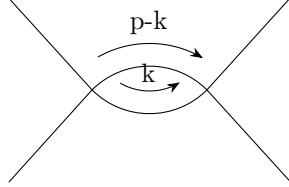


# Homework: Gauge Field Theory #1

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1.  $\phi^4$  theory ( $\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$ ). Verify optical theorem in the lowest order.



$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - m^2 + i\epsilon}$$

For simplicity, we ignore the mass term.

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(p-k)^2}$$

Apply feynnman parameterization

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[x(p-k)^2 + (1-x)k^2]^2}$$

$$k \rightarrow k + xp$$

$$= \frac{(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + x(1-x)p^2 + i\epsilon]^2}$$

Set  $\Delta \equiv -x(1-x)p^2 + i\epsilon$ , and apply wick rotation

$$i\mathcal{M}_2 = \frac{i(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2}$$

Dimensional regularization

$$\begin{aligned} i\mathcal{M}_2 &= \frac{i(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[k_E^2 + \Delta]^2} \\ &= \frac{i(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d\Omega_d}{(2\pi)^d} dk_E \frac{k_E^{d-1}}{[k_E^2 + \Delta]^2} \\ &= \frac{i(-i\lambda)^2}{2} \int_0^1 dx \frac{\pi^{d/2} \Gamma(2-d/2)}{\Gamma(2)(2\pi)^d} \Delta^{d/2-2} \\ &\xrightarrow{d \rightarrow 4} -i\lambda^2 \frac{\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)}{32\pi^2} \int_0^1 dx \left(\frac{\Delta}{4\pi}\right)^{-\epsilon/2} \\ &= -i\lambda^2 \frac{\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)}{32\pi^2} \int_0^1 dx \left(1 - \frac{\epsilon}{2} \ln \frac{\Delta}{4\pi}\right) \\ &= \frac{-i\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + 2 - \ln(-p^2) + \ln(4\pi) + \mathcal{O}(\epsilon)\right) \end{aligned}$$

where  $\epsilon = 4 - d$ .

So

$$i\mathcal{M}(s) = -i\lambda + \frac{-i\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + 2 - \ln(-s) + \ln(4\pi)\right)$$

$$\mathcal{M}(s) = -\lambda - \frac{\lambda^2}{32\pi^2} \left( \frac{2}{\epsilon} - \gamma + 2 - \ln(-s) + \ln(4\pi) \right) = -\lambda - \frac{\lambda^2}{32\pi^2} \left( \frac{2}{\epsilon} - \ln(-s) + \text{finite terms} \right)$$

where  $\text{finite terms} = \ln(4\pi) + 2 - \gamma$ .

$$\lambda_R = \lambda + \frac{\lambda^2}{32\pi^2} \left( \frac{2}{\epsilon} - \ln(-s_0) + \text{finite terms} \right)$$

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2} \left( \frac{2}{\epsilon} - \ln(-s_0) + \text{finite terms} \right)$$

$$\begin{aligned} \mathcal{M}(s) &= -\lambda - \frac{\lambda^2}{32\pi^2} \left( \frac{2}{\epsilon} - \ln(-s) + \text{finite terms} \right) \\ &= -\lambda_R + \frac{\lambda_R^2}{32\pi^2} \left( \frac{2}{\epsilon} - \ln(-s_0) + \text{finite terms} \right) - \frac{\lambda_R^2}{32\pi^2} \left( \frac{2}{\epsilon} - \ln(-s) + \text{finite terms} \right) \\ &= -\lambda_R - \frac{\lambda_R^2}{32\pi^2} \ln \frac{s_0}{s} \end{aligned}$$

As the lowest order, the results are always  $-\lambda$ .

Optical theorem concludes that

$$\frac{\lambda^2}{16\pi} = \int d\Pi \lambda^2$$

where

$$\begin{aligned} \int d\Pi \lambda^2 &= \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^6 4E_1 E_2} (2\pi)^4 \delta^4(p - p_1 - p_2) \lambda^2 \\ &= \frac{1}{16\pi} \lambda^2 \end{aligned}$$

2. Proca field, QED with massive photon. Calculate the leading order of  $e^- e^- \rightarrow e^- e^-$ .

The propagator

$$\langle 0 | T \{ A_{in}^\mu(x) A_{in}^\nu(y) \} | 0 \rangle = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x-y)} \frac{i(-g^{\mu\nu} + \frac{k^\mu k^\nu}{\mu^2})}{k^2 - \mu^2 + i\epsilon} + \frac{i}{\mu^2} \delta^4(x-y) \delta^{\mu 0} \delta^{\nu 0}$$

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \mu^2 A^\mu A_\mu + \bar{\psi}(i\not{D} - m)\psi$$

and the interaction part

$$\mathcal{L}_I = e\bar{\psi}\gamma^\mu\psi A_\mu$$

( $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \mu^2 A^\mu A_\mu + \bar{\psi}(i\not{D} - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$ ). The corresponding Hamiltonian is

$$\mathcal{H} =$$