

# Local Operator Divergence

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## 1 NRQED local matrix elements

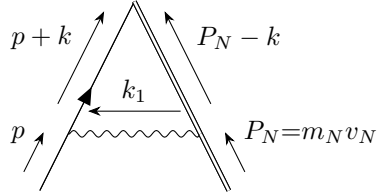
Since what we want is an operator equation independent of states, we can choose  $p^\mu = (0, \mathbf{p})$  as our outstate electron momentum, and put the nucleus on-shell. This way, we can eliminate possible IR divergence and make the integral much easier (especially at NNLO where the first integral will give a  $\sinh^{-1} \left( \frac{1}{\sqrt{\frac{\mathbf{k}_1^2}{2Em} - 1}} \right)$  if no constrain condition put at external momentum).

At NLO:

$$\begin{aligned}
 \langle 0 | \psi_e(0) N(0) (-ie) \int d^4 y \bar{\psi}_e \psi_e A^0 (-ie) \int d^4 z \bar{N} N A^0 | e N \rangle &= \\
 & \text{Diagram: A triangle loop with a wavy line. The left vertical line is labeled } p \text{ at the bottom and } p+k \text{ at the top. The right vertical line is labeled } P_N \text{ at the top and } P_N=m_N v_N \text{ at the bottom. The top horizontal line is labeled } k_1. \text{ The wavy line connects the two vertical lines.} \\
 &= e^2 u_N(v_N) \left[ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p})^2 (E - \frac{\mathbf{k}^2}{2m})} \left( 1 - \frac{\mathbf{k}^4}{8m^3 (E - \frac{\mathbf{k}^2}{2m})} \right) \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} \\
 &= -e^2 u_N(v_N) \left[ \frac{\pi}{v} + \mathcal{O}(v^2) \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}}
 \end{aligned}$$

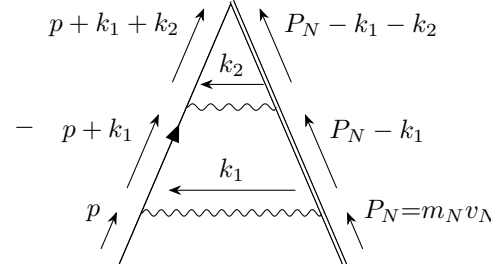
# Appendices

NRQED matrix element at NLO



$$\begin{aligned}
 \langle 0 | \psi_e(0) N(0) (-ie) \int d^4 y \bar{\psi}_e \psi_e A^0 (-ie) \int d^4 z \bar{N} N A^0 | e N \rangle &= \\
 = i e^2 u_N(v_N) \left[ \int [dk] \frac{1}{\mathbf{k}^2 (-k^0 + i\epsilon) (0 + k^0 - m - \frac{(\mathbf{p}+\mathbf{k})^2}{2m} + \frac{(\mathbf{p}+\mathbf{k})^4}{8m^3} + i\epsilon)} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} \\
 = e^2 u_N(v_N) \left[ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p})^2 (E - \frac{\mathbf{k}^2}{2m} + \frac{\mathbf{k}^4}{8m^3})} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} \\
 = e^2 u_N(v_N) \left[ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p})^2 (E - \frac{\mathbf{k}^2}{2m})} \left( 1 - \frac{\mathbf{k}^4}{8m^3 (E - \frac{\mathbf{k}^2}{2m})} \right) \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} \\
 = -2m e^2 u_N(v_N) \frac{i}{E - \frac{\mathbf{p}^2}{2m}} \left[ \frac{\pi}{2p} + \mathcal{O}(p^2) \right] = -e^2 u_N(v_N) \frac{i}{E - \frac{\mathbf{p}^2}{2m}} \left[ \frac{\pi}{v} + \mathcal{O}(v^2) \right]
 \end{aligned}$$

At NNLO (where we're only interested in divergent parts)



$$= e^4 \left[ \int [dk_1][dk_2] \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{p^0 + k_1^0 - m - \frac{(\mathbf{p}+\mathbf{k}_1)^2}{2m} + i\epsilon} \frac{1}{p^0 + k_1^0 + k_2^0 - m - \frac{(\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)^2}{2m} + i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)$$

do the shift as above

$$= -e^4 \left[ \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{E - \frac{|\mathbf{k}_1|^2}{2m} + 2i\epsilon} \frac{1}{E - \frac{|\mathbf{k}_2|^2}{2m} + 2i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)$$

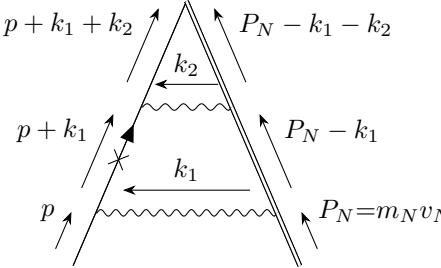
drop  $\mathbf{p}$

$$= -e^4 \left[ \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{-\frac{|\mathbf{k}_1|^2}{2m} + 2i\epsilon} \frac{1}{-\frac{|\mathbf{k}_2|^2}{2m} + 2i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)$$

if we add higher relativistic correction

$$\begin{aligned}
 &= -e^4 \left[ \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{2m}{|\mathbf{k}_1|^2 - \frac{|\mathbf{k}_1|^4}{4m^2}} \frac{2m}{|\mathbf{k}_2|^2 - \frac{|\mathbf{k}_2|^4}{4m^2}} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N) \\
 &= -4m^2 e^4 \left[ \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_1|^2} \left( 1 + \frac{|\mathbf{k}_1|^2}{4m^2} \right) \frac{1}{|\mathbf{k}_2|^2} \left( 1 + \frac{|\mathbf{k}_2|^2}{4m^2} \right) \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)
 \end{aligned}$$

The integral (we can verify that  $\mathbf{k}^0$  part is not UV divergent, and we don't care about  $\mathbf{k}^8$  term for now) for



$$= \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{|\mathbf{k}_1|^4/4m^2}{[|\mathbf{k}_1|^2 - 2mE]^2} \frac{1}{|\mathbf{k}_2|^2 - 2mE}$$

$$= \int_0^1 dx \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{|\mathbf{k}_1|^4/4m^2}{[|\mathbf{k}_1|^2 - 2mE]^2} \frac{\left(\frac{4\pi}{\Delta_2}\right)^{2-\frac{d}{2}} \Gamma(2-\frac{d}{2})}{(4\pi)^2 \Gamma(2)}$$

where  $\Delta_2 = (1-x)(|\mathbf{k}_1|^2 x - 2Em)$

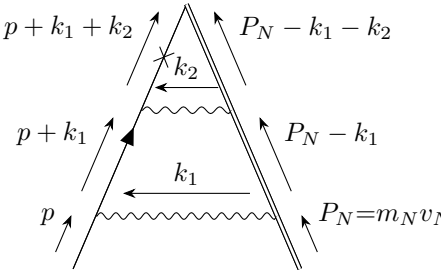
$$= \frac{1}{(4\pi)^2} \int_0^1 dx \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{|\mathbf{k}_1|^4/4m^2}{[|\mathbf{k}_1|^2 - 2mE]^2} \frac{1}{(|\mathbf{k}_1|^2 - 2mE/x)^{2-d/2}} \left(\frac{4\pi}{x(1-x)}\right)^{2-d/2} \Gamma(2-d/2)$$

$$= \frac{1}{(4\pi)^2} \int_0^1 dx \int_0^1 dy dz dt \delta(y+z+t-1) \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{z t^{1-d/2} |\mathbf{k}_1|^4/4m^2}{[|\mathbf{k}_1|^2 + \Delta_1]^{5-d/2}} \frac{\Gamma(5-d/2)}{\Gamma(2-d/2)} \left(\frac{4\pi}{x(1-x)}\right)^{2-d/2} \Gamma(2-d/2)$$

where  $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mE(z+t/x)$

$$= \frac{1}{4m^2(4\pi)^2} \int_0^1 dx \int_0^1 dy dz dt \delta(y+z+t-1) z t^{1-d/2} \frac{d(d+2)}{4} \frac{\Gamma(3-d)}{(4\pi)^{3-d/2}} \left(\frac{4\pi}{\Delta_1}\right)^{3-d} \left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}$$

$$= -\frac{1}{128\pi^2(d-3)m^2} + \text{finite terms}$$



$$= \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_1|^2 - 2mE} \frac{|\mathbf{k}_2|^4/4m^2}{[|\mathbf{k}_2|^2 - 2mE]^2}$$

$$= \frac{1}{4m^2} \int_0^1 dx \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_1|^2 - 2mE} \frac{(1-x)\Gamma(1-d/2)}{8\pi} \left(\frac{4\pi}{\Delta_2}\right)^{1-d/2} \frac{d(d+2)}{4}$$

where  $\Delta_2 = x(1-x)|\mathbf{k}_1|^2 - 2mE(1-x)$

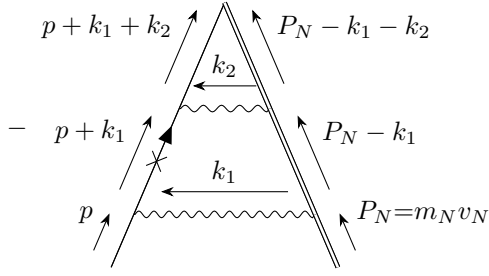
$$= \frac{1}{4m^2} \int_0^1 dx \int_0^1 dy dz dt \frac{t^{-d/2}}{[|\mathbf{k}_1|^2 + \Delta_1]^{3-d/2}} \frac{\Gamma(3-d/2)}{\Gamma(1-d/2)} \delta(y+z+t-1) \frac{\Gamma(1-d/2)}{8\pi} \left(\frac{4\pi}{x(1-x)}\right)^{1-d/2} \frac{d(d+2)x}{4}$$

where  $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mEz - 2mE\frac{t}{x}$

$$= \frac{1}{4m^2} \int_0^1 dx \int_0^1 dy dz dt \delta(y+z+t-1) \frac{1}{(4\pi)^{3-d/2}} \left(\frac{4\pi}{\Delta_1}\right)^{3-d} \frac{\Gamma(3-d)}{8\pi} \left(\frac{4\pi}{x(1-x)}\right)^{1-d/2} \frac{d(d+2)x}{4} t^{-d/2}$$

$$= \frac{15}{8192\pi^2(d-3)m^2} + \text{finite terms}$$

Check the contour integral



$$\begin{aligned}
&= e^4 \left[ \int [dk_1][dk_2] \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{(\mathbf{p} + \mathbf{k})^4 / 8m^3}{[p^0 + k_1^0 - m - \frac{(\mathbf{p} + \mathbf{k}_1)^2}{2m} + i\epsilon]^2} \frac{1}{p^0 + k_1^0 + k_2^0 - m - \frac{(\mathbf{p} + \mathbf{k}_1 + \mathbf{k}_2)^2}{2m} + i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N) \\
&= -e^4 \left[ \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{|\mathbf{k}_1|^4 / 8m^3}{[E - \frac{|\mathbf{k}_1|^2}{2m} + 2i\epsilon]^2} \frac{1}{E - \frac{|\mathbf{k}_2|^2}{2m} + 2i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)
\end{aligned}$$

which is the same as previous (by expansion) one.

Contact (Darwin) term

$$\begin{aligned}
&- \left[ \text{Diagram 1} + \text{Diagram 2} \right] \\
&= \frac{4\pi e^4}{3m} \left[ \int \frac{d^4k}{(2\pi)^4} \frac{1}{p^0 + k^0 - m - \frac{(\mathbf{p} + \mathbf{k})^2}{2m} + i\epsilon} \left( 1 + \frac{(\mathbf{p} + \mathbf{k})^4 / 8m^3}{p^0 + k^0 - m - \frac{(\mathbf{p} + \mathbf{k})^2}{2m} + i\epsilon} \right) \frac{1}{-k^0 + i\epsilon} \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N) \\
&= \frac{4\pi e^4}{3m} \left[ \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{E - \frac{\mathbf{k}^2}{2m} + 2i\epsilon} \left( 1 + \frac{|\mathbf{k}|^4 / 8m^3}{E - \frac{\mathbf{k}^2}{2m} + i\epsilon} \right) \right] \frac{i}{E - \frac{\mathbf{p}^2}{2m}} u_N(v_N)
\end{aligned}$$

if the dispersion relation is up to  $\mathbf{k}^4$  then

$$= \frac{4e^4 m}{3}$$