Euler-Lagarange equation (面上 $\delta\phi=0$): $\delta S=\int \mathrm{d}^4x\delta\mathcal{L}(x)=\int \mathrm{d}^4x[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)]=\int \mathrm{d}^4x\left\{(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta\phi+\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi)\right\}$ $\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})=0$ 全局对称性：时空对称性：庞加莱群，分立对称性：P，T；内禀对称性：连续：电荷，同位旋，相位对称性：$\phi(x)\to e^{-i\alpha}\phi(x)$；分立：$\phi\to-\phi$. Noether's theorom: $\delta\mathcal{L}(x)=0\xrightarrow{U(1)}0=\frac{\delta\mathcal{L}}{\delta\alpha}=\sum_n\left\{\frac{\partial\mathcal{L}}{\partial\phi_n}\frac{\delta\phi_n}{\delta\alpha}+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta(\partial_\mu\phi_n)}{\delta\alpha}\right\}=\sum_n[\frac{\partial\mathcal{L}}{\partial\phi_n}-\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)})]\frac{\delta\phi_n}{\delta\alpha}+\partial_\mu J^\mu$ $\partial_\mu J^\mu=0$ where $J^\mu=\sum_n\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\alpha}$, $Q=\int \mathrm{d}^3xJ^0(x)$. General: $\delta S=\int \mathrm{d}^4x\partial_\mu[(\mathcal{L}g^\mu_\rho-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi)\delta x^\rho+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi]=0$,$f(x)\to f'(x')$,$\delta f\equiv f'(x')-f(x)=\delta_0f+\delta x^\mu\partial_\mu f$,$x'^\mu=x^\mu+\epsilon^\mu$,$\delta f=0\Longrightarrow\delta_0f=-\epsilon^\mu\partial_\mu f$, $\delta S=\int \mathrm{d}^4x\delta\mathcal{L}+\int\delta(\mathrm{d}^4x)\mathcal{L}=0$,$\mathrm{d}^4x'=\left|\frac{\partial x'^\mu}{\partial x^\nu}\right|\mathrm{d}^4x=\left|\frac{\partial(x^\mu+\delta x^\mu)}{\partial x^\nu}\right|\mathrm{d}^4x=(1+\frac{\partial\delta x^\mu}{\partial x^\mu})\mathrm{d}^4x$, $\delta\mathcal{L}=\delta_0\mathcal{L}+\delta x^\mu\partial_\mu\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0(\partial_\mu\phi)+\delta x^\mu\partial_\mu\mathcal{L}=\delta x^\mu\partial_\mu\mathcal{L}+(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta_0\phi+\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi)$. 能动张量：$T^{\mu\nu}=\sum_n\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n-g_{\mu\nu}\mathcal{L}$,$\partial^\mu T_{\mu\nu}=0$,$T_{00}=\mathcal{H}$,$Q_\nu=\int \mathrm{d}^3xT_{0\nu}\Longrightarrow Q_0=H$. **K-G 场算符**：$\phi(\mathbf{x})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}+a^\dagger_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}})$, $\pi(\mathbf{x})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}i\sqrt{\frac{\omega_\mathbf{p}}{2}}(a^\dagger_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}}-a_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}})$, $\phi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_\mathbf{p}e^{-ip\cdot x}+b^\dagger_\mathbf{p}e^{ip\cdot x})$ 等时对易关系：$[a_\mathbf{p},a^\dagger_{\mathbf{p}'}]=(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$, $[\phi(\mathbf{x}),\pi(\mathbf{x}')]=i\delta^3(\mathbf{x}-\mathbf{x}')$, $|\mathbf{p}\rangle=\sqrt{2\omega_\mathbf{p}}a^\dagger_\mathbf{p}|0\rangle$, $\langle\mathbf{p}|\psi\rangle=e^{-i\mathbf{p}\cdot\mathbf{x}}$, $D_F(x-y)=\theta(x^0-y^0)\langle0|\phi(x)\phi(y)|0\rangle+\theta(y^0-x^0)\langle0|\phi(y)\phi(x)|0\rangle\equiv\langle0|T\phi(x)\phi(y)|0\rangle$. 留数定理：$\oint_\gamma f(z)\,dz=2\pi i\sum_{k=1}^n\mathrm{Res}(f,a_k)$. **K-G 传播子**：$[\phi(x),\phi(y)]=\langle0|[\phi(x),\phi(y)]|0\rangle=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}p^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip\cdot(x-y)}$, $D_F(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$. 阶跃函数：$\langle0|\theta(x^0-y^0)\phi(x)\phi(y)|0\rangle=\langle0|\frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p^0\frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_\mathbf{p}e^{-ip\cdot x}+a^\dagger_\mathbf{p}e^{ip\cdot x})\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{q}}}(a_\mathbf{q}e^{-iq\cdot y}+a^\dagger_\mathbf{q}e^{iq\cdot y})|0\rangle=\langle0|\frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p^0\frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\frac{1}{\sqrt{2\omega_\mathbf{q}}}[a_\mathbf{p},a^\dagger_\mathbf{q}]e^{-ip\cdot x}e^{iq\cdot y}|0\rangle=\frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p^0\frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_\mathbf{p}}e^{-ip\cdot(x-y)}=\frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p^0\frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{-iE_\mathbf{p}(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}=\frac{i}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^0\mathrm{d}^3p}{(2\pi)^4}\frac{1}{E_\mathbf{p}}\frac{e^{-i(p^0+E_\mathbf{p})(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{p^0+i\epsilon}$make $p^0=(p^0+E_\mathbf{p})=\frac{i}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^0\mathrm{d}^3p}{(2\pi)^4}\frac{e^{-ip^0(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(E_\mathbf{p})(p^0-E_\mathbf{p}+i\epsilon)}$.

Dirac 场：$\sigma^0=1,\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\sigma^2=\begin{pmatrix}0&-1\\i&0\end{pmatrix},\sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. 主动变换：$x^\mu\to x'^\mu=\Lambda^\mu_\nu$, $\phi(x)\to\phi'(x)=\phi(\Lambda^{-1}x)$, $\partial_\mu\phi(x)\to\partial_\mu\phi(\Lambda^{-1}x)=(\Lambda^{-1})^\nu_\mu(\partial_\nu\phi)(\Lambda^{-1}x)$, $(\Lambda^{-1})^\rho_\mu(\Lambda^{-1})^\sigma_\nu g^{\mu\nu}=g^{\rho\sigma}$, $(\partial_\mu\phi(x))^2\to(\partial_\mu\phi)^2(\Lambda^{-1}x)$, $\mathcal{L}(x)\to\mathcal{L}'(x)=\mathcal{L}(\Lambda^{-1}x)$, $V^\mu(x)\to\Lambda^\mu_\nu V^\nu(\Lambda^{-1}x)$, $\Phi_a(x)\to M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x)$, $\psi(x)\to\Lambda_{\frac12}\psi(\Lambda^{-1}x)$. Lorentz 生成元：$J^{\mu\nu}=i(x^\mu\partial^\nu-x^\nu\partial^\mu)$, $[J^{\mu\nu},J^{\rho\sigma}]=i(g^{\nu\rho}J^{\mu\sigma}-g^{\mu\rho}J^{\nu\sigma}-g^{\nu\sigma}J^{\mu\rho}+g^{\mu\sigma}J^{\nu\rho})$, $(\mathcal{J}^{\mu\nu})_{\alpha\beta}=i(\delta^\mu_\alpha\delta^\nu b-\delta^\mu_\beta\delta^\nu_\alpha)$, $V\to e^{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}V$, $V^\alpha=(\delta^\alpha_\beta-\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta)V^\beta$. $S^{\mu\nu}=\frac{i}{4}[\gamma^\mu,\gamma^\nu]$, $\{\sigma^i,\sigma^j\}=2\delta^{ij}$, $\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$, **Weyl rep**: $\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}$.

	γ^0	γ^1	γ^2	γ^3	γ^5
T	1	-1	1	-1	1
-1	1	-1	-1	-1	1

	γ^0	γ^1	γ^2	γ^3	γ^5
*	1	1	-1	1	1
†	1	-1	-1	-1	1

, $\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3=\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$, $\gamma_\mu\not{p}\gamma^\mu=-2\not{p}$, $\gamma_\mu\not{p}\not{q}\not{p}\gamma^\mu=-2\not{p}\not{q}\not{p}$, $\{\gamma^5,\gamma^\mu\}=0$. $[\gamma^\mu,S^{\rho\sigma}]=(\mathcal{J}^{\rho\sigma})^\mu_\nu\gamma^\nu$, $\Lambda^{-1}_{\frac12}\gamma^\mu\Lambda_{\frac12}=\Lambda^\mu_\nu\gamma^\nu$, $\sigma^2\boldsymbol{\sigma}^*=-\boldsymbol{\sigma}\sigma^2$, $\bar\sigma^\mu\sigma_\mu=4$. $\mathcal{L}_{Dirac}=\bar\psi(i\not\partial-m)\psi$, $\mathcal{H}=\bar\psi(-i\boldsymbol{\gamma}\cdot\nabla+m)\psi$, $j^\mu=\bar\psi\gamma^\mu\psi$, $j^{5\mu}=\bar\psi\gamma^\mu\gamma^5\psi$. **Dirac 方程**：$i\not\partial\psi=m\psi$, $\bar\psi(i\overleftarrow{\not\partial}-m)=0$. $(\not{p}-m)u(p)=0$, $\bar u(p)(\not{p}-m)=0$, $(\not{p}+m)v(p)=0$, $\bar v(p)(\not{p}+m)=0$. **解**：$u^s=\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\\sqrt{p\cdot\bar\sigma}\xi^s\end{pmatrix}$, $v^s=\begin{pmatrix}\sqrt{p\cdot\sigma}\eta^s\\-\sqrt{p\cdot\bar\sigma}\eta^s\end{pmatrix}$, $\bar u^s(p)u^{s'}(p)=2m\delta^{ss'}$, $u^{s\dagger}(p)u^{s'}(p)=2E_\mathbf{p}\delta ss'$, $\bar v^s(p)v^{s'}(p)=-2m\delta^{ss'}$, $v^{s\dagger}(p)v^{s'}(p)=2E_\mathbf{p}\delta ss'$, $\bar u^r(p)v^s(p)=\bar v^r(p)u^s(p)=0$, $u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p})=v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p})=0$, others uncertain. $\sum_s u_s(p)\bar u_s(p)=\not{p}+m$, $\sum_s v^s(p)\bar v^s(p)=\not{p}-m$. $\bar u_\sigma(p)\gamma^\mu u_{\sigma'}(p)=2\delta_{\sigma\sigma'}p^\mu$, $\bar u(p')\gamma^\mu u(p)=\bar u(p')\left[\frac{p'^\mu+p^\mu}{2m}+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$ (Gordon identity, $q=p'-p$). **Dirac 场量子化**：$\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(a^s_\mathbf{p}u^s(p)e^{-ip\cdot x}+b^{s\dagger}_\mathbf{p}v^s(p)e^{ip\cdot x})$, $\bar\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(b^s_\mathbf{p}\bar v^s(p)e^{-ip\cdot x}+a^{s\dagger}_\mathbf{p}\bar u^s(p)e^{ip\cdot x})$, **条件**：$\{\psi_a(\mathbf{x}),\psi^\dagger_b(\mathbf{y})\}=\delta^3(\mathbf{x}-\mathbf{y})\delta_{ab}$, $\{\psi_a(\mathbf{x}),\psi_b(\mathbf{y})\}=\{\psi^\dagger_a(\mathbf{x}),\psi^\dagger_b(\mathbf{y})\}=0$, $\{a^r_\mathbf{p},a^{s\dagger}_\mathbf{k}\}=\{b^r_\mathbf{p},b^{s\dagger}_\mathbf{k}\}=(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta^{rs}$, others are zero. $H=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_s E_\mathbf{p}(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p}+b^{s\dagger}_\mathbf{p}b^s_\mathbf{p})$, $P=\int\mathrm{d}^3x\psi^\dagger(-i\nabla)\psi$, $J_z=\int\mathrm{d}^3x\int\frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}e^{-i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{x}}\sum_{r,s}(a^{r\dagger}_\mathbf{q}u^{r\dagger}(\mathbf{q})+b^r_{-\mathbf{q}}v^{r\dagger}(-\mathbf{q}))\frac{\Sigma^3}{2}(a^s_\mathbf{p}u^s(\mathbf{p})+b^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p}))$. 量子守恒荷：$\hat Q=\int\mathrm{d}^3x\hat j^0(x)=\int\mathrm{d}^3x\psi^\dagger(x)\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_s(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p}-b^{s\dagger}_\mathbf{p}b^s_\mathbf{p})$. 单粒子态：$|p,s\rangle=\sqrt{2E_\mathbf{p}}a^{s\dagger}_\mathbf{p}|0\rangle$.

H,P: In Schrödinger picture $\psi(\mathbf{x})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(a^s_\mathbf{p}u^s(p)e^{i\mathbf{p}\cdot\mathbf{x}}+b^{s\dagger}_\mathbf{p}v^s(p)e^{-i\mathbf{p}\cdot\mathbf{x}})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(a^s_\mathbf{p}u^s(p)+b^{s\dagger}_{-\mathbf{p}}v^s(-p))e^{i\mathbf{p}\cdot\mathbf{x}}$, $\bar\psi(\mathbf{x})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(b^s_\mathbf{p}\bar v^s(p)e^{i\mathbf{p}\cdot\mathbf{x}}+a^{s\dagger}_\mathbf{p}\bar u^s(p)e^{-i\mathbf{p}\cdot\mathbf{x}})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(b^s_\mathbf{p}\bar v^s(p)+a^{s\dagger}_{-\mathbf{p}}\bar u^s(-p))e^{i\mathbf{p}\cdot\mathbf{x}}$. $\nabla\psi=\nabla\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(a^s_\mathbf{p}u^s(p)+b^{s\dagger}_{-\mathbf{p}}v^s(-p))e^{i\mathbf{p}\cdot\mathbf{x}}=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\mathbf{p}}{\sqrt{2E_\mathbf{p}}}\sum_s(a^s_\mathbf{p}u^s(p)+b^{s\dagger}_{-\mathbf{p}}v^s(-p))e^{i\mathbf{p}\cdot\mathbf{x}}$. $H=\int\mathrm{d}^3x\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}2E_\mathbf{p}}}\sum_{s,r}[(b^s_\mathbf{p}\bar v^s(p)+a^{s\dagger}_{-\mathbf{p}}\bar u^s(-p))(\gamma\cdot\mathbf{k}+m)(a^r_\mathbf{k}u^r(k)+b^{r\dagger}_{-\mathbf{k}}v^r(-k))]e^{i(\mathbf{p}+\mathbf{k})\cdot\mathbf{x}}=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{p}}}\sum_{s,r}[-(b^s_\mathbf{p}\bar v^s_p+a^{s\dagger}_{-\mathbf{p}}\bar u^s_{-p})\gamma\cdot\mathbf{p}(a^r_{-\mathbf{p}}u^r_{-p}+b^{r\dagger}_\mathbf{p}v^r_p)+m((b^s_\mathbf{p}\bar v^s_pb^{r\dagger}_\mathbf{p}v^r_p+a^{s\dagger}_{-\mathbf{p}}\bar u^s_{-p}a^r_{-\mathbf{p}}u^r_{-p})]$. $m((b^s_\mathbf{p}\bar v^s_pb^{r\dagger}_\mathbf{p}v^r_p+a^{s\dagger}_{-\mathbf{p}}\bar u^s_{-p}a^r_{-\mathbf{p}}u^r_{-p})=-2m^2(b^s_\mathbf{p}b^{s\dagger}_\mathbf{p}-a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})$, $(b^s_\mathbf{p}\bar v^s_p+a^{s\dagger}_{-\mathbf{p}}\bar u^s_{-p})\gamma\cdot\mathbf{p}(a^r_{-\mathbf{p}}u^r_{-p}+b^{r\dagger}_\mathbf{p}v^r_p)=(b^s_\mathbf{p}\bar v^s_p+a^{s\dagger}_{-\mathbf{p}}\bar u^s_{-p})\gamma^ip_i(a^r_{-\mathbf{p}}u^r_{-p}+b^{r\dagger}_\mathbf{p}v^r_p)=2\mathbf{p}^2(b^s_\mathbf{p}b^{s\dagger}_\mathbf{p}-a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})$. Use $\bar u_\sigma(p)\gamma^\mu u_{\sigma'}(p)=2\delta_{\sigma\sigma'}p^\mu$ and $\bar v_\sigma(p)\gamma^\mu v_{\sigma'}(p)=2\delta_{\sigma\sigma'}p^\mu$, $H=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_s[-2\mathbf{p}^2(b^s_\mathbf{p}b^{s\dagger}_\mathbf{p}-a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})-2m^2(b^s_\mathbf{p}b^{s\dagger}_\mathbf{p}-a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})]=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_sE_\mathbf{p}(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p}-b^s_\mathbf{p}b^{s\dagger}_\mathbf{p})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_sE_\mathbf{p}(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p}+b^{s\dagger}_\mathbf{p}b^s_\mathbf{p})$. $P=\int\mathrm{d}^3x\psi^\dagger(-i\nabla)\psi$, $P=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(b^s_\mathbf{p}\bar v^s(p)+a^{s\dagger}_{-\mathbf{p}}\bar u^s(-p))\gamma^0\frac{-\mathbf{p}}{\sqrt{2E_\mathbf{p}}}\sum_r(a^r_{-\mathbf{p}}u^r(-p)+b^{r\dagger}_\mathbf{p}v^r(p))=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{-i\mathbf{p}}{2E_\mathbf{p}}\sum_{s,r}(b^s_\mathbf{p}v^{s\dagger}(p)+a^{s\dagger}_{-\mathbf{p}}u^{s\dagger}(-p))(a^r_{-\mathbf{p}}u^r(-p)+b^{r\dagger}_\mathbf{p}v^r(p))=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{-\mathbf{p}}{2E_\mathbf{p}}\sum_s2E_\mathbf{p}(b^s_\mathbf{p}b^{s\dagger}_\mathbf{p}+a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})=-\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathbf{p}\sum_s(b^s_\mathbf{p}b^{s\dagger}_\mathbf{p}+a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathbf{p}\sum_s(-b^s_\mathbf{p}b^{s\dagger}_\mathbf{p}+a^{s\dagger}_\mathbf{p}a^s_\mathbf{p})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathbf{p}\sum_s(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p}+b^{s\dagger}_\mathbf{p}b^s_\mathbf{p}-(2\pi)^3\delta(0))=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathbf{p}\sum_s(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p}+b^{s\dagger}_\mathbf{p}b^s_\mathbf{p})$.

CPT 对称性：**P**: $P\psi P^{-1}=\eta\gamma^0\psi(t,-\mathbf{x})$, $P\bar\psi P^{-1}=\eta^*\bar\psi(t,-\mathbf{x})\gamma^0$. **T**: $\mathscr{T}\equiv i\gamma^1\gamma^3$ and $T\psi T^{-1}=\mathscr{T}\psi$, $T\bar\psi T^{-1}=\bar\psi\mathscr{T}^{-1}$, $\mathscr{T}(\gamma^\mu)^*\mathscr{T}^{-1}=\gamma_\mu=(-1)^\mu\gamma^\mu$, $\mathscr{T}(\gamma^5)^*\mathscr{T}^{-1}=\gamma^5$, $\mathscr{T}=\mathscr{T}^{-1}=\mathscr{T}^\dagger$. **C**: $\mathscr{C}\equiv i\gamma^2\gamma^0$ and $C\psi C^{-1}=\mathscr{C}\bar\psi^T$, $C\bar\psi C^{-1}=\psi^T\mathscr{C}$, $\mathscr{C}(\gamma^\mu)^T\mathscr{C}^{-1}=-\gamma^\mu$, $\mathscr{C}(\gamma^5)^T\mathscr{C}^{-1}=\gamma^5$, $\mathscr{C}^\dagger=\mathscr{C}^{-1}=-\mathscr{C}=\mathscr{C}^T$, $(\mathscr{C}(\gamma^\mu)^T\mathscr{C}^{-1})^\dagger=-(\gamma^\mu)^\dagger=\mathscr{C}(\gamma^\mu)^*\mathscr{C}^{-1}=-(\gamma^\mu)^\dagger$, $\mathscr{C}\gamma^5\mathscr{C}^{-1}=\gamma^5$.

结果：$P\bar\psi\psi P^{-1}=+\bar\psi\psi(t,-\mathbf{x})$, $T\bar\psi\psi T^{-1}=+\bar\psi\psi(-t,\mathbf{x})$, $C\bar\psi\psi C^{-1}=+\bar\psi\psi(t,\mathbf{x})$, $P\bar\psi\gamma^5\psi P^{-1}=-\bar\psi\gamma^5\psi(t,-\mathbf{x})$, $T\bar\psi\gamma^5\psi T^{-1}=-\bar\psi\gamma^5\psi(-t,\mathbf{x})$, $C\bar\psi\gamma^5\psi C^{-1}=+\bar\psi\gamma^5\psi(t,\mathbf{x})$, $P\bar\psi\gamma^\mu\psi P^{-1}=(-1)^\mu\bar\psi\gamma^\mu\psi(t,-\mathbf{x})$, $T\bar\psi\gamma^\mu\psi T^{-1}=(-1)^\mu\bar\psi\gamma^\mu\psi(-t,\mathbf{x})$, $C\bar\psi\gamma^\mu\psi C^{-1}=-\bar\psi\gamma^\mu\psi(t,\mathbf{x})$, $CPT\bar\psi\gamma^5\psi CPT^{-1}=+\bar\psi\gamma^5\psi(-t,-\mathbf{x})$, $CPT\bar\psi\psi CPT^{-1}=+\bar\psi\psi(-t,-\mathbf{x})$, $CPT\bar\psi\gamma^\mu\psi CPT^{-1}=-\bar\psi\gamma^\mu\psi(-t,-\mathbf{x})$, $P\bar\psi\gamma^\mu\gamma^5\psi P^{-1}=|\eta|^2\bar\psi\gamma^0\gamma^\mu\gamma^5\gamma^0\psi=-(-1)^\mu\bar\psi\gamma^\mu\gamma^5\psi$, $T\bar\psi\gamma^\mu\gamma^5\psi T^{-1}=\bar\psi\mathscr{T}^{-1}(\gamma^\mu\gamma^5)^*\mathscr{T}\psi=\bar\psi\mathscr{T}^{-1}\gamma^{\mu*}\mathscr{T}^{-1}\mathscr{T}\gamma^{5*}\mathscr{T}\psi=\bar\psi\gamma_\mu\gamma^5\psi=(-1)^\mu\bar\psi\gamma^\mu\gamma^5\psi$, $C\bar\psi\gamma^\mu\gamma^5\psi C^{-1}=\psi^T\mathscr{C}\gamma^\mu\gamma^5\mathscr{C}\bar\psi^T=\psi^T\gamma^{\mu T}\gamma^{5T}\bar\psi^T=-(\bar\psi\gamma^5\gamma^\mu\psi)^T=\bar\psi\gamma^\mu\gamma^5\psi$, $P\bar\psi\sigma^{\mu\nu}\psi P^{-1}=\frac{i}{2}\bar\psi\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0\psi=\frac{i}{2}(-1)^\mu(-1)^\nu\bar\psi[\gamma^\mu,\gamma^\nu]\psi=(-1)^\mu(-1)^\nu\bar\psi\sigma^{\mu\nu}\psi$, $T\bar\psi\sigma^{\mu\nu}\psi T^{-1}=-\frac{i}{2}T\bar\psi[\gamma^\mu,\gamma^\nu]\psi T^{-1}=-\frac{i}{2}\bar\psi\mathscr{T}[\gamma^\mu,\gamma^\nu]^*\mathscr{T}^{-1}\psi=-(-1)^\mu(-1)^\nu\bar\psi\sigma^{\mu\nu}\psi$, $C\bar\psi\sigma^{\mu\nu}\psi C^{-1}=\frac{i}{2}\psi^T\mathscr{C}[\gamma^\mu,\gamma^\nu]\mathscr{C}\bar\psi^T=-\frac{i}{2}\psi^T[\gamma^{\mu T},\gamma^{\nu T}]\bar\psi^T=\frac{i}{2}(\bar\psi[\gamma^\nu,\gamma^\mu]\psi)^T=-\bar\psi\sigma^{\mu\nu}\psi$, $P\bar\psi\partial_\mu\psi P^{-1}=(-1)^\mu\bar\psi\partial_\mu\psi$, $T\bar\psi\partial_\mu\psi T^{-1}=-(-1)^\mu\bar\psi\partial_\mu\psi$, $C\bar\psi\partial_\mu\psi C^{-1}=\bar\psi\partial_\mu\psi$. $(-1)^\mu=1,\mu=0;(-1)^\mu=-1,\mu=1,2,3$.

Dirac 传播子：$\langle0|\psi_a(x)\bar\psi_b(y)|0\rangle=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_su^s_a(p)\bar u^s_b(p)e^{-ip\cdot(x-y)}=(i\not\partial_x+m)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{-ip\cdot(x-y)}$, $\langle0|\bar\psi_b(y)\psi_a(x)|0\rangle=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_sv^s_a(p)\bar v^s_b(p)e^{-ip\cdot(x-y)}=-(i\not\partial_x+m)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{ip\cdot(x-y)}$. $S_F(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$

矢量场: $\mathcal{L}_{Maxwell}=-\frac14(F_{\mu\nu})^2-A_\mu(x)J^\mu(x)=-\frac12(\partial_\mu A_\nu)^2+\frac12(\partial_\mu A^\mu)^2-A_\mu J^\mu$. $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$. E-L eq: $-J_\nu-\partial_\mu(-\partial^\mu A_\nu)-\partial_\nu(\partial^\mu A_\mu)=0$. **Failed**: $\mathcal{L}=-\frac12A_\mu(\Box+m^2)A^\mu$. E-L eq: $(\Box+m^2)A_\mu=0$. $\pi^\mu(x)=-\dot A^\mu(x)$. 能量密度: $\varepsilon=\pi^\mu\dot A_\mu-\mathcal{L}=-(\dot A_\mu)^2+\frac12\dot A_\mu\dot A^\mu-\frac12\nabla A_\mu\cdot\nabla A^\mu-\frac12m^2(A^0)^2+\frac12m^2\mathbf{A}^2=-\frac12[\dot A_0^2+(\nabla A_0)^2+m^2A_0^2]+\frac12[\mathbf{A}^2+(\partial_i\mathbf{A})^2+m^2\mathbf{A}^2]$ 能量不囿于下. **General Proca Lagrangian**: $\mathcal{L}=-\frac{a}{2}\partial_\nu A_\mu\partial^\nu A^\mu-\frac{b}{2}\partial_\mu A_\nu\partial^\nu A^\mu+\frac12m^2A^2-A_\mu J^\mu$. $\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}=-F^{\mu\nu}$ E-L eq: $-a\Box A^\mu-b\partial^\mu(\partial_\nu A^\nu)-m^2A^\mu=-J$. 两边求 ∂_μ: $\left\{(a+b)\Box+m^2\right\}\partial\cdot A=\partial_\mu J^\mu$. $(\partial\cdot A)$ 是标量场，自旋为 0，令 $a+b=0$，$m^2\partial\cdot A=0$（无源），除掉自旋为 0 的自由度。取 $a=1,b=-1$，$\mathcal{L}_{Proca}=-\frac14(F_{\mu\nu})^2+\frac12m^2A^2-A_\mu J^\mu$. E-L eq: $\Box A^\mu+m^2A^\mu=J^\mu\Longrightarrow(\Box+m^2)A^\mu=0,\partial\cdot A=0$. 极化矢量：纵向：$\epsilon^3=(\frac{p_z}{m},0,0,\frac{E}{m})^T$ 横向：$\epsilon^1=(0,1,0,0)^T,\epsilon^2=(0,0,1,0)^T$ 右旋圆极化：$\epsilon^\mu_{(+1)}=-\frac{1}{\sqrt2}(0,1,i,0)^T$ 左旋圆极化：$\epsilon^\mu_{(-1)}=\frac{1}{\sqrt2}(0,1,-i,0)^T$, $\lambda=0$ 不存在（已去除），主要纵向贡献. 正交性：$\epsilon^\lambda(p)\epsilon^{*(\lambda)}(p)=-\delta^{\lambda\lambda'}=g^{\lambda\lambda'}(\lambda=1,2,3)$ 完备性：$\sum_{\lambda=1,2,3}\epsilon^{(\lambda)}_\mu\epsilon^{*(\lambda)}_\nu=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}$

量子化 Proca 场: $A_\mu(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{E_\mathbf{p}}\sum_{\lambda=1,2,3}[a^{(\lambda)}_\mathbf{p}\epsilon^{(\lambda)}_\mu(p)e^{-ip\cdot x}+a^{(\lambda)\dagger}_\mathbf{p}\epsilon^{*(\lambda)}_\mu(p)e^{ip\cdot x}]$ 条件: $[a^{(\lambda)}_\mathbf{p},a^{(\lambda')\dagger}_{\mathbf{p}'}]=(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta^{\lambda\lambda'}$ 单粒子态: $|\mathbf{p},\lambda\rangle=\sqrt{2E_\mathbf{p}}a^{(\lambda)\dagger}_\mathbf{p}|0\rangle$. $[A^i(t,\mathbf{x}),\pi^j(t,\mathbf{y})]=-i\delta^{ij}\delta^3(\mathbf{x}-\mathbf{y})\Longrightarrow[A^\mu(t,\mathbf{x}),\pi^\nu(t,\mathbf{y})]=ig^{\mu\nu}\delta^3(\mathbf{x}-\mathbf{y})$, $[A_\mu(x),A_\nu(y)]=[-g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}]\Delta(x-y)$ where $\Delta(x-y)=[\phi(x),\phi(y)]$.

$\pi_i(x)=-\dot A_i-\partial_iA_0=i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_\mathbf{p}}{2}}\sum_\lambda[a^\lambda_\mathbf{p}\epsilon^\lambda_i(p)e^{-ip\cdot x}-a^{\lambda\dagger}_\mathbf{p}\epsilon^{\lambda*}_i(p)e^{ip\cdot x}]-i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p_i}{\sqrt{2E_\mathbf{p}}}\sum_\lambda[a^\lambda_\mathbf{p}\epsilon^\lambda_0(p)e^{-ip\cdot x}-a^{\lambda\dagger}_\mathbf{p}\epsilon^{\lambda*}_0(p)e^{ip\cdot x}]$.

$$[A^i(x),\pi^j(y)]=i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{\lambda,\lambda'}\left\{\sqrt{\frac{E_\mathbf{k}}{4E_\mathbf{p}}}(-2)[a^\lambda_\mathbf{p},a^{\lambda'\dagger}_\mathbf{k}]\epsilon^\lambda_i(p)\epsilon^{\lambda'*}_j(k)e^{-ip\cdot x}e^{ik\cdot y}+\frac{k_j}{2\sqrt{E_\mathbf{p}E_\mathbf{k}}}[a^\lambda_\mathbf{p},a^{\lambda'\dagger}_\mathbf{k}]\epsilon^\lambda_i(p)e^{-ip\cdot x}\epsilon^{\lambda'*}_0(k)e^{ik\cdot y}+\frac{k_j}{2\sqrt{E_\mathbf{p}E_\mathbf{k}}}[a^{\lambda'}_\mathbf{k},a^{\lambda\dagger}_\mathbf{p}]\epsilon^{\lambda'}_0(k)e^{-ik\cdot y}\epsilon^{\lambda*}_i(p)e^{ip\cdot x}\right\}$$
$$=i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_\lambda\left\{-\epsilon^\lambda_i(p)\epsilon^{\lambda*}_j(p)e^{-ip\cdot(x-y)}+\frac{p_j}{2E_\mathbf{p}}[\epsilon^\lambda_i(p)\epsilon^{\lambda*}_0(p)e^{-ip\cdot(x-y)}+\epsilon^\lambda_0(p)\epsilon^{\lambda*}_i(p)e^{ip\cdot(x-y)}]\right\}=i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{g_{ij}-\frac{p_ip_j}{m^2}+\frac{p_j}{2E_\mathbf{p}}[\delta_{i0}+\frac{p_ip_0}{m^2}-\delta_{i0}+\frac{p_ip_0}{m^2}]\right\}e^{-ip\cdot(x-y)}$$
$$=i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{g_{ij}-\frac{p_ip_j}{m^2}[1-\frac{p_0}{E_\mathbf{p}}]\right\}=-i\delta^{ij}\delta^3(\mathbf{x}-\mathbf{y})$$

Maxwell 场：$L_{Max}=-\frac14F_{\mu\nu}F^{\mu\nu}-J_\mu A^\mu$ 困难：$\epsilon^\mu_3=(\frac{p_z}{m},0,0,\frac{E}{m})\xrightarrow{m\to0}\infty$, $\sum\epsilon^\mu\epsilon^\nu=-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\to\infty$, 物理极化 $3\neq2$.

量子化 Maxwell 场 (协变规范 $\partial\cdot A=0$)：$\mathcal{L}_{Max}=-\frac14F^2_{\mu\nu}-\frac{\lambda}{2}(\partial\cdot A)^2-J\cdot A$ E-L eq: $\Box A_\mu-(1-\lambda)\partial_\mu(\partial\cdot A)=0$ 费曼规范：$\lambda=1$, $\Box A_\mu=0$, $\mathcal{L}=-\frac12(\partial_\nu A_\mu)(\partial^\nu A^\mu)$, 正则动量：$\pi^\mu=\frac{\partial\mathcal{L}}{\partial\dot A_\mu}=-\dot A^\mu(x)$. $\mathcal{H}=-\frac12[\dot A_0^2+(\nabla A_0)^2]+\frac12[\dot{\mathbf{A}}^2+(\partial_i\mathbf{A})^2]$.

矢量场算符：$A_\mu(x)=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{k}|}}\sum_\lambda(a^\lambda_\mathbf{k}\epsilon^\lambda_\mu(k)e^{-ik\cdot x}+a^{\lambda\dagger}_\mathbf{k}\epsilon^{\lambda*}_\mu(k)e^{ik\cdot x})$. $[a^\lambda(k),a^{\lambda'\dagger}(p)]=-g^{\lambda\lambda'}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})$, $[A^\mu(t,\mathbf{x}),\pi^\nu(t,\mathbf{y})]=ig^{\mu\nu}\delta^3(\mathbf{x}-\mathbf{y})$, others are zero. $\mathbf{k}$ 沿 z 轴，$k^\mu=(|\mathbf{k}|,0,0,k)$, $\mathbf{k}\cdot\boldsymbol{\epsilon}=0$（物理极化方向与波矢正交），$\partial\cdot A\neq0\Longrightarrow k\cdot\epsilon^{(3)}\neq0,k\cdot\epsilon^{(0)}\neq0$（非物理极化方向不正交）. $\epsilon^{(0)\mu}(\mathbf{k})\equiv n^\mu=(1,\mathbf{0})$, $\epsilon^{(3)\mu}(\mathbf{k})=(0,\mathbf{k})=\frac{k^\mu-(k\cdot n)n^\mu}{k\cdot n}\to\frac{k^\mu-(k\cdot n)n^\mu}{\sqrt{(k\cdot n)^2-k^2}}$(off-shell). $\epsilon^\mu_R=\frac{1}{\sqrt2}(0,1,i,0)$, $\epsilon^\mu_L=\frac{1}{\sqrt2}(0,1,-i,0)$. 正交性：$\epsilon^{(\lambda)}(k)\cdot\epsilon^{(\lambda')}(k)=g^{\lambda\lambda'}$, 完备性：$\sum_{\lambda=0}^3g_{\lambda\lambda}\epsilon^{(\lambda)}_\mu\epsilon^{(\lambda)*}_\nu=g_{\mu\nu}$（$\lambda$ 不求和）. $H=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{\lambda=0}^3|\mathbf{k}|(-g_{\lambda\lambda})a^{(\lambda)\dagger}_\mathbf{k}a^{(\lambda)}_\mathbf{k}+constant$ 真空：$a^{(\lambda)}_\mathbf{k}|0\rangle=0$, 单光子态：$|\mathbf{k},\lambda\rangle=a^{(\lambda)\dagger}_\mathbf{k}|0\rangle$. Number 算符：$N^{(\lambda)}(\mathbf{k})=-g_{\lambda\lambda}a^{(\lambda)\dagger}_\mathbf{k}a^{(\lambda)}_\mathbf{k}$. 负模态：$|1\rangle=\int\mathrm{d}^3kf(\mathbf{k})a^{(\lambda)\dagger}_\mathbf{k}|0\rangle$,

$\langle 1|1\rangle = \int \mathrm{d}^3k\mathrm{d}^3k' f(k)f(k')\langle 0|[a_{\mathbf{k}}^{(\lambda)}, a_{\mathbf{k}'}^{(\lambda')\dagger}]|0\rangle = -g_{\lambda\lambda}\langle 0|0\rangle\int \mathrm{d}^3k|f(\mathbf{k})|^2$，$\lambda = 0$ 时出现负模态. G-B 方案：初末态要求允许的态：$\partial_\mu A^{(+)}(x)|\psi\rangle = 0 = \langle\psi|\partial_\mu^T A^{\mu(-)}(x) \Longrightarrow (a^{(3)}(\mathbf{k}) - a^{(0)}(\mathbf{k}))|\psi\rangle = 0 \Longrightarrow k\cdot\epsilon^{(0)} = k\cdot n = |\mathbf{k}|, k\cdot\epsilon^{(3)} = -\frac{(k\cdot n)^2}{k\cdot n} = -|\mathbf{k}|$. **矢量场传播子**：$D_F^{\mu\nu}(x-y) \equiv \langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle$. $\langle 0|A^\mu(x)A^\nu(y)|0\rangle = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2|\mathbf{p}|}e^{-ip\cdot(x-y)}(-g^{\mu\nu})$. $D_F^{\mu\nu}(x-y) = -g^{\mu\nu}\{\theta(x^0-y^0)\int\frac{\mathrm{d}^3p}{(2\pi)^3 2|\mathbf{p}|}e^{-ip\cdot(x-y)} + \theta(y^0-x^0)\int\frac{\mathrm{d}^3p}{(2\pi)^3|\mathbf{p}|}e^{ip\cdot(x-y)}\} = -g_{\mu\nu}D_F(x-y)$, $\tilde{D}^{\mu\nu}(k) = (-g^{\mu\nu})\frac{i}{k^2+i\epsilon}$.

S 矩阵元: $S_{\beta\alpha} = \langle\beta_{out}|\alpha_{in}\rangle_{Heisenberg}$, $S_{fi} = \langle f|\psi(\infty)\rangle = \langle f|U(\infty,-\infty)|i\rangle = \langle f|S_I|i\rangle$, interaction picture: $i\frac{\mathrm{d}}{\mathrm{d}t}U(t_f,t_i) = H_I(t_f)U(t_f,t_i)$, $|\psi(t)\rangle_I = |i\rangle + (-i)\int_{-\infty}^t \mathrm{d}t_1 H_I(t_1)|\psi(t_1)\rangle_I = |i\rangle + (-i)\int_{-\infty}^t \mathrm{d}t_1 H_I(t_1)(|i\rangle + (-i)\int_{-\infty}^{t_1}\mathrm{d}t_2 H_I(t_2)|\psi(t_2)\rangle_I)$, $S = \sum_{n=0}^\infty (-i)^n\int_{-\infty}^t \mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2\cdots\int_{-\infty}^{t_{n-1}}\mathrm{d}t_n H_I(t_1)H_I(t_2)\cdots H_I(t_n)$, $\int_{t_0}^t\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2 H_I(t_1)H_I(t_2)$, $t_1 > t_2 = \int_{t_0}^t \mathrm{d}t_2\int_{t_0}^{t_2}\mathrm{d}t_1 H_I(t_2)H_I(t_1)$, $t_2 > t_1 = \frac{1}{2}\int_{t_0}^t\mathrm{d}t_1\int_{t_0}^t\mathrm{d}t_2 T\{H_I(t_1)H_I(t_2)\}$, $S = \sum_{n=0}^\infty\frac{(-i)^n}{2^n}\int_{-\infty}^t\mathrm{d}t_1\int_{-\infty}^t\mathrm{d}t_2\cdots\int_{-\infty}^t\mathrm{d}t_n T\{H_I(t_1)H_I(t_2)\cdots H_I(t_n)\}|_{t=\infty} = Te^{-i\int_{-\infty}^\infty \mathrm{d}tH_I(t)} = Te^{i\int_{-\infty}^\infty \mathrm{d}^4x\mathcal{L}_I(x)}$.

LIPS: $\mathrm{d}\Phi_{N_\beta} = \prod_{f=1}^{N_\beta}\frac{\mathrm{d}^3p_f}{(2\pi)^3 2E_{\mathbf{p}}}\delta^4(\sum_{N_\alpha}p_\alpha - \sum_{N_\beta}p_\beta)$. $\mathrm{d}\Phi = \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 2E_{\mathbf{p_1}}2E_{\mathbf{p_2}}}\delta^4(\sum_{N_\alpha}p_\alpha - \sum_{N_\beta}p_\beta) = \frac{p_1^2\mathrm{d}p_1\mathrm{d}\Omega}{(2\pi)^6 2E_1 2E_2}\delta(M-E_1-E_2)$ and $\delta(M-E_1-E_2) = \left|\frac{\mathrm{d}(M-E_1-E_2)}{\mathrm{d}p_1}\right|_{p_1=p_0}^{-1}\delta(p_1-p_0) = \frac{E_1E_2}{p_0(E_1+E_2)}\delta(p_1-p_0)$ where p_0 is the solution of $f(p_1) = M - E_1 - E_2 = 0$ $\mathrm{d}\Phi = \frac{p_0^2\mathrm{d}\Omega}{(2\pi)^6 2E_1 2E_2}\frac{E_1E_2}{p_0(E_1+E_2)} = \frac{p_0}{(2\pi)^3 4M}\mathrm{d}\Omega$

Wick 定理：$\overbracket{\phi(x)|\mathbf{p}}\rangle = e^{-ip\cdot x}$，$\langle\overbracket{\mathbf{p}|\phi}(x) = e^{ip\cdot x}$. $\overbracket{\psi(x)|\mathbf{p}},s\rangle = e^{-ip\cdot x}u^s(p)|0\rangle$，$\langle\overbracket{\mathbf{p},s|\bar{\psi}}(x) = \langle 0|\bar{u}^s(p)e^{ip\cdot x}$. Fermion: $N(\psi_1\overbracket{\psi_2\bar{\psi}_3}\bar{\psi}_4) = -\overbracket{\psi_1\bar{\psi}_3}N(\psi_2\bar{\psi}_4)$ 以 Yukawa 为例：$\langle\mathbf{p}'\mathbf{k}'|(\bar{\psi}\psi)_x(\bar{\psi}\psi)_y|\mathbf{pk}\rangle \sim \langle 0|a_{\mathbf{p}'}a_{\mathbf{k}'}(\bar{\psi}\psi)_x(\bar{\psi}\psi)_y a_{\mathbf{p}}^\dagger a_{\mathbf{k}}^\dagger|0\rangle$ move to $+\langle 0|a_{\mathbf{k}'}a_{\mathbf{p}'}\bar{\psi}_y\bar{\psi}_x\psi_x\psi_y a_{\mathbf{p}}^\dagger a_{\mathbf{k}}^\dagger|0\rangle$ and $\langle\mathbf{p}'\mathbf{k}'|(\bar{\psi}\psi)_x(\bar{\psi}\psi)_y|\mathbf{pk}\rangle$ with a minus sign.

费曼规则：Scalar: in: $\overbracket{\phi|\mathbf{p}}\rangle = 1$ out: $\langle\overbracket{\mathbf{p}|\phi} = 1$. Fermion: in: $\overbracket{\psi|\mathbf{p}},s\rangle = u^s(p)$ out: $\langle\overbracket{\mathbf{p},s|\bar{\psi}} = \bar{u}^s(p)$ Anti-fermion: in: $\overbracket{\psi|\mathbf{p}},s\rangle = \bar{v}^s(p)$ out: $\langle\overbracket{\mathbf{p},s|\bar{\psi}} = v^s(p)$ Photon: in: $\overbracket{A_\mu|\mathbf{p}}\rangle = \epsilon_\mu(p)$ out: $\langle\overbracket{\mathbf{p}|A_\mu} = \epsilon_\mu^*(p)$.

Trace Technology: $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$, $\mathrm{tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] = 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta})$, $\mathrm{tr}\{\gamma^5\} = 0$, $\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\} = 0$, $\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = -4i\epsilon^{\mu\nu\rho\sigma}$, $\gamma^\mu\gamma_\mu = 4$, $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$, $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}$, $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$, $\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu - i\epsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma^5$, trace of γ^5 times a product of an odd number of γ^μ is still zero, $\mathrm{tr}(\gamma^{\mu 1}\ldots\gamma^{\mu n}) = \mathrm{tr}(\gamma^{\mu n}\ldots\gamma^{\mu 1})$, $\not{a}\not{b} = a\cdot b - ia_\mu\sigma^{\mu\nu}b_\nu$, $\not{a}\not{a} = a^\mu a^\nu\gamma_\mu\gamma_\nu = \frac{1}{2}a^\mu a^\nu(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu}a^\mu a^\nu = a^2$, $\mathrm{tr}(\not{a}\not{b}) = 4(a\cdot b)$, $\mathrm{tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$, $\mathrm{tr}(\gamma_5\not{a}\not{b}\not{c}\not{d}) = -4i\epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma$, $\gamma_\mu\not{a}\gamma^\mu = -2\not{a}$, $\gamma_\mu\not{a}\not{b}\gamma^\mu = 4a\cdot b$, $\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$, $(\bar{v}\gamma^\mu u)^* = \bar{u}\gamma^\mu v$, $\bar{u}u = \mathrm{tr}\{u\bar{u}\}$.

两体散射：$e^-\mu^- \to e^-\mu^-$：$i\mathcal{M} = (ie^2)\bar{u}^s(p)\gamma^\mu u^{s'}(p')\frac{g_{\mu\nu}}{q^2}\bar{u}^r(k)\gamma^\nu u^r(k')$, $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|_{CM} = \frac{1}{2E_p2E_k|v_p-v_k|}\frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$, 质量全等：$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2E_{CM}^2}$.

$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\mathrm{tr}\{(\not{p}+m_e)\gamma^\mu(\not{p}'+m_e)\gamma^\nu\}\mathrm{tr}\{(\not{k}+m_\mu)\gamma_\mu(\not{k}'+m_\mu)\gamma_\nu\} = \frac{e^4}{4q^4}[4(p^\mu p'^\nu + p'^\mu p^\nu - p\cdot p'g^{\mu\nu}) + 4m_e^2g^{\mu\nu}][4(k^\mu k'^\nu + k'^\mu k^\nu - k\cdot k'g_{\mu\nu}) + 4m_\mu^2g_{\mu\nu}] = \frac{4e^4}{q^4}[p^\mu p'^\nu + p'^\mu p^\nu - p\cdot p'g^{\mu\nu} + m_e^2g^{\mu\nu}][k^\mu k'^\nu + k'^\mu k^\nu - k\cdot k'g_{\mu\nu} + m_\mu^2g_{\mu\nu}] = \frac{4e^4}{q^4}[p^\mu p'^\nu + p'^\mu p^\nu - p\cdot p'g^{\mu\nu}][k^\mu k'^\nu + k'^\mu k^\nu - k\cdot k'g_{\mu\nu} + m_\mu^2g_{\mu\nu}] = \frac{4e^4}{q^4}[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) - (p\cdot p')(k\cdot k') + m_\mu^2(p\cdot p') + (p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) - (p'\cdot p)(k\cdot k') + m_\mu^2(p'\cdot p) - (p\cdot p')(k\cdot k') - (p\cdot p')(k\cdot k') + 4(p\cdot p')(k\cdot k') - 4m_\mu^2(p\cdot p')] = \frac{8e^4}{q^4}[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) - m_\mu^2(p\cdot p')]$ $p = (\omega,\omega\hat{z}), k = (E_k,-\omega\hat{z}), p' = (\omega,-\omega\sin\theta,0,-\omega\cos\theta), k' = (E_k,\omega\sin\theta,0,\omega\cos\theta)$, $p\cdot k = \omega E_k + \omega^2, p'\cdot k' = \omega(\omega+E_k), E_k^2 = \omega^2 + m_\mu^2, p\cdot k' = \omega E_k - \omega^2\cos\theta, p'\cdot k = \omega E_k - \omega^2\cos\theta, p\cdot p' = \omega^2(1+\cos\theta), q^2 = (p'-p)^2 = 2\omega^2(1-\cos\theta)$

$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|_{CM} = \frac{1}{2(\omega+E_k)^2}\frac{\alpha^2}{\omega^2(1-\cos\theta)^2}[(E_k+\omega)^2 + (E_k-\omega\cos\theta)^2 - m_\mu^2(1+\cos\theta)] \xrightarrow{\text{high energy limit}} \frac{1}{2E_{CM}^2}\frac{\alpha^2}{(1-\cos\theta)^2}[4+(1-\cos\theta)^2]$

Compton 散射 $(pk\to p'k')$：$p^2 = p'^2 = m^2, k^2 = k'^2 = 0$, $(p+k)^2 - m^2 = 2p\cdot k, (p-k')^2 - m^2 = -2p\cdot k', (\not{p}+m)\gamma^\nu u(p) = (2p^\nu - \gamma^\nu\not{p} + m\gamma^\nu)u(p) = 2p^\nu u(p)$.

$i\mathcal{M} = \bar{u}(p')(-ie\gamma^\mu)\epsilon_\mu^*(k')\frac{i(\not{q}+m)}{q^2-m^2}(-ie\gamma^\nu)u(p)\epsilon_\nu(k) + \bar{u}(p')(-ie\gamma^\nu)\epsilon_\nu(k)\frac{i(\not{q}+m)}{q^2-m^2}(-ie\gamma^\mu)u(p)\epsilon_\mu^*(k') = -ie^2\epsilon_\mu^*(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p)$,

$i\mathcal{M} = -ie^2\epsilon_\mu^*(k')\epsilon_\nu(k)\bar{u}(p')[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}]u(p)$

费曼参数化：$\frac{1}{AB} = \int_0^1\mathrm{d}x\frac{1}{[xA+(1-x)B]^2}$，$\frac{1}{AB^n} = \int_0^1\mathrm{d}x\frac{n(1-x)^{n-1}}{[xA+(1-x)B]^{n+1}}$，$\frac{1}{A_1A_2\cdots A_n} = \int_0^1\mathrm{d}x_1\cdots\mathrm{d}x_n\delta(\sum x_i - 1)\frac{(n-1)!}{[x_1A_1+x_2A_2+\cdots x_nA_n]^n}$.

$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{1}{[x(p-k)^2+(1-x)k^2]^2} = \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{1}{[xp^2-2xp\cdot k+k^2]^2} \xrightarrow{k\to k+xp} = \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{1}{[xp^2-2xp\cdot(k+xp)+(k+xp)^2]^2} = \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{1}{[k^2+x(1-x)p^2+i\epsilon]^2}$. Wick rotation: $k^0\to ik_E^0, \mathbf{k} = \mathbf{k_E}, k^2 = -k_E^2$. $\Delta \equiv -x(1-x)p^2 - i\epsilon$. $\int_0^1\mathrm{d}x\frac{-x(1-x)p^2-i\epsilon}{\Lambda} = \frac{p^2}{3\Lambda} - \frac{p^2}{2\Lambda} - \frac{i\epsilon}{\Lambda}, \int_0^1\mathrm{d}x\ln(-x(1-x)p^2 - i\epsilon) = -2 + \ln(p^2) + i\pi$

维数正规化：Replace the dimension with d: $\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{1}{[k_E^2+\Delta]^2}$, $\int\mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, $\int_0^1\mathrm{d}xx^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$,

$\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{1}{[k_E^2+\Delta]^2} = \int\frac{\mathrm{d}\Omega_d}{(2\pi)^d}\mathrm{d}k_E\frac{k_E^{d-1}}{[k_E^2+\Delta]^2} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty\mathrm{d}k_E\frac{k_E^{d/2-1}}{[k_E+\Delta]^2} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_1^0\mathrm{d}l\frac{-\Delta}{l^2}\frac{l^2}{\Delta^2}(\Delta\frac{1-l}{l})^{d/2-1} = \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\Gamma(2)}\Delta^{d/2-2} \xrightarrow{d\to 4} \frac{\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)}{(4\pi)^2}(1 - \frac{1}{2}\ln\frac{\Delta}{4\pi}\epsilon + \mathcal{O}(\epsilon^2)) = \frac{1}{(4\pi)^2}(\frac{2}{\epsilon} - \gamma - \ln\Delta + \ln 4\pi + \mathcal{O}(\epsilon))$ where $l = \Delta/(k_E+\Delta)$, $\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$.