Homework: Quantum Field Theory #3

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1. We know that

$$\gamma_W^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_W^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{1}$$

$$\gamma_D^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_D^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2}$$

and it must have the unitary transformation relation

$$\gamma_W^\mu = U\gamma_D^\mu U^\dagger \tag{3}$$

Now we start with γ^0. Form linear algebra, we know how to diagonalize a unitary matrix. It's easy to find

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and after verifying, it consists with the other three matrices.

2. Verify $[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu_\nu\gamma^\nu$.

From the definition of $(\mathcal{J}^{\rho\sigma})_{\mu\nu}$, we have

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha) \tag{4}$$

which means

$$(\mathcal{J}^{\rho\sigma})^\mu_\nu = i(g^{\rho\mu}\delta^\sigma_\nu - g^{\sigma\mu}\delta^\rho_\nu)$$

With $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and $\rho \neq \sigma$ (otherwise the entire term vanished and it's a trival situation) which means $\gamma^\rho\gamma^\sigma = -\gamma^\sigma\gamma^\rho$ (Of course we can compute the g metrice but it would take time and this shall do the trick.)

$$\begin{aligned}
[\gamma^\mu, S^{\rho\sigma}] &= \frac{i}{4}[\gamma^\mu, [\gamma^\rho, \gamma^\sigma]] \\
&= \frac{i}{4}(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho + \gamma^\sigma\gamma^\rho\gamma^\mu - \gamma^\rho\gamma^\sigma\gamma^\mu) \\
&= \frac{i}{4}(2g^{\mu\rho}\gamma^\sigma - \gamma^\rho\gamma^\mu\gamma^\sigma - 2g^{\mu\sigma}\gamma^\rho + \gamma^\sigma\gamma^\mu\gamma^\rho + 2g^{\mu\rho}\gamma^\sigma - \gamma^\sigma\gamma^\mu\gamma^\rho - 2g^{\mu\sigma}\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\sigma) \\
&= i(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho) \\
&= i(g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu)\gamma^\nu \\
&= (\mathcal{J}^{\rho\sigma})^\mu_\nu\gamma^\nu
\end{aligned}$$

3. Derive the Schrödinger-Pauli equation.

For electron in an EM field, Dirac equation can be written as

$$[i\frac{\partial}{\partial t} + e\phi - \boldsymbol{\alpha}\cdot(\mathbf{P}+e\mathbf{A}) - m\beta]\psi = 0 \tag{5}$$

where $\mathbf{P} = -i\nabla$. Set

$$\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} e^{-imt}$$

so that the certain part of electron rest mass can be removed. Then we have

$$\begin{aligned} i\frac{\partial}{\partial t}\varphi &= \boldsymbol{\sigma}\cdot(\mathbf{P}+e\mathbf{A})\chi - e\phi\varphi \\ i\frac{\partial}{\partial t}\chi &= \boldsymbol{\sigma}\cdot(\mathbf{P}+e\mathbf{A})\varphi - e\phi\chi - 2m\chi \end{aligned}$$

At nonrelativistic limit, we have

$$\chi \approx \frac{1}{2m}\boldsymbol{\sigma}\cdot(\mathbf{P}+e\mathbf{A})\varphi \tag{6}$$

where $\chi/\varphi \ll 1$. Then

$$i\frac{\partial}{\partial t}\varphi = \frac{1}{2m}[\boldsymbol{\sigma}\cdot(\mathbf{P}+e\mathbf{A})]^2\varphi - e\phi\varphi$$

Using

$$\begin{aligned} [\boldsymbol{\sigma}\cdot(\mathbf{P}+e\mathbf{A})]^2 &= (\mathbf{P}+e\mathbf{A})^2 + i\boldsymbol{\sigma}\cdot[(\mathbf{P}+e\mathbf{A})\times(\mathbf{P}+e\mathbf{A})] \\ &= (\mathbf{P}+e\mathbf{A})^2 + ie\boldsymbol{\sigma}\cdot[\mathbf{P}\times\mathbf{A}+\mathbf{A}\times\mathbf{P}] \\ &= (\mathbf{P}+e\mathbf{A})^2 + e\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{A}) \\ &= (\mathbf{P}+e\mathbf{A})^2 + e\boldsymbol{\sigma}\cdot\boldsymbol{B} \end{aligned}$$

Then (6) becomes

$$\begin{aligned} i\frac{\partial}{\partial t}\varphi &= \left[\frac{1}{2m}(\mathbf{P}+e\mathbf{A})^2 + \frac{e}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B} - e\phi\right]\varphi \qquad (7) \\ &= \left[\frac{1}{2m}(\mathbf{P}+e\mathbf{A})^2 - \boldsymbol{\mu}\cdot\boldsymbol{B} - e\phi\right]\varphi \qquad (8) \end{aligned}$$

where $\boldsymbol{\mu} = -\frac{e\hbar}{2mc}\boldsymbol{\sigma}$ (in our previous calculation $\hbar = c = 1$) is the intrinsic magnetic moment of electron. And (7) is the standard form of the Schrödinger-Pauli equation.