Coulomb Resummation Near $t\bar{t}$ Threshold in $e^+e^- \to HZ$ Process

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Abstract

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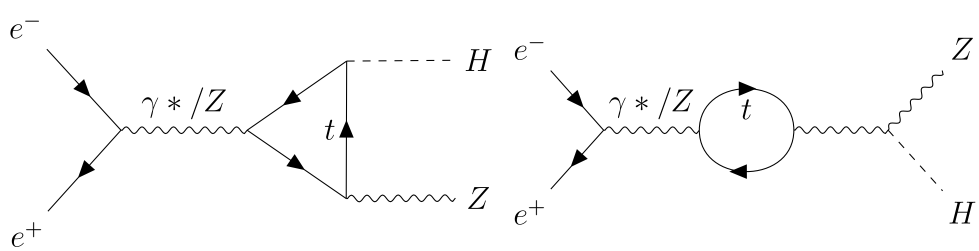
I. Introduction

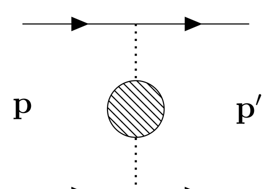
The first relevant calculation should be Fadin et al. in 1987 [1]. The imaginary part of the Coulomb Green function is given explicitly. The whole expression is given by Melnikov et al. in 1994 [2]. While this being true, there's a real constant D related to renormalization scheme that remains to be fixed. In 1991, Strassler and Peskin details the top quark production near threshold at the leading order in α_s. The high order correction effects considered are restrict to running α_s (to two loop order), Higgs static potential and electroweak correction though. Kats and Schwartz gave a review about annihilation decays of bound states at LHC [3]. There're also a number of papers involving the threshold effects of diphoton resonance[], in which Chway et al. explicitly expressed that the 1-loop amplitude can be well separated into relativistic and non-relativistic parts near the threshold. For exclusive processes,

II. Coulomb Part

We have two diagrams:



The leading order Coulomb resummation (which refers to Coulomb resummation in the following since no higher order correction is involved) can be expressed diagrammatically as



where the blob stands for the resummation of all Coulomb-mode gluon exchange. This Green function can be expressed as (including the diagram without Coulomb-gluon exchange) $G(\mathbf{p},\mathbf{p}';E) = G_0^{(1)}(\mathbf{p},\mathbf{p}';E)$ which obeys a Lippmann-Schwinger equation[4]

$$\left(\frac{\mathbf{p}^2}{m} - E\right) G_0^{(R)}(\mathbf{p},\mathbf{p}';E) + \tilde{\mu}^{2\epsilon} \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \frac{4\pi D_R \alpha_s}{\mathbf{k}^2} G_0^{(R)}(\mathbf{p}-\mathbf{k},\mathbf{p}';E) = (2\pi)^{d-1}\delta^{(d-1)}(\mathbf{p}-\mathbf{p}') . \tag{1}$$

In our case, it's a color-singlet state, thus $D_1 = -C_F = \frac{4}{3}$. The coordinate space Coulomb Green function $G\left(\mathbf{r}, \mathbf{r}'; E\right)$ is related to $G(\mathbf{p}, \mathbf{p}'; E)$ via a Fourier transform

$$G(\mathbf{p}, \mathbf{p}'; E) = \int \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}' G\left(\mathbf{r}, \mathbf{r}'; E\right) e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{r}} e^{-\mathrm{i}\mathbf{p}'\cdot\mathbf{r}'} \tag{2}$$

and $G\left(\mathbf{r}, \mathbf{r}'; E\right)$ which obeys a Schrödinger equation

$$\left(-\frac{\nabla^2_{(r)}}{m} + \frac{D_R \alpha_s}{r} - E\right) G\left(\mathbf{r}, \mathbf{r}'; E\right) = \delta^{(3)}\left(\mathbf{r} - \mathbf{r}'\right) \tag{3}$$

What we want is actually a spatially local Green function $G(0, 0; E)$, of which the result is given[1, 2]

$$G(0,0;E) = -\frac{m_t p}{4\pi} + \frac{m_t p_0}{2\pi} \log\left(\frac{m_t}{p} D\right) + \frac{m_t p_0^2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n\left(np - p_0\right)} \tag{4}$$

where D is a real constant depends on renormalization scheme and is taken to be unity here (if we're to pursuit two-loop precision, this constant must be determined by fixed order calculation), $p_0 = \frac{2}{3} m_t \alpha_s$ and $p = \sqrt{m_t\left(-E - i\epsilon\right)}$. Considering

$$\sum_{i=1}^{n} \frac{1}{n\left(np - p_0\right)} = -\frac{\psi^{(0)}\left(1 - \frac{p_0}{p}\right) + \gamma_E}{p_0} = -\frac{H_{-p_0/p}}{p_0} \tag{5}$$

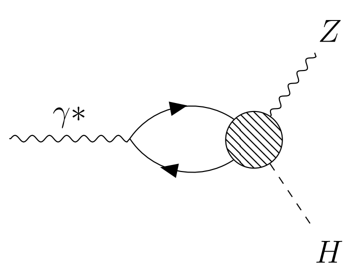
where $H_n = \sum_{i=1}^{n} \frac{1}{i}$ is the harmonic number, the Coulomb Green function becomes

$$G(0,0;E) = -\frac{m_t p}{4\pi} + \frac{m_t p_0}{2\pi} \log\left(\frac{m_t}{p} D\right) - \frac{m_t p_0^2}{2\pi} \frac{H_{-p_0/p}}{p_0} \tag{6}$$

To include the finite width of top quark, one can perform the following replacement

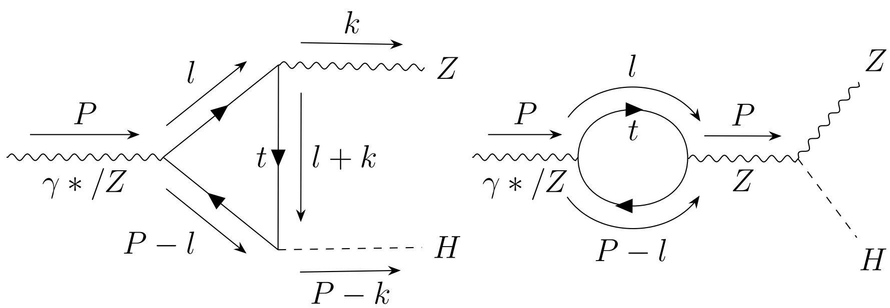
$$E \to E + i\Gamma_t;\ p \to \sqrt{m_t(-E - i\Gamma_t)} \tag{7}$$

Now let's start with a simpler diagram



Simply put, once we can arrive at a top bubble as in the second diagram in the beginning after nonrelativistic approximation, this top bubble can be replaced by $G(0, 0; E)$.

To the lowest order of those couplings between top quarks and external lines, we're considering



The second one is well-discussed in [4]. Beneke et al. gave the explicit relation between the Coulomb Green function and the total cross section via optical theorem

$$\begin{aligned}\Pi_{\mu\nu}^{(X)}\left(q^2\right) &= i\int d^4 x e^{iq\cdot x}\left\langle 0\left|T\left(j_\mu^{(X)}(x)j_\nu^{(X)}(0)\right)\right|0\right\rangle \\ &= \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi^{(X)}\left(q^2\right) + q_\mu q_\nu \Pi_L^{(X)}\left(q^2\right)\end{aligned} \tag{8}$$

where the vector current $j_\mu^{(v)} = \bar{t}\gamma_\mu t$ and the axial vector current $j_\mu^{(v)} = \bar{t}\gamma_\mu\gamma_5 t$

$$\begin{aligned}\sigma_{t\bar{t}X} = \sigma_0 \times 12\pi\,\mathrm{Im}[&e_t^2\Pi^{(v)}\left(q^2\right) - \frac{2q^2}{q^2-M_Z^2}v_e v_t e_t \Pi^{(v)}\left(q^2\right) \\ &+ \left(\frac{q^2}{q^2-M_Z^2}\right)^2\left(v_e^2+a_e^2\right)\left(v_t^2\Pi^{(v)}\left(q^2\right)+a_t^2\Pi^{(a)}\left(q^2\right)\right)\Bigg]\end{aligned} \tag{9}$$

However, we can directly access the amplitude. Here we hold all conventions of signs implicitly as η[5].

The amplitude of the second diagram is

$$\begin{aligned}&\bar{u}(p_1)(-i\eta_e e\gamma_\mu)u(p_2)\frac{-ig^{\mu\nu}}{P^2+i0}\,\mathrm{tr}\Bigg\{(-i\eta_e e Q_t\gamma_\nu)\frac{1+\gamma^0}{2} \\ &\left(-i\eta\eta_Z\frac{g}{\cos\theta_W}\gamma_\rho\big(g_V^t - g_A^t\gamma^5\big)\right)\frac{1-\gamma^0}{2}\Bigg\}\frac{-ig^{\rho\sigma}}{P^2-M_Z^2+i0}\frac{igm_Z g_{\sigma\omega}}{\cos\theta_W}\epsilon^\omega(k)\end{aligned}$$

We'll start to expand the first diagram. The triangle loop is expressed as

$$\begin{aligned}-\int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\mathrm{tr}\Bigg\{&(-i\eta_e e Q_t\gamma_\mu)\frac{i\big(\not{P}-\not{l}+m_t\big)}{(P-l)^2-m_t^2+im_t\Gamma_t}\left(-i\frac{g}{2}\frac{m_t}{m_W}\right)\frac{i\big(\not{l}+\not{k}+m_t\big)}{(l+k)^2-m_t^2+im_t\Gamma_t} \\ &\left(-i\eta\eta_Z\frac{g}{\cos\theta_W}\gamma_\nu\big(g_V^t-g_A^t\gamma^5\big)\right)\frac{i\big(\not{l}+m_t\big)}{l^2-m_t^2+im_t\Gamma_t}\Bigg\}\end{aligned} \tag{10}$$

We have the following regions

$$\begin{aligned} \text{hard(h)}:&\ell^0 \sim m, & \ell \sim m \\ \text{soft(s)}:&\ell^0 \sim mv, & \ell \sim mv \\ \text{potential(p)}:&\ell^0 \sim mv^2, & \ell \sim mv \\ \text{ultrasoft(us)}:&\ell^0 \sim mv^2, & \ell \sim mv^2 \end{aligned} \tag{11}$$

and $P = (2m_t + E, \mathbf{0}) \sim (m_t + m_t v^2, 0)$, $k \sim (m_t, m_t)$. We're to put the loop momentum in potential region. The propagators in the integrand is then simplified to (with a shift in $l^0 \to m_t + \epsilon$)

$$\frac{i(\not{l} + m_t)}{l^2 - m_t^2 + im_t\Gamma_t} = \frac{i((m_t+\epsilon)\gamma^0 + l^i\gamma_i + m_t)}{(m_t+\epsilon)^2 - \mathbf{l}^2 - m_t^2 + im_t\Gamma_t} \to \frac{1+\gamma^0}{2}\frac{i}{\epsilon - \frac{\mathbf{l}^2}{2m_t} + \frac{i\Gamma_t}{2}} \tag{12}$$

$$\begin{aligned} \frac{i(\not{l} - \not{P} + m_t)}{(P-l)^2 - m_t^2 + im_t\Gamma_t} &= \frac{i((m_t + \epsilon - 2m_t - E)\gamma^0 + l^i\gamma_i + m_t)}{(2m_t + E - m_t - \epsilon)^2 - \mathbf{l}^2 - m_t^2 + im_t\Gamma_t} \quad (13) \\ &= \frac{i((\epsilon - m_t - E)\gamma^0 - l^i\gamma_i + m_t)}{(m_t + E - \epsilon)^2 - \mathbf{l}^2 - m_t^2 + im_t\Gamma_t} \to \frac{1-\gamma^0}{2}\frac{i}{E - \epsilon - \frac{\mathbf{l}^2}{2m_t} + \frac{i\Gamma_t}{2}} \quad (14) \end{aligned}$$

$$\frac{i(\not{l} + \not{k} + m_t)}{(l+k)^2 - m_t^2 + im_t\Gamma_t} = \frac{i((m_t + \epsilon + k^0)\gamma^0 + (l^i + k^i)\gamma_i + m_t)}{(m_t + \epsilon + k^0)^2 - (\mathbf{l}+\mathbf{k})^2 - m_t^2 + im_t\Gamma_t} \to \frac{i\left(\tilde{\not{k}} + m_t\right)}{\tilde{k}^2 - m_t^2 + im_t\Gamma_t} \tag{15}$$

where $\tilde{k} = (k^0 + m_t, \mathbf{k})$. To calculate the amplitude, a more appropriate way is to define a substitution rule for the top loop. We already know that in leading power

$$\begin{aligned} G(E) &= \int \frac{\mathrm{d}^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{\mathrm{d}^{d-1}\mathbf{p}'}{(2\pi)^{d-1}} \frac{-(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{p}' - \mathbf{p})}{E + i\epsilon - \frac{\mathbf{p}^2}{m}} \xrightarrow{\epsilon \to \Gamma_t} \int \frac{\mathrm{d}^{d-1}l}{(2\pi)^{d-1}} \frac{-1}{E - \frac{\mathbf{l}^2}{m_t} + i\Gamma_t} \quad (16) \\ &= i\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{i}{\epsilon - \frac{\mathbf{l}^2}{2m_t} + \frac{i\Gamma_t}{2}} \frac{i}{E - \epsilon - \frac{\mathbf{l}^2}{2m_t} + \frac{i\Gamma_t}{2}} \quad (17) \end{aligned}$$

For the top bubble type diagram

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{i(\not{l} - \not{P} + m_t)}{(P-l)^2 - m_t^2 + im_t\Gamma_t} \Gamma \frac{i(\not{l} + m_t)}{l^2 - m_t^2 + im_t\Gamma_t} \to \frac{1-\gamma^0}{2}\Gamma\frac{1+\gamma^0}{2} iG(E)$$

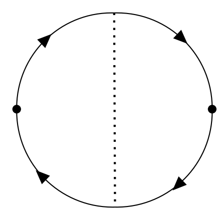
where Γ is arbitrary coupling sandwiched between two top propagators.

For the top triangle diagram

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{i\left(\not{l} - \not{P} + m_t\right)}{(P-l)^2 - m_t^2 + i m_t \Gamma_t} \Gamma_1 \frac{i\left(\not{l} + \not{k} + m_t\right)}{(l+k)^2 - m_t^2 + i m_t \Gamma_t} \Gamma_2 \frac{i\left(\not{l} + m_t\right)}{l^2 - m_t^2 + i m_t \Gamma_t}$$
$$\to \frac{1-\gamma^0}{2} \Gamma_1 \frac{i\left(\tilde{\not{k}} + m_t\right)}{\tilde{k}^2 - m_t^2 + i m_t \Gamma_t} \Gamma_2 \frac{1+\gamma^0}{2} i G(E)$$

III. One Loop Subtraction and the Determination of the Renormalization Artifact

To get the exact contribution from Coulomb gluon exchanges and to avoid double counting, one must calculate the following diagram:



(18)

and the full amplitude with this correction is obtained via replacing the leading order one.

$$\mu^{2(3-d)} \int \frac{\mathrm{d}^{d-1} l_1}{(2\pi)^{d-1}} \frac{\mathrm{d}^{d-1} l_2}{(2\pi)^{d-1}} \frac{1}{E - \frac{\mathbf{l}_1^2}{m_t} + i\Gamma_t} \frac{1}{E - \frac{\mathbf{l}_2^2}{m_t} + i\Gamma_t} \frac{1}{(\mathbf{l}_1 - \mathbf{l}_2)^2} \tag{19}$$
$$= -\frac{m_t^2}{32\pi^2(d-3)} - \frac{m_t^2(\log(-m_t(E+i\Gamma)) - 2\log(\mu) + \gamma_E - 1 - \log(\pi))}{32\pi^2} + O\left(d-3\right) \tag{20}$$

Multiply the coupling $-g_s^2 C_F = -\frac{16\pi\alpha_s}{3}$, the first logarithm is exactly what appears in (6) in this order (with an extra $-2\log m$ term to fix the dimension in the logarithm, in [2] they appears to have chosen a scheme with no μ presence). Now based on which scheme we choose, we may now determine the renormalization artifact D and subtract the one loop level Coulomb contribution in the full QCD calculation to avoid double counting.

A. Pseudoscalar Higgs Decay

B. $\gamma * \to ZH$

We then have an overall factor[6]

$$
\begin{aligned}
&-\operatorname{tr}\left\{(-i\eta_e eQ_t\gamma^\mu)\frac{1-\gamma^0}{2}\left(-i\frac{g}{2}\frac{m_t}{m_W}\right)\frac{i\left(\tilde{\not{k}}+m_t\right)}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\left(-i\eta\eta_Z\frac{g}{\cos\theta_W}\gamma^\nu\left(g_V^t-g_A^t\gamma^5\right)\right)\frac{1+\gamma^0}{2}\right\}\\
&=-(-i\eta_e eQ_t)\left(-i\frac{g}{2}\frac{m_t}{m_W}\right)\frac{i}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\left(-i\eta\eta_Z\frac{g}{\cos\theta_W}\right)\\
&\qquad\qquad\operatorname{tr}\left\{\gamma^\mu\frac{1-\gamma^0}{2}\left(\tilde{\not{k}}+m_t\right)\gamma^\nu\left(g_V^t-g_A^t\gamma^5\right)\frac{1+\gamma^0}{2}\right\}\\
&=\frac{g^2m_t}{2m_W\cos\theta_W}\frac{\eta\eta_Z\eta_e eQ_t}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\operatorname{tr}\left\{\gamma^\mu\frac{1-\gamma^0}{2}\left(\tilde{\not{k}}+m_t\right)\gamma^\nu\left(g_V^t-g_A^t\gamma^5\right)\frac{1+\gamma^0}{2}\right\}\\
&=\frac{g^2m_t}{2m_W\cos\theta_W}\frac{\eta\eta_Z\eta_e eQ_t}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\operatorname{tr}\left\{\gamma^\mu\frac{1-\gamma^0}{2}\left(\tilde{\not{k}}+m_t\right)\gamma^\nu\left(g_V^t-g_A^t\gamma^5\right)\right\}\\
&=\frac{g^2m_t}{m_W\cos\theta_W}\frac{\eta\eta_Z\eta_e eQ_t}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\left(-g_V^tg^{0\mu}\tilde{k}^\nu+g_V^tg^{0\nu}\tilde{k}^\mu-g_V^t\tilde{k}^0g^{\mu\nu}+g_V^tm_tg^{\mu\nu}-ig_A^t\epsilon^{0\mu\nu\rho}\tilde{k}_\rho\right)
\end{aligned}
\tag{B1}
$$

Considering the external states, applying Feynman gauge, and counting the electron-positron pair in, we have

$$
\begin{aligned}
&\frac{g^2m_t}{m_W\cos\theta_W}\frac{\eta\eta_Z\eta_e eQ_t}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\left(-g_V^tg^{0\mu}\tilde{k}^\nu+g_V^tg^{0\nu}\tilde{k}^\mu-g_V^t\tilde{k}^0g^{\mu\nu}+g_V^tm_tg^{\mu\nu}-ig_A^t\epsilon^{0\mu\nu\rho}\tilde{k}_\rho\right)\\
&\qquad\qquad\epsilon_\nu(k)\frac{-i}{P^2+i0}\bar{u}(p_1)(-ie\gamma_\mu)u(p_2)\\
&=\frac{g^2m_t}{m_W\cos\theta_W}\frac{\eta\eta_Z\eta_e eQ_t}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\frac{-i}{P^2+i0}(-ie)\\
&\qquad\bar{u}(p_1)\left(-g_V^t\gamma^0m_t\epsilon^0(k)+g_V^t\epsilon^0(k)\tilde{\not{k}}-g_V^t\tilde{k}^0\not{\epsilon}(k)+g_V^tm_t\not{\epsilon}(k)-ig_A^t\epsilon^{0\mu\nu\rho}\tilde{k}_\rho\gamma_\mu\epsilon_\nu(k)\right)u(p_2)\\
&=\frac{g^2m_t(-ie)}{m_W\cos\theta_W}\frac{\eta\eta_Z\eta_e eQ_t}{\tilde{k}^2-m_t^2+im_t\Gamma_t}\frac{-i}{P^2+i0}\bar{u}(p_1)\left(g_V^t\epsilon^0(k)\not{k}-g_V^tk^0\not{\epsilon}(k)-ig_A^t\epsilon^{0\mu\nu\rho}\tilde{k}_\rho\gamma_\mu\epsilon_\nu(k)\right)u(p_2)
\end{aligned}
\tag{B2}
$$

The loop part is

$$
\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{i}{\epsilon-\frac{\mathbf{l}^2}{2m_t}+\frac{i\Gamma_t}{2}}\frac{i}{E-\epsilon-\frac{\mathbf{l}^2}{2m_t}+\frac{i\Gamma_t}{2}}=\int\frac{\mathrm{d}^{d-1}l}{(2\pi)^{d-1}}\frac{i}{E-\frac{\mathbf{l}^2}{m_t}+i\Gamma_t}
\tag{B3}
$$

and this is the leading order of $G(E)$, differed by a overall sign. While this integral is divergent in 3-dimension, by integrating it in $(d-1)$-dimension then take the limit, the divergence disappeared and the result in (6) is obtained.

Acknowledgments

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[2] K. Melnikov, M. Spira, and O. I. Yakovlev, Z. Phys. **C64**, 401 (1994), arXiv:hep-ph/9405301 [hep-ph].

[3] Y. Kats and M. D. Schwartz, JHEP **04**, 016 (2010), arXiv:0912.0526 [hep-ph].

[4] M. Beneke, Y. Kiyo, and K. Schuller, (2013), arXiv:1312.4791 [hep-ph].

[5] J. C. Romao and J. P. Silva, Int. J. Mod. Phys. **A27**, 1230025 (2012), arXiv:1209.6213 [hep-ph].

[6] Convention follows [5]. One only needs to check the sign convention η.