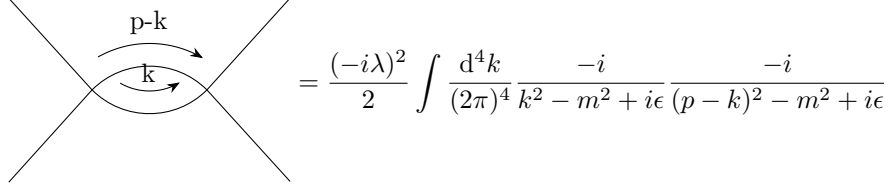# Homework: Gauge Field Theory #1

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1. ϕ^4 theory ($\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$). Verify optical theorem in the lowest order.

$$= \frac{(-i\lambda)^2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{-i}{k^2-m^2+i\epsilon}\frac{-i}{(p-k)^2-m^2+i\epsilon}$$

For simplicity, we ignore the mass term.

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{k^2(p-k)^2}$$

Apply feynman parameterization

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{[x(p-k)^2+(1-x)k^2]^2}$$

$k \to k+xp$

$$= \frac{(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{[k^2+x(1-x)p^2+i\epsilon]^2}$$

Set $\Delta \equiv -x(1-x)p^2+i\epsilon$, and apply wick rotation

$$i\mathcal{M}_2 = \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^4 k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2}$$

Dimensional regularization

$$i\mathcal{M}_2 = \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^d k_E}{(2\pi)^d}\frac{1}{[k_E^2+\Delta]^2}$$

$$= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\mathrm{d}k_E\frac{k_E^{d-1}}{[k_E^2+\Delta]^2}$$

$$= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x\frac{\pi^{d/2}\Gamma(2-d/2)}{\Gamma(2)(2\pi)^d}\Delta^{d/2-2}$$

$$\xrightarrow{d\to 4} -i\lambda^2\frac{\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)}{32\pi^2}\int_0^1 \mathrm{d}x(\frac{\Delta}{4\pi})^{-\epsilon/2}$$

$$= -i\lambda^2\frac{\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)}{32\pi^2}\int_0^1 \mathrm{d}x(1-\frac{\epsilon}{2}\ln\frac{\Delta}{4\pi})$$

$$= \frac{-i\lambda^2}{32\pi^2}(\frac{2}{\epsilon}-\gamma+2-\ln(-p^2)+\ln(4\pi)+\mathcal{O}(\epsilon))$$

where $\epsilon = 4-d$.

So

$$i\mathcal{M}(s) = -i\lambda - \frac{-i\lambda^2}{32\pi^2}(\frac{2}{\epsilon}-\gamma+2-\ln(-s)+\ln(4\pi))$$

1

$$\mathcal{M}(s) = -\lambda - \frac{-\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + 2 - \ln(-s) + \ln(4\pi)\right) = -\lambda - \frac{-\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(-s) + finite\ terms\right)$$

where $finite\ terms = \ln(4\pi) + 2 - \gamma$.

$$\lambda_R = \lambda - \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms\right)$$

$$\lambda = \lambda_R + \frac{\lambda_R^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms\right)$$

$$\mathcal{M}(s) = -\lambda + \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(-s) + finite\ terms\right)$$
$$= -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms\right) + \frac{\lambda_R^2}{32\pi^2}\left(\frac{2}{\epsilon} - \ln(-s) + finite\ terms\right)$$
$$= -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{s}$$

As the lowest order, the results are always $-\lambda$.

2. Proca field, QED with massive photon. Calculate the leading order of $e^-e^- \to e^-e^-$.

The propagator

$$\langle 0|T\{A_{in}^\mu(x)A_{in}^\nu(y)\}|0\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{\mu^2}\right)}{k^2 - \mu^2 + i\epsilon} + \frac{i}{\mu^2}\delta^4(x-y)\delta^{\mu 0}\delta^{\nu 0}$$

The Lagarangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu + \bar{\psi}(i\slashed{D} - m)\psi$$

and the interaction part

$$\mathcal{L}_I = e\bar{\psi}\gamma^\mu\psi A^\mu$$