Homework: Particle Physics #1

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1. $\gamma^0(\gamma^\mu)^\dagger\gamma^0=\gamma^\mu$

$$\gamma^0{\gamma^0}^\dagger\gamma^0=\gamma^0, \gamma^0{\gamma^i}^\dagger\gamma^0=2\delta_i^0-{\gamma^i}^\dagger=-{\gamma^i}^\dagger=\gamma^i$$

$$\Longrightarrow \gamma^0(\gamma^\mu)^\dagger\gamma^0=\gamma^\mu \qquad (\text{Used } {\gamma^i}^\dagger=\gamma_i=-\gamma^i)$$

2. From the orthogonal condition, we have the orthogonal condition of spinor ξ: $\xi^\dagger\xi=1$, then

$$\begin{aligned}\sum_s u_s(p)\bar{u}_s(p)&=\sum_s\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar\sigma}\xi^s\end{pmatrix}\begin{pmatrix}\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \xi^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\\ &=\sum_s\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\end{aligned}$$

(Use $\sum_s\xi^s\xi^{s\dagger}=\mathbb{1}$)

$$\begin{aligned}&=\begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar\sigma & m\end{pmatrix}\\ &=\not{p}+m\end{aligned}$$

Similarly, $\sum_s v_s(p)\bar{v}_s(p)=\not{p}-m$.

3. $m=0$, $k^\mu=\{k_0,k_x,k_y,k_z\}$, there's two transverse polarization vector, we can choose (note that the orthogonal and completeness conditions are also considered)

$$k\cdot\epsilon=0\Longrightarrow \epsilon^1=\left(0,\frac{\sqrt{\mathrm{ky}^2+\mathrm{kz}^2}}{\sqrt{\mathrm{kx}^2+\mathrm{ky}^2+\mathrm{kz}^2}},-\frac{\mathrm{kxky}}{\sqrt{\mathrm{ky}^2+\mathrm{kz}^2}\sqrt{\mathrm{kx}^2+\mathrm{ky}^2+\mathrm{kz}^2}},-\frac{\mathrm{kxkz}}{\sqrt{\mathrm{ky}^2+\mathrm{kz}^2}\sqrt{\mathrm{kx}^2+\mathrm{ky}^2+\mathrm{kz}^2}}\right),\epsilon^2=\left(0,0,\frac{\mathrm{kz}}{\sqrt{\mathrm{ky}^2+\mathrm{kz}^2}},-\frac{\mathrm{ky}}{\sqrt{\mathrm{ky}^2+\mathrm{kz}^2}}\right)$$

4. Consider a general transformation

$$f(x)\to f'(x')$$

and define

$$\delta f\equiv f'(x')-f(x)=f'(x+\delta x^\mu)-f(x)\cong f'(x)-f(x)+\delta x^\mu\partial_\mu f+\mathcal{O}(\delta x^2)$$

and also define

$$\delta_0 f=f'(x)-f(x)$$

then

$$\delta f=\delta_0 f+\delta x^\mu\partial_\mu f$$

Now we deal with this problem from the least action principle first

$$\begin{aligned}\delta S&=0\\ &=\int \mathrm{d}^4x\delta\mathcal{L}+\int\delta(\mathrm{d}^4x)\mathcal{L}\end{aligned}$$

And

$$\begin{aligned}\delta\mathcal{L} &= \delta_0\mathcal{L} + \delta x^\mu\partial_\mu\mathcal{L} \\ &= \frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0(\partial_\mu\phi) + \delta x^\mu\partial_\mu\mathcal{L} \\ &= \delta x^\mu\partial_\mu\mathcal{L} + (\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta_0\phi + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi)\end{aligned}$$

The other part

$$\delta(\mathrm{d}^4x) = \frac{\partial\delta x^\mu}{\partial x^\mu}\mathrm{d}^4x = \partial_\mu(\delta x^\mu)\mathcal{L}$$

which can be derived by

$$\mathrm{d}^4x' = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right|\mathrm{d}^4x = \left|\frac{\partial(x^\mu + \delta x^\mu)}{\partial x^\nu}\right|\mathrm{d}^4x = (1 + \frac{\partial\delta x^\mu}{\partial x^\mu})\mathrm{d}^4x$$

And the whole part of δS becomes (applying Euler-Lagarange equation)

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\delta x^\mu\partial_\mu\mathcal{L} + (\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta_0\phi + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi) + \partial_\mu(\delta x^\mu)\mathcal{L} \\ &= \int \mathrm{d}^4x\partial_\mu(\delta x^\mu\mathcal{L}) + (\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)})\delta_0\phi + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi) \\ &= \int \mathrm{d}^4x\partial_\mu(\delta x^\mu\mathcal{L}) + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi)\end{aligned}$$

Note that $\delta_0\phi = \delta\phi - \delta x^\mu\partial_\mu\phi$, and

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x\partial_\mu(\delta x^\mu\mathcal{L}) + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta x^\rho\partial_\rho\phi) \\ &= \int \mathrm{d}^4x\partial_\mu\{\delta x^\mu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta x^\rho\partial_\rho\phi\} \\ &= \int \mathrm{d}^4x\partial_\mu\{(\mathcal{L}g^\mu_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi)\delta x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\}\end{aligned}$$

5. Given

$$\begin{aligned}P\psi P^{-1} &= \eta\gamma^0\psi(t, -\mathbf{x}) \\ P\bar{\psi}P^{-1} &= \eta^*\bar{\psi}(t, -\mathbf{x})\gamma^0\end{aligned}$$

The field operators are

$$\begin{aligned}\psi(x) &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(a^s_\mathbf{p}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_\mathbf{p}v^s(p)e^{ip\cdot x}) \\ \bar{\psi}(x) &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_s(b^s_\mathbf{p}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_\mathbf{p}\bar{u}^s(p)e^{ip\cdot x})\end{aligned}$$

To satisfy the first two equations when acted on by P operator, the creation & annihilation operators must obey

$$Pa^s_\mathbf{p}P^{-1} = \eta a^s_{-\mathbf{p}}, Pb^s_\mathbf{p}P^{-1} = -\eta^*b^s_{-\mathbf{p}}$$

(Note that $u(p^0, \mathbf{p}) = \gamma^0u(p^0, -\mathbf{p})$, $v(p^0, \mathbf{p}) = -\gamma^0v(p^0, -\mathbf{p})$)

6. Given

$$\begin{aligned}C\psi C^{-1} &= -i(\bar{\psi}\gamma^0\gamma^2)^T = -i\gamma^2\psi^* \\ C\bar{\psi}C^{-1} &= -i(\gamma^0\gamma^2\psi)^T = i\bar{\psi}^*\gamma^2\end{aligned}$$

Similarly, the creation & annihilation operators must obey

$$Ca^s_\mathbf{p}C^{-1} = b^s_\mathbf{p}, Cb^s_\mathbf{p}C^{-1} = a^s_\mathbf{p}$$

(Note that $u^s(p) = -i\gamma^2(v^s(p))^*$, $v^s(p) = -i\gamma^2(u^s(p))^*$)

7. Prove Landau-Yang theorem.

For any vector particles,we can always write the field operator as a single vector field.

$$A_\mu(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\mathbf{k}|}}\sum_\lambda (a^\lambda_{\mathbf{k}}\epsilon^\lambda_\mu(k)e^{-ik\cdot x} + a^{\lambda\dagger}_{\mathbf{k}}\epsilon^{\lambda *}_\mu(k)e^{ik\cdot x})$$

Then the feynman rules can be easily derived. The amplitude of $vector \to \gamma\gamma$ is

$$i\mathcal{M} = \epsilon_1^{*\mu}(p_1)\epsilon_2^{*\nu}(p_2)\epsilon^\alpha(p)\Gamma_{\mu\nu\sigma}$$

since it must obey Lorentz-invariant

$$= (\epsilon_1\cdot\epsilon_2)(a_1\epsilon\cdot p_1 + a_2\epsilon\cdot p_2) + a_3(\epsilon_1\cdot\epsilon)(\epsilon_2\cdot p_1) + a_4(\epsilon_2\cdot\epsilon)(\epsilon_1\cdot p_2)$$

final states symmetry (identical), $a_1 = a_2$, first term vanishes. And $\epsilon_2\cdot p_1 = \epsilon_1\cdot p_2 = 0$

$$= 0$$

8. Tensor field.

At lowest order, we can assume there's no propagator.

$$i\mathcal{M} = \varepsilon_{\mu\nu}(p)(i\Gamma^{\mu\nu\rho\sigma}(k_1,k_2))\epsilon^*_{1\rho}(k_1)\epsilon^*_{2\sigma}(k_2)$$

Γ is the effective vertex.

Use Lorenz covariance

$$i\mathcal{M} = \varepsilon\cdot(ak_1+bk_2)(ck_1+dk_2)(\epsilon_1\cdot\epsilon_2) + e(\epsilon_1\cdot k_2)(\varepsilon\cdot\epsilon_2 k_1) + f(\epsilon_2\cdot k_1)(\varepsilon\cdot\epsilon_1 k_2) + j\varepsilon^\mu_\mu(\epsilon_1\cdot\epsilon_2)$$

with $\epsilon_1\cdot k_2 = \epsilon_2\cdot k_1 = 0$,

$$i\mathcal{M} = \varepsilon\cdot(ak_1+bk_2)(ck_1+dk_2)(\epsilon_1\cdot\epsilon_2) + j\varepsilon^\mu_\mu(\epsilon_1\cdot\epsilon_2)$$

9. $n \to p + e^- + \bar{\nu}_e$.

In centre-of-mass frame, the range is $0 \sim \sqrt{(m_n-m_p)^2 - m_e^2}$.

10. (a). Use natual units:

$$\psi(r) = \frac{g_0}{4\pi r}e^{-rm}$$

and the equation

$$(\nabla^2 - m^2)\psi = 0$$

Obviously if we put ψ in the equation there's a m^2 factor cancelled with the original m^2, so ψ is a root of the equation.

If closer look is taken, the Laplacian part should be (the coefficients are neglected)

$$\frac{m^2e^{-mr}}{r} + \frac{2e^{-mr}}{r^3} + \frac{2me^{-mr}}{r^2} + \frac{2\left(-\frac{e^{-mr}}{r^2} - \frac{me^{-mr}}{r}\right)}{r} = \frac{m^2e^{-mr}}{r}$$

(b). Add the source

$$(\nabla^2 - m^2)\psi = J^\mu$$

11. $\pi^0 \to \gamma\gamma$.

The quantum numbers J^{PC} of π is 0^{-+} with zero spin and orbital angular momentum while for γ it's 1^{--} with spin 1. Assuming we don't know anything about π, the final state angular momentum gives $J(\pi) = J(\gamma\gamma) = L + S$ is even. Since L is odd, the final state system parity is $P = (-)^{L=1} = -$. And $J(\pi) = 0$ (can also be proved by Furry's theorem, $J(\pi)$ can't be 1).

12. $e^+e^- \to \gamma^* \to J/\psi$.

Assuming $\gamma - J/\psi$ vertex is $(-iag^{\mu\nu})$

$$i\mathcal{M}^{sr\lambda} = i\frac{ea}{M^2}\bar{v}^s(p_1)\not{\epsilon}^{\lambda}(p_1+p_2)u^r(p_2)$$

Use kinematics (centre-of-mass), and choose circular polarization

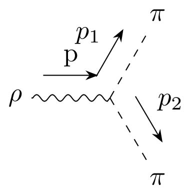
$$\epsilon^0 = (0,0,0,1), \epsilon^+ = (0,1,i,1), \epsilon^- = (0,1,-i,1)$$

and only

$$M^{12+} = M^{21-} = \frac{2ae}{M}$$

don't vanish. So J/ψ is circular polarized.

13. $\rho \to \pi\pi$. ρ is vector particle and π is pesudoscalar particle.

$i\mathcal{M} =$ 

$= i(ap_1 + bp_2)\cdot\epsilon_\rho(p)$

So

$$\sum|\mathcal{M}|^2 = \sum[(ap_1+bp_2)\cdot\epsilon_\rho(p)]^2 = a(p_1-p_2)^2 \times (I, I_3)$$

14. In $2 \to 3$ decay, there're total 4 independent 4-momentums, so 5 Lorenz scalars.

For n particles, there're $\frac{n(n-3)}{2}$.

15. 3-particle decay phase space integrals can be derived as follows:

First we know that the gengeral non-relativistic expression for N-body phase space is

$$\mathrm{d}n = (2\pi)^3\prod_{i=1}^{N}\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3}\delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N}\mathbf{p}_i\right) \tag{1}$$

where $\mathbf{p}_a$ is the momentum of the decaying particle. (This expression can be derived easily from the phase space volume of each particle.) According to Fermi's golden rule (and notice the $(2E_i)^{1/2}$ ratio difference between $\mathcal{M}_{fi}$ and T_{fi}), the decay rate can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\int|\mathcal{M}_{fi}|^2\delta(E_a - \sum_{i=1}^{N}E_i)\delta^3(\mathbf{p}_a - \sum_{i=1}^{N}\mathbf{p}_i)\prod_{i=1}^{N}\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i} \tag{2}$$

So for 3-particle decay the decay rate can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\int|\mathcal{M}_{fi}|^2\delta(E_a - E_1 - E_2 - E_3)\delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3}$$

Now we consider it in the centre-of-mass frame of the decaying particle A, which means $E_a = m_a$ and $\mathbf{p}_a = 0$. Through the integration of delta function and $\mathrm{d}^3\mathbf{p}_3$, we have

$$\begin{aligned}
\Gamma_{fi} &= \frac{(2\pi)^4}{2m_a}\int|\mathcal{M}_{fi}|^2\delta(m_a - E_1 - E_2 - E_3)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3} \\
&= \frac{(2\pi)^4}{2m_a}\int|\mathcal{M}_{fi}|^2\delta(m_a - E_1 - E_2 - E_3)\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2}\frac{1}{(2\pi)^3 2E_3} \\
&= \frac{1}{2^4 m_a(2\pi)^5}\int\frac{|\mathcal{M}_{fi}|^2}{E_1E_2E_3}\delta(m_a - E_1 - E_2 - E_3)\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2 \\
&= \frac{1}{2^4 m_a(2\pi)^5}\int\frac{|\mathcal{M}_{fi}|^2}{E_1E_2E_3}\delta(m_a - E_1 - E_2 - E_3)\mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2 \\
&= \frac{1}{2^4 m_a(2\pi)^5}\int\frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1|^2|\mathbf{p}_2|^2\mathrm{d}p_1\,\mathrm{d}(\cos\theta_1)\,\mathrm{d}\phi_1\mathrm{d}p_2\,\mathrm{d}(\cos\theta_2)\,\mathrm{d}\phi_2}{E_1E_2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}} \\
&\qquad\qquad \delta(m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2})
\end{aligned}$$

where θ_1, ϕ_1 and ϕ_2 are independent of the integral and therefore can be integrated first. Note that

$$\frac{\mathrm{d}|\mathbf{p}_i|}{\mathrm{d}E_i} = \frac{E_i}{|\mathbf{p}_i|}$$

which means

$$\mathrm{d}|\mathbf{p}_i| = \frac{E_i}{|\mathbf{p}_i|}\mathrm{d}E_i$$

and mark the kernel of δ function as

$$f(\cos\theta_2) \equiv m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}$$

we have

$$f'(\cos\theta_2) = -\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}}$$

and the real root of $f(\cos\theta_2) = 0$ is

$$\cos\theta_2' = \frac{(m_a - E_1 - E_2)^2 - m_3^2 - |\mathbf{p}_1|^2 - |\mathbf{p}_2|^2}{2|\mathbf{p}_1||\mathbf{p}_2|}$$

So we have

$$\begin{aligned}
&\delta(m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}) \\
=&\frac{\delta(\cos\theta_2 - \cos\theta_2')}{\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}}
\end{aligned}$$

And the original formula becomes

$$\begin{aligned}
\Gamma_{fi} &= \frac{8\pi^2}{2^4 m_a (2\pi)^5}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1|^2|\mathbf{p}_2|^2\mathrm{d}p_1\mathrm{d}p_2\,\mathrm{d}(\cos\theta_2)}{E_1E_2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}}\frac{\delta(\cos\theta_2 - \cos\theta_2')}{\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}} \\
&= \frac{1}{2^3 m_a (2\pi)^3}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1|^2|\mathbf{p}_2|^2\mathrm{d}p_1\mathrm{d}p_2}{E_1E_2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}\frac{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}{2|\mathbf{p}_1||\mathbf{p}_2|} \\
&= \frac{1}{2^3 m_a (2\pi)^3}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1||\mathbf{p}_2|\mathrm{d}p_1\mathrm{d}p_2}{E_1E_2} \\
&= \frac{1}{2^3 m_a (2\pi)^3}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1||\mathbf{p}_2|\frac{E_1}{|\mathbf{p}_1|}\mathrm{d}E_1\frac{E_2}{|\mathbf{p}_2|}\mathrm{d}E_2}{E_1E_2} \\
&= \frac{1}{8 m_a (2\pi)^3}\int |\mathcal{M}_{fi}|^2\mathrm{d}E_1\mathrm{d}E_2
\end{aligned}$$

which is exactly the form of square Dalitz plot. Transform it a little bit and we have the standard form of Dalitz plot (note that $s_2 = (p_2 + p_3)^2 = (p_a - p_1)^2 \to \mathrm{d}s_2 = -2m_a\mathrm{d}E_1$ and similar for s_3)

$$\Gamma_{fi} = \frac{1}{32 m_a (2\pi)^3}\int |\mathcal{M}_{fi}|^2\mathrm{d}s_2\mathrm{d}s_3$$

Now let's review another form of the standard Dalitz form

$$\frac{\mathrm{d}\Gamma_{fi}}{\mathrm{d}s_2\mathrm{d}s_3} = \frac{1}{32 m_a (2\pi)^3}|\mathcal{M}_{fi}|^2$$

and its physical meaning is obvious: the density of data points on a Dalitz plot is proportional to the decay matrix element.

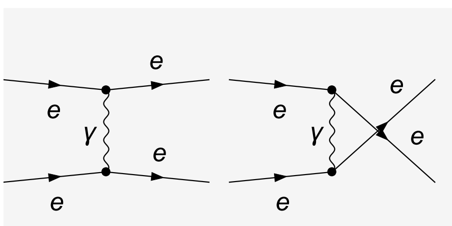
16. $J^{PC}(^{2S+1}L_J)$ of $q\bar{q}$ system. ($P=(-)^{L+1}$, $C=(-)^{L+S}$)

$L\backslash S$	0	1
0	$0^{-+}(^1S_0)$	$1^{--}(^3S_1)$
1	$1^{+-}(^1P_1)$	$0^{++}(^3P_0)$
		$1^{++}(^3P_1)$
		$2^{++}(^3P_2)$
2	$2^{-+}(^1D_2)$	$1^{--}(^3D_1)$
		$2^{--}(^3D_2)$
		$3^{--}(^3D_3)$

17. Moller scattering.

Use FeynCalc.

```
<<FeynCalc`;
topMoeller = CreateTopologies[0, 2 -> 2];
diagsMoeller = InsertFields[topMoeller, {F[2, {1}], F[2, {1}]} -> {F[
                   2, {1}], F[2, {1}]}, InsertionLevel -> {Classes}, Model -> "SM",
                   ExcludeParticles -> {S[1], S[2], V[2]}];
Paint[diagsMoeller, ColumnsXRows -> {2, 1}, Numbering -> None,
  SheetHeader->None,ImageSize->{512,256}];
```




```
ampMoeller=FCFAConvert[CreateFeynAmp[diagsMoeller, Truncated -> False],
  IncomingMomenta->{p1,p2},OutgoingMomenta->{k1,k2},UndoChiralSplittings->True,
  ChangeDimension->4,List->False,SMP->True]
```

$$i\mathcal{M}=\frac{\bar{g}^{\text{Lor1Lor2}}\left(\varphi(\overline{\text{k1}},m_e)\right).\left(ie\bar{\gamma}^{\text{Lor1}}\right).\left(\varphi(\overline{\text{p1}},m_e)\right)\left(\varphi(\overline{\text{k2}},m_e)\right).\left(ie\bar{\gamma}^{\text{Lor2}}\right).\left(\varphi(\overline{\text{p2}},m_e)\right)}{\left(\overline{\text{k2}}-\overline{\text{p2}}\right)^2}$$
$$-\frac{\bar{g}^{\text{Lor1Lor2}}\left(\varphi(\overline{\text{k1}},m_e)\right).\left(ie\bar{\gamma}^{\text{Lor2}}\right).\left(\varphi(\overline{\text{p2}},m_e)\right)\left(\varphi(\overline{\text{k2}},m_e)\right).\left(ie\bar{\gamma}^{\text{Lor1}}\right).\left(\varphi(\overline{\text{p1}},m_e)\right)}{\left(\overline{\text{k1}}-\overline{\text{p2}}\right)^2}$$

```
SetMandelstam[s, t, u, p1, p2, -k1, -k2, SMP["m_e"], SMP["m_e"], SMP["m_e"], SMP["m_e"]];
sqAmpMoeller =(ampMoeller (ComplexConjugate[ampMoeller]//FCRenameDummyIndices))//
    PropagatorDenominatorExplicit//Contract//FermionSpinSum[#, ExtraFactor -> 1/2^2]&//
    ReplaceAll[#, DiracTrace :> Tr] & // Contract//Simplify
```

$$\sum|\mathcal{M}|^2=\frac{2\mathrm{e}^4\left(-4m_e^2\left(s\left(t^2+3tu+u^2\right)+t^3-2t^2u-2tu^2+u^3\right)+8m_e^4\left(t^2+tu+u^2\right)+s^2(t+u)^2+t^4+u^4\right)}{t^2u^2}$$

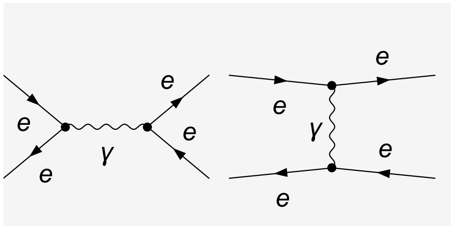
And the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{\left|\mathcal{M}^2\right|}{64\pi^2E_{cm}^2}$$

18. Bhabha scattering. Similarily, use FeynCalc.

```
<<FeynCalc`;
topBhabha = CreateTopologies[0, 2 -> 2];
diagsBhabha = InsertFields[topBhabha, {F[2, {1}], -F[2, {1}]} ->
```

```
        {F[ 2, {1}], -F[2, {1}]}, InsertionLevel -> {Classes},
          Model -> "SM", ExcludeParticles -> {S[1], S[2], V[2]}];
Paint[diagsBhabha, ColumnsXRows -> {2, 1}, Numbering -> None, SheetHeader->None,
   ImageSize->{512,256}];
```



;

```
ampBhabha=FCFAConvert[CreateFeynAmp[diagsBhabha, Truncated -> False],
IncomingMomenta->{p1, p2}, OutgoingMomenta->{k1, k2}, UndoChiralSplittings->True, ChangeDim
ension->4, List->False, SMP->True]
```

$$i\mathcal{M} = \frac{\bar{g}^{\text{Lor1Lor2}}\left(\varphi(\overline{\text{k1}}, m_e)\right).\left(ie\bar{\gamma}^{\text{Lor2}}\right).\left(\varphi(-\overline{\text{k2}}, m_e)\right)\left(\varphi(-\overline{\text{p2}}, m_e)\right).\left(ie\bar{\gamma}^{\text{Lor1}}\right).\left(\varphi(\overline{\text{p1}}, m_e)\right)}{\left(\overline{\text{k1}}+\overline{\text{k2}}\right)^2} - \frac{\bar{g}^{\text{Lor1Lor2}}\left(\varphi(\overline{\text{k1}}, m_e)\right).\left(ie\bar{\gamma}^{\text{Lor1}}\right).\left(\varphi(\overline{\text{p1}}, m_e)\right)\left(\varphi(-\overline{\text{p2}}, m_e)\right).\left(ie\bar{\gamma}^{\text{Lor2}}\right).\left(\varphi(-\overline{\text{k2}}, m_e)\right)}{\left(\overline{\text{k2}}-\overline{\text{p2}}\right)^2}$$

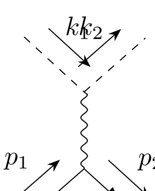
```
SetMandelstam[s, t, u, p1, p2, -k1, -k2, SMP["m_e"], SMP["m_e"], SMP["m_e"], SMP["m_e"]];
sqAmpBhabha =
          (ampBhabha (ComplexConjugate[ampBhabha]//FCRenameDummyIndices))
//PropagatorDenominatorExplicit//
          Contract//FermionSpinSum[#, ExtraFactor -> 1/2^2] & //ReplaceAll[#,
  DiracTrace :> Tr]&//Contract//Simplify
masslessSqAmpBhabha = (sqAmpBhabha /. {SMP["m_e"] -> 0})//Simplify
```

$$\sum |\mathcal{M}|^2 = \frac{2\mathrm{e}^4\left(s^4+s^2u^2+2stu^2+t^4+t^2u^2\right)}{s^2t^2}$$

And the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\left|\mathcal{M}^2\right|}{64\pi^2 E_{cm}^2}$$

19. Mott scattering. (Assuming the scalar-photon vertex is $ig(k_1-k_2)^\mu$, ignore electron mass.)

$i\mathcal{M} =$ 

$$= ige\bar{u}(p_2)\gamma^\mu u(p_1)\frac{1}{(p_2-p_1)^2+i\epsilon}(k_1-k_2)_\mu$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{g^2e^2}{(p_2-p_1)^4}[2(k_1-k_2)\cdot p_1(k_1-k_2)\cdot p_2-(k_1-k_2)\cdot(k_1-k_2)p_1\cdot p_2]$$

Use kinematics:

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2 &= \frac{g^2e^2}{(p_2-p_1)^4}[2(k_1-k_2)\cdot p_1(k_1-k_2)\cdot(k_1-k_2+p_1)] - \frac{g^2e^2}{(p_2-p_1)^2}[p_1\cdot(k_1+p_1-k_2)] \\ &= \frac{2g^2e^2}{(p_2-p_1)^4}[p_2\cdot p_1]^2 + \frac{g^2e^2}{(p_2-p_1)^2}[p_2\cdot p_1] \\ &= \frac{2g^2e^2}{(p_2-p_1)^2}[p_2\cdot p_1] \\ &= 4g^2e^2\end{aligned}$$

The cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} = \frac{1}{2k_1^0 2p_1^0 |v_{k1} - vp1|} \frac{1}{16\pi} \frac{2|p_1|}{E_{cm}} |\mathcal{M}|^2$$

20. Form factor.

$$F^E(q^2) = \frac{1}{e}\int \mathrm{d}^3 r \rho(r) e^{iq\cdot r} = \frac{2\pi}{e}\int \mathrm{d}r\mathrm{d}\theta \rho(r) e^{iqr\cos\theta} r^2 \sin\theta$$

$$F^E(q^2) = 1 - \frac{|q|^2}{6}\left\langle r^2 \right\rangle$$

if no angular part, $\left\langle r^2 \right\rangle \equiv \frac{1}{e}\int \mathrm{d}^3 r\rho(r) r^2 = \frac{4\pi}{e}\int \mathrm{d}r \rho(r) r^4$.

$\rho(r) = \delta(r)$:

$$F^E(q^2) = 1$$

$\rho(r) = \frac{\alpha^2}{4\pi}\frac{e^{-\alpha r}}{r}$:

$$F^E(q^2) = \frac{\alpha^2}{\alpha^2 + q^2} = 1 - \frac{|q|^2}{\alpha^2}$$

$\rho(r) = \frac{m^3}{8\pi} e^{-mr}$:

$$F^E(q^2) = \frac{m}{(m^2+q^2)^2} = 1 - \frac{2|q|^2}{m^2}$$

21. CP eigenvalues of $K^0 \to \pi\pi$ & $K^0 \to \pi\pi\pi$ ($CP = (-)^{S-2s}$ for neutral system).

For 2π system obviously they're all positive.

For 3π system, consider $\pi^0\pi^0\pi^0$, $CP = (+)(-)^L(-) = (-)^{L+1}$ where L is the orbital angular momentum between a $\pi^0\pi^0$ system and the other π^0. Also consider $\pi^+\pi^-\pi^0$, $CP = (-)^{L+1}$ where L is the orbital angular momentum between a $\pi^+\pi^-$ system and π^0.

22. Meson mass.

From Gell-Mann-Okubo formula

$$M(I,Y) = A + CY + B(I(I+1) - \frac{1}{4}Y^2)$$

Take $J^P = 0^-$, we have

$$\begin{aligned} M(K) = M(\frac{1}{2},1) &= A + \frac{1}{2}B \\ M(\pi) = M(1,0) &= A + 2B \\ M(\bar{K}) = M(\frac{1}{2},-1) &= A + \frac{1}{2}B \\ M(\eta) = M(0,0) &= A \end{aligned}$$

so

$$4m_K^2 = 3m_{\eta_s}^2 + m_\pi^2$$

Experiments give a deviation of 6%.

23. $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to J/\psi \to \mu^+\mu^-$.

Check Peskin 5.1, the unpolarized cross section is

$$\sigma = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}(1 + \frac{1}{2}\frac{m_\mu^2}{E^2})$$

and for J/ψ production we need to work with the propagator:

add resonance structure to the propagator, it becomes (in Feynman gauge)

$$\frac{-g^{\mu\nu}}{p^2 - (m_{J/\psi} + i\frac{\Gamma}{2})^2}$$

24. Use CP invariance (if not violated) to determine P parity of strange particles via weak interaciton.