Homework: Quantum Field Theory #6

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1. Derive equal-time commutation relations $[A^i(x), \pi^j(y)]$ and $[A_\mu(x), A_\nu(y)]$.

The quantized Proca field is

$$A_\mu(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_\lambda [a^\lambda_\mathbf{p} \epsilon^\lambda_\mu(p) e^{-ip\cdot x} + a^{\lambda\dagger}_\mathbf{p} \epsilon^{\lambda *}_\mu(p) e^{ip\cdot x}]$$

where λ can only be $1, 2, 3$. Thus

$$\pi_i(x) = -\dot{A}_i - \partial_i A_0 = i\int \frac{\mathrm{d}^3p}{(2\pi)^3} \sqrt{\frac{E_\mathbf{p}}{2}} \sum_\lambda [a^\lambda_\mathbf{p} \epsilon^\lambda_i(p) e^{-ip\cdot x} - a^{\lambda\dagger}_\mathbf{p} \epsilon^{\lambda *}_i(p) e^{ip\cdot x}] - i\int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{p_i}{\sqrt{2E_\mathbf{p}}} \sum_\lambda [a^\lambda_\mathbf{p} \epsilon^\lambda_0(p) e^{-ip\cdot x} - a^{\lambda\dagger}_\mathbf{p} \epsilon^{\lambda *}_0(p) e^{ip\cdot x}]$$

And naturally

$$[A^i(x), \pi^j(y)] = i\int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_{\lambda,\lambda'} \left\{ \sqrt{\frac{E_\mathbf{k}}{4E_\mathbf{p}}} (-2) [a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_\mathbf{k}] \epsilon^\lambda_i(p) \epsilon^{\lambda' *}_j(k) e^{-ip\cdot x} e^{ik\cdot y} \right.$$
$$\left. + \frac{k_j}{2\sqrt{E_\mathbf{p} E_\mathbf{k}}} [a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_\mathbf{k}] \epsilon^\lambda_i(p) e^{-ip\cdot x} \epsilon^{\lambda' *}_0(k) e^{ik\cdot y} + \frac{k_j}{2\sqrt{E_\mathbf{p} E_\mathbf{k}}} [a^{\lambda'}_\mathbf{k}, a^{\lambda\dagger}_\mathbf{p}] \epsilon^{\lambda'}_0(k) e^{-ik\cdot y} \epsilon^{\lambda *}_i(p) e^{ip\cdot x} \right\}$$

we have $[a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_\mathbf{k}] = (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k}) \delta^{\lambda\lambda'}$ and $\sum_\lambda \epsilon^\lambda_\mu \epsilon^{\lambda *}_\nu = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$

$$= i\int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_\lambda \left\{ -\epsilon^\lambda_i(p) \epsilon^{\lambda *}_j(p) e^{-ip\cdot(x-y)} + \frac{p_j}{2E_\mathbf{p}} [\epsilon^\lambda_i(p) \epsilon^{\lambda *}_0(p) e^{-ip\cdot(x-y)} + \epsilon^\lambda_0(p) \epsilon^{\lambda *}_i(p) e^{ip\cdot(x-y)}] \right\}$$
$$= i\int \frac{\mathrm{d}^3p}{(2\pi)^3} \left\{ g_{ij} - \frac{p_i p_j}{m^2} + \frac{p_j}{2E_\mathbf{p}} [\delta_{i0} + \frac{p_i p_0}{m^2} - \delta_{i0} + \frac{p_i p_0}{m^2}] \right\} e^{-ip\cdot(x-y)}$$
$$= i\int \frac{\mathrm{d}^3p}{(2\pi)^3} \left\{ g_{ij} - \frac{p_i p_j}{m^2} [1 - \frac{p_0}{E_\mathbf{p}}] \right\}$$
$$= -i\delta^{ij} \delta^3(\mathbf{x}-\mathbf{y})$$

Now

$$[A_\mu(x), A_\nu(y)] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\sqrt{4E_\mathbf{p} E_\mathbf{k}}} \sum_{\lambda,\lambda'} \left\{ [a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_\mathbf{k}] \epsilon^\lambda_\mu(p) e^{-ip\cdot x} \epsilon^{\lambda' *}_\nu(k) e^{ik\cdot y} - [a^{\lambda'}_\mathbf{k}, a^{\lambda\dagger}_\mathbf{p}] \epsilon^{\lambda'}_\nu(k) e^{-ik\cdot y} \epsilon^{\lambda *}_\mu(k) e^{ip\cdot x} \right\}$$
$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_\lambda \left\{ \epsilon^\lambda_\mu(p) \epsilon^{\lambda *}_\nu(p) e^{-ip\cdot(x-y)} - \epsilon^\lambda_\nu(p) \epsilon^{\lambda *}_\mu(p) e^{ip\cdot(x-y)} \right\}$$
$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left\{ (-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}) e^{-ip\cdot(x-y)} - (-g_{\nu\mu} + \frac{p_\nu p_\mu}{m^2}) e^{ip\cdot(x-y)} \right\}$$

set $\Delta(x-y) = [\phi(x), \phi(y)]$

$$= [-g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2}] \Delta(x-y)$$

2. Prove $\theta(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}s \frac{e^{-isx}}{s+i\epsilon} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}s \frac{e^{+isx}}{s-i\epsilon}$.

The Heaviside step function

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Use contour integral (given $x > 0$, the contour is closed below),

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}s \frac{e^{-isx}}{s+i\epsilon} = -\frac{i}{2\pi}2\pi i e^{-\epsilon x} = e^{-\epsilon x} = 1$$

and if $x < 0$, the contour is closed above and therefore equals to 0. Then we have the Heaviside step function.

Similarly, we can perform the same analysis on the other representation.

3. Calculate $\langle 0|T\phi(x)\phi(y)|0\rangle$.

From the definition of time-ordering operator, we have

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)|0\rangle$$

and we take a look at the first term

$$\begin{aligned}
\langle 0|\theta(x^0 - y^0)\phi(x)\phi(y)|0\rangle &= \langle 0|\frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p^0 \frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x})\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}(a_{\mathbf{q}}e^{-iq\cdot y} + a_{\mathbf{q}}^\dagger e^{iq\cdot y})|0\rangle \\
&= \langle 0|\frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p^0 \frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]e^{-ip\cdot x}e^{iq\cdot y}|0\rangle
\end{aligned}$$

we knew that $[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$, so

$$\begin{aligned}
&= \frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p^0 \frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}e^{-ip\cdot(x-y)} \\
&= \frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p^0 \frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-iE_{\mathbf{p}}(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \frac{i}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0\mathrm{d}^3p}{(2\pi)^4}\frac{1}{E_{\mathbf{p}}}\frac{e^{-i(p^0+E_{\mathbf{p}})(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{p^0+i\epsilon}
\end{aligned}$$

make the new $p^0 = (p^0 + E_{\mathbf{p}})$

$$= \frac{i}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0\mathrm{d}^3p}{(2\pi)^4}\frac{e^{-ip^0(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(E_{\mathbf{p}})(p^0 - E_{\mathbf{p}} + i\epsilon)}$$

The next term is similar and then we have the whole propagator

$$\begin{aligned}
\langle 0|T\phi(x)\phi(y)|0\rangle &= \frac{i}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0\mathrm{d}^3p}{(2\pi)^4}\frac{e^{-ip^0(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(E_{\mathbf{p}})(p^0 - E_{\mathbf{p}} + i\epsilon)} - \frac{i}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0\mathrm{d}^3p}{(2\pi)^4}\frac{e^{-ip^0(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(E_{\mathbf{p}})(p^0 + E_{\mathbf{p}} + i\epsilon)} \\
&= \frac{i}{2E_{\mathbf{p}}}\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0\mathrm{d}^3p}{(2\pi)^4}\frac{2E_{\mathbf{p}}e^{-ip\cdot(x-y)}}{p^{0\,2} - E_{\mathbf{p}}^2 + i\epsilon} \\
&= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^{0\,2} - E_{\mathbf{p}}^2 + i\epsilon}e^{-ip\cdot(x-y)}
\end{aligned}$$

and now we have the Klein-Gordon propagator.

4. Calculate $\langle 0|T\psi_a(x)\bar{\psi}_b(x)|0\rangle$.

From before, we knew that

$$\langle 0|T\psi_a(x)\bar{\psi}_b(x)|0\rangle = \langle 0|\theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(x) - \theta(x^0 - y^0)\bar{\psi}_b(x)\psi_a(x)|0\rangle$$

Like before we take a look at the first term

$$\begin{aligned}
\langle 0|\theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(x)|0\rangle &= \frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p^0 \frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s u_a^s(p)\bar{u}_b^s(p)e^{-ip\cdot(x-y)} \\
&= \frac{i}{2\pi}(i\partial\!\!\!/_x + m)_{ab}\int_{-\infty}^{\infty} \mathrm{d}p^0 \frac{e^{-ip^0(x^0-y^0)}}{p^0+i\epsilon}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}
\end{aligned}$$

so

$$\langle 0|T\psi_a(x)\bar{\psi}_b(x)|0\rangle = (i\partial\!\!\!/_x + m)_{ab}\,\langle 0|\theta(x^0-y^0)\phi(x)\phi(y)|0\rangle + (i\partial\!\!\!/_x + m)_{ab}\,\langle 0|\theta(y^0-x^0)\phi(y)\phi(x)|0\rangle$$

and similarly we can derive the propagator.

5. $|\phi\rangle = c_0\,|0\rangle + c_1\,|\phi_1\rangle$

$|\phi_1\rangle = \int \mathrm{d}^3q f(\mathbf{q})[a^{3\dagger}(\mathbf{q}) - a^{0\dagger}(\mathbf{q})]\,|0\rangle$

Calculate $\langle\phi|A_\mu|\phi\rangle = \partial_\mu\Lambda(x)$.

Given the commutation relation

$$[a^\lambda(k), a^{\lambda'\dagger}(p)] = -g^{\lambda\lambda'}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})$$

and the field operator

$$A_\mu(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{k}|}}\sum_\lambda (a^\lambda_{\mathbf{k}}\epsilon^\lambda_\mu(k)e^{-ik\cdot x} + a^{\lambda\dagger}_{\mathbf{k}}\epsilon^{\lambda *}_\mu(k)e^{ik\cdot x})$$

we can see that only terms with the structure of $aa^\dagger$ are non-zero, $aaa^\dagger$ or $a/a^\dagger$ vanishes by applying simple commutation relations, and the rest can be annihilated straight forward.

$$\begin{aligned}
\langle 0|A_\mu|\phi_1\rangle &= c_0c_1\int\frac{\mathrm{d}^3k\mathrm{d}^3q}{(2\pi)^3}f(\mathbf{q})\frac{1}{\sqrt{2|\mathbf{k}|}}\,\langle 0|a^0_{\mathbf{k}}a^{0\dagger}_{\mathbf{q}}\epsilon^0_\mu(k) - a^3_{\mathbf{k}}a^{3\dagger}_{\mathbf{q}}\epsilon^3_\mu(k)|0\rangle\, e^{-ik\cdot x} \\
&= -c_0c_1\int\frac{\mathrm{d}^3k}{\sqrt{2|\mathbf{k}|}}f(\mathbf{k})\,\langle 0|\epsilon^0_\mu(k) + \epsilon^3_\mu(k)|0\rangle\, e^{-ik\cdot x} \\
&= -c_0c_1\int\frac{\mathrm{d}^3k}{\sqrt{2|\mathbf{k}|}}f(\mathbf{k})\,\langle 0|n_\mu + \frac{k_\mu-(k\cdot n)n_\mu}{k\cdot n}|0\rangle\, e^{-ik\cdot x} \\
&= c_0c_1\int\frac{\mathrm{d}^3k}{\sqrt{2|\mathbf{k}|}}f(\mathbf{k})\frac{k_\mu}{k\cdot n}e^{-ik\cdot x} \\
&= \partial_\mu\left\{c_0c_1\int\frac{\mathrm{d}^3k}{\sqrt{2|\mathbf{k}|}}f(\mathbf{k})\frac{1}{k\cdot n}e^{-ik\cdot x}\right\}
\end{aligned}$$

and $\langle\phi_1|A_\mu|0\rangle$ is exactly the complex conjugate of $\langle 0|A_\mu|\phi_1\rangle$.

$$\therefore \Lambda = c_0c_1\int\frac{\mathrm{d}^3k}{\sqrt{2|\mathbf{k}|}}f(\mathbf{k})\frac{1}{k\cdot n}(e^{-ik\cdot x} - e^{ik\cdot x})$$

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