Scalar QED

Yingsheng Huang and Rui Yu

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1 Hydrogen Wavefunction Divergence in Dirac Equation and Schrödinger Equation

1.1 The Dirac part

The Klein-Gordon Hydrogen Equation is

$$((i\partial_0 + \frac{Z\alpha}{r})^2 + \nabla^2 - m^2)\Psi = 0 \tag{1}$$

For the bound state, the eigen value and the wave function are

$$E = m\frac{1}{\sqrt{1+\frac{\alpha^2 Z^2}{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}}} \tag{2}$$

$$\Psi = \frac{c}{\sqrt{4\pi}} e^{-kr} r^{\lambda} \tag{3}$$

where

$$\lambda = -\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2\alpha^2} \qquad c = \sqrt{\frac{(2k)^{2(1+\sqrt{\frac{1}{4}-Z^2\alpha^2})}}{\Gamma(2+2\sqrt{\frac{1}{4}-Z^2\alpha^2}))}} \qquad k = \frac{m}{\sqrt{1+\frac{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}{\alpha^2 Z^2}}} \tag{4}$$

c is the normaliztion factor for $\int d^3r|\Psi|^2 = 1$. For convenience, define

$$\Psi' = \frac{\Psi}{2(mZ\alpha)^{\frac{3}{2}}} \tag{5}$$

Now Ψ' is dimensionless and expand it in α, we get the origin divergence comes from a term

$$-(Z\alpha)^2 \log(mr) \tag{6}$$

the m in log could be interpreted as a subtraction point μ.

1.2 The Schrödinger part

The Hamiltonian is

$$H = H_0 + H_{int} \tag{7}$$

$$H_0 = -\frac{\nabla^2}{2m} - \frac{Z\alpha}{r}, \quad H_{int} = \frac{\nabla^4}{8m^3} + \frac{1}{32m^4}[-\nabla^2, [-\nabla^2, -\frac{Z\alpha}{r}]] \tag{8}$$

The first term of H_{int} is the relativistic kinematic v^2 correction, the second one is the Darwin term.
The H_0 gives the radial wave functions as follows

$$R_{n0} = \frac{2(mZ\alpha)^{\frac{3}{2}}}{n^{\frac{3}{2}}} e^{-\frac{mZ\alpha}{n}r} F(1-n, 2, \frac{2mZ\alpha r}{n}), \quad E_n = -\frac{Z^2\alpha^2 m}{2n^2} \tag{9}$$

$$R_{k0} = \sqrt{\frac{2}{\pi}}(mZ\alpha)^{\frac{3}{2}} k e^{\frac{\pi}{2k}} |\Gamma(1-\frac{i}{k})| e^{-imz\alpha kr} F(1+\frac{i}{k}, 2, 2imZ\alpha kr), \quad E_k = \frac{mZ^2\alpha^2 k^2}{2} \tag{10}$$

Within perturbation theory,$E_1^{(1)} =< \phi|H_{int}|\phi >$, in quantum mechanics, the NLO energy correction is

$$E_1^{(1)} = E_1 Z^2 \alpha^2 \tag{11}$$

The NLO corretion of the bound state wave function is

$$\sum_{n\neq 1} a_{n1}\phi_{n00} + \int dk a_{k1}\phi_{k00} \tag{12}$$

with

$$a_{n1} = \frac{< \phi_{n00}|H_{int}|\phi_{100} >}{E_1 - E_n} \tag{13}$$

the discrete part of (12) is not divergent at $r = 0$. We now focus on the integration part and seperate the relativstic kinematic term and the Darwin term. Since we are only interested in the divergent part, here we give a hard cutoff $\frac{\Lambda}{m}$ as the up-limit of the integration and a also a down-limit λ, with $\lambda >> 1$ (note that the following wave function have been multiplied by $2(mZ\alpha)^{\frac{3}{2}}$)

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk \frac{2Z^2\alpha^2 k^{\frac{3}{2}}}{2\pi(\sqrt{1-\exp(-\frac{2\pi}{k})})}(1 - \frac{2}{1+k^2}\exp\left(-\frac{2\arctan(k)}{k}\right)) e^{\frac{\pi}{2k}} |\Gamma(1-\frac{i}{k})| \tag{14}$$

with the integral region we defined($k >> 1$), it would be OK to expand the integrand in $\frac{1}{k}$ (I havn't prove it yet), then the UV divergent term is

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk (Z\alpha)^2 (\frac{1}{\pi} + \frac{1}{k}) \tag{15}$$

$$\sim (\alpha Z)^2 (\frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right)) \tag{16}$$

The UV divergent part of Darwin term is

$$\Phi^{(1)}(0)_D = -\frac{(Z\alpha)^4}{8\pi}\int_{\lambda}^{\frac{\Lambda}{m}} dkk^2 e^{\frac{\pi}{k}}|\Gamma(1-\frac{i}{k})|^2 \tag{17}$$

with the same trick as (15), the UV divergen part is

$$\Phi^{(1)}(0)_D = -(\alpha Z)^4 \int_{\lambda}^{\frac{\Lambda}{m}} dk \frac{k^2}{8\pi} + \frac{k}{8} + \frac{1}{24}\pi \tag{18}$$

$$\sim -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m}) \tag{19}$$

Now collect all the results we get as follow.
The K-G wave function's origin UV divergence is

$$K - G \;\; UV : -(Z\alpha)^2 \log(mr) \tag{20}$$

The purterbative Schrödinger wave function's origin UV divergence, with a k cutoff $\frac{\Lambda}{m}$, is

$$Kin \;\; UV : (\alpha Z)^2(\frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right)) \tag{21}$$

$$Darwin \;\; UV : -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m}) \tag{22}$$

All the m, under Λ or in a log, can be interpreted as a subtraction point μ.

2 Non-relativistic Scalar QED (NRSQED) Matching

2.1 Feynman Rules

2.1.1 QED

Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi + \Phi_v^* iv \cdot D\Phi_v \tag{23}$$

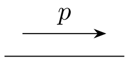
with

$$D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi$$

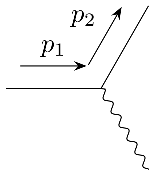
and

$$D_\mu \Phi_v = \partial_\mu \Phi - iZeA_\mu \Phi_v$$

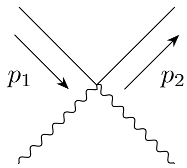
But note that no **A** can appear in actual calculation because here only static scalar potential exists. And the Feynman rules



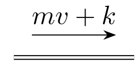
$$= \frac{i}{p^2 - m^2 + i\epsilon}$$



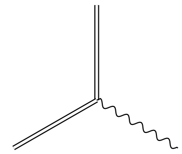
$$= -ie(p_1^\mu + p_2^\mu)$$



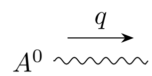
$$= 2ie^2 g^{\mu\nu}$$



$$= \frac{i}{v \cdot k}$$



$$= iZev^\mu$$



$$= \frac{i}{\mathbf{q}^2}$$

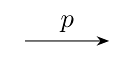
2.1.2 NRSQED

Using the transformation $\phi \to \frac{e^{-imt}}{\sqrt{2m}} \varphi$, we can have the Lagrangian

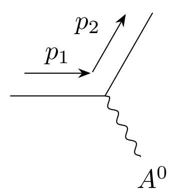
$$\mathcal{L}_{NRSQED} = \varphi^* \left(iD_0 + \frac{\mathbf{D}^2}{2m} \right) \varphi + \delta\mathcal{L} + \Phi_v^* iv \cdot D \Phi_v \tag{24}$$

with the same notation above. Here $\mathbf{D} = \nabla - ie\mathbf{A}$.

Feynman rules are also the same except for the scalar electron side which becomes



$$= \frac{i}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon}$$



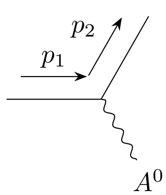
$$= -ie$$

We can ignore all interacting terms involving $\mathbf{A}$.

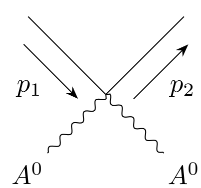
Since we need to match it to $\mathcal{O}(v^2)$ order

$$\delta\mathcal{L} = \frac{(D_0\varphi)^*(D_0\varphi)}{2m} = \frac{\dot{\varphi}^* \dot{\varphi}}{2m} + \frac{e^2 \varphi^* \varphi A_0^2}{2m} - \frac{ie}{2m} A_0 (\varphi^* \dot{\varphi} - \dot{\varphi}^* \varphi) \tag{25}$$

and it changes the Feynman rules to[1]



$$= -ie\left(1 + \frac{E_1 + E_2}{2m}\right)$$



$$= \frac{ie^2}{2m}$$

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Since we rescaled ϕ by $\frac{1}{\sqrt{2m}}$ to get φ, the in/out states are also changed. We must multiply them by $\sqrt{2p^0}$ to compensate that change.

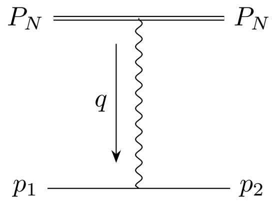
Another way to achieve it is to use the transform rules of heavy scalar effective theory (HSET).

[1] In this note, p^0 is the zero component of relativistic four momentum, and $E = p^0 - m$.

2.2 LO

2.2.1 SQED

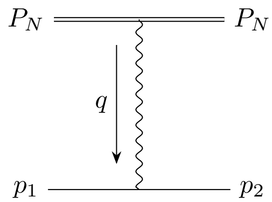
$$i\mathcal{M}^{(0)}_{SQED} =$$



$$= -e^2 v^0 \frac{i(p_1^0 + p_2^0)}{\mathbf{q}^2} = -e^2 v^0 \frac{i}{\mathbf{q}^2}(2m + 2E_1)$$

2.2.2 NRSQED

$$i\mathcal{M}^{(0)}_{NRSQED} =$$

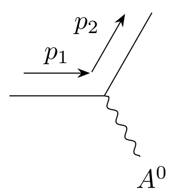


$$= -2\sqrt{p_1^0 p_2^0} e^2 v^0 \frac{i(1+\delta)}{\mathbf{q}^2} = -e^2 v^0 \times 2p_1^0 \frac{i(1+\delta)}{\mathbf{q}^2}$$

which gives

$$\delta = \frac{p_1^0 + p_2^0}{2\sqrt{p_1^0 p_2^0 i}} - 1 \approx \frac{\mathbf{p}_1^4 - 2\mathbf{p}_1^2\mathbf{p}_2^2 + \mathbf{p}_2^4}{32m^4}$$

The new electron-photon vertex is



$$= i + i\frac{\mathbf{p}_1^4 - 2\mathbf{p}_1^2\mathbf{p}_2^2 + \mathbf{p}_2^4}{32m^4}$$

We can write the correction interacting term corresponding to δ as

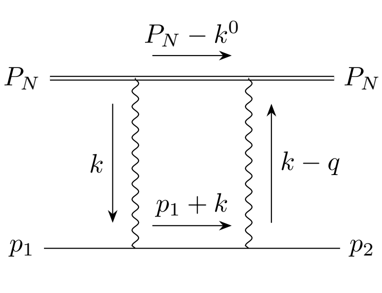
$$\begin{aligned} A_0\varphi\nabla^4\varphi^* - 2A_0\nabla^2\varphi^*\nabla^2\varphi + \varphi^* A_0\nabla^4\varphi &= \varphi^*\nabla^4(A_0\varphi) - 2\varphi^*\nabla^2(A_0\nabla^2\varphi) + \varphi^* A_0\nabla^4\varphi \\ &= \varphi^*(\nabla^2[\nabla^2, A_0] - [\nabla^2, A_0]\nabla^2)\varphi \\ &= \varphi^*[\nabla^2, [\nabla^2, A_0]]\varphi \end{aligned}$$

which is exactly the Darwin term in Holstein's *Advanced Topics in QM* with coefficient $1/32m^4$.

2.3 NLO

2.3.1 SQED

$$i\mathcal{M}^{(1)}_{SQED} =$$



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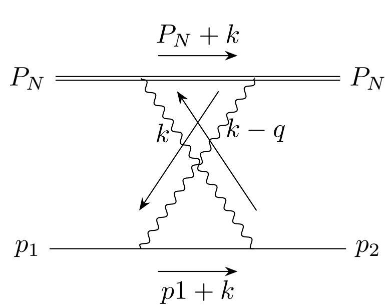

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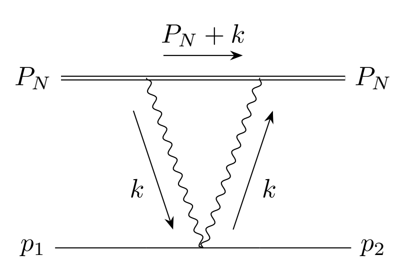

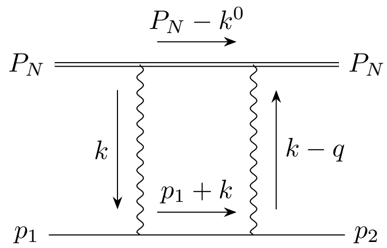

$$= -e^2 v^0 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{\mathbf{k}^2} \; <+content+>$$

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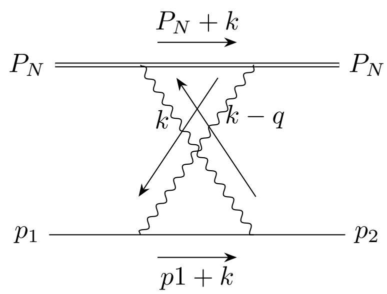

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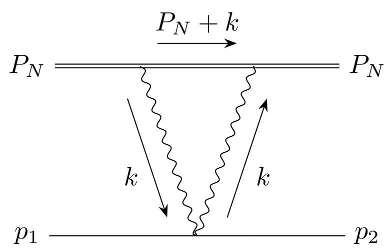

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2.3.2 NRSQED