i\epsilon)}\\
&= -\frac{g_s^2 C_F}{2}\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{p-l^z+i\epsilon}\frac{2i}{p+l^z+i\epsilon}
\end{aligned}$$

According to Tong, one can manually add a regulating momentum and then take the derivative to eliminate the effect. [3]

Diagram 2h is

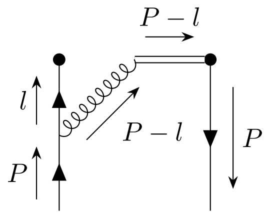
$$\frac{1}{2}\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \frac{\mathcal{P}\left[ig_s\int_0^\infty \mathrm{d}z' A^{a,z}(z'+z)\,\mathrm{t}^a\right]\left[ig_s\int_0^\infty \mathrm{d}z'' A^{a,z}(z''+z)\,\mathrm{t}^a\right]}{2}\psi(0)|P,S\rangle \tag{81}$$

and it behaves exactly like 2g, since those extra $(+z)$s will be cancelled in the Wick contraction. These three diagrams are combined to form a plus function of diagram 2d. [4]

C Comparing two different prescription of Feynman integrals

Take diagram 2b as an example (we use a boldface font with subscript 3 to symbolize the first three components of a four vector: $\tilde{\mathbf{l}}^\mu = (l^0, \mathbf{l}_T)$)

$$\frac{1}{2}\sum_s$$



$$
\begin{aligned}
&= \frac{i g_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\mathrm{Tr}\left\{(\not{P}+m)\gamma^z(\not{l}+m)\gamma^z\right\}}{l^2-m^2+i\epsilon}\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z) \\
&= i g_s^2 C_F \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+l^z P^z-m^2}{\tilde{\mathbf{l}}^2-(l^z)^2-m^2+i\epsilon}\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(\tilde{\mathbf{P}}-\tilde{\mathbf{l}})^2-(P^z-l^z)^2+i\epsilon}\delta(l^z-xP^z) \\
&= \frac{i g_s^2 C_F}{2\pi P^z(x-1)}\int \frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+x(P^z)^2-m^2}{\tilde{\mathbf{l}}^2-x^2(P^z)^2-m^2+i\epsilon}\frac{1}{(\tilde{\mathbf{l}}-\tilde{\mathbf{P}})^2-(P^z)^2(1-x)^2+i\epsilon}
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
&y\Big(\tilde{\mathbf{l}}^2-x^2(P^z)^2-m^2+i\epsilon\Big)+(1-y)\Big((\tilde{\mathbf{l}}-\tilde{\mathbf{P}})^2-(P^z)^2(1-x)^2+i\epsilon\Big) \\
=&\tilde{\mathbf{l}}^2-x^2y(P^z)^2-ym^2-2(1-y)\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+(1-y)\tilde{\mathbf{P}}^2-(1-y)(P^z)^2(1-x)^2+i\epsilon \\
=&(\tilde{\mathbf{l}}-(1-y)\tilde{\mathbf{P}})^2+y(1-y)\tilde{\mathbf{P}}^2-x^2y(P^z)^2-ym^2-(1-y)(P^z)^2(1-x)^2+i\epsilon
\end{aligned}
\tag{83}
$$

$$\Delta = -y(1-y)\tilde{\mathbf{P}}^2+x^2y(P^z)^2+ym^2+(1-y)(P^z)^2(1-x)^2 \tag{84}$$

The integral is

$$
\begin{aligned}
&\int_0^1 \mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+x(P^z)^2-m^2}{\tilde{\mathbf{l}}^2-x^2(P^z)^2-m^2+i\epsilon}\frac{1}{(\tilde{\mathbf{l}}-\tilde{\mathbf{P}})^2-(P^z)^2(1-x)^2+i\epsilon} \\
=&\int_0^1 \mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{(\tilde{\mathbf{l}}+(1-y)\tilde{\mathbf{P}})\cdot\tilde{\mathbf{P}}+x(P^z)^2-m^2}{\left[\tilde{\mathbf{l}}^2-\Delta+i\epsilon\right]^2} \\
=&\int_0^1 \mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{\left[\tilde{\mathbf{l}}^2-\Delta+i\epsilon\right]^2}
\end{aligned}
\tag{85}
$$

The first term in the numerator vanishes

$$=\int_0^1 \mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{\left[\tilde{\mathbf{l}}^2-\Delta+i\epsilon\right]^2}$$

after Wick rotation

$$
\begin{aligned}
=&\frac{i}{(-1)^2}\int_0^1 \mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{\left[\tilde{\mathbf{l}}^2+\Delta-i\epsilon\right]^2} \\
=&i\int_0^1 \mathrm{d}y\frac{(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{8\pi\sqrt{\Delta}}
\end{aligned}
\tag{86}
$$

The final result agrees with what we got from integrating l^0 first:

$$
\begin{aligned}
&\frac{C_F g_s^2}{32\pi^2 P^z(x-1)\sqrt{m^2+P^{z2}}}\Bigg\{P^{z2}x\Bigg\{3\log\left(|x-1|\sqrt{m^2+P^{z2}}+P^z(x-1)\right)-\log\left(|x-1|\sqrt{m^2+P^{z2}}+P^z(-x)+P^z\right) \\
&-3\log\left(\sqrt{\left(m^2+P^{z2}\right)\left(m^2+P^{z2}x^2\right)}+m^2+P^{z2}x\right)+\log\left(\sqrt{\left(m^2+P^{z2}\right)\left(m^2+P^{z2}x^2\right)}-m^2-P^{z2}x\right)+2\log(P^z)\Bigg\} \\
&-2P^z|x-1|\sqrt{m^2+P^{z2}}+2\sqrt{\left(m^2+P^{z2}\right)\left(m^2+P^{z2}x^2\right)}\Bigg\}
\end{aligned}
$$

D Diagram 1b Comparing (Defuncted)

Let's start with

$$
\begin{aligned}
&\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\,(-ig_st^a\gamma^\mu)\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}(-ig_st^a\gamma^\nu)\tilde{D}^A_{G\mu\nu}(P-l)u(P)\Big|_{l^z=xP^z}\\
=&-g_s^2C_F\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\,\gamma^\mu\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}\gamma^\nu\tilde{D}^A_{G\mu\nu}(P-l)u(P)\Big|_{l^z=xP^z}\\
=&-ig_s^2C_F\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\gamma^\mu\frac{\not{l}+m}{l^2-m^2}\gamma^z\frac{\not{l}+m}{l^2-m^2}\gamma^\nu\frac{1}{(P-l)^2+i\epsilon}u(P)\\
&\left[\bar{g}^{\mu\nu}-\frac{n^\nu\left(P^\mu-l^\mu\right)+n^\mu\left(P^\nu-l^\nu\right)}{n\cdot(P-l)}+\frac{n^2\left(P^\mu-l^\mu\right)\left(P^\nu-l^\nu\right)}{\left(n\cdot P-n\cdot l\right)^2}\right]\Bigg|_{l^z=xP^z}
\end{aligned}
\tag{87}
$$

We consider the numerator as a first step

$$
\bar{u}(P)\gamma^\mu(\not{l}+m)\gamma^z(\not{l}+m)\gamma^\nu\left[(P-l)^2\tilde{D}^A_{G\mu\nu}(P-l)\right]u(P)
\tag{88}
$$

We can separate the gluon propagator into there parts. The first one gives a metric tensor and the final result

$$
4l^3\left(m\bar{u}(P)u(P)-\bar{u}(P)\not{l}u(P)\right)-2\left(m^2-l^2\right)\bar{u}(P)\gamma^3u(P)
\tag{89}
$$

The combined result can be further separated with respect to the structure of gamma matrices. The first one is for $\bar{u}(P)\not{l}u(P)$:

$$
\begin{aligned}
&\frac{2l^z\left(2l^z\left(P^z-l^z\right)-l^2+m^2\right)}{\left(l^2-m^2\right)^2(P-l)^2\left(l^z-P^z\right)^2}\\
=&-\frac{4(l^z)^2}{\left(l^2-m^2\right)^2(P-l)^2\left(l^z-P^z\right)}-\frac{2l^z}{\left(l^2-m^2\right)(P-l)^2\left(l^z-P^z\right)^2}
\end{aligned}
$$

for $\bar{u}(P)u(p)$:

$$
\begin{aligned}
&\frac{2ml^z\left(-6l^zP^z+4\left(l^z\right)^2+2\left(P^z\right)^2+l^2-m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{2ml^z\left(-4l^zP^z+4\left(l^z\right)^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2ml^z\left(-2l^zP^z+2\left(P^z\right)^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2ml^z\left(l^2-m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{8m\left(l^z\right)^2\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}-\frac{4ml^zP^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2ml^z\left(l^2-m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{8m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{4ml^zP^z}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4m\left(l^z\right)^2-4ml^zP^z}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4ml^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4ml^z}{\left(l^2-m^2\right)^2(l-P)^2}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}
\end{aligned}
$$

for $\bar{u}(P)\gamma^zu(p)$:

$$
\frac{(l-P)^2\left(2l^z\left(P^z-l^z\right)-l^2+m^2\right)+2\left(m^2-l^2\right)P^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}
$$

$$
\begin{aligned}
&=\frac{(l-P)^2\left(2l^z\left(P^z-l^z\right)-l^2+m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2\left(m^2-l^2\right)P^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
&=\frac{2l^z\left(P^z-l^z\right)-l^2+m^2}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)^2}-\frac{2P^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\\
&=\frac{2l^z\left(P^z-l^z\right)}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)^2}-\frac{l^2-m^2}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)^2}-\frac{2P^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\\
&=-\frac{2l^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{1}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}-\frac{2P^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}
\end{aligned}
$$

The total result is

$$
\begin{aligned}
&\bar{u}(P)\Bigg\{-\frac{4(l^z)^2\not{l}}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\not{l}}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
&\quad+\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4ml^z}{\left(l^2-m^2\right)^2(l-P)^2}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2} \qquad (90)\\
&\quad-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}-\frac{2P^z\gamma^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\Bigg\}u(P)\\
&=\bar{u}(P)\Bigg\{\frac{-4\left(l^z\right)^2\left(\not{l}-m\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\left(\not{l}-m\right)}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}+\frac{4ml^z}{\left(l^2-m^2\right)^2(l-P)^2}\\
&\quad-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}-\frac{2P^z\gamma^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\Bigg\}u(P) \qquad (91)\\
&=\bar{u}(P)\Bigg\{\frac{-4\left(l^z\right)^2\left(\not{l}-m\right)+4ml^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\left(\not{l}-m\right)+2P^z\gamma^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
&\quad-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}\Bigg\}u(P) \qquad (92)\\
&=\bar{u}(P)\Bigg\{-\frac{4\left(l^z\right)^2\left(\not{l}-2m\right)+4ml^zP^z}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\left(\not{l}-m+P^z\gamma^z\right)-2(P^z)^2\gamma^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
&\quad-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}\Bigg\}u(P) \qquad (93)
\end{aligned}
$$

Xiong's result is

$$
\begin{aligned}
&-ig_s^2C_F\int\frac{d^4k}{(2\pi)^4}\bar{u}(P)\Bigg[\frac{2\gamma^z}{\left(k^2-m^2\right)(P-k)^2}+\frac{4\left(2m-k^z\right)\not{k}}{\left(k^2-m^2\right)^2(P-k)^2}\\
&+\frac{2\left(k^z\gamma^z+\not{k}-m\right)}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)}-\frac{\gamma^z}{(P-k)^2\left(P^z-k^z\right)^2}\Bigg]P^z\delta\left(k^z-xP^z\right)u(P) \qquad (94)
\end{aligned}
$$

As we discussed earlier, it can be dissected into

$$
\frac{4\left(2m-k^z\right)\not{k}}{\left(k^2-m^2\right)^2(P-k)^2}+\frac{2\not{k}}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)} \qquad \bar{u}(P)\not{k}u(P) \qquad (95)
$$

$$
\frac{-2m}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)} \qquad \bar{u}(P)u(P) \qquad (96)
$$

$$
\frac{2\gamma^z}{\left(k^2-m^2\right)(P-k)^2}+\frac{2k^z\gamma^z}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)}-\frac{\gamma^z}{(P-k)^2\left(P^z-k^z\right)^2} \qquad \bar{u}(P)\gamma^zu(P) \qquad (97)
$$

Notes

1. The constant mentioned above is

$$-\frac{C_F g_s^2\left(-\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}-\frac{2P_3^2\left(\log\left(2\left(m^2+P_3^2\right)\right)\right)}{\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}+m^2+P_3^2}+\Lambda\sqrt{m^2+P_3^2}+m^2+P_3^2\right)}{16\pi^2\left(P_3-l_3\right)\sqrt{m^2+P_3^2}}-\frac{C_F g_s^2}{16\pi^2\operatorname{sgn}\left(l_3-P_3\right)}$$

$$\frac{P_3 C_F g_s^2\left(\log\left(l_3-P_3\right)^2-\frac{2\left(\log\left(2\left(\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}-m^2-P_3^2\right)\right)\right)}{\Lambda}\right)}{16\pi^2\sqrt{m^2+P_3^2}}+\frac{m^4 P_3 C_F g_s^2\left(\Lambda-\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}\right)}{16\pi^2\Lambda\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}$$

$$-\frac{m^2 P_3^3 C_F g_s^2\left(\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}+\sqrt{m^2+P_3^2}\right)}{8\pi^2\Lambda\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}-\frac{P_3^5 C_F g_s^2\left(\Lambda+\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}+2\sqrt{m^2+P_3^2}\right)}{16\pi^2\Lambda\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}$$

$$-\frac{\Lambda^2 m^2 P_3 C_F g_s^2\left(\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}-1\right)}{16\pi^2\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}-\frac{P_3^3 C_F g_s^2\left(\Lambda\left(\Lambda+\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}+2\sqrt{m^2+P_3^2}\right)-2\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}\right)}{16\pi^2\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}$$

multiplied by the delta function and integration.

2. Wrong prescription: Having $\left(\int_0^\infty-\int_1^\infty\right)\mathrm{d}xF(x)[G(x)-G(1)]=\int_0^1\mathrm{d}xF_+(x)G(x)$

$$\int\mathrm{d}y\frac{\delta(y-x)-\delta(1-x)}{|y-1|}=\frac{\theta(1-x)\theta(x)}{(1-x)_+}+\frac{\theta(-x)}{2(x-1)}+\frac{3\theta(x-1)}{2(x-1)}+\frac{\theta(1-x)\theta(x)}{x-1} \tag{98}$$

$$\int\mathrm{d}y\left(\frac{\log\frac{y-1}{\Lambda/P^z}}{y-1}\right)[\delta(y-x)-\delta(1-x)]=\left(\frac{\log\frac{y-1}{\Lambda/P^z}}{y-1}\right)_+ +\frac{\log\left(\frac{x-1}{\Lambda/P^z}\right)\theta(x-1)}{x-1}+\frac{\log(1-x)\theta(1-x)}{x-1} \tag{99}$$

3. According to Tong, one can choose

$$\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{0+i\epsilon}\frac{i}{-l^z+i\epsilon}=\lim_{p\to 0}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{p+i\epsilon}\frac{i}{p-l^z+i\epsilon} \tag{100}$$

$$=\lim_{p\to 0}\frac{i}{p+i\epsilon}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{\tilde{\mathbf{l}}^2-l^{z\,2}+i\epsilon}\frac{i}{p-l^z+i\epsilon} \tag{101}$$

where

$$\mathcal{I}\equiv\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{\tilde{\mathbf{l}}^2-l^{z\,2}+i\epsilon}\frac{i}{p-l^z+i\epsilon} \tag{102}$$

With partial derivative operator

$$\frac{\partial}{\partial p}\mathcal{I}=-\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{\tilde{\mathbf{l}}^2-l^{z\,2}+i\epsilon}\frac{i}{[p-l^z+i\epsilon]^2} \tag{103}$$

$$\mathcal{I}=\frac{\partial}{\partial p}\mathcal{I}(l^z-p) \tag{104}$$

We can evaluate the value of $\frac{\partial}{\partial p}\mathcal{I}l^z$

$$\begin{aligned}\frac{i}{p}\mathcal{I}l^z&=\frac{i}{p}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{\tilde{\mathbf{l}}^2-l^{z\,2}+i\epsilon}\frac{il^z}{p-l^z+i\epsilon}\\&=\frac{i}{p}\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{-i/2}{\sqrt{\tilde{\mathbf{l}}^2}+p+i\epsilon}\\&=\frac{2p^2(\log(p)-\log(p+i\Lambda))+\Lambda(\Lambda+2ip)}{8\pi^2p}\end{aligned}$$

$$-\frac{i}{p}\frac{\partial\mathcal{I}}{\partial p}p=-i\frac{\partial\mathcal{I}}{\partial p}$$

With Dim-Reg $\mathcal{I}l^z/p\to 0$. The original diagram gives

$$-i\frac{\partial\mathcal{I}}{\partial p}=i\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{\tilde{\mathbf{l}}^2-l^{z\,2}+i\epsilon}\frac{i}{[l^z-i\epsilon]^2} \tag{105}$$

4. Take only one combination of the theta function/one possible path

$$\begin{aligned}
&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \int_0^\infty \mathrm{d}z' A^{a,z}\left(z'\right) \int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''\right) \theta(z'-z'')\psi(0)|P,S\rangle \\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \int \mathrm{d}z' A^{a,z}\left(z'\right) \int \mathrm{d}z'' A^{a,z}\left(z''\right) \psi(0)|P,S\rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz''}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z'-z'')}}{h+i\epsilon} \\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \psi(0)|P,S\rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) e^{-il\cdot(z''-z')} \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz''}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z'-z'')}}{h+i\epsilon} \\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \psi(0)|P,S\rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \int \frac{\mathrm{d}w}{2\pi} \frac{i}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{i}{h+i\epsilon} e^{-i(l+h)z'} e^{-i(w-h-l)z''}
\end{aligned}$$

The other one is

$$\begin{aligned}
&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''\right) \int_0^\infty \mathrm{d}z' A^{a,z}\left(z'\right) \theta(z''-z')\psi(0)|P,S\rangle \\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \int \mathrm{d}z'' A^{a,z}\left(z''\right) \int \mathrm{d}z' A^{a,z}\left(z'\right) \psi(0)|P,S\rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z''-z')}}{h+i\epsilon} \\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \psi(0)|P,S\rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) e^{-il\cdot(z'-z'')} \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z''-z')}}{h+i\epsilon} \\
=&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \psi(0)|P,S\rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \int \frac{\mathrm{d}w}{2\pi} \frac{i}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{i}{h+i\epsilon} e^{-i(l+h)z''} e^{-i(w-h-l)z'}
\end{aligned}$$

References

[Collins(2009)] J. Collins, *Foundations of Perturbative QCD* (Cambridge University Press, 2009).

[Srednicki(2007)] M. Srednicki, *Quantum Field Theory* (Cambridge University Pr., 2007).