Homework: Gauge Field Theory

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1. Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda^2}{4}\phi^4$$

which satisfies

$$\phi \to -\phi$$

For such symmetry to break, we perform the following presedure:

First, the minimum of $V(\phi)$ can be found in $\phi = \pm\frac{\mu^2}{\lambda^2}$, and we can define $v^2 = |\langle 0|\phi|0\rangle|^2 = \frac{\mu^2}{\lambda}$, which yields the broken symmetry of vacuum.

By redefining the field $\phi(x) = \rho(x) + v$ such that $\rho(x)$ has the right vacuum, the Lagrangian is now

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 - \mu^2\rho^2 - \lambda^2\rho^3 v - \frac{\lambda^2}{4}\rho^4 + \frac{\mu^4}{4\lambda^2}$$

and we can see that there is no massless Goldstone particle. That's because that although the symmetry $\phi \to -\phi$ has broken, but it's discrete symmetry, therefore can't produce Goldstone particles.

2. **R$_\xi$ Gauge.** The Lagrangian is

$$\mathcal{L}(\phi, A^\mu) = (D^\mu\phi)^\dagger(D_\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where $D^\mu = \partial^\mu + igA^\mu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Also we have $|\langle 0|\phi|0\rangle| = v$, $v^2 = \frac{\mu^2}{\lambda}$.

R_ξ gauge

$$\mathcal{L} \to \mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu - \xi g v b)^2$$

Choose ϕ to be $\phi = \frac{1}{\sqrt{2}}(v + h(x) + ib(x))$,

$$D^\mu\phi = \frac{1}{\sqrt{2}}[\partial^\mu h + i\partial^\mu b + igA^\mu(v+h) - gbA^\mu] = \frac{1}{\sqrt{2}}[(\partial^\mu h - gbA^\mu) + i(\partial^\mu b + g(v+h)A^\mu)]$$

so the kinetic term

$$(D^\mu\phi)^\dagger(D_\mu\phi) = \frac{1}{2}[(\partial^\mu h - gbA^\mu)^2 + (\partial^\mu b + g(v+h)A^\mu)^2]$$

this gives

$$\begin{aligned}(D^\mu\phi)^\dagger(D_\mu\phi) &= \frac{1}{2}\partial^\mu h\partial_\mu h - gb\partial^\mu hA_\mu + \frac{1}{2}g^2b^2A^\mu A_\mu + \frac{1}{2}\partial^\mu b\partial_\mu b + g(v+h)\partial^\mu bA_\mu + \frac{1}{2}g^2(v+h)^2A^\mu A_\mu \\ &= \frac{1}{2}\partial^\mu h\partial_\mu h + \frac{1}{2}\partial^\mu b\partial_\mu b + \frac{1}{2}g^2v^2A^\mu A_\mu + gv\partial^\mu bA_\mu + g^2vhA^\mu A_\mu + \frac{1}{2}g^2(b^2+h^2)A^\mu A_\mu + g(h\partial^\mu b - b\partial^\mu h)A_\mu\end{aligned}$$

now we got the kinetic terms of scalar fields $h(x)$ and $b(x)$, mass term for gauge field A^μ, crossing term of b and A^μ, and some interacting terms in the end.

The mass term of original scala field gives

$$\mu^2\phi^\dagger\phi = \frac{1}{2}\mu^2(v+h)^2 - \frac{1}{2}\mu^2b^2$$

so the rest part of scalar field is

$$-\frac{b^4\lambda}{4}-\frac{1}{2}b^2h^2\lambda-b^2h\lambda v-\frac{h^4\lambda}{4}-h^3\lambda v-h^2\mu^2+\frac{\mu^4}{4\lambda}$$

Now the gauge fixing term is

$$-\frac{1}{2\xi}\partial^\mu A_\mu\partial^\nu A_\nu+gvb\partial_\mu A^\mu-\frac{\xi g^2v^2}{2}b^2$$

we know that $F^{\mu\nu}F_{\mu\nu}$ can always be written in two terms, so

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}=-\frac{1}{2}(\partial^\mu A_\nu)^2+\frac{1}{2}(1-\xi^{-1})(\partial^\mu A_\mu)^2+gvb\partial^\mu A_\mu-\frac{\xi g^2v^2}{2}b^2$$

and

$$gvb\partial^\mu A_\mu=-gvA_\mu\partial^\mu b$$

the crossing term is cancelled. The last term also gives b field mass $\frac{\xi g^2v^2}{2}$.

The Lagrangian is now

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}\partial^\mu h\partial_\mu h+\frac{1}{2}\partial^\mu b\partial_\mu b-\frac{1}{2}(\partial^\mu A_\nu)^2+\frac{1}{2}(1-\xi^{-1})(\partial^\mu A_\mu)^2+\frac{1}{2}g^2v^2A^\mu A_\mu-\mu^2h^2-\frac{\xi g^2v^2}{2}b^2(+\frac{\mu^4}{4\lambda})\\&+g^2vhA^\mu A_\mu+\frac{1}{2}g^2(b^2+h^2)A^\mu A_\mu+g(h\partial^\mu b-b\partial^\mu h)A_\mu-\frac{b^4\lambda}{4}-b^2h\lambda v-\frac{h^4\lambda}{4}-h^3\lambda v\end{aligned}$$

Then we have some standard 3 and 4 particle vertexs. Now we just need to deal with the propagators and the vertex with derivative.

The propagators of both scalar fields are trival, with $m_h=\sqrt{2}\mu$, $m_b=\sqrt{\xi}gv$. The propagator of the vector field is, however, a bit more complicated.

$$\Delta_A^{\mu\nu}(x-y)=\frac{g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}}{k^2-m^2+i\epsilon}+\frac{\xi\frac{k^\mu k^\nu}{k^2}}{k^2-\xi m^2+i\epsilon}$$

where the mass of vector field $m=gv$.

Now we'll show how to derive the propagator: Define $\mathcal{L}_0$

$$\mathcal{L}_0=-\frac{1}{2}\partial_\mu A^\nu\partial^\mu A_\nu+\frac{1}{2}(1-\xi^{-1})\partial^\nu A_\mu\partial^\mu A_\nu+\frac{1}{2}m^2A^\nu A_\nu$$

and

$$S_0=\int \mathrm{d}^4x\mathcal{L}_0$$

Transform to momentum space

$$S_0=-\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Big\{\tilde{A}_\mu(k)\big(g^{\mu\nu}k^2-(1-\xi^{-1})k^\mu k^\nu-m^2g^{\mu\nu}\big)\tilde{A}_\nu(-k)-\tilde{J}^\mu(k)\tilde{A}_\mu(-k)-\tilde{J}^\mu(-k)\tilde{A}_\mu(k)\Big\}$$

Define $\tilde{D}^{\mu\nu}(k)=g^{\mu\nu}k^2-(1-\xi^{-1})k^\mu k^\nu-m^2g^{\mu\nu}$

$$\begin{aligned}\tilde{D}^{\mu\nu}(k)&=g^{\mu\nu}k^2-(1-\xi^{-1})k^\mu k^\nu-m^2g^{\mu\nu}\\&=(k^2-m^2)g^{\mu\nu}-(1-\xi^{-1})k^\mu k^\nu\\&=(k^2-m^2)(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2})+(k^2-m^2)\frac{k^\mu k^\nu}{k^2}-(1-\xi^{-1})k^\mu k^\nu\\&=(k^2-m^2)(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2})+\xi^{-1}(k^2-\xi m^2)\frac{k^\mu k^\nu}{k^2}\end{aligned}$$

then to have the result

$$S_0=-\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{J}_\mu(k)\tilde{\Delta}_F^{\mu\nu}(k)\tilde{J}_\nu(-k)$$

we must have

$$\tilde{D}_{\mu\nu}\tilde{\Delta}_F^{\nu\rho}=\delta_\mu^\rho$$

that is

$$
\begin{aligned}
&\tilde{D}_{\mu\nu}(k)\tilde{\Delta}_F^{\nu\rho}(k) = \delta_\mu^\rho \\
=&\left\{(k^2-m^2)(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}) + \xi^{-1}(k^2-\xi m^2)\frac{k_\mu k_\nu}{k^2}\right\}\{Ag^{\nu\rho}+Bk^\nu k^\rho\} \\
=&A(k^2-m^2)\delta_\mu^\rho - A(k^2-m^2)\frac{k_\mu k^\rho}{k^2} + \xi^{-1}(k^2-\xi m^2)Ak_\mu k^\rho + \xi^{-1}(k^2-\xi m^2)Bk_\mu k^\rho
\end{aligned}
$$

such that $A=\frac{1}{k^2-m^2+i\epsilon}$ and $B=\frac{\xi}{(k^2-\xi m^2)k^2}-\frac{1}{k^2(k^2-m^2)}$ (with the Feynman prescription). The propagator is now

$$
\tilde{\Delta}_F^{\mu\nu}(k) = \frac{g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}}{k^2-m^2+i\epsilon} + \frac{\xi k^\mu k^\nu/k^2}{k^2-\xi m^2+i\epsilon}
$$

We still have to deal with the term with derivative.

3. $Z^0 \to l\bar{l}$.