

# Hydrogen

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## 1 Matching

QED Lagrangian is

$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \quad (1)$$

Set the NRQED Lagrangian as (take large  $M$  limit where  $M$  is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \quad (2)$$

In tree level

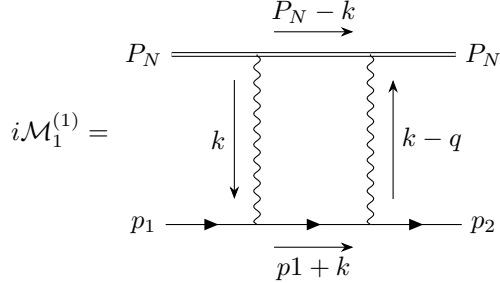
$$i\mathcal{M}_{QED}^{(0)} = \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \text{---} \text{---} p_2 \end{array} = -e^2 \bar{u}_N(P_N) v^\mu u_N(P_N) \frac{i}{\mathbf{q}^2} \bar{u}_e(p_2) \gamma_\mu u_e(p_1)$$

$$i\mathcal{M}_{NRQED}^{(0)} = \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \text{---} \text{---} p_2 \end{array} = -e^2 \bar{u}_N(P_N) v^\mu u_N(P_N) \frac{i}{\mathbf{q}^2} \psi^\dagger(p_2) \gamma_\mu \psi(p_1)$$

The box diagram for NRQED process is

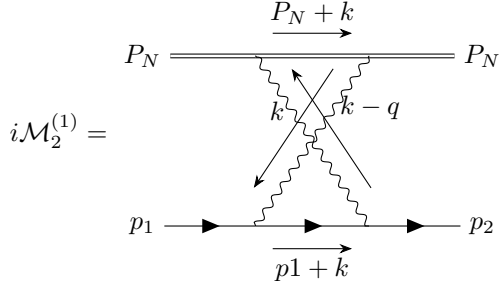
$$\begin{aligned} i\mathcal{M}_{NRQED}^{(1)} &= \begin{array}{c} P_N \text{---} \text{---} P_N \\ \begin{array}{c} \xrightarrow{P_N - k} \\ \downarrow k \\ \uparrow k - q \end{array} \\ p_1 \text{---} \text{---} p_2 \\ \xrightarrow{p_1 + k} \end{array} \\ &= e^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int [dk] \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (-k^0 + i\epsilon)(p_1^0 + k^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\ &= -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (E_1 - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m})} \psi(p_1) \\ &= -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 (E_1 - \frac{\mathbf{k}^2}{2m})} \psi(p_1) \end{aligned}$$

The box and crossed box diagram for QED process is



$$\begin{aligned}
i\mathcal{M}_1^{(1)} &= \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 - p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_1^{(1)}$  has infrared log divergence and no ultraviolet divergence.



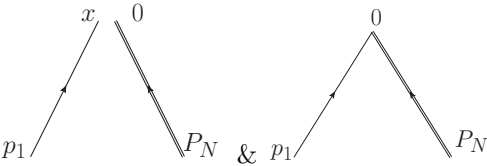
$$\begin{aligned}
i\mathcal{M}_2^{(1)} &= \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 - \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 + p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 - \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 + p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_2^{(1)}$  has no infrared or ultraviolet divergence.

$$\begin{aligned}
i\mathcal{M}_1^{(1)} + i\mathcal{M}_2^{(1)} &= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^{0^2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 - p_1^{0^2}]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^{0^2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 - \mathbf{p}_1^2]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1)
\end{aligned}$$

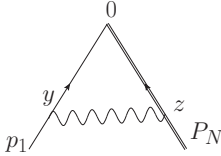
Note that after the expansion over external momentum,  $k^i$  can be converted into  $p^i$  so it's actually at  $p^1$  order.

Now consider operator product expansion.

Tree level matching: 

$$\begin{aligned} \langle 0|T\psi(x)N(0)|pP_N\rangle &= \text{diagram} = u_e(p)u_N(P_N)e^{-ip\cdot x} \\ \langle 0|T\psi_e(x)N(0)|pP_N\rangle &= \text{diagram} = \psi_e(p)u_N(P_N) \end{aligned}$$

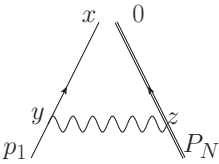
At leading order  $u_e(p) = \begin{pmatrix} \psi_e(p) \\ 0 \end{pmatrix}$ .

One loop scenario for NRQED case: 

$$\begin{aligned} \langle 0|\psi_e(0)N(0)e \int d^4y \bar{\psi}_e \psi_e A^0 e \int d^4z \bar{N} N A^0 |eN\rangle &= e^2 u_N(P_N) \int [dk] \frac{1}{\mathbf{k}^2(-k^0 + i\epsilon)(p_1^0 + k^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\ &= -ie^2 u_N(P_N) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(E_1 - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\ &= -ie^2 u_N(P_N) \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p}_1)^2(E_1 - \frac{\mathbf{k}^2}{2m} + i\epsilon)} \psi(p_1) \end{aligned}$$

drop  $p_1$

$$= -ie^2 u_N(P_N) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(E_1 - \frac{\mathbf{k}^2}{2m} + i\epsilon)} \psi(p_1) = \pi i e^2 \sqrt{\frac{2m}{E_1}} u_N(P_N) \psi(p_1)$$

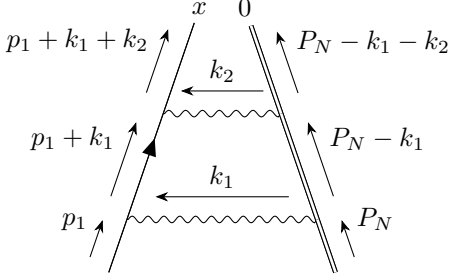
For QED case: 

$$\begin{aligned} \langle 0|\psi(x)N(0)e \int d^4y \bar{\psi} \gamma^0 \psi A^0 e \int d^4z \bar{N} N A^0 |eN\rangle &= e^2 u_N(P_N) \int [dk] e^{-i(\mathbf{k} + \mathbf{p}_1) \cdot \mathbf{x}} \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2[(p_1 + k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\ &= e^2 u_N(P_N) \int [dk] e^{-i(\mathbf{k} + \mathbf{p}_1) \cdot \mathbf{x}} \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2[(p_1 + k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\ &= ie^2 u_N(P_N) \int \frac{d^3k}{(2\pi)^3} e^{-i(\mathbf{k} + \mathbf{p}_1) \cdot \mathbf{x}} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 - p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\ &= ie^2 u_N(P_N) \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1) \end{aligned}$$

drop  $p_1$

$$= ie^2 u_N(P_N) \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{p_1^0 + \sqrt{\mathbf{k}^2 + m^2}}{2\mathbf{k}^2 [\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)$$

Two loop scenario for QED case  $\langle 0 | T \psi(x) N(0) e \int d^4 y_1 \bar{\psi} \gamma^0 \psi A^0 e \int d^4 z_1 \bar{N} N A^0 e \int d^4 y_2 \bar{\psi} \gamma^0 \psi A^0 e \int d^4 z_2 \bar{N} N A^0 | 0 \rangle$ :



$$= e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \frac{1}{\mathbf{k}_1^2} \frac{1}{\mathbf{k}_2^2} \frac{1}{-k_1 - k_2 + i\epsilon} \frac{1}{-k_1 + i\epsilon} \frac{1}{(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon} \frac{1}{(p_1 + k_1)^2 - m^2 + i\epsilon} u_N(P_N) u_e(p)$$

$$= e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} e^{-i(\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \frac{1}{-k_1 - k_2 + i\epsilon} \frac{1}{-k_1 + i\epsilon} \frac{1}{(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon} \frac{1}{(p_1 + k_1)^2 - m^2 + i\epsilon} u_N(P_N) u_e(p)$$

## 2 HSET

### 2.1 Lagrangian

For scalar QED

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi$$

where

$$D_\mu = \partial_\mu + ieA_\mu$$

In Schwartz's QFT (Chap. 35) he mentioned a choice of  $\chi_v$  and  $\tilde{\chi}_v$ :

$$\phi(x) = e^{-imv \cdot x} \frac{1}{\sqrt{2m}} (\chi_v(x) + \tilde{\chi}_v(x)) \quad (3)$$

$$\chi_v(x) = e^{imv \cdot x} \frac{1}{\sqrt{2m}} (iv \cdot D + m) \phi(x), \quad \tilde{\chi}_v(x) = e^{imv \cdot x} \frac{1}{\sqrt{2m}} (-iv \cdot D + m) \phi(x) \quad (4)$$

Put (3) into (4), a simple relation is derived:

$$(-iv \cdot D) \chi_v(x) = (2m + iv \cdot D) \tilde{\chi}_v(x)$$

It can also be written as

$$2m \tilde{\chi}_v = (-iv \cdot D) (\chi_v + \tilde{\chi}_v)$$

Use this result

$$\mathcal{L} = \frac{1}{2m} \left\{ \left\{ [D^\mu (\chi_v + \tilde{\chi}_v)]^\dagger + imv^\mu (\chi_v + \tilde{\chi}_v)^\dagger \right\} \left\{ [D_\mu (\chi_v + \tilde{\chi}_v)] - imv_\mu (\chi_v + \tilde{\chi}_v) \right\} - m^2 (\chi_v + \tilde{\chi}_v)^\dagger (\chi_v + \tilde{\chi}_v) \right\}$$

$$= (\chi_v + \tilde{\chi}_v)^\dagger (iv \cdot D) (\chi_v + \tilde{\chi}_v) + \frac{1}{2m} [D^\mu (\chi_v + \tilde{\chi}_v)]^\dagger D_\mu (\chi_v + \tilde{\chi}_v) \quad (5)$$

$$= (\chi_v(x) + \tilde{\chi}_v(x))^\dagger (iv \cdot D) (\chi_v(x) + \tilde{\chi}_v(x)) + \mathcal{O}\left(\frac{1}{m}\right) \quad (6)$$

(note that  $D_\mu \phi = e^{-imv \cdot x} [D_\mu (\chi_v + \tilde{\chi}_v) - imv_\mu (\chi_v + \tilde{\chi}_v)]$  and  $-imv^\mu [D_\mu (\chi_v + \tilde{\chi}_v)]^\dagger (\chi_v + \tilde{\chi}_v) = imv^\mu (\chi_v + \tilde{\chi}_v)^\dagger D_\mu (\chi_v + \tilde{\chi}_v) - \text{total derivative term}$ )

Use the leading order of (5)

$$\begin{aligned}
\mathcal{L}^{(0)} &= (\chi_v + \tilde{\chi}_v)^\dagger (iv \cdot D)(\chi_v + \tilde{\chi}_v) \\
&= \chi_v^\dagger iv \cdot D \chi_v + \tilde{\chi}_v^\dagger iv \cdot D(\chi_v + \tilde{\chi}_v) + \chi_v^\dagger iv \cdot D \tilde{\chi}_v \\
&= \chi_v^\dagger iv \cdot D \chi_v - 2m \tilde{\chi}_v^\dagger \tilde{\chi}_v + (iv \cdot D \chi_v)^\dagger \tilde{\chi}_v \\
&= \chi_v^\dagger iv \cdot D \chi_v - 2m \tilde{\chi}_v^\dagger \tilde{\chi}_v + [(-2m - iv \cdot D) \tilde{\chi}_v]^\dagger \tilde{\chi}_v \\
&= \chi_v^\dagger iv \cdot D \chi_v - \tilde{\chi}_v^\dagger (iv \cdot D + 4m) \tilde{\chi}_v
\end{aligned}$$

We can have the final form

$$\mathcal{L} = \chi_v^\dagger iv \cdot D \chi_v - \tilde{\chi}_v^\dagger (iv \cdot D + 4m) \tilde{\chi}_v + \mathcal{O}\left(\frac{1}{m}\right)$$

## 2.2 Quantization

### 2.2.1 HQET as an example

The leading term of HQET Lagrangian is

$$\mathcal{L} = \bar{Q}_v (iv \cdot D) Q_v$$

$$Q_v(x) = e^{imv \cdot x} \frac{1 + \not{v}}{2} \psi(x)$$

In free Dirac fermion theory we know that

$$\begin{aligned}
\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\} &= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_{ab} \\
\{a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger\} &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')
\end{aligned}$$

also the plane wave expansion of  $\psi$  is

$$\begin{aligned}
\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}} u(p) e^{-ip \cdot x} \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2mv^0}} a_v \sqrt{m} u(v) e^{-imv \cdot x - ik \cdot x}
\end{aligned}$$

using normalization of states  $u(k) = \sqrt{m} u(v)$ <sup>1</sup>,  $\langle p' | p \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\mathbf{p}' - \mathbf{p})$  and  $\langle v', k' | v, k \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta^{(3)}(\mathbf{k}' - \mathbf{k})$  we have  $|p\rangle = \sqrt{m} |v\rangle$  ( $|p\rangle = \sqrt{2E_p} a_{\mathbf{p}}^\dagger |0\rangle$  while  $|v, k\rangle = \sqrt{2v^0} a_{v, \mathbf{k}}^\dagger |0\rangle$ )

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v u(v) e^{-imv \cdot x - ik \cdot x}$$

Using the definition of  $Q_v(x)$

$$\begin{aligned}
Q_v(x) &= e^{imv \cdot x} \frac{1 + \not{v}}{2} \psi(x) \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v \frac{1 + \not{v}}{2} u(v) e^{-ik \cdot x} \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v u(v) e^{-ik \cdot x}
\end{aligned}$$

The commutation relation should be

$$\{Q_{va}(\mathbf{x}), Q_{v'b}(\mathbf{x}')\} = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{4v^0 v'^0}} \{a_v, a_{v'}^\dagger\} u_a(v) u_b^\dagger(v') e^{-ik \cdot x + ik' \cdot x'}$$

<sup>1</sup>The relation  $\bar{u}^s(p) \gamma^\mu u^s(p) = 2p^\mu$  can be derived using Gordon identity, same for  $\bar{u}^s(v) \gamma^\mu u^s(v) = 2v^\mu$ , but it's actually  $\bar{u}u$ .

using  $\sum_s u_a(v)u_b^\dagger(v) = \frac{1}{m} \sum_s u_a(p)u_b^\dagger(p) = [(\not{v} + 1)\gamma^0]_{ab}$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{4v^0v'^0}} \{a_v, a_{v'}^\dagger\} [(\not{v} + 1)\gamma^0]_{ab} e^{-ik \cdot x + ik' \cdot x'}$$

assuming  $\{a_v, a_{v'}\} = (2\pi)^3 \delta_{vv'} \delta^{(3)}(\mathbf{k} - \mathbf{k}')$

$$\begin{aligned} &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2v^0} [(\not{v} + 1)\gamma^0]_{ab} e^{-ik \cdot (x - x')} \delta_{vv'} \\ &= \left[ \frac{(\not{v} + 1)\gamma^0}{2v^0} \right]_{ab} \delta_{vv'} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \end{aligned}$$

### 2.2.2 HSET

The anti-particle field is decoupled so we don't have to consider that for now. The equation-of-motion is

$$\begin{cases} v \cdot D \chi_v^\dagger = 0 \\ v \cdot D \chi_v = 0 \end{cases}$$

By definition

$$\begin{aligned} \chi_v(x) &= \frac{e^{imv \cdot x}}{\sqrt{2m}} (iv \cdot D + m) \phi(x) \\ &= \frac{1}{\sqrt{2m}} (iv \cdot D + 2m) e^{imv \cdot x} \phi(x) \end{aligned}$$