Homework: Quantum Field Theory #4

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1. Derive $\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$.

We know the "quantized" dirac field operators (here and in Heisenberg picture) are

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^s_{\mathbf{p}}v^s(p)e^{ip\cdot x})$$

$$\bar{\psi}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^{s\dagger}_{\mathbf{p}}\bar{u}^s(p)e^{ip\cdot x} + b^{s\dagger}_{\mathbf{p}}\bar{v}^s(p)e^{-ip\cdot x})$$

So

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\sum_r (a^{r\dagger}_{\mathbf{k}}\bar{u}^r_b(k)e^{ik\cdot y} + b^{r\dagger}_{\mathbf{k}}\bar{v}^r_b(k)e^{-ik\cdot y})\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^s_{\mathbf{p}}u^s_a(p)e^{-ip\cdot x} + b^s_{\mathbf{p}}v^s_a(p)e^{ip\cdot x})$$

since we only want the positive-frequency terms of $\bar{\psi}_b(y)$ and the negative-frequency terms of $\psi_a(x)$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\sum_r (b^{r\dagger}_{\mathbf{k}}\bar{v}^r_b(k)e^{-ik\cdot y})\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (b^s_{\mathbf{p}}v^s_a(p)e^{ip\cdot x})$$

we knew that $\langle 0|b^{r\dagger}_{\mathbf{k}}b^s_{\mathbf{p}}|0\rangle = e^{-i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}}\langle 0|b^{r\dagger}_{\mathbf{k}}b^s_{\mathbf{p}}|0\rangle$ and it implies $\mathbf{k}=\mathbf{p}$ and similarly $r=s$, so $\langle 0|b^{r\dagger}_{\mathbf{k}}b^s_{\mathbf{p}}|0\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta^{rs}\cdot B(\mathbf{p})$ and

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s \bar{v}^s_b(p)v^s_a(p)e^{ip(x-y)}B(\mathbf{p})$$
$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not{p}-m)_{ba}e^{ip(x-y)}B(\mathbf{p})$$

from the sign of x in the exponential factor, we can rewrite $\not{p}$ as $-i\not{\partial}_x$, and note that B is a constant (from Lorentz invariance)

$$= -(i\not{\partial}_x + m)_{ab}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{ip(x-y)}B$$

2. Derive H and P from the proper quantized Dirac field.

The field operators are

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s (a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x})$$

$$\bar{\psi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x})$$

And in Schrödinger picture

$$\psi(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} u^s(p) e^{i\mathbf{p}\cdot\mathbf{x}} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{-i\mathbf{p}\cdot\mathbf{x}}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} u^s(p) + b^{s\dagger}_{-\mathbf{p}} v^s(-p)) e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\bar{\psi}(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (b^s_{\mathbf{p}} \bar{v}^s(p) e^{i\mathbf{p}\cdot\mathbf{x}} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{-i\mathbf{p}\cdot\mathbf{x}}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (b^s_{\mathbf{p}} \bar{v}^s(p) + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s(-p)) e^{i\mathbf{p}\cdot\mathbf{x}}$$

The Hamiltionian is

$$H = \int \mathrm{d}^3 x \bar{\psi}(-i\gamma\cdot\nabla + m)\psi$$

We can write down

$$\begin{aligned}
\nabla\psi &= \nabla \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} u^s(p) + b^{s\dagger}_{-\mathbf{p}} v^s(-p)) e^{i\mathbf{p}\cdot\mathbf{x}} \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{i\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}} \sum_s (a^s_{\mathbf{p}} u^s(p) + b^{s\dagger}_{-\mathbf{p}} v^s(-p)) e^{i\mathbf{p}\cdot\mathbf{x}}
\end{aligned}$$

So

$$\begin{aligned}
H &= \int \mathrm{d}^3 x [\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s (b^s_{\mathbf{p}} \bar{v}^s(p) + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s(-p)) e^{i\mathbf{p}\cdot\mathbf{x}} (\gamma\cdot\mathbf{k} + m) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \sum_r (a^r_{\mathbf{k}} u^r(k) + b^{r\dagger}_{-\mathbf{k}} v^r(-k)) e^{i\mathbf{k}\cdot\mathbf{x}}] \\
&= \int \mathrm{d}^3 x \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}} 2E_{\mathbf{p}}}} \sum_{s,r} [(b^s_{\mathbf{p}} \bar{v}^s(p) + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s(-p))(\gamma\cdot\mathbf{k} + m)(a^r_{\mathbf{k}} u^r(k) + b^{r\dagger}_{-\mathbf{k}} v^r(-k))] e^{i(\mathbf{p}+\mathbf{k})\cdot\mathbf{x}} \\
&= \int \mathrm{d}^3 x \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}} 2E_{\mathbf{p}}}} \sum_{s,r} [(b^s_{\mathbf{p}} \bar{v}^s_p + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s_{-p}) \gamma\cdot\mathbf{k} (a^r_{\mathbf{k}} u^r_k + b^{r\dagger}_{-\mathbf{k}} v^r_{-k}) + m((b^s_{\mathbf{p}} \bar{v}^s_p b^{r\dagger}_{-\mathbf{k}} v^r_{-k} + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s_{-p} a^r_{\mathbf{k}} u^r_k)] e^{i(\mathbf{p}+\mathbf{k})\cdot\mathbf{x}}
\end{aligned}$$

integrate by x to remove the exponential term and

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{p}}}} \sum_{s,r} [-(b^s_{\mathbf{p}} \bar{v}^s_p + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s_{-p}) \gamma\cdot\mathbf{p} (a^r_{-\mathbf{p}} u^r_{-p} + b^{r\dagger}_{\mathbf{p}} v^r_p) + m((b^s_{\mathbf{p}} \bar{v}^s_p b^{r\dagger}_{\mathbf{p}} v^r_p + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s_{-p} a^r_{-\mathbf{p}} u^r_{-p})]$$

View it term by term: the mass term

$$m((b^s_{\mathbf{p}} \bar{v}^s_p b^{r\dagger}_{\mathbf{p}} v^r_p + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s_{-p} a^r_{-\mathbf{p}} u^r_{-p}) = -2m^2 (b^s_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}} - a^{s\dagger}_{-\mathbf{p}} a^s_{-\mathbf{p}})$$

the γ term

$$\begin{aligned}
(b^s_{\mathbf{p}} \bar{v}^s_p + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s_{-p}) \gamma\cdot\mathbf{p} (a^r_{-\mathbf{p}} u^r_{-p} + b^{r\dagger}_{\mathbf{p}} v^r_p) &= (b^s_{\mathbf{p}} \bar{v}^s_p + a^{s\dagger}_{-\mathbf{p}} \bar{u}^s_{-p}) \gamma^i p_i (a^r_{-\mathbf{p}} u^r_{-p} + b^{r\dagger}_{\mathbf{p}} v^r_p) \\
&= 2\mathbf{p}^2 (b^s_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}} - a^{s\dagger}_{-\mathbf{p}} a^s_{-\mathbf{p}})
\end{aligned}$$

here the property $\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2\delta_{\sigma\sigma'} p^\mu$ and the corresponding one for $v^s(p)$ are used ($\bar{v}_\sigma(p)\gamma^\mu v_{\sigma'}(p) = 2\delta_{\sigma\sigma'} p^\mu$, note that in the derivation both minus sign of Dirac equation and the normalization canceled and made no sign change from the result of u). Combine these and we have

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s[-2\mathbf{p}^2(b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}} - a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}}) - 2m^2(b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}} - a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})]$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-2E^2_{\mathbf{p}}}{2E_{\mathbf{p}}}\sum_s[(b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}} - a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}}) + (b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}} - a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})]$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\sum_s E_{\mathbf{p}}(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} - b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}})$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\sum_s E_{\mathbf{p}}(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}} - (2\pi)^3\delta(0))$$

ignore the infinity term

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\sum_s E_{\mathbf{p}}(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}})$$

Similarly

$$P = \int \mathrm{d}^3 x \psi^\dagger(-i\nabla)\psi$$

Use the given field operator

$$P = \int \mathrm{d}^3 x \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s(b^s_{\mathbf{p}}\bar{v}^s(p) + a^{s\dagger}_{-\mathbf{p}}\bar{u}^s(-p))e^{i\mathbf{p}\cdot\mathbf{x}}\gamma^0\int\frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{\mathbf{k}}{\sqrt{2E_{\mathbf{k}}}}\sum_r(a^r_{\mathbf{k}}u^r(k) + b^{r\dagger}_{-\mathbf{k}}v^r(-k))e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s(b^s_{\mathbf{p}}\bar{v}^s(p) + a^{s\dagger}_{-\mathbf{p}}\bar{u}^s(-p))\gamma^0\frac{-\mathbf{p}}{\sqrt{2E_{\mathbf{p}}}}\sum_r(a^r_{-\mathbf{p}}u^r(-p) + b^{r\dagger}_{\mathbf{p}}v^r(p))$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-\mathbf{p}}{2E_{\mathbf{p}}}\sum_{s,r}(b^s_{\mathbf{p}}\bar{v}^s(p) + a^{s\dagger}_{-\mathbf{p}}\bar{u}^s(-p))\gamma^0(a^r_{-\mathbf{p}}u^r(-p) + b^{r\dagger}_{\mathbf{p}}v^r(p))$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-i\mathbf{p}}{2E_{\mathbf{p}}}\sum_{s,r}(b^s_{\mathbf{p}}v^{s\dagger}(p) + a^{s\dagger}_{-\mathbf{p}}u^{s\dagger}(-p))(a^r_{-\mathbf{p}}u^r(-p) + b^{r\dagger}_{\mathbf{p}}v^r(p))$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-\mathbf{p}}{2E_{\mathbf{p}}}\sum_s 2E_{\mathbf{p}}(b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}} + a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})$$

$$= -\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\mathbf{p}\sum_s(b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}} + a^{s\dagger}_{-\mathbf{p}}a^s_{-\mathbf{p}})$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\mathbf{p}\sum_s(-b^s_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}} + a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}})$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\mathbf{p}\sum_s(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}} - (2\pi)^3\delta(0))$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\mathbf{p}\sum_s(a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}})$$

3. Derive $J_z b^{s\dagger}_0|0\rangle$.

First

$$J_z = \int \mathrm{d}^3 x\int\frac{\mathrm{d}^3 p\mathrm{d}^3 q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}e^{-i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{x}}\sum_{r,s}(a^{r\dagger}_{\mathbf{q}}u^{r\dagger}(\mathbf{q}) + b^r_{-\mathbf{q}}v^{r\dagger}(-\mathbf{q}))\frac{\Sigma^3}{2}(a^s_{\mathbf{p}}u^s(\mathbf{p}) + b^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p}))$$

Since J_z must annihilate the vacuum

$$J_z b_0^{s\dagger}|0\rangle = [J_z, b_0^{s\dagger}]|0\rangle$$

We know that the only nonzero term must like

$$[b^r_{-\mathbf{q}}b^{s\dagger}_{-\mathbf{p}}, b_0^{s\dagger}] = (2\pi)^3\delta^3(\mathbf{q})\delta^{rs}b^{s\dagger}_{-\mathbf{p}}$$

(there must not be two terms without dagger or it annihilates the vacuum.) then

$$\begin{aligned}
J_z b_0^{s\dagger}|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_{r,s}(a^{r\dagger}_\mathbf{p}u^{r\dagger}(\mathbf{p}) + b^r_{-\mathbf{p}}v^{r\dagger}(-\mathbf{p}))\frac{\Sigma^3}{2}(a^s_\mathbf{p}u^s(\mathbf{p}) + b^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p}))b_0^{s\dagger}|0\rangle \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_{r,s}(2\pi)^3\delta^3(\mathbf{p})\delta^{rs}b^{s\dagger}_{-\mathbf{p}}v^{r\dagger}(-\mathbf{p})\frac{\Sigma^3}{2}v^s(-\mathbf{p})|0\rangle \\
&= \frac{1}{2E_0}\sum_s v^{s\dagger}(0)\frac{\Sigma^3}{2}v^s(0)b_0^{s\dagger}|0\rangle
\end{aligned}$$

note that

$$v^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s\end{pmatrix}$$

so

$$v^s(0) = \begin{pmatrix}\sqrt{E}\eta^s \\ -\sqrt{E}\eta^s\end{pmatrix}$$

and

$$\begin{aligned}
&= \frac{1}{2E}\sum_s \begin{pmatrix}\sqrt{E}\eta^{s\dagger} & -\sqrt{E}\eta^{s\dagger}\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}\begin{pmatrix}\sqrt{E}\eta^s \\ -\sqrt{E}\eta^s\end{pmatrix}b_0^{s\dagger}|0\rangle|_{\mathbf{p}=0} \\
&= \frac{1}{4E}\sum_s \begin{pmatrix}\sqrt{E}\eta^{s\dagger}\sigma^3 & -\sqrt{E}\eta^{s\dagger}\sigma^3\end{pmatrix}\begin{pmatrix}\sqrt{E}\eta^s \\ -\sqrt{E}\eta^s\end{pmatrix}b_0^{s\dagger}|0\rangle|_{\mathbf{p}=0} \\
&= \frac{1}{2}\sum_s \eta^{s\dagger}\sigma^3\eta^s b_0^{s\dagger}|0\rangle
\end{aligned}$$

For $\eta^s = \begin{pmatrix}1\\0\end{pmatrix}$, we have the eigenvalue for J_z is $\frac{1}{2}$. For $\eta^s = \begin{pmatrix}0\\1\end{pmatrix}$, we have the eigenvalue for J_z is $-\frac{1}{2}$.

4. Derive charge operator Q.

First we know that

$$Q = \int \mathrm{d}^3x\psi^\dagger(x)\psi(x)$$

From the field operators given above, we have (due to the *delta* that bound to appear in the normalization of u and v I write all spin indices into s)

$$\begin{aligned}
Q &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_s(b^s_\mathbf{p}\bar v^s(p) + a^{s\dagger}_{-\mathbf{p}}\bar u^s(-p))\gamma^0(a^s_{-\mathbf{p}}u^s(-p) + b^{s\dagger}_\mathbf{p}v^s(p)) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_s(b^s_\mathbf{p}v^{s\dagger}(p) + a^{s\dagger}_{-\mathbf{p}}u^{s\dagger}(-p))(a^s_{-\mathbf{p}}u^s(-p) + b^{s\dagger}_\mathbf{p}v^s(p)) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_s(b^s_\mathbf{p}v^{s\dagger}(p)b^{s\dagger}_\mathbf{p}v^s(p) + a^{s\dagger}_{-\mathbf{p}}u^{s\dagger}(-p)a^s_{-\mathbf{p}}u^s(-p)) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_s(a^{s\dagger}_\mathbf{p}a^s_\mathbf{p} + b^s_\mathbf{p}b^{s\dagger}_\mathbf{p})
\end{aligned}$$

cancel the infinity term

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s)$$