Note on Braaten's Paper

Yingsheng Huang

June 14, 2020

1 Intro

Hamiltonian [1]:

$$\mathcal{H} = \sum_\sigma \frac{1}{2m} \nabla\psi_\sigma^\dagger \cdot \nabla\psi_\sigma^{(\Lambda)} + \frac{g(\Lambda)}{m} \psi_1^\dagger \psi_2^\dagger \psi_3 \psi_4^{(\Lambda)} + \mathcal{V} \tag{1}$$

where the renormalized coupling

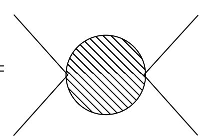
$$g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \tag{2}$$

has dimension of $[M]^{-1}$.

2 Amplitude

Consider:

$$i\mathcal{A} = \langle 34|\psi^\dagger\psi|12\rangle = \tag{3}$$



Define $P = p_1 + p_2 = (E, \mathbf{0})$, and $E = p^2/m$. The integral equation is

$$i\mathcal{A} = -\frac{ig(\Lambda)}{m}\left(1 + i\mathcal{A}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon} \frac{i}{P^0 - k^0 - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + i\epsilon}\right) \tag{4}$$

The integral gives (redefine $\epsilon \to 2m\epsilon$)

$$\mathcal{I} = \frac{im}{2\pi^2}\left(-\Lambda + \sqrt{-mE - i\epsilon}\tan^{-1}\left(\frac{\Lambda}{\sqrt{-mE - i\epsilon}}\right)\right) = -\frac{i\Lambda m}{2\pi^2} + \frac{mp}{4\pi} \tag{5}$$

(the last step isn't carefully checked and might have sign error concerning the contour on the complex plane) and

$$i\mathcal{A} = \frac{-1}{\mathcal{I} + \frac{m}{ig(\Lambda)}} = -\left[\frac{im\sqrt{-mE - i\epsilon}\tan^{-1}\left(\frac{\Lambda}{\sqrt{-mE-i\epsilon}}\right)}{2\pi^2} - \frac{im}{4\pi a}\right]^{-1} \tag{6}$$

$$\xlongequal{\Lambda\to\infty} \frac{4i\pi/m}{-1/a+\sqrt{-mE-i\epsilon}} \tag{7}$$

Note that by definition, scattering length is the leading order momentum expansion of $1/\mathcal{A}$, which gives

$$\frac{1}{a} = \frac{4i\pi}{m}\left(\mathcal{I}+\frac{m}{ig(\Lambda)}\right)^{(0)} \tag{8}$$

$$= \frac{4\pi}{g(\Lambda)} + \frac{2\Lambda}{\pi} \tag{9}$$

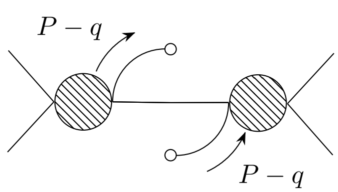
$$\Rightarrow g(\Lambda) = \frac{4\pi a}{1-2a\Lambda/\pi} \tag{10}$$

and this is actually how we get the form of (2).

3 OPE

3.1 l.h.s.

Take what we got in the last section as a new non-perturbative vertex, we only need to deal with tree diagram this way. First we have Figure 2(a) in Braaten's paper:



$$= \langle 34|\psi^\dagger\left(-\frac{\mathbf{r}}{2}\right)\psi\left(\frac{\mathbf{r}}{2}\right)|12\rangle \tag{11}$$

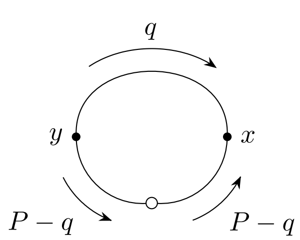
$$= (i\mathcal{A})^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^0-\frac{\mathbf{q}^2}{2m}+i\epsilon}\frac{i}{\left[E-q^0-\frac{\mathbf{q}^2}{2m}+i\epsilon\right]^2}e^{i\mathbf{q}\cdot\mathbf{r}} \tag{12}$$

$$= \mathcal{A}^2\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{m^2e^{i\mathbf{q}\cdot\mathbf{r}}}{\left(\mathbf{q}^2-p^2-i\epsilon\right)^2} \tag{13}$$

$$= \frac{im^2\mathcal{A}^2e^{ipr}}{8\pi p} \tag{14}$$

3.2 r.h.s.

For simplicity, we drop the external lines and focus on the internal subgraph. Consider Figure 2(b):



$$= \langle 34|\psi^\dagger\psi\,(0)|12\rangle_{amp} \tag{15}$$

$$= \int\mathrm{d}^4x\int\mathrm{d}^4y\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}e^{iP\cdot y}e^{-iP\cdot x}e^{-il_1\cdot y}e^{il_2\cdot x}e^{iq\cdot(x-y)}\tilde{D}(l_1)\tilde{D}(l_2)\tilde{D}(q) \tag{16}$$

$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \tilde{D}(P-q)\tilde{D}(P-q)\tilde{D}(q) \tag{17}$$

$$= -\int \frac{\mathrm{d}^3 \mathbf{q}}{(2\pi)^3} \frac{m^2}{\left(\mathbf{q}^2 - p^2 - i\epsilon\right)^2} \tag{18}$$

$$= -\frac{im^2}{8\pi p} \tag{19}$$

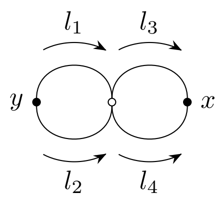
where $\tilde{D}$ marks momentum space propagator and two external vertexes give an $(i\mathcal{A})^2$ factor. The total contribution is

$$\frac{im^2\mathcal{A}^2}{8\pi p}, \tag{20}$$

the first order Fourier expansion of the l.h.s.. The Wilson coefficient of this order is 1.

3.3 Higher dimensional operators

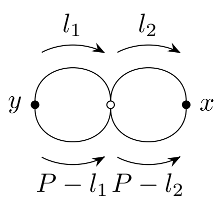
Figure 2(c) gives



$$= \langle 34|\psi^\dagger\psi^\dagger\psi\psi\,(0)|12\rangle_{amp} \tag{21}$$

$$= \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{\mathrm{d}^4 l_3}{(2\pi)^4}\frac{\mathrm{d}^4 l_4}{(2\pi)^4} e^{iP\cdot y} e^{-iP\cdot x} e^{-i(l_1+l_2)\cdot y} e^{i(l_3+l_4)\cdot x}$$
$$\tilde{D}(l_1)\tilde{D}(l_2)\tilde{D}(l_3)\tilde{D}(l_4) \tag{22}$$

which becomes

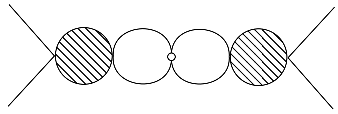


$$= \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\frac{\mathrm{d}^4 l_2}{(2\pi)^4} \tilde{D}(l_1)\tilde{D}(P-l_1)\tilde{D}(l_2)\tilde{D}(P-l_2) \tag{23}$$

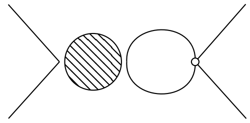
$$= -\int \frac{\mathrm{d}^3 \mathbf{l_1}}{(2\pi)^3}\frac{\mathrm{d}^3 \mathbf{l_2}}{(2\pi)^3} \frac{m^2}{\left(\mathbf{l_1}^2 - p^2 - i\epsilon\right)\left(\mathbf{l_2}^2 - p^2 - i\epsilon\right)} \tag{24}$$

$$= -\mathcal{I}^2 \tag{25}$$

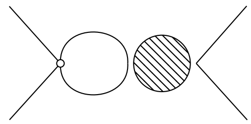
There're four diagrams in total:



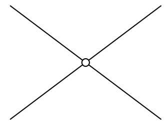
$$= \mathcal{A}^2\mathcal{I}^2 \tag{26}$$



$$= \mathcal{A}\mathcal{I} \tag{27}$$



$$= \mathcal{A}\mathcal{I} \tag{28}$$



$$= 1 \tag{29}$$

We have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm\mathbf{p}} = (\mathcal{A}\mathcal{I} + 1)^2 \tag{30}$$

in total. Plug in

$$\mathcal{I} = -\frac{m}{ig(\Lambda)} - \frac{1}{\mathcal{A}} \tag{31}$$

we have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm\mathbf{p}} = m^2 g^{-2}(\Lambda) \mathcal{A}^2 \tag{32}$$

The Wilson coefficient must be

$$-\frac{r}{8\pi} g^2(\Lambda) \tag{33}$$

4 Contact

4.1 Definition

$$C = \int \mathrm{d}^3 R \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2(R) \right\rangle \tag{34}$$

4.2 Energy Relation

According to the Hamiltonian:

$$\mathcal{H} = \left(\sum_\sigma \frac{1}{2m} \nabla \psi_\sigma^\dagger \cdot \nabla \psi_\sigma^{(\Lambda)} - \frac{\Lambda}{2\pi^2 m} g^2(\Lambda) \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 \right) + \frac{1}{4\pi m a} g^2(\Lambda) \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 + \mathcal{V} \tag{35}$$

where the matrix elements of those three operators are finite. The $\nabla \psi_\sigma^\dagger \cdot \nabla \psi_\sigma^{(\Lambda)}$ part gives a linear divergence $2 \times \frac{\Lambda m \mathcal{A}^2}{4\pi^2}$ for two spin states in total while the other one gives $-\frac{\Lambda m \mathcal{A}^2}{2\pi^2}$, we can see that the linear divergence is cancelled. Integrating over positions, we obtain

$$\int \mathrm{d}^3 R \left\langle \nabla \psi_\sigma^\dagger \cdot \nabla \psi_\sigma \right\rangle = \int \mathrm{d}^3 R \mathrm{d}^3 r \delta^{(3)}(\mathbf{r}) \left\langle \nabla \psi_\sigma^\dagger (\mathbf{R} - \frac{\mathbf{r}}{\mathbf{2}}) \cdot \nabla \psi_\sigma (\mathbf{R} + \frac{\mathbf{r}}{\mathbf{2}}) \right\rangle = \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} k^2 \rho_\sigma(k) \tag{36}$$

$$\frac{1}{4\pi m a} \int \mathrm{d}^3 R \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 \right\rangle = \frac{1}{4\pi m a} C \tag{37}$$

also notice

$$\int^{\Lambda} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{\mathbf{k}^2} = \frac{\Lambda}{2\pi^2} \tag{38}$$

we have

$$\int \mathrm{d}^3 R \frac{\Lambda}{2\pi^2 m} \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 \right\rangle = \sum_\sigma \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{k^2}{2m} \frac{C}{\mathbf{k}^4} \tag{39}$$

we achieve

$$E = \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m} \left(\rho_\sigma(k) - \frac{C}{k^4} \right) + \frac{C}{4\pi m a} + \int d^3 R \langle V \rangle \tag{40}$$

4.3 Adiabatic relation

Using F-H theorem

$$\mathrm{d}E/\mathrm{d}a = \int \mathrm{d}^3 R \langle \partial \mathcal{H} / \partial a \rangle \tag{41}$$

it's straightforward that

$$\partial \mathcal{H} / \partial a = g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 / (4\pi m a^2) \tag{42}$$

We then have

$$\frac{\mathrm{d}E}{\mathrm{d}(1/a)} = -\frac{1}{4\pi m} C \tag{43}$$

using (34).

4.4 Viral Theorem

Given a harmonic trapping potential:

$$V(\mathbf{R}) = \frac{m}{2} \omega^2 R^2 \tag{44}$$

Dimensional analysis requires

$$\left[\omega \frac{\partial}{\partial \omega} - \frac{1}{2} a \frac{\partial}{\partial a} \right] \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle = \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle \tag{45}$$

Together with F-H theorem

$$\frac{a}{2} \frac{\partial}{\partial a} \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle = \frac{C}{8\pi m a} \tag{46}$$

$$\frac{\partial}{\partial \omega} \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle = \frac{\partial}{\partial \omega} \int \mathrm{d}^3 R \, \langle \mathcal{V} \rangle = 2 \int \mathrm{d}^3 R \, \langle \mathcal{V} \rangle \tag{47}$$

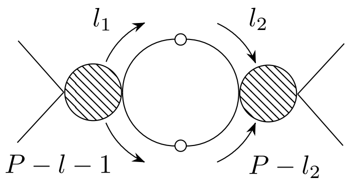
$$E = 2 \int d^3 R \langle \mathcal{V} \rangle - C/(8\pi m a) \tag{48}$$

4.5 OPE for number density operators

We have a pair of number density operators

$$\psi_1^\dagger\psi_1(\mathbf{R}-\frac{1}{2}\mathbf{r}) \;\&\; \psi_2^\dagger\psi_2(\mathbf{R}+\frac{1}{2}\mathbf{r}) \tag{49}$$

and the diagram is



(50)

$$= (i\mathcal{A})^2 \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{i}{l_1^0-\frac{\mathbf{l_1}^2}{2m}+i\epsilon}\frac{i}{E-l_1^0-\frac{\mathbf{l_1}^2}{2m}+i\epsilon}\frac{i}{l_2^0-\frac{\mathbf{l_2}^2}{2m}+i\epsilon}\frac{i}{E-l_2^0-\frac{\mathbf{l_2}^2}{2m}+i\epsilon}e^{i\mathbf{q}\cdot\mathbf{r}} \tag{51}$$

$$= \frac{\mathcal{A}^2 m^2}{16\pi^2 r^2}e^{2ipr} \tag{52}$$

Compare with the result of Figure 2(c) (32) we have the Wilson coefficient

$$\frac{g^2(\Lambda)}{16\pi^2 r^2} \tag{53}$$

5 Reformulate in Dimensional Regularization

5.1 Naïve Dim-Reg

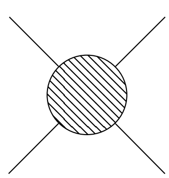
The renormalized Hamiltonian reads

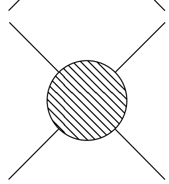
$$\mathcal{H} = \sum_\sigma \frac{1}{2m}\nabla\psi_\sigma^\dagger\cdot\nabla\psi_\sigma + Z_g\frac{g}{m}\psi_1^\dagger\psi_2^\dagger\psi_3\psi_4 \tag{54}$$

as well as

$$S = \int \mathrm{d}^{d+1}x\left[\sum_\sigma \frac{1}{2m}\nabla\psi_\sigma^\dagger\cdot\nabla\psi_\sigma + \mu^{3-d}\frac{g}{m}\psi_1^\dagger\psi_2^\dagger\psi_3\psi_4 + \delta_g\frac{g}{m}\psi_1^\dagger\psi_2^\dagger\psi_3\psi_4\right] \tag{55}$$

We have

$i\mathcal{A} =$  (56)

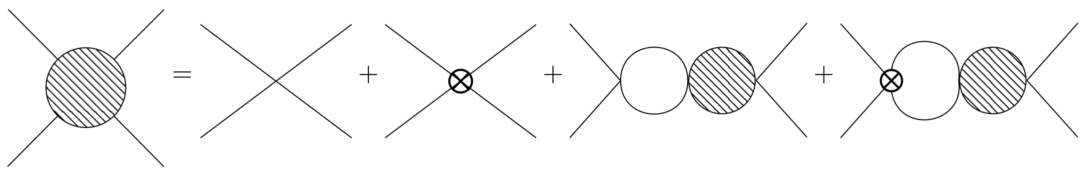
$\mathcal{I} \equiv$  (57)

The integral equation is

$$i\mathcal{A} = -\frac{iZ_g(\mu)g}{m}(1 + i\mathcal{A}\mathcal{I}) \tag{58}$$

$$\mathcal{I} \equiv \mu^{3-d} \int \frac{\mathrm{d}^{d+1}k}{(2\pi)^{d+1}} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\eta} \frac{i}{P^0 - k^0 - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + i\eta} \tag{59}$$

Graphically we have



The integral equals to

$$\begin{aligned}
\mathcal{I} &= \mu^{3-d} \int \frac{\mathrm{d}^{d+1}k}{(2\pi)^{d+1}} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\eta} \frac{i}{k^0 - P^0 - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + i\eta} && (60)\\
&= \mu^{3-d} \int \frac{\mathrm{d}^{d}\mathbf{k}}{(2\pi)^{d}} \frac{i}{P^0 - \frac{\mathbf{k}^2}{2m} - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + 2i\eta} && (61)\\
&= \mu^{3-d} \int \frac{\mathrm{d}^{d}\mathbf{k}}{(2\pi)^{d}} \frac{im}{p^2 - \mathbf{k}^2 + i\eta} && (62)\\
&= -\mu^{3-d} \frac{1}{2\pi^2} \frac{im}{2} \pi \left(-p^2 - i\eta\right)^{\frac{d-2}{2}} \csc\left(\frac{\pi d}{2}\right) && (63)\\
&\xrightarrow{d\to 4} \frac{im\left(p^2 + i\eta\right)}{2\pi^2(d-4)\mu} + \frac{im\left(p^2 + i\eta\right)\log\left(-\frac{p^2+i\eta}{\mu^2}\right)}{4\pi^2\mu} + \mathcal{O}(d-4) && (64)
\end{aligned}$$

When we set $\mu = p$, the logarithm disappears and the finite term goes to our result with cutoff regularization

$$\frac{im\left(p^2 + i\eta\right)\log\left(-\frac{p^2+i\eta}{\mu^2}\right)}{4\pi^2\mu} \xrightarrow{\mu\to p,\eta\to 0} -\frac{mp}{4\pi} \tag{65}$$

Similar to our previous calculation, we have

$$\mathcal{I} = -\frac{imp^2}{2\pi^2\epsilon\mu} + \frac{im\left(p^2 + i\eta\right)\log\left(-\frac{p^2+i\eta}{\mu^2}\right)}{4\pi^2\mu} \tag{66}$$

with $\epsilon \equiv 4 - d$. We put it back to the integral equation

$$i\mathcal{A} = \frac{-1}{\mathcal{I} + \frac{m}{iZ_g(\mu)g}} \equiv \frac{4i\pi/m}{-1/a + \sqrt{-p^2 - i\eta}} \tag{67}$$

With this definition we have

$$\begin{aligned}
\frac{1}{a} &= \frac{4i\pi}{m}\left(\mathcal{I} + \frac{m}{iZ_g(p)g}\right)^{(0)} && (68)\\
&= \frac{2p}{\pi\epsilon} + \frac{4\pi}{Z_g(p)g} && (69)
\end{aligned}$$

$$\Rightarrow Z_g(p)g = \frac{4\pi a}{1 - 2ap/(\pi\epsilon)} \tag{70}$$

We could see that p/ϵ is effectively Λ.

Now without pre-determined scale μ we have

$$\frac{1}{a} = \frac{2p^2}{\pi\mu\epsilon} + \left(\frac{4\pi}{g} + \sqrt{-p^2 - i\eta} - \frac{p^2 \log\left(-\frac{p^2+i\eta}{\mu^2}\right)}{\pi\mu} - \frac{i\eta \log\left(-\frac{p^2+i\eta}{\mu^2}\right)}{\pi\mu} \right) \tag{71}$$

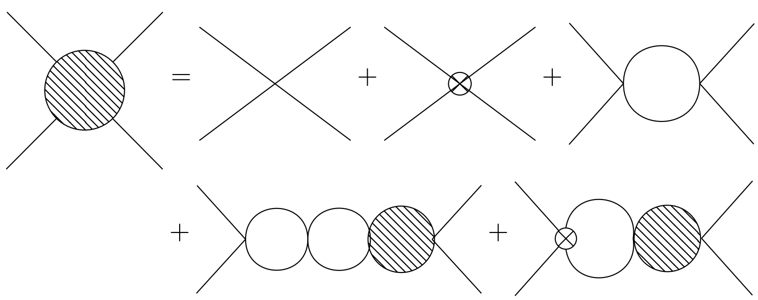
$$\Rightarrow Z_g(\mu)g = \frac{4\pi^2\mu}{p^2 \log\left(-\frac{p^2+i\eta}{\mu^2}\right) + i\eta \log\left(-\frac{p^2+i\eta}{\mu^2}\right) - \pi\mu\sqrt{-p^2 - i\eta} - 2p^2/\epsilon + \pi\mu/a} \tag{72}$$

5.2 PDS scheme

Here we adopt the power divergence subtraction (PDS) scheme by Kaplan, Savage and Wise in 1998 [2].

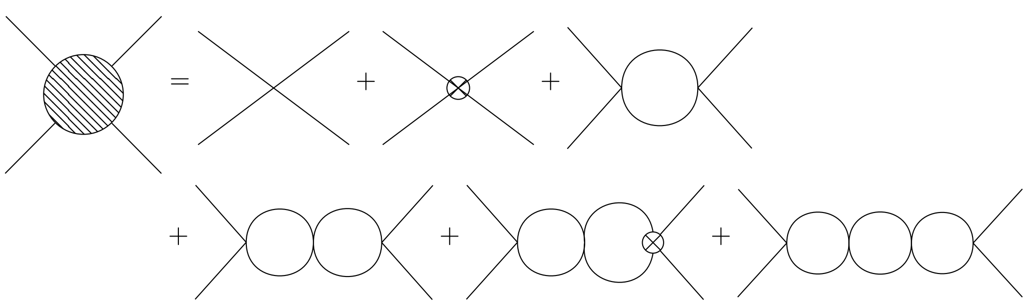
A Perturbative view of integral equation

If we're to have the integral equation only with perturbative counterterms, for a start we can write

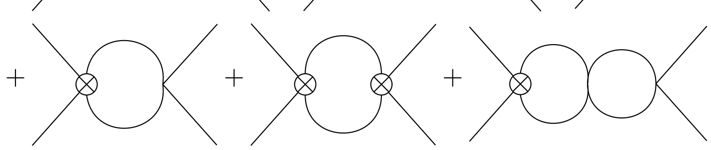


with one-loop counterterm.

The iterative form is

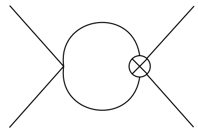


(73)



(74)

It'd appeared that we missed one diagram during the iteration:



If we add this one, the r.h.s. of the integral equation won't be finite again. Thus, only the counterterm from one loop is not sufficient enough to cancel all the divergences.

References

[1] E. Braaten and L. Platter, Phys. Rev. Lett. **100** (2008).

[2] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. **B424**, 390 (1998).