$\bar{c}\gamma^\mu c$ matrix element

Yingsheng Huang

July 28, 2020

1 Kinematics and Conventions

Quark and antiquark momenta are

$$p_1 = P/2 + p = (E, \mathbf{p}) \tag{1}$$
$$p_2 = P/2 - p = (E, -\mathbf{p}) \tag{2}$$

where in rest frame

$$P = (2E(p), 0) \tag{3}$$
$$p = (0, \mathbf{p}) \tag{4}$$

Dirac spinors are normalized as following

$$u(\mathbf{p}) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \xi \\ \frac{\mathbf{p}\cdot\sigma}{E+m}\xi \end{pmatrix} \tag{5}$$
$$v(\mathbf{p}) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \frac{-\mathbf{p}\cdot\sigma}{E+m}\eta \\ \eta \end{pmatrix} \tag{6}$$

where

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$
$$\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{8}$$

2 State Projection

The bound state is [Weinberg(2015)]

$$|P, E; J, m_j; L; S\rangle = \int \mathrm{d}\Omega_{\mathbf{p_1}} \sum_{\lambda_1\lambda_2 s_z m_l} Y_L^{m_l}(\hat{\mathbf{p_1}}) \langle S_1\lambda_1 S_2\lambda_2 | S s_z\rangle \langle S s_z L m_l | J m_J\rangle |\mathbf{p_1}, \lambda_1\rangle |\mathbf{P} - \mathbf{p_1}, \lambda_2\rangle \tag{9}$$

3 Bilinears [Bodwin and Petrelli(2002)]

$$\Pi_0(P,p) \equiv -\sum_{\lambda_1,\lambda_2} u(\mathbf{p},\lambda_1)\bar{v}(-\mathbf{p},\lambda_2) \left\langle \frac{1}{2}\lambda_1 \frac{1}{2}\lambda_2 \middle| 00 \right\rangle \tag{10}$$
$$= \frac{1}{2\sqrt{2}E(E+m)} \left(\frac{1}{2}\not{P} + m + \not{p}\right) \frac{\not{P} + 2E}{4E} \gamma_5 \left(\frac{1}{2}\not{P} - m - \not{p}\right) \tag{11}$$
$$\Pi_1(P,p) \equiv \sum_{\lambda_1,\lambda_2} u(\mathbf{p},\lambda_1)\bar{v}(-\mathbf{p},\lambda_2) \left\langle \frac{1}{2}\lambda_1 \frac{1}{2}\lambda_2 \middle| 1\epsilon \right\rangle \tag{12}$$
$$= \frac{-1}{2\sqrt{2}E(E+m)} \left(\frac{1}{2}\not{P} + m + \not{p}\right) \frac{\not{P} + 2E}{4E} \not{\epsilon} \left(\frac{1}{2}\not{P} - m - \not{p}\right) \tag{13}$$

4 $^3S_1$

Average over $\mathbf{p}$ with Bodwin's convention (extra $1/(4\pi)$):

$$\left\langle 0 \middle| \bar{c}\gamma^\mu c \middle| {}^3S_1 \right\rangle^{(0)} = \frac{1}{4\pi}\int \mathrm{d}\Omega\, \mathrm{tr}[\Pi_1\gamma^\mu] = \sqrt{2}\left(\frac{m}{3E}+\frac{2}{3}\right)\epsilon^\mu$$

5 $^3D_1$

The matrix element reads:

$$\left\langle 0 \middle| \bar{c}\gamma^\mu c \middle| {}^3D_1 \right\rangle^{(0)} = \int \mathrm{d}\Omega \sum_{\lambda_1\lambda_2 s_z m_l} \mathrm{tr}\{\Pi_1\gamma^\mu\} \left\langle 2m_l; 1s_z | 1J_z \right\rangle Y_{2m_l}(\theta,\phi)$$

while the trace part is the same as $^3S_1$:

$$\mathrm{tr}\{\Pi_1\gamma^\mu\} = \frac{\sqrt{2}p^\mu(p\cdot\epsilon)}{E(E+m)} + \epsilon^\mu$$

Chosen polarization vectors:

$$\epsilon^{(-)} = \frac{1}{\sqrt{2}}(0,1,-i,0), \epsilon^{(0)} = (0,0,0,1), \epsilon^{(+)} = \frac{1}{\sqrt{2}}(0,-1,-i,0)$$

Result (the first row and the last are orthogonal):

$$\left(\begin{array}{c|cccc} \epsilon^{(-)} & 0 & \frac{2\sqrt{2\pi}p^2}{3E(m+E)} & -\frac{2i\sqrt{2\pi}p^2}{3E(m+E)} & 0 \\ \hline \epsilon^{(0)} & 0 & 0 & 0 & \frac{4\sqrt{\pi}p^2}{3E(m+E)} \\ \hline \epsilon^{(+)} & 0 & -\frac{2\sqrt{2\pi}p^2}{3E(m+E)} & -\frac{2i\sqrt{2\pi}p^2}{3E(m+E)} & 0 \end{array}\right)$$

and the decay constant is $\frac{4\sqrt{\pi}p^2}{3E(E+m)}$ where $p^2 = \mathbf{p}^2 = E^2 - m^2$.

6 NRQCD $^3D_1$

$$\langle 0 | \frac{1}{2m^2}\chi^\dagger D^{\{i}D^{j\}}\sigma^j\psi | Q\overline{Q}[^3D_1(\varepsilon)]\rangle^{(0)} = \int \mathrm{d}\Omega \sum_{\lambda_1\lambda_2 s_z m_l} \frac{-p^{\{i}p^{j\}}}{2m^2}\eta^\dagger_{\lambda_2}\sigma^j\xi_{\lambda_1} \left\langle 2m_l; 1s_z | 1J_z \right\rangle \left\langle \frac{1}{2}\lambda_1\frac{1}{2}\lambda_2 \middle| 1s_z \right\rangle Y_{2m_l}(\theta,\phi) \tag{14}$$

$$= \sqrt{\pi}\frac{4p^2}{3m^2}\varepsilon^i \tag{15}$$

if $\partial^{\{i}\partial^{j\}} = 2\partial^i\partial^j$.

References

[Weinberg(2015)] S. Weinberg, *Lectures on Quantum Mechanics* (Cambridge University Pr., 2015).

[Bodwin and Petrelli(2002)] G. T. Bodwin and A. Petrelli, Phys. Rev. **D66**, 094011 (2002), [Erratum: Phys. Rev.D87,no.3,039902(2013)], arXiv:hep-ph/0205210 [hep-ph] .