Day 3:

A. NLO ($\mathcal{O}(\alpha_s)$) QCD corrections for top total width

easier way, virtual corrections + real corrections

$\Gamma_V = \frac{1}{2m_t}\int d\Pi_{2,d}\cdot 2\,\mathrm{Re}\Big($  $\otimes$  $\Big)$

$+$

$\Gamma_R = \frac{1}{2m_t}\int d\Pi_{3,d}\cdot \Big|$  $+$  $\Big|^2$

KLN theorem ensures cancellation of IR/collinear divergences in real and virtual corrections.

[ref. C.S. Li in 1990', $\mathcal{O}(\alpha_s)$]

We need inputs of one-loop integral,

$$B_0(p^2,0,0) \equiv \frac{\mu^{4-d}}{i\pi^{d/2}\cdot r_\Gamma}\int d^dk \cdot \frac{1}{[k^2+i\epsilon][(k+p)^2+i\epsilon]}$$

$$= \left(\frac{\mu^2}{-p^2-i\epsilon}\right)^{\epsilon}\cdot\frac{1}{\epsilon(1-2\epsilon)} .$$

$$\frac{\mu^{4-d}}{i\pi^{d/2}\cdot r_\Gamma}\int d^dk \cdot \frac{k^\mu}{[k^2+i\epsilon][(k+p)^2+i\epsilon]} \equiv B_1(p^2,0,0)\cdot p^\mu$$

and

$$B_1(p^2,0,0) = -\frac{1}{2}B_0(p^2,0,0) ,$$

Recall

$$r_\Gamma = \frac{1}{\Gamma(1-\epsilon)} + O(\epsilon^3)$$

Alternative way. calculating two-loop self-energy diagrams (one-loop EW + one-loop QCD)

E.g., need imaginary part of two-loop diagrams

(a)

$$-i\Sigma_a(p) = \mu^{4\epsilon}\cdot\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\cdot g_s^2\cdot\frac{g^2}{2}\cdot C_F\cdot i$$

$$\frac{1}{[k_1^2]^2[k_2^2][(k_1-k_2)^2][(k_1-p)^2-m_W^2]}\cdot\Big\{\Big[g_{\mu\rho} - \frac{(k_1-p)_\mu(k_1-p)_\rho}{m_W^2}\Big]\gamma^\mu\not{k}_1\gamma^\nu(\not{k}_1-\not{k}_2)\gamma_\nu\not{k}_1\gamma^\rho\Big\}P_L,$$

$$\Downarrow$$

$$N$$

[ref. 2000', $O(\alpha_s^2)$]

it must can be expressed in a form

$$-i\Sigma_a(p) = i\cdot(F_a \not{p} + G_a),$$

thus using Dirac trace to project out,

$$\boxed{\begin{aligned} F_a &= \frac{\mu^{4\epsilon}}{i}\int d^dk_1\, d^dk_2 \cdots \frac{1}{d\cdot p^2}\cdot \mathrm{Tr}\left[\not{p}\cdot N\right] \\ G_a &= \frac{\mu^{4\epsilon}}{i}\int d^dk_1\, d^dk_2 \cdots \frac{1}{d}\cdot \mathrm{Tr}\left[N\right], \end{aligned}}$$

we arrive at

$$G_a = 0, \quad F_a = \frac{g_s^2\cdot g^2\cdot C_F}{2}\cdot\frac{1}{d\cdot p^2}\cdot \mu^{4\epsilon}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\cdot$$

$$(2-d)\cdot\Big\{-\left(d-2+\frac{1}{a}\right)\cdot 4k_2\cdot p\cdot Q(1,1,1,1)$$

$$-\frac{1}{a}\cdot\frac{1}{p^2}\cdot(4k_1\cdot p - 4k_2\cdot p - 4p^2)\cdot Q(0,1,1,1)\Big\}.$$

define $a = m_W^2/p^2$.

with

$$\boxed{Q(n_1,n_2,n_3,n_4) = \frac{1}{[k_1^2]^{n_1}[k_2^2]^{n_2}[(k_1-k_2)^2]^{n_3}[(k_1-p)^2-m_w^2]^{n_4}}},$$

note one can always integrate k_2 out first, giving $B_0(k_1^2,0,0)$ and $B_1(k_1^2,0,0) = -\frac{1}{2}B_0(k_1^2,0,0)$

also using $\int d^dk \frac{1}{[k^2]^n} = 0$. In this sense

$$\boxed{\begin{aligned} &k_1\cdot p\, Q(0,1,1,1) \to -\frac{1}{2}\left[(m_w^2-p^2)\,Q(0,1,1,1) - Q(-1,1,1,1)\right], \\ &-k_2\cdot p\, Q(0,1,1,1) \to -\frac{1}{2}\,k_1\cdot p\, Q(0,1,1,1) \\ &k_2\cdot p\cdot Q(1,1,1,1) \to \frac{1}{4}\left[(p^2-m_w^2)\,Q(1,1,1,1) + Q(0,1,1,1)\right]. \end{aligned}}$$

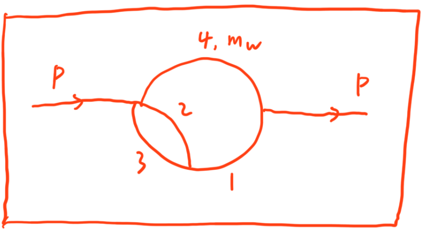
now

$$F_a = \frac{g_s^2 g^2 C_F}{2} \cdot \frac{2-d}{d\,p^2} \cdot \frac{-\pi^d r_\Gamma^2}{(2\pi)^{2d}} \cdot \left(\frac{\mu^{2\epsilon}}{i\pi^{d/2} r_\Gamma}\right)^2 \int d^d k_1\, d^d k_2$$

$$\cdot \Big\{ p^2 \cdot \left(2-d-\frac{1}{a}\right)(1-a) \cdot Q(1,1,1,1)$$

$$- \frac{1}{p^2 \cdot a} \cdot Q(-1,1,1,1) + \left(3-d+\frac{2}{a}\right) Q(0,1,1,1) \Big\}$$

with $a = m_w^2/p^2$. We arrive at calculation of master scalar integrals.



fortunately only the imaginary part is needed.

define

$$K_{n111} = \left(\frac{\mu^{2\epsilon}}{i\pi^{d/2} r_\Gamma}\right)^2 \cdot \int d^d k_1 \, d^d k_2 \cdot \frac{1}{[k_1^2]^n \, [k_2^2] \, [(k_1-k_2)^2] \, [(k_1-p)^2 - m_w^2]}$$

integrating out k_2,

$$K_{n111} = \frac{\mu^{2\epsilon}}{i\pi^{d/2} r_\Gamma} \cdot \int d^d k_1 \cdot \frac{(-1)^n}{[-k_1^2]^{n+\epsilon} \, [(k_1-p)^2 - m_w^2]} \cdot \mu^{2\epsilon} \cdot \left(\frac{1}{\epsilon(1-2\epsilon)}\right).$$

with Feynman parameters,

$$K_{n111} = \frac{\mu^{4\epsilon}}{i\pi^{d/2} r_\Gamma} \cdot \left(\frac{1}{\epsilon(1-2\epsilon)}\right) \cdot (-1)^{n+1} \cdot \int_0^1 dy \int d^d k$$

$$(1-y)^{n+\epsilon-1} \cdot \left(-(k_1-yp)^2 - y(1-y)p^2 + y m_w^2 - i\epsilon\right)^{-n-1-\epsilon}$$

$$\cdot \frac{\Gamma(n+1+\epsilon)}{\Gamma(n+\epsilon)}$$

integrating out k_1,

$$K_{n111} = \frac{\mu^{4\epsilon}}{i\pi^{d/2} r_\Gamma} \left(\frac{1}{\epsilon(1-2\epsilon)} \right) \cdot (-1)^{n+1} \cdot (n+\epsilon) \cdot \frac{(2\pi)^d}{(4\pi)^{d/2}} \cdot \frac{i\,\Gamma(n-1+2\epsilon}{\Gamma(n+1+\epsilon)}$$

$$\cdot \int_0^1 dy \cdot (1-y)^{n+\epsilon-1} \cdot y^{1-n-2\epsilon} \cdot (a-(1-y))^{1-n-2\epsilon} \cdot (p^2)^{1-n-2\epsilon}$$

$$= \frac{(-1)^{n+1}}{\epsilon(1-2\epsilon)} \frac{\Gamma(n-1+2\epsilon)}{\Gamma(n+\epsilon)} \frac{(p^2)^{1-n}}{r_\Gamma} \cdot \left(\frac{\mu^2}{p^2}\right)^{2\epsilon} \cdot \int_0^1 dy \left(\frac{(a-y)(1-y)}{y}\right)^{1-n-2\epsilon} \cdot y^{-\epsilon}$$

Considering $n = 0, \pm 1$ and $\epsilon < 0$, imaginary part only arise when y crosses a,

$n=0$, $\mathrm{Im} \int_0^1 dy \cdots = \int_a^1 \frac{(a-y)(1-y)}{y} dy \cdot (2\epsilon \cdot \pi)$

$$= -\epsilon\pi \cdot \left((1-a^2) + 2a \ln a \right)$$

$n=-1$, $\mathrm{Im} \int_0^1 dy \cdots = \int_a^1 \left[\frac{(a-y)(1-y)}{y} \right]^2 dy \cdot (2\epsilon\pi)$

$$= 2\epsilon\pi \cdot \Big(2a \ln a \cdot (1+a) + \frac{1}{3}(1-a^3) + 3a(1-a) \Big)$$

$n=1$, $\mathrm{Im}\int_0^1 dy \cdots = \mathrm{Im}\int_0^1 dy \left(\frac{(a-y)(1-y)}{y}\right)^{-2\epsilon} \cdot y^{-\epsilon}$

$$= \mathrm{Im}\int_0^1 dy \cdot \Big[1 - 2\epsilon\left[\ln(a-y) + \ln(1-y) - \tfrac{1}{2}\ln y\right] + 2\epsilon^2 \cdot \left[\ln(a-y) + \ln(1-y) - \tfrac{1}{2}\ln y\right]^2\Big]$$

$$= 2\epsilon\pi(1-a) - 4\epsilon^2\pi \cdot \Big(2(1-a)\ln(1-a) + \tfrac{1}{2}a\ln a - \tfrac{3}{2}(1-a)\Big).$$

finally

$$\mathrm{Im}\, K_{0111} = -\tfrac{1}{2}p^2\pi \cdot \left((1-a^2) + 2a\ln a\right).$$

$$\mathrm{Im}\, K_{-1111} = \tfrac{1}{4}(p^2)^2\pi \cdot \Big(a\ln a \cdot (1+a) + \tfrac{1}{6}(1-a^3) + \tfrac{3}{2}a(1-a)\Big).$$

$$\mathrm{Im}\, K_{1111} = \left(\frac{M^2}{p^2}\right)^{2\epsilon} \cdot \pi(1-a) \cdot \Big(\frac{1}{\epsilon} - 4\ln(1-a) - \frac{a}{1-a}\ln a + 5\Big).$$

and

$$\mathrm{Im}\, F_a = -\frac{g^2 C_F}{128\pi}\cdot\frac{\alpha_s}{4\pi}\cdot\Big\{\frac{3}{2}(3-a)\ln a$$

$$+\frac{1}{12}\left(11a^2-33a-3+\frac{25}{a}\right)$$

$$+4(1-a)^2\cdot\left(1+\frac{1}{2a}-\epsilon\right)\cdot\Big[\frac{1}{\epsilon}-4\ln(1-a)$$

$$-\frac{a}{1-a}\ln a+\frac{9}{2}\Big)\Big]\Big\}\cdot\left(\frac{4\pi\mu^2}{p^2}\right)^{2\epsilon} r_\Gamma^2$$

note the remaining IR/Collinear divergence that is propotional to the tree level. Can not gurantee correctness of finite terms.

$$\Gamma_a = m_t\cdot \mathrm{Im}\, F_a$$

$$= \Gamma_0\cdot C_F\cdot\frac{\alpha_s}{4\pi}\cdot\Big\{-\frac{1}{\epsilon}+\cdots\Big\}.$$

the second diagram

(b)

$$-i\Sigma_b(\not p) = \mu^{4\epsilon} \cdot \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \cdot g_s^2 \cdot \frac{g^2}{2} \cdot C_F \cdot i$$

$$\frac{1}{[k_1^2 - m_w^2][k_2^2 - m_t^2]^2[(k_1-k_2)^2][(k_2-p)^2]} \cdot \left\{ \left(g^{\mu\nu} - \frac{k_1^\mu k_1^\nu}{m_w^2}\right) \right.$$

$$\cdot\ \gamma^\rho (\not k_2 + m_t) \gamma_\mu (\not k_2 - \not k_1) \gamma_\nu \cdot \left[\not k_2 \gamma_\rho P_L \right.$$

$$\left.\left. + m_t \gamma_\rho P_R \right] \right\}.$$

define

$$-i\Sigma_b^+(\not p) = i(F_b^+ \not p + G_b^+),$$

$$-i\Sigma_b^-(\not p) = i(F_b^- \not p + G_b^-)$$

e.g.,

$$F_b^{\dagger} = \frac{g_s^2 \cdot g^2 \cdot C_F}{2} \cdot \frac{1}{d \cdot p^2} \cdot \mu^{4\epsilon} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \cdot \text{DEN} \cdot$$

$$\cdot (2-d)\, m_t^2 \cdot \Big\{ (2-d)(4 p\cdot k_2 - 4 p\cdot k_1)$$

$$- \frac{1}{m_W^2} \left[(2 k_1 \cdot k_2 - k_1^2)\, 4 p\cdot k_1 - k_1^2 \cdot 4 p\cdot k_2 \right] \Big\} .$$

with

$$\text{DEN} = \frac{1}{[k_1^2 - m_W^2][k_2^2 - m_t^2]^2 [(k_1-k_2)^2][(k_2-p)^2]}$$

$$\equiv R(1,2,1,1)$$

recall Cutkosky rule, since (b) only have one possible on-shell cut.

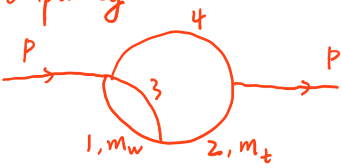
$$k_1^2 = m_W^2 \,, \quad (k_2-p)^2 = 0 \,, \quad (k_1-k_2)^2 = 0$$

we can replace in numerator,

$$k_1^2 \to m_W^2 \,, \quad p\cdot k_2 \to \frac{1}{2}(k_2^2 + p^2) \,, \quad 2 k_1\cdot k_2 - k_1^2 \to k_2^2$$

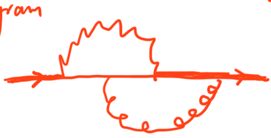
if only need imaginary part.

We arrive at computing



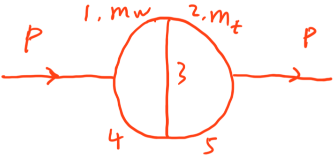
which can be done as first integrating out k_1, then k_2

The last diagram



(c)

related to scalar integrals of



Full results, $\Gamma_1/\Gamma_0 \sim -8\%$,

$$\frac{\Gamma_1}{\Gamma_0} = -\frac{\alpha_s}{2\pi} C_F \left[\frac{2\pi^2}{3} + 4\mathrm{Li}_2(a) - \frac{3}{2} - 2\ln\left(\frac{a}{1-a}\right) + 2\ln a \ln(1-a) - \frac{4}{3(1-a)} + \frac{(22-34a)}{9(1-a)^2}\ln a + \frac{(3+27\ln(1-a)-4\ln a)}{9(1+2a)}\right].$$