Renormalization and effective theory for Schrodinger equation

Yingsheng Huang

Contents

1 Introduction

Renormalization and effective theory can be used for non-relativistic quantum mechanics,for Schrodinger equation,according to Lepage [1].

The basic idea is to mimic the high-energy physics we yet don't know from low-energy data.First,we introduce a momentum cutoff Λ as the renormalization process in QFT.This cutoff preserve only those physics that we know of.Then we add local corrections,and use low energy data to fit it.

2 Constructing a effective theory for Schrodinger equation

2.1 The "true" Theory

Starting with a simple Coulomb potential,we attach a short-distance potential to it.

$$H = \frac{\mathbf{p}^2}{2m} + V(r) \tag{1}$$

$$\text{where} \quad V(r) = -\frac{\alpha}{r} + V_s(r) \tag{2}$$

The exact form of this short-distance potential in Lepage's paper is unknown.With Lepage's original data of one of his figures,I have this potential as my short distance potential:

$$V_s(r) = -\frac{1.04152e^{-0.9991r}}{r} \tag{3}$$

We choose $\alpha = 1$ and $m = 1$,so the fullform of my potential is

$$V(r) = -\frac{1}{r} - \frac{1.04152e^{-0.9991r}}{r} \tag{4}$$

Solve it and we get

energy level	my energy	Lepage's energy	relative error	energy level	my energy	Lepage's energy	relative error
1S	1.33732	1.28711542	3.90057%	6S	0.0156184	0.0155492598	0.444587%
2S	0.186434	0.183325753	1.69544%	. . .			
3S	0.0710575	0.0703755485	0.968963%	10S	0.00535929	0.00534541931	0.259539%
4S	0.0374072	0.0371495726	0.693401%	20S	0.0012937	0.00129205010	0.127336%
5S	0.023051	0.0229268241	0.541494%				

Table 1: Comparison in S-wave binding energy of my potential and Lepage's

And the S-wave phase shift can also be computed.Here I choose the phase shift at $r = 50$ as Lepage did.So now we have s set of data can be used as experimental data and from now on we pretend that the exact form of V_s is completely unknown.

2.2 Try delta function and perturbation theory

We can choose delta function $\delta^3(\mathbf{r})$ to approximate the unknown short-distance potential.The Hamiltonian then is

$$H_{app} = \frac{\mathbf{p}^2}{2m} + V(r) + c\delta^3(\mathbf{r}) \tag{5}$$

Using 1st-order perturbation theory and we have the binding energy

$$E_{2n} = -\frac{1}{2n^2} + c\frac{\delta_{l,0}}{\sqrt{\pi}\, n^3} \tag{6}$$

The idea is to fit (6) to the lowest energy we have (in our case,it's the 20S binding energy).Here I calculated based on both Lepage's binding energy and my own data,which respectively give $c = -.596255$ and $c = -.619584$.The relative error is given in figure 1.

But this delta function and perturbation approximation blows up when we ask for more accuracy.The 2nd-order perturbation

$$\sum_{m\neq n}\frac{\langle n|c\delta^3(\mathbf{r})|m\rangle\ \langle m|c\delta^3(\mathbf{r})|n\rangle}{E_n-E_m} \tag{7}$$

obviously gives a divergent term when the scattering momentum $\mathbf{p}\to\infty$.The delta function is too singular to be meaningful beyond 1st-order perturbation theory.

Another problem is,we can Fourier transform V_s into momentum space and Taylor expand it.The first term can be transformed back and becomes a delta function.The 2nd term is the Laplacian of delta function.It seems we can get another correction,but the first order perturbation of the 2nd term $\langle n|\nabla^2\delta^3(\mathbf{r})|n\rangle$ is again too singular to have any physical meaning.

2.3 Constructing effective theory

Now we construct a effective theory in the guidance of renormalization theory (using potential model,of course).This effective theory is specified by this Hamiltonian

$$H_{eff}=\frac{\mathbf{p}^2}{2m}+V_{eff}(\mathbf{r}) \tag{8}$$

First we Fourier transform the Coulomb potential $\dfrac{1}{r}$ into $\dfrac{4\pi}{q^2}$.Then we introduce a cutoff $\Lambda=\dfrac{1}{a}$.Potential
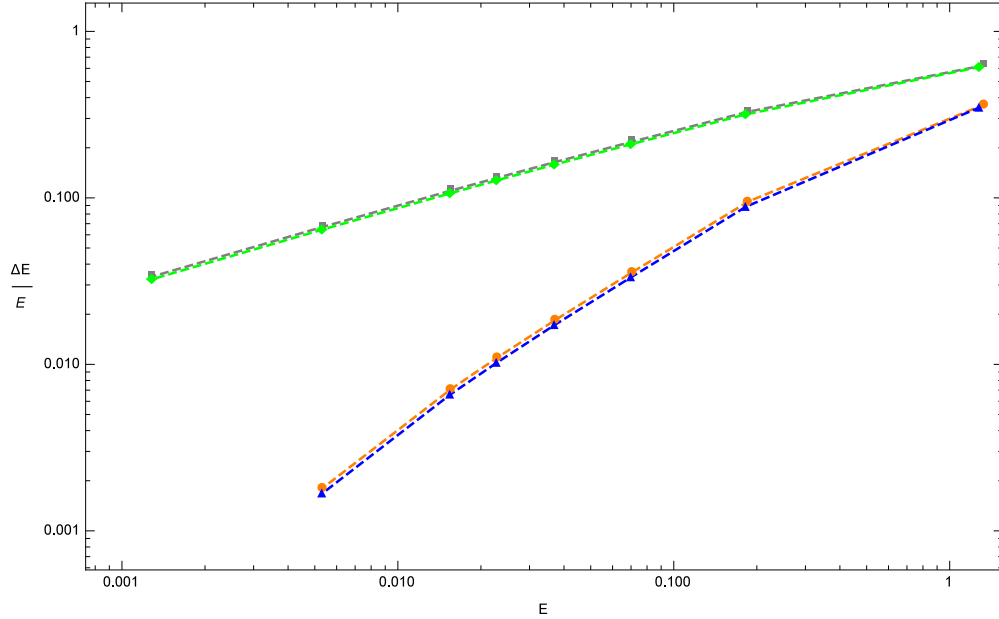

Figure 1: Relative errors in S-wave binding energy plotted versus binding energy for the Coulomb theory,the two curves upper are for only the Coulomb theory and the other two are for E_{2n}.

in momentum space becomes $\dfrac{4\pi}{q^2}e^{-q^2a^2/2}$.Fourier transform it back into coordinate space and we have the

first term of our effective potential $-\dfrac{erf\left(\frac{r}{\sqrt{2}a}\right)}{r}$ where $erf(z)=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z\mathrm{d}t\ e^{-t^2}$.

3

Now we add local corrections.First we "smear" the delta function to make it less singular.Define "smeared" delta function by

$$\delta_a^3(\mathbf{r}) = \frac{e^{-\frac{r^2}{2a^2}}}{(2\pi)^{\frac{3}{2}} \, a^3}. \tag{9}$$

The effective potential with $\mathcal{O}(a^2)$ is

$$V_{eff}^{(a^2)} = -\frac{\alpha}{r} \, erf\left(\frac{r}{\sqrt{2}a}\right) + c \, a^2 \, \delta_a^3(\mathbf{r}) + \mathcal{O}(a^2) \tag{10}$$

then.Similarily,the effective potential with $\mathcal{O}(a^4)$ is

$$V_{eff}^{(a^4)} = -\frac{\alpha}{r} \, erf\left(\frac{r}{\sqrt{2}a}\right) + c \, a^2 \, \delta_a^3(\mathbf{r}) + d_1 \, a^4 \, \nabla^2 \delta_a^3(\mathbf{r}) + d_2 \, a^4 \, \boldsymbol{\nabla} \cdot \delta_a^3(\mathbf{r})\boldsymbol{\nabla} + \mathcal{O}(a^4) \tag{11}$$

Using phase shift in $E = 10^{-10}$ and $E = 10^{-5}$,we can determine the values of c and d in both effective potential.We then have $c^{(a^2)} = -44.294$,$c^{(a^4)} = -39.9477$ and $d_1{}^{(a^4)} = 3.26552$.Note that in S-wave,$d_2{}^{(a^4)} = 0$,and here we just choose $a = 1$,we'll discuss why in the following section.

The relative error in S-wave binding energy is as following: And the error in phase shift of all four theory
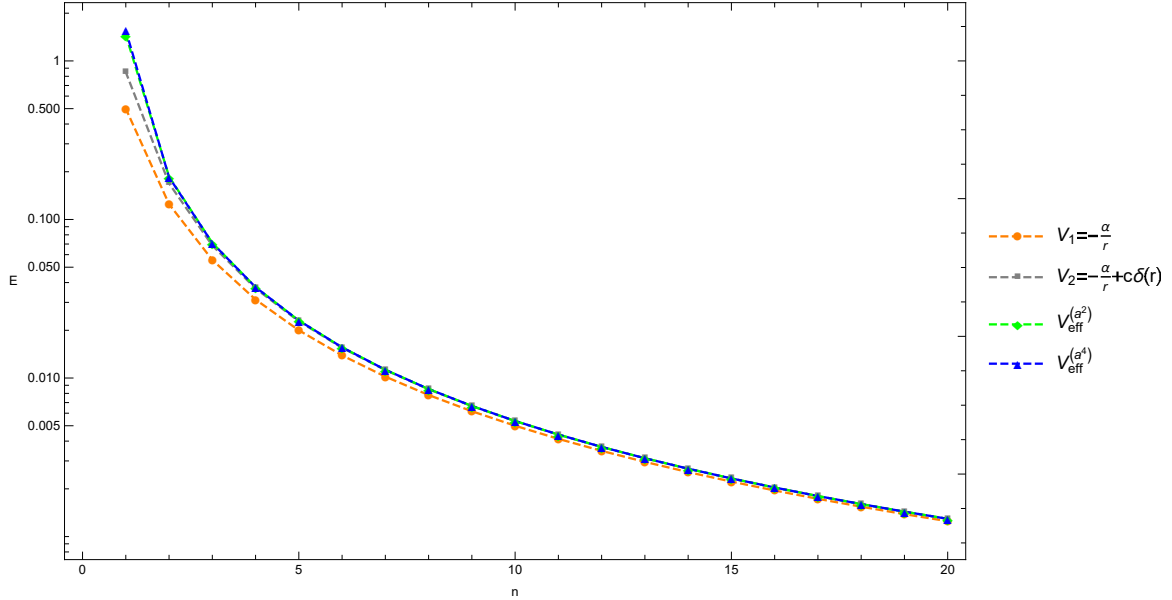

Figure 2: Binding energy for only the Coulomb theory,delta function and perturbative theory,$V_{eff}^{(a^2)}$ and $V_{eff}^{(a^4)}$.

is

2.4   The influence of cutoff a

In our "ture" theory with an additional short-distance potential,if we choose different value of a,the final result can be different.It's showed in figure 5.

If we change the value of α and m as well,the best cutoff a sticks to 1.But if we change the "true" theory into only a Coulomb potential,when a decreases,the error decreases with it and doesn't stop.

4
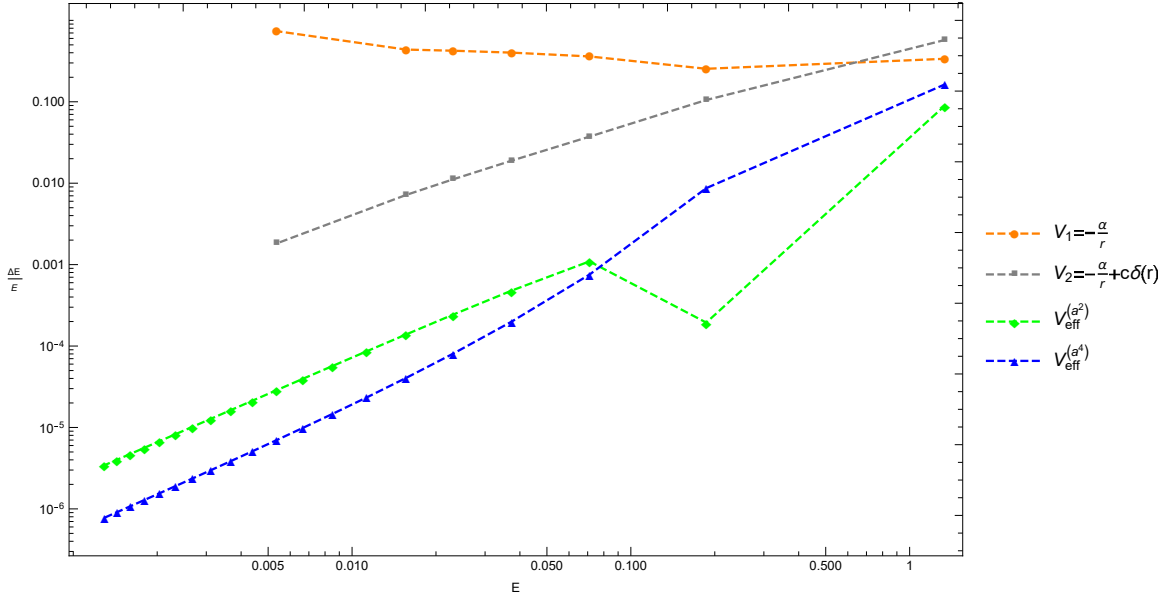
Figure 3: Relative error in binding energy for only the Coulomb theory,delta function and perturbative theory,$V_{eff}^{(a^2)}$ and $V_{eff}^{(a^4)}$.
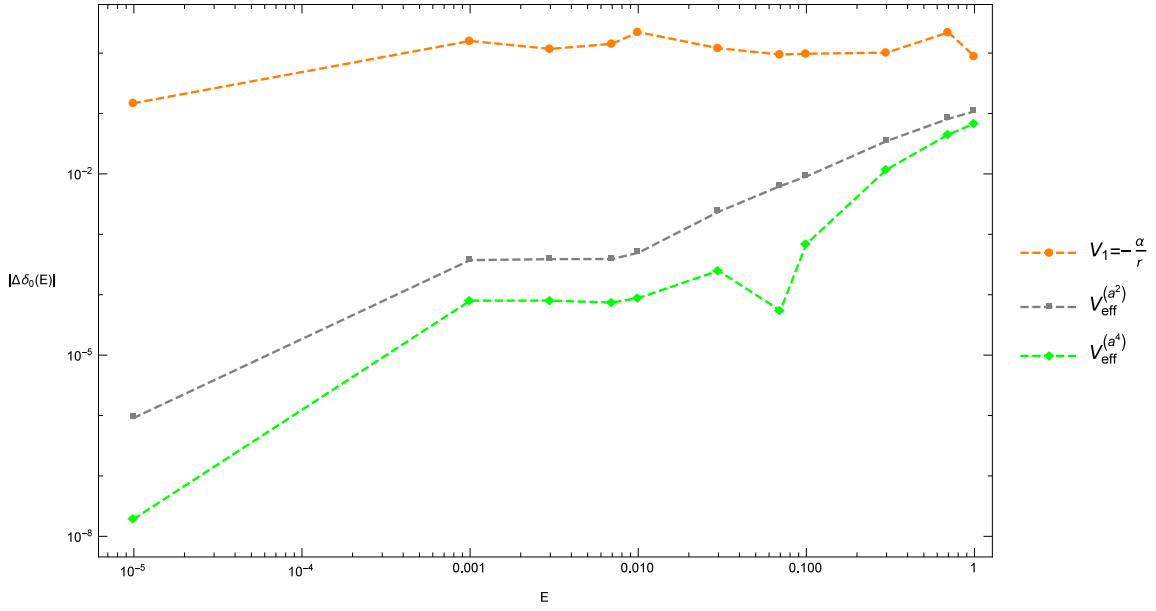
Figure 4: Absolute error in phase shift for only the Coulomb theory,delta function and perturbative theory,$V_{eff}^{(a^2)}$ and $V_{eff}^{(a^4)}$.
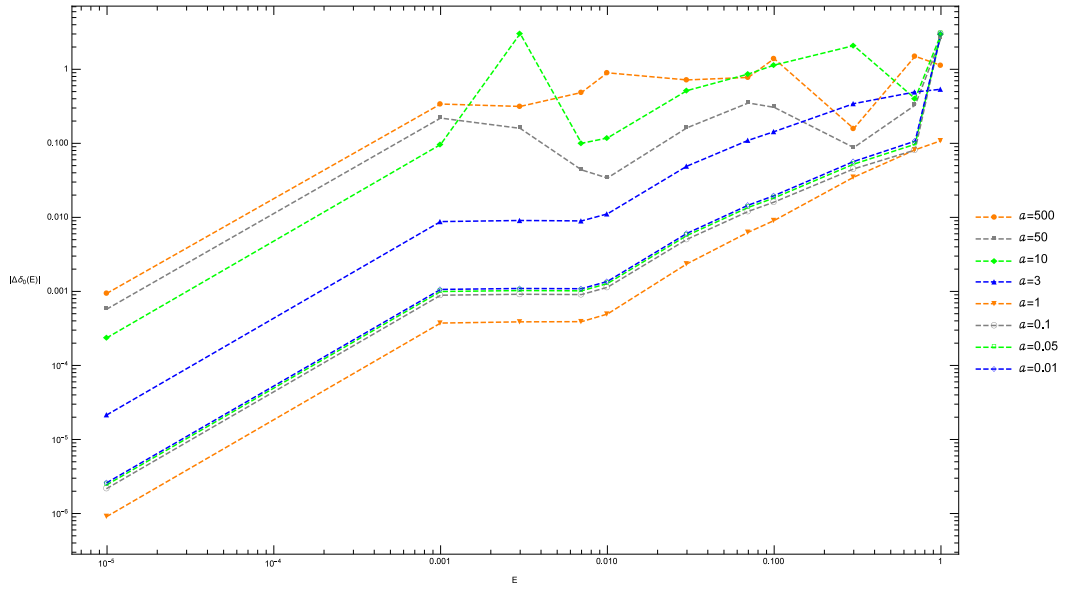
Figure 5: The absolute error of phase shift using effective potential $V_{eff}^{(a^2)}$ in our "true" theory with different cutoff a.
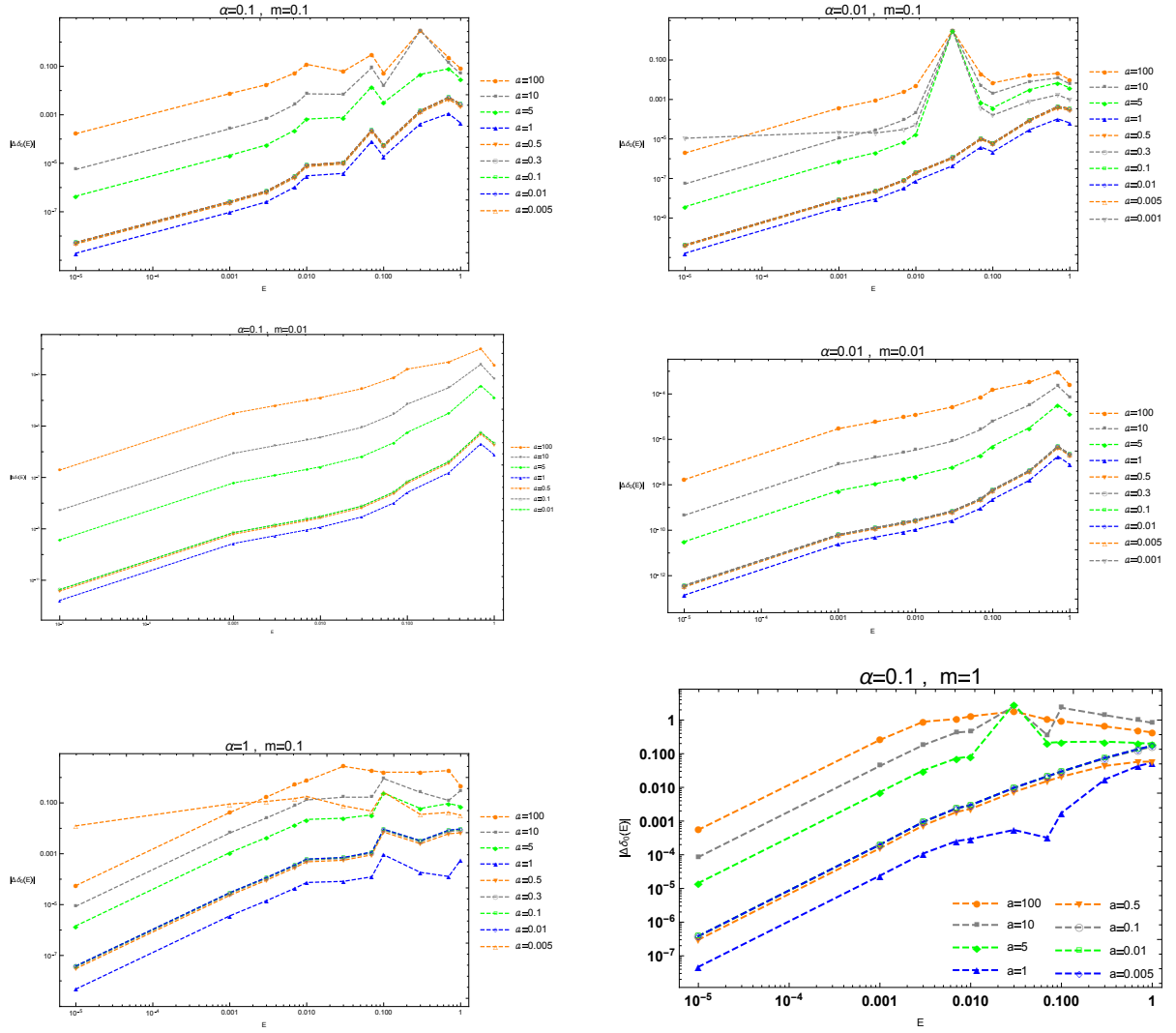
Figure 6: The absolute error of phase shift using effective potential $V_{eff}^{(a^2)}$ in our "true" theory with different cutoff a and different α and m as well.
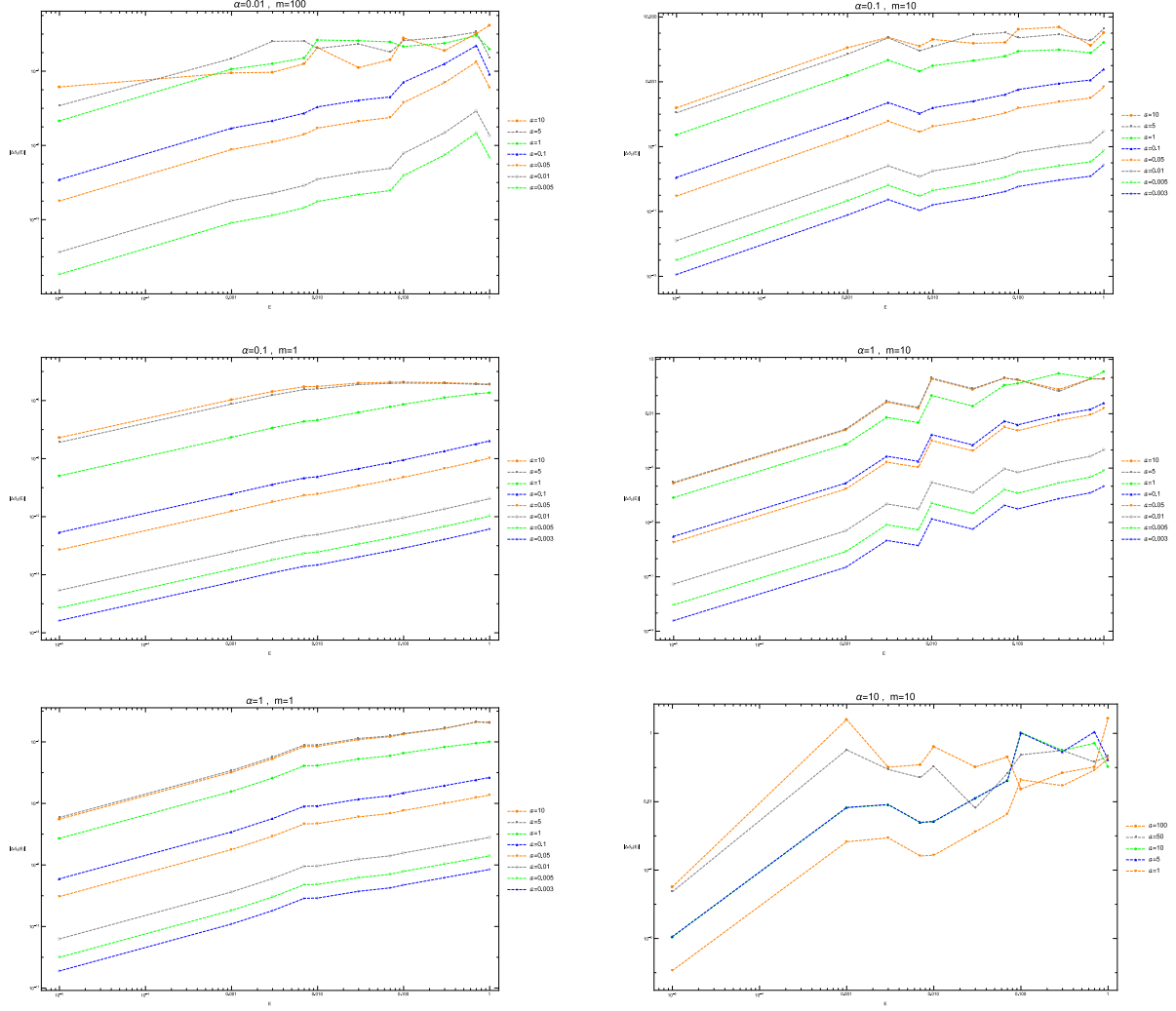
Figure 7: The absolute error of phase shift using effective potential $V_{eff}^{(a^2)}$ in only Coulomb potential with different cutoff a and different α and m as well.

2.5 Misconceptions

Firstly,the effective theory can never be the true theory.The odd that we happen to choose the true theory as our effective theory is just too small.

Secondly,the Fourier transform of our effective theory V_{eff} can't converge to that of the true theory.

In my effective theory,the potentials in coordinate space are

$$V(\mathbf{r}) \quad = \quad -\frac{1}{r} - \frac{1.04152e^{-0.9991r}}{r} \tag{12}$$

$$V_{eff}^{(a^2)}(\mathbf{r}) \quad = \quad -\frac{\text{erf}\left(\frac{r}{\sqrt{2}}\right)}{r} - 2.81241e^{-\frac{r^2}{2}} \tag{13}$$

$$V_{eff}^{(a^4)}(\mathbf{r}) \quad = \quad -\frac{\text{erf}\left(\frac{r}{\sqrt{2}}\right)}{r} + 3.26552\left(\frac{e^{-\frac{r^2}{2}}r^2}{2\sqrt{2}\pi^{3/2}} - \frac{3e^{-\frac{r^2}{2}}}{2\sqrt{2}\pi^{3/2}}\right) - 2.53643e^{-\frac{r^2}{2}} \tag{14}$$

They look like this: And can be Fourier transformed into momentum space and become
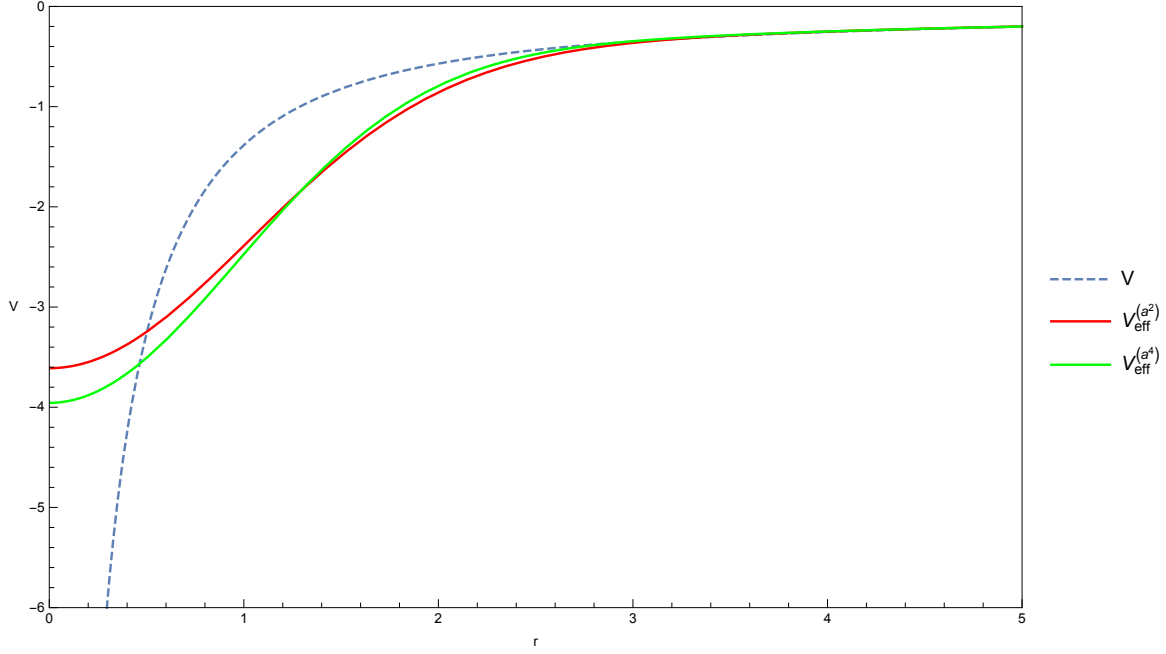
Figure 8: "True" theory and effective theory in coordinate space.

$$v(\mathbf{k}) \quad = \quad -\frac{12.5438}{k^2\left(k^2 + 0.998201\right)} - \frac{25.6545}{k^2 + 0.998201} \tag{15}$$

$$v_{eff}^{(a^2)}(\mathbf{k}) \quad = \quad -\frac{12.5664e^{-0.5k^2}}{k^2} - 44.2944e^{-0.5k^2} \tag{16}$$

$$v_{eff}^{(a^4)}(\mathbf{k}) \quad = \quad -\frac{12.5664e^{-0.5k^2}}{k^2} - 3.26552e^{-0.5k^2}k^2 - 39.9477e^{-0.5k^2} \tag{17}$$

We can divide those in momentum space into two parts:the first term, and the rest.
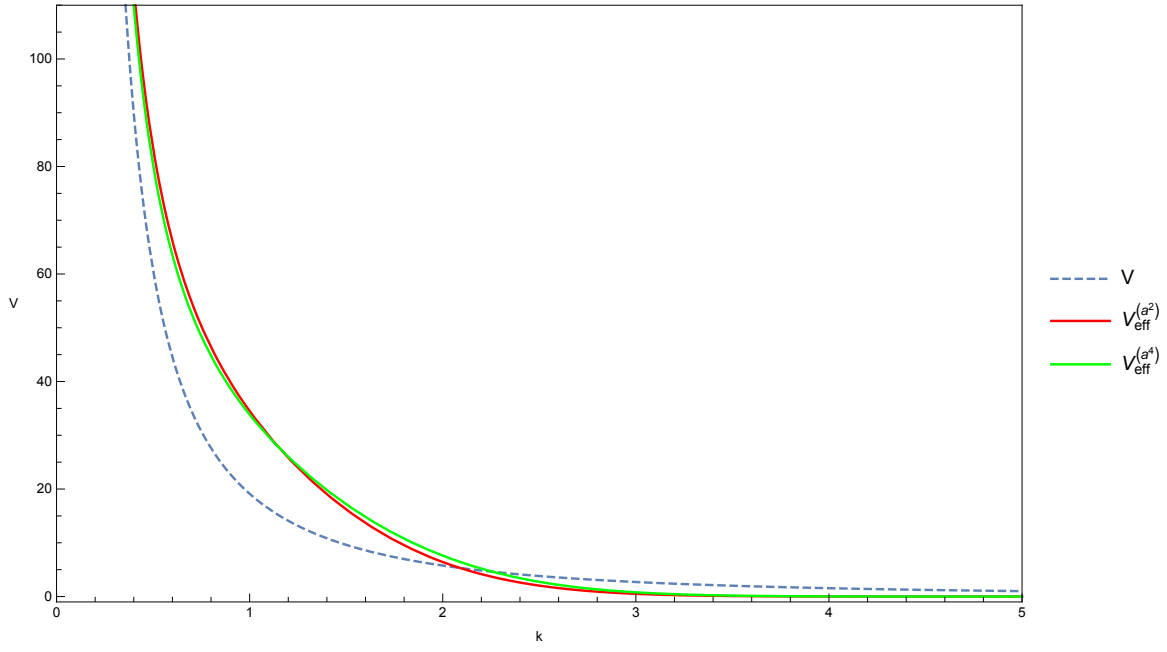
Figure 9: "True" theory and effective theory in momentum space.
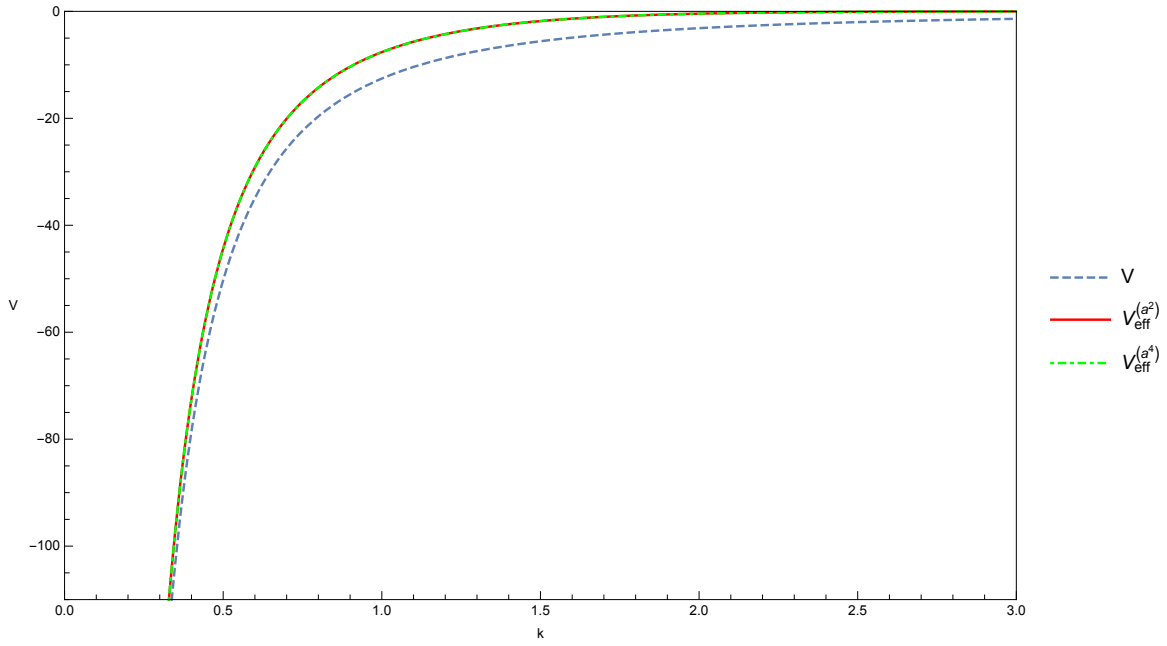
Figure 10: The first term of "true" theory and effective theory in momentum space.

10
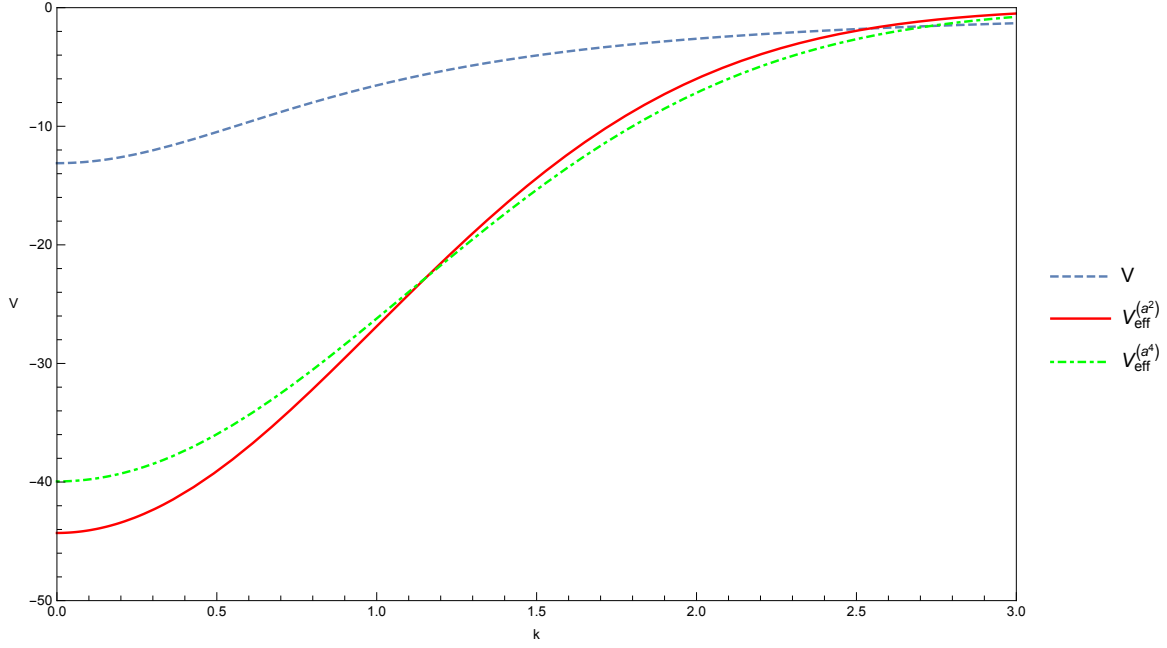
Figure 11: The rest terms of "true" theory and effective theory in momentum space.

The rest terms are not identical obviously,but the first terms are not identical when close to zero. The

Figure 12: The first term of "true" theory minus that of effective theory in momentum space.

last one,it's curve fitting,but it's more systematical and highly optimized.We can improve the accuracy only though adding more correction terms,and the error is removed order-by-order in a.

2.6 matrix element

2.6.1 $\left\langle \mathbf{p}^4 \right\rangle$

When calculating the matrix element $\left\langle \mathbf{p}^4 \right\rangle$,we found that we can't do the integration directly.It's because of $\mathbf{p}^4$ in effective t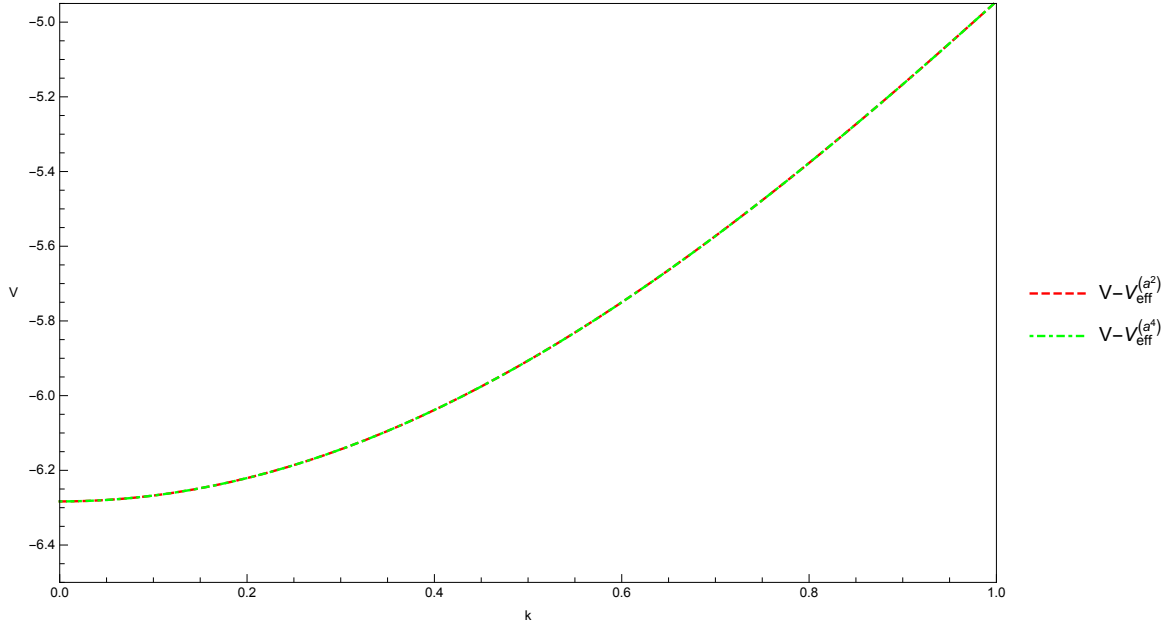heory stands for different physical meaning as in our "true" theory.So we need corrections.Lepage give this as his corrections:

$$\left\langle \mathbf{p}^4 \right\rangle_{true} = Z \left\langle \mathbf{p}^4 \right\rangle_{eff} + \frac{\gamma}{a} \left\langle \delta_a^3(\mathbf{r}) \right\rangle_{eff} + \eta a \left\langle \nabla^2 \delta_a^3(\mathbf{r}) \right\rangle_{eff} + \mathcal{O}(a^3) \tag{18}$$

level	$\left\langle \mathbf{p}^4 \right\rangle$	$\left\langle \mathbf{p}^4 \right\rangle_{eff}$	Relative error	$\left\langle Z\mathbf{p}^4 + \gamma\delta_a^3/a + \ldots \right\rangle_{eff}$	Relative error after correction
1S	75.0651	6.39016	0.914872	86.2584	0.149114
2S	5.89805	1.80467	0.694022	5.63427	0.0447230
3S	1.38388	0.459182	0.668193	1.37834	0.00400119
4S	0.533537	0.181115	0.660539	0.533006	0.000995149
5S	0.259685	0.0892336	0.656377	0.259594	0.000349898
6S	0.145438	0.0503636	0.653711	0.145417	0.000142768
7S	0.0895058	0.0311615	0.651849	0.895003	0.0000608579
8S	0.0589478	0.0206038	0.650473	0.0589463	0.0000247520
9S	0.0408594	0.0143248	0.649413	0.0408591	7.89362×10^{-6}
10S	0.0294762	0.0103588	0.648572	0.0294762	$0. \times 10^{-50}$
11S	0.0219577	0.00773158	0.647888	0.0219578	2.86723×10^{-6}
12S	0.0167936	0.00592277	0.647320	0.0167937	3.65482×10^{-6}
13S	0.0131299	0.00463692	0.646842	0.0131299	3.06033×10^{-6}
14S	0.0104589	0.00369791	0.646433	0.0104589	1.68057×10^{-6}
15S	0.00846592	0.00299626	0.646080	0.00846592	$0. \times 10^{-50}$
16S	0.00694880	0.00246146	0.645772	0.00694878	1.78021×10^{-6}
17S	0.00577356	0.00204673	0.645500	0.00577354	3.67384×10^{-6}
18S	0.00484908	0.00172016	0.645260	0.00484906	5.47461×10^{-6}
19S	0.00411194	0.00145962	0.645029	0.00411195	2.37062×10^{-6}
20S	0.00503698	0.00278627	0.446837	0.00503698	$0. \times 10^{-50}$

Table 2: $\left\langle \mathbf{p}^4 \right\rangle$ matrix element in "true" theory,effective theory without correction and effective theory with correction

2.6.2 $\left\langle \frac{1}{r} \right\rangle$

$$\left\langle \frac{1}{r} \right\rangle_{true} = Z \left\langle \frac{1}{r} \right\rangle_{eff} + \frac{\gamma}{a} \left\langle \delta_a^3(\mathbf{r}) \right\rangle_{eff} + \eta a \left\langle \nabla^2 \delta_a^3(\mathbf{r}) \right\rangle_{eff} + \mathcal{O}(a^3) \tag{19}$$

2.7 $\left\langle \frac{1}{r^2} \right\rangle$

$$\left\langle \frac{1}{r^2} \right\rangle_{true} = Z \left\langle \frac{1}{r^2} \right\rangle_{eff} + \frac{\gamma}{a} \left\langle \delta_a^3(\mathbf{r}) \right\rangle_{eff} + \eta a \left\langle \nabla^2 \delta_a^3(\mathbf{r}) \right\rangle_{eff} + \mathcal{O}(a^3) \tag{20}$$

level	$\langle 1/r \rangle$	$\langle 1/r \rangle_{eff}$	Relative error	$\langle Z/r + \gamma\delta_a^3/a + \ldots \rangle_{eff}$	Relative error after correction
1S	1.92548	1.28960	0.330244	1.7642	0.0837596
2S	0.351169	0.344153	0.0199786	0.345096	0.0172951
3S	0.136034	0.134796	0.00909973	0.135905	0.00094791
4S	0.0724346	0.0719137	0.00719131	0.0724226	0.000165167
5S	0.0449404	0.0446755	0.00589443	0.0449384	0.0000438437
6S	0.0305861	0.0304343	0.00496313	0.0305856	0.0000143115

Table 3: $\left\langle \dfrac{1}{r} \right\rangle$ matrix element in "true" theory,effective theory without correction and effective theory with correction

level	$\langle 1/r^2 \rangle$	$\langle 1/r^2 \rangle_{eff}$	Relative error	$\langle Z/r^2 + \gamma\delta_a^3/a + \ldots \rangle_{eff}$	Relative error after correction
1S	7.54675	2.64833	0.649077	-15.8996	3.10682
2S	0.513020	0.338657	0.339876	0.519597	0.0128214
3S	0.117680	0.0787820	0.330541	0.11814	0.00391092
4S	0.0448732	0.0299838	0.331810	0.0449209	0.00106295
5S	0.0217022	0.0144860	0.332509	0.02171	0.000359689
6S	0.0121042	0.00807484	0.332891	0.0121059	0.000136607

Table 4: $\left\langle \dfrac{1}{r^2} \right\rangle$ matrix element in "true" theory,effective theory without correction and effective theory with correction

2.8 $\left\langle \dfrac{1}{r^3} \right\rangle$

$$\left\langle \frac{1}{r^3} \right\rangle_{true} = Z\left\langle \frac{1}{r^3} \right\rangle_{eff} + \frac{\gamma}{a}\left\langle \delta_a^3(\mathbf{r}) \right\rangle_{eff} + \eta a\left\langle \nabla^2\delta_a^3(\mathbf{r}) \right\rangle_{eff} + \mathcal{O}(a^3) \tag{21}$$

level	$\langle 1/r^3 \rangle$	$\langle 1/r^3 \rangle_{eff}$	Relative error	$\langle Z/r^3 + \gamma\delta_a^3/a + \ldots \rangle_{eff}$	Relative error after correction
1S	21261.3	2612.36	0.877131	30560.7	0.437390
2S	1365.28	336.763	0.753337	1302.27	0.0461507
3S	309.748	78.1585	0.747671	308.562	0.00382939
4S	117.801	29.6996	0.747882	117.700	0.000849579
5S	56.9110	14.3375	0.748072	56.8958	0.000265940
6S	31.7243	7.98857	0.748187	31.7212	0.0000974506

Table 5: $\left\langle \dfrac{1}{r^3} \right\rangle$ matrix element in "true" theory,effective theory without correction and effective theory with correction

2.9 $\psi(0)$

Just like $\langle \mathbf{p}^4 \rangle$,$\psi(0)$ also needs correction.But it's more interesting because now we know even effective theory,after those correction and fitting,can not have the exact same wavefunction.

$$\psi_{true}(0) = \overline{\gamma}\int \mathrm{d}^3r\psi_{eff}\delta_a^3(\mathbf{r}) + \overline{\eta}a^2\int \mathrm{d}^3r\psi_{eff}\nabla^2\delta_a^3(\mathbf{r}) + \mathcal{O}(a^3) \tag{22}$$

level	$\psi(0)$	$\psi_{eff}(0)$	$\psi_{eff}(0)$ Relative error	$\overline{\gamma}\int\psi_{eff}\delta_a^3+\dots$	Relative error after correction
1S	-1.54924	-0.542484	0.64984	4.11644	3.65707
2S	-0.392598	-0.194805	0.503805	-0.377193	0.0392382
3S	-0.187001	-0.0938498	0.498132	-0.186328	0.00360212
4S	0.115323	0.0578526	0.498341	0.115242	0.000698381
5S	0.0801566	0.0401962	0.498529	0.0801444	0.000151733
6S	-0.0598462	-0.0300043	0.498643	-0.0598454	0.0000141098

Table 6: $\psi(0)$in "true" theory,effective theory without correction and effective theory with correction

3 Using effective theory in perturbative theory

Effective theory can be used to understand those high-energy physics that we have no knowledge of,and it can be used to simplify calculation of low energy behavior as well.

We use only simple Coulomb potential,

$$H = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r} \tag{23}$$

where $\alpha = 0.01$,$m = 100$,and the effective theory is

$$H_{eff} = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r}erf(\frac{r}{\sqrt{2}a}) - 2\pi\alpha ca^2\delta_a^3(\mathbf{r}) \tag{24}$$

Using 1st-order Born approximation

$$f^{(1)}(\mathbf{q}) = \int \mathrm{d}^3x' e^{-i\mathbf{q}\cdot\mathbf{x}'}V = -\frac{4\pi\alpha}{q^2} \tag{25}$$

$\mathbf{q} = \mathbf{k_0} - \mathbf{k}'$ is the momentum transfer.For H_{eff}:

$$f_{eff}^{(1)}(\mathbf{q}) = -\frac{4\pi\alpha}{q^2}e^{-q^2a^2/2}(1 + cq^2a^2/2) \tag{26}$$

$$= -\frac{4\pi\alpha}{q^2}1 + c - 1)q^2a^2/2 + \mathcal{O}(q^4a^4)) \tag{27}$$

when $c = 0$,the relative error in the 1S binding energy is 0.0387861%.If $c = 1$,the effective potential is

$$V_{eff} = -\frac{\alpha}{r}erf(\frac{r}{\sqrt{2}a}) - 2\pi\alpha a^2\delta_a^3(\mathbf{r}) \tag{28}$$

$\delta_a^3(\mathbf{r})$ is given before.

When $a = 0.01$,

When calculating 2nd-order Born approximation

$$T = V + V\frac{1}{E_i - E_m + i\hbar\epsilon}V \tag{29}$$

we have

$$V\left|\psi^{(+)}\right\rangle = T\left|\mathbf{k}\right\rangle = V\left|\mathbf{k}\right\rangle + V\frac{1}{E_i - E_m + i\hbar\epsilon}V\left|\mathbf{k}\right\rangle \tag{30}$$

14

level	binding energy	relative error	level	binding energy	relative error
1S	0.00499998	3.61412×10^{-6}	11S	0.0000413223	3.27197×10^{-7}
2S	0.00125	1.83241×10^{-6}	12S	0.0000347222	2.68281×10^{-7}
3S	0.000555556	1.77179×10^{-8}	13S	0.0000295858	2.78941×10^{-7}
4S	0.0003125	9.04534×10^{-7}	14S	0.0000255102	2.57908×10^{-7}
5S	0.0002	7.10634×10^{-7}	15S	0.0000222222	2.30776×10^{-7}
6S	0.000138889	5.8436×10^{-7}	16S	0.0000195313	2.25173×10^{-7}
7S	0.000102041	5.13932×10^{-7}	17S	0.000017301	2.1558×10^{-7}
8S	0.000078125	4.49199×10^{-7}	18S	0.0000154321	2.04102×10^{-7}
9S	0.0000617284	3.98123×10^{-7}	19S	0.0000138504	1.92648×10^{-7}
10S	0.00005	3.54257×10^{-7}	20S	0.0000125	1.85357×10^{-7}

Table 7: Relative error in binding energy of 1st-perturbation approximation

and

$$f(\mathbf{k_0}, \mathbf{k'}) = (2\pi)^3 \left\langle \mathbf{k'} \left| V \right| \psi^{(+)} \right\rangle \tag{31}$$

$$= (2\pi)^3 (\langle \mathbf{k'}|V|\mathbf{k_0}\rangle + \langle \mathbf{k'}|V \frac{1}{E_i - E_m + i\hbar\epsilon} V|\mathbf{k_0}\rangle) \tag{32}$$

$$= f^{(1)}(\mathbf{k_0}, \mathbf{k'}) + f^{(2)}(\mathbf{k_0}, \mathbf{k'}) \tag{33}$$

$f^{(1)}(\mathbf{k_0}, \mathbf{k'})$ as above, and $f^{(2)}(\mathbf{k_0}, \mathbf{k'})$ can be the integration form of $f^{(1)}(\mathbf{k_0}, \mathbf{k'})$.

$$f^{(2)}(\mathbf{k_0}, \mathbf{k'}) = (2\pi)^3 \langle \mathbf{k'}|V \frac{1}{E_i - E_m + i\hbar\epsilon} V|\mathbf{k_0}\rangle \tag{34}$$

$$= \int \frac{\mathrm{d}^3 k''}{(2\pi)^3} f^{(1)}(\mathbf{k''}, \mathbf{k'}) \frac{1}{E_i - E_m + i\hbar\epsilon} f^{(1)}(\mathbf{k_0}, \mathbf{k''}) \tag{35}$$

$$= \int \frac{\mathrm{d}^3 k''}{(2\pi)^3} f^{(1)}(\mathbf{k''}, \mathbf{k'}) \frac{2m}{k_0^2 - k''^2 + 2mi\hbar\epsilon} f^{(1)}(\mathbf{k_0}, \mathbf{k''}) \tag{36}$$

Assuming $\mathbf{k} = \mathbf{k_0} - \mathbf{k''}$, and $\mathbf{k} - \mathbf{q} = \mathbf{k'} - \mathbf{k''}$, redefine ϵ as $2m\epsilon$, and

$$f^{(2)} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} f^{(1)}(\mathbf{k} - \mathbf{q}) \frac{2m}{k_0^2 - (\mathbf{k} - \mathbf{k_0})^2 + i\hbar\epsilon} f^{(1)}(\mathbf{k}) \tag{37}$$

For the 1st-order, the "true" theory is almost identical to the effective one. In the 2nd-order, the difference in scattering amplitude of these two is mainly contribute by $f^{(2)}$ and $f_{eff}^{(2)}$, and p and q can be ignored comparing to k.

$$f_{eff}^{(2)} - f^{(2)} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\left(\frac{-4\pi\alpha}{k^2} \frac{-2m}{k^2} \frac{-4\pi\alpha}{k^2} \right) e^{-k^2 a^2} (1 + k^2 a^2/2)^2 - \left(\frac{-4\pi\alpha}{k^2} \frac{-2m}{k^2} \frac{-4\pi\alpha}{k^2} \right) \right] \tag{38}$$

$$= \frac{10}{3} \sqrt{\pi} \alpha^2 a^3 m \tag{39}$$

making correction to c

$$- \frac{4\pi\alpha}{q^2} (c-1) q^2 a^2/2 = - \frac{10}{3} \sqrt{\pi} \alpha^2 a^3 m \tag{40}$$

and we have $c = 1 + \frac{5}{3\sqrt{\pi}} \alpha a m = 1.00940316$. It's very close to the result we get from non-perturbative effective theory, which have c equal to 1.0094723.

level	binding energy	relative error	level	binding energy	relative error
1S	0.005	2.12769×10^{-8}	11S	0.0000413223	4.18928×10^{-9}
2S	0.00125	1.33472×10^{-8}	12S	0.0000347222	4.21076×10^{-9}
3S	0.000555556	2.5859×10^{-9}	13S	0.0000295858	2.66263×10^{-10}
4S	0.0003125	3.1513×10^{-8}	14S	0.0000255102	8.8104×10^{-12}
5S	0.0002	1.04359×10^{-9}	15S	0.0000222222	2.89155×10^{-10}
6S	0.000138889	9.14987×10^{-9}	16S	0.0000195313	2.68476×10^{-9}
7S	0.000102041	1.30144×10^{-8}	17S	0.000017301	2.15443×10^{-9}
8S	0.000078125	6.59177×10^{-9}	18S	0.0000154321	2.08666×10^{-9}
9S	0.0000617284	7.4271×10^{-9}	19S	0.0000138504	2.99971×10^{-9}
10S	0.00005	5.1792×10^{-9}	20S	0.0000125	2.81353×10^{-9}

Table 8: Relative error in binding energy of 2nd-perturbation approximation

References

[1] Lepage P. How to Renormalize the Schrodinger Equation[J]. Nuclear Theory, 1997.

[2] Sakurai J J, Tuan S F, Commins E D. Modern Quantum Mechanics, Second Edition. Pearson Schweiz Ag, 2013.

[3] Griffiths, David Jeffery. Introduction to quantum mechanics. Pearson Education India, 2013.

[4] Korsch H J, Glck M. Computing quantum eigenvalues made easy[J]. European Journal of Physics, 2002, 23(23):413-426.

[5] Weinberg S. Lectures on Quantum Mechanics[J]. Lectures on Quantum Mechanics by Steven Weinberg Cambridge Uk Cambridge University Press, 2012, 83(9):xiv,528.