Hydrogen

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November 3, 2017

1 Matching

QED Lagrangian is

$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \tag{1}$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \tag{2}$$

In tree level

$$iM_{QED}^{(0)} = \quad\quad\quad = -e^2 \bar{u}_N(P_N)v^\mu u_N(P_N)\frac{i}{\mathbf{q}^2}\bar{u}_e(p_2)\gamma_\mu u_e(p1)$$

with diagram labels P_N, P_N on top, q on the wavy line, p_1, p_2 on bottom.

$$iM_{NRQED}^{(0)} = \quad\quad\quad = -e^2 \bar{u}_N(P_N)v^\mu u_N(P_N)\frac{i}{\mathbf{q}^2}\psi^\dagger(p_2)\gamma_\mu \psi(p1)$$

with diagram labels P_N, P_N on top, q on the wavy line, p_1, p_2 on bottom.

The box diagram for NRQED process is

$$iM_{NRQED}^{(1)} = \quad\quad\quad$$

with diagram labels $P_N - k$ (top arrow), P_N, P_N, k, $k-q$, p_1, p_2, $p1+k$.

$$= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)$$

$$= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m})}\psi(p_1)$$

$$= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2(E_1-\frac{\mathbf{k}^2}{2m})}\psi(p_1)$$

The box and crossed box diagram for QED process is

$$i\mathcal{M}_1^{(1)} =$$

$$= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)$$

$$= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)$$

$$= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)$$

$$= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.

$$i\mathcal{M}_2^{(1)} =$$

$$= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)$$

$$= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)$$

$$= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0-\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2+p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)$$

$$= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0-\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2+p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)$$

$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

$$i\mathcal{M}_1^{(1)}+i\mathcal{M}_2^{(1)} = ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-{p_1^0}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)$$

$$= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2-\mathbf{p_1}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)$$
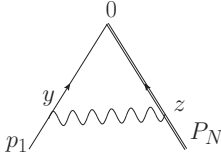
Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order.

Now consider operator product expansion.

Tree level matching:

$$\langle 0|T\psi(x)N(0)|pP_N\rangle = $$ $$= u_e(p)u_N(P_N)e^{-ip\cdot x}$$

$$\langle 0|T\psi_e(x)N(0)|pP_N\rangle = $$ $$= \psi_e(p)u_N(P_N)$$
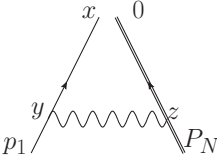
At leading order $u_e(p) = \begin{pmatrix} \psi_e(p) \\ 0 \end{pmatrix}$.

One loop scenario for NRQED case:

$$\langle 0|\psi_e(0)N(0)e\int \mathrm{d}^4y\,\bar\psi_e\psi_e A^0 e\int \mathrm{d}^4z\,\bar N N A^0|eN\rangle = e^2 u_N(P_N)\int [\mathrm{d}k]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)$$

$$= -ie^2 u_N(P_N)\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)$$

$$= -ie^2 u_N(P_N)\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1)$$

drop p_1

$$= -ie^2 u_N(P_N)\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1) = \pi ie^2\sqrt{\frac{2m}{E_1}}u_N(P_N)\psi(p_1)$$

For QED case:

$$\langle 0|\psi(x)N(0)e\int \mathrm{d}^4y\,\bar\psi\gamma^0\psi A^0 e\int \mathrm{d}^4z\,\bar N N A^0|eN\rangle = e^2 u_N(P_N)\int [\mathrm{d}k]e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{(\slashed{p}_1+\slashed{k}+m)\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)$$

$$= e^2 u_N(P_N)\int [\mathrm{d}k]e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{2p_1^0+\slashed{k}\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)$$

$$= ie^2 u_N(P_N)\int \frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)$$

$$= ie^2 u_N(P_N)\int \frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{p_1})\cdot\mathbf{x}}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)$$

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drop p_1

$$= ie^2 u_N(P_N) \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{p_1^0 + \sqrt{\mathbf{k}^2 + m^2}}{2\mathbf{k}^2[\mathbf{k}^2 + m^2 - p_1^0\sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)$$

2   HSET

2.1   Lagrangian

For scalar QED

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi$$

where

$$D_\mu = \partial_\mu + ieA_\mu$$

In Schwartz's QFT (Chap. 35) he mentioned a choice of χ_v and $\tilde{\chi}_v$:

$$\phi(x) = e^{-imv\cdot x} \frac{1}{\sqrt{2m}} (\chi_v(x) + \tilde{\chi}_v(x)) \tag{3}$$

$$\chi_v(x) = e^{imv\cdot x} \frac{1}{\sqrt{2m}} (iv\cdot D + m)\phi(x), \ \tilde{\chi}_v(x) = e^{imv\cdot x} \frac{1}{\sqrt{2m}} (-iv\cdot D + m)\phi(x) \tag{4}$$

Put (3) into (4), a simple relation is derived:

$$(-iv\cdot D)\chi_v(x) = (2m + iv\cdot D)\tilde{\chi}_v(x)$$

It can also be writen as

$$2m\tilde{\chi}_v = (-iv\cdot D)(\chi_v + \tilde{\chi}_v)$$

Use this result

$$\mathcal{L} = \frac{1}{2m}\Big\{\{[D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger + imv^\mu(\chi_v + \tilde{\chi}_v)^\dagger\}\{[D_\mu(\chi_v + \tilde{\chi}_v)] - imv_\mu(\chi_v + \tilde{\chi}_v)\} - m^2(\chi_v + \tilde{\chi}_v)^\dagger(\chi_v + \tilde{\chi}_v)\Big\}$$

$$= (\chi_v + \tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v + \tilde{\chi}_v) + \frac{1}{2m}[D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger D_\mu(\chi_v + \tilde{\chi}_v) \tag{5}$$

$$= (\chi_v(x) + \tilde{\chi}_v(x))^\dagger(iv\cdot D)(\chi_v(x) + \tilde{\chi}_v(x)) + \mathcal{O}(\frac{1}{m}) \tag{6}$$

(note that $D_\mu\phi = e^{-imv\cdot x}[D_\mu(\chi_v + \tilde{\chi}_v) - imv_\mu(\chi_v + \tilde{\chi}_v)]$ and $-imv^\mu[D_\mu(\chi_v + \tilde{\chi}_v)]^\dagger(\chi_v + \tilde{\chi}_v) = imv^\mu(\chi_v + \tilde{\chi}_v)^\dagger D_\mu(\chi_v + \tilde{\chi}_v) - $ *total derivative term*)

Use the leading order of (5)

$$\begin{aligned}
\mathcal{L}^{(0)} &= (\chi_v + \tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v + \tilde{\chi}_v) \\
&= \chi_v^\dagger iv\cdot D\chi_v + \tilde{\chi}_v^\dagger iv\cdot D(\chi_v + \tilde{\chi}_v) + \chi_v^\dagger iv\cdot D\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + (iv\cdot D\chi_v)^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + [(-2m - iv\cdot D)\tilde{\chi}_v]^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D + 4m)\tilde{\chi}_v
\end{aligned}$$

We can have the final form

$$\mathcal{L} = \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D + 4m)\tilde{\chi}_v + \mathcal{O}(\frac{1}{m})$$

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2.2 Quantization

2.2.1 HQET as an example

The leading term of HQET Lagrangian is

$$\mathcal{L} = \bar{Q}_v(iv \cdot D)Q_v$$

$$Q_v(x) = e^{imv \cdot x}\frac{1 + \slashed{v}}{2}\psi(x)$$

In free Dirac fermion theory we know that

$$\left\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\right\} = \delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{ab}$$

$$\{a_\mathbf{p}, a_{\mathbf{p}'}^\dagger\} = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

also the plane wave expansion of ψ is

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_\mathbf{p}u(p)e^{-ip \cdot x}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2mv^0}}a_v\sqrt{m}u(v)e^{-imv \cdot x - ik \cdot x}$$

using normalization of states $u(k) = \sqrt{m}u(v)$[1], $\langle p'|p\rangle = 2E_p(2\pi)^3\delta^{(3)}(\mathbf{p}' - \mathbf{p})$ and $\langle v', k'|v, k\rangle = 2v^0\delta_{vv'}(2\pi)^3\delta^{(3)}(\mathbf{k}' - \mathbf{k})$ we have $|p\rangle = \sqrt{m}|v\rangle$ ($|p\rangle = \sqrt{2E_p}a_\mathbf{p}^\dagger|0\rangle$ while $|v, k\rangle = \sqrt{2v^0}a_{v,\mathbf{k}}^\dagger|0\rangle$)

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v u(v)e^{-imv \cdot x - ik \cdot x}$$

Using the definition of $Q_v(x)$

$$Q_v(x) = e^{imv \cdot x}\frac{1 + \slashed{v}}{2}\psi(x)$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v\frac{1 + \slashed{v}}{2}u(v)e^{-ik \cdot x}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v u(v)e^{-ik \cdot x}$$

The commutation relation should be

$$\{Q_{va}(\mathbf{x}), Q_{v'b}(\mathbf{x}')\} = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0v'^0}}\{a_v, a_{v'}^\dagger\}u_a(v)u_b^\dagger(v')e^{-ik \cdot x + ik' \cdot x'}$$

using $\sum_s u_a(v)u_b^\dagger(v) = \frac{1}{m}\sum_s u_a(p)u_b^\dagger(p) = [(\slashed{v} + 1)\gamma^0]_{ab}$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0v'^0}}\{a_v, a_{v'}^\dagger\}[(\slashed{v} + 1)\gamma^0]_{ab}e^{-ik \cdot x + ik' \cdot x'}$$

assuming $\{a_v, a_{v'}\} = (2\pi)^3\delta_{vv'}\delta^{(3)}(\mathbf{k} - \mathbf{k}')$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2v^0}[(\slashed{v} + 1)\gamma^0]_{ab}e^{-ik \cdot (x - x')}\delta_{vv'}$$

$$= [\frac{(\slashed{v} + 1)\gamma^0}{2v^0}]_{ab}\delta_{vv'}\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

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[1] The relation $\bar{u}^s(p)\gamma^\mu u^s(p) = 2p^\mu$ can be derived using Gordon identity, same for $\bar{u}^s(v)\gamma^\mu u^s(v) = 2v^\mu$, but it's actually $\bar{u}u$.

2.2.2 HSET

The anti-particle field is decoupled so we don't have to consider that for now. The equation-of-motion is

$$\begin{cases} v \cdot D\chi_v^\dagger = 0 \\ v \cdot D\chi_v = 0 \end{cases}$$

By definition

$$\chi_v(x) = \frac{e^{imv \cdot x}}{\sqrt{2m}}(iv \cdot D + m)\phi(x)$$
$$= \frac{1}{\sqrt{2m}}(iv \cdot D + 2m)e^{imv \cdot x}\phi(x)$$