Homework: Quantum Field Theory #3

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11.1

(a). $(\gamma^5)^2 = \mathbb{1}$

$$\begin{aligned}
(\gamma^5)^2 &= \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= (-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= (-1)^3\gamma^0\gamma^0(-1)^2\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= (-1)^3\gamma^0\gamma^0(-1)^2\gamma^1\gamma^1(-1)\gamma^2\gamma^2\gamma^3\gamma^3 \\
&= \mathbb{1}
\end{aligned}$$

(b). $\gamma_\mu \not{p}\gamma^\mu = -2\not{p}$

$$\begin{aligned}
\gamma_\mu \not{p}\gamma^\mu &= g_{\mu\alpha}\gamma^\alpha\gamma^\nu p_\nu\gamma^\mu \\
&= (2g^{\alpha\nu} - \gamma^\nu\gamma^\alpha)\gamma^\mu g_{\mu\alpha}p_\nu \\
&= 2\gamma^\nu p_\nu - \gamma^\nu\gamma_\mu\gamma^\mu p_\nu \\
&= -2\not{p}
\end{aligned}$$

(c). $\gamma_\mu \not{p}\not{q}\not{p}\gamma^\mu = -2\not{p}\not{q}\not{p}$

$$\begin{aligned}
\gamma_\mu \not{p}\not{q}\not{p}\gamma^\mu &= \gamma_\mu\gamma^\nu p_\nu\gamma^\alpha q_\alpha\gamma^\beta p_\beta\gamma^\mu \\
&= g_{\mu\tau}\gamma^\tau\gamma^\nu p_\nu\gamma^\alpha q_\alpha\gamma^\beta p_\beta\gamma^\mu \\
&= (2g^{\nu\tau} - \gamma^\nu\gamma^\tau)g_{\mu\tau}\gamma^\alpha\gamma^\beta\gamma^\mu p_\nu q_\alpha p_\beta \\
&= 2\gamma^\alpha\gamma^\beta\gamma^\nu p_\nu q_\alpha p_\beta - \gamma^\nu\gamma^\tau\gamma^\alpha\gamma^\beta\gamma^\mu g_{\mu\tau}p_\nu q_\alpha p_\beta \\
&= 2\gamma^\alpha\gamma^\beta\gamma^\nu p_\nu q_\alpha p_\beta - \gamma^\nu(2g^{\alpha\tau} - \gamma^\alpha\gamma^\tau)\gamma^\beta\gamma^\mu g_{\mu\tau}p_\nu q_\alpha p_\beta \\
&= 2\gamma^\alpha\gamma^\beta\gamma^\nu p_\nu q_\alpha p_\beta - 2\gamma^\nu\gamma^\beta\gamma^\alpha p_\nu q_\alpha p_\beta + \gamma^\nu\gamma^\alpha\gamma^\tau\gamma^\beta\gamma^\mu g_{\mu\tau}p_\nu q_\alpha p_\beta \\
&= 2\gamma^\alpha\gamma^\beta\gamma^\nu p_\nu q_\alpha p_\beta - 2\gamma^\nu\gamma^\beta\gamma^\alpha p_\nu q_\alpha p_\beta + \gamma^\nu\gamma^\alpha(2g^{\tau\beta} - \gamma^\beta\gamma^\tau)\gamma^\mu g_{\mu\tau}p_\nu q_\alpha p_\beta \\
&= 2\gamma^\alpha\gamma^\beta\gamma^\nu p_\nu q_\alpha p_\beta - 2\gamma^\nu\gamma^\beta\gamma^\alpha p_\nu q_\alpha p_\beta + 2\gamma^\nu\gamma^\alpha\gamma^\beta p_\nu q_\alpha p_\beta - 4\gamma^\nu\gamma^\alpha\gamma^\beta p_\nu q_\alpha p_\beta \\
&= -2\not{p}\not{q}\not{p}
\end{aligned}$$

(d). $\left\{\gamma^5, \gamma^\mu\right\} = 0$

$$\begin{aligned}
\gamma^5\gamma^0 &= (-1)^3\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5 \\
\gamma^5\gamma^1 &= (-1)^3\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^5 \\
\gamma^5\gamma^2 &= (-1)^3\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma^5
\end{aligned}$$

$$\gamma^5\gamma^3 = (-1)^3\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^3\gamma^5$$
$$\Longrightarrow \left\{\gamma^5, \gamma^\mu\right\} = 0$$

(e). $\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] = 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta})$

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}[2g^{\mu\nu}\cdot\mathbb{1} - \gamma^\nu\gamma^\mu]\\ &= 8g^{\mu\nu} - \mathrm{Tr}[\gamma^\mu\gamma^\nu]\\ &= 4g^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] &= \mathrm{Tr}\left[(2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha)\gamma^\beta\gamma^\nu\right]\\ &= \mathrm{Tr}\left[2g^{\alpha\mu}\gamma^\beta\gamma^\nu - 2g^{\alpha\beta}\gamma^\mu\gamma^\nu + 2g^{\alpha\nu}\gamma^\mu\gamma^\beta - \gamma^\mu\gamma^\beta\gamma^\nu\gamma^\alpha\right]\\ &= g^{\alpha\mu}\,\mathrm{Tr}\left[\gamma^\beta\gamma^\nu\right] - g^{\gamma\beta}\,\mathrm{Tr}[\gamma^\mu\gamma^\nu] + g^{\alpha\nu}\,\mathrm{Tr}\left[\gamma^\mu\gamma^\beta\right]\\ &= 4(g^{\alpha\mu}g^{\beta\nu} - g^{\gamma\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta})\end{aligned}$$

2. Spinor identity:

(a). Show that $\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m$ and $\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$.

$$\begin{aligned}\sum_s u_s(p)\bar{u}_s(p) &= \sum_s \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}\begin{pmatrix}\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \xi^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\\ &= \sum_s \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\end{aligned}$$

(Use $\sum_s \xi^s\xi^{s\dagger} = \mathbb{1}$)

$$\begin{aligned}&= \begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar{\sigma} & m\end{pmatrix}\\ &= \not{p} + m\end{aligned}$$

Similarly, $\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$.

(b). Show that $\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2\delta_{\sigma\sigma'}p^\mu$.

$$\begin{aligned}&\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2\delta_{\sigma\sigma'}p^\mu\\ \Longrightarrow &p_\mu\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2\delta_{\sigma\sigma'}p_\mu p^\mu\end{aligned}$$

From the dirac equation for $\bar{u}$, we have

$$\bar{u}_\sigma\gamma^\mu p_\mu = m\bar{u}_\sigma$$

So

$$\begin{aligned}&p_\mu\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2\delta_{\sigma\sigma'}p_\mu p^\mu\\ \Longrightarrow &m\bar{u}_\sigma u_{\sigma'} = 2\delta_{\sigma\sigma'}p_\mu p^\mu\\ \Longrightarrow &2m^2\xi_\sigma^\dagger\xi_{\sigma'} = 2\delta_{\sigma\sigma'}m^2\\ \Longrightarrow &2\delta_{\sigma\sigma'}m^2 = 2\delta_{\sigma\sigma'}m^2\end{aligned}$$

More strict prove can be done by involving Gordon identity, which is shown in the last of problem **11.4**.

We can also compute only the third component of p^i, then do a coordinate transformation to the actual p^μ with all 4 component.

$$
\begin{aligned}
\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) &= \begin{pmatrix}\xi^{\sigma\dagger}\sqrt{p\cdot\bar{\sigma}} & \xi^{\sigma\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\gamma^\mu\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^{\sigma'}\\ \sqrt{p\cdot\bar{\sigma}}\xi^{\sigma'}\end{pmatrix}\\
&= \begin{pmatrix}\xi^{\sigma\dagger}\sqrt{p\cdot\bar{\sigma}} & \xi^{\sigma\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^{\sigma'}\\ \sqrt{p\cdot\bar{\sigma}}\xi^{\sigma'}\end{pmatrix}\\
&= \begin{pmatrix}\xi^{\sigma\dagger}\sqrt{p\cdot\sigma}\bar{\sigma}^\mu & \xi^{\sigma\dagger}\sqrt{p\cdot\bar{\sigma}}\sigma^\mu\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^{\sigma'}\\ \sqrt{p\cdot\bar{\sigma}}\xi^{\sigma'}\end{pmatrix}\\
&= \xi^{\sigma\dagger}\sqrt{p\cdot\sigma}\bar{\sigma}^\mu\sqrt{p\cdot\sigma}\xi^{\sigma'} + \xi^{\sigma\dagger}\sqrt{p\cdot\bar{\sigma}}\sigma^\mu\sqrt{p\cdot\bar{\sigma}}\xi^{\sigma'}
\end{aligned}
$$

Insert $p_\mu = \left(E, 0, 0, -p^3\right)$ condition

$$
\begin{aligned}
&= \xi^{\sigma\dagger}\begin{pmatrix}\sqrt{E-p_3} & \\ & \sqrt{E+p_3}\end{pmatrix}\bar{\sigma}^\mu\begin{pmatrix}\sqrt{E-p_3} & \\ & \sqrt{E+p_3}\end{pmatrix}\xi^{\sigma'}\\
&\quad + \xi^{\sigma\dagger}\begin{pmatrix}\sqrt{E+p_3} & \\ & \sqrt{E-p_3}\end{pmatrix}\sigma^\mu\begin{pmatrix}\sqrt{E+p_3} & \\ & \sqrt{E-p_3}\end{pmatrix}\xi^{\sigma'}
\end{aligned}
$$

if only p_3 component exists, all component of μ but 3 are zero.

$$
\begin{aligned}
&= \xi^{\sigma\dagger}\begin{pmatrix}\sqrt{E-p_3} & \\ & \sqrt{E+p_3}\end{pmatrix}\begin{pmatrix}-1 & \\ & 1\end{pmatrix}\begin{pmatrix}\sqrt{E-p_3} & \\ & \sqrt{E+p_3}\end{pmatrix}\xi^{\sigma'}\\
&\quad + \xi^{\sigma\dagger}\begin{pmatrix}\sqrt{E+p_3} & \\ & \sqrt{E-p_3}\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{pmatrix}\sqrt{E+p_3} & \\ & \sqrt{E-p_3}\end{pmatrix}\xi^{\sigma'}\\
&= \xi^{\sigma\dagger}\begin{pmatrix}2p_3 & \\ & 2p_3\end{pmatrix}\xi^{\sigma'}\\
&= 2p_3\xi^{\sigma\dagger}\xi^{\sigma'}\\
&= 2p_3\delta_{\sigma\sigma'}
\end{aligned}
$$

Choose a different coordinate system and we have a set of new p_μ, with gives $\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2\delta_{\sigma\sigma'}p^\mu$.

3.2 Derive the *Gordon identity*

$$
\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{1}
$$

where $q = (p' - p)$.

From the standard covariant form of Dirac equation

$$
(i\gamma^\mu\partial_\mu - m)\psi(x) = 0
$$

and can be written as

$$
\gamma^\mu p_\mu u(p) = mu(p) \tag{2}
$$

From previous definition

$$
\bar{u}(p) \equiv u^\dagger(p)\gamma^0
$$

and

$$u^\dagger(p)p_\mu^\dagger(\gamma^\mu)^\dagger = mu^\dagger(p)$$

So we have

$$\bar{u}(p)\gamma^0 p_\mu^\dagger(\gamma^\mu)^\dagger\gamma^0 = m\bar{u}(p)$$

Then

$$\begin{aligned}\bar{u}(p')\gamma^\mu u(p) &= \frac{\bar{u}(p')\gamma^0 p'^{\dagger}_{\mu'}(\gamma^{\mu'})^\dagger\gamma^0}{m}\gamma^\mu\frac{\gamma^{\mu''}p_{\mu''}u(p)}{m} \\ &= \bar{u}(p')\frac{\gamma^0 p'^{\dagger}_{\mu'}(\gamma^{\mu'})^\dagger\gamma^0\gamma^\mu\gamma^{\mu''}p_{\mu''}}{m^2}u(p)\end{aligned}$$

Note that p_μ and γ commute, and

$$\begin{aligned}\gamma^0(\gamma^\mu)^\dagger\gamma^0 &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ -\sigma^\mu & 0\end{pmatrix}^\dagger\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\ &= \begin{pmatrix}0 & \sigma^\mu\\ -\sigma^\mu & 0\end{pmatrix} \\ &= \gamma^\mu\end{aligned}$$

which means

$$\bar{u}(p)\gamma^\mu p_\mu = m\bar{u}(p)$$

and

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\frac{\gamma^\nu p'_\nu\gamma^\mu\gamma^\nu p_\nu}{m^2}u(p)$$

Now we observe

$$\begin{aligned}i\sigma^{\mu\nu}q_\nu &= -\frac{1}{2}[\gamma^\mu,\gamma^\nu](p'_\nu - p_\nu) \\ &= -\frac{1}{2}(\gamma^\mu\gamma^\nu p'_\nu - \gamma^\nu\gamma^\mu p'_\nu - \gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p_\nu)\end{aligned}$$

and

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$$

We have

$$\begin{aligned}i\sigma^{\mu\nu}q_\nu &= -\frac{1}{2}(2\gamma^\mu\gamma^\nu p'_\nu - 2g^{\mu\nu}p'_\nu - 2\gamma^\mu\gamma^\nu p_\nu + 2g^{\mu\nu}p_\nu) \\ &= (p'^\mu - p^\mu) - \gamma^\mu\gamma^\nu(p'_\nu - p_\nu)\end{aligned}$$

With this (1) becomes

$$\begin{aligned}\bar{u}(p')\gamma^\mu u(p) &= \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{(p'^\mu - p^\mu) - \gamma^\mu\gamma^\nu(p'_\nu - p_\nu)}{2m}\right]u(p) \\ &= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{\gamma^\mu\gamma^\nu(p'_\nu - p_\nu)}{2m}\right]u(p) \\ &= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{\gamma^\mu\gamma^\nu(p'_\nu - p_\nu)}{2m}\right]u(p)\end{aligned}$$

We know that

$$\begin{aligned}
\bar{u}(p')\frac{\gamma^\nu p'_\nu \gamma^\mu \gamma^\nu p_\nu}{m^2}u(p) &= \frac{1}{2}\left\{\bar{u}(p')\frac{-\gamma^\nu p'_\nu \gamma^\nu \gamma^\mu p_\nu + 2\gamma^\nu p'_\nu g^{\mu\nu} p_\nu - \gamma^\mu p'_\nu \gamma^\nu \gamma^\nu p_\nu + 2p'_\nu g^{\mu\nu}\gamma^\nu p_\nu}{m^2}u(p)\right\} \\
&= \frac{1}{2}\left\{\bar{u}(p')\frac{-m\gamma^\nu \gamma^\mu p_\nu + 2\gamma^\nu p'_\nu g^{\mu\nu} p_\nu - \gamma^\mu p'_\nu \gamma^\nu m + 2p'_\nu g^{\mu\nu}\gamma^\nu p_\nu}{m^2}u(p)\right\} \\
&= \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{m} - \frac{\gamma^\nu \gamma^\mu p_\nu + \gamma^\mu p'_\nu \gamma^\nu}{2m}\right]u(p) \\
&= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{-\gamma^\mu \gamma^\nu p'_\nu + \gamma^\mu p'_\nu \gamma^\nu}{2m}\right]u(p) \\
&= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{\gamma^\mu \gamma^\nu (p'_\nu - p_\nu)}{2m}\right]u(p)
\end{aligned}$$

And it consists with the former one.

From Gorden identity $\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$, we can derive $(p' = p)$

$$\begin{aligned}
\bar{u}(p)\gamma^\mu u(p) &= \bar{u}(p)\frac{p^\mu}{m}u(p) \\
&= 2\delta_{\sigma\sigma'}p^\mu
\end{aligned}$$