Local Operator Divergence

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1   NRQED local matrix elements
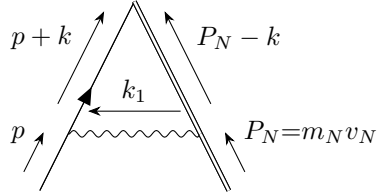
Since what we what is an operator equation independent of states, we can choose $p^\mu = (0, \mathbf{p})$ as our outstate electron momentum, and put the nucleus on-shell. This way, we can eliminate possible IR divergence and make the integral much easier (especially at NNLO where the first integral will give a $\sinh^{-1}\left(\frac{1}{\sqrt{\frac{\mathbf{k_1}^2}{2Em} - 1}}\right)$ if no constrain condition put at external momentum).

At NLO:

$$
\langle 0|\psi_e(0)N(0)(-ie)\int \mathrm{d}^4 y \bar{\psi}_e \psi_e A^0 (-ie)\int \mathrm{d}^4 z \bar{N}N A^0 |eN\rangle =
$$

$$
= e^2 u_N(v_N)\left[\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p})^2(E - \frac{\mathbf{k}^2}{2m})}(1 - \frac{\mathbf{k}^4}{8m^3(E - \frac{\mathbf{k}^2}{2m})})\right]\frac{i}{E - \frac{\mathbf{p}^2}{2m}}
$$

$$
= -e^2 u_N(v_N)[\frac{\pi}{v} + \mathcal{O}(v^2)]\frac{i}{E - \frac{\mathbf{p}^2}{2m}}
$$

Appendices

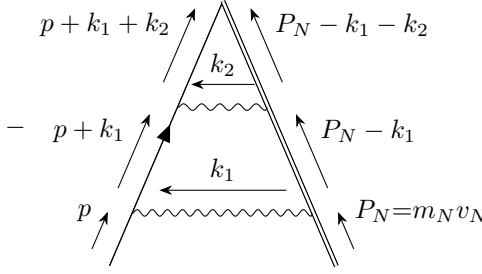NRQED matrix element at NLO

$$\langle 0|\psi_e(0)N(0)(-ie)\int \mathrm{d}^4 y\bar\psi_e\psi_e A^0(-ie)\int \mathrm{d}^4 z\bar{N}NA^0|eN\rangle =$$

$$=ie^2 u_N(v_N)\left[\int [\mathrm{d}k]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(^0+k^0-m-\frac{(\mathbf{p}+\mathbf{k})^2}{2m}+\frac{(\mathbf{p}+\mathbf{k})^4}{8m^3}+i\epsilon)}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}$$

$$=e^2 u_N(v_N)\left[\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p})^2(E-\frac{\mathbf{k}^2}{2m}+\frac{\mathbf{k}^4}{8m^3})}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}$$

$$=e^2 u_N(v_N)\left[\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p})^2(E-\frac{\mathbf{k}^2}{2m})}(1-\frac{\mathbf{k}^4}{8m^3(E-\frac{\mathbf{k}^2}{2m})})\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}$$

$$=-2me^2 u_N(v_N)\frac{i}{E-\frac{\mathbf{p}^2}{2m}}[\frac{\pi}{2p}+\mathcal{O}(p^2)]=-e^2 u_N(v_N)\frac{i}{E-\frac{\mathbf{p}^2}{2m}}[\frac{\pi}{v}+\mathcal{O}(v^2)]$$

At NNLO (where we're only interested in divergent parts)

$$=e^4\left[\int [dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{1}{p^0+k_1^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{1}{p^0+k_1^0+k_2^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

do the shift as above

$$=-e^4\left[\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{E-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{1}{E-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

drop $\mathbf{p}$

$$=-e^4\left[\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{1}{-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$
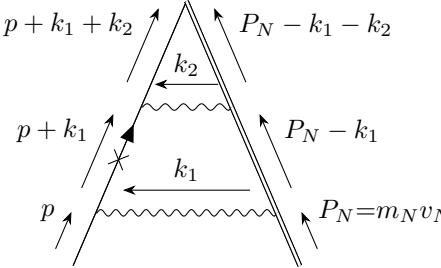
if we add higher reltivistic correction

$$=-e^4\left[\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{2m}{|\mathbf{k_1}|^2-\frac{|\mathbf{k_1}|^4}{4m^2}}\frac{2m}{|\mathbf{k_2}|^2-\frac{|\mathbf{k_2}|^4}{4m^2}}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

$$=-4m^2 e^4\left[\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_1}|^2}(1+\frac{|\mathbf{k_1}|^2}{4m^2})\frac{1}{|\mathbf{k_2}|^2}(1+\frac{|\mathbf{k_2}|^2}{4m^2})\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

The integral (we can verify that $\mathbf{k}^0$ part is not UV divergent, and we don't care aobut $\mathbf{k}^8$ term for now) for

$$= \int \frac{d^3\mathbf{k_1}}{(2\pi)^3} \frac{d^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}-\mathbf{p}|^2} \frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2} \frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2} \frac{1}{|\mathbf{k_2}|^2-2mE}$$

$$= \int_0^1 dx \int \frac{d^3\mathbf{k_1}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}-\mathbf{p}|^2} \frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2} \frac{\left(\frac{4\pi}{\Delta_2}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2 \Gamma(2)}$$

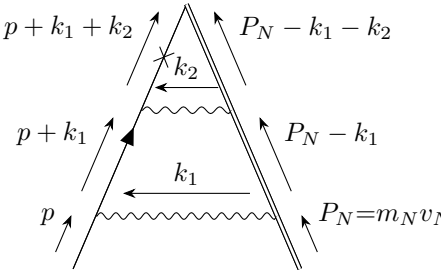where $\Delta_2 = (1-x)\left(|\mathbf{k_1}|^2 x - 2Em\right)$

$$= \frac{1}{(4\pi)^2} \int_0^1 dx \int \frac{d^3\mathbf{k_1}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}-\mathbf{p}|^2} \frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2} \frac{1}{(|\mathbf{k_1}|^2-2mE/x)^{2-d/2}} \left(\frac{4\pi}{x(1-x)}\right)^{2-d/2} \Gamma(2-d/2)$$

$$= \frac{1}{(4\pi)^2} \int_0^1 dx \int_0^1 dydzdt \delta(y+z+t-1) \int \frac{d^3\mathbf{k_1}}{(2\pi)^3} \frac{zt^{1-d/2}|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2+\Delta_1]^{5-d/2}} \frac{\Gamma(5-d/2)}{\Gamma(2-d/2)} \left(\frac{4\pi}{x(1-x)}\right)^{2-d/2} \Gamma(2-d/2)$$

where $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mE(z+t/x)$

$$= \frac{1}{4m^2(4\pi)^2} \int_0^1 dx \int_0^1 dydzdt \delta(y+z+t-1) zt^{1-d/2} \frac{d(d+2)}{4} \frac{\Gamma(3-d)}{(4\pi)^{3-d/2}} \left(\frac{4\pi}{\Delta_1}\right)^{3-d} \left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}$$

$$= -\frac{1}{128\pi^2(d-3)m^2} + \text{finite terms}$$

$$= \int \frac{d^3\mathbf{k_1}}{(2\pi)^3} \frac{d^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}-\mathbf{p}|^2} \frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_1}|^2-2mE} \frac{|\mathbf{k_2}|^4/4m^2}{[|\mathbf{k_2}|^2-2mE]^2}$$

$$= \frac{1}{4m^2} \int_0^1 dx \int \frac{d^3\mathbf{k_1}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}-\mathbf{p}|^2} \frac{1}{|\mathbf{k_1}|^2-2mE} \frac{(1-x)\Gamma(1-d/2)}{8\pi} \left(\frac{4\pi}{\Delta_2}\right)^{1-d/2} \frac{d(d+2)}{4}$$

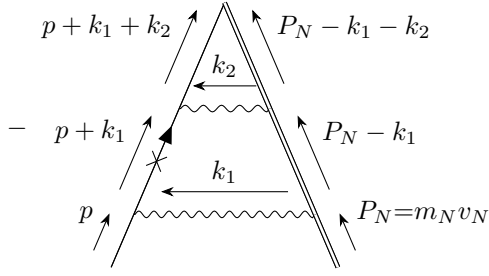where $\Delta_2 = x(1-x)|\mathbf{k_1}|^2 - 2mE(1-x)$

$$= \frac{1}{4m^2} \int_0^1 dx \int_0^1 dydzdt \frac{t^{-d/2}}{[|\mathbf{k_1}|^2+\Delta_1]^{3-d/2}} \frac{\Gamma(3-d/2)}{\Gamma(1-d/2)} \delta(y+z+t-1) \frac{\Gamma(1-d/2)}{8\pi} \left(\frac{4\pi}{x(1-x)}\right)^{1-d/2} \frac{d(d+2)x}{4}$$

where $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mEz - 2mE\frac{t}{x}$

$$= \frac{1}{4m^2} \int_0^1 dx \int_0^1 dydzdt \delta(y+z+t-1) \frac{1}{(4\pi)^{3-d/2}} \left(\frac{4\pi}{\Delta_1}\right)^{3-d} \frac{\Gamma(3-d)}{8\pi} \left(\frac{4\pi}{x(1-x)}\right)^{1-d/2} \frac{d(d+2)x}{4} t^{-d/2}$$

$$= \frac{15}{8192\pi^2(d-3)m^2} + \text{finite terms}$$

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Check the contour integral

$$=e^4\left[\int[dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{(\mathbf{p}+\mathbf{k})^4/8m^3}{[p^0+k_1^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}+i\epsilon]^2}\frac{1}{p^0+k_1^0+k_2^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

$$=-e^4\left[\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{|\mathbf{k_1}|^4/8m^3}{[E-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon]^2}\frac{1}{E-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

which is the same as previous (by expansion) one.

Contact (Darwin) term

$$=\frac{4\pi e^4}{3m}\left[\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{p^0+k^0-m-\frac{(\mathbf{p}+\mathbf{k})^2}{2m}+i\epsilon}\left(1+\frac{(\mathbf{p}+\mathbf{k})^4/8m^3}{p^0+k^0-m-\frac{(\mathbf{p}+\mathbf{k})^2}{2m}+i\epsilon}\right)\frac{1}{-k^0+i\epsilon}\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

$$=\frac{4\pi e^4}{3m}\left[\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{E-\frac{\mathbf{k}^2}{2m}+2i\epsilon}\left(1+\frac{|\mathbf{k}|^4/8m^3}{E-\frac{\mathbf{k}^2}{2m}+i\epsilon}\right)\right]\frac{i}{E-\frac{\mathbf{p}^2}{2m}}u_N(v_N)$$

if the dispersion relation is up to $\mathbf{k}^4$ then

$$=\frac{4e^4m}{3}$$