Local Operator Divergence

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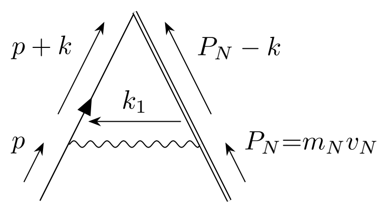
1 NRQED local matrix elements

Since what we what is an operator equation independent of states, we can choose $p^\mu = (0, \mathbf{p})$ as our outstate electron momentum, and put the nucleus on-shell. This way, we can eliminate possible IR divergence and make the integral much easier (especially at NNLO where the first integral will give a $\sinh^{-1}\left(\frac{1}{\sqrt{\frac{\mathbf{k1}^2}{2Em}-1}}\right)$ if no constrain condition put at external momentum).

At NLO:

$$\langle 0|\psi_e(0)N(0)(-ie)\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0(-ie)\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle =$$



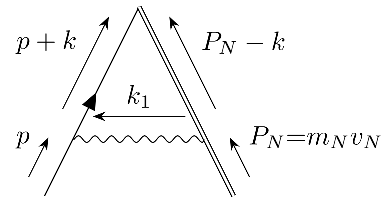
$$= e^2 u_N(v_N)\left[\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p})^2(E-\frac{\mathbf{k}^2}{2m})}(1-\frac{\mathbf{k}^4}{8m^3(E-\frac{\mathbf{k}^2}{2m})})\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}$$

$$= -e^2 u_N(v_N)[\frac{\pi}{v}+\mathcal{O}(v^2)]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}$$

Appendices

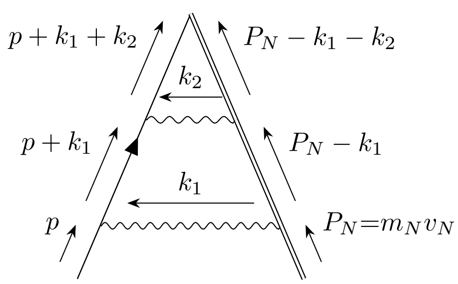
NRQED matrix element at NLO

$$\langle 0|\psi_e(0)N(0)(-ie)\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0(-ie)\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle =$$



$$\begin{aligned}
=&ie^2u_N(v_N)\left[\int [\mathrm{d}k]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(^0+k^0-m-\frac{(\mathbf{p}+\mathbf{k})^2}{2m}+\frac{(\mathbf{p}+\mathbf{k})^4}{8m^3}+i\epsilon)}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}\\
=&e^2u_N(v_N)\left[\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p})^2(E-\frac{\mathbf{k}^2}{2m}+\frac{\mathbf{k}^4}{8m^3})}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}\\
=&e^2u_N(v_N)\left[\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p})^2(E-\frac{\mathbf{k}^2}{2m})}(1-\frac{\mathbf{k}^4}{8m^3(E-\frac{\mathbf{k}^2}{2m})})\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}\\
=&-2me^2u_N(v_N)\frac{i}{E-\frac{\mathbf{p}^2}{2m}}[\frac{\pi}{2p}+\mathcal{O}(p^2)]=-e^2u_N(v_N)\frac{i}{E-\frac{\mathbf{p}^2}{2m}}[\frac{\pi}{v}+\mathcal{O}(v^2)]
\end{aligned}$$

At NNLO (where we're only interested in divergent parts)

$-$ 

$$=e^4\left[\int[dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{1}{p^0+k_1^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{1}{p^0+k_1^0+k_2^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

do the shift as above

$$=-e^4\left[\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{E-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{1}{E-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

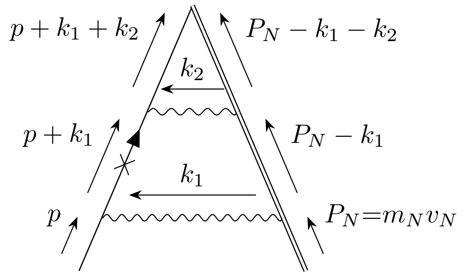
drop **p**

$$=-e^4\left[\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{1}{-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

if we add higher reltivistic correction

$$\begin{aligned}
=&-e^4\left[\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{2m}{|\mathbf{k_1}|^2-\frac{|\mathbf{k_1}|^4}{4m^2}}\frac{2m}{|\mathbf{k_2}|^2-\frac{|\mathbf{k_2}|^4}{4m^2}}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)\\
=&-4m^2e^4\left[\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_1}|^2}(1+\frac{|\mathbf{k_1}|^2}{4m^2})\frac{1}{|\mathbf{k_2}|^2}(1+\frac{|\mathbf{k_2}|^2}{4m^2})\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)
\end{aligned}$$

The integral (we can verify that $\mathbf{k}^0$ part is not UV divergent, and we don't care aobut $\mathbf{k}^8$ term for now) for



$$= \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{1}{|\mathbf{k_2}|^2-2mE}$$

$$= \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{\left(\frac{4\pi}{\Delta_2}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2\Gamma(2)}$$

where $\Delta_2 = (1-x)\left(|\mathbf{k_1}|^2 x - 2Em\right)$

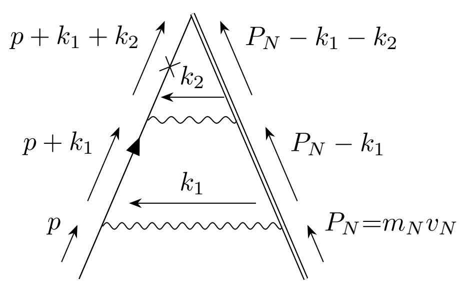
$$=\frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{1}{(|\mathbf{k_1}|^2-2mE/x)^{2-d/2}}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}\Gamma(2-d/2)$$

$$=\frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\mathrm{d}z\mathrm{d}t\,\delta(y+z+t-1)\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{zt^{1-d/2}|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2+\Delta_1]^{5-d/2}}\frac{\Gamma(5-d/2)}{\Gamma(2-d/2)}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}\Gamma(2-d/2)$$

where $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mE(z+t/x)$

$$=\frac{1}{4m^2(4\pi)^2}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\mathrm{d}z\mathrm{d}t\,\delta(y+z+t-1)zt^{1-d/2}\frac{d(d+2)}{4}\frac{\Gamma(3-d)}{(4\pi)^{3-d/2}}\left(\frac{4\pi}{\Delta_1}\right)^{3-d}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}$$

$$=-\frac{1}{128\pi^2(d-3)m^2}+\text{finite terms}$$



$$= \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_1}|^2-2mE}\frac{|\mathbf{k_2}|^4/4m^2}{[|\mathbf{k_2}|^2-2mE]^2}$$

$$=\frac{1}{4m^2}\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_1}|^2-2mE}\frac{(1-x)\Gamma(1-d/2)}{8\pi}\left(\frac{4\pi}{\Delta_2}\right)^{1-d/2}\frac{d(d+2)}{4}$$

where $\Delta_2 = x(1-x)|\mathbf{k_1}|^2 - 2mE(1-x)$

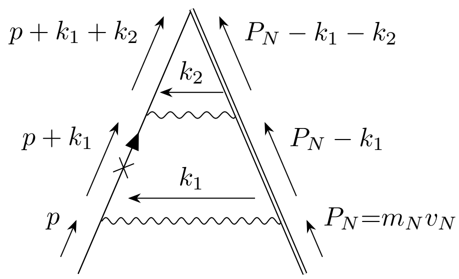
$$=\frac{1}{4m^2}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\mathrm{d}z\mathrm{d}t\frac{t^{-d/2}}{[|\mathbf{k_1}|^2+\Delta_1]^{3-d/2}}\frac{\Gamma(3-d/2)}{\Gamma(1-d/2)}\delta(y+z+t-1)\frac{\Gamma(1-d/2)}{8\pi}\left(\frac{4\pi}{x(1-x)}\right)^{1-d/2}\frac{d(d+2)x}{4}$$

where $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mEz - 2mE\frac{t}{x}$

$$=\frac{1}{4m^2}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\mathrm{d}z\mathrm{d}t\,\delta(y+z+t-1)\frac{1}{(4\pi)^{3-d/2}}\left(\frac{4\pi}{\Delta_1}\right)^{3-d}\frac{\Gamma(3-d)}{8\pi}\left(\frac{4\pi}{x(1-x)}\right)^{1-d/2}\frac{d(d+2)x}{4}t^{-d/2}$$

$$=\frac{15}{8192\pi^2(d-3)m^2}+\text{finite terms}$$

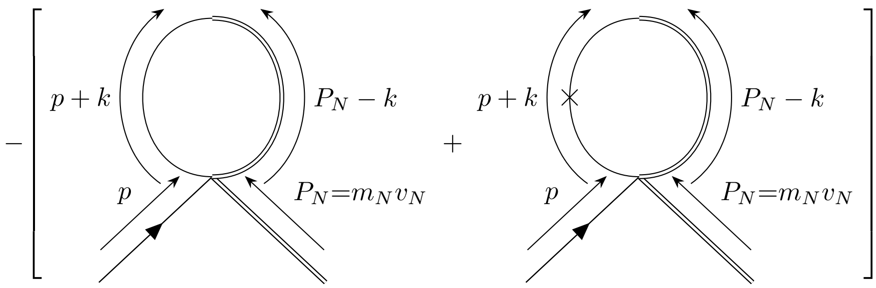
Check the contour integral

$-$ 

$$=e^4 \left[\int [dk_1][dk_2] \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{(\mathbf{p}+\mathbf{k})^4/8m^3}{[p^0 + k_1^0 - m - \frac{(\mathbf{p}+\mathbf{k_1})^2}{2m} + i\epsilon]^2} \frac{1}{p^0 + k_1^0 + k_2^0 - m - \frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m} + i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)$$
$$= -e^4 \left[\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1} - \mathbf{p}|^2} \frac{1}{|\mathbf{k_2} - \mathbf{k_1}|^2} \frac{|\mathbf{k_1}|^4/8m^3}{[E - \frac{|\mathbf{k_1}|^2}{2m} + 2i\epsilon]^2} \frac{1}{E - \frac{|\mathbf{k_2}|^2}{2m} + 2i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)$$

which is the same as previous (by expansion) one.

Contact (Darwin) term



$$= \frac{4\pi e^4}{3m} \left[\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{p^0 + k^0 - m - \frac{(\mathbf{p}+\mathbf{k})^2}{2m} + i\epsilon} \left(1 + \frac{(\mathbf{p}+\mathbf{k})^4/8m^3}{p^0 + k^0 - m - \frac{(\mathbf{p}+\mathbf{k})^2}{2m} + i\epsilon} \right) \frac{1}{-k^0 + i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)$$
$$= \frac{4\pi e^4}{3m} \left[\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{E - \frac{\mathbf{k}^2}{2m} + 2i\epsilon} \left(1 + \frac{|\mathbf{k}|^4/8m^3}{E - \frac{\mathbf{k}^2}{2m} + i\epsilon} \right) \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)$$

if the dispersion relation is up to $\mathbf{k}^4$ then

$$= \frac{4e^4 m}{3}$$