One Loop Gauge Link Self Energy

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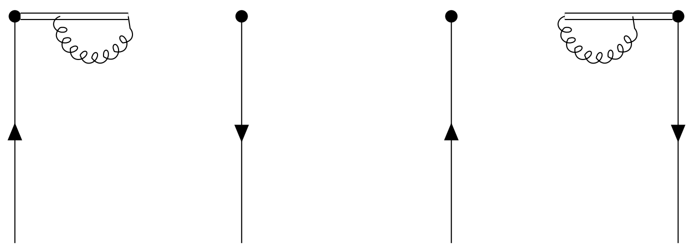
1 One Loop



(i) g (ii) h

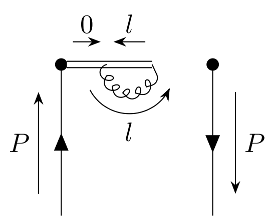
Figure 1: Diagrams of quasi PDF in Feynman gauge.

The definition of the gauge link self energy diagram (diagram g) is

$$\frac{1}{2}\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z\frac{\mathcal{P}\left[-ig_s\int_0^\infty \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right]\left[-ig_s\int_0^\infty \mathrm{d}z'' A^{a,z}(z'')\,\mathrm{t}^a\right]}{2}\psi(0)|P,S\rangle \tag{1}$$

Applying Feynman rule straightaway gives (the overall 1/2 factor has been counted in)

$$\Gamma_g(l) =$$



$$= -g_s^2 C_F\delta(1-x)\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{i\epsilon}\frac{i}{-l^z+i\epsilon} \tag{2}$$

1.1 Direct Contraction

1.1.1 Left

$$\begin{aligned}
&\frac{1}{2!}\mathcal{P}\left[\int_0^\infty \mathrm{d}z' A^{a,z}(z')\int_0^\infty \mathrm{d}z'' A^{a,z}(z'')\right]\\
=&\int_0^\infty \mathrm{d}z' A^{a,z}(z')\int_0^\infty \mathrm{d}z'' A^{a,z}(z'')\,\theta(z'-z'')\\
=&\int \mathrm{d}z' A^{a,z}(z')\int \mathrm{d}z'' A^{a,z}(z'')\,\theta(z'-z'')\theta(z'')\\
=&\int \mathrm{d}z' A^{a,z}(z')\int \mathrm{d}z'' A^{a,z}(z'')\,\theta(z'-z'')\theta(z'')\\
=&\int \mathrm{d}z'\int \mathrm{d}z''\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}e^{-il\cdot(z''-z')}\theta(z'-z'')\theta(z'')
\end{aligned}$$

1.1.2 Right

$$
\begin{aligned}
&\frac{1}{2!}\mathcal{P}\left[\int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''+z\right)\int_0^\infty \mathrm{d}z' A^{a,z}\left(z'+z\right)\right]\\
&=\int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''+z\right)\int_0^\infty \mathrm{d}z' A^{a,z}\left(z'+z\right)\theta(z''-z')\\
&=\int \mathrm{d}z'' A^{a,z}\left(z''+z\right)\int \mathrm{d}z' A^{a,z}\left(z'+z\right)\theta(z''-z')\theta(z')\\
&=\int \mathrm{d}z'' A^{a,z}\left(z''+z\right)\int \mathrm{d}z' A^{a,z}\left(z'+z\right)\theta(z''-z')\theta(z')\\
&=\int \mathrm{d}z'\int \mathrm{d}z''\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}e^{-il\cdot(z''-z')}\theta(z''-z')\theta(z')
\end{aligned}
$$

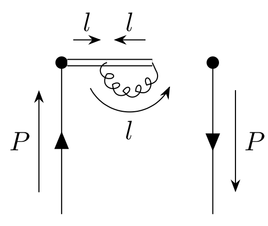
1.1.3 Summing together

$$
\int \mathrm{d}z'\int \mathrm{d}z''\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}e^{-il\cdot(z''-z')}[\theta(z'-z'')\theta(z'')+\theta(z''-z')\theta(z')]
$$

1.2 Adding two path order together

$$
\begin{aligned}
&\mathcal{P}\left[\int_0^\infty \mathrm{d}z' A^{a,z}\left(z'\right)\int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''\right)\right]\\
&=\int_0^\infty \mathrm{d}z' A^{a,z}\left(z'\right)\int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''\right)[\theta(z'-z'')+\theta(z''-z')]\\
&=\int_0^\infty \mathrm{d}z' A^{a,z}\left(z'\right)\int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''\right)\\
&=\int \mathrm{d}z' A^{a,z}\left(z'\right)\int \mathrm{d}z'' A^{a,z}\left(z''\right)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
&=\int \mathrm{d}z' A^{a,z}\left(z'\right)\int \mathrm{d}z'' A^{a,z}\left(z''\right)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
&=\int \mathrm{d}z'\int \mathrm{d}z''\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}e^{-il\cdot(z''-z')}\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
&=\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\int \mathrm{d}z'\int \mathrm{d}z''\int\frac{\mathrm{d}w}{2\pi}\frac{i}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{i}{h+i\epsilon}e^{-i(w-l)\cdot z'}e^{-i(h+l)\cdot z''}\\
&=\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}
\end{aligned}
$$

The amplitude is



$$
= -\frac{g_s^2 C_F}{2}\delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon} \tag{3}
$$

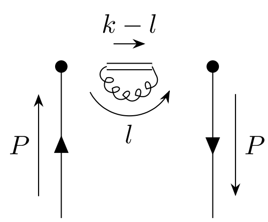
1.3 Adding $\Gamma_g(l)$ and $\Gamma_g(-l)$

$$
\begin{aligned}
\Gamma_g(l) = \frac{\Gamma_g(l)+\Gamma_g(-l)}{2} &= -\frac{1}{2}\left[g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{i\epsilon}\frac{i}{-l^z+i\epsilon} + g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{i\epsilon}\frac{i}{l^z+i\epsilon}\right] \\
&= -\frac{1}{2} g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{i\epsilon}\left[\frac{i}{-l^z+i\epsilon}+\frac{i}{l^z+i\epsilon}\right] \\
&= -\frac{1}{2} g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{i\epsilon}\frac{2\epsilon}{l^{z2}+\epsilon^2} \\
&= g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{l^z+i\epsilon}\frac{i}{l^z-i\epsilon}
\end{aligned}
$$

There's an overall factor of 1/2 missing.

1.4 Taking derivatives

Add a small momentum to the gauge link line and consider an actual self energy diagram



$$
= -g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{k^z-l^z+i\epsilon} \tag{4}
$$

take the derivative

$$
-\, g_s^2 C_F \delta(1-x)\lim_{k^z\to 0}\frac{\partial}{\partial k^z}\left[(i)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{k^z-l^z}\right] \tag{5}
$$

$$
= i g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{(l^z)^2} \tag{6}
$$

Adding $i\epsilon$ by hand and we got

$$
g_s^2 C_F \delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^2+i\epsilon}\frac{i}{l^z+i\epsilon}\frac{i}{l^z-i\epsilon} \tag{7}
$$

2 Two Loop

In coordinate space

$$
\langle P,S|\bar\psi(z)\gamma^z \mathcal{P}\frac{\left[-ig_s n_\mu\int_0^\infty \mathrm{d}z_1 A^{a,\mu}(z_1)t^a\right]\left[-ig_s n_\nu\int_0^\infty \mathrm{d}z_2 A^{b,\nu}(z_2)t^b\right]\left[-ig_s n_\rho\int_0^\infty \mathrm{d}z_3 A^{c,\rho}(z_3)t^c\right]\left[-ig_s n_\sigma\int_0^\infty \mathrm{d}z_4 A^{d,\sigma}(z_4)t^d\right]}{4!} \psi(0)|P,S\rangle \tag{8}
$$

$$
\begin{aligned}
&\frac{1}{4!}\mathcal{P}\left[\int_0^\infty \mathrm{d}z_1 A^{a,\mu}(z_1)\right]\left[\int_0^\infty \mathrm{d}z_2 A^{b,\nu}(z_2)\right]\left[\int_0^\infty \mathrm{d}z_3 A^{c,\rho}(z_3)\right]\left[\int_0^\infty \mathrm{d}z_4 A^{d,\sigma}(z_4)\right] \\
&= \left[\int_0^\infty \mathrm{d}z_1 A^{a,\mu}(z_1)\right]\left[\int_0^\infty \mathrm{d}z_2 A^{b,\nu}(z_2)\right]\left[\int_0^\infty \mathrm{d}z_3 A^{c,\rho}(z_3)\right]\left[\int_0^\infty \mathrm{d}z_4 A^{d,\sigma}(z_4)\right]\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)
\end{aligned}
$$

The coefficient of the above expression is

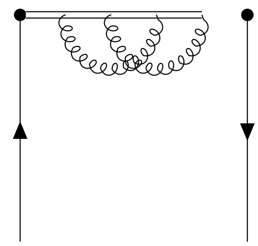
$$
\langle P|\bar\psi(z)\gamma^z\psi(0)|P\rangle\,(-ig_s n^\mu)(-ig_s n^\nu)(-ig_s n^\rho)(-ig_s n^\sigma)t^a t^b t^c t^d \tag{9}
$$

take a trace

$$
e^{-iP^z z}\,\mathrm{Tr}\{(\not{P}+m)\gamma^z\}\,\mathrm{Tr}\{t^a t^b t^c t^d\} g_s^4 n^\mu n^\nu n^\rho n^\sigma \tag{10}
$$

2.1 Diag. 50

The amplitude for



is related to the color ordering $t^a t^b t^a t^b$.

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) A^{d,\sigma}(z_4) \theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4) \\
&= \int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{-ig^{\mu\rho}\delta^{ac}}{l_1^2+i\epsilon} e^{-il_1\cdot(z_3-z_1)} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-ig^{\nu\sigma}\delta^{bd}}{l_2^2+i\epsilon} e^{-il_2\cdot(z_4-z_2)} \\
&\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4) \\
&= \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}z_4 \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{-ig^{\mu\rho}\delta^{ac}}{l_1^2+i\epsilon} e^{-il_1\cdot(z_3-z_1)} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-ig^{\nu\sigma}\delta^{bd}}{l_2^2+i\epsilon} e^{-il_2\cdot(z_4-z_2)} \\
&\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4)
\end{aligned}
$$

The exponent is (for simplicity we assume vectors $z_i = (0,0,0,z_i)$ and $k_i = (0,0,0,-k_i)$)

$$
\begin{aligned}
&-il_1\cdot(z_3-z_1) - il_2\cdot(z_4-z_2) - ik_1\cdot(z_1-z_2) - ik_2\cdot(z_2-z_3) - ik_3\cdot(z_3-z_4) - ik_4\cdot z_4 \\
&= -iz_3\cdot(l_1+k_3-k_2) - iz_1\cdot(k_1-l_1) - iz_4\cdot(l_2+k_4-k_3) - iz_2\cdot(k_2-k_1-l_2)
\end{aligned}
$$

which gives 4 delta functions. The propagators involved are then

$$
\int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-ig^{\mu\rho}\delta^{ac}}{l_1^2+i\epsilon} \frac{-ig^{\nu\sigma}\delta^{bd}}{l_2^2+i\epsilon} \frac{i}{-l_1^z+i\epsilon} \frac{i}{-l_1^z-l_2^z+i\epsilon} \frac{i}{-l_2^z+i\epsilon} \frac{i}{i\epsilon}
$$

$$
\begin{aligned}
&\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}z_4 A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) A^{d,\sigma}(z_4) \theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4) \\
&= \frac{1}{2} \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}z_4 A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) A^{d,\sigma}(z_4) [\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4) \\
&+ \theta(z_3-z_4)\theta(z_4-z_1)\theta(z_1-z_2)\theta(z_2)] \\
&= \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-ig^{\mu\rho}\delta^{ac}}{l_1^2+i\epsilon} \frac{-ig^{\nu\sigma}\delta^{bd}}{l_2^2+i\epsilon} \frac{-\left({l_1^z}^2 + 3l_1^z l_2^z + {l_2^z}^2 - \epsilon^2\right)}{(l_1^z-i\epsilon)(l_1^z+i\epsilon)(l_2^z+i\epsilon)(l_2^z-i\epsilon)(l_1^z+l_2^z-i\epsilon)(l_1^z+l_2^z+i\epsilon)}
\end{aligned}
$$

Taking the derivative of the former expression, we can also arrive at a divergence-free form

$$
\frac{i}{2} \lim_{p\to 0} \frac{\partial}{\partial p} \left[\frac{i}{p+l_1^z} \frac{i}{p+l_1^z+l_2^z} \frac{i}{p+l_2^z} \right] = \frac{-\left({l_1^z}^2 + 3l_1^z l_2^z + {l_2^z}^2\right)}{{l_1^z}^2 {l_2^z}^2 (l_1^z+l_2^z)^2}
$$

which is equivalent to above expression.

Adding ϵ in the definition so that the definition is $\mathcal{P} e^{-ig_s \int_0^\infty e^{-z\epsilon} n\cdot A^a(z) t^a}$, the expression becomes

$$
\begin{aligned}
&\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}z_4 A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) A^{d,\sigma}(z_4) e^{-(z_1+z_2+z_3+z_4)\epsilon} \theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4) \\
&= \frac{1}{2} \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}z_4 A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) A^{d,\sigma}(z_4) e^{-(z_1+z_2+z_3+z_4)\epsilon} [\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4)
\end{aligned}
$$

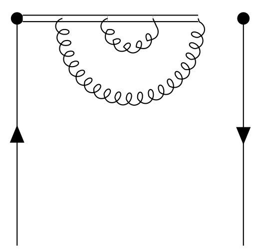
$$+\,\theta(z_3-z_4)\theta(z_4-z_1)\theta(z_1-z_2)\theta(z_2)]$$
$$=\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{-ig^{\mu\rho}\delta^{ac}}{l_1^2+i\epsilon}\frac{-ig^{\nu\sigma}\delta^{bd}}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-6l_1^zl_2^z-{l_2^z}^2}{4(l_1^z-i\epsilon)(l_1^z+i\epsilon)(3\epsilon-il_2^z)(3\epsilon+il_2^z)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}$$

The full amplitude in coordinate space is

$$4P^ze^{-iP^zz}g_s^4\,\mathrm{Tr}\{t^at^bt^ct^d\}n^\mu n^\nu n^\rho n^\sigma \tag{11}$$
$$\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{-ig^{\mu\rho}\delta^{ac}}{l_1^2+i\epsilon}\frac{-ig^{\nu\sigma}\delta^{bd}}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-6l_1^zl_2^z-{l_2^z}^2}{4(l_1^z-i\epsilon)(l_1^z+i\epsilon)(3\epsilon-il_2^z)(3\epsilon+il_2^z)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}$$
$$=4P^ze^{-iP^zz}g_s^4\,\mathrm{Tr}\{t^at^bt^at^b\} \tag{12}$$
$$\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{-in^2}{l_1^2+i\epsilon}\frac{-in^2}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-6l_1^zl_2^z-{l_2^z}^2}{4(l_1^z-i\epsilon)(l_1^z+i\epsilon)(3\epsilon-il_2^z)(3\epsilon+il_2^z)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}$$
$$=4P^ze^{-iP^zz}g_s^4\,\mathrm{Tr}\{t^at^bt^at^b\} \tag{13}$$
$$\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{i}{l_1^2+i\epsilon}\frac{i}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-6l_1^zl_2^z-{l_2^z}^2}{4(l_1^z-i\epsilon)(l_1^z+i\epsilon)(l_2^z+3i\epsilon)(l_2^z-3i\epsilon)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}$$

2.2 Diag. 37

The amplitude for



is related to the color ordering $t^at^bt^bt^a$.

$$\int_0^\infty\mathrm{d}z_1\int_0^\infty\mathrm{d}z_2\int_0^\infty\mathrm{d}z_3\int_0^\infty\mathrm{d}z_4A^{a,\mu}(z_1)A^{b,\nu}(z_2)A^{c,\rho}(z_3)A^{d,\sigma}(z_4)\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)$$
$$=\int\mathrm{d}z_1\int\mathrm{d}z_2\int\mathrm{d}z_3\int\mathrm{d}z_4\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ad}}{l_1^2+i\epsilon}e^{-il_1\cdot(z_3-z_1)}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{-ig^{\nu\rho}\delta^{bc}}{l_2^2+i\epsilon}e^{-il_2\cdot(z_4-z_2)}$$
$$\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4)$$

The propagators involved are

$$\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ad}}{l_1^2+i\epsilon}\frac{-ig^{\nu\rho}\delta^{bc}}{l_2^2+i\epsilon}\frac{i}{-l_1^z+i\epsilon}\frac{i}{-l_1^z-l_2^z+i\epsilon}\frac{i}{-l_1^z+i\epsilon}\frac{i}{i\epsilon}$$

$$\frac{1}{2}\int\mathrm{d}z_1\int\mathrm{d}z_2\int\mathrm{d}z_3\int\mathrm{d}z_4A^{a,\mu}(z_1)A^{b,\nu}(z_2)A^{c,\rho}(z_3)A^{d,\sigma}(z_4)[\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4)$$
$$+\theta(z_4-z_3)\theta(z_3-z_2)\theta(z_2-z_1)\theta(z_1)]$$
$$=\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ad}}{l_1^2+i\epsilon}\frac{-ig^{\nu\rho}\delta^{bc}}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-2l_1^zl_2^z+\epsilon^2}{\left({l_1^z}^2+\epsilon^2\right)^2\left({l_1^z}^2+2l_1^zl_2^z+{l_2^z}^2+\epsilon^2\right)}$$

Taking the derivative of the former expression, we can also arrive at a divergence-free form

$$\frac{i}{2}\lim_{p\to0}\frac{\partial}{\partial p}\left[\frac{i}{p+l_1^z}\frac{i}{p+l_1^z+l_2^z}\frac{i}{p+l_1^z}\right]=\frac{-(3l_1^z+2l_2^z)}{{l_1^z}^3(l_1^z+l_2^z)^2}$$

which is equivalent to above expression.

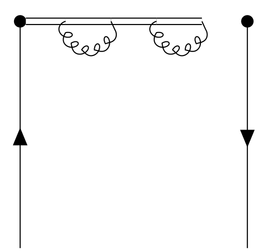
$$\frac{-3{l_1^z}^2-2l_1^z l_2^z+3\epsilon^2}{2(l_1^z-i\epsilon)(l_1^z+i\epsilon)(l_1^z-3i\epsilon)(l_1^z+3i\epsilon)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}$$

The full amplitude in coordinate space is

$$\begin{aligned}
&4P^z e^{-iP^z z} g_s^4 \operatorname{Tr}\{t^a t^b t^c t^d\} n^\mu n^\nu n^\rho n^\sigma && (14)\\
&\int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\int \frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ad}}{l_1^2+i\epsilon}\frac{-ig^{\nu\rho}\delta^{bc}}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-2l_1^z l_2^z+3\epsilon^2}{2(l_1^z-i\epsilon)(l_1^z+i\epsilon)(l_1^z-3i\epsilon)(l_1^z+3i\epsilon)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}\\
=&4P^z e^{-iP^z z} g_s^4 \operatorname{Tr}\{t^a t^b t^b t^a\} && (15)\\
&\int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\int \frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{-in^2}{l_1^2+i\epsilon}\frac{-in^2}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-2l_1^z l_2^z+3\epsilon^2}{2(l_1^z-i\epsilon)(l_1^z+i\epsilon)(l_1^z-3i\epsilon)(l_1^z+3i\epsilon)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}\\
=&4P^z e^{-iP^z z} g_s^4 \operatorname{Tr}\{t^a t^b t^b t^a\} && (16)\\
&\int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\int \frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{i}{l_1^2+i\epsilon}\frac{i}{l_2^2+i\epsilon}\frac{-3{l_1^z}^2-2l_1^z l_2^z}{2(l_1^z-i\epsilon)(l_1^z+i\epsilon)(l_1^z-3i\epsilon)(l_1^z+3i\epsilon)(l_1^z+l_2^z-2i\epsilon)(l_1^z+l_2^z+2i\epsilon)}
\end{aligned}$$

2.3 Diag. 43

The amplitude for



is related to the color ordering $t^a t^a t^b t^b$.

$$\begin{aligned}
&\int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 \overbracket{A^{a,\mu}(z_1)A}^{b,\nu}(z_2)\overbracket{A^{c,\rho}(z_3)A}^{d,\sigma}(z_4)\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\\
&=\int \mathrm{d}z_1\int \mathrm{d}z_2\int \mathrm{d}z_3\int \mathrm{d}z_4 \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ad}}{l_1^2+i\epsilon}e^{-il_1\cdot(z_2-z_1)}\int \frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{-ig^{\nu\rho}\delta^{bc}}{l_2^2+i\epsilon}e^{-il_2\cdot(z_4-z_3)}\\
&\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4)
\end{aligned}$$

The propagators involved are

$$\int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\int \frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ad}}{l_1^2+i\epsilon}\frac{-ig^{\nu\rho}\delta^{bc}}{l_2^2+i\epsilon}\frac{i}{-l_1^z+i\epsilon}\frac{i}{i\epsilon}\frac{i}{-l_2^z+i\epsilon}\frac{i}{i\epsilon}$$

$$\begin{aligned}
&\frac{1}{4}\int \mathrm{d}z_1\int \mathrm{d}z_2\int \mathrm{d}z_3\int \mathrm{d}z_4 \overbracket{A^{a,\mu}(z_1)A}^{b,\nu}(z_2)\overbracket{A^{c,\rho}(z_3)A}^{d,\sigma}(z_4)[\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4)\\
&+\theta(z_2-z_1)\theta(z_1-z_3)\theta(z_3-z_4)\theta(z_4)+\theta(z_1-z_2)\theta(z_2-z_4)\theta(z_4-z_3)\theta(z_3)+\theta(z_2-z_1)\theta(z_1-z_4)\theta(z_4-z_3)\theta(z_3)]\\
&=\int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\int \frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ab}}{l_1^2+i\epsilon}\frac{-ig^{\nu\rho}\delta^{cd}}{l_2^2+i\epsilon}\frac{1}{\left({l_1^z}^2+\epsilon^2\right)\left({l_2^z}^2+\epsilon^2\right)}
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 \overbracket{A^{a,\mu}(z_1)A}^{b,\nu}(z_2)\overbracket{A^{c,\rho}(z_3)A}^{d,\sigma}(z_4)e^{-(z_1+z_2+z_3+z_4)\epsilon}\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\\
&=\int \mathrm{d}z_1\int \mathrm{d}z_2\int \mathrm{d}z_3\int \mathrm{d}z_4 \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\frac{-ig^{\mu\sigma}\delta^{ab}}{l_1^2+i\epsilon}e^{-il_1\cdot(z_2-z_1)}\int \frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{-ig^{\nu\rho}\delta^{cd}}{l_2^2+i\epsilon}e^{-il_2\cdot(z_4-z_3)}e^{-(z_1+z_2+z_3+z_4)\epsilon}
\end{aligned}$$

$$\theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3-z_4)\theta(z_4)$$

and the eikonal part is

$$\frac{3}{8(l_1^z-i\epsilon)(l_1^z+i\epsilon)(3\epsilon-il_2^z)(3\epsilon+il_2^z)}$$

The full amplitude in coordinate space is

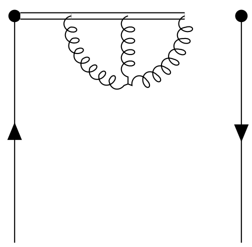
$$4P^z e^{-iP^z z} g_s^4 \operatorname{Tr}\{t^a t^b t^c t^d\} n^\mu n^\nu n^\rho n^\sigma \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-ig^{\mu\sigma}\delta^{ab}}{l_1^2+i\epsilon} \frac{-ig^{\nu\rho}\delta^{cd}}{l_2^2+i\epsilon} \frac{3}{8(l_1^z-i\epsilon)(l_1^z+i\epsilon)(3\epsilon-il_2^z)(3\epsilon+il_2^z)} \tag{17}$$

$$=4P^z e^{-iP^z z} g_s^4 \operatorname{Tr}\{t^a t^a t^b t^b\} \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-in^2}{l_1^2+i\epsilon} \frac{-in^2}{l_2^2+i\epsilon} \frac{3}{8(l_1^z-i\epsilon)(l_1^z+i\epsilon)(3\epsilon-il_2^z)(3\epsilon+il_2^z)} \tag{18}$$

$$=4P^z e^{-iP^z z} g_s^4 \operatorname{Tr}\{t^a t^a t^b t^b\} \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{i}{l_1^2+i\epsilon} \frac{i}{l_2^2+i\epsilon} \frac{3}{8(l_1^z-i\epsilon)(l_1^z+i\epsilon)(l_2^z-3i\epsilon)(l_2^z+3i\epsilon)} \tag{19}$$

2.4 Diag. 47

The amplitude for



is related to

$$\langle P,S|\bar{\psi}(z)\gamma^z V_3 \mathcal{P} \frac{\left[-ig_s n_\mu \int_0^\infty \mathrm{d}z_1 A^{a,\mu}(z_1) t^a\right]\left[-ig_s n_\nu \int_0^\infty \mathrm{d}z_2 A^{b,\nu}(z_2) t^b\right]\left[-ig_s n_\rho \int_0^\infty \mathrm{d}z_3 A^{c,\rho}(z_3) t^c\right]}{3!} \psi(0)|P,S\rangle \tag{20}$$

where

$$V_3 = -\frac{g_s}{2} f^{def} \int \mathrm{d}^4 t \left(\partial^\alpha A_d^\beta - \partial^\beta A_d^\alpha\right) A_\alpha^e A_\beta^f$$

The gluon related contraction (one out of three, others can be obtained by exchanging d,e,f.)

$$\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \big(\partial^\alpha A^{d,\beta} - \partial^\beta A^{d,\alpha}\big) A_\alpha^e A_\beta^f A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) \theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3)$$

$$= \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{-i\delta^{dc}}{l_1^2+i\epsilon} \Big[g^{\beta\rho}\partial^\alpha e^{-il_1\cdot(z_3-t)} - g^{\alpha\rho}\partial^\beta e^{-il_1\cdot(z_3-t)}\Big] \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-ig_\alpha^\mu \delta^{ea}}{l_2^2+i\epsilon} e^{-il_2\cdot(z_1-t)}$$

$$\int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-ig_\beta^\nu \delta^{fb}}{l_3^2+i\epsilon} e^{-il_3\cdot(z_2-t)} \theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3)$$

$$= \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{-i\delta^{dc}}{l_1^2+i\epsilon} \Big[g^{\beta\rho} i l_1^\alpha - g^{\alpha\rho} i l_1^\beta\Big] e^{-il_1\cdot(z_3-t)} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-ig_\alpha^\mu \delta^{ea}}{l_2^2+i\epsilon} e^{-il_2\cdot(z_1-t)}$$

$$\int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-ig_\beta^\nu \delta^{fb}}{l_3^2+i\epsilon} e^{-il_3\cdot(z_2-t)} \theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3)$$

$$= \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{-i\delta^{dc}}{l_1^2+i\epsilon} [g^{\nu\rho} i l_1^\mu - g^{\mu\rho} i l_1^\nu] e^{-il_1\cdot(z_3-t)} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-i\delta^{ea}}{l_2^2+i\epsilon} e^{-il_2\cdot(z_1-t)}$$

$$\int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-i\delta^{fb}}{l_3^2+i\epsilon} e^{-il_3\cdot(z_2-t)} \theta(z_1-z_2)\theta(z_2-z_3)\theta(z_3)$$

Multiplied by n

$$n_\mu n_\nu n_\rho [g^{\nu\rho} i l_1^\mu - g^{\mu\rho} i l_1^\nu] = i n^2 [l_1^z - l_1^z] = 0$$

Contracting with different fields in the parenthesis

$$\partial^\alpha A^{d,\beta} A^e_\alpha A^f_\beta A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) - \partial^\beta A^{d,\alpha} A^e_\alpha A^f_\beta A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3)$$

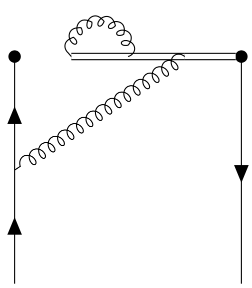
$$\begin{aligned}
&\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-i\delta^{dc}}{l_3^2 + i\epsilon} \frac{-i g^\mu_\alpha \delta^{ea}}{l_1^2 + i\epsilon} e^{-i l_1 \cdot (z_1 - t)} \frac{-i g^\nu_\beta \delta^{fb}}{l_2^2 + i\epsilon} \\
&\left[e^{-i l_2 \cdot (z_2 - t)} g^{\beta\rho} \partial^\alpha e^{-i l_3 \cdot (z_3 - t)} - e^{-i l_3 \cdot (z_3 - t)} g^{\alpha\rho} \partial^\beta e^{-i l_2 \cdot (z_2 - t)} \right] \theta(z_1 - z_2)\theta(z_2 - z_3)\theta(z_3) \\
&= \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-i\delta^{dc}}{l_3^2 + i\epsilon} \frac{-i\delta^{ea}}{l_1^2 + i\epsilon} \frac{-i\delta^{fb}}{l_2^2 + i\epsilon} e^{-i l_1 \cdot (z_1 - t)} e^{-i l_2 \cdot (z_2 - t)} e^{-i l_3 \cdot (z_3 - t)} \\
&[g^{\nu\rho} i l_3^\mu - g^{\mu\rho} i l_2^\nu] \theta(z_1 - z_2)\theta(z_2 - z_3)\theta(z_3)
\end{aligned}$$

Make $z_1 \to -z_1, t \to -t, l_2 \to -l_2$,

$$\begin{aligned}
&\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-i\delta^{dc}}{l_3^2 + i\epsilon} \frac{-i\delta^{ea}}{l_1^2 + i\epsilon} \frac{-i\delta^{fb}}{l_2^2 + i\epsilon} e^{-i l_1 \cdot (z_1 - t)} e^{-i l_2 \cdot (z_2 - t)} e^{-i l_3 \cdot (z_3 - t)} \\
&[g^{\nu\rho} i l_3^\mu - g^{\mu\rho} i l_2^\nu] \theta(-z_1 - z_2)\theta(z_2 - z_3)\theta(z_3) \\
&= \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-i\delta^{dc}}{l_3^2 + i\epsilon} \frac{-i\delta^{ea}}{(l_3 + l_2)^2 + i\epsilon} \frac{-i\delta^{fb}}{l_2^2 + i\epsilon} \frac{i}{-l_3^z + i\epsilon} \frac{i}{-2l_3^z + i\epsilon} \frac{i}{-2l_3^z - l_2^z + i\epsilon} [g^{\nu\rho} i l_3^\mu - g^{\mu\rho} i l_2^\nu]
\end{aligned}$$

$$\partial^\alpha A^{d,\beta} A^e_\alpha A^f_\beta A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) - \partial^\beta A^{d,\alpha} A^e_\alpha A^f_\beta A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3)$$

$$\begin{aligned}
&\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \mathrm{d}^4 t \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{-i\delta^{dc}}{l_3^2 + i\epsilon} \frac{-i g^\mu_\alpha \delta^{ea}}{l_1^2 + i\epsilon} e^{-i l_1 \cdot (z_1 - t)} \frac{-i g^\nu_\beta \delta^{fb}}{l_2^2 + i\epsilon} \\
&\left[e^{-i l_2 \cdot (z_2 - t)} g^{\beta\rho} \partial^\alpha e^{-i l_3 \cdot (z_3 - t)} - e^{-i l_3 \cdot (z_3 - t)} g^{\alpha\rho} \partial^\beta e^{-i l_2 \cdot (z_2 - t)} \right] \theta(z_1 - z_2)\theta(z_2 - z_3)\theta(z_3)
\end{aligned}$$



is related to

$$\langle P, S | \bar{\psi}(z) \gamma^z Q_3 \mathcal{P} \frac{\left[-i g_s n_\mu \int_0^\infty \mathrm{d}z_1 A^{a,\mu}(z_1) t^a\right] \left[-i g_s n_\nu \int_0^\infty \mathrm{d}z_2 A^{b,\nu}(z_2) t^b\right] \left[-i g_s n_\rho \int_0^\infty \mathrm{d}z_3 A^{c,\rho}(z_3) t^c\right]}{3!} \psi(0) | P, S \rangle \tag{21}$$

where

$$Q_3 = (-i g_s \gamma_\sigma) \int \mathrm{d}^4 t \bar{\psi} \psi A^{d,\sigma}$$

$$\begin{aligned}
&\int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 A^{d,\sigma}(t) A^{a,\mu}(z_1) A^{b,\nu}(z_2) A^{c,\rho}(z_3) \theta(z_1 - z_2)\theta(z_2 - z_3)\theta(z_3) \\
&= \int \mathrm{d}z_1 \int \mathrm{d}z_2 \int \mathrm{d}z_3 \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{-i g^{\mu\nu} \delta^{ab}}{l_1^2 + i\epsilon} \frac{-i g^{\sigma\rho} \delta^{dc}}{l_2^2 + i\epsilon} e^{-i l_1 \cdot (z_2 - z_1)} e^{-i l_2 \cdot (z_3 - t)} \theta(z_1 - z_2)\theta(z_2 - z_3)\theta(z_3)
\end{aligned}$$

The eikonal propagators are

$$\frac{i}{-l_1^z + i\epsilon} \frac{i}{i\epsilon} \frac{i}{l_2^z + i\epsilon}$$

Flip z_1 and z_2

$$-\frac{i}{-l_1^z + i\epsilon} \frac{i}{l_1^z + i\epsilon} \frac{i}{l_2^z + i\epsilon} \tag{22}$$

This covers for all diagrams involved one gauge field contracting with the would-be-divergent three gauge link fields, and the extra eikonal line behaves exactly as in one loop level.