Internal Note of 2019 PKU Summer School: Derivation of the master formula on $a_\mu^{\mathbf{HVP,LO}}$

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This internal note is written for the derivation of the master formula on $a_\mu^{\mathrm{HVP,LO}}$ given in the page 90 of Prof. Luchang Jin's slide for his lecture on muon $g-2$.

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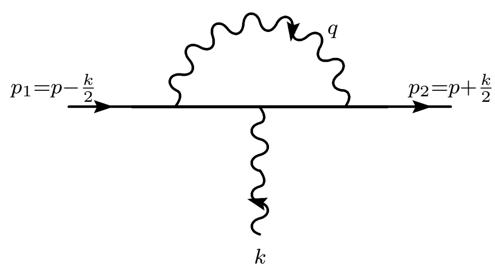
CONTENTS

I. INTRODUCTION

Among the quantum field theory, QED is used here. We start from



(1)

where

$$p_1 = p - \frac{k}{2}\,, \quad p_2 = p + \frac{k}{2} \tag{2}$$

are the incoming and outgoing momentum of muon, respectively. Here, $k = p_2 - p_1$ is the momentum of photon. Hereafter we use

$$p_1^2 = m_\mu^2\,, \quad p_2^2 = m_\mu^2\,, \tag{3}$$

$$k = p_2 - p_1\,, \quad p = \frac{p_1 + p_2}{2}\,, \tag{4}$$

$$p^2 = m_\mu^2 - \frac{1}{4}k^2\,, \quad p \cdot k = 0 \tag{5}$$

where m_μ is mass of muon. The vertex function with the form factors is given as

$$\begin{aligned}\Gamma^\rho\left(p_2, p_1\right) &= \gamma^\rho F_1\left(k^2\right) + \frac{i\sigma^{\rho\nu}k_\nu}{2m_\mu}F_2\left(k^2\right) \\ &= \gamma^\rho F_1\left(k^2\right) + \frac{i}{2m_\mu}\left(\frac{i}{2}\left[\gamma^\rho, \gamma^\nu\right]\right)k_\nu F_2\left(k^2\right) \\ &= \gamma^\rho F_1\left(k^2\right) - \frac{1}{4m_\mu}\left(\gamma^\rho \not{k} - \not{k}\gamma^\rho\right)F_2\left(k^2\right)\,.\end{aligned} \tag{6}$$

Here, the trace projection of vertex function is given as

$$\mathrm{Tr}\left(P_\rho \Gamma^\rho\right) = \left[4\left(k^2+2m_\mu^2\right)g_1 - 2\left(k^2-4m_\mu^2\right)g_2\right]F_1\left(k^2\right) + k^2\left[6g_1 - \frac{k^2-4m_\mu^2}{2m_\mu^2}g_2\right]F_2\left(k^2\right) \tag{7}$$

where

$$P_\rho = \left(\not{p}_1 + m_\mu\right)\left(g_1\gamma_\rho + \frac{1}{m_\mu}g_2 p_\rho\right)\left(\not{p}_2 + m_\mu\right) \tag{8}$$

is the projection operator [1].

If we let g_1 and g_2 as

$$g_1 = \frac{1}{4\left(k^2-4m_\mu^2\right)}\,, \quad g_2 = \frac{3m_\mu^2}{\left(k^2-4m_\mu^2\right)^2}\,, \tag{9}$$

then we get $F_1\left(k^2\right)$ as

$$\begin{aligned} F_1\left(k^2\right) &= \mathrm{Tr}\left(\left(\not{p}_1+m_\mu\right)\left(\frac{1}{4\left(k^2-4m_\mu^2\right)}\gamma_\rho + \frac{1}{m_\mu}\frac{3m_\mu^2}{\left(k^2-4m_\mu^2\right)^2}p_\rho\right)\left(\not{p}_2+m_\mu\right)\Gamma^\rho\right) \\ &= \frac{1}{4\left(k^2-4m_\mu^2\right)}\mathrm{Tr}\left(\left(\not{p}_1+m_\mu\right)\left(\gamma_\rho + \frac{12m_\mu^2}{k^2-4m_\mu^2}\frac{p_\rho}{m_\mu}\right)\left(\not{p}_2+m_\mu\right)\Gamma^\rho\right)\,. \end{aligned} \tag{10}$$

If we let g_1 and g_2 as

$$g_1 = -\frac{m_\mu^2}{k^2\left(k^2-4m_\mu^2\right)}\,, \quad g_2 = -\frac{2m_\mu^2\left(k^2+2m_\mu^2\right)}{k^2\left(k^2-4m_\mu^2\right)^2}\,, \tag{11}$$

then we get $F_2\left(k^2\right)$ as

$$\begin{aligned} F_2\left(k^2\right) &= -\mathrm{Tr}\left(\left(\not{p}_1+m_\mu\right)\left(\frac{m_\mu^2}{k^2\left(k^2-4m_\mu^2\right)}\gamma_\rho + \frac{1}{m_\mu}\frac{2m_\mu^2\left(k^2+2m_\mu^2\right)}{k^2\left(k^2-4m_\mu^2\right)^2}p_\rho\right)\left(\not{p}_2+m_\mu\right)\Gamma^\rho\right) \\ &= -\frac{m_\mu^2}{k^2\left(k^2-4m_\mu^2\right)}\mathrm{Tr}\left(\left(\not{p}_1+m_\mu\right)\left(\gamma_\rho + 2\frac{k^2+2m_\mu^2}{m_\mu\left(k^2-4m_\mu^2\right)}p_\rho\right)\left(\not{p}_2+m_\mu\right)\Gamma^\rho\right)\,. \end{aligned} \tag{12}$$

II. PROJECTION OF THE ONE-LOOP VERTEX FUNCTION INTO THE $F_2(k^2)$

Hereafter we calculate the vertex function at the one-loop level from

$$\begin{aligned} &\bar{u}\left(p_2\right)\delta\Gamma^\rho\left(p_2,p_1\right)u\left(p_1\right) \\ &= \int\frac{d^4q}{(2\pi)^4}\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\bar{u}\left(p_2\right)\left(-ie\gamma^\mu\right)\frac{i\left(\not{p}_2-\not{q}+m_\mu\right)}{\left(p_2-q\right)^2-m_\mu^2+i\varepsilon}\gamma^\rho\frac{i\left(\not{p}_1-\not{q}+m_\mu\right)}{\left(p_1-q\right)^2-m_\mu^2+i\varepsilon}\left(-ie\gamma^\nu\right)u\left(p_1\right) \\ &= \bar{u}\left(p_2\right)\left[-e^2 i\int\frac{d^4q}{(2\pi)^4}\frac{\gamma^\mu\left(\not{p}_2-\not{q}+m_\mu\right)\gamma^\rho\left(\not{p}_1-\not{q}+m_\mu\right)\gamma_\mu}{\left(q^2+i\varepsilon\right)\left[\left(p_2-q\right)^2-m_\mu^2+i\varepsilon\right]\left[\left(p_1-q\right)^2-m_\mu^2+i\varepsilon\right]}\right]u\left(p_1\right)\,. \end{aligned} \tag{13}$$

Here, we are interested in Eq. (12)

$$F_2\left(k^2\right)=e^2 i\int\frac{d^4q}{(2\pi)^4}\frac{1}{\left(q^2+i\varepsilon\right)\left[\left(p_2-q\right)^2-m^2+i\varepsilon\right]\left[\left(p_1-q\right)^2-m^2+i\varepsilon\right]}\times\frac{m_\mu^2}{k^2\left(k^2-4m_\mu^2\right)} \tag{14}$$

$$\times\,\mathrm{Tr}\left[\left(\not p_1+m_\mu\right)\left\{\gamma_\rho+2\left(\frac{k^2+2m_\mu^2}{k^2-4m_\mu^2}\right)\frac{p_\rho}{m_\mu}\right\}\left(\not p_2+m_\mu\right)\gamma^\mu\left(\not p_2-\not q+m_\mu\right)\gamma^\rho\left(\not p_1-\not q+m_\mu\right)\gamma_\mu\right].$$

The trace is given as

$$\mathrm{Tr}\left[\left(\not p_1+m_\mu\right)\left(\gamma_\rho+\frac{k^2+2m_\mu^2}{k^2-4m_\mu^2}\cdot\frac{2p_\rho}{m_\mu}\right)\left(\not p_2+m_\mu\right)\gamma^\mu\left(\not p_2-\not q+m_\mu\right)\gamma^\rho\left(\not p_1-\not q+m_\mu\right)\gamma_\mu\right] \tag{15}$$

$$=\mathrm{Tr}\left[\left(\not p_1+m_\mu\right)\gamma_\rho\left(\not p_2+m_\mu\right)\gamma^\mu\left(\not p_2-\not q+m_\mu\right)\gamma^\rho\left(\not p_1-\not q+m_\mu\right)\gamma_\mu\right] \tag{16}$$

$$+\left(\frac{k^2+2m_\mu^2}{k^2-4m_\mu^2}\right)\frac{2}{m_\mu}\mathrm{Tr}\left[\left(\not p_1+m_\mu\right)\left(\not p_2+m_\mu\right)\gamma^\mu\left(\not p_2-\not q+m_\mu\right)\not p\left(\not p_1-\not q+m_\mu\right)\gamma_\mu\right] \tag{17}$$

$$=\langle 1\rangle+\left(\frac{k^2+2m_\mu^2}{k^2-4m_\mu^2}\right)\frac{2}{m_\mu}\times\langle 2\rangle \tag{18}$$

where $\langle 1\rangle$ and $\langle 2\rangle$ are calculated below. We start from $\langle 1\rangle$

$$\begin{aligned}\langle 1\rangle&=\mathrm{Tr}\left[\left(\not p_1+m_\mu\right)\gamma_\rho\left(\not p_2+m_\mu\right)\gamma^\mu\left(\not p_2-\not q+m_\mu\right)\gamma^\rho\left(\not p_1-\not q+m_\mu\right)\gamma_\mu\right]\\&=-32p^4+16p^2k^2-2k^4+16q^2m_\mu^2+96p^2m_\mu^2-8k^2m_\mu^2-32m_\mu^4+64p^2\left(q\cdot p\right)\\&\quad-16k^2\left(q\cdot p\right)-96\left(q\cdot p\right)m_\mu^2-32\left(q\cdot p\right)^2+16\left(q\cdot k\right)\left(q\cdot p\right)-16\left(q\cdot p\right)\left(q\cdot k\right)+8\left(q\cdot k\right)^2\\&=32m_\mu^4-8k^4+16q^2m_\mu^2-32m_\mu^2\left(q\cdot p\right)-32k^2\left(q\cdot p\right)-32\left(q\cdot p\right)^2+8\left(q\cdot k\right)^2\end{aligned} \tag{19}$$

where we used

$$p^2=m_\mu^2-\frac{1}{4}k^2\,. \tag{20}$$

Next we calculate $\langle 2\rangle$ as

$$\begin{aligned}\langle 2\rangle&=\mathrm{Tr}\left[\left(\not p_1+m_\mu\right)\left(\not p_2+m_\mu\right)\gamma^\mu\left(\not p_2-\not k+m_\mu\right)\not p\left(\not p_1-\not k+m_\mu\right)\gamma_\mu\right]\\&=16q^2p^2m_\mu+16p^4m_\mu-12p^2k^2m_\mu+16p^2m_\mu^3+8k^2\left(q\cdot p\right)m_\mu-32\left(q\cdot p\right)m_\mu^3-32\left(q\cdot p\right)m_\mu\left(q\cdot p\right)\\&\quad+8m_\mu\left(p\cdot k\right)\left(p\cdot k\right)\end{aligned} \tag{21}$$

so that

$$\begin{aligned}\frac{2}{m_\mu}\times\langle 2\rangle&=32q^2p^2+32p^4-24p^2k^2+32p^2m_\mu^2+16k^2\left(q\cdot p\right)-64\left(q\cdot p\right)m_\mu^2-64\left(q\cdot p\right)^2+16\left(p\cdot k\right)^2\\&=32q^2\left(m_\mu^2-\frac{1}{4}k^2\right)+32\left(m_\mu^2-\frac{1}{4}k^2\right)^2-24\left(m_\mu^2-\frac{1}{4}k^2\right)k^2\end{aligned}$$

$$+32\left(m_\mu^2-\frac{1}{4}k^2\right)m_\mu^2+16k^2\left(q\cdot p\right)-64\left(q\cdot p\right)m_\mu^2-64\left(q\cdot p\right)^2$$
$$=32m_\mu^2q^2-8q^2k^2+64m_\mu^4-48k^2m_\mu^2+8k^4+16k^2\left(q\cdot p\right)-64m_\mu^2\left(q\cdot p\right)-64\left(q\cdot p\right)^2\,. \tag{22}$$

Therefore, the trace is

$$\mathrm{Tr}\left[\left(\not{p}_1+m_\mu\right)\left(\gamma_\rho+\frac{k^2+2m_\mu^2}{k^2-4m_\mu^2}\cdot\frac{2p_\rho}{m_\mu}\right)\left(\not{p}_2+m_\mu\right)\gamma^\mu\left(\not{p}_2-\not{q}+m_\mu\right)\gamma^\rho\left(\not{p}_1-\not{q}+m_\mu\right)\gamma_\mu\right]$$
$$=8\left[-2k^2\left(p\cdot q\right)-q^2k^2-\frac{12k^2}{k^2-4m_\mu^2}\left(p\cdot q\right)^2+\left(q\cdot k\right)^2\right]\,. \tag{23}$$

Hence, the form factor $F_2\left(k^2\right)$ is

$$F_2\left(k^2\right)=e^2i\int\frac{d^4q}{\left(2\pi\right)^4}\frac{\frac{8m^2}{k^2-4m_\mu^2}\left(-2\left(p\cdot q\right)-q^2-12\frac{\left(p\cdot q\right)^2}{k^2-4m_\mu^2}+\frac{\left(q\cdot k\right)^2}{k^2}\right)}{\left(q^2+i\varepsilon\right)\left[\left(p_2-q\right)^2-m_\mu^2+i\varepsilon\right]\left[\left(p_1-q\right)^2-m_\mu^2+i\varepsilon\right]}\,. \tag{24}$$

III. THE $k^2\to 0$ LIMIT OF $F_2(k^2)$

Following Refs. [1, 2], from now, we take $k^2\to 0$ limit to get $a_\mu=F_2\left(k^2=0\right)$. We expand $\Gamma^\rho\left(p_2,p_1\right)=\Gamma^\rho\left(p,k\right)$ for $k\ll 1$,

$$\Gamma^\rho\left(p,k\right)\simeq\Gamma^\rho\left(p,0\right)+k^\mu\frac{\partial}{\partial k^\mu}\Gamma^\rho\left(p,k\right)\bigg|_{k=0}\equiv V^\rho\left(p\right)+k^\mu T_{\mu\rho}\left(p\right)\,, \tag{25}$$

where $p=p_1=p_2$ in $k\to 0$ limit.

Here, note that $p\cdot k=0$, from the on-shell condition $p_1^2=p_2^2=m_\mu^2$. The trace projection given in Eq. (23) is explicitly scalar. Hence, we may average the residual k dependence over all spatial directions and this does not change our final result. Here we define the average integration of k as

$$\overline{f(k)}=\int\frac{\Omega\left(p,k\right)}{4\pi}f(k)=g(p)\,, \tag{26}$$

where $f(k)$ and $g(p)$ are n-th rank tensor in k and p, respectively. Here, note that the result of average integration should be the function of p, from $p{\cdot}k=0$, that is, p and k are orthogonal and hence independent to each other. In the expansion of Eq. (25), we may only consider the terms proportional to k^μ and $k^\mu k^\nu$. We first consider the average of $f(k)=k^\mu$. In this case, the integrand is the odd function and therefore,

$$\overline{k^\mu}=\int\frac{d\Omega\left(p,k\right)}{4\pi}k^\mu=0\,. \tag{27}$$

Next, we consider the average of $f(k)=k^\mu k^\nu$. The result might be

$$\overline{k^\mu k^\nu}=\int\frac{d\Omega\left(p,k\right)}{4\pi}k^\mu k^\nu=\alpha g^{\mu\nu}+\beta\frac{p^\mu p^\nu}{p^2}\,, \tag{28}$$

that is, should be the second rank tensor in p. We determine the values of α and β. From $p\cdot k=0$, The inner product between p_μ and $\overline{k^\mu k^\nu}$ must be vanished, that is,

$$\beta=-\alpha\,. \tag{29}$$

Next, we put $g_{\mu\nu}$ on both side and take the summation on μ and ν. This should return k^2 as

$$\int\frac{d\Omega\left(p,k\right)}{4\pi}k^2=k^2=4\alpha+\beta=3\alpha\,. \tag{30}$$

For this, it should be

$$\alpha=\frac{k^2}{3}\,. \tag{31}$$

Therefore, we conclude that the average on $k^\mu k^\nu/k^2$ should be

$$\frac{\overline{k^\mu k^\nu}}{k^2}=\frac{1}{3}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\,. \tag{32}$$

Hence, the numerator of the integrand of Eq. (24) becomes

$$\begin{aligned}
&\lim_{k^2\to 0}\frac{8m_\mu^2}{k^2-4m_\mu^2}\left(-2\left(p\cdot q\right)-q^2-12\frac{\left(p\cdot q\right)^2}{k^2-4m_\mu^2}+\frac{\left(q\cdot k\right)^2}{k^2}\right)\\
&=\frac{8m_\mu^2}{-4m_\mu^2}\left(-2\left(p\cdot q\right)-q^2-12\frac{\left(p\cdot q\right)^2}{-4m_\mu^2}+q_\mu q_\nu\frac{\overline{k^\mu k^\nu}}{k^2}\right)\\
&=-2\left(-2\left(p\cdot q\right)-q^2+\frac{3}{m_\mu^2}\left(p\cdot q\right)^2+\frac{q_\mu q_\nu}{3}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\right)\\
&=-2\left(-2\left(p\cdot q\right)-q^2+3\frac{\left(p\cdot q\right)^2}{m_\mu^2}+\frac{1}{3}q^2-\frac{1}{3}\frac{\left(p\cdot q\right)^2}{m_\mu^2}\right)\\
&=4\left(p\cdot q\right)+\frac{4}{3}q^2-\frac{16}{3}\frac{\left(p\cdot q\right)^2}{m_\mu^2}\,.
\end{aligned} \tag{33}$$

For the denominator of the integrand of Eq. (24), we use k-expansion ($\because k\ll 1$)

$$\frac{1}{\left(p_1-q\right)^2-m_\mu^2+i\varepsilon}\sim\frac{1}{\left(p-q\right)^2-m_\mu^2+i\varepsilon}\left(1+\frac{q\cdot k}{\left(p-q\right)^2-m_\mu^2+i\varepsilon}\right)\,, \tag{34}$$

$$\frac{1}{\left(p_2-q\right)^2-m_\mu^2+i\varepsilon}\sim\frac{1}{\left(p-q\right)^2-m_\mu^2+i\varepsilon}\left(1-\frac{q\cdot k}{\left(p-q\right)^2-m_\mu^2+i\varepsilon}\right)\,, \tag{35}$$

so that

$$\frac{1}{\left[\left(p_2-q\right)^2-m_\mu^2+i\varepsilon\right]\left[\left(p_1-q\right)^2-m_\mu^2+i\varepsilon\right]}=\frac{1}{\left[\left(p-q\right)^2-m_\mu^2+i\varepsilon\right]^2}+\mathcal{O}\left(k^2\right)\,. \tag{36}$$

Now the anomalous magnetic moment of muon with photon one-loop vertex is written as

$$a_\mu=F_2\left(k^2=0\right)=e^2 i\int\frac{d^4q}{\left(2\pi\right)^4}\frac{4\left(p\cdot q\right)+\frac{4}{3}q^2-\frac{16}{3m_\mu^2}\left(p\cdot q\right)^2}{\left[\left(p-q\right)^2-m_\mu^2+i\varepsilon\right]^2\left(q^2+i\varepsilon\right)}\,, \tag{37}$$

where we note that this agrees with the Eq. (2) of Ref. [3].

IV. THE ANGULAR INTEGRATION USING GEGENBAUER POLYNOMIAL

Now we calculate the 4-dimensional angular integration. For it, we first take Wick rotation as

$$\begin{aligned} a_\mu &= -\frac{e^2}{(2\pi)^4}\int d^4q_E \frac{-4\left(p_E\cdot q_E\right)-\frac{4}{3}q_E^2-\frac{16}{3m_\mu^2}\left(p_E\cdot q_E\right)^2}{\left[-\left(p_E-q_E\right)^2-m_\mu^2\right]^2\left(-q_E^2\right)} \\ &= \frac{e^2}{(2\pi)^4}\int d^4q_E \frac{-4\left(p_E\cdot q_E\right)-\frac{4}{3}q_E^2-\frac{16}{3m_\mu^2}\left(p_E\cdot q_E\right)^2}{\left[\left(p_E-q_E\right)^2+m_\mu^2\right]^2 q_E^2} . \end{aligned} \tag{38}$$

We next separate the measure of integration into radial and angular part as

$$\begin{aligned} e^2\int\frac{d^4q_E}{(2\pi)^4} &= e^2\int\frac{d\Omega_{\hat q}}{(2\pi)^4}\cdot\int_0^\infty dq_E q_E^3 = \frac{e^2}{8\pi^2}\int\frac{d\Omega_{\hat q}}{2\pi^2}\cdot\int_0^\infty dq_E q_E^3 \\ &= \frac{\alpha}{2\pi}\int_0^\infty dq_E q_E^3\cdot\int\frac{d\Omega_{\hat q}}{2\pi^2}\quad\left(\because \alpha=\frac{e^2}{4\pi}\right) . \end{aligned} \tag{39}$$

so that

$$\begin{aligned} a_\mu &= \frac{e^2}{(2\pi)^4}\int d^4q_E \frac{-4\left(p_E\cdot q_E\right)-\frac{4}{3}q_E^2-\frac{16}{3m_\mu^2}\left(p_E\cdot q_E\right)^2}{\left[\left(p_E-q_E\right)^2+m_\mu^2\right]^2 q_E^2} \\ &= \frac{\alpha}{4\pi}\int_0^\infty dq_E^2\cdot\int\frac{d\Omega_{\hat q}}{2\pi^2}\frac{-4\left(p_E\cdot q_E\right)-\frac{4}{3}q_E^2-\frac{16}{3m_\mu^2}\left(p_E\cdot q_E\right)^2}{\left[\left(p_E-q_E\right)^2+m_\mu^2\right]^2} . \end{aligned} \tag{40}$$

Hereafter we use Gegenbauer polynomial which is defined as

$$\frac{1}{\left(1-2xt+t^2\right)^\alpha} = \sum_{n=0}^\infty C_n^{(\alpha)}\left(x\right)t^n . \tag{41}$$

To use Gegenbauer polynomial, we have to describe the denominator with t which satisfies

$$\left|p_E\right|\left|q_E\right|t^2-\left(p_E^2+q_E^2+m^2\right)t+\left|p_E\right|\left|q_E\right|=0 \tag{42}$$

so that

$$\frac{1}{\left(p_E-q_E\right)^2+m^2} = \frac{t}{\left|p_E\right|\left|q_E\right|}\left[\frac{1}{1-2\left(\hat p_E\cdot\hat q_E\right)t+t^2}\right] = \frac{t}{\left|p_E\right|\left|q_E\right|}\sum_{n=0}^\infty C_n^{(1)}\left(\hat p_E\cdot\hat q_E\right)t^n . \tag{43}$$

The recursion formulas of Gegenbauer polynomial are [1]

$$C_0^{(1)}\left(\hat p_E\cdot\hat q_E\right)=1 , \tag{44}$$

$$C_1^{(1)}\left(\hat p_E\cdot\hat q_E\right)=2\left(\hat p_E\cdot\hat q_E\right) , \tag{45}$$

$$C_n^{(1)}\left(\hat p_E\cdot\hat q_E\right)=2\left(\hat p_E\cdot\hat q_E\right)\cdot C_{n-1}^{(1)}\left(\hat p_E\cdot\hat q_E\right)-C_{n-2}^{(1)}\left(\hat p_E\cdot\hat q_E\right) , \tag{46}$$

where we need to use

$$C_0^{(1)}\left(\hat{p}_E\cdot\hat{q}_E\right)=1 \tag{47}$$

$$C_1^{(1)}\left(\hat{p}_E\cdot\hat{q}_E\right)=2\left(\hat{p}_E\cdot\hat{q}_E\right) \tag{48}$$

$$C_2^{(1)}\left(\hat{p}_E\cdot\hat{q}_E\right)=4\left(\hat{p}_E\cdot\hat{q}_E\right)^2-1\,. \tag{49}$$

in our case.

Next, we challenge to

$$\begin{aligned}\frac{1}{\left[\left(p_E-q_E\right)^2+m^2\right]^2}&=\frac{1}{\left[-2\left(p_E\cdot q_E\right)+\left(p_E^2+q_E^2+m^2\right)\right]^2}\\&=\frac{1}{\left[-2\left(p_E\cdot q_E\right)+\left|p_E\right|\left|q_E\right|\left(t+t^{-1}\right)\right]^2}\,.\end{aligned} \tag{50}$$

The result is

$$\frac{1}{\left[\left(p_E-q_E\right)^2+m^2\right]^2}=\frac{t^2}{p_E^2q_E^2}\frac{1}{1-t^2}\sum_{n=0}^{\infty}\left(n+1\right)C_n^{(1)}\left(\hat{p}_E\cdot\hat{q}_E\right)t^n\,. \tag{51}$$

Let us prove this. We start from

$$\frac{1}{1-2\left(\hat{p}_E\cdot\hat{q}_E\right)t+t^2}=\sum_{n=0}^{\infty}C_n^{(1)}\left(\hat{p}_E\cdot\hat{q}_E\right)t^n\,. \tag{52}$$

Differentiating Eq. (52) with respect to t, we have

$$\frac{2\left(\hat{p}_E\cdot\hat{q}_E\right)-2t}{\left(1-2\left(\hat{p}_E\cdot\hat{q}_E\right)t+t^2\right)^2}=\sum_{n=1}^{\infty}nC_n^{(1)}\left(\hat{p}_E\cdot\hat{q}_E\right)t^{n-1}\,. \tag{53}$$

Multiplying by t and adding by Eq. (52) to Eq. (53), we have

$$\frac{1-t^2}{\left(1-2\left(\hat{p}_E\cdot\hat{q}_E\right)t+t^2\right)^2}=\sum_{n=0}^{\infty}\left(n+1\right)C_n^{(1)}\left(\hat{p}_E\cdot\hat{q}_E\right)t^n\,. \tag{54}$$

With this, Eq. (51) is proven. Now we describe the numerator of Eq. (40) with Gegenbauer terms

$$\begin{aligned}&-4\left(p_E\cdot q_E\right)-\frac{4}{3}q_E^2-\frac{16}{3m_\mu^2}\left(p_E\cdot q_E\right)^2\\&=-2\left|p_E\right|\left|q_E\right|C_1\left(\hat{p}_E\cdot\hat{q}_E\right)-\frac{4}{3}q_E^2C_0\left(\hat{p}_E\cdot\hat{q}_E\right)-\frac{4}{3}\frac{p_E^2}{m_\mu^2}q_E^2\left[C_2\left(\hat{p}_E\cdot\hat{q}_E\right)+C_0\left(\hat{p}_E\cdot\hat{q}_E\right)\right]\\&=-\frac{4}{3}\left(q_E^2+\frac{p_E^2}{m_\mu^2}q_E^2\right)C_0\left(\hat{p}_E\cdot\hat{q}_E\right)-2\left|p_E\right|\left|q_E\right|C_1\left(\hat{p}_E\cdot\hat{q}_E\right)-\frac{4}{3}\frac{p_E^2}{m_\mu^2}q_E^2C_2\left(\hat{p}_E\cdot\hat{q}_E\right)\,.\end{aligned} \tag{55}$$

For the angular integral, we use

$$\int\frac{d\Omega_{\hat{q}}}{2\pi^2}C_n^{(1)}\left(\hat{a}\cdot\hat{b}\right)C_m^{(1)}\left(\hat{b}\cdot\hat{c}\right)=\frac{\delta_{nm}}{n+1}C_n^{(1)}\left(\hat{a}\cdot\hat{c}\right) \tag{56}$$

so that

$$\int \frac{d\Omega_{\hat{q}}}{2\pi^2} C_0^{(1)}\left(\hat{p}_E \cdot \hat{q}_E\right) C_0^{(1)}\left(\hat{p}_E \cdot \hat{q}_E\right) = \frac{1}{1+0} C_0^{(1)}\left(\hat{p}_E \cdot \hat{p}_E\right) = 1\,, \tag{57}$$

$$\int \frac{d\Omega_{\hat{q}}}{2\pi^2} C_1^{(1)}\left(\hat{p}_E \cdot \hat{q}_E\right) C_1^{(1)}\left(\hat{p}_E \cdot \hat{q}_E\right) = \frac{1}{1+1} C_1^{(1)}\left(\hat{p}_E \cdot \hat{p}_E\right) = \frac{1}{2} \cdot 2\left(\hat{p}_E \cdot \hat{p}_E\right) = 1\,, \tag{58}$$

$$\int \frac{d\Omega_{\hat{q}}}{2\pi^2} C_2^{(1)}\left(\hat{p}_E \cdot \hat{q}_E\right) C_2^{(1)}\left(\hat{p}_E \cdot \hat{q}_E\right) = \frac{1}{1+2} C_2^{(1)}\left(\hat{p}_E \cdot \hat{p}_E\right) = \frac{1}{3}\left\{4\left(\hat{p}_E \cdot \hat{p}_E\right)^2 - 1\right\} = 1\,. \tag{59}$$

Now we have

$$\begin{aligned}
&\int \frac{d\Omega_{\hat{q}}}{2\pi^2} \frac{-4\left(p_E \cdot q_E\right) - \frac{4}{3} q_E^2 - \frac{16}{3m_\mu^2}\left(p_E \cdot q_E\right)^2}{\left[\left(p_E - q_E\right)^2 + m_\mu^2\right]^2} \\
&= \frac{t^2}{p_E^2 q_E^2} \frac{1}{1-t^2}\left[-\frac{4}{3}\left(q_E^2 + \frac{p_E^2}{m_\mu^2} q_E^2\right) - 2\left|p_E\right|\left|q_E\right| \cdot 2t - \frac{4}{3} \cdot \frac{p_E^2}{m^2} q_E^2 \cdot 3t^2\right] \\
&= -\frac{t^2}{m_\mu^2 q_E^2} \frac{1}{1-t^2}\left[-\frac{4}{3}\left(q_E^2 - \frac{m_\mu^2}{m_\mu^2} q_E^2\right) - 4\left|p_E\right|\left|q_E\right| \cdot t + 4 \cdot \frac{m_\mu^2}{m_\mu^2} q_E^2 \cdot t^2\right] \\
&= -\frac{t^2}{q_E^2 m_\mu^2} \frac{1}{1-t^2}\left[-4\left|p_E\right|\left|q_E\right| \cdot t + 4 q_E^2 \cdot t^2\right] \\
&= -\frac{t^2}{1-t^2}\left[-4 \frac{\left|p_E\right|\left|q_E\right|}{m_\mu^2 q_E^2} \cdot t + \frac{4}{m_\mu^2} \cdot t^2\right]\,,
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
t &= \frac{p_E^2 + q_E^2 + m_\mu^2 - \sqrt{\left(p_E^2 + q_E^2 + m_\mu^2\right)^2 - 4 p_E^2 q_E^2}}{2\left|p_E\right|\left|q_E\right|} \\
&= \frac{-m_\mu^2 + q_E^2 + m_\mu^2 - \sqrt{\left(-m_\mu^2 + q_E^2 + m_\mu^2\right)^2 + 4 m_\mu^2 q_E^2}}{2\left|p_E\right|\left|q_E\right|} \\
&= \frac{q_E^2 - \sqrt{q_E^4 + 4 m_\mu^2 q_E^2}}{2\left|p_E\right|\left|q_E\right|}
\end{aligned} \tag{61}$$

and where we analytically continued $p_E^2 \to -m_\mu^2$. Hereafter, we use

$$Z = \frac{t}{\left|p_E\right|\left|q_E\right|} \tag{62}$$

so that

$$Z = \frac{q_E^2 - \sqrt{q_E^4 + 4 m_\mu^2 q_E^2}}{2 p_E^2 q_E^2} = -\frac{q_E^2 - \sqrt{q_E^4 + 4 m_\mu^2 q_E^2}}{2 m_\mu^2 q_E^2}\,. \tag{63}$$

Now the 4-dimensional angular integration becomes

$$\int \frac{d\Omega_{\hat{q}}}{2\pi^2} \frac{-4\left(p_E \cdot q_E\right) - \frac{4}{3} q_E^2 - \frac{16}{3m^2}\left(p_E \cdot q_E\right)^2}{\left[\left(p_E - q_E\right)^2 + m_\mu^2\right]^2}$$

$$
\begin{aligned}
&= -\frac{\left(|p_E|\,|q_E|\,Z\right)^2}{1-\left(|p_E|\,|q_E|\,Z\right)^2}\left[-4\frac{|p_E|\,|q_E|}{m_\mu^2\,|q_E|^2}\cdot\left(|p_E|\,|q_E|\,Z\right)+\frac{4}{m_\mu^2}\cdot\left(|p_E|\,|q_E|\,Z\right)^2\right]\\
&= -\frac{p_E^2q_E^2Z^2}{1-p_E^2q_E^2Z^2}\left[-4\frac{p_E^2Z}{m_\mu^2}+\frac{4p_E^2q_E^2Z^2}{m_\mu^2}\right]\\
&= \frac{m_\mu^2q_E^2Z^2}{1+m_\mu^2q_E^2Z^2}\left[4\frac{m_\mu^2Z}{m_\mu^2}-\frac{4m_\mu^2q_E^2Z^2}{m_\mu^2}\right]\\
&= \frac{m_\mu^2q_E^2Z^2\left(4Z-4q_E^2Z^2\right)}{1+m_\mu^2q_E^2Z^2}\\
&= \frac{4m_\mu^2q_E^2Z^3\left(1-q_E^2Z\right)}{1+m_\mu^2q_E^2Z^2}\,.
\end{aligned}
\tag{64}
$$

Hereafter we represent the radial momentum q_E^2 as q^2 so that

$$
\int\frac{d\Omega_{\hat{q}}}{2\pi^2}\frac{-4\left(p_E\cdot q_E\right)-\frac{4}{3}q_E^2-\frac{16}{3m^2}\left(p_E\cdot q_E\right)^2}{\left[\left(p_E-q_E\right)^2+m_\mu^2\right]^2}=4\frac{m_\mu^2q^2Z^3\left(1-q^2Z\right)}{1+m_\mu^2q^2Z^2}\,.
\tag{65}
$$

Finally we get

$$
\begin{aligned}
a_\mu = F_2\left(k^2=0\right) &= e^2 i\int\frac{d^4q}{\left(2\pi\right)^4}\frac{4\left(p\cdot q\right)+\frac{4}{3}q^2+\frac{16}{3m_\mu^2}\left(p\cdot q\right)^2}{\left[\left(p-q\right)^2-m_\mu^2+i\varepsilon\right]^2\left(q^2+i\varepsilon\right)}\\
&= \frac{\alpha}{\pi}\int_0^\infty dq^2\cdot f\left(q^2\right)
\end{aligned}
\tag{66}
$$

where

$$
f\left(q^2\right)=\frac{m_\mu^2q^2Z^3\left(1-q^2Z\right)}{1+m_\mu^2q^2Z^2}\,,
\tag{67}
$$

$$
Z=-\frac{q^2-\sqrt{q^4+4m_\mu^2q^2}}{2m_\mu^2q^2}\,.
\tag{68}
$$

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