Hydrogen

Yingsheng Huang

October 23, 2017

QED Lagrangian is

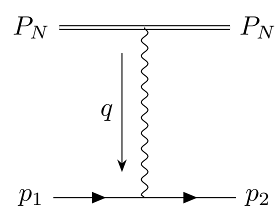
$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \tag{1}$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \tag{2}$$

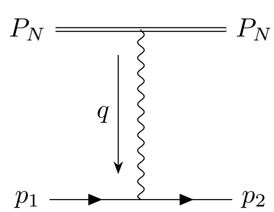
In tree level

$$i\mathcal{M}^{(0)}_{QED} =$$



$$= -e^2\bar{u}_N(P_N)v^\mu u_N(P_N)\frac{i}{\mathbf{q}^2}\bar{u}_e(p_2)\gamma_\mu u_e(p1)$$

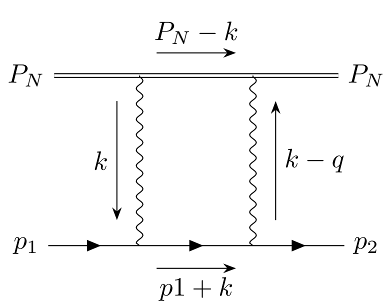
$$i\mathcal{M}^{(0)}_{NRQED} =$$



$$= -e^2\bar{u}_N(P_N)v^\mu u_N(P_N)\frac{i}{\mathbf{q}^2}\psi^\dagger(p_2)\gamma_\mu\psi(p1)$$

The box diagram for NRQED process is

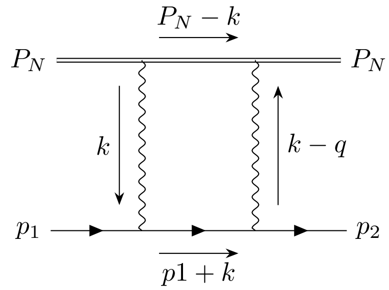
$$i\mathcal{M}^{(1)}_{NRQED} =$$



$$\begin{aligned}
&= e^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1) \\
&= -ie^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m})}\psi(p_1) \\
&= -ie^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2(E_1-\frac{\mathbf{k}^2}{2m})}\psi(p_1)
\end{aligned}$$

The box and crossed box diagram for QED process is

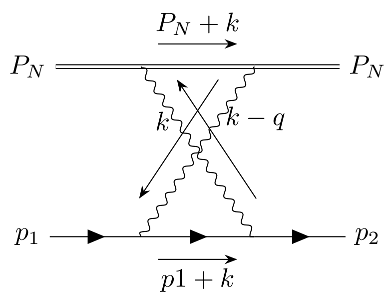
$i\mathcal{M}_1^{(1)} =$



$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.

$i\mathcal{M}_2^{(1)} =$



$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0-\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2+p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0-\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2+p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

$$
\begin{aligned}
i\mathcal{M}_1^{(1)}+i\mathcal{M}_2^{(1)} &= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-{p_1^0}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2-\mathbf{p_1}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)
\end{aligned}
$$

Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order.

Now consider operator product expansion.

One loop scenario for NRQED case:

$$
\begin{aligned}
\langle 0|\psi_e(0)N(0)e\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle &= e^2u_N(P_N)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\
&= -ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\
&= -ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1)
\end{aligned}
$$

drop p_1

$$
= ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1) = \pi ie^2\sqrt{\frac{2m}{E_1}}u_N(P_N)\psi(p_1)
$$

For QED case:

$$
\begin{aligned}
\langle 0|\psi(x)N(0)e\int \mathrm{d}^4y\bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle &= e^2u_N(P_N)\int[\mathrm{d}k]e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= e^2u_N(P_N)\int[\mathrm{d}k]e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{p_1})\cdot\mathbf{x}}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$