Chiral Symmetry and Lattice Fermions: Lectures 1-4

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Chiral symmetry and anomalies in d=1+1.

1.1 Introduction

Chiral symmetries play an important role in the spectrum and phenomenology of both the standard model and various theories for physics beyond the standard model. In many cases chiral symmetry is associated with nonperturbative physics which can only be quantitatively explored in full on a lattice. It is therefore important to implement chiral symmetry on the lattice, which turns out to be less than straightforward. In these lectures I discuss what chiral symmetry is, why it is important, how it is broken, and ways to implement it on the lattice. There have been many hundreds of papers on the subject and this is not an exhaustive review; the limited choice of topics I cover reflects on the scope of my own understanding and not the value of the omitted work.

1.2 Dirac fermions in 1+1 dimensions

1.2.1 γ-matrices

Consider a free, massive Dirac fermion in $d = 1 + 1$ dimensions, whose coordinates we will call $x^0 = t$, $x^1 = x$. The Lagrange density is

$$\mathcal{L} = \bar{\psi}\left(i\partial_\mu \gamma^\mu - m\right)\psi \ , \tag{1.1}$$

where ψ is a 2-component spinor, and the γ-matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \ . \tag{1.2}$$

A convenient representation for the γ-matrices in terms of the Pauli matrices is

$$\gamma^0 = \sigma_1 \ , \qquad \gamma^1 = -i\sigma_2 \ . \tag{1.3}$$

Lorentz transformation of ψ takes the form

$$\psi(x) \to e^{i\frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi(\Lambda^{-1}x) \ , \qquad \sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \ , \tag{1.4}$$

where ${\Lambda_\mu}^\nu(\omega)$ is the corresponding Lorentz transformation matrix for a 2-vector. This is a bit heavy-handed: life in $d = 1 + 1$ dimensions is life on a wire, and the only Lorentz transformations are boosts in the x direction; for

$$\omega_{01} = -\omega_{10} = \theta \tag{1.5}$$

we have

$$\Lambda_\mu{}^\nu = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} , \qquad \sigma^{01} = i\frac{\sigma_3}{2} . \tag{1.6}$$

Note that we can define a third γ-matrix I will denote Γ which is Hermitian and which anticommutes with both γ^μ, the analogue of γ_5 in $d = 3+1$:

$$\Gamma = \Gamma^\dagger , \qquad \Gamma^2 = 1 , \qquad \{\Gamma, \gamma^\mu\} = 0 \quad \Longrightarrow \quad [\Gamma, \sigma^{\mu\nu}] = 0 . \tag{1.7}$$

In the above basis, we can take $\Gamma = \sigma_3$. Since Γ commutes with Lorentz transformations, we conclude that we have a *reducible* representation of the Lorentz group. We can define projection operators

$$P_\pm = \frac{1\pm\Gamma}{2} , \qquad P_+^2 = P_+ , \qquad P_-^2 = P_- , \qquad P_+ + P_- = 1 . \tag{1.8}$$

(where "1" means the 2×2 unit matrix). Then we define

$$\psi_R = P_+\psi , \quad \psi_L = P_-\psi , \quad \bar\psi_L = \psi_L^\dagger\gamma^0 = \bar\psi P_+ , \quad \bar\psi_R = \psi_R^\dagger\gamma^0 = \bar\psi P_- , \tag{1.9}$$

and we know that $\psi_{L,R}$ will not mix under Lorentz transformations. With the above definitions, writing $\psi = \psi_L + \psi_R$ and plugging back into our Lagrange density in eqn. (1.1) we find we can rewrite it as

$$\mathcal{L} = \bar\psi_L i\partial\!\!\!/\psi_L + \bar\psi_R i\partial\!\!\!/\psi_R - m\left(\bar\psi_L\psi_R + \bar\psi_R\psi_L\right) = \bar\psi_L i\partial\!\!\!/\psi_L + \bar\psi_R i\partial\!\!\!/\psi_R - \left(m\bar\psi_L\psi_R + \text{h.c.}\right) \tag{1.10}$$

1.2.2 The massless case and chiral symmetry

Let's set the fermion mass to zero. We see that the above Lagrange density has two $U(1)$ symmetries: we can rotate the fermions with the two independent phases α and β as

$$\begin{aligned} \psi_L &\to e^{i\alpha}\psi_L , & \bar\psi_L &\to e^{-i\alpha}\bar\psi_L , \\ \psi_R &\to e^{i\beta}\psi_R , & \bar\psi_R &\to e^{-i\beta}\bar\psi_R , \end{aligned} \tag{1.11}$$

without affecting $\mathcal{L}$ in eqn. (1.10), provided that $m = 0$. Especially interesting is that the symmetry persists even if we add gauge interactions, replacing ∂_μ by a gauge covariant derivative D_μ. Therefore, even with gauge interactions, there are apparently two conserved currents

$$j_R^\mu = \bar\psi\gamma^\mu P_+\psi , \qquad j_L^\mu = \bar\psi\gamma^\mu P_-\psi . \tag{1.12}$$

Evidently these are both symmetry currents only for massless fermions, since the mass term in eqn. (1.10) couples ψ_L to ψ_R, and the Lagrange density is not invariant under independent phase rotations. It is useful, therefore, to consider two different linear combinations of these currents, referred to as the vector and axial currents,

$$j^\mu = \bar\psi\gamma^\mu\psi \ , \qquad j^\mu_A = \bar\psi\gamma^\mu\Gamma\psi \ . \tag{1.13}$$

These two currents correspond to the two independent transformations

$$\psi \to e^{i\theta}\psi \ , \qquad \psi \to e^{i\omega\Gamma}\psi \tag{1.14}$$

respectively. The first conserved quantity is just fermion number, the second, which counts right minus left number, is called axial charge. The fact that $U(1)_A$ axial transformations are a symmetry of the kinetic operator is a consequence of the property

$$\{\Gamma, \not{D}\} = 0 \ . \tag{1.15}$$

Later we will see that on the lattice, even for massless fermions it is not possible to define a kinetic operator analogous to $\not{D}$ which anti-commutes with Γ, but that the above equation will have to be modified.

The reason why both $j^\mu_{L,R}$ currents are conserved for massless fermions is easy to see if we look at the equation of motion for the free massless fermion:

$$0 = i\not\partial\psi = i\begin{pmatrix} 0 & \partial_t - \partial_x \\ \partial_t + \partial_x & 0 \end{pmatrix}\psi \ , \tag{1.16}$$

which has plane wave solutions

$$\psi_R = e^{-ik(t-x)}\begin{pmatrix}1\\0\end{pmatrix} \qquad \psi_L = e^{-ik(t+x)}\begin{pmatrix}0\\1\end{pmatrix} \ , \tag{1.17}$$

with $P_+\psi_R = \psi_R$ and $P_-\psi_L = \psi_L$. We see that ψ_R corresponds to fermions moving at the speed of light to the right (positive x-direction) and ψ_L corresponds to particles moving to the left. Clearly, Lorentz boosts cannot change the number of either, which is therefore conserved quantities. Thus we expect to be able to write

$$\partial_\mu j^\mu_{L,R} = \partial_\mu j^\mu = \partial_\mu j^\mu_A = 0 \ . \tag{1.18}$$

We will show below, however, that this is not the case, and that quantum effects spoil some of these conservation laws through "anomalies".

Because ψ_L and ψ_R transform under independent irreducible representations of the Lorentz group, we can consider a theory with just one of them, such as

$$\mathcal{L} = \bar\psi_R i\not{D}\psi_R \ , \tag{1.19}$$

which, if gauged as in the above example, would be called an example of a "chiral gauge theory". Note that ψ_R is a 2-component spinor, where the lower component equals zero. So we could have written this as a 1-component fermion, but it is convenient often to write it as a Dirac spinor, with one component projected out by P_+. This theory looks like it should make sense because the gauge field appears to be coupled to a conserved current, but again, anomalies will spoil this assumption.

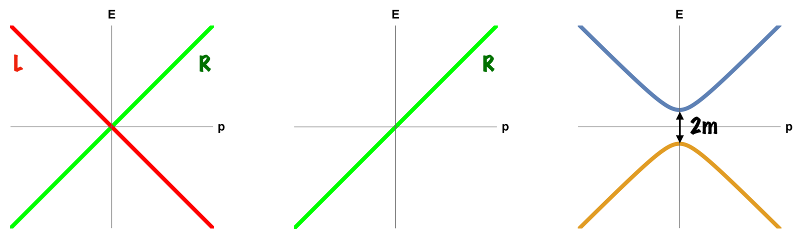

Fig. 1.1 *The spectrum of (i) a free massless Dirac fermion in $d = 1 + 1$, (ii) a free massless RH chiral fermion, (iii) a free massive Dirac fermion. In the first case both LH and RH fermion numbers are independently conserved — or equivalently, both fermion number and axial charge are conserved; in the second case there is a conserved RH fermion number; for the massive case there is only a conserved fermion number.*

1.2.3 The massive Dirac fermion

In contrast to the massless case, if the fermion is massive we can always boost between frames where a left-mover in one frame is a right-mover in the other, and so the number of either cannot be individually conserved. Figure 1.1 shows the spectrum for of free fermions for the cases we have discussed. If you take at the Lagrange density in eqn. (1.10) and perform the axial transformation in eqn. (1.14), you find

$$\mathcal{L} \to \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R - \left(me^{2i\omega}\bar{\psi}_L\psi_R + \mathrm{h.c.}\right) \ , \tag{1.20}$$

which is equivalent to rotating the mass term by a phase, $m \to me^{2i\omega}$. (This shows that the phase of the fermion mass has no physical meaning in a theory where it is the only source of axial symmetry violation, since we can change the phase at will with a change of variables.) Noether's theorem then tells us that for the massive case we have

$$\partial_\mu j^\mu = 0, \qquad \partial_\mu j^\mu_A = 2im\bar{\psi}\Gamma\psi \ . \tag{1.21}$$

1.3 The $U(1)_A$ anomaly in $d = 1 + 1$

So far we have blithely assumed that symmetries of the Lagrange density imply symmetries of the quantum theory. However, one of the fascinating features of chiral symmetry is that sometimes it is not a symmetry of the quantum field theory even when it is a symmetry of the Lagrangian. In particular, Noether's theorem can be modified in a theory with an infinite number of degrees of freedom; the modification is called "an anomaly". Anomalies turn out to be very relevant both for phenomenology, and central for understanding the challenges for implementing chiral symmetry in lattice field theory. The reason anomalies affect chiral symmetries is that regularization requires a cut-off on the infinite number of modes above some mass scale, while chiral symmetry is incompatible with fermion masses[1].

[1]Dimensional regularization is not a loophole, since chiral symmetry cannot be analytically continued away from odd space dimensions.

A simple way to derive anomalies (and in some ways, overly simple) is to look at what happens to the ground state of a theory with a single flavor of massless Dirac fermion in $(1+1)$ dimensions in the presence of an electric field. Suppose one adiabatically turns on a constant positive electric field $E(t)$, then later turns it off; the equation of motion for the fermion is [2] $\frac{dp}{dt} = eE(t)$ and the total change in momentum is

$$\Delta p = e \int E(t)\, dt \ . \tag{1.22}$$

Thus the momenta of both left- and right-moving modes increase; if one starts in the ground state of the theory with filled Dirac sea, after the electric field has turned off, both the right-moving and left-moving sea levels have shifted to the right as in Fig. 1.2. The the final state differs from the original by the creation of particle- antiparticle pairs: right moving particles and left moving antiparticles. Thus while there is a fermion current in the final state, fermion number has not changed. This is what one would expect from conservation of the $U(1)$ current:

$$\partial_\mu j^\mu = 0 \ , \tag{1.23}$$

However, recall that right-moving and left-moving particles have positive and negative chirality respectively; therefore the final state in Fig. 1.2 has net axial charge, even though the initial state did not. This is peculiar, since the coupling of the electromagnetic field in the Lagrangian does not violate chirality. We can quantify the effect: if we place the system in a box of size L with periodic boundary conditions, momenta are quantized as $p_n = 2\pi n/L$. The change in axial charge is then

$$\Delta Q_A = 2\frac{\Delta p}{2\pi/L} = \frac{e}{\pi}\int d^2x\, E(t) = \frac{e}{2\pi}\int d^2x\, \epsilon_{\mu\nu}F^{\mu\nu} \ , \tag{1.24}$$

where I expressed the electric field in terms of the field strength F, where $F^{01} = -F^{10} = E$. This can be converted into the local equation using $\Delta Q_A = \int d^2x\, \partial_\mu j^\mu_A$, a modification of eqn. (1.18):

$$\partial_\mu j^\mu_A = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ , \tag{1.25}$$

where in the above equation I have included the classical violation due to a mass term as well. The second term is the axial anomaly in $1+1$ dimensions.

Exercise 1.1 Use the above arguments to derive the anomaly that results if one gauges the axial current instead of the vector current.

[2] While in much of these lectures I will normalize gauge fields so that $D_\mu = \partial_\mu + iA_\mu$, in this section I need to put the gauge coupling back in. If you want to return to the nicer normalization, rescale the gauge field by e so that there is no coupling constant in the covariant derivative and a $1/e^2$ factor appears in front of the gauge action.

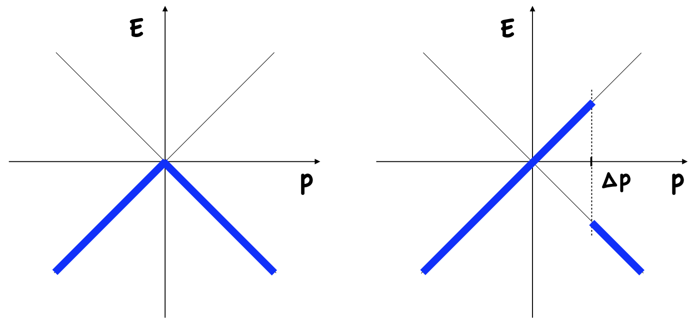

Fig. 1.2 *On the left: the ground state for a theory of a single massless Dirac fermion in* $(1 + 1)$ *dimensions; on the right: the theory after application of an adiabatic electric field with all states shifted to the right by* Δp*, given in eqn.* (1.22). *Filled states are indicated by the heavier blue lines.*

So how did an electric field end up violating chiral charge? Note that this analysis relied on the Dirac sea being infinitely deep. If there had been a finite number of negative energy states, then they would have shifted to higher momentum, but there would have been no change in the axial charge. With an infinite number of degrees of freedom, though, one can have a "Hilbert Hotel": the infinite hotel which can always accommodate another visitor, even when full, by moving each guest to the next room and thereby opening up a room for the newcomer. This should tell you that it will not be straightforward to represent chiral anomalies on the lattice: a lattice field theory approximates quantum field theory with a finite number of degrees of freedom — the lattice may be a big hotel, but it is quite conventional. In such a hotel there can be no anomaly, since there is no ambiguity about how many occupants it has.

This method of deriving the anomaly gives the correct answer, but is a bit too simplistic. For one thing, there is no need to assume that the gauge field must change adiabatically. For another, it doesn't help one figure out what happens in the case where there is a fermion mass and a gap, where the correct answer is that one just adds together the anomalous and explicit symmetry violation, modifying eqn. (1.21) to read

$$\partial_\mu j^\mu_A = 2im\overline{\psi}\Gamma\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ , \tag{1.26}$$

We can derive the anomaly in other ways, such as by computing the anomaly diagram Fig. 1.3, or by following Fujikawa (Fujikawa, 1979; Fujikawa, 1980) and carefully accounting for the Jacobian from the measure of the path integral when performing a chiral transformation. It is particularly instructive for our later discussion of lattice fermions to compute the anomaly in perturbation theory using Pauli-Villars regulators of mass M. Consider the fermion determinant obtained from the path integral in Euclidian spacetime:

$$\det\left(\not{D} + m\right) \ . \tag{1.27}$$

Here we assume hermitian γ-matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and so $\not{D}$ is an anti-hermitian operator with unbounded imaginary eigenvalues. The determinant is formally given by

$$\det\left(\not{D} + m\right) = \prod_i (i\lambda_i + m) \ , \tag{1.28}$$

but this is ill defined. To better define it, we consider instead a regulated version,

$$\lim_{M\to\infty} \frac{\det\left(\not{D} + m\right)}{\det\left(\not{D} + M\right)} = \lim_{M\to\infty} \prod_i \frac{(i\lambda_i + m)}{(i\lambda_i + M)} \ . \tag{1.29}$$

Note that at fixed M, for $\lambda_i \gg M \gg m$ the contributions to the regulated determinant all go to factors of 1, so the effect of the regulator is to cancel off contributions from those states. Of course, in the end we take $M \to \infty$ and recover the theory we are interested in. The Feynman rules for this regulated determinant are simple: we just add a heavy "Pauli-Villars" fermion Φ with a Dirac action with mass M, but instead of having a factor of -1 from each closed loop, we get a $+1$ in order to obtain the inverse determinant, as we would get from a boson field. It is important that the Φ have all the same couplings as the fermion, including to external sources. As a result, we should consider the divergence of the regulated axial current

$$j^\mu_{A,\mathrm{reg}} = \overline{\psi}\gamma^\mu\Gamma\psi + \overline{\Phi}\gamma^\mu\Gamma\Phi \ , \tag{1.30}$$

where it follows from Noether's theorem that

$$\partial_\mu j^\mu_{A,\mathrm{reg}} = 2im\overline{\psi}\Gamma\psi + 2iM\overline{\Phi}\Gamma\Phi \ . \tag{1.31}$$

note that Φ contributes a new contribution to axial symmetry breaking proportional to M. However, *this* current does not have any additional anomalous divergence, because we have essentially removed all high λ_i states from consideration and have a "conventional hotel". Therefore, if we are to recover the anaomaly, it must come the Pauli-Villars contribution somehow. As we are interested in matrix elements of $j^\mu_{A,\mathrm{reg}}$ in a background gauge field between states that do not contain any Pauli-Villars particles, we need to evaluate the expectation value $\langle 2iM\overline{\Phi}\Gamma\Phi\rangle$ in a background gauge field and take the limit $M \to \infty$, in order to see if $\partial_\mu j^\mu_{A,\mathrm{reg}}$ picks up any anomalous contributions that do not decouple as we remove the cutoff $M \to \infty$.

To compute $\langle 2iM\overline{\Phi}\Gamma\Phi\rangle$ we need to consider all Feynman diagrams with a Pauli-Villars loop, and insertion of the $\overline{\Phi}\Gamma\Phi$ operator, and any number of external $U(1)$ gauge fields. By gauge invariance, a graph with n external photon lines will contribute n powers of the field strength tensor $F^{\mu\nu}$. For power counting, it is convenient that we normalize the gauge field so that the covariant derivative is $D_\mu = (\partial_\mu + iA_\mu)$; then the gauge field has mass dimension 1, and $F^{\mu\nu}$ has dimension 2. In $(1 + 1)$ dimensions $\langle 2iM\overline{\Phi}\Gamma\Phi\rangle$ has dimension 2, and so simple dimensional analysis implies that the graph with n photon lines must make a contribution proportional to $(F^{\mu\nu})^n/M^{2(n-1)}$. Therefore only the graph in Fig. 1.3 with one photon insertion can make a contribution that survives the $M \to \infty$ limit (the graph with zero photons vanishes). Calculation of this diagram yields the same result for the divergence of the regulated axial current as we found in eqn. (1.26); to show this is an exercise.

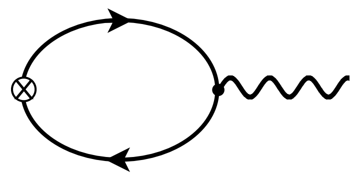

Fig. 1.3 *The anomaly diagram in 1+1 dimensions, with one Pauli-Villars loop and an insertion of $2iM\overline{\Phi}\Gamma\Phi$ at the X.*

Exercise 1.2 Compute the diagram in Fig. 1.3 using the conventional normalization of the gauge field $D_\mu = (\partial_\mu + ieA_\mu)$ and verify that $2iM\langle\overline{\Phi}\Gamma\Phi\rangle = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}$ when $M \to \infty$. Note that you are looking for a contribution proportional to $i\epsilon_{\mu\nu}k^\nu$, where k^ν is the momentum of the external gauge boson and μ is its polarization.

Note that in this description of the anomaly we (i) effectively rendered the number of degrees of freedom finite by introducing the regulator; (ii) the regulator explicitly broke the chiral symmetry; (iii) as the regulator was removed, the symmetry breaking effects of the regulator never decoupled, indicating that the anomaly arises when the two vertices in Fig. 1.3 sit at the same spacetime point. While we used a Pauli-Villars regulator here, the use of a lattice regulator will have qualitatively similar features, with the inverse lattice spacing playing the role of the Pauli-Villars mass, and we can turn these observations around: A lattice theory will not correctly reproduce anomalous symmetry currents in the continuum limit, unless that symmetry is broken explicitly by the lattice regulator. This means we would be foolish to expect that a continuum field theory with anomalies could ever be represented by a lattice theory with exact chiral symmetry.

1.4 A lattice Hamiltonian for d=1+1 fermions

1.4.1 Doubling of as chiral fermion

Let's reconsider the theory of a single gauged right-handed fermion, as in eqn. (1.19). We now know that the current will have an anomalous divergence, which means that the theory is not gauge invariant! Such theories are known to be sick, so it should not be possible to give them a definition on the lattice. To keep things simple, I will first consider a latticized version of the Hamiltonian for the free theory, which only involves discretizing space, not time. The continuum Hamiltonian in our γ-matrix basis dfor the free fermion is simply

$$H = -i\partial_x \tag{1.32}$$

with naive discretization

$$H = -i\frac{1}{2a}\sum_n c_n^\dagger \left(c_{n+1} - c_{n-1}\right) \tag{1.33}$$

where a is the lattice spacing, and the $c_n, c_n^\dagger$ are fermionic ladder operators at site n:

$$\{c_m, c_n\} = 0 \ , \qquad \{c_m, c_n^\dagger\} = \delta_{mn} \ . \tag{1.34}$$

This theory has an exact $U(1)$ symmetry, which is fermion number:

$$Q = \sum_n c_n^\dagger c_n \ , \qquad [Q, H] = 0 \ . \tag{1.35}$$

This is the symmetry we can gauge. The single-particle eigenstates of H are

$$|p\rangle = \sum_n e^{iapn} c_n^\dagger |0\rangle \tag{1.36}$$

with energy eigenvalue

$$H|p\rangle = E_p|p\rangle \ , \qquad E_p = \frac{\sin ap}{a} \ , \qquad -\frac{\pi}{a} \leq p \leq \frac{\pi}{a} \ . \tag{1.37}$$

Note from the construction of the state $|p\rangle$ that shifting $p \to p + 2\pi a$ gives back the same state, so p-space is a circle and so taking the above range for p (the Brillouin zone) accounts for all states.

What is the continuum limit of this theory? Naively, the continuum limit $ap \to 0$ gives $E_p = p$, the desired continuum result corresponding to a single right-mover, shown in Fig. 1.1. However, if we rewrite $p = \pi - k$, then the $ak \to 0$ limit gives $E_k = -k$, a left-mover! We see that the continuum theory describes a single massless Dirac fermion in the continuum, with both right and left modes, not a single right-mover. That is because the dispersion relation $E_p = \sin ap/a$ crosses the p-axis in two places, $p = 0$ and $p = \pm\pi$, so there will always be two low energy modes, even as $a \to 0$. Furthermore, the exact $U(1)$ symmetry of the lattice is just fermion number in the continuum theory, so if we gauge it, the result looks like QED in $d = 1 + 1$, a sensible theory with a conserved gauge current, unlike the chiral gauge theory in eqn. (1.19).

Can we add a local interaction that will gap the spectrum at $p = \pm\pi/a$, to get rid of the continuum left-mover? Obviously not: the function E_p will be a continuous function of p and therefore must be periodic; a periodic function of p cannot cross the p-axis an odd number of times.

And what about the anomalous $U(1)_A$ global symmetry in $d = 1 + 1$ QED? How does the lattice model realize the anomaly? The answer is that the lattice theory does not have a second $U(1)$ symmetry that we can identify with $U(1)_A$ in the continuum...that would require rotating states with $p \sim 0$ with the opposite phase from states near $p \sim \pm\pi/a$, which is not a symmetry of H. Consider an eigenstate of H at finite lattice spacing which we will call the vacuum when $a \to 0$, with every negative energy state occupied and every positive energy state empty, as shown on the left in Fig. 1.4. Now consider what happens when we turn on an electric field for some time in the x direction: all states will move to the right (increasing p) and we end up with the state shown on the right in that figure. In the continuum theory that corresponds to a state that still has no net fermion number, but a nonzero axial charge. The lattice correctly reproduces the axial anomaly by having no exact axial symmetry.

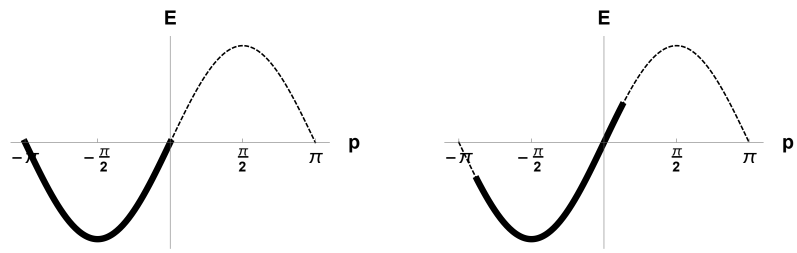

Fig. 1.4 *The ground state of a lattice Dirac fermion (left) and how it has evolved after application of an electric field in the x direction (right). The solid line denotes occupied 1-particle states, and the dashed line vacant states. In the continuum limit, it will appear as if an anomaly has violated axial charge, giving rise to right moving particles and left moving anti-particles.*

Note that it would have been wrong to consider the lattice theory to be similar to a continuum Dirac theory with a momentum cutoff at $p \sim 1/a$: such a theory would have an exact $U(1)_A$ symmetry (since a momentum cutoff is a regulator that does not violate axial rotations) but could not be gauged (since a momentum cutoff violates gauge symmetry). It behaves much more like a continuum theory with a Pauli-Villars regulator with mass $M \sim 1/a$, a regulator that preserves gauge symmetry while breaking axial symmetry.

It seems we should be happy that the lattice was "smart enough" to not give us a sick theory in the continuum, a gauge theory whose gauge symmetry was broken by an anomaly. However there are problems we can see even with very simple variants of this model. The first has to do with the role chiral symmetry plays in the Standard Model, protecting fermion masses from additive mass renormalization. The second has to do with creating a lattice regulator for chiral gauge theories that are not sick....like the Standard Model itself.

1.4.2 Problems for chiral gauge theories

It is possible to construct a theory with several chiral fermions that has an anomaly-free $U(1)$ symmetry that can be gauged. If the fermion representation is such that one cannot write down gauge-invariant mass terms for the fermions, then the theory is called a chiral gauge theory. From our discussion of the anomaly we see that an example of a chiral $U(1)$ gauge theory in $d = 1 + 1$ dimensions is the $3 - 4 - 5$ model which consists of three fermions, right-movers $\psi_{3,4}$ with electric charges 3 and 4, and a left-mover χ_5 with charge 5,

$$\mathcal{L} = \bar{\psi}_3 i\not{D} P_+ \psi_3 + \bar{\psi}_4 i\not{D} P_+ \psi_4 + \bar{\chi}_5 i\not{D} P_- \chi_5 \ . \tag{1.38}$$

Note that unlike in QED, it is impossible to write a gauge invariant fermion mass. That requires both a left- and a right-moving particle...however the two right movers we have in the theory have charges 3 and 4, while the only left-mover has charge 5, and neither $\bar{\chi}_5 P_+ \psi_3$ nor $\bar{\chi}_5 P_+ \psi_4$ operators are invariant under the gauge symmetry.

It appears that there are three conserved currents in this theory, one for each type of fermion number:

$$j_3^\mu = \bar\psi_3 \gamma^\mu P_+ \psi_3 \ , \qquad j_4^\mu = \bar\psi_4 \gamma^\mu P_+ \psi_4 \ , \qquad j_5^\mu = \bar\chi_5 \gamma^\mu P_- \chi_5 \ , \tag{1.39}$$

Note that because $P_\pm = (1 \pm \Gamma)/2$ these current can be written as a sum or difference of a vector and an axial current, with a factor of 1/2. We have seen that the vector currents are conserved in the presence of a background electric field, while the axial currents are anomalous, so that we get:

$$\partial_\mu j_3^\mu = 3\frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ , \qquad \partial_\mu j_4^\mu = 4\frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ , \qquad \partial_\mu j_5^\mu = -5\frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ . \tag{1.40}$$

However, the current that the gauge field couples to is divergenceless, by construction:

$$j^\mu = 3ej_3^\mu + 4ej_4^\mu + 5ej_5^\mu \ , \qquad \partial_\mu j^\mu = \frac{e^2}{4\pi}\left(3^2 + 4^2 - 5^2\right)\epsilon_{\mu\nu}F^{\mu\nu} = 0 \ . \tag{1.41}$$

Therefore we do not expect this to be a sick theory and would like to study it on a computer.

What happens when we try to construct this theory on the lattice, using a copy of H from eqn. (1.33) for each fermion and adding gauge fields with appropriate charges? We get a sensible theory in the continuum limit, but not the one we wanted: a theory of three Dirac fermions with Lagrangian

$$\mathcal{L} = \bar\psi_3 i\not{D}\psi_3 + \bar\psi_4 i\not{D}\psi_4 + \bar\chi_5 i\not{D}\chi_5 \ . \tag{1.42}$$

Note that this theory has no chiral projection operators in the kinetic terms, and that it is possible to write down gauge invariant Dirac mass terms for each field...this is not a chiral gauge theory. Obviously there are an infinite number of healthy chiral gauge theories and we do not seem to have a way to regulate them on the lattice. This is not just an academic problem because the Standard Model is a chiral gauge theory in $d = 3 + 1$. There have been ideas on how to construct lattice gauge theories which I will discuss later, but it remains an open problem.

1.5 Doubling of a Dirac fermion and the need for fine tuning

Some operators in a Lagrangian suffer from additive renormalizations, such as the unit operator (cosmological constant) and scalar mass terms, such as the Higgs mass in the Standard Model, $|H|^2$. Therefore, the mass scales associated with such operators will naturally be somewhere near the UV cutoff of the theory, unless the bare couplings of the theory are fine-tuned to cancel radiative corrections. Such fine tuning problems have obsessed particle theorists since the work of Wilson and 't Hooft on renormalization and naturalness in the 1970s. However, such intemperate behavior will not occur for operators which violate a symmetry respected by the rest of the theory: if the bare couplings for such operators were set to zero, the symmetry would ensure they could not be generated radiatively in perturbation theory. Fermion mass operators generally fall into this benign category.

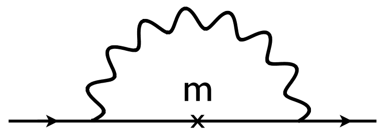

Fig. 1.5 *One-loop renormalization of the electron mass in QED due to photon exchange. A mass operator flips chirality, while gauge interactions do not. A contribution to the electron mass requires an odd number of chirality flips, and so there has to be at least one insertion of the electron mass in the diagram: the electron mass is multiplicatively renormalized. A scalar interaction flips chirality when the scalar is emitted, and flips it back when the scalar is absorbed, so replacing the photon with a scalar in the above graph again requires a fermion mass insertion to contribute to mass renormalization.*

Consider the following toy model: QED with a charge-neutral complex scalar field coupled to the electron:

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi + |\partial\phi|^2 - \mu^2|\phi|^2 - g|\phi|^4 + y\left(\overline{\psi}_R\phi\psi_L + \overline{\psi}_L\phi^*\psi_R\right) \ . \tag{1.43}$$

Note that in the limit $m \to 0$ this Lagrangian respects a chiral symmetry $\psi \to e^{i\alpha\gamma_5}\psi$, $\phi \to e^{-2i\alpha}\phi$. The symmetry ensures that if $m = 0$, a mass term for the fermion would not be generated radiatively in perturbation theory. With $m \neq 0$, this means that any renormalization of m must be proportional to m itself (i.e. m is "multiplicatively renormalized"). This is evident if one traces chirality through the Feynman diagrams; see Fig. 1.5. Multiplicative renormalization implies that the fermion mass can at most depend logarithmically on the cutoff (by dimensional analysis): $\delta m \sim (\alpha/4\pi)m\ln m/\Lambda$. Try plugging in some numbers here: with $\alpha = 1/137$ and $\Lambda = M_{\mathrm{Planck}} = 10^{19}$ GeV we get a radiative correction to the electron mass δm which is about 3% of the electron mass, not a shift that requires fine tuning.

In contrast, the scalar mass operator $|\phi|^2$ does not violate any symmetry and therefore suffers from additive renormalizations, such as through the graph in Fig. 1.6. By dimensional analysis, the scalar mass operator can have a coefficient that scales quadratically with the cutoff: $\delta\mu^2 \sim (y^2/16\pi^2)\Lambda^2$. This is called an additive renormalization, since $\delta\mu^2$ is not proportional to μ^2. It is only possible in general to have a scalar in the spectrum of this theory with mass much lighter than $y\Lambda/4\pi$ if the bare couplings are finely tuned to cause large radiative corrections to cancel. For $\Lambda = M_{\mathrm{Planck}}$, we require a bare mass term to cancel this one-loop radiative contribution to one part in 10^{30} to get a 100 GeV Higgs. When referring to the Higgs mass in the Standard Model, this is called the hierarchy problem.

Let's return to the problem of lattice Hamiltonians for fermions in $d = 1 + 1$. Suppose we want a lattice model to describe a massive Dirac fermion in $d = 1 + 1$. The continuum Hamiltonian is

$$H = \gamma^0(-i\gamma^1\partial_x + m) = -i\sigma_3\partial_x + m\sigma_1 \ , \tag{1.44}$$

and so we again naively write down a Hamiltonian for a free fermion, this time of the form

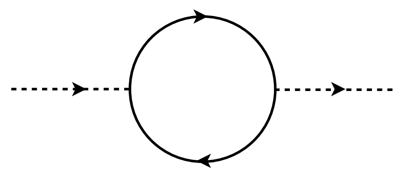

Fig. 1.6 *One-loop additive renormalization of the scalar mass due to a quadratically divergent fermion loop.*

$$H = \sum_n \psi_n^\dagger \left[-\frac{i}{2a}\sigma_3 \left(\psi_{n+1} - \psi_n\right) + m\sigma_1 \psi_n \right] \tag{1.45}$$

where $\psi_{n,i}$ is a 2-component fermion ladder operator at site n with $\{\psi_{m,i}^\dagger, \psi_{n,j}\} = \delta_{mn}\delta_{ij}$, with all other anticommutators vanishing. The eigenvalues of H are the eigenvalues of the matrix

$$\left[\sigma_3 \frac{\sin ap}{a} + m\sigma_1 \right] \ , \tag{1.46}$$

or

$$E_p = \pm\sqrt{\left(\frac{\sin ap}{a}\right)^2 + m^2} \tag{1.47}$$

Expanding about $a = 0$ we get $E_p \simeq \pm\sqrt{p^2 + m^2}$ for $p = O(1)$, while writing $p = -\pi/a + k$ and expanding about $a = 0$ we get $E_k = \pm\sqrt{k^2 + m^2}$... so we find *two* Dirac fermions in the continuum. This is the same doubling of the spectrum we saw when we tried to construct a lattice model for just a right moving 1-component fermion. In thaty case the doubling kep us from creating a sick theory....here it is just annoying, because a single massive Dirac fermion is a perfectly fine theory, and the one we want.

Can we make the mode near $p \sim \pi/a$ very heavy and get rid of it? Yes, now the spectrum never crosses the p axis and there is no reason a periodic function could exhibit a small gap $\sim m$ at $p = 0$ and a large gap $\sim 1/a$ at $p = \pi/a$. We can do that by adding to H a term of the form

$$-a\bar{\psi}\partial_x^2\psi \tag{1.48}$$

which looks like a mass term that will only affect modes with wavenumber $p \sim 1/a$ in the $a \to 0$ limit. The lattice realization of this operator is

$$H_w = -\frac{ra}{2}\sum_n \frac{1}{a^2}\bar{\psi}_n \left(\psi_{n+1} - 2\psi_n + \psi_{n-1}\right) \ , \tag{1.49}$$

where the $r/2$ factor is a parameter we can adjust. Now the energy is given by the eigenvalues of

$$\frac{\sin ap}{a}\sigma_3 + \left(m + r\frac{1 - \cos ap}{a^2}\right)\sigma_1 \ , \tag{1.50}$$

or

$$E_p = \pm\sqrt{\left(\frac{\sin ap}{a}\right)^2 + \left(m + r\frac{1-\cos ap}{a^2}\right)^2} \tag{1.51}$$

Now if we expand about $ap = 0$ we get $E_p = \pm\sqrt{p^2+m^2}$ as before, the desired dispersion relation for a massive Dirac fermion in the continuum. However, when we set $p = -\pi/a + k$ and expand about $ak = 0$ we get $E_k = 2r/a + O(1)$. So we see that the unwanted mode at the edge of the Brillouin zone ($p = \pm\pi/a$) becomes infinitely heavy as $a \to 0$ and decouples from low energy physics. This is Wilson's solution for getting rid of the unwanted "doubler".

There has been a cost, however. Now we have two terms violating chiral symmetry in the Lagrangian: the $m\bar{\psi}\psi$ term, and the $a\bar{\psi}\nabla^2\psi$ term. Thus when we add interactions (e.g. by gauging fermion number) the Wilson interaction term $\bar{\psi}D^2\psi$ will renormalize the mass term through gauge boson loops, and we expect radiative corrections of size

$$\delta m \sim r\frac{\alpha}{a} \ , \tag{1.52}$$

which looks very similar to the Higgs mass fine tuning problem: the bare mass will have to be fine tuned to one part in $\sim ma/\alpha$ in order to obtain a physical mass m, which becomes harder to do the smaller a becomes. In getting rid of our doubler fermion we lost our approximate chiral symmetry, and have to fine tune parameters to recover it in the continuum limit. And the problem seemed to arise from the need for the lattice to properly account for anomalies.

Exercise 1.3 Compute the spectrum of the Wilson-Dirac Hamiltonian, $H + H_w$ and plot it for various values of m, r with $a = 1$.

1.6 What we have found

- Chiral symmetries appear in theories of massless fermions
- Chiral symmetries can be broken in the continuum by quantum effects called anomalies
- Anomalies cannot exist in a finite system; lattice theories of fermions break potentially anomalous symmetries explicitly
- ...or else double the fermions so that the exact lattice symmetries are vector symmetries in the continuum.
- Lattice doubling of the spectrum makes it problematic to construct sensible chiral gauge theories in the continuum limit
- Eliminating doubling in vector-like gauge theories eliminates global chiral symmetry and requires fine tuning in order to find light fermions in the continuum limit.

What we would like is a lattice fermion formulation which at the very least correctly accounts for anomalies, while protecting fermion masses from additive renormalization, like in the continuum. Better: we would like to know how to construct lattice models for chiral gauge theories.

2

Chiral symmetry, parity, and anomalies in higher dimensions.

2.1 Spinor representations of the Lorentz group

To understand chiral symmetry in $d = 3+1$ it is helpful to understand representations of the Lorentz group. Since we will be discussing fermions in various dimensions of spacetime, consider the generalization of the usual Lorentz group to d dimensions. The Lorentz group is defined by the real matrices Λ which preserve the form of the d-dimensional metric

$$\Lambda^T \eta\, \Lambda = \eta\ , \qquad \eta = \mathrm{diag}\,(1, -1, \ldots -1)\,. \tag{2.1}$$

With this definition, the inner product between two 4-vectors, $v^\mu \eta_{\mu\nu} w^\mu = v^T \eta w$, is preserved under the Lorentz transformations $v \to \Lambda v$ and $w \to \Lambda w$. This defines the group $SO(d-1,1)$, which — like $SO(d)$ — has $d(d-1)/2$ linearly independent generators, which may be written as $M^{\mu\nu} = -M^{\nu\mu}$, where the indices $\mu, \nu = 0, \ldots, (d-1)$ and

$$\Lambda = e^{i\theta_{\mu\nu} M^{\mu\nu}}\ , \tag{2.2}$$

with $\theta_{\mu\nu} = -\theta_{\nu\mu}$ being $d(d-1)/2$ real parameters. Note that μ, ν label the $d(d-1)/2$ generators, while in a representation R each M is a $d_R \times d_R$ matrix, where d_R is the dimension of R. By expanding eqn. (2.1) to order θ one sees that the generators M must satisfy

$$(M^{\mu\nu})^T \eta + \eta M^{\mu\nu} = 0. \tag{2.3}$$

It is really easy to find all the solutions to this equation in the defining representation (eg for the d-vector)! Given the form of η, the M^{ij} matrices (all spatial indices) have to be antisymmetric, while the M^{0i} matrices have to be symmetric. We can find a simple basis by just scattering factors of 1 and -1 in appropriate places, and then put in an overall factor of i to ensure that the transformation Λ is real. For example, in $d = 3+1$ we can find a representation for the generators of rotations and boosts in the z direction of the familiar form:

$$M^{12} = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\ , \qquad M^{03} = i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

and exponentiating them gives the familiar form for Λ,

$$e^{i\theta M^{12}} = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}, \quad e^{i\theta M^{03}} = \begin{pmatrix} \cosh\theta & & & \sinh\theta \\ & 1 & & \\ & & 1 & \\ \sinh\theta & & & \cosh\theta \end{pmatrix} \tag{2.5}$$

Note that we find a Hermitian generator for rotations, but an anti-Hermitian generator for boosts. The transformation Λ is therefore not a unitary matrix. This is a hallmark of the Lorentz group being noncompact: if you keep boosting, you never return to your original frame, unlike if you keep rotating. In fact, there are no finite-dimensional unitary representations of the Lorentz group. (There are, however, infinite dimensional unitary representations, such as the ones that act on our Hilbert space and conserve probability!).

Once one knows the defining representation that solves eqn. (2.3) on can determine the commutation relations for the algebra,

$$\left[M^{\alpha\beta}, M^{\gamma\delta}\right] = i\left(\eta^{\beta\gamma}M^{\alpha\delta} - \eta^{\alpha\gamma}M^{\beta\delta} - \eta^{\beta\delta}M^{\alpha\gamma} + \eta^{\alpha\delta}M^{\beta\gamma}\right). \tag{2.6}$$

By finding all representations of this algebra, one can construct all representations of the group by exponentiating the generators. A Dirac spinor representation can be constructed as

$$M^{\alpha\beta} \equiv \Sigma^{\alpha\beta} = \frac{i}{4}\left[\gamma^\alpha, \gamma^\beta\right] \tag{2.7}$$

where the gamma matrices satisfy the Clifford algebra:

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} \tag{2.8}$$

You can check explicitly that if you can find γ matrices satisfying eqn. (2.8), then Σ satisfies the commutation relations eqn. (2.6).

Solutions to the Clifford algebra are easy to find by making use of direct products of Pauli matrices. In a direct product space we can write a matrix as $M = a \otimes A$ where a and A are matrices of dimension d_a and d_A respectively, acting in different spaces; the matrix M then has dimension $(d_a \times d_A)$. Matrix multiplication is defined as $(a \otimes A)(b \otimes B) = (ab) \otimes (AB)$. It is usually much easier to construct a representation when you need one rather than to look one up and try to keep the conventions straight! One finds that solutions for the γ matrices in d Minkowski dimensions obey the following properties:

1. For both $d = 2k$ and $d = 2k + 1$, the γ-matrices are 2^k dimensional;
2. For even spacetime dimension $d = 2k$ (such as our own with $k = 2$) one can define a generalization of γ_5 to be

$$\Gamma = i^{k-1} \prod_{\mu=0}^{2k-1} \gamma^\mu \tag{2.9}$$

with the properties

$$\{\Gamma, \gamma^\mu\} = 0\,, \quad \Gamma = \Gamma^\dagger = \Gamma^{-1}\,, \quad \mathrm{Tr}\left(\Gamma\gamma^{\alpha_1}\cdots\gamma^{\alpha_{2k}}\right) = 2^k i^{-1-k}\epsilon^{\alpha_1\ldots\alpha_{2k}}, \tag{2.10}$$

where $\epsilon_{012\ldots 2k-1} = +1 = -\epsilon^{012\ldots 2k-1}$.

3. In $d = 2k+1$ dimensions one needs one more γ-matrix than in $d = 2k$, and one can take it to be $\gamma^{2k} = i\Gamma$.

Sometimes it is useful to work in a specific basis for the γ-matrices; a particularly useful choice is a "chiral basis", defined to be one where Γ is diagonal. For example, for $d = 2$ and $d = 4$ (Minkowski spacetime, i.e. $d = 1+1$ and $d = 3+1$) one can choose

$$d = 2: \quad \gamma^0 = \sigma_1 \ , \quad \gamma^1 = -i\sigma_2 \ , \quad \Gamma = \sigma_3 \tag{2.11}$$

$$d = 4: \quad \gamma^0 = -\sigma_1 \otimes 1 \ , \quad \gamma^i = i\sigma_2 \otimes \sigma_i \ , \quad \Gamma = \sigma_3 \otimes 1 \ . \tag{2.12}$$

It is easy to convert these direct product matrices into ordinary 4×4 matrices, for example one can write

$$\gamma^1 = i\sigma_2 \otimes \sigma_1 = i \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.13}$$

For spinors in $d = 3$ and $d = 5$ we can just take the above matrices for $d = 2$ and $d = 4$ respectively, and tack on $i\Gamma$ as the extra matrix:

$$d = 3: \quad \gamma^0 = \sigma_1 \ , \quad \gamma^1 = -i\sigma_2 \ , \quad \gamma^2 = i\sigma_3 \tag{2.14}$$

$$d = 5: \quad \gamma^0 = -\sigma_1 \otimes 1 \ , \quad \gamma^{i=1,2,3} = i\sigma_2 \otimes \sigma_i \ , \quad \gamma^5 = i\sigma_3 \otimes 1 \ . \tag{2.15}$$

It is evident that there does not exist a fourth 2×2 matrix in $d = 3$ or a sixth 4×4 matrix in $d = 5$ that anticommutes with all the other γ^μ matrices, and so there is no notion of chirality and Dirac fermions are irreducible for odd d.

2.1.1 γ-matrices in Euclidian spacetime

In Feynman diagrams one performs a Wick rotation $\int_{-\infty}^{\infty} dk_0 \to \int_{-i\infty}^{i\infty} dk_0 \equiv i\int_{-\infty}^{\infty} dk_0^E$, where the last step is a change of variables $k_0 = ik_0^E$. This implies that to go to Euclidian spacetime we should replace $\partial_0 \to i\partial_0^E$ and $x^0 \to -ix_0^E$. Therefore, to go to Euclidian spacetime with metric $\eta_E^{\mu\nu} = \delta_{\mu\nu}$, we take

$$\partial_0^M \to i\partial_0^E \ , \quad \partial_i^M \to \partial_i^E \tag{2.16}$$

and defines

$$\gamma_M^0 = \gamma_E^0 \ , \quad \gamma_M^i = i\gamma_E^i \ , \tag{2.17}$$

so that

$$(\gamma_E^\mu)^\dagger = \gamma_E^\mu \ , \qquad \{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta_{\mu\nu} \tag{2.18}$$

and $\not{D}_M \to i\not{D}_E$, with

$$\not{D}_E = -\not{D}_E^\dagger \tag{2.19}$$

and the Euclidian Dirac operator is $(\not{D}_E + m)$. In the path integral

$$e^{iS_M} \to e^{-S_E}, \quad S_M = \int d^dx\, \bar\psi(i\not{D} - m)\psi, \quad S_E = \int d^4x_E\, \bar\psi(\not{D}_E + m)\psi. \tag{2.20}$$

With the Euclidian metric there is no difference between upper and lower indices. (The annoying thing about working with the mostly minus metric is that one has to also flip the sign of scalar products, such as $x_\mu x^\mu \to -x^E_\mu x^E_\mu$...this can be avoided by working with the mostly plus metric, the drawback of that metric being that the non-relativistic limit is less convenient.) The matrix $\Gamma^{(2k)}$ in $2k$ dimensions is taken to equal γ_E^{2k} in $(2k+1)$ dimensions:

$$\Gamma_E^{(2k)} = \gamma_E^{2k} = \Gamma_M^{(2k)}\ , \qquad \mathrm{Tr}\,(\Gamma_E\, \gamma_E^{\alpha_1} \cdots \gamma_E^{\alpha_{2k}}) = -2^k i^k \epsilon^{\alpha_1 \ldots \alpha_{2k}} \tag{2.21}$$

where $\epsilon_{012\ldots 2k-1} = +1 = +\epsilon^{012\ldots 2k-1}$.

2.2 Chirality in even dimensions

As we saw in $d = 2$, the existence of Γ means that Dirac spinors are reducible representations of the Lorentz group, which in turn means we can have symmetries ("chiral symmetries") which transform different parts of Dirac spinors in different ways. To see this, define the projection operators

$$P_\pm = \frac{(1 \pm \Gamma)}{2}\ , \tag{2.22}$$

which have the properties

$$P_+ + P_- = \mathbf{1}\ , \qquad P_\pm^2 = P_\pm\ , \qquad P_+P_- = 0\ . \tag{2.23}$$

Since in odd spatial dimensions $\{\Gamma, \gamma^\mu\} = 0$ for all μ, it immediately follows that Γ commutes with the Lorentz generators $\Sigma^{\mu\nu}$ in eqn. (2.7): $[\Gamma, \Sigma^{\mu\nu}] = 0$. Therefore we can write $\Sigma^{\mu\nu} = \Sigma_+^{\mu\nu} + \Sigma_-^{\mu\nu}$ where

$$\Sigma_\pm^{\mu\nu} = P_\pm \Sigma^{\mu\nu} P_\pm\ , \tag{2.24}$$

Thus $\Sigma^{\mu\nu}$ is reducible: spinors $\psi_\pm$ which are eigenstates of Γ with eigenvalue ± 1 respectively transform independently under Lorentz transformations.

The word "chiral" comes from the Greek word for hand. We saw in $d = 1+1$ that massless $\Gamma = +1$ chirality fermions correspond to right movers, while $\Gamma = -1$ chirality fermions correspond to left movers. In $d = 3+1$ one finds that for massless fermions, chirality is equal to helicity, and again $\Gamma = \pm 1$ corresponds to RH and LH helicity respectively.

Exercise 2.1 You should perform the same exercise in $3+1$ dimensions and find that solutions $\psi_\pm$ to the massless Dirac equation satisfying $\Gamma\psi_\pm = \pm\psi_\pm$ must also satisfy $|\vec p| = E$ and $(2\vec p \cdot \vec S/E)\psi_\pm = \pm\psi_\pm$, where $S_i = \frac{1}{2}\epsilon_{0ijk}\Sigma^{jk}$ are the generators of rotations. Thus $\psi_\pm$ correspond to states with positive or negative helicity respectively, and are called right- and left-handed particles.

2.3 Chiral symmetry and fermion mass in four dimensions

In many ways, chiral symmetry is very similar in $d = 3 + 1$ dimensions to what we saw for $d = 1 + 1$. Consider the Lagrangian for a single flavor of Dirac fermion in 3+1 dimensions, coupled to a background gauge field

$$\mathcal{L} = \left(\overline{\psi}_L i\not{D}\psi_L + \overline{\psi}_R i\not{D}\psi_R\right) - \left(m\overline{\psi}_L\psi_R + \text{h.c.}\right) \tag{2.25}$$

where I have defined

$$\psi_L = P_-\psi\ , \quad \bar{\psi}_L = \psi_L^\dagger\gamma^0 = \bar{\psi}P_+\ , \quad \psi_R = P_+\psi\ , \quad \bar{\psi}_R = \bar{\psi}P_-\ . \tag{2.26}$$

For now I am assuming that $\psi_{L,R}$ are in the same complex representation of the gauge group, where D_μ is the gauge covariant derivative appropriate for that representation. It is important to note the the property $\{\gamma_5, \gamma^\mu\} = 0$ ensured that the kinetic terms in eqn. (2.26) do not couple left-handed and right-handed fermions; on the other hand, the mass terms do[1]. The above Lagrangian has an exact $U(1)$ symmetry, associated with fermion number, $\psi \to e^{i\alpha}\psi$. Under this symmetry, left-handed and right-handed components of ψ rotate with the same phase; this is often called a "vector symmetry". In the case where $m = 0$, it apparently has an additional symmetry where the left- and right-handed components rotate with the opposite phase, $\psi \to e^{i\alpha\gamma_5}\psi$; this is called an "axial symmetry", $U(1)_A$.

Symmetries are associated with Noether currents, and symmetry violation appears as a nonzero divergence for the current. Recall the Noether formula for a field ϕ and infinitesimal transformation $\phi \to \phi + \epsilon\delta\phi$:

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\ , \qquad \partial_\mu j^\mu = -\delta\mathcal{L}\ . \tag{2.27}$$

In the Dirac theory, the vector symmetry corresponds to $\delta\psi = i\psi$, and the axial symmetry transformation is $\delta\psi = i\gamma_5\psi$, so that the Noether formula yields the vector and axial currents:

$$U(1): \qquad j^\mu = \overline{\psi}\gamma^\mu\psi\ , \qquad \partial_\mu j^\mu = 0 \tag{2.28}$$

$$U(1)_A: \qquad j^\mu_A = \overline{\psi}\gamma^\mu\gamma_5\psi\ , \qquad \partial_\mu j^\mu_A = 2im\overline{\psi}\gamma_5\psi\ . \tag{2.29}$$

Some comments are in order:

- Eqn. (2.29) is not the whole story! As in $d = 1 + 1$, the axial current will also have an anomalous divergence from quantum effects.
- As in $d = 1 + 1$, the fact that the fermion mass explicitly breaks chiral symmetry means that fermion masses get multiplicatively renormalized, so that fermions can naturally be light.
- The variation of a general fermion bilinear $\overline{\psi}X\psi$ under chiral symmetry is

  $$\delta\overline{\psi}X\psi = i\overline{\psi}\{\gamma_5, X\}\psi\ . \tag{2.30}$$

  This will vanish if X can be written as the product of an odd number of γ^μ matrices. In any even dimension the chirally invariant bilinears include currents,

[1] I will use the familiar γ_5 in $3 + 1$ dimensions instead of Γ when there is no risk of ambiguity.

with $X = \gamma^\mu$ or $X = \gamma^\mu\Gamma$, and so gauge interactions are always invariant under chiral symmetry (up to anomalies). The bilinears which transform nontrivially under the chiral symmetry include not only mass terms, $X = \mathbf{1}, \Gamma$, but in $d = 4$ $X = \sigma^{\mu\nu}, \sigma^{\mu\nu}\Gamma$ as well. The latter operators when coupled to $F^{\mu\nu}$ give rise to contributions to magnetic and electric dipole moments, respectively.

The Lagrangian for N_f flavors of massive Dirac fermions in even d, coupled to some background gauge field may be written as

$$\mathcal{L} = \left(\overline{\psi}^a_L i\not{D}\psi^a_L + \overline{\psi}^a_R i\not{D}\psi^a_R\right) - \left(\overline{\psi}^a_L M_{ab}\psi^b_R + \overline{\psi}^a_R M^\dagger_{ab}\psi^b_L\right) . \tag{2.31}$$

The index on ψ denotes flavor, with $a, b = 1, \ldots N_f$, and M_{ab} is a general complex mass matrix (no distinction between upper and lower flavor indices). Again assuming the fermions to be in a complex representation of the gauge group, this theory is invariant under independent chiral transformations if the mass matrix vanishes:

$$\psi^a_R \to U_{ab}\psi^b_R , \quad \psi^a_L \to V_{ab}\psi^b_L , \quad U^\dagger U = V^\dagger V = \mathbf{1} . \tag{2.32}$$

where U and V are independent $U(N_f)$ matrices. Since $U(N_f) = SU(N_f) \times U(1)$, it is convenient to write

$$U = e^{i(\alpha+\beta)}R , \quad V = e^{i(\alpha-\beta)}L , \quad R^\dagger R = L^\dagger L = \mathbf{1} , \quad |R| = |L| = 1 , \tag{2.33}$$

so that the symmetry group is $SU(N_f)_L \times SU(N_f)_R \times U(1) \times U(1)_A$ with $L \in SU(N_f)_L$, $R \in SU(N_f)_R$.

If we turn on the mass matrix, the chiral symmetry is explicitly broken, since the mass matrix couples left- and right-handed fermions to each other. If $M_{ab} = m\delta_{ab}$ then the "diagonal" or "vector" symmetry $SU(N_f) \times U(1)$ remains unbroken, where $SU(N_f) \subset SU(N_f)_L \times SU(N_f)_R$ corresponding to the transformation eqn. (2.32), eqn. (2.33) with $L = R$. If M_{ab} is diagonal but with unequal eigenvalues, the symmetry may be broken down as far as $U(1)^{N_f}$, corresponding to independent phase transformations of the individual flavors. With additional flavor-dependent interactions, these symmetries may be broken as well.

2.4 Lorentz group in d=3+1: SU(2) x SU(2) and Weyl fermions

We have seen that Dirac fermions in even dimensions form a reducible representation of the Lorentz group. Dirac notation is convenient when both LH and RH parts of the Dirac spinor transform as the same complex representation under a gauge group, and when there is a conserved fermion number. This sounds restrictive, but applies to QED and QCD. For other applications — such as chiral gauge theories (where LH and RH fermions carry different gauge charges, as under $SU(2) \times U(1)$), or when fermion number is violated (as is the case for neutrinos with a Majorana mass), or when fermions transform as a real representation of gauge group — then it is much more convenient to use irreducible fermion representations, called Weyl fermions.

The six generators of the Lorentz group may be chosen to be the three Hermitian generators of rotations J_i, and the three anti-Hermitian generators of boosts K_i, so that an arbitrary Lorentz transformation takes the form

$$\Lambda = e^{i(\theta_i J_i + \omega_i K_i)} \ . \tag{2.34}$$

In terms of the $M_{\mu\nu}$ generators in §2.1,

$$J_i = \tfrac{1}{2}\epsilon_{0i\mu\nu} M^{\mu\nu} \ , \qquad K_i = M^{0i} \ . \tag{2.35}$$

These generators have the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ , \quad [J_i, K_j] = i\epsilon_{ijk} K_k \ , \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \ . \tag{2.36}$$

It is convenient to define different linear combinations of generators

$$A_i = \frac{J_i + iK_i}{2} \ , \quad B_i = \frac{J_i - iK_i}{2} \ , \quad \Longrightarrow \quad J_i = (A_i + B_i) \ , \quad K_i = -i(A_i - B_i) \tag{2.37}$$

with

$$\Lambda = e^{i\left[(\vec{\theta}_i - i\vec{\omega})\cdot\vec{A} + (\vec{\theta}_i + i\vec{\omega})\cdot\vec{B})\right]} \ . \tag{2.38}$$

Life simplifies now since eqn. (2.36) implies that $\vec{A}$ and $\vec{B}$ are the six Hermitian generators of an $SU(2)\times SU(2)$ algebra:

$$[A_i, A_j] = i\epsilon_{ijk} A_k \ , \quad [B_i, B_j] = i\epsilon_{ijk} B_k, \quad [A_i, B_j] = 0 \ . \tag{2.39}$$

We already know all about representations of $SU(2)$! They are labeled by non-negative half-integer j. Thus Lorentz representations may be labelled with two $SU(2)$ spins, $j_{A,B}$ corresponding to the two $SU(2)$s: $(j_A,\, j_B)$, transforming as

$$\Lambda(\vec{\theta}, \vec{\omega}) = D^{j_A}(\vec{\theta} - i\vec{\omega}) \times D^{j_B}(\vec{\theta} + i\vec{\omega}) \tag{2.40}$$

where the D^j is the usual $SU(2)$ rotation in the spin j representation; boosts appear as imaginary parts to the rotation angle; the D^{j_A} and D^{j_B} matrices act in different spaces and therefore commute. For example, under a general Lorentz transformation, a LH Weyl fermion $\psi = (\frac{1}{2}, 0)$ has $\vec{A} = \frac{1}{2}\vec{\sigma}$ and $\vec{B} = 0$, so that it transforms as

$$\psi \to e^{i(\vec{\theta} - i\vec{\omega})\cdot\vec{\sigma}/2}\psi \ . \tag{2.41}$$

Similarly, a RH Weyl fermion $\chi = (0, \frac{1}{2})$ transforms under Lorentz transformations as

$$\chi \to e^{i(\vec{\theta} + i\vec{\omega})\cdot\vec{\sigma}/2}\chi \ . \tag{2.42}$$

Evidently the two types of fermions transform the same way under rotations, but differently under boosts.

The dimension of the $(j_A,\, j_B)$ representation is $(2j_A+1)(2j_B+1)$. In this notation, the smaller irreducible Lorentz representations are labelled as:

$$\begin{aligned}(0,0):&\quad \text{scalar}\\ (\tfrac{1}{2},0)\ ,\ (0,\tfrac{1}{2}):&\quad \text{LH and RH Weyl fermions}\\ (\tfrac{1}{2},\tfrac{1}{2}):&\quad \text{four-vector}\\ (1,0)\ ,\ (0,1):&\quad \text{self-dual and anti-self-dual antisymmetric tensors}\end{aligned}$$

A Dirac fermion is the reducible representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ consisting of a LH and a RH Weyl fermion.

Parity takes $\vec{J}\to\vec{J}$ and $\vec{K}\to-\vec{K}$, and therefore interchanges $\vec{A}\leftrightarrow\vec{B}$, transforming a $(j_1,\, j_2)$ representation into $(j_2,\, j_1)$. So parity will change a LH Weyl fermion into a RH one, and vice versa, but takes a 4-vector into a 4-vector.

Charge conjugation taqkes a field to its complex conjugate, and therefore also effectively flips the sign of K_i in eqn. (2.40) due to the factor of i in front of ω, implying that if a field ϕ transforms as $(j_1,\, j_2)$, then $\phi^\dagger$ transforms as (j_2, j_1); for this reason, the combined symmetry CP does not alter the particle content of a chiral theory, so that CP violation must arise from complex coupling constants. This is in contrast to P violation, which will occur whenever RH and LH Weyl fermions do not have the same gauge charges, or when one of them is missing from the theory. Therefore a theory of N_L flavors of LH Weyl fermions ψ_i, and N_R flavors of RH Weyl fermions χ_a may be recast as a theory of (N_L+N_R) LH fermions by defining $\chi_a\equiv\omega_a^\dagger$. The fermion content of the theory can be described entirely in therms of LH Weyl fermions then, $\{\psi_i,\omega_a\}$; this often simplifies the discussion of parity violating theories, such as the Standard Model or Grand Unified Theories. Note that if the RH χ_a transformed under a gauge group as representation R, the conjugate fermions ω_a transform under the conjugate representation $\overline{R}$.

For example, QCD written in terms of Dirac fermions has the Lagrangian:

$$\mathcal{L}=\sum_{i=u,d,s\ldots}\overline{\psi}_n(i\not{D}-m_n)\psi_n\ , \tag{2.43}$$

where D_μ is the $SU(3)_c$ covariant derivative, and the ψ_n fields (both LH and RH components) transform as a 3 of $SU(3)_c$. However, we could just as well write the theory in terms of the LH quark fields ψ_n and the LH anti-quark fields χ_n. Using the γ-matrix basis in eqn. (2.12), we write the Dirac spinor ψ in terms of two-component LH spinors ψ and χ as

$$\psi=\begin{pmatrix}-\sigma_2\chi^\dagger\\ \psi\end{pmatrix}\ . \tag{2.44}$$

Note that ψ transforms as a 3 of $SU(3)_c$, while χ transforms as a $\overline{3}$. Then the kinetic operator becomes (up to a total derivative)

$$\overline{\psi}i\not{D}\psi=\psi^\dagger iD_\mu\sigma^\mu\psi+\chi^\dagger iD_\mu\sigma^\mu\chi\ ,\qquad \sigma^\mu\equiv\{\mathbf{1},-\vec{\sigma}\}\ , \tag{2.45}$$

and the mass terms become

$$\overline{\psi}_R\psi_L=\chi\sigma_2\psi=\psi\sigma_2\chi$$

$$\overline{\psi}_L \psi_R = \psi^\dagger \sigma_2 \chi^\dagger = \chi^\dagger \sigma_2 \psi^\dagger \ , \tag{2.46}$$

where I used the fact that fermion fields anti-commute. Thus a Dirac mass in terms of Weyl fermions is just

$$m\overline{\psi}\psi = m(\psi\sigma_2\chi + h.c) \ , \tag{2.47}$$

and preserves a fermion number symmetry where ψ has charge $+1$ and χ has charge -1. On the other hand, one can also write down a Lorentz invariant mass term of the form

$$m(\psi\sigma_2\psi + h.c.) \tag{2.48}$$

which violates fermion number by two units; this is a Majorana mass, which is clumsy to write in Dirac notation. Experimentalists are trying to find out which form neutrino masses have — Dirac, or Majorana? If the latter, lepton number is violated by two units and could show up in neutrinoless double beta decay, where a nucleus decays by emitting two electrons and no anti-neutrinos.

The Standard Model is a relevant example of a chiral gauge theory. Written in terms of LH Weyl fermions, the quantum numbers of a single family under $SU(3) \times SU(2) \times U(1)$ are:

$$\begin{aligned} Q &= (3,2)_{+\frac{1}{6}} & L &= (1,2)_{-\frac{1}{2}} \\ U^c &= (\bar{3},1)_{-\frac{2}{3}} & E^c &= (1,1)_{+1} \\ D^c &= (\bar{3},1)_{+\frac{1}{3}} \ . \end{aligned} \tag{2.49}$$

Evidently this is a complex representation and chiral. If neutrino masses are found to be Dirac in nature (i.e. lepton number preserving) then a partner for the neutrino must be added to the theory, the "right handed neutrino", which can be described by a LH Weyl fermion which is neutral under all Standard Model gauge interactions, $N = (1,1)_0$.

If unfamiliar with two-component notation, you can find all the details in Appendix A of Wess and Bagger's classic book on supersymmetry (Wess and Bagger, 1992); the notation used here differs slightly as I use the metric and γ-matrix conventions of Itzykson and Zuber (Itzykson and Zuber, 1980), and write out the σ_2 matrices explicitly.

Exercise 2.2 Consider a theory of N_f flavors of Dirac fermions in a real or pseudo-real representation of some gauge group. (Real representations combine symmetrically to form an invariant, such as a triplet of $SU(2)$; pseudo-real representations combine anti-symmetrically, such as a doublet of $SU(2)$). Show that if the fermions are massless the action exhibits a $U(2N_f) = U(1) \times SU(2N_f)$ flavor symmetry at the classical level (the $U(1)$ subgroup being anomalous in the quantum theory). If the fermions condense as in QCD, what is the symmetry breaking pattern? How do the resultant Goldstone bosons transform under the unbroken subgroup of $SU(2N_f)$?

Exercise 2.3 To see how the $(\frac{1}{2},\frac{1}{2})$ representation behaves like a four-vector, consider the 2×2 matrix $P = P_\mu\sigma^\mu$, where σ^μ is given in eqn. (2.45). Show that the transformation $P \to LPL^\dagger$ (with $\det L = 1$) preserves the Lorentz invariant inner product $P_\mu P^\mu = (P_0^2 - P_i P_i)$. Show that with L given by eqn. (2.41), P_μ transforms properly like a four-vector.

Exercise 2.4 Is it possible to write down an anomalous electric or magnetic moment operator in a theory of a single charge-neutral Weyl fermion?

2.5 Anomalies in 3+1 dimensions

2.5.1 The $U(1)_A$ anomaly

An analogous violation of the $U(1)_A$ current occurs in $3+1$ dimensions as well [2]. One might guess that the analogue of $\epsilon_{\mu\nu}F^{\mu\nu} = 2E$ in the anomalous divergence eqn. (1.26) would be the quantity $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 8\vec{E}\cdot\vec{B}$, which has the right dimensions and properties under parity and time reversal. So we should consider the behavior a massless Dirac fermion in $(3+1)$ in parallel constant E and B fields. First turn on a B field pointing in the $\hat{z}$ direction: this gives rise to Landau levels, with energy levels E_n characterized by non-negative integers n as well as spin in the $\hat{z}$ direction S_z and momentum p_z, where

$$E_n^2 = p_z^2 + (2n+1)eB - 2eBS_z \ . \tag{2.50}$$

The dispersion relation looks like that of an infinite number of one-dimensional fermions of mass $m_{n,\pm}$, where

$$m_{n\pm}^2 = (2n+1)eB - 2eBS_z \ , \quad S_z = \pm\tfrac{1}{2} \ . \tag{2.51}$$

The state with $n = 0$ and $S_z = +\frac{1}{2}$ is distinguished by having $m_{n,+} = 0$; it behaves like a *massless* one-dimensional Dirac fermion (with transverse density of states g_0) moving along the $\hat{z}$ axis with dispersion relation $E = |p_z|$. If we now turn on an electric field also pointing along the $\hat{z}$ direction we know what to expect from our analysis in 1+1 dimensions: we find an anomalous divergence of the axial current equal to $g_0 eE/\pi$ where g_0 is the transverse density of states in the $n = 0$ and $S_z = +\frac{1}{2}$ state.

How do we compute g_0? We need to recover the free fermion result as $B \to 0$. For free fermions we have the density of states is $d^3p/(2\pi)^3 = (dp_z/2\pi)(dp_x dp_y/(2\pi)^2)$, where the second factor is the transverse density of states. If we convert to radial coordinates and integrate over angle in the $x-y$ plane we are left with a density of states $g = p_\perp dp_\perp/(2\pi)$, where $E = \sqrt{p_z^2 + p_\perp^2}$. If we equate this formula with eqn. (2.5.1) we get

$$p_\perp^2 = [(2n+1)eB - 2eBS_z] \qquad \Longrightarrow \qquad 2dp_\perp dp_\perp = 2eB \ , \tag{2.52}$$

where $dp_\perp$ is the increment as $n \to n+1$. Therefore the transverse density of states is independent of n with

[2] Part of the content of this section comes directly from John Preskill's class notes on the strong interactions, available at his web page: http://www.theory.caltech.edu/~preskill/notes.html.

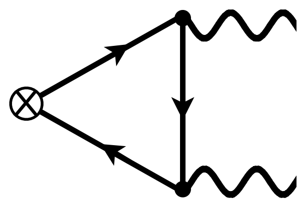

Fig. 2.1 *The $U(1)_A$ anomaly diagram in 3+1 dimensions, with one Pauli-Villars loop and an insertion of $2iM\overline{\Phi}\Gamma\Phi$.*

$$g_n = \frac{dp_\perp dp_\perp}{2\pi} = \frac{eB}{2\pi} \ . \tag{2.53}$$

and so we conclude that

$$\partial_\mu j^\mu_A = g_0 \frac{eE}{\pi} = \frac{e^2}{2\pi^2} EB = \frac{e^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \ . \tag{2.54}$$

If we include explicit breaking from the fermion mass term, we get

$$\partial_\mu j^\mu_A = 2im\overline{\psi}\Gamma\psi + \left(\frac{e^2}{16\pi^2}\right)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \ . \tag{2.55}$$

One can derive this result by computing $\langle M\overline{\Phi}i\Gamma\Phi\rangle$ for a Pauli-Villars regulator as in the $1+1$ dimensional example; now the relevant graph is the triangle diagram of Fig. 2.1.

If the external fields are nonabelian, the analogue of eqn. (2.55) is

$$\partial_\mu j^\mu_A = 2im\overline{\psi}\Gamma\psi + \left(\frac{g^2}{16\pi^2}\right)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}_a F^{\rho\sigma}_b \,\mathrm{Tr}\, T_a T_b \ . \tag{2.56}$$

If the fermions transform in the defining representation of $SU(N)$, it is conventional to normalize the coupling g so that $\mathrm{Tr}\, T_a T_b = \frac{1}{2}\delta_{ab}$. This is still called an "Abelian anomaly", since j^μ_A generates a $U(1)$ symmetry.

2.5.2 Anomalies in Euclidian spacetime

Continuing to Euclidian spacetime by means of eqns. (2.16)-(2.21) changes the anomaly equations simply by eliminating the factor of i from in front of the fermion mass:

$$2d: \quad \partial_\mu j^\mu_A = 2m\overline{\psi}\Gamma\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \tag{2.57}$$

$$4d: \quad \partial_\mu j^\mu_A = 2m\overline{\psi}\Gamma\psi + \left(\frac{g^2}{16\pi^2}\right)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}_a F^{\rho\sigma}_b \,\mathrm{Tr}\, T_a T_b \ . \tag{2.58}$$

2.5.3 The index theorem in four dimensions

For nonabelian gauge theories the quantity on the far right of eqn. (2.58) is a topological charge density, with

$$\nu = \frac{g^2}{64\pi^2}\int d^4x_E\,\epsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_a^{\rho\sigma} \tag{2.59}$$

being the winding number associated with $\pi_3(G)$, the homotopy group of maps of S_3 (spacetime infinity) into the gauge group G. Instantons are specific gauge configurations with nontrivial winding number. You should recognize that topology is involved by the epsilon tensor. Recall that topology deals with how manifolds are connected, with no reference to a metric, the domain of differential geometry. The way you can see that the above operator is related to topology is that if we go to curved spacetime it does not depend on the metric. That is because in curved spacetime, the diffeomorphism invariant integration measure picks up a factor of $\sqrt{g}$, where g is the determinant of the metric. This term is required for diffeomorphism invariance because it cancels the Jacobean from a change of variables. On the other hand, the epsilon "tensor" is not really a tensor unless it is accompanied by a factor of $1/\sqrt{g}$, and the two factors cancel, and so no metric appears in the above intewgral, even in curved spacetime. In contrast, typical terms one encounters in the action of quantum field theories, such as mass terms, will depend on the metric and are not topological in nature.

It is striking that the above integrand looks just like the anomalous divergence in the axial current, and this has an interesting implication for fermions: that the Dirac operator in Euclidian spacetime must have exact zero eigenvalues in the presence of gauge fields with nonzero winding number. The exact connection between gauge field topology and eigenvalues of the Dirac operator is the index theorem.

Consider then continuing the anomaly equation eqn. (2.56) to Euclidian space and integrating over spacetime its vacuum expectation value in a background gauge field (assuming the fermions to be in the N-dimensional representation of $SU(N)$ so that $\mathrm{Tr}\ T_aT_b = \frac{1}{2}\delta_{ab}$). The integral of $\partial_\mu\langle j_A^\mu\rangle$ vanishes because it is a pure divergence, so we get

$$\int d^4x_E\, m\langle\overline{\psi}\Gamma\psi\rangle = -\nu\ . \tag{2.60}$$

The matrix element above on the right equals

$$\frac{\int[d\psi][d\overline{\psi}]\,e^{-S_E}\,\int d^4x_E\, m\,\overline{\psi}\Gamma\psi}{\int[d\psi][d\overline{\psi}]\,e^{-S_E}}\ . \tag{2.61}$$

where $S_E = \overline{\psi}(\not{D}_E + m)\psi$. We can expand ψ and $\overline{\psi}$ in terms of eigenstates of the anti-hermitian operator $\not{D}_E$, where

$$\not{D}_E\psi_n = i\lambda_n\psi_n\ , \qquad \int d^4x_E\,\psi_m^\dagger\psi_n = \delta_{mn}\ , \tag{2.62}$$

with

$$\psi = \sum c_n\psi_n\ , \qquad \overline{\psi} = \sum \bar{c}_n\psi_n^\dagger\ . \tag{2.63}$$

Then

$$S_E = \sum_n (i\lambda_n + m)\bar{c}_n c_n \ , \quad e^{-S_E} = \prod_n \left[1 - \bar{c}_n c_n (i\lambda_n + m)\right] \ . \tag{2.64}$$

With the notation $\langle m|\Gamma|n\rangle = \int d^4x \psi_m^\dagger \Gamma \psi_n$, we have

$$\int d^4x_E \, m\langle\bar{\psi}\Gamma\psi\rangle = m \frac{\int \left(\prod_k dc_k d\bar{c}_k\right) \prod_n \left[1 - \bar{c}_n c_n (i\lambda_n + m)\right] \sum_{pq} \langle p|\Gamma|q\rangle \bar{c}_p c_q}{\int \left(\prod_k dc_k d\bar{c}_k\right) \prod_n \left[1 - \bar{c}_n c_n (i\lambda_n + m)\right]}$$

$$= m \sum_n \frac{\langle n|\Gamma|n\rangle}{-(i\lambda_n + m)} \ , \tag{2.65}$$

where I used the basic facts about a Grassmann variable c that $\int dc = 0$, $\int dc\, c = 1$, $c^2 = 0$. To compute this recall that $\{\Gamma, \not{D}\} = 0$; thus

$$\not{D}\psi_n = i\lambda_n \psi_n \quad \text{implies} \quad \not{D}(\Gamma\psi_n) = -i\lambda_n(\Gamma\psi_n) \ . \tag{2.66}$$

Thus for $\lambda_n \neq 0$, the eigenstates ψ_n and $(\Gamma\psi_n)$ must be orthogonal to each other (they are both eigenstates of the anti-hermitian operator $\not{D}$ with different eigenvalues), and so $\psi_n^\dagger \Gamma \psi_n$ vanishes for $\lambda_n \neq 0$ and does not contribute to the sum in eqn. (2.65). In contrast, modes with $\lambda_n = 0$ can simultaneously be eigenstates of $\not{D}$ and of Γ; we refer to such solutions as "zeromodes". Let n_+, n_- be the number of RH ($\Gamma = +1$) and LH ($\Gamma = -1$) zeromodes respectively. Therefore we arrive at the index equation

$$m \sum_n \frac{\langle n|\Gamma|n\rangle}{-(i\lambda_n + m)} = -(n_+ - n_-) \tag{2.67}$$

which when combined with eqn. (2.60) gives us the index theorem,

$$n_+ - n_- = \nu \ , \tag{2.68}$$

which states that the difference in the number of RH and LH zeromode solutions to the Euclidian Dirac equation in a background gauge field equals the winding number of the gauge field. With N_f flavors, the index equation is trivially modified to read

$$n_+ - n_- = N_f \nu \ . \tag{2.69}$$

This link between eigenvalues of the Dirac operator and the topological winding number of the gauge field provides a precise definition for the topological winding number of a gauge field on the lattice (where there is no obvious topology) — provided we have a definition of a lattice Dirac operator which exhibits exact zeromodes. We will see that the overlap operator is such an operator.

2.5.4 More general anomalies

Even more generally, one can consider the 3-point correlation function of three arbitrary currents as in Fig. 2.2,

$$\langle j_a^\alpha(k) j_b^\beta(p) j_c^\gamma(q)\rangle \ , \tag{2.70}$$

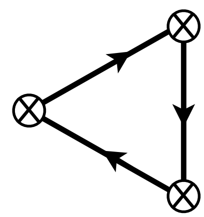

Fig. 2.2 *Anomalous three-point function of three currents.*

and show that the divergence with respect to any of the indices is proportional to a particular group theory factor

$$k_\mu \langle j^\mu_a(k) j^\alpha_b(p) j^\beta_c(q)\rangle \propto \mathrm{Tr}\ Q_a\{Q_b, Q_c\}\Big|_{R-L} \epsilon^{\alpha\beta\rho\sigma} k_\rho k_\sigma \ , \tag{2.71}$$

where the Qs are the generators associated with the three currents in the fermion representation, the symmetrized trace being computed as the difference between the contributions from RH and LH fermions in the theory. The anomaly $\mathcal{A}$ for the fermion representation is defined by the group theory factor

$$\mathrm{Tr}\ (Q_a\{Q_b, Q_c\})\Big|_{R-L} \equiv \mathcal{A}\, d_{abc} \ , \tag{2.72}$$

with d_{abc} being the totally symmetric invariant tensor of the symmetry group. For a simple group G (implying G is not $U(1)$ and has no factor subgoups), d_{abc} is only nonzero for $G = SU(N)$ with $N \geq 3$; even in the case of $SU(N)$, d_{abc} will vanish for real irreducible representations (for which $Q_a = -Q_a^*$), or for judiciously chosen reducible complex representations, such as $\bar{5} \oplus 10$ in $SU(5)$. For a semi-simple group $G_1 \times G_2$ (where G_1 and G_2 are themselves simple) there are no mixed anomalies since the generators are all traceless, implying that if $Q \in G_1$ and $\mathcal{Q} \in G_2$ then $\mathrm{Tr}\ (Q_a\{\mathcal{Q}_b, \mathcal{Q}_c\}) \propto \mathrm{Tr}\ Q_a = 0$. When considering groups with $U(1)$ factors there can be nonzero mixed anomalies of the form $U(1)G^2$ and $U(1)^3$ where G is simple; the $U(1)^3$ anomalies can involve different $U(1)$ groups. With a little group theory it is not difficult to compute the contribution to the anomaly of any particular group representation.

If a current with an anomalous divergence is gauged, then the theory does not make sense. That is because the divergenceless of the current is required for the unphysical modes in the gauge field A_μ to decouple; if they do not decouple, their propagator has a piece that goes as $k_\mu k_\nu/k^2$ which does not fall off at large momentum, and the theory is not renormalizable.

When global $U(1)$ currents have anomalous divergences, that is interesting. We have seen that the $U(1)_A$ current is anomalous, which explains the η' mass; the divergence of the axial isospin current explains the decay $\pi^0 \to \gamma\gamma$; the anomalous divergence of the baron number current in background $SU(2)$ in the Standard Model predicts baryon violation in the early universe and the possibility of weak-scale baryogenesis.

Exercise 2.5 Verify that all the gauge currents are anomaly-free in the standard model with the representation in eqn. (2.49). The only possible G^3 anomalies are for $G = SU(3)$ or $G = U(1)$; for the $SU(3)^3$ anomaly use the fact that a LH Weyl fermion contributes $+1$ to $\mathcal{A}$ if it transforms as a 3 of $SU(3)$, and contributes -1 to $\mathcal{A}$ if it is a $\bar{3}$. There are two mixed anomalies to check as well: $U(1)SU(2)^2$ and $U(1)SU(3)^2$.

This apparently miraculous cancellation is suggestive that each family of fermions may be unified into a spinor of $SO(10)$, since the vanishing of anomalies which happens automatically in $SO(10)$ is of course maintained when the symmetry is broken to a smaller subgroup, such as the Standard Model.

Exercise 2.6 Show that the global B (baryon number) and L (lepton number) currents are anomalous in the Standard Model eqn. (2.49), but that $B - L$ is not.

2.6 Parity and fermion mass in odd dimensions

Despite these lectures being about chiral fermions, it turns out that we will not only be interested in $d = 2, 4$ but also $d = 3, 5$! In these lectures I will be discussing fermions in $(2k + 1)$ dimensions with a spatially varying mass term which vanishes in some $2k$-dimensional region; in such cases we find chiral modes of a $2k$-dimensional effective theory bound to this mass defect. Such an example could arise dynamically when fermions have a Yukawa coupling to a real scalar ϕ which spontaneously breaks a discrete symmetry, where the surface with $\phi = 0$ forms a domain wall between two different phases; for this reason such fermions are called domain wall fermions, even though we will be putting the spatially dependent mass in by hand and not through spontaneous symmetry breaking.

In odd dimensions there is no analogue of Γ and therefore there is no such thing as chiral symmetry. Nevertheless, fermion masses still break a symmetry: parity. In a theory with parity symmetry one has extended the Lorentz group to include improper rotations: spatial rotations R for which the determinant of R is negative. Parity can be defined as a transformation where an odd number of the spatial coordinates flip sign. In even dimensions parity can be the transformation $\mathbf{x} \to -\mathbf{x}$ and

$$\psi(\mathbf{x}, t) \to \gamma^0 \psi(-\mathbf{x}, t) \qquad \text{(parity, } d \text{ even)} . \tag{2.73}$$

The role of the γ^0 is to transform the kinetic term correctly to realize $\vec{x} \to -\vec{x}$:

$$\gamma^0(\partial_0 \gamma^0)\gamma^0 = \partial_0 \gamma^0 , \qquad \gamma^0(\nabla_i \gamma^i)\gamma^0 = -\nabla_i \gamma^i . \tag{2.74}$$

Since $\{\gamma^0, \Gamma\} = 0$, ψ_L and ψ_R are exchanged under parity and a Dirac mass term is parity invariant.

However, in odd spacetime dimensions the transformation $\mathbf{x} \to -\mathbf{x}$ is just a proper rotation; instead we must define parity as the transformation which just flips the sign of one coordinate x^1 (or an odd number), and

$$\psi(\mathbf{x}, t) \to \gamma^1 \psi(\tilde{\mathbf{x}}, t) , \qquad \bar{\psi}(\mathbf{x}, t) \to -\bar{\psi}(\tilde{\mathbf{x}}, t)\gamma^1 , \qquad \tilde{\mathbf{x}} = (-x^1, x^2, \ldots, x^{2k}) \tag{2.75}$$

since

$$-\gamma^1(\partial_\mu\gamma^\mu)\gamma^1 = \begin{cases} +\partial_\mu\gamma^\mu & \mu \neq 1 \\ -\partial_\mu\gamma^\mu & \mu = 1 \end{cases} \qquad \text{(no sum on } \mu) \ . \tag{2.76}$$

Remarkably, a Dirac mass term flips sign under parity in this case; and since there is no chiral symmetry in odd d to rotate the phase of the mass matrix, the sign of the quark mass has physical meaning. Note that it is still possible to define a parity invariant theory of massive fermions, however, provided that they come in pairs with masses $\pm M$, and parity is defined to interchange the two, while flipping the sign of M...so long as the regulator also respects parity.

2.7 The non-decoupling of parity violation in odd dimensions

We have seen that fermion masses break chiral symmetry for even d and that they can break parity for odd d. One might then that in a in odd d with a massless fermion, which is parity invariant, we might get anomalous parity violation from a massive regulator, such as a Pauli-Villars fermion with mass M. Indeed that occurs as can be seen both by power counting and explicit calculation. In $2k+1$ spacetime dimensions the Chern Simons form, which for an Abelian gauge field is proportional to

$$\epsilon^{\alpha_1\cdots\alpha_{2k+1}} A_{\alpha_1} F_{\alpha_2\alpha_3} \cdots F_{\alpha_{2k}\alpha_{2k+1}} \ . \tag{2.77}$$

Not that this operator violates parity and time reversal. This operator is not gauge invariant, but transforms into a total derivative under a gauge transformation, so that its inclusion in the Lagrangian does not spoil the gauge invariance of the action. The Chern Simons form for nonabelian gauge fields is more complicated, and requires a quantized coefficient in order for the action to be gauge invariant. let's count dimensions: with $D_\mu = (\partial_\mu + iA_\mu)$ we have the mass dimension of A_μ is 1, and therefore the mass dimension of the above operator is d. That means that its coefficient in the action must be dimensionless. On the other hand, its coefficient must be proportional to explicit parity symmetry violation. That suggests that on integrating out a Pauli-Villars fermion with mass M, this operator can and should be generated with a coefficient proportional to $M/|M|$, which flips sign when M flips sign. Such a coefficient can arise naturally from an integral such as

$$\int \frac{d^3k}{(2\pi)^3} \frac{M}{(k^2+M^2)^2} = \frac{1}{8\pi}\frac{M}{|M|} \ , \tag{2.78}$$

relevant for the $d=3$ case. Because the coefficient is proportional to $M/|M|$, it survives the limit $M \to \infty$. This effect is called a "parity anomaly" and arises because we have to break parity to regulate the theory.

For domain wall fermions we will be interested in a closely related but slightly different problem: the generation of a Chern Simons operator on integrating out a heavy fermion of mass m. In 2+1 dimensions with an Abelian gauge field one computes the graph in Fig. 2.3, which contributes to the low-energy Lagrangian

$$\mathcal{L}_{CS} = \frac{e^2}{8\pi}\frac{m}{|m|}\epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \ . \tag{2.79}$$

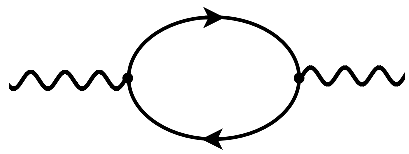

Fig. 2.3 *Integrating out a heavy fermion in three dimensions gives rise to the Chern Simons term in the effective action of eqn.* (2.79).

It is striking how related the parity and chiral anomalies look. In particular, suppose one differentiates $\mathcal{L}_{CS}$ in eqn. (2.79) with respect to the gauge field to find the current this operator will contribute to Maxwell's equations. For example, in $d = 2+1$ one finds:

$$j_\mu = \frac{1}{e}\frac{\partial \mathcal{L}_{CS}}{\partial A_\mu} = \frac{e}{8\pi}\frac{m}{|m|}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \tag{2.80}$$

where, if I pick $\mu = 2$, what I get is related to the chiral anomaly we found in eqn. (1.26) for $1+1$ dimensions. This connection was made clear with a very physical model by Callan and Harvey, and is the basis for domain wall fermions on the lattice, the subject of the next lecture.

Exercise 2.7 Verify the coefficient in eqn. (2.79) by computing the diagram Fig. 2.3. By isolating the part that is proportional to $\epsilon_{\mu\nu\alpha}p^\alpha$ before performing the integral, one can make the diagram very easy to compute.

2.8 What we have found

- Chirality exists only in even spacetime dimensions;
- Representations of the Lorentz group in $d = 4$ were derived, showing where the LH and RH Weyl fermions fit in, and how they transform;
- RH Weyl fermion fields can be replaced by charge conjugated LH Weyl fermion fields, and we saw how to represent the content of one family of the Standard Model entirely in terms of LH Weyl fermion fields;
- The chiral $U(1)_A$ anomaly in $d = 3+1$ was derived and shown to be proportional to $\epsilon_{\mu]\nu\rho\sigma}F_a^{\mu\nu}F_a^{\rho\sigma}$;
- using the anomaly, a connection was made between gauge field topology and zeromodes of the Dirac operator through the index theorem;
- It was shown that in even spacetime dimensions a Dirac fermion mass violates chirality...
- ... and that for a single massless Dirac fermion there is a parity anomaly proportional to $\epsilon_{\alpha\beta\gamma\ldots}A^\alpha\partial^\beta A^\gamma\ldots$, the Chern-Simons term;
- A correllary of the parity anomaly discussion was that a Chern-Simons term will be generated when any massive fermion is integrated out of the theory, and the effect will not decouple as $m \to \infty$.

3

Domain Wall Fermions

3.1 Chirality, anomalies and fermion doubling

So far we have seen that in order to simulate Standard Model physics, we would like a lattice regulator that has chiral symmetries with the same nice properties that we see in the continuum :

- the chiral symmetry ensures multiplicative mass renormalization for fermions (and more complicated protection against radiative corrections, such as ensuring the CP-violating electric dipole moments are proportional to the fermion mass, that electroweak operators with different chiral symmetry properties don't mix under renormalization, etc.);
- we would also like to reproduce the behavior of the pseudoscalar meson octet in QCD as approximate Nambu-Goldstone bosons of spontaneously broken chiral symmetry;
- we would like to be able to regulate chiral gauge theories for which the gauge anomalies cancel, (such as the Standard Model)

However, we have also seen that anomalies in the continuum imply that chiral symmetries must be explicitly broken on the lattice, or fermion doubling such that the exact symmetries of the lattice that were *intended* to be chiral, end up being vector symmetries in the continuum. These observations are codified in the Nielsen-Ninomiya theorem: it states that a fermion action in $2k$ Euclidian spacetime dimensions

$$S = \int_{\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^4}\, \overline{\psi}_{-\mathbf{p}} \tilde{D}(\mathbf{p})\psi(\mathbf{p}) \tag{3.1}$$

cannot have the operator $\tilde{D}$ satisfy all four of the following conditions simultaneously:

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_μ;
2. $\tilde{D}(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$;
3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_\mu = 0$;
4. $\{\Gamma, \tilde{D}(\mathbf{p})\} = 0$.

The first condition is required for locality of the Fourier transform of $\tilde{D}(\mathbf{p})$ in coordinate space, instead of just involving nearest neighbor interactions. The next two state that we want a single flavor of conventional Dirac fermion in the continuum limit. The last item is the statement of chiral symmetry. One can try keeping that and eliminating one or more of the other conditions; for example, the SLAC derivative took $\tilde{D}(\mathbf{p}) = \gamma_\mu p_\mu$ within the Brillouin zone (BZ), which violates the first condition

— if taken to be periodic, it is discontinuous at the edge of the BZ. This causes problems — for example, the QED Ward identity states that the photon vertex Γ_μ is proportional to $\partial\tilde{D}(\mathbf{p})/\partial p_\mu$, which is infinite at the BZ boundary. Naive fermions satisfy all the conditions except (3): there $\tilde{D}(\mathbf{p})$ vanishes at the 2^4 corners of the BZ, and so we have 2^4 flavors of Dirac fermions in the continuum. Staggered fermions are somewhat less redundant, producing four flavors in the continuum for each lattice field. The discussion in any even spacetime dimension is analogous. Wilson fermions just do away with (4), which is the problem we wish to solve.

Again, I want to emphasize the idea that the roadblock to developing a lattice theory with chirality is the existence of anomalies in the continuum. Any symmetry that is exact on the lattice will be exact in the continuum limit, while any symmetry anomalous in the continuum limit must be broken explicitly on the lattice. In principle though this should not be a fatal obstacle. After all, in the continuum, chiral symmetry can be broken by an anomaly, while still protecting a fermion from additive mass renormalization, at least in a theory like QED. So the question we should be posing is: can we violate (5) in just the right way to reproduce the anomalies, while preserving all the desired features of chiral symmetry?

It turns out that the answer is “yes” and that the key lies in a very simple model of fermions in the continuum: a free massive Dirac fermion living in odd spacetime dimensions which has a discontinuity in the value of its mass — either the fermion lives on a space with a boundary from which it cannot escape, or else it lives in a space where its mass changes sign somewhere. In discussing this, we will see that all of the things we have been learning about anomalies in different dimensions will come together.

3.2 Domain wall fermions in the continuum

3.2.1 Motivation

To be concrete, consider a fermion in infinite, continuum three dimensions (coordinates (x_0, x_1, x_2)), where the fermion has a mass which depends on x_2 and switches sign at $x_2 = 0$. For simplicity, I will take $m(x_2) = m\epsilon(x_2) = mx_2/|x_2|$. Curiously enough, we will show that a massless fermion mode exists bound to this 2-dimensional surface, and that it is chiral: there exist a RH mode, and not a LH one. Thus the low energy limit of this theory looks like a $d = 2$ theory of a Weyl fermion with a chiral symmetry, even though we started with a 3-dimensional theory, where there is no chiral symmetry at all.

This should sound impossible: we know that the low energy effective theory I just described is anomalous. The fermion current for a right-handed fermion can be written as a current made from a Dirac fermion with a chiral projection operator

$$j_R^\mu = \bar{\psi}\gamma^\mu P_+\psi = \tfrac{1}{2}\left(j^\mu + j_A^\mu\right) \tag{3.2}$$

and so if the fermion carries a $U(1)$ gauge charge, there is an anomalous divergence,

$$\partial_\mu j_R^\mu = \tfrac{1}{2}\partial_\mu j_A^\mu = \frac{e}{2\pi}E\ . \tag{3.3}$$

If we turn on an electric field pointing in the x_1 direction, then the charge on the mass defect must increase with time. Yet in the full 3-dimensional theory with a Dirac fermion, the fermion current is $j^\mu = \bar{\psi}\gamma^\mu\psi$, and it is conserved. So it looks like it would be a sick gauge theory from the point of view of the low energy theory of the surface modes, but a healthy theory when we look at the full $d = 3$ theory including the heavy fermions. This apparent contradiction needs to be resolved!

You might also be suspicious that there is some hidden fine tuning required to keep the chiral mode on the domain wall surface massless, but that cannot be: the low energy effective theory would need a LH mode as well as a RH mode in order for there to be a mass, and quantum corrections cannot make a single LH mode suddenly appear. In this lecture I show how the continuum theory works, and then how it can be transcribed to the lattice. We will see that the theory has distinct topological phases which determine the spectrum of massless fermions on the surface and protects them from becoming massive, while nondecoupling effects related to the parity anomaly ensure current conservation throughout the lattice. In the next lecture I will discuss how the effective theory can be described directly using the overlap formulation, without any reference to the higher dimensional parent theory.

3.2.2 The model

Consider the Dirac equation in odd spacetime dimension with a space-dependent mass term. One can think of a space-dependent mass as arising from a Higgs mechanism, for example, where there is a topological defect trapped in the classical Higgs field, such as a domain wall or a vortex. A domain wall can arise dynamically when the Higgs field breaks a discrete symmetry; a vortex when the Higgs field breaks a $U(1)$ symmetry (Preskill, 1985). Domain wall defects are pertinent to putting chiral fermions on the lattice, so I will consider that example. Here I am putting it into the theory by hand, without any dynamical explanation.

Consider a fermion in Euclidian spacetime with dimension $d = 2k + 1$, where the coordinates are written as $\{x_0, x_1, \dots x_{2k-1}, s\} \equiv \{x_\mu, s\}$, where $\mu = 0, \dots, 2k - 1$ and s, which is what I call the coordinate x_{2k}. The $(2k + 1)$ γ matrices are written as $\{\gamma_0, \dots, \gamma_{2k}, \Gamma\}$. This fermion is assumed to have an s-dependent mass with the simple form

$$m(s) = m\,\epsilon(s) = \begin{cases} +m & s > 0 \\ -m & s < 0 \end{cases}, \qquad m > 0\ . \tag{3.4}$$

This mass function explicitly breaks the Poincaré symmetry of $2k + 1$ dimensional spacetime, but preserves the Euclidian Poincaré symmetry of $2k$ dimensional spacetime (Poincaré symmetry is Lorentz symmetry and spacetime translations combined). The fermion is also assumed to interact with $2k$-dimensional background gauge fields $A_\mu(x_\mu)$ which are independent of s. The Dirac equation may be written as:

$$\left[\not{D} + \Gamma\partial_s + m(s)\right]\Psi(x_\mu, s) = 0\ , \tag{3.5}$$

where $\not{D}$ is the lower dimension ($d = 2k$) covariant Dirac operator. We can perform separation of variables and expand the spinor Ψ as the product of functions of s times spinors $\psi(x_\mu)$,

$$\Psi(x_\mu, s) = \sum_n \left[b_n(s)P_+ + f_n(s)P_-\right]\psi_n(x_\mu)\ , \qquad P_\pm = \frac{1\pm\Gamma}{2}\ , \tag{3.6}$$

satisfying the equations

$$\begin{aligned}[][\partial_s + m(s)]b_n(s) &= \mu_n f_n(s)\ ,\\ [-\partial_s + m(s)]f_n(s) &= \mu_n b_n(s)\ ,\end{aligned} \tag{3.7}$$

where the ψ_n are general functions of x_μ, which can later be expanded themselves in some useful basis[1]. That we can write can motivated by finite matrices. Consider a non-hermitian matrix M; it will in general have different left- and right-eigenvectors. Let the matrices L and R diagonalize the Hermitian matrices $MM^\dagger$ and $M^\dagger M$ respectively. For a finite matrix M, it is guaranteed that $MM^\dagger$ and $M^\dagger M$ have the same, non-negative real eigenvalues μ_n^2. The matrices L and R are only determined up to phases, and it is possible to choose the phases of L and R such that the matrix $L^d aggerMR$ is diagonal, with entries equal to $|\mu_n|$. (This fact is used in the Standard Model when we diagonalize the quark masses, and $L^\dagger R$ plays the role of the CKM matrix, while the $|\mu_n|$ are what we take for the quark masses.) In the present case, the differential operator $(\partial_s + m(s))$ plays the role of M, while $(-\partial_s + m(s))$ plays the role of $M^\dagger$; an important difference though is that it is possible for infinite matrices $MM^\dagger$ and $M^\dagger M$ to have a different number of zero eigenvalues, as we will see explicitly. Since the $b_n(s)$ and $f_n(s)$ functions are eigenstates of (different) hermitian operators, we know that the b_n functions form a complete orthogonal basis, as do the f_n.

One might expect all the eigenvalues in eqn. (3.2.2) to satisfy $|\mu_n| \gtrsim O(m)$, since that is the only scale in the problem. However, there is also a solution to eqn. (3.2.2) with eigenvalue $\mu = 0$ given by

$$b_0 = Ne^{-\int_0^s m(s')ds'} = Ne^{-m|s|}\ . \tag{3.8}$$

This solution is localized near the defect at $s = 0$, falling off exponentially fast away from it. There is no analogous solution to eqn. (3.2.2) of the form

$$f_0 \sim e^{+\int_0^s m(s')ds'}\ ,$$

since that would be exponentially growing in $|s|$ and not normalizable. In terms of this expansion, the $2k+1$-dimensional Dirac action can be written as an infinite sum of $2k$-dimensional Dirac actions:

$$\begin{aligned} S &= \int d^{2k}x \int ds\, \bar\Psi\left(\not{D} + \Gamma\partial_s + m\epsilon(s)\right)\Psi \\ &= \sum_{k,\ell}\int d^{2k}x\int ds\, \bar\psi_k(x)\left[b_k^\dagger(s)P_- + f_n^\dagger(s)P_+\right]\left(\not{D} + \Gamma\partial_s + m\epsilon(s)\right)\left[b_\ell(s)P_+ + f_\ell(s)P_-\right]\psi_\ell(x)\end{aligned}$$

[1]In the present example with infinite extra dimension, the index n will be continuous; in the next section I will solve this on a compact extra dimension where the label n is discrete; here I will use the discrete notation for simplicity, but sums over the index should actually be integrals in this section.

$$= \int \int d^{2k}x \left[\bar{\psi}_0 \not{D} P_+ \psi_0 + \sum_{k \neq 0} \bar{\psi}_k \left(\not{D} + \mu_n \right) \psi_k \right] \tag{3.9}$$

So we see that the spectrum consists of an infinite tower of a single massless right-handed $d = 2k$ chiral fermion, and an infinite tower of massive Dirac fermions with mass $O(m)$ and higher. The massless fermion is localized at the defect at $s = 0$, whose profile in the transverse extra dimension is given by eqn. (3.8); the massive fermions are not localized in the extra dimension. Because of the gap in the spectrum, at low energy the accessible part of the spectrum consists only of the massless RH chiral fermion. Since the states in the bulk are gapped, this is what would be referred to as an insulator by condensed matter physicists. However, we will see that the bulk is characterized by different topological phases, so in fact this is an example of a *topological insulator* (Hasan and Kane, 2010; Qi and Zhang, 2011).

Exercise 3.1 Construct three different $d = 2k + 1$ theories which, when continued back to Minkowski spacetime, have a low energy spectrum consisting of a single massless $d = 2k$ *Dirac* fermion localized at $s = 0$. (Recall that in Minkowski spacetime we can represent a Dirac fermion as a LH Weyl fermion plus a RH Weyl fermion; or two LH Weyl fermions; or two RH Weyl fermions.)

Some comments are in order:

- It is not a paradox that the low energy theory of a single right-handed chiral fermion ψ_0 violates parity in $d = 2k$, since the mass for Ψ breaks parity in $d = 2k + 1$;
- Furthermore, nothing is special about right-handed fermions, and a left handed mode would have resulted if we had chosen the opposite sign for the mass in eqn. (3.4). This makes sense because choosing the opposite sign for the mass can be attained by flipping the sign of all the space coordinates: a rotation in the $(2k + 1)$ dimensional theory, but a parity transformation from the point of view of the $2k$-dimensional fermion zeromode.
- The fact that a chiral mode appeared at all is a consequence of the normalizability of $\exp(-\int_0^s m(s')ds')$, which in turn follows from the two limits $m(\pm\infty)$ being nonzero with opposite signs. Any function $m(s)$ with that boundary condition will support a single chiral mode, although in general there may also be a number of very light fermions localized in regions wherever $|m(s)|$ is small — possibly extremely light if $m(s)$ crosses zero a number of times, so that there are widely separated defects and anti-defects.
- Gauge boson loops will generate contributions to the fermion mass function which are even in s. If the coupling is sufficiently weak, it cannot effect the masslessness of the chiral mode. However if the gauge coupling is strong, or if the mass m is much below the cutoff of the theory, the radiative corrections could cause the fermion mass function to never change sign, and the chiral mode would not exist. Or it could still change sign, but become small in magnitude in places, causing the chiral mode to significantly delocalize. An effect like this can cause trouble with lattice simulations at finite volume and lattice spacing; more later.

3.3 Domain wall fermions and the Callan-Harvey mechanism

Now turn on background gauge fields and see how the anomaly works, following (Callan and Harvey, 1985). To do this, I integrate out the heavy modes in the presence of a background gauge field. Although I will be interested in having purely $2k$-dimensional gauge fields in the theory, I will for now let them be arbitrary $2k+1$ dimensional fields. Since it is hard to integrate out the heavy modes exactly, because of their complicate $b_n(s)$ and $f_n(s)$ wave functions, I will assume perform the calculation as if the mass m was constant, and then substitute $m(s)$; this is not valid where $m(s)$ is changing rapidly (near the domain wall) but should be adequate farther away. Also — in departure from the work of (Callan and Harvey, 1985), I will include a Pauli-Villars field with constant mass $M<0$, independent of s; this is necessary to regulate fermion loops in the wave function renormalization for the gauge fields, for example.

When one integrates out the heavy fields, one generates a Chern Simons operator in the effective Lagrangian, as discussed in §2.7:

$$\mathcal{L}_{\rm CS} = \left(\frac{m(s)}{|m(s)|} + \frac{M}{|M|}\right)\mathcal{O}_{CS} = (\epsilon(s)-1)\,\mathcal{O}_{CS} \tag{3.10}$$

Note that with $M<0$, the coefficient of the operator equals -2 on the side where $m(s)$ is negative, and equals zero on the side where it is positive. For a background $U(1)$ gauge field one finds in Euclidian spacetime:

$$d=3: \quad \mathcal{O}_{CS} = -\frac{e^2}{8\pi}\epsilon_{abc}(A_a\partial_a A_c)\ , \tag{3.11}$$

$$d=5: \quad \mathcal{O}_{CS} = -\frac{e^3}{48\pi^2}\epsilon_{abcde}(A_a\partial_b A_c\partial_d A_e)\ . \tag{3.12}$$

Differentiating $\mathcal{L}_{\rm CS}$ by A_μ and dividing by e gives the particle number current:

$$j_a^{(CS)} = (\epsilon(s)-1)\begin{cases} -\frac{e}{8\pi}\epsilon_{abc}(F_{bc}) & d=3 \\ \\ -\frac{e^2}{64\pi^2}\epsilon_{abcde}(F_{bc}F_{de}) & d=5 \end{cases} \tag{3.13}$$

where I use Latin letters to denote the coordinates in $2k+1$ dimensions, while Greek letters will refer to indices on the $2k$-dimensional defect. So when we turn on background $2k$ dimensional gauge fields, particle current flows either onto or off of the domain wall along the transverse s direction on the left side (where $m(s)=-m$). If we had regulated with a positive mass Pauli Villars field, the current would flow on the right side.

Why does this current in the bulk seem bizarre? Because the spectrum in the bulk is gapped – there are no light excitations there. In particle physics we are accustomed to the fact that if we want to see heavy particles we have to spend billions of dollars! Yet in the bulk we can generate currents with very cheap (weak, smooth) gauge fields...how is this possible? Furthermore, the currents are transverse to the applied electric field! Well, you have hopefully seen this before: the phenomenon is identical to the Integer Quantum Hall effect. In that case the UV theory is quite different (magnetic field, nonrelativistic electrons, Landau levels) but the underlying topological phase structure and description in terms of Chern-Simons currents is identical. And where the condensed matter system relies on the B field to provide P and T violation, here the Ψ mass m performs the same role. More on this connection later.

But in either case, this bizarre current exactly accounts for the anomaly. Consider the case of a 2-dimensional domain wall embedded in 3-dimensions. If we turn on an E field we know that from the point of view of a 2d creature, RH Weyl particles are created, where from eqn. (2.58),

$$\partial_\mu j_{\mu,R} = \tfrac{1}{2}\partial_\mu j_{\mu,A} = \frac{e}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu} \ . \tag{3.14}$$

We see from eqn. (3.13) this current is exactly compensated for by the Chern Simons current $j_2^{(CS)} = \frac{e}{4\pi}\epsilon_{2\mu\nu}F_{\mu\nu}$ which flows onto the domain wall from the $-s = -x_2$ side. The total particle current is divergenceless as we expect, since the current is just our usual conserved particle number current in the parent $d = 3$ theory.

This is encouraging for finding an optimal solution for chiral fermions within the confines of the Nielsen-Ninomiya theorem:

(i) we managed to obtain a fermion whose mass is zero due to topology and not fine tuning;
(ii) the low energy theory therefore has an almost exact chiral symmetry, even though the full 3d theory does not;
(iii) the only remnant of the explicit chiral symmetry breaking of the full theory is the anomalous divergence of the chiral symmetry in the presence of gauge fields.

One drawback though is the infinite dimension in the s direction, since we will eventually want to simulate this on a finite lattice; besides, it is always disturbing to see currents streaming in from $s = -\infty$! One solution is to work in finite $(2k+1)$ dimensions, in which case we end up with a massless RH mode stuck to the boundary on one side and a LH mode on the other (which is great for a vector like theory of massless Dirac fermions, but not for chiral gauge theories). This is what one does when simulating domain wall fermions for QCD. The other solution is more devious, deriving the exact effective theory for the surface modes in the limit infinite extra dimension, leading to the "overlap operator"; this will be the subject of the next lecture.

3.3.1 Domain wall fermions on a compact extra dimension

To get a better understanding for how the theory works, it is useful to consider a compact extra dimension. In particular, consider the case of periodic boundary conditions $\psi(x_\mu, s+2L) = \psi(x_\mu, s)$; we define the theory on the interval $-L \le s \le L$ with $\psi(x_\mu, -L) = \psi(x_\mu, L)$ and mass $m(s) = m\frac{s}{|s|}$. Note the the mass function $m(s)$ now has a domain wall kink at $s = 0$ and an anti-kink at $s = \pm L$. There are now two exact zeromode solutions to the Dirac equation,

$$b_0(s) = Ne^{-\int_0^s m(s')ds'} \ , \qquad f_0(s) = N'e^{+\int_0^s m(s')ds'} \ . \tag{3.15}$$

Both solutions are normalizable since the transverse direction is finite; b_0 corresponds to a right-handed chiral fermion located at $s = 0$, and f_0 corresponds to a left-handed chiral fermion located at $s = \pm L$. For completeness, I note here that all of the b_n and f_n can be found explicitly, and they are given by

$$\begin{aligned} b_{n1} &= \frac{\cos(k_n|s| + \alpha_n)}{\sqrt{L}}, \quad & f_{n1} &= \frac{\sin k_n s}{\sqrt{L}}, \\ b_{n2} &= \frac{\sin k_n s}{\sqrt{L}}, & f_{n2} &= -\frac{\cos(k_n|s| - \alpha_n)}{\sqrt{L}} \end{aligned} \tag{3.16}$$

where

$$k_n = \frac{\pi n}{L}\,, \qquad \mu_n = \sqrt{\Lambda^2 + k_n^2}\,, \qquad \alpha_n = \cot^{-1}\frac{k_n}{\Lambda}\,. \tag{3.17}$$

The two solutions for each n correspond to even and odd solutions under $s \to -s$.

The existence of two exactly massless modes for finite L is a result of the fact that $\int_{-L}^{+L} m(s)\,ds = 0$, which is not a topological condition and not robust. For example, turning on weakly coupled gauge interactions will cause a shift the mass by $\delta m(s) \propto \alpha\Lambda$ (assuming Λ is the cutoff) which ruins this property. However: remember that to get a mass in the $2k$-dimensional defect theory, the RH and LH chiral modes have to couple to each other. The induced residual mass will be

$$m_{\mathrm{res}} \sim \delta m \int ds\, b_0(s)\, f_0(s) = \delta m\, N N' \sim \alpha m \times \frac{2mL}{\cosh[mL]} \tag{3.18}$$

which vanishes exponential fast as $(mL) \to \infty$.

As seen in eqn. (3.10) and corroborated below in an analogous calculation on the lattice, the Chern-Simons current flows onto the defect from only one side, and therefore it is a waste of resources to simulate the system described above with periodic boundary conditions. Instead one can simulate massive Dirac fermions on a finite interval in the extra dimension with a constant mass. The zeromodes then exist on the surfaces. Now even without extra interactions, one finds

$$m_{\mathrm{res}} \sim 2me^{-2mL} \tag{3.19}$$

Any matrix element of a chiral symmetry violating operator will be proportional to the overlap of the LH and RH zeromode wave functions, which is proportional to m_{res}. On the lattice the story of m_{res} is more complicated, both because of the discretization of the fermion action, and because of the presence of rough gauge fields. Rough gauge field configurations should be rare when one is near the continuum limit (α small) but they can have an outsized effect on m_{res} by delocalizing the zeromodes, and the choice of gauge action can make a difference. Lattice computations with domain wall fermions need to balance the cost of simulating a large extra dimension near the continuum limit versus the need to make m_{res} small enough to attain chiral symmetry.

3.3.2 The (almost) chiral propagator

Before moving to the lattice, I want to mention an illuminating calculation by Lüscher (Luscher, 2000*a*) who considered noninteracting domain wall fermions with a semi-infinite fifth dimension, negative fermion mass, and and LH Weyl fermion zeromode bound to the boundary at $s = 0$. He computed the Green function for propagation of the zeromode from $(x, s = 0)$ to $(y, s = 0)$ and examined the chiral properties of this propagator. The differential operator to invert should be familiar now:

$$D_5 = \not{\partial}_4 + \gamma_5 \partial_s - m \ , \qquad s \geq 0 \ . \tag{3.20}$$

We wish to look at the Green function G which satisfies

$$D_5 G(x, s; y, t) = \delta^{(4)}(x - y)\delta(s - t) \ , \qquad P_+ G(x, 0; y, t) = 0 \ . \tag{3.21}$$

The solution Lüscher found for propagation along the boundary was

$$G(x, s; y, t)\Big|_{s=t=0} = 2P_- D^{-1} P_+ \ , \tag{3.22}$$

where D is the peculiar looking operator

$$D = [1 + \gamma_5 \epsilon(H)] \ , \qquad H \equiv \gamma_5(\not{\partial}_4 - m) = H^\dagger \ , \qquad \epsilon(\mathcal{O}) \equiv \frac{\mathcal{O}}{\sqrt{\mathcal{O}^\dagger \mathcal{O}}} \ . \tag{3.23}$$

This looks pretty bizarre! Since H is hermitian, in a basis where H is diagonal, $\epsilon(H) = \pm 1$! But don't conclude that in this basis the operator is simply $D = (1 \pm \gamma_5)$ — you must remember, that in the basis where H is diagonal, $\epsilon(H)\gamma_5$ is not (by which I mean $\langle m|\epsilon(H)\gamma_5|n\rangle$ is in general nonzero for $m \neq n$ in the H eigenstate basis). In fact, eqn. (3.23) looks very much like the overlap operator discovered some years earlier and which we will be discussing soon.

A normal Weyl fermion in four dimensions would have a propagator $P_-(\not{\partial}_4)^{-1}P_+$; here we see that the domain wall fermion propagator looks like the analogous object arising from the fermion action $\overline{\psi} D \psi$, with D playing the role of the four-dimensional Dirac operator $\not{\partial}_4$. So what are the properties of D?

- For long wavelength modes (e.g. $k \ll m$) we can expand D in powers of $\not{\partial}_4$ and find

$$D = \frac{1}{m}\left(\not{\partial}_4 - \frac{\partial_4^2}{2m} + \dots\right) \ , \tag{3.24}$$

  which is reassuring: we knew that at long wavelengths we had a garden variety Weyl fermion living on the boundary of the extra dimension (the factor of $1/m$ is an unimportant normalization).
- A massless Dirac action is chirally invariant because $\{\gamma_5, \not{\partial}_4\} = 0$. However, the operator D does not satisfy this relationship, but rather:

$$\{\gamma_5, D\} = D\gamma_5 D \ , \tag{3.25}$$

  or equivalently,

$$\{\gamma_5, D^{-1}\} = \gamma_5 \ . \tag{3.26}$$

  This is the famous Ginsparg-Wilson equation which will be discussed in the next lecture, first introduced in context of the lattice (but not solved) many years earlier

(Ginsparg and Wilson, 1982). Note the right hand side of the above equations encodes the violation of chiral symmetry that our Weyl fermion experience; the fact that the right side of eqn. (3.26) is local in spacetime implies that violations of chiral symmetry will be seen in Green functions *only* when operators are sitting at the same spacetime point. We know from our previous discussion, the only chiral symmetry violation that survives to low energy in the domain wall model is the anomaly, and so it must be that the chiral symmetry violation in eqns. (3.25)-(3.26) encode the anomaly and nothing else, at low energy [2].

3.4 Domain wall fermions on the lattice

The next step is to transcribe this theory onto the lattice. If you replace continuum derivatives with the usual lattice operator $D \to \frac{1}{2}(\nabla^* + \nabla)$ (where ∇ and ∇^* are the forward and backward lattice difference operators respectively) then one discovers...doublers! Not only are the chiral modes doubled in the $2k$ dimensions along the domain wall, but there are two solutions for the transverse wave function of the zero mode, $b_0(s)$, one of which alternates sign with every step in the s direction and which is a LH mode. So this ends up giving us a theory of naive fermions on the lattice, only in a much more complicated and expensive way!

However, when we add Wilson terms $\frac{r}{2}\nabla^*\nabla$ for each of the dimensions, things get interesting. You can think of these as mass terms which are independent of s but which are dependent on the wave number k of the mode, vanishing for long wavelength. What happens of we add a k-dependent spatially constant mass $\Delta m(k)$ to the step function mass $m(s) = m\epsilon(s)$? The solution for $b_0(s)$ in eqn. (3.8) for an infinite extra dimension becomes

$$b_0 = Ne^{-\int^L [m(s')+\Delta m(k)]ds'} \ , \tag{3.27}$$

which is a normalizable zeromode solution — albeit, distorted in shape — so long as $|\Delta m(k)| < m$. However, for $|\Delta m(k)| > m$, the chiral mode vanishes. What happens to it? It becomes more and more extended in the extra dimension until it ceases to be normalizable. What is going on is easier to grasp for a finite extra dimension: as $|\Delta m(k)|$ increases with increasing k, eventually the b_0 zeromode solution extends to the opposing boundary of the extra dimension, when $|\Delta m(k)| \sim (m - 1/L)$. At that point it can pair up with the LH mode and become heavy.

So the idea is: add a Wilson term, with strength such that the doublers at the corners of the Brillouin zone have $|\Delta m(k)|$ too large to support a zeromode solution. Under separation of variables, one looks for zeromode solutions with $\psi(x,s) = e^{ipx}\phi_\pm(s)\psi_\pm$ with $\Gamma\psi_\pm = \pm\psi$. One then finds (for $r = 1$)

$$\not{p}_4\psi_\pm = 0 \ , \qquad -\phi_\pm(s \mp 1) + (m_{\text{eff}}(s) + 1)\phi_\pm(s) = 0 \ \ , \tag{3.28}$$

[2] A lattice solution to eqn. (3.25) (the only solution in existence) is the overlap operator discovered by Neuberger (Neuberger, 1998*a*; Neuberger, 1998*b*); it was a key reformulation of earlier work (Narayanan and Neuberger, 1993; Narayanan and Neuberger, 1995) on how to represent domain wall fermions with an infinite extra dimension (and therefore exact chiral symmetry) in terms of entirely lower dimensional variables. We will discuss overlap fermions and the Ginsparg-Wilson equation further in the next lecture.

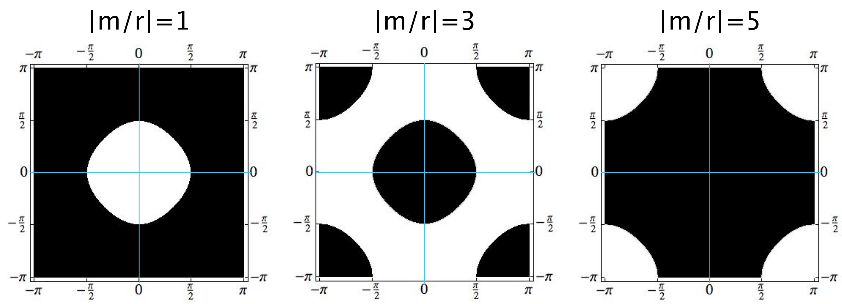

Fig. 3.1 *Domain wall fermions in $d = 2$ on the lattice: dispersion relation plotted in the Brillouin zone. Chiral modes exist in white regions only. For $0 < |m/r| < 2$ there exists a single RH mode centered at $(k_1, k_2) = (0, 0)$. for $2 < |m/r| < 4$ there exist two LH modes centered at $(k_1, k_2) = (\pi, 0)$ and $(k_1, k_2) = (0, \pi)$; for $4 < |m/r| < 6$ there exists a single RH mode centered at $(k_1, k_2) = (\pi, \pi)$. For $|m/r| > 6$ there are no chiral mode solutions.*

where

$$m_{\text{eff}}(s) = m\epsilon(s) + \sum_{\mu}(1 - \cos p_\mu) \equiv m\epsilon(s) + F(p) \ . \tag{3.29}$$

Solutions of the form $\phi_\pm(s) = z_\pm^s$ are found with

$$z_\pm = (1 + m_{\text{eff}}(s))^{\mp 1} = (1 + m\epsilon(s) + F(p))^{\mp 1} \ ; \tag{3.30}$$

they are normalizable if $|z|^{\epsilon(s)} < 1$. Solutions are found for ψ_+ only, and then provided that m is in the range $F(p) < m < F(p)+2$. (For $r \neq 1$, this region is found by replacing $m \to m/r$.) However, even though the solution is only found for ψ_+, the chirality of the solutions will alternate with corners of the Brillouin zone, just as we found for naive fermions. The picture for the spectrum in 2d is shown in Fig. 3.1. It was first shown in (Kaplan, 1992) that doublers could be eliminated for domain wall fermions on the lattice; the rich spectrum in Fig. 3.1 was worked out in (Jansen and Schmaltz, 1992), where for 4d they found the number of zeromode solutions to be the Pascal numbers $(1, 4, 6, 4, 1)$ with alternating chirality, the critical values for $|m/r|$ being $0, 2, \ldots, 10$. One implication of their work is that the Chern Simons currents must also change discontinuously on the lattice at these critical values of $|m|/r$; indeed that is the case, and the lattice version of the Callan-Harvey mechanism was verified analytically in (Golterman, Jansen and Kaplan, 1993), discussed below.

Fig. 3.1 suggests that chiral fermions will exist in two spacetime dimensions so long as $0 < |m/r| < 6$, with critical points at $|m/r| = 0, 2, \ldots, 6$ where the numbers of massless flavors and their chiralities change discontinuously. In four spacetime dimensions a similar calculation leads to chiral fermions for $0 < |m/r| < 10$ with critical points at $|m/r| = 0, 2, \ldots, 10$. However, this reasoning ignores the gauge fields. In perturbation theory one would expect the bulk fermions to obtain a radiative mass correction of size $\delta m \sim O(\alpha)$ in lattice units, independent of the extra dimension s. Extrapolating shamelessly to strong coupling, one then expects the domain wall form

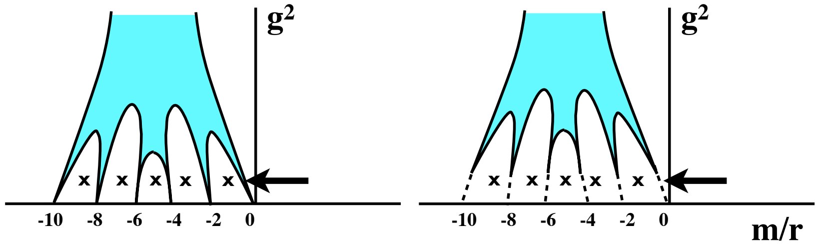

Fig. 3.2 *A sketch of the possible phase structure of QCD with Wilson fermions where the shaded region is the Aoki phase — pictured extending to the continuum limit (left) or not (right). When using Wilson fermions one attempts to tune the fermion mass to the phase boundary (arrow) to obtain massless pions; this is only possible in the continuum limit of the picture on the left is correct. For domain wall fermions chiral symmetry results at infinite L when one simulates in any of the regions marked with an "X". There are six "fingers" in this picture instead of five due to the discretization of the fifth dimension.*

of the mass to be ruined when $\alpha \sim 1$ for $|m/r| \sim (2n+1)$, $n = 0, \ldots, 4$ causing a loss of chiral symmetry; near the critical points in $|m/r|$ the critical gauge coupling which destroys chiral symmetry will be smaller.

While qualitatively correct, this argument ignores the discrete nature of the lattice. On the lattice, the exponential suppression $m_{\rm res} \sim \exp(-2mL)$ found in eqn. (3.18) is replaced by $\hat{T}^L = \exp(-L\hat{h})$, where $\hat{T}$ is a transfer matrix in the fifth dimension which is represented by L lattice sites. Good chiral symmetry is attained when $\hat{h}$ exhibits a "mass gap", i.e. when all its eigenvalues are positive and bounded away from zero. However one finds that at strong coupling, rough gauge fields can appear which give rise to near zero-modes of $\hat{h}$, destroying chiral symmetry, with $m_{\rm res} \propto 1/L$. To avoid this problem, one needs to work at weaker coupling and with an improved gauge action which suppresses the appearance of rough gauge fields.

At finite lattice spacing the phase diagram is expected to look something like in Fig. 3.2 where I have plotted m versus g^2, the strong coupling constant. On this diagram, $g^2 \to 0$ is the continuum limit. Domain wall fermions do not require fine tuning so long as the mass is in one of the distinct topological phases marked by an "X", which yield $\{1, 4, 6, 4, 1\}$ chiral flavors from left to right. The shaded region is a phase called the Aoki phase (Aoki, 1984); it is presently unclear whether the phase extends to the continuum limit (left side of Fig. 3.2) or not (right side) (Golterman, Sharpe and Singleton, 2005). In either case, the black arrow indicates how for Wilson fermions on tunes the mass from the right to the boundary of the Aoki phase to obtain massless pions and chiral symmetry; if the Aoki phase extends down to $g^2 = 0$ than the Wilson program will work in the continuum limit, but not if the RH side of Fig. 3.2 pertains. See (Golterman and Shamir, 2000; Golterman and Shamir, 2003) for a sophisticated discussion of the physics behind this diagram.

Of course, in the real world we do not see exact chiral symmetry, since quarks and leptons do have mass. A mass for the domain wall fermion can be included as a coupling between the LH mode at $s = 1$, and the RH mode at $s = N_s$:

$$m_q \left[\overline{\psi}(\mathbf{x},1)P_+\psi(\mathbf{x},N_s) + \overline{\psi}(\mathbf{x},N_s)P_-\psi(\mathbf{x},1)\right] \tag{3.31}$$

and correlation functions are measured by sewing together propagators from one boundary to itself for chiral symmetry preserving operators, or from one boundary to the other for operators involving a chiral flip. The latter will require insertions of the mass operator above to be nonzero (assuming a neglible $m_{\rm res}$) — just like it should be in the continuum.

3.4.1 Shamir's formulation

Domain wall fermions are used by a number of lattice collaborations these days, using the formulation of Shamir (Shamir, 1993; Furman and Shamir, 1995), which is equivalent to the continuum version of domain wall fermions on a slab described above. The lattice action is given by:

$$\sum_{b=1}^{5}\sum_{\mathbf{x}}\sum_{s=1}^{N_s}\left[\tfrac{1}{2}\overline{\psi}\gamma_b(\partial_b{}^* + \partial_b)\psi - m\overline{\psi}\psi - \frac{r}{2}\overline{\psi}\partial_b{}^*\partial_b\psi\right] \tag{3.32}$$

where the lattice coordinate on the 5d lattice is $\mathbf{n} = \{\mathbf{x}, s\}$, $\mathbf{x}$ and s being the 4d and fifth dimension lattice coordinates respectively. The difference operators are

$$\partial_b\psi(\mathbf{n}) = \psi(\mathbf{n}+\hat{\mu}_b) - \psi(\mathbf{n}) \ , \qquad \partial_b{}^*\psi(\mathbf{n}) = \psi(\mathbf{n}) - \psi(\mathbf{n}-\hat{\mu}_b) \tag{3.33}$$

where $\hat{\mu}_b$ is a unit vector in the x_b direction. In practice of course, these derivatives are gauged in the usual way by inserting gauge link variables. The boundary conditions are defined by setting fields to zero on sites with $s = 0$ and $s = N_s + 1$. I have reversed the sign of m and r from Shamir's original paper, since the above sign for r appears to be relatively standard now. For domain wall fermions, m has the opposite sign from standard Wilson fermions, which is physics, not convention. The above action gives rise to a RH chiral mode bound to the $s = 1$ boundary of the lattice, and a LH chiral mode bound at the $s = N_s$ boundary. The original papers discuss how to define Green functions of interest.

4

Overlap fermions and the Ginsparg-Wilson equation

4.1 Overlap fermions

We have seen that the low energy limit of a domain wall fermion in the limit of large extra dimension is a single massless Dirac fermion, enjoying the full extent of the chiral symmetry belonging to massless fermions in the continuum. In this low energy limit, the effective theory is four-dimensional if the original domain wall fermion lived in five dimensions. One might wonder whether one could dispense with the whole machinery of the extra dimension and simply write down the low energy four-dimensional theory to start with. Furthermore, one would like a four dimensional formulation with exact chiral symmetry, which could only occur for domain wall fermions with infinite extent in the time direction, which is not very practical numerically!

Neuberger and Narayanan found an extremely clever way to do this, leading to the four-dimensional "overlap operator" which describes lattice fermions with perfect chiral symmetry. The starting point is to consider a five dimensional fermion in the continuum with a single domain wall, and to consider the fifth dimension to be time (after all, it makes no difference in Euclidian space). Then $\gamma_5(\not{D}_4+m(s))$ looks like the Hamiltonian, where s is the new time coordinate, and $m(-\infty)=-m_1$, $m(\infty)=+m_2$, where $m_{1,2}>0$. The path integral projects onto ground states, and so the partition function for this system is $Z=\langle\Omega,-m_1|\Omega,+m_2\rangle$, where the state $|0,m\rangle$ is the ground state of $\mathcal{H}_4(m)=\gamma_5(\not{D}_4+m)$ We know that this should describe a massless Weyl fermion. Note that the partition function is in general complex with an ill-defined phase (we can redefine the phase of $|\Omega,-m_1\rangle$ and $|\Omega,m_2\rangle$ separately and arbitrarily). If we now instead imagine that the fermion mass function $m(s)$ exhibits a wall-antiwall pair, with the two defects separated infinitely far apart, we recognize a system that will have a massless Dirac fermion in the spectrum, and $Z=\left|\langle\Omega,-m_1|\Omega,+m_2\rangle\right|^2$, which is real, positive, and independent on how we chose the phase for the groundstates.

We can immediately transcribe this to the lattice, where we replace $\not{D}_4$ with the four dimensional Wilson operator,

$$\mathcal{H}(m)=\gamma_5(D_w+m)=\gamma_5\left(D_\mu\gamma_\mu-\frac{r}{2}D_\mu^2+m\right) \tag{4.1}$$

with D_μ being the symmetric covariant derivative on the lattice, and D_μ^2 being the covariant lattice Laplacian. Note that $\mathcal{H}(m)$ is Hermitian, and so its eigenvalues are real. Furthermore, one can show that it has equal numbers of positive and negative eigenvalues.

We can account for the γ-matrix structure of $\mathcal{H}(m)$ explicitly in a chiral basis where $\gamma_5 = \sigma_3 \otimes 1$:

$$\mathcal{H}(m) = \begin{pmatrix} B+m & C \\ C^\dagger & -B-m \end{pmatrix} \tag{4.2}$$

where $B = -\frac{r}{2}\nabla^2$ is the Wilson operator and $C = D_\mu \sigma_\mu$ where $\sigma_\mu = \{i, \vec{\sigma}\}$. For simplicity for $\langle \Omega, m_2|$ one can take $m_2 \to \infty$, in which case $\mathcal{H} \sim +m_2\gamma_5$.

We know that $Z = \left|\langle \Omega, -m_1 | \Omega, +m_2\rangle\right|^2$ will represent a massless Dirac fermion on the lattice, so long as $0 < m_1 < 2r$, with m_2 arbitrary. The groundstates of interest may be written as Slater determinants of all the one-particle wave functions with negative energy. Let us designate the one-particle energy eigenstates of $\mathcal{H}(-m_1)$ and $\mathcal{H}(m_2)$ to be $|n, -m_1\rangle$ and $|n, m_2\rangle$ respectively, with

$$\langle n, m_2 | n', -m_1\rangle \equiv U_{nn'} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_{nn'} , \qquad U^\dagger U = 1 , \tag{4.3}$$

where the block structure of U is in the same γ-matrix space that we introduced in writing $\mathcal{H}$ in block form, eqn. (4.2). Now, we want to only fill negative energy eigenstates, so it is convenient to introduce the sign function

$$\varepsilon(\lambda) \equiv \frac{\lambda}{\sqrt{\lambda^\dagger \lambda}} . \tag{4.4}$$

With $m_2 \to \infty$ we have

$$\varepsilon(\mathcal{H}(m_2)) \xrightarrow[m_2 \to \infty]{} \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.5}$$

Assuming $\mathcal{H}(-m_1)$ has no exact zeromodes then, it follows that all eigenvalues come in $\pm$ pairs (just like the operator γ_5) and we can choose our basis $|n, -m_1\rangle$ so that

$$\varepsilon(\mathcal{H}(m_2)) = U\gamma_5 U^\dagger = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^\dagger . \tag{4.6}$$

Therefore the Slater determinant we want is

$$\begin{aligned} Z &= \left|\langle \Omega, m_2 | \Omega, -m_1\rangle\right|^2 \\ &= \left|\det U_{22}\right|^2 \\ &= \det \delta^\dagger \det \delta \\ &= \det\left(\frac{1 + \gamma_5 \varepsilon(\mathcal{H}(-m_1))}{2}\right) . \end{aligned} \tag{4.7}$$

Some steps have been omitted from this derivation (Narayanan, 2001); see exercise 4.2.

Exercise 4.1 Prove the assertion that if $\mathcal{H}(-m_1)$ has no zeromodes, it has equal numbers of positive and negative eigenvalues.

Exercise 4.2 You should prove the last step in eqn. (4.7), breaking it down to the following steps:

(a) Show that $\det \delta^\dagger = \det \alpha \det U^\dagger$;

(b) ...so that $\det \delta^\dagger \det \delta = \det \delta \det \alpha \det U^\dagger = \det \left[\frac{1}{2}(U + \gamma_5 U \gamma_5) U^\dagger\right]$;

(c) ...which combines with eqn. (4.6) to yield eqn. (4.7).

On the other hand, $Z \propto \det D$, where D is the fermion operator. So we arrive at the overlap operator (dropping the subscript from m_1):

$$\begin{aligned} D &= 1 + \gamma_5 \varepsilon(\mathcal{H}(-m)) \\ &= 1 + \gamma_5 \frac{\mathcal{H}(-m)}{\sqrt{\mathcal{H}(-m)^2}} \\ &= 1 + \frac{D_w - m}{\sqrt{(D_w - m)^\dagger (D_w - m)}} \ . \end{aligned} \tag{4.8}$$

a remarkable result. It was subsequently shown explicitly that this fermion operator can be derived directly from lattice domain wall fermions at infinite wall separation (Neuberger, 1998*c*). In this paper it was shown that the overlap operator for the surface modes could be written as

$$D = 2 \lim_{L\to\infty} \frac{\mathcal{D}_f}{\mathcal{D}_{PV}} \ , \tag{4.9}$$

where $\mathcal{D}_f$ is a contribution from the fermions in the domain wall theory, and $\mathcal{D}_{PV}$ is a lattice Pauli-Villars contribution put in to cancel the massive bulk modes that aren't interesting for low energy physics. These $\mathcal{D}_f$ and $\mathcal{D}_{PV}$ operators can be expressed in terms of transfer matrices as

$$\mathcal{D}_f = \frac{1 + \gamma_5 \tanh \frac{L}{2}\mathcal{H}(-m)}{1 - \gamma_5 \tanh \frac{L}{2}\mathcal{H}(-m)} \ , \qquad \mathcal{D}_{PV} = \mathcal{D}_f + 1 \ . \tag{4.10}$$

From these expressions and the fact that $\lim_{L\to\infty} \tanh L\lambda/2 = \mathrm{sign}(\lambda)$ one can derive the result in eqn. (4.8). Recall from our discussion of domain wall fermions that at least for weak gauge fields, we need $0 < m < 2r$ in order to obtain one flavor of massless Dirac fermion (where I have set the lattice spacing $a = 1$).

Exercise 4.3 Compute the spectrum for a fermion with mass Λ on a cylinder of circumference $2L$, finding solutions analogous to those in eqn. (3.16). If we treat this fermion as a Pauli-Villars field, show that in the absence of gauge fields, they will cancel the effects of the entire massive tower of states we found there, leaving behind only the chiral surface zeromodes and a single massive Pauli-Villars mode with $\tilde{\mu}_0 = \Lambda$ to regulate them.

4.1.1 Eigenvalues of the overlap operator

Recall that the eigenvalues of the Dirac operator in the continuum are $\pm i\lambda_n$ for real nonzero λ_n, plus n_+ RH and n_- LH zero modes, where the difference is constrained by the index theorem to equal the topological winding number of the gauge field. Thus the spectrum looks like a line on the imaginary axis. What does the spectrum of the overlap operator look like? Consider

$$(D-1)^\dagger(D-1) = \epsilon(\mathcal{H})^2 = 1\ . \tag{4.11}$$

Thus $(D-1)$ is a unitary matrix and the eigenvalues of D are constrained to lie on a circle of unit radius in the complex plane, with the center of the circle at $z=1$. If you put the lattice spacing back into the problem, $D \to aD$ in the above expression to get the dimensions right, and so the eigenvalues sit on a circle of radius $1/a$ centered at $1/a$. Thus, as $a \to 0$ the circle gets bigger, and the eigenvalues with small magnitude almost lie on the imaginary axis, like the continuum eigenvalues. See the problem below, where you are to show that the eigenfunctions of D with real eigenvalue are chiral.

4.1.2 Locality of the overlap operator

If just presented with the overlap operator eqn. (4.7) without knowing how it was derived, one might worry that its unusual structure could entail momentum space singularities corresponding to unacceptable nonlocal behavior in coordinate space. (From its derivation from domain wall fermions this would be very surprising for sufficiently weakly coupled gauge fields, since the domain wall theory looks well defined and local with a mass gap.) The locality of the overlap operator (i.e. that it falls off exponentially in coordinate space) was proven analytically in (Hernandez, Jansen and Luscher, 1999), under the assumption of sufficiently smooth gauge link variables, namely that $|1-U| < 1/30$. They also claimed numerical evidence for locality that was less restrictive.

4.1.3 Simulating the overlap operator

The overlap operator has exact chiral symmetry, in the sense that it is an exact solution to the Ginsparg Wilson relation, which cannot be said for domain wall fermions at finite N_s; furthermore, it is a four-dimensional operator, which would seem to be easier to simulate than a 5d theory. However, the inverse square root of an operator is expensive to compute, and requires some approximations. The algorithms for computing it are described in detail in an excellent review by A. Kennedy (Kennedy, 2006). Amusingly, he explains that the method for computing the overlap operator can be viewed as simulating a five-dimensional theory, albeit one with more general structure than the domain wall theory. For a review comparing the computational costs of different lattice fermions, see (Jansen, 2008).

Exercise 4.4 Show that the overlap operator in eqn. (4.8) has the following properties:

(a) At zero gauge field and acting on long wavelength fermion modes, $D \simeq \not{\partial}_4$, the ordinary Dirac operator for a massless fermion.

(b) It satisfies the Ginsparg-Wilson equation, eqn. (3.25):

$$\{\gamma_5, D\} = D\gamma_5 D\ . \tag{4.12}$$

Exercise 4.5

(a) Show that one can write $D = 1+V$ where $V^\dagger V = 1$, and that therefore D can be diagonalized by a unitary transformation, with its eigenvalues lying on the circle $z = 1 + e^{i\phi}$.

(b) Show that, despite D being non-hermitian, normalized eigenstates satisfying $D|z\rangle = z|z\rangle$ with different eigenvalues are orthogonal, satisfying $\langle z'|z\rangle = \delta_{z'z}$

(c) Show that if $D|z\rangle = z|z\rangle$ then $D^\dagger|z\rangle = z^*|z\rangle$

(d) Using the property of "γ_5-hermiticity", namely that $\gamma_5 D\gamma_5 = D^\dagger$, show that $\langle z|\gamma_5|z\rangle = 0$ unless $z = 0$ or $z = 2$, in which case $\langle z|\gamma_5|z\rangle = \pm 1$.

(e) Compare the above result with what happens with eigenstates of the Euclidian Dirac operator $\not{D}$ in the continuum: given states $|n\rangle$ which are eigenstates of $\not{D}$, $\not{D}|n\rangle = i\lambda_n|n\rangle$, compute $\langle n|\gamma_5|n\rangle$. Which of the $|z\rangle$ eigenstates of the overlap correspond to the $|n\rangle$ states in the continuum limit?

4.2 The Ginsparg-Wilson equation and its consequences

In 1982 Paul Ginsparg and Kenneth Wilson wrote a paper about chiral lattice fermions which was immediately almost completely forgotten, accruing 10 citations in the first ten years and none in the subsequent five; today it is marching toward 700 citations. The reason for this peculiar history is that they wrote down an equation they speculated should be obeyed by a fermion operator in the fixed point action of a theory tuned to the chiral point — but they did not solve it. After domain wall and overlap fermions were discovered in the early 1990s, it was realized that they provided a solution to this equation (the domain wall solution only being exact in the limit of infinite extra dimension). Shortly afterward, M. Lüscher elaborated on how the salient features of chirality flowed from the Ginsparg-Wilson equation — in particular, how anomalies and multiplicative mass renormalization were consequences of the equation, which provided a completely explicit four-dimensional explanation for the success of the overlap and domain wall fermions.

4.2.1 Motivation

A free Wilson fermion with its mass tuned to the critical value describes a chiral fermion in the continuum. As we have seen, chiral symmetry does not exist on the lattice, but its violation is not evident at low energy, except through correctly reproducing the anomaly. However, imagine studying this low energy effective theory by repeatedly performing block spin averages. One would eventually have a lattice theory with all the properties one would desire: chiral fermions and chiral anomalies. What is the fermion operator in this low energy theory, and how does it realize chiral symmetry? Motivated by this question, Ginsparg and Wilson performed a somewhat simpler calculation: they took a continuum theory with chiral symmetry and anomalies, and performed a average of spacetime cells to create a lattice theory, and asked how the chiral symmetry in the original theory was expressed in the resulting lattice theory.

The starting point is the continuum theory

$$Z = \int [d\psi]\, [d\overline{\psi}]\, e^{-S(\psi,\overline{\psi})} \tag{4.13}$$

I assume there are N_f identical flavors of fermions, and that S is invariant under the full $U(N_f) \times U(N_f)$ chiral symmetry. We define $\psi_{\mathbf{n}}$ to be localized averages of ψ,

$$\psi_{\mathbf{n}} = \int d^4x\, \psi(x) f(\mathbf{x} - a\mathbf{n}) \tag{4.14}$$

where $f(\mathbf{x})$ is some function with support in the region of $|\mathbf{x}| \lesssim a$. Then up to an irrelevant normalization, we can rewrite

$$\begin{aligned} Z &= \int [d\psi]\, [d\overline{\psi}] \int \prod_{\mathbf{n}} d\chi_{\mathbf{n}}\, d\overline{\chi}_{\mathbf{n}}\, e^{-\left[\sum_{\mathbf{n}} \alpha(\overline{\chi}_{\mathbf{n}} - \overline{\psi}_{\mathbf{n}})(\chi_{\mathbf{n}} - \psi_{\mathbf{n}}) - S(\psi,\overline{\psi})\right]} \\ &\equiv \int \prod_{\mathbf{n}} d\chi_{\mathbf{n}}\, d\overline{\chi}_{\mathbf{n}}\, e^{-S_{\mathrm{lat}}(\overline{\chi}_{\mathbf{n}},\chi_{\mathbf{n}})} \equiv e^{-\overline{\chi} D \chi} \ , \end{aligned} \tag{4.15}$$

where α is a dimensionful parameter, where D is the resulting lattice fermion operator. Since there are N_f copies of all the fields, the operator D is invariant under the vector $U(N_f)$ symmetry, so that if T is a $U(N_f)$ generator, $[T, D] = 0$. The lattice action is therefore defined as

$$e^{-\overline{\chi} D \chi} = \int [d\psi]\, [d\overline{\psi}]\, e^{-\left[\sum_{\mathbf{n}} \alpha\, (\overline{\chi}_{\mathbf{n}} - \overline{\psi}_{\mathbf{n}})(\chi_{\mathbf{n}} - \psi_{\mathbf{n}}) - S(\psi,\overline{\psi})\right]} \ , \tag{4.16}$$

Note that explicit chiral symmetry breaking has crept into our definition of S_{lat} through the fermion bilinear we have introduced in the Gaussian in order to change variables.

Now consider a chiral transformation on the lattice variables, $\chi_{\mathbf{n}} \to e^{i\epsilon\gamma_5 T}\chi_{\mathbf{n}}$, $\overline{\chi}_{\mathbf{n}} \to \overline{\chi}_{\mathbf{n}} e^{i\epsilon\gamma_5 T}$, where T is a generator for a $U(N_f)$ flavor transformation. This is accompanied by a corresponding change of integration variables ψ, $\overline{\psi}$:

$$e^{-\overline{\chi} e^{i\epsilon\gamma_5 T} D e^{i\epsilon\gamma_5 T} \chi} = \int [d\psi]\, [d\overline{\psi}]\, e^{i \int \epsilon \mathcal{A}\ \mathrm{Tr}\ T}\, e^{-\left[\sum_{\mathbf{n}} \alpha\, (\overline{\chi}_{\mathbf{n}} - \overline{\psi}_{\mathbf{n}}) e^{2i\epsilon\gamma_5 T} (\chi_{\mathbf{n}} - \psi_{\mathbf{n}}) - S(\psi,\overline{\psi})\right]}. \tag{4.17}$$

where $\mathcal{A}$ is the anomaly due to the non-invariance of the measure $[d\psi]\, [d\overline{\psi}]$ as computed by Fujikawa (Fujikawa, 1979):

$$\mathcal{A} = \frac{1}{8\pi^2} \epsilon_{\alpha\beta\gamma\delta}\, \mathrm{Tr}\ F_{\alpha\beta} F_{\gamma\delta} \tag{4.18}$$

with

$$\int \mathcal{A} = 2\nu \ , \tag{4.19}$$

ν being the topological charge of the gauge field.

Expanding to linear order in ϵ gives

$$
\begin{aligned}
-\overline{\chi}\{\gamma_5, D\}T\chi\, e^{-\overline{\chi}D\chi} &= \int [d\psi]\,[d\overline{\psi}]\,\left(2\nu\ \mathrm{Tr}\ T + \sum_{\mathbf{n}}\left[(\overline{\chi}_{\mathbf{n}} - \overline{\psi}_{\mathbf{n}})2\alpha\gamma_5 T(\chi_{\mathbf{n}} - \psi_{\mathbf{n}})\right]\right)\\
&\qquad \times \exp\left[-\alpha\sum_{\mathbf{m}}(\overline{\chi}_{\mathbf{m}} - \overline{\psi}_{\mathbf{m}})(\chi_{\mathbf{m}} - \psi_{\mathbf{m}}) - S(\psi,\overline{\psi})\right]\\
&= \sum_{\mathbf{n}}\left(2\nu\ \mathrm{Tr}\ T - \frac{2}{\alpha}\frac{\delta}{\delta\chi_{\mathbf{n}}}\gamma_5 T\frac{\delta}{\delta\overline{\chi}_{\mathbf{n}}}\right)e^{-\overline{\chi}D\chi}\\
&= \left(\mathrm{Tr}\ \gamma_5 DT + 2\nu\ \mathrm{Tr}\ T - \frac{2}{\alpha}\overline{\chi}_{\mathbf{n}}D\gamma_5 DT\chi_{\mathbf{n}}\right)e^{-\overline{\chi}D\chi}
\end{aligned}
\tag{4.20}
$$

Defining $\alpha \equiv 2/a$ this yields the operator identity

$$
(\{\gamma_5, D\} - a\, D\gamma_5 D)\,T = (\mathrm{Tr}\ \gamma_5 DT + 2\nu\ \mathrm{Tr}\ T)\ . \tag{4.21}
$$

If T is taken to be a traceless generator of $U(N_f)$, multiplying both sides by T and taking the trace yields the Ginsparg-Wilson equation:

$$
\{\gamma_5, D\} = a\, D\gamma_5 D\ . \tag{4.22}
$$

If on the other hand we take T to be the unit matrix and use eqn. (4.22) we find

$$
-\,\mathrm{Tr}\ \gamma_5 D = 2N_f\nu\ , \tag{4.23}
$$

where I snuck in a factor of N_f, for N_f flavors of fermions. This latter equation was not derived in the original Ginsparg-Wilson paper; from our discussion of the index theorem eqn. (2.69), it follows from eqn. (4.23) that we should have $\mathrm{Tr}\ \gamma_5 D = -2(n_+ - n_-)$, where $n_\pm$ are the number of $\pm$ chirality zeromodes. We will see that that is indeed the case.

Note that the GW relation eqn. (4.22) is the same equation satisfied by the overlap operator (Neuberger, 1998*b*) — and therefore by the domain wall propagator at infinite wall separation on the lattice, being equivalent as shown in (Neuberger, 1998*a*; Neuberger, 1998*c*) — as well as by the infinitely separated domain wall propagator in the continuum (Luscher, 2000*a*). In fact, the general overlap operator derived by Neuberger

$$
D = 1 + \gamma_5\epsilon(\mathcal{H}) \tag{4.24}
$$

is the only explicit solution to the GW equations that is known.

4.2.2 Exact lattice chiral symmetry

Missing from the discussion so far is how the overlap operator is able to ensure multiplicative renormalization of fermion masses (and similarly, multiplicative renormalization of pion masses). In the continuum, both phenomena follow from the fact that fermion masses are the only operators breaking an otherwise good symmetry. The GW relation states exactly how chiral symmetry is broken on the lattice, but does not specify a symmetry that *is* exact on the lattice and capable of protecting fermion masses from additive renormalization.

Lüscher was able to solve this problem by discovering the GW relation implied the existence of an exact symmetry of the lattice action: $\int \overline{\psi} D \psi$ is invariant under the transformation

$$\delta\psi = \gamma_5 \left(1 + \frac{a}{2} D\right) \psi \ , \qquad \delta\overline{\psi} = \overline{\psi} \left(1 - \frac{a}{2} D\right) \gamma_5 \ . \tag{4.25}$$

Note that this becomes ordinary chiral symmetry in the $a \to 0$ limit, and that it is broken explicitly by a mass term for the fermions.

4.2.3 Anomaly

If this symmetry were an exact symmetry of the path integral, we would run afoul of all the arguments we have made so far: it becomes the anomalous $U(1)_A$ symmetry in the continuum, so it cannot be an exact symmetry on the lattice! The answer is that this lattice chiral transformation is not a symmetry of the measure of the lattice path integral:

$$\begin{aligned} \delta[d\psi][d\overline{\psi}] &= [d\psi][d\overline{\psi}] \left(\mathrm{Tr} \left[\gamma_5 \left(1 + \frac{a}{2} D \right) \right] + \mathrm{Tr} \left[\left(1 - \frac{a}{2} D \right) \gamma_5 \right] \right) \\ &= [d\psi][d\overline{\psi}] \times a \, \mathrm{Tr} \ \gamma_5 D \ , \end{aligned} \tag{4.26}$$

where I used the relation $d \det M/dx = \det[M] \, \mathrm{Tr} \ M^{-1} dM/dx$. Unlike the tricky non-invariance of the fermion measure in the continuum under a $U(1)_A$ transformation – which only appears when the measure is properly regulated — here we have a perfectly ordinary integration measure and a transformation that gives rise to a Jacobean with a nontrivial phase (unless, of course, $\mathrm{Tr} \ \gamma_5 D = 0$). To make sense, $\mathrm{Tr} \ \gamma_5 D$ must map into the continuum anomaly...and we have already seen that it does, from eqn. (4.23).

What remains is to prove the index theorem (Hasenfratz, Laliena and Niedermayer, 1998; Luscher, 1998), the lattice equivalent of eqn. (2.68). From exercise 4.5 it follows that for states $|z\rangle$ satisfying $D|z\rangle = z|z\rangle$

$$\mathrm{Tr} \ \gamma_5 D = \sum_z \langle z|\gamma_5 D|z\rangle = 2(n_+^{(2)} - n_-^{(2)}) \ , \tag{4.27}$$

where $n_\pm^{(2)}$ are the number of positive and negative chirality states with eigenvalue $z = 2$. We also know that

$$0 = \mathrm{Tr} \ \gamma_5 = \sum_z \langle z|\gamma_5|z\rangle = (n_+ - n_-) + (n_+^{(2)} - n_-^{(2)}) \ , \tag{4.28}$$

where $n_\pm$ are the number of $\pm$ chirality zeromodes at $z = 0$. Therefore we can write

$$\mathrm{Tr} \ \gamma_5 D = -2N_f(n_+ - n_-) \ , \tag{4.29}$$

where again I snuck in a factor of N_f for the case of N_f flavors (copies) of fermions. Substituting into eqn. (4.23) we arrive at the lattice index theorem,

$$(n_+ - n_-) = \nu N_f \tag{4.30}$$

which is equivalent to the continuum result eqn. (2.69), and provides an interesting definition for the topological charge of a lattice gauge field. Note that $(n_+ - n_-)$ is always an integer, even at finite lattice spacing. However ν entered the discussion in eqn. (4.19) assuming continuous gauge fields; on a lattice, there will not in general be a definition for integer winding number of the gauge field, so in fact the index of the overlap operator gives a nice lattice definition of gauge topology, that yields the desired result in the continuum limit. In other words, gauge topology on the lattice can be defined by the spectrum of the overlap operator.

This is a desirable feature of the overlap operator is the existence of exact zeromode solutions in the presence of topology; it is also a curse for realistic simulations, since the zeromodes make it difficult to sample different global gauge topologies. And while it cannot matter what the global topology of the Universe is, fixing the topology in a lattice QCD simulation gives rise to spurious effects which only vanish with a power of the volume (Edwards, 2002).

4.2.4 What we have found

- By taking the $L \to \infty$ limit Neuberger and Narayanan were able to derive the remarkable overlap operator, which is a 4-dimensional (or $d = 2$) operator at finite lattice spacing that describes the physics of the chiral surface modes in that limit.
- Neuberger showed that the overlap operator was the solution to the Ginsparg-Wilson equation, derived years earlier to describe optimal representation of chiral symmetry on the lattice
- Using the Ginsparg-Wilson equation, Lüscher showed that the lattice action possessed an exact symmetry at finite lattice spacing that becomes chiral symmetry in the continuum limit, but forbids fermion masses even at finite lattice spacing. However, the fermion measure is not invariant, and the lack of invariance reproduces the anomaly. This looks a lot like what happens in the continuum, but does not depend on there being an infinite number of states! With this, we finally understand how ideal chiral symmetry is realized on the lattice without violating the Nielsen-Ninomiya theorem.

5
Practical applications of chiral fermions

As discussed above, practical implementation of the overlap operator reduces it to a five-dimensional action with only approximate chiral symmetry, similar to the domain wall formulation at finite L but with a more generalized action. Simulations with domain wall or overlap fermions typically cost 10-100 times more computer time for the same level of accuracy, as Wilson or staggered fermions, and so it is not usually practical to employ them unless looking at physics that is particularly sensitive to chiral symmetry.

An examples of where having good chiral symmetry iinclude simulating $N = 1$ supersymmetric gauge theory, and determining the $K \to \pi\pi$ amplitude.

5.1 Supersymmetric Yang Mills theory

In contrast to QCD, $N = 1$ super Yang Mills theory has a single Weyl fermion (the gaugino) transforming as an adjoint under the gauge group; the theory has a $U(1)_A$ symmetry at the classical level — phase rotations of the gaugino — but it is broken by anomalies to a discrete symmetry. This discrete symmetry is then spontaneously broken by a gluino condensate, but without any continuous symmetries, no Goldstone bosons are produced. What should the spectrum of this theory look like? Presumably a bunch of massive boson and fermion glueball-like states. They will form degenerate supersymmetric multiplets when the gluino mass is tuned to zero. In QCD one can tune the bare Wilson fermion mass to the chiral symmetric point by finding what value sets the pion masses to zero, since they are Nambu-Goldsotne bosons of spontaneously broken chiral symmetry. In the supersymmetric Yang-Mills theory, however, there are no continuous symmetries that appear when the gluino mass is set to zero and the theory becomes supersymmetric – just a discrete chiral symmetry that survives the gauge anomalies, and there is no particle that becomes massless in the symmetry limit in this case, making tuning the bare mass is difficult. After tuning away the $O(1/a)$ mass correction, there remain for Wilson fermions the dimension-5 chiral symmetry violating operators in the Symanzik action which require $O(a)$ tuning. In contrast, chiral fermions receive finite lattice corrections only at $O(a^2)$, simply because one cannot write down a dimension-5 chiral symmetry preserving operator in QCD. Preliminary studies of the theory using domain wall fermions have been carried out, but not to date with sufficiently small m_{res} to be called a success (Endres, 2009). The domain wall theory had to modified in this case, to produce Majorana edge states suitable for describing the gluino (Kaplan and Schmaltz, 2000).

5.2 Operator mixing

One also encounters un wanted $1/a$ fine tuning that can be avoided with chiral lattice fermions when computing weak processes. One of the most curious feature of the strong interactions is the $\Delta I = 1/2$ rule, which is the observation that $\Delta s = 1$ transitions in nature are greatly enhanced when they change isospin by $\Delta I = 1/2$, in comparison to $\Delta I = 3/2$. For example, one requires for the amplitudes for kaon decay $K \to \pi\pi$:

$$\frac{\mathcal{A}(\Delta I = 1/2)}{\mathcal{A}(\Delta I = 3/2)} \simeq 20 \ . \tag{5.1}$$

To compute this in the standard model, one starts with four-quark operators generated by W-exchange, which can be written as the linear combination of two operators

$$\begin{aligned} \mathcal{L}_{\Delta S=1} &= -V_{ud}V_{us}^* \frac{G_F}{\sqrt{2}} \left[C_+(\mu, M_w)\mathcal{O}^+ + C_-(\mu, M_w)\mathcal{O}^-\right] \ , \\ \mathcal{O}^\pm &= [(\bar{s}d)_L(\bar{u}u)_L \pm (\bar{s}u)_L(\bar{u}d)_L] - [u \leftrightarrow c] \ , \end{aligned} \tag{5.2}$$

where $(\bar{q}q')_L \equiv (\bar{q}\gamma^\mu P_L q')$. If one ignores the charm quark contribution, the $\mathcal{O}^-$ transforms as an 8 under $SU(3)_f$, while $\mathcal{O}^+$ transforms as a 27; therefore $\mathcal{O}^-$ is pure $I = 1/2$, while $\mathcal{O}^+$ is a mix of $I = 3/2$ and $I = 1/2$. The full $\mathcal{O}^\pm$ operators are in $SU(4)_f$ multiplets; while $SU(4)_f$ is not a good symmetry of the spectrum (especially since the charm quark mass is heavier than the scale of spontaneous chiral symmetry breaking), it is only broken by quark masses which do not effect the log divergences of the theory. Thus the running of the operators respect $SU(4)_f$ down to $\mu = m_c$, and there is no mixing between $\mathcal{O}^\pm$. At the weak scale $\mu = M_W$, one finds $|C^+/C^-| = 1 + O(\alpha_s(M_W)$, showing that there is no $\Delta I = 1/2$ enhancement intrinsic to the weak interactions. One then scales these operators down to $\mu \sim 2$ GeV in order to match onto the lattice theory; using the renormalization group to sum up leading $\alpha_s \ln \mu/M_W$ corrections gives an enhancement $|C^+/C^-| \simeq 2$ — which is in the right direction, but not enough to explain eqn. (5.1), which should then either be coming from QCD at long distances, or else new physics! This is a great problem for the lattice to resolve.

A wonderful feature about using dimensional regularization and $\overline{MS}$ in the continuum is that an operator will never mix with another operator of lower dimension. This is because there is no UV mass scale in the scheme which can make up for the mismatch in operator dimension. This is not true on the lattice, where powers of the inverse lattice spacing $1/a$ can appear. In particular, the the dimension-6 four fermion operators $\mathcal{O}^\pm$ could in principle mix with dimension-3 two fermion operators. The only $\Delta S = 1$ dimension-3 operator that could arise is $\bar{s}\gamma_5 d$, which is also $\Delta I = \frac{1}{2}$ [1]. If the quarks were massless, the lattice theory would possess an exact discrete "SCP" symmetry under which on interchanges $s \leftrightarrow d$ and performs a CP transformation to change LH quarks into LH anti-quarks; the operators $\mathcal{O}_\pm$ are even under SCP while $\bar{s}\gamma_5 d$ is odd, to the operator that could mix on the lattice is

$$\mathcal{O}_p = (m_s - m_d)\,\bar{s}\gamma_5 d \ . \tag{5.3}$$

[1] The operator $\bar{s}d$ is removed by rediagonalizing the quark mass matrix and does not give rise to $K\pi\pi$.

In a theory where the quark masses are the only source of chiral symmetry breaking, then $\mathcal{O}_p = \partial_\mu A_\mu^{\bar{s}d}$, the divergence of the $\Delta S = 1$ axial current. Therefore on-shell matrix elements of this operator vanish, since the derivative gives $(p_K - p_{2\pi}) = 0$, i.e. no momentum is being injected by the weak interaction. We can ignore $\mathcal{O}_p$ then when the $K \to \pi\pi$ amplitude is measured with chiral lattice fermions with on-shell momenta.

For a lattice theory without chiral symmetry, $\mathcal{O}_p = \partial_\mu A_\mu^{\bar{s}d} + O(a)$ and so has a nonvanishing $O(a)$ matrix element. In this case operators $\mathcal{O}_\pm$ from eqn. (5.2) in the continuum match onto the lattice operators

$$\mathcal{O}^\pm(\mu) = Z^\pm(\mu a, g_0^2)\left[\mathcal{O}^\pm(a) + \frac{C_p^\pm}{a^2}\mathcal{O}_p\right] + \mathcal{O}(a) \ . \tag{5.4}$$

In general then one would need to determine the coefficient $C_p^\pm$ to $O(a)$ in order to determine the $\Delta I = \frac{1}{2}$ amplitude for $K \to \pi\pi$ to leading order in an a-expansion, which is not really feasible. Other weak matrix elements such as B_K and ϵ'/ϵ similarly benefit from the use of lattice fermions with good chiral symmetry.

6

Chiral gauge theories: the challenge

6.0.1 Why chiral gauge theories are interesting

Practical applications of chiral fermions have centered on producing Dirac fermions whose symmetry is enhanced as the mass goes to zero, as desired for studies of quarks in QCD, or similar theories. Chiral gauge theories remain an interesting challenge, however. A chiral gauge theory in Minkowski $d = 4$ can be defined as a theory where the fermions, represented as left-handed Weyl fermions, do not form a real representation of the gauge group. An immediate consequence of this is that it is impossble to give gauge-invariant masses to all of the fermions without spontaneously breaking the gauge symmetry. The quarks in QCD can be represented as pairs of LH Weyl fermions transforming as $3\oplus\bar{3}$ under $SU(3)_c$, which is a real representation, and so QCD is not an example of a chiral gauge theory. When $SU(2)\times U(1)$ gauge interactions are included, however, the Standard Model is an example of a chiral gauge theory. Obviously, that should be motivation enough to search for a lattice regularization for chiral gauge theories, but it is not very urgent, since the weak interactions are never very strong, and nonperturbative effects in the theory are pretty small, so there is little demand for numerical simulation. Theorists can cook up other chiral gauge theories that would be interesting to simulate, however. Chiral $U(1)$ gauge theories in $d = 2$, like the $3-4-5$ model discussed earlier, are particularly interesting when supersymmetric: in that case there are scalar fields with a potential whose flat directions describe Calabi-Yau manifolds, which would be of interest to explore numerically.

There are also strongly coupled $d = 4$ chiral gauge theories that can be argued to contain massless composite fermions in the spectrum, which have never been seen before. Perhaps the quarks and leptons are composite particles such as these? A nice toy example of a strongly coupled chiral gauge theory is $SU(5)$ with LH fermions

$$\psi = \bar{5}\ , \qquad \chi = 10\ . \tag{6.1}$$

It so happens that the ψ and the χ contribute with opposite signs to the $(\mathrm{SU}(5))^3$ anomaly $\mathcal{A}$ in eqn. (2.72), so this seems to be a well defined gauge theory. Furthermore, the $SU(5)$ gauge interactions are asymptotically free, meaning that interactions becomes strong at long distances. One might therefore expect the theory to confine as QCD does. However, unlike QCD, there are no gauge invariant fermion bilinear condensates which could form, and which in QCD are responsible for baryon masses. That being the case, might there be any massless composite fermions in the spectrum of this theory? 't Hooft came up with a nice general argument involving global anomalies that suggests there will be.

In principle there are two global $U(1)$ chiral symmetries in this theory corresponding to independent phase rotations for ψ and χ; however both of these rotations have global$\times$SU(5)^2 anomalies, similar to the global$\times$SU(3)^2 of the $U(1)_A$ current in QCD. This anomaly can only break one linear combination of the two $U(1)$ symmetries, and one can choose the orthogonal linear combination which is anomaly-free. With a little group theory you can show that the anomaly-free global $U(1)$ symmetry corresponds to assigning charges

$$\psi = \bar{5}_3 \ , \qquad \chi = 10_{-1} \ , \tag{6.2}$$

where the subscript gives the global $U(1)$ charge. This theory has a nontrivial global $U(1)^3$ anomaly, $\mathcal{A} = 5 \times (3)^3 + 10 \times (-1)^3 = 125$. 't Hooft's argument is that this(global)^3 anomaly restricts — and helps predict — the low energy spectrum of the theory. Applied to the present model, his argument goes as follows: imagine weakly gauging this $U(1)$ symmetry. This would be bad news as the theory stands, since a (gauge)^3 anomaly leads to a sick theory, but one can add a LH "spectator fermion" $\omega = 1_{-5}$ which is a singlet under $SU(5)$ but has charge -5 under this $U(1)$ symmetry, canceling the $U(1)^3$ anomaly. This weak $U(1)$ gauge interaction plus the SU(5)-singlet ω fermion should not interfere with the strong $SU(5)$ dynamics. If that dynamics leads to confinement and no $U(1)$ symmetry breaking, then the weak $U(1)$ gauge theory must remain anomaly free at low energy, implying that there has to be one or more massless composite fermions to cancel the $U(1)^3$ anomaly of the ω. A good candidate massless composite LH fermion is $(\psi\psi\chi)$ which is an $SU(5)$-singlet (as required by confinement), and which has $U(1)$ charge of $(3 + 3 - 1) = 5$, exactly canceling the $U(1)^3$ anomaly of the ω. Now forget the thought experiment: do not gauge the $U(1)$ and do not include the ω spectator fermion. It should still be true that this $SU(5)$ gauge theory produces a single massless composite fermion $(\psi\psi\chi)$ [1].

While it is hard to pin down the spectrum of general strongly coupled chiral gauge theories using 't Hooft's anomaly matching condition alone, a lot is known about strongly coupled supersymmetric chiral gauge theories, and they typically have a very interesting spectrum of massless composite fermions, which can be given small masses and approximate the quarks and leptons we see by tweaking the theory. See for example (Kaplan, Lepeintre and Schmaltz, 1997), which constructs a theory with three families of massless composite fermions, each with a different number of constituents.

6.1 Why chiral gauge theories are complicated

Lattice computations employ Monte Carlo integration, which requires a positive integrand that can be interpreted as a probability distribution. While not sufficient for a lattice action to yield a positive measure, it is certainly necessary for the continuum theory one is approximating to have this property. Luckily, the fermion determinant

[1] You may wonder about whether fermion condensates form which break the global $U(1)$ symmetry. Perhaps, but it seems unlikely. The lowest dimension gauge invariant fermion condensates involve four fermion fields — such as $\langle\chi\chi\chi\psi\rangle$ or $\langle(\chi\psi)(\chi\psi)^\dagger\rangle$ — which are all neutral under the $U(1)$ symmetry. Furthermore, there are arguments that a Higgs phase would not be distinguishable from a confining phase for this theory.

for vector-like gauge theories (such as QCD), $\det(\not{D}+m)$, has this property in Euclidian space. Since $\not{D}^\dagger = -\not{D}$ and $\{\Gamma, \not{D}\} = 0$, it follows that there exist eigenstates ψ_n of $\not{D}$ such that

$$\not{D}\psi_n = i\lambda_n\psi_n \ , \qquad \not{D}\Gamma\psi_n = -i\lambda_n\Gamma\psi_n \ , \qquad \lambda_n \text{ real.} \tag{6.3}$$

For nonzero λ, ψ_m and $\Gamma\psi_n$ are all mutually orthogonal and we see that the eigenvalue spectrum contains $\pm i\lambda_n$ pairs. On the other hand, if $\lambda_n = 0$ then ψ_n can be an eigenstate of Γ as well, and $\Gamma\psi_n$ is not an independent mode. Therefore

$$\det(\not{D}+m) = \prod_{\lambda_n>0}(\lambda_n^2+m^2) \times \prod_{\lambda_n=0} m \tag{6.4}$$

which is real and for positive m is positive for all gauge fields.

What about a chiral gauge theory? The fermion Lagrangian for a LH Weyl fermion in Euclidian space looks like $\overline{\psi}D_L\psi$ with $D_L = D_\mu\sigma_\mu$ and (in the chiral basis eqn. (2.12), continued to Euclidian space) $\sigma_\mu = \{1, i\vec{\sigma}\}$. Note that D_L has no nice hermiticity properties, which means its determinant will be complex, its right eigenvectors and left eigenvectors will be different, and its eigenvectors will not be mutually orthogonal. Furthermore, D_L is an operator which maps vectors from the space $\mathcal{L}$ of LH Weyl fermions to the space $\mathcal{R}$ of RH Weyl fermions. In Euclidian space, these spaces are unrelated and transform independently under the $SU(2)\times SU(2)$ Lorentz transformations. Suppose we have an orthonormal basis $|n,\mathcal{R}\rangle$ for the RH Hilbert space and $|n,\mathcal{L}\rangle$ for the LH Hilbert space; we can expand our fermion integration variables as

$$\psi = \sum_n c_n|n,\mathcal{L}\rangle \ , \qquad \overline{\psi} = \sum_n \bar{c}_n\langle n,\mathcal{R}| \tag{6.5}$$

so that

$$\int [d\psi][d\overline{\psi}]\, e^{-\int \overline{\psi}D_L\psi} = \det_{mn}\langle m,\mathcal{R}|D_L|n,\mathcal{L}\rangle \ . \tag{6.6}$$

However, the answer we get will depend on the basis we choose. For example, we could have chosen a different orthonormal basis for the $\mathcal{L}$ space $|n',\mathcal{L}\rangle = \mathcal{U}_{n'n}|n,\mathcal{L}\rangle$ which differed from the first by a unitary transformation $\mathcal{U}$; the resultant determinant would differ by a factor $\det\mathcal{U}$, which is a phase. If this phase were a number, it would not be an issue — but it can in general be a functional of the background gauge field, so that different choices of phase for $\det D_L$ lead to completely different theories.

We do know that in a chiral basis where Γ is diagonal,

$$\not{D} = \begin{pmatrix} 0 & \not{D}_L \\ \not{D}_R & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \det\not{D} = \det\not{D}_R \det\not{D}_L = \left|\not{D}_L\right|^2 \ . \tag{6.7}$$

Therefore the the norm of $|\det D_L|$ can be defined as $\sqrt{\det\not{D}}$ and

$$\det D_L = \sqrt{\det\not{D}}\,e^{iW[A]} \tag{6.8}$$

where $W[A]$ is a real functional of the gauge fields. What do we know about $W[A]$?

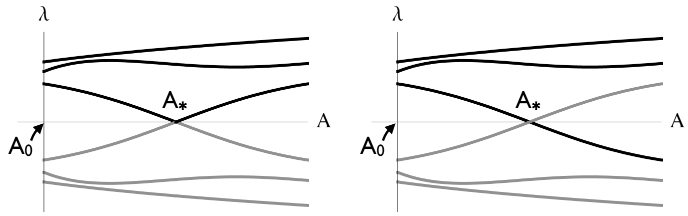

Fig. 6.1 *The eigenvalue flow of the Euclidian Dirac operator as a function of gauge fields, and two unsatisfactory ways to define the Weyl fermion determinant* $\det D_L$ *as a square root of* $\det \not{D}$. *The expression* $\sqrt{|\det \not{D}|}$ *corresponds to the picture on the left, where* $\det D_L$ *is defined as the product of positive eigenvalues of* $\not{D}$*; this definition is nonanalytic at* A_*. *The picture on the right corresponds to the product of half the eigenvalues, following those which were positive at some reference gauge field* A_0. *This definition is analytic, but not necessarily local. Both definitions are gauge invariant, which is incorrect for an anomalous fermion representation.*

1. Since $\det \not{D}$ describes a vector-like gauge theory, it is gauge invariant; therefore $W[A]$ should be gauge invariant if $\not{D}_L$ is anomaly-free, or should correctly reproduce the anomaly if not;
2. It should be analytic in the gauge fields, so that the computation of gauge field correlators (or the gauge current) are well defined.
3. It should be a local functional of the gauge fields since $\not{D}_L$ is a local operator.

The phase $W[A]$ was shown in the continuum to be equal to a quantity called the η-invariant of a certain 5-dimensional operator, the η-invariant being a sort of generalization of the Chern-Simons operator on a manifold with a boundary (Alvarez-Gaume, Della Pietra and Della Pietra, 1986). Once again, extra dimensions and boundaries seem to be central to our understanding of chirality. This definition does not directly lead to an algorithm for computing $W[A]$ on the lattice, however, even though there are apparent connections with domain wall fermions (Kaplan and Schmaltz, 1996).

In Fig. 6.1 I show two possible ways to define $\det D_L$, neither of which satisfy the above criteria. The naive choice of just setting $W[A] = 0$ not only fails to reproduce the anomaly (if the fermion representation is anomalous) but is also nonanalytic and nonlocal. It corresponds to taking the product of all the positive eigenvalues λ_n of $\not{D}$ (up to an uninteresting overall constant phase), as shown on the left in Fig. 6.1. This definition is seen to be nonanalytic where eigenvalues cross zero. Another definition might be to take the product of positive eigenvalues at some reference gauge field A_0, following those eigenvalues as they cross zero; this definition is analytic and is always gauge invariant, but is presumably nonlocal. At the level of Feynman diagrams, a perturbative definition is problematic with Pauli-Villars, since it is impossible to give the Pauli-Villars fields a gauge-invariant mass, and dimensional regularization suffers the problem that since γ_5 has no analogue in odd d, there is no obviously correct way to analytically continue Feynman rules in a way that can be shown to work to all orders in perturbation theory.

Lüscher's discovery of the exact chiral symmetry with the overlap operator suggests one can gauge that theory, but there remains the problem of how to define the fermion measure, even on the lattice. In several papers he has shown how to do it for both $U(1)$ and non-abelian gauge symmetries, although there remain some technical issues with the latter (Luscher, 1999; Luscher, 2000*b*; **?**). Interestingly enough, his definition of the fermion measure again leads one to consider extra dimensions. This work has not let to actual simulations that I am aware of.

One of the original motivations for domain wall fermions was to regulate chiral gauge theories, where the paradigm of having positive and negative chirality surface modes physically separated was deemed to be a first step (Kaplan, 1992). The goal then was to make interactions at the two surfaces different in a way that would eliminate the unwanted "mirror" modes at one of the surfaces decoupled from the physical spectrum. This required choosing fermion representations that were anomaly-free, so that the only Chern-Simons currents traversing the bulk were anomalous flavor currents and not gauge currents. Along with the classification of topological phases in this theory (Golterman, Jansen and Kaplan, 1993) it was a concrete realization of what condensed matter physicists call the Spin Quantum Hall effect, although it predated the famous Kane-Mele paper (Kane and Mele, 2005) by over a decade.

The problem is how to decouple the modes on the far surface. An idea proposed in (Grabowska and Kaplan, 2016; **?**) was to give the unwanted fermions infinitely soft form factors, so that the gauge fields did not couple to them. This proposal may produce nonlocal physics and be sick; it has not been thoroughly investigated though. Another approach by Eichten and Preskill that predated domain wall fermions (Eichten and Preskill, 1985) suggested that unwanted mirror fermions could be decoupled by strong, multi-particle interactions that were carefully selected to break all anomalous symmetries explicitly, and which gapped the mirror fermions as constituents of massive multi-particle bound states without giving them a symmetry breaking single-particle mass term. The problem with this proposal is finding a theory that dynamically realizes such a continuum phase, with all the desired features, such as Lorentz symmetry. To date, no Monte-Carlo simulation has revealed such a symmetric gapped phase arising, although it remains an attractive possibility. This approach has been pursued recently in the context of domain wall fermions by Wen, and simulations of his theories would be very interesting to see. See, for example (Wang and Wen, 2018) and reference therein. The actual simulation of chiral gauge theories would be an exciting prospect, although such theories should be expected to have a complex fermion determinant, thereby entailing all the challenges of simulating theories with sign problems.

I think it is fair to say that while there has been promising progress toward the nonperturbative definition of chiral gauge theories, and that they all entail extra dimensions so that the domain wall fermion formulation seems to be a natural fit, the problem has not received a definitive solution for an actual simulation, and there is much room for further progress.

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