One Loop Matching for Quasi PDF

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1 Background

The definition of parton distribution function (PDF) is

$$q\left(x, \mu_f\right)=\frac{1}{2} \int \frac{d \eta^{-}}{2 \pi} e^{-i x P^{+} \eta^{-}}\left\langle P, S\left|\bar{\psi}\left(0, \eta^{-}, \mathbf{0}_T\right) \Gamma \mathcal{W}\left[\eta^{-} ; 0\right] \psi(0)\right| P, S\right\rangle \tag{1}$$

where with this unpolarized PDF case, $\Gamma=\gamma^{+}$. $\mathcal{W}$ is the gauge link defined as [Collins(2009)]

$$\mathcal{W}\left[w^{-}, 0\right]=P\left\{e^{-i g_0 \int_0^{w^{-}} \mathrm{d} y^{-} A_{(0) \sigma}^{+}\left(0, y^{-}, \mathbf{0}_{\mathrm{T}}\right) t_\sigma}\right\} \tag{2}$$

The definition of quasi PDF is

$$\tilde{q}(x)=\frac{1}{2} \int \frac{\mathrm{d} z}{2 \pi} e^{i x P^z z}\langle P, S|\bar{\psi}(z) \tilde{\Gamma} \tilde{\mathcal{W}}[z, 0] \psi(0)| P, S\rangle \tag{3}$$

where

$$\tilde{\mathcal{W}}[z, 0]=\mathcal{P} \exp \left[i g \int_0^z \mathrm{d} z^{\prime} n \cdot A^a\left(z^{\prime}\right) \mathrm{t}^a\right], n^\mu=(0,0,0,-1) \tag{4}$$

and $\tilde{\Gamma}=\gamma^z$ in our case. This means $n^2=-1$ and $n \cdot P=P^z$.

To make the gauge links equal to unity, we choose light cone gauge for PDF and axial gauge for quasi PDF.

2 Tree Level Matching

In axial gauge, the quasi PDF is

$$\tilde{q}(x) = \frac{1}{4\pi}\int \mathrm{d}z e^{ixP^z z}\langle P|\bar{\psi}(z)\gamma^z\psi(0)|P\rangle \tag{5}$$

The frame is chosen such that $P^\mu = (P^0, \mathbf{0}, P^z)$.

$$P^0 = \sqrt{m^2 + {P^z}^2} \tag{6}$$

Up to one loop, we can use quark state as the external state to complete the matching process. The quark field ψ reads

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}\left[u(k)e^{-ik\cdot x}b_k + v(k)e^{ik\cdot x}d_k^\dagger\right] \tag{7}$$

Insert it to (5)

$$\tilde{q}^{(0)}(x) = \int \frac{\mathrm{d}z}{4\pi}e^{ixP^z z}\langle 0|b_P \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\left[\bar{u}(p)e^{ip\cdot x}b_p^\dagger + \bar{v}(p)e^{-ip\cdot x}d_p\right]\gamma^z \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}\left[u(k)e^{-ik\cdot x}b_k + v(k)e^{ik\cdot x}d_k^\dagger\right]b_P^\dagger|0\rangle \tag{8}$$

Look at the creation-annihilation operators, we have the following combinations:

$$b_P b_p^\dagger b_k b_P^\dagger,\ b_P d_p b_k b_P^\dagger,\ b_P b_p^\dagger d_k^\dagger b_P^\dagger,\ b_P d_p d_k^\dagger b_P^\dagger \tag{9}$$

Apparently the latter three all go to zero by moving the anti-quark operators to the side:

$$\begin{aligned}
\tilde{q}^{(0)}(x) &= \int \frac{\mathrm{d}z}{4\pi}e^{ixP^z z}\langle 0|\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\bar{u}(p)e^{ip\cdot z}b_P b_p^\dagger\gamma^z \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}u(k)e^{-ik\cdot 0}b_k b_P^\dagger|0\rangle \\
&= \int \frac{\mathrm{d}z}{4\pi}e^{ixP^z z}\langle 0|\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{e^{ip\cdot z}}{2E_p}\bar{u}(p)(2\pi)^3 2E_{\mathbf{P}}\delta^{(3)}(\mathbf{p}-\mathbf{P})\gamma^z \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{e^{-ik\cdot 0}}{2E_k}u(k)(2\pi)^3 2E_{\mathbf{P}}\delta^{(3)}(\mathbf{k}-\mathbf{P})|0\rangle \\
&= \int \frac{\mathrm{d}z}{4\pi}e^{ixP^z z + iP\cdot z}\bar{u}(P)\gamma^z u(P)
\end{aligned} \tag{10}$$

Using Gordon identity

$$\begin{aligned}
\tilde{q}^{(0)}(x) &= \int \frac{\mathrm{d}z}{4\pi}e^{ixP^z z - iP^z z}\bar{u}(P)\frac{P^z}{m}u(P) \\
&= \int \frac{\mathrm{d}z}{2\pi}e^{ixP^z z - iP^z z}P^z \\
&= \delta(1-x)
\end{aligned} \tag{11}$$

3 One Loop Quasi PDF (Axial Gauge)

First we consider the matrix element in the definition of quasi PDF

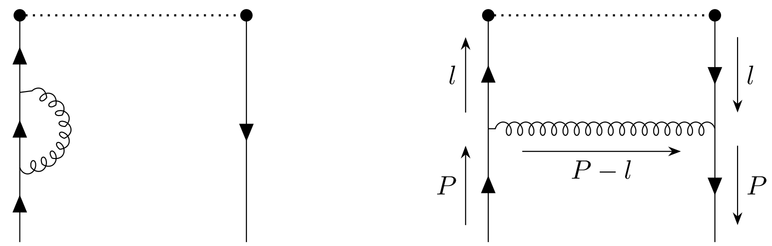
$$\langle P|\bar{\psi}(z)\gamma^z\psi(0)|P\rangle \tag{12}$$

and in leading order this one gives

$$e^{-iP^z z}\bar{u}(P)\gamma^z u(P) \tag{13}$$

as mentioned above. This, in higher orders, is embedded via a Fourier transform. The full form of quasi PDF can be considered as a momentum space matrix element with an $1/4\pi$ factor.

Two diagrams are required with one loop corrections to quasi PDF. Detailed derivation with rigorous Wick contraction is to be found in Section B.



(a) (b)

Figure 1:

The Feynman rule for the composite operator is

$$\overset{p_1,0}{\bullet}\cdots\cdots\cdots\overset{p_2,\text{z}}{\bullet} = e^{-ip_2^z z}\gamma^z \tag{14}$$

and two external lines give $\bar{u}(P)$ and $u(P)$ respectively.

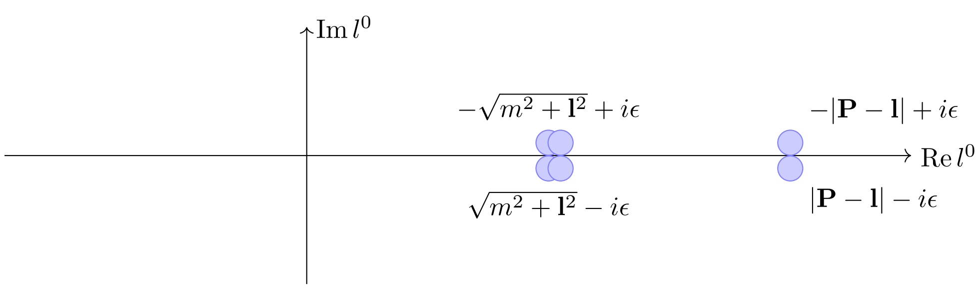
The first one is a quark self-energy correction

$$\bar{u}(P)e^{-iP^z z}\gamma^z\frac{i(\not{P}+m)}{P^2-m^2}(-i\Sigma_2(P))u(P) \tag{15}$$

The second one is

$$\begin{aligned}&\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\,(-ig_s t^a\gamma^\mu)\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}(-ig_s t^a\gamma^\nu)\tilde{D}^A_{G\mu\nu}(P-l)u(P)\Big|_{l^z=xP^z}\\ &=-g_s^2C_F\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\,\gamma^\mu\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}\gamma^\nu\tilde{D}^A_{G\mu\nu}(P-l)u(P)\Big|_{l^z=xP^z}\end{aligned} \tag{16}$$

For the definition of $\tilde{D}^A_{G\mu\nu}$, see Section A. There're in total 6 poles:



For the result of numerator simplification, see Section E

4 One Loop Quasi PDF (Feynman Gauge)

In Feynman gauge, we must have the full definition of quasi PDF. For unpolarized quasi PDF

$$\tilde{q}(x) = \frac{1}{2}\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z \mathcal{P}\exp\left[ig\int_0^z \mathrm{d}z' A^{a,z}\left(z'\right)\mathrm{t}^a\right]\psi(0)|P,S\rangle \tag{17}$$

There're following 8 diagrams.

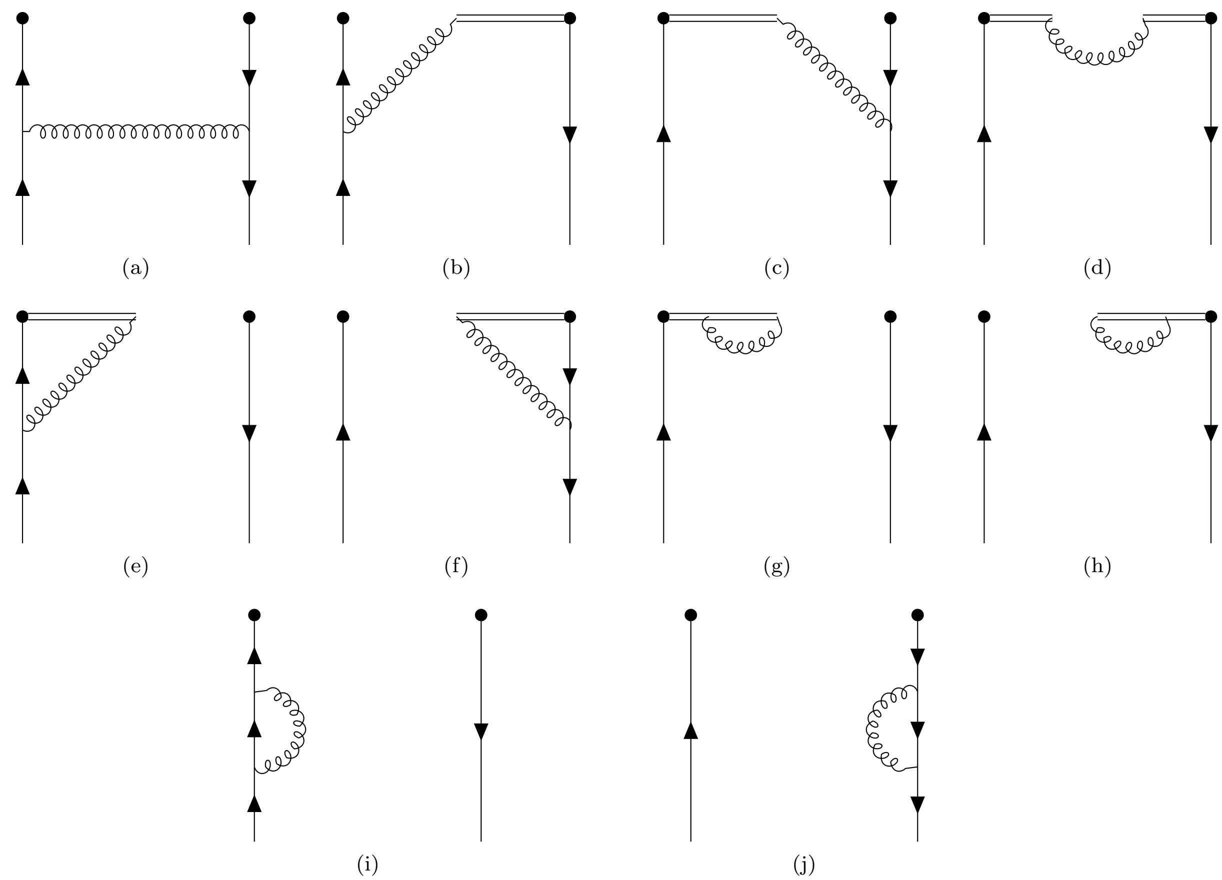
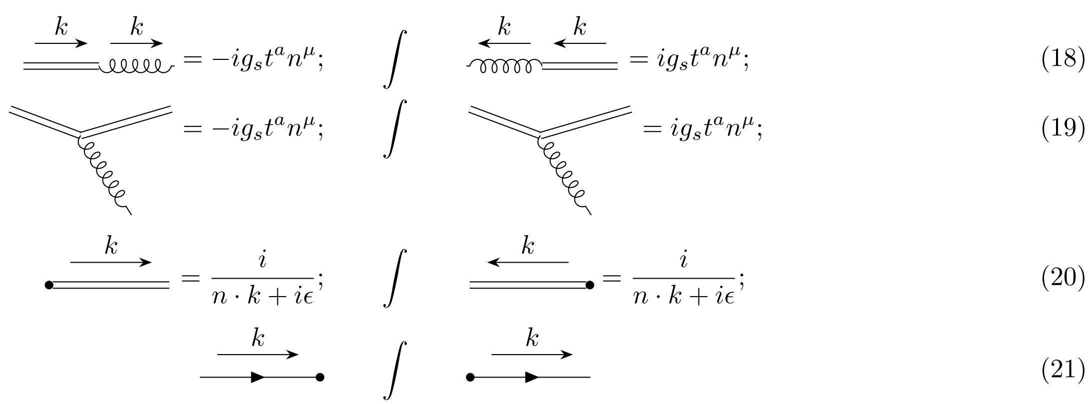


Figure 2: Diagrams of quasi PDF in Feynman gauge.

The corresponding Feynman rules are:



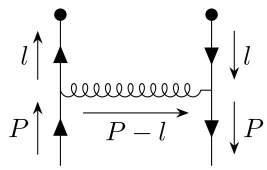
The last line stands for the momentum conservation between two dots. There're also an extra 1/2 factor on the outside and a Dirac delta function to eliminate all z-direction loop momenta. The delta function confines the sum of all momenta flow in z equals to xP^z.

Let's take the spin sum of external states.

4.1 Real corrections

First we must deal with those real diagrams.

Diagram 2a gives

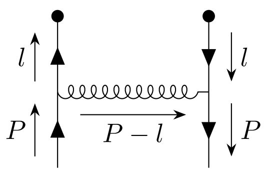


$$= \frac{1}{2}\bar{u}(P)(-ig_s t^a \gamma_\nu)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{\not{l} - m + i\epsilon}\gamma^z \frac{-ig^{\mu\nu}}{(P-l)^2 + i\epsilon}\frac{i}{\not{l} - m + i\epsilon}(-ig_s \gamma_\mu t^a)u(P)\delta(l^z - xP^z)$$

$$= -i\frac{g_s^2 C_F}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^\mu \frac{\not{l} + m}{l^2 - m^2 + i\epsilon}\gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon}\gamma_\mu u(P)\frac{1}{(P-l)^2 + i\epsilon}\delta(l^z - xP^z) \tag{22}$$

After spin sum:

$$\frac{1}{2}\sum_s$$

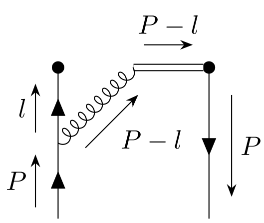


$$= \frac{-ig_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\sum_s \bar{u}(P)\gamma^\mu \frac{\not{l} + m}{l^2 - m^2 + i\epsilon}\gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon}\gamma_\mu u(P)\frac{1}{(P-l)^2 + i\epsilon}\delta(l^z - xP^z)$$

$$= \frac{-ig_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\mathrm{Tr}\big\{(\not{P} + m)\gamma^\mu(\not{l} + m)\gamma^z(\not{l} + m)\gamma_\mu\big\}}{(l^2 - m^2 + i\epsilon)^2 (P-l)^2}\delta(l^z - xP^z) \tag{23}$$

the numerator and the denominator of the integrand is checked out in this step with Xiong's result.

Diagram 2b gives

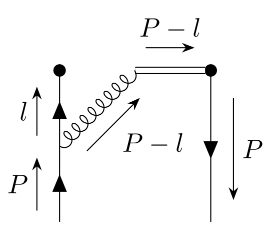


$$= \frac{1}{2}\bar{u}(P)\gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^z - P^z + i\epsilon}(ig_s t^a)\frac{-ig^{\mu z}}{(P-l)^2 + i\epsilon}\frac{i}{\not{l} - m + i\epsilon}(-ig_s \gamma_\mu t^a)u(P)\delta(l^z - xP^z)$$

$$= \frac{ig_s^2 C_F}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon}\gamma^z u(P)\frac{1}{l^z - P^z + i\epsilon}\frac{1}{(P-l)^2 + i\epsilon}\delta(l^z - xP^z) \tag{24}$$

Take the spin sum

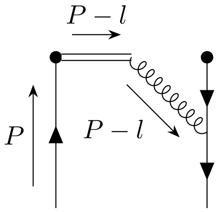
$$\frac{1}{2}\sum_s$$



$$= \sum_s \frac{ig_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon}\gamma^z u(P)\frac{1}{l^z - P^z + i\epsilon}\frac{1}{(P-l)^2 + i\epsilon}\delta(l^z - xP^z)$$

$$= \frac{ig_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\mathrm{Tr}\big\{(\not{P} + m)\gamma^z(\not{l} + m)\gamma^z\big\}}{l^2 - m^2 + i\epsilon}\frac{1}{l^z - P^z + i\epsilon}\frac{1}{(P-l)^2 + i\epsilon}\delta(l^z - xP^z) \tag{25}$$

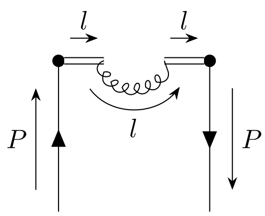
Diagram 2c should be the same with Diagram 2b.



$$
= \frac{1}{2}\bar{u}(P)(-ig_s\gamma_\mu t^a)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{\not{l}-m+i\epsilon}\frac{-ig^{\mu z}}{(P-l)^2+i\epsilon}\frac{i}{P^z-l^z+i\epsilon}(-ig_s t^a)\gamma^z u(P)\delta(l^z-xP^z)
$$

$$
= \frac{ig_s^2 C_F}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P)\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z) \tag{26}
$$

Diagram 2d is

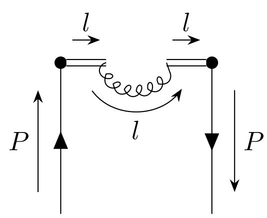


$$
= \frac{1}{2}\bar{u}(P)\gamma^z\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{i}{l^z+i\epsilon}(ig_s t^a)\frac{-ig^{zz}}{l^2+i\epsilon}(-ig_s t^a)\frac{i}{-l^z+i\epsilon}u(P)\delta(l^z-(1-x)P^z)
$$

$$
= \frac{-ig_s^2 C_F}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^z u(P)\frac{1}{l^z+i\epsilon}\frac{1}{l^2+i\epsilon}\frac{1}{-l^z+i\epsilon}\delta(l^z-(1-x)P^z) \tag{27}
$$

Take the spin sum

$$\frac{1}{2}\sum_s$$



$$
= \sum_s \frac{-ig_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\bar{u}(P)\gamma^z u(P)\frac{1}{l^z+i\epsilon}\frac{1}{l^2+i\epsilon}\frac{1}{-l^z+i\epsilon}\delta(l^z-(1-x)P^z)
$$

$$
= \frac{-ig_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\operatorname{Tr}\big\{(\not{P}+m)\gamma^z\big\}\frac{1}{l^z+i\epsilon}\frac{1}{l^2+i\epsilon}\frac{1}{-l^z+i\epsilon}\delta(l^z-(1-x)P^z) \tag{28}
$$

For the final result see my Mathematica notebook (and Xiong's).

4.2 Compare amplitudes of real corrections with Feng's code

Diagram 2a gives

$$
-\frac{i\,\delta_{\mathrm{CI(3)\,CI(4)}}\, g^{\mathrm{LI(3)\,LI(4)}}\, g_s^2\,\mathrm{MomC(k1+pe)}\,\mathrm{ColorLine}(T_{\mathrm{CI(4)}}.T_{\mathrm{CI(3)}},\{p,p\})\,\mathrm{SpinLine}\big(\gamma^{\mathrm{LI(4)}}.(\gamma\cdot\mathrm{k1}+m).(\gamma\cdot n).(\gamma\cdot\mathrm{k1}+m).\gamma^{\mathrm{LI(3)}},\{p,p\}\big)}{\big(\mathrm{k1}^2-m^2\big)^2(\mathrm{k1}-p)^2}
$$

which we may work out to be

$$
-i\delta_{ab}\operatorname{Tr}\big\{t^a t^b\big\}g_s^2\frac{\bar{u}(p)\gamma^\mu(\not{k}_1+m)\not{n}(\not{k}_1+m)\gamma_\mu u(p)}{\big[k_1^2-m^2\big]^2\big[k_1-p\big]^2}\delta(k_1+p_e)\Bigg|_{p_e=-xP^z,p=P,n\to z,k_1=l} \tag{29}
$$

And what we got earlier is

$$
-i\frac{g_s^2 C_F}{2}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\bar{u}(P)\gamma^\mu\big(\not{l}+m\big)\gamma^z\big(\not{l}+m\big)\gamma_\mu u(P)}{\big[l^2-m^2+i\epsilon\big]^2\big[(P-l)^2+i\epsilon\big]}\delta(l^z-xP^z) \tag{30}
$$

As we can see, apart from an overall 1/2 factor, these two are in perfect agreement. According to tree level result, we know that the 1/2 factor wasn't implanted in the code.

Let's test out diagrams with gauge link. Diagram 2b gives

$$\frac{i\,\delta_{\mathrm{CI}(7)\,\mathrm{CI}(8)}\,g^{\mathrm{LI}(7)\,\mathrm{LI}(8)}\,g_s^2\,\mathrm{n2}^{\mathrm{LI}(8)}\,\mathrm{MomC}(\mathrm{pe}-\mathrm{k1})\,\mathrm{ColorLine}(T_{\mathrm{CI}(8)}.T_{\mathrm{CI}(7)},\{p,\,p\})\,\mathrm{SpinLine}((\gamma\cdot n).(\gamma\cdot -\mathrm{k1}+m).\gamma^{\mathrm{LI}(7)},\{p,\,p\})}{(\mathrm{k1}^2-m^2)\,(-\mathrm{k1}-p)^2\,\mathrm{n2}\cdot(p+\mathrm{pe})}$$

Our previous result is

$$\frac{ig_s^2C_F}{4}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{\mathrm{Tr}\{(\not{P}+m)\gamma^z(\not{l}+m)\gamma^z\}}{l^2-m^2+i\epsilon}\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z) \tag{31}$$

Clearly the momentum of gauge link propagators don't consist with each other. By applying the Delta function, we'll see that they agree with each other.

Diagram 2d gives

$$-\frac{i\,\delta_{\mathrm{CI}(9)\,\mathrm{CI}(10)}\,g^{\mathrm{LI}(9)\,\mathrm{LI}(10)}\,g_s^2\,\mathrm{MomC}(-\mathrm{k1})\,\mathrm{n1}^{\mathrm{LI}(9)}\,\mathrm{n2}^{\mathrm{LI}(10)}\,\mathrm{ColorLine}(T_{\mathrm{CI}(10)}.T_{\mathrm{CI}(9)},\{p,\,p\})\,\mathrm{SpinLine}(\gamma\cdot n,\{p,\,p\})}{(-p-\mathrm{pe})^2\,\mathrm{n1}\cdot(p+\mathrm{pe})\,\mathrm{n2}\cdot(p+\mathrm{pe})}$$

and it gives

$$-ig_s^2C_F\delta(-k_1)n_1\cdot n_2\bar{u}(p)\not{n}u(p)\frac{1}{(-p-p_e)^2n_1\cdot(p+p_e)n_2\cdot(p+p_e)} \tag{32}$$

$$=-ig_s^2C_F\delta(k_1-p-p_e)n_1\cdot n_2\bar{u}(p)\not{n}u(p)\frac{1}{k_1^2n_1\cdot k_1n_2\cdot k_1} \tag{33}$$

$$=ig_s^2C_F\delta(k_1-p-p_e)\bar{u}(p)\not{n}u(p)\frac{1}{k_1^2n_1\cdot k_1n_2\cdot k_1} \tag{34}$$

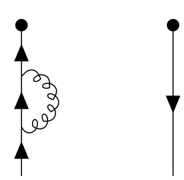
Our previous result is

$$\frac{-ig_s^2C_F}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\bar{u}(P)\not{n}u(P)\frac{1}{l^2+i\epsilon}\frac{1}{n\cdot l+i\epsilon}\frac{1}{-n\cdot l+i\epsilon}\delta(l^z-(1-x)P^z) \tag{35}$$

$$=\frac{ig_s^2C_F}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\delta(l^z-p^z-p_e^z)\bar{u}(P)\not{n}u(P)\frac{1}{l^2+i\epsilon}\frac{1}{n\cdot l+i\epsilon}\frac{1}{n\cdot l-i\epsilon} \tag{36}$$

4.3 Virtual corrections

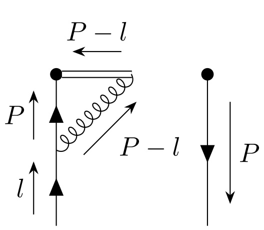
The quark self energy diagram gives



(37)

and the divergence it brings should exactly cancel the divergence of all other diagrams since the operator is conserved current and all external legs are on-shell. With standard prescription, we perform a derivative operation on the self energy loop to extract the Z_F factor. In the end we have $Z_F\delta(1-x)$.

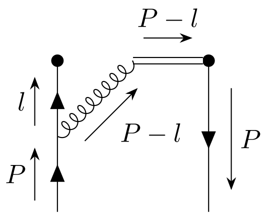
Diagram 2e corresponds to the vertex correction of a heavy-light current for HQET



$$=\frac{1}{2P^z}\bar{u}(P)\gamma^z\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{l^z-P^z+i\epsilon}(-ig_st^a)\frac{-ig^{\mu z}}{(P-l)^2+i\epsilon}\frac{i}{\not{l}-m+i\epsilon}(-ig_s\gamma_\mu)u(P)\delta(1-x)$$

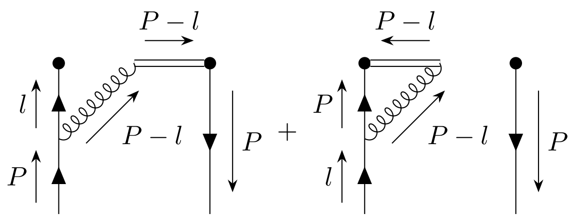
$$= \frac{-ig_s^2 C_F}{2P^z} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P) \gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon} \gamma^z u(P) \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} \delta(1-x) \tag{38}$$

The other diagram 2b is



$$= \frac{ig_s^2 C_F}{2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P) \gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon} \gamma^z u(P) \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} \delta(l^z - xP^z) \tag{39}$$

The sum of both diagrams is



$$= \frac{ig_s^2 C_F}{2} \int \mathrm{d}z \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P) \gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon} \gamma^z u(P) \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} \left[e^{i(l^z - xP^z)z} - e^{i(P^z - xP^z)z} \right] \tag{40}$$

$$= \frac{ig_s^2 C_F}{2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P) \gamma^z \frac{\not{l} + m}{l^2 - m^2 + i\epsilon} \gamma^z u(P) \frac{1}{l^z - P^z + i\epsilon} \frac{1}{(P-l)^2 + i\epsilon} [\delta(l^z - xP^z) - \delta(P^z - xP^z)] \tag{41}$$

Here we're to take the spin sum. After integrated out l^0 and $\mathbf{l}_T$, the remaining integrand is [1]

$$\frac{g_s^2 C_F P^{z3}}{8\pi^2 (m^2 + P^{z2})} \int \mathrm{d}l^z \frac{\delta(l^z - xP^z) - \delta(P^z - xP^z)}{|l^z - P^z|} + \frac{g_s^2 C_F P^{z2}}{8\pi^2 \sqrt{m^2 + P^{z2}}} \int \mathrm{d}l^z \left(\frac{\log \frac{l^z - P^z}{\Lambda}}{l^z - P^z} \right) [\delta(l^z - xP^z) - \delta(P^z - xP^z)] + \text{Constant}$$

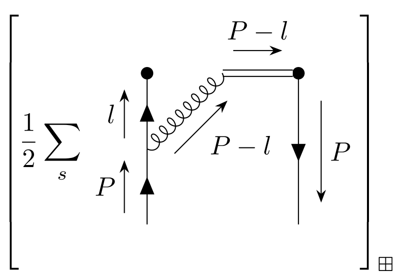
by setting $l^z = yP^z$, $\mathrm{d}l^z(\delta(l^z - xP^z) - \delta(P^z - xP^z)) = \mathrm{d}y(\delta(y-x) - \delta(1-x))$

$$= \frac{g_s^2 C_F P^{z3}/|P^z|}{8\pi^2 (m^2 + P^{z2})} \int \mathrm{d}y \frac{\delta(y-x) - \delta(1-x)}{|y-1|} + \frac{g_s^2 C_F P^z}{8\pi^2 \sqrt{m^2 + P^{z2}}} \int \mathrm{d}y \left(\frac{\log \frac{y-1}{\Lambda/P^z}}{y-1} \right) [\delta(y-x) - \delta(1-x)] + \text{Constant}$$

Now we can transform the integration on y to plus functions[2]. By redefining the plus function to an extended version:

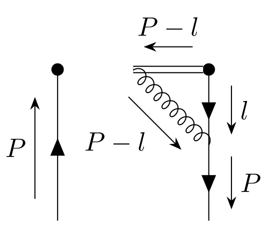
$$f_{\boxplus}(x) = f(x) - \delta(1-x) \int_{-\infty}^{\infty} \mathrm{d}y f(y) \tag{42}$$

The full result is



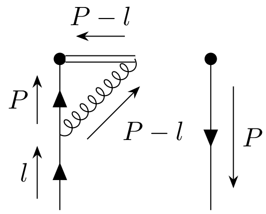
(43)

Diagram 2f is

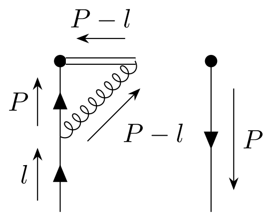


$$= \frac{1}{2P^z} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P) (-ig_s \gamma_\mu t^a) \frac{i(\not{l} + m)}{l^2 - m^2 + i\epsilon} \gamma^z \frac{-ig^{\mu z}}{(P-l)^2 + i\epsilon} \frac{i}{P^z - l^z} (ig_s t^a) u(P) \delta(1-x)$$

$$= \frac{-ig_s^2 C_F}{2P^z} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \bar{u}(P)\gamma^z \frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P) \frac{1}{l^z-P^z-i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(1-x)$$

$=$  (44)

After spin sum diagram 2e

$\frac{1}{2}\sum_s$  $= \frac{-ig_s^2 C_F}{4P^z}\delta(1-x) \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not{P}+m)\gamma^z(\not{l}+m)\gamma^z\}}{l^2-m^2+i\epsilon} \frac{1}{n\cdot(l-P)+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}$ (45)

First we apply Feynman parametrization to the normal propagators

$$\frac{1}{l^2-m^2+i\epsilon}\frac{1}{(P-l)^2+i\epsilon} = \int_0^1 \mathrm{d}y \frac{1}{\left[(l+P(y-1))^2 - ym^2 - P^2(y-1)y\right]^2} \tag{46}$$

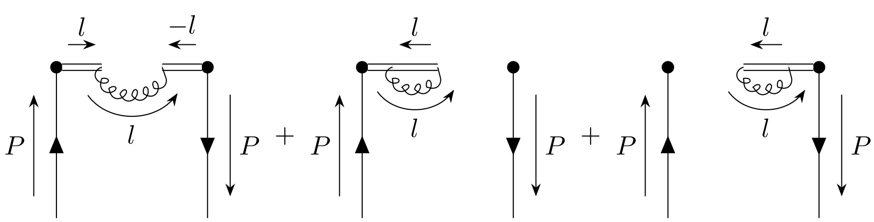
According to identity

$$\frac{1}{a^r b^s} = 2^s \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^\infty \mathrm{d}\lambda \frac{\lambda^{s-1}}{(a+2b\lambda)^{r+s}} \tag{47}$$

the whole denominator becomes

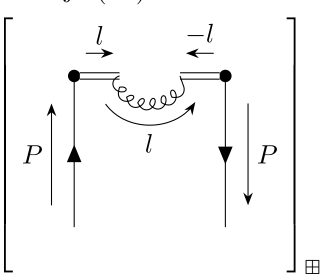
$$\frac{2\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 \mathrm{d}y \int_0^\infty \mathrm{d}\lambda \frac{1}{\left[(l+\lambda n + P(y-1))^2 - ym^2 - \lambda^2 n^2 - 2\lambda n P y - P^2(y-1)y\right]^3} \tag{48}$$

The combination of diagrams 2d, 2g and 2h is

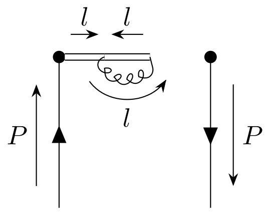


$$= \frac{g_s^2 C_F}{2}\bar{u}(P)\gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}\delta(l^z-(1-x)P^z)u(P) - \frac{2g_s^2 C_F}{2}\delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}$$

$$= g_s^2 C_F \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}\left[\delta(l^z/P^z-(1-x)) - \delta(1-x)\right]$$

$=$  (49)

For the gauge link self energy diagram 2g/2h,



$$= -\frac{g_s^2 C_F}{2}\delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon} \tag{50}$$

$$= \frac{i g_s^2 C_F}{2}\delta(1-x)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{l^2+i\epsilon}\frac{1}{n\cdot l+i\epsilon}\frac{1}{-n\cdot l+i\epsilon} \tag{51}$$

As mentioned before, a parametrization scheme is available for only one eikonal propagator. If there's two:

$$\frac{1}{a^r e_1^{s_1} e_2^{s_2}} = 2^{s_1}\frac{\Gamma(r+s_1)}{\Gamma(r)\Gamma(s_1)}\int_0^\infty \mathrm{d}\lambda_1 \frac{\lambda_1^{s_1-1}}{(a+2e_1\lambda_1)^{r+s_1}}\frac{1}{e_2^{s_2}} = 2^{s_1+s_2}\frac{\Gamma(r+s_1)}{\Gamma(r)\Gamma(s_1)}\frac{\Gamma(r+s_1+s_2)}{\Gamma(r+s_1)\Gamma(s_2)}\int_0^\infty \mathrm{d}\lambda_1\mathrm{d}\lambda_2\frac{\lambda_1^{s_1-1}\lambda_2^{s_2-1}}{(a+2e_1\lambda_1+2e_2\lambda_2)^{r+s_1+s_2}}$$

$$= 2^{s_1+s_2}\frac{\Gamma(r+s_1+s_2)}{\Gamma(r)\Gamma(s_1)\Gamma(s_2)}\int_0^\infty \mathrm{d}\lambda_1\mathrm{d}\lambda_2\frac{\lambda_1^{s_1-1}\lambda_2^{s_2-1}}{(a+2e_1\lambda_1+2e_2\lambda_2)^{r+s_1+s_2}} \tag{52}$$

and naturally

$$\frac{1}{a^r\prod e_i^{s_i}} = 2^{\sum s_i}\frac{\Gamma(r+\sum s_i)}{\Gamma(r)\prod\Gamma(s_i)}\int_0^\infty \left(\prod \mathrm{d}\lambda_i\right)\frac{\prod\lambda_i^{s_i-1}}{(a+2\sum e_i\lambda_i)^{r+\sum s_i}} \tag{53}$$

The amplitude becomes

$$4 i g_s^2 C_F\delta(1-x)\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda_1\mathrm{d}\lambda_2\frac{1}{(l^2+i\epsilon)^3} \tag{54}$$

$$= i g_s^2 C_F\delta(1-x)(2\pi)^2\int\frac{\mathrm{d}^6 l_{D+2}}{(2\pi)^6}\frac{1}{(l^2+i\epsilon)^3} \tag{55}$$

Clearly this is not right. [3]

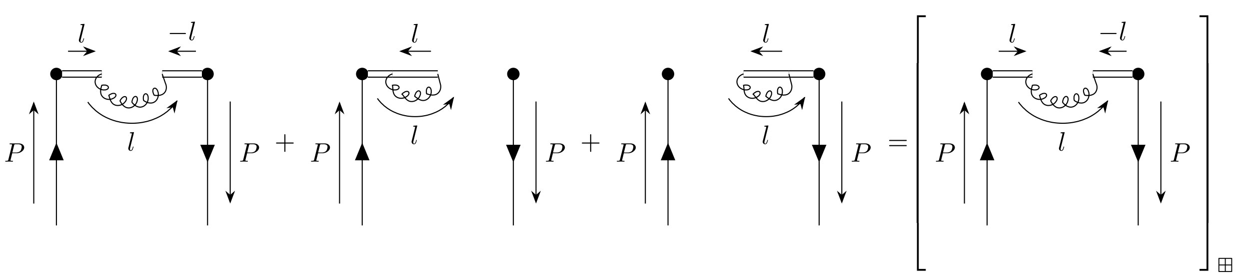
We can also parametrize this as

$$-4 i g_s^2 C_F\delta(1-x)\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\int_0^\infty \mathrm{d}\lambda_1\mathrm{d}\lambda_2\frac{1}{\left[l^2+2(\lambda_1+\lambda_2)n\cdot l+(\lambda_1-\lambda_2)i\epsilon\right]^3} \tag{56}$$

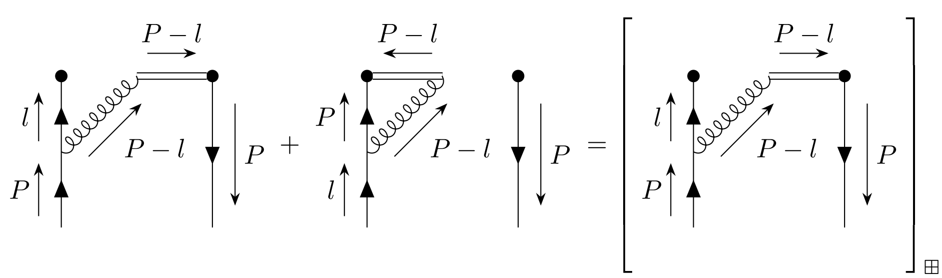
then we can add small gluon mass to subtract the UV divergence. This way we get

$$\frac{C_F\alpha_s}{4\pi\epsilon} - \frac{C_F\alpha_s\log m}{2\pi} \tag{57}$$

The final result is



(58)



(59)

4.4 Compare amplitudes of virtual corrections with Feng's code

Diagram 2e gives

$$\frac{i\,\delta_{\mathrm{CI}(7)\,\mathrm{CI}(8)}\,g^{\mathrm{LI}(7)\,\mathrm{LI}(8)}\,g_s^2\,\mathrm{n1}^{\mathrm{LI}(8)}\,\mathrm{MomC}(p+\mathrm{pe})\,\mathrm{ColorLine}\big(T_{\mathrm{CI}(8)}.T_{\mathrm{CI}(7)},\,\{p,\,p\}\big)\,\mathrm{SpinLine}\big((\gamma\cdot n).(\gamma\cdot\mathrm{k1}+m).\gamma^{\mathrm{LI}(7)},\,\{p,\,p\}\big)}{\big(\mathrm{k1}^2-m^2\big)\,(\mathrm{k1}-p)^2\,(\mathrm{n1}\cdot(\mathrm{k1}+\mathrm{pe})+i\,\varepsilon)}$$

and it's

$$ig_s^2C_F\delta(p+p_e)\frac{\not{n}(\not{k}_1+m)\not{n}_1}{(k_1^2-m^2)(k_1-p)^2(n_1\cdot(k_1+p_e))}\Big|_{p_e=-xP^z\to-p,p=P,n\to z,k_1=l,n_1=n} \tag{60}$$

Our previous result is

$$\frac{-ig_s^2C_F}{2P^z}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\bar{u}(P)\gamma^z\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^z u(P)\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(1-x) \tag{61}$$

these two are in agreement.

Let's check the gauge link self energy diagram 2g:

$$-\frac{i\,\delta_{\mathrm{CI}(7)\,\mathrm{CI}(8)}\,g^{\mathrm{LI}(7)\,\mathrm{LI}(8)}\,g_s^2\,\mathrm{n1}^{\mathrm{LI}(7)}\,\mathrm{n1}^{\mathrm{LI}(8)}\,\mathrm{MomC}(p+\mathrm{pe})\,\mathrm{ColorLine}\big(T_{\mathrm{CI}(8)}.T_{\mathrm{CI}(7)},\,\{p,\,p\}\big)\,\mathrm{SpinLine}(\gamma\cdot n,\,\{p,\,p\})}{(\mathrm{k1}-p-\mathrm{pe})^2\,(\mathrm{SmVar}(\mathrm{n1},\,1)+\mathrm{k1}\cdot\mathrm{n1})}$$

and it gives

$$-ig_s^2C_Fn_1^2\bar{u}(p)\not{n}u(p)\delta(p+p_e)\frac{1}{(k_1-p-p_e)^2(k_1\cdot n_1+SmVar(n_1,1))}\Big|_{p_e=-xP^z\to-p,p=P,n\to z,k_1=l,n_1=n} \tag{62}$$

After taking derivative against $SmVar$

$$\frac{i\,\delta_{\mathrm{CI}(7)\,\mathrm{CI}(8)}\,g^{\mathrm{LI}(7)\,\mathrm{LI}(8)}\,g_s^2\,\mathrm{n1}^{\mathrm{LI}(7)}\,\mathrm{n1}^{\mathrm{LI}(8)}\,\mathrm{MomC}(p+\mathrm{pe})\,\mathrm{ColorLine}\big(T_{\mathrm{CI}(8)}.T_{\mathrm{CI}(7)},\,\{p,\,p\}\big)\,\mathrm{SpinLine}(\gamma\cdot n,\,\{p,\,p\})}{(\mathrm{k1}\cdot\mathrm{n1})^2\,(\mathrm{k1}-p-\mathrm{pe})^2}$$

it gives (applying the delta function)

$$-ig_s^2C_Fn_1^2\bar{u}(p)\not{n}u(p)\delta(p+p_e)\frac{1}{(k_1-p-p_e)^2(k_1\cdot n_1)^2}\Big|_{p_e=-xP^z\to-p,p=P,n\to z,k_1=l,n_1=n} \tag{63}$$

$$=-ig_s^2C_F2P^z\delta(P^z(1-x))\frac{1}{l^2(n\cdot l)^2} \tag{64}$$

Our previous result is

$$\frac{ig_s^2C_F}{2}\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{l^2+i\epsilon}\frac{1}{n\cdot l+i\epsilon}\frac{1}{-n\cdot l+i\epsilon} \tag{65}$$

and there's a twice times difference (apart from the 1/2 from definition).

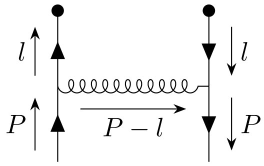
The real diagram 2d generated by code is (34)

$$ig_s^2C_F\delta(k_1-p-p_e)\bar{u}(p)\not{n}u(p)\frac{1}{k_1^2n_1\cdot k_1n_2\cdot k_1} \tag{66}$$

$$\to ig_s^2C_F2P^z\delta(l-(1-x)P^z)\frac{1}{l^2n_1\cdot ln_2\cdot l} \tag{67}$$

5 Light Cone PDF

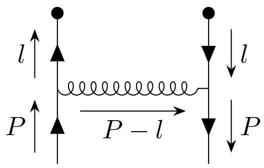
Diagram 2a gives



$$
\begin{aligned}
&= \frac{1}{2}\bar{u}(P)(-ig_s t^a\gamma_\nu)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{i}{\not{l}-m+i\epsilon}\gamma^+\frac{-ig^{\mu\nu}}{(P-l)^2+i\epsilon}\frac{i}{\not{l}-m+i\epsilon}(-ig_s\gamma_\mu t^a)u(P)\delta(l^z-xP^z)\\
&= -i\frac{g_s^2C_F}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\bar{u}(P)\gamma^\mu\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^+\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma_\mu u(P)\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z) \qquad (68)
\end{aligned}
$$

After spin sum:

$$\frac{1}{2}\sum_s$$



$$
\begin{aligned}
&= \frac{-ig_s^2C_F}{4}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\sum_s\bar{u}(P)\gamma^\mu\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma^+\frac{\not{l}+m}{l^2-m^2+i\epsilon}\gamma_\mu u(P)\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z)\\
&= \frac{-ig_s^2C_F}{4}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{\mathrm{Tr}\big\{(\not{P}+m)\gamma^\mu(\not{l}+m)\gamma^+(\not{l}+m)\gamma_\mu\big\}}{(l^2-m^2+i\epsilon)^2(P-l)^2}\delta(l^z-xP^z) \qquad (69)
\end{aligned}
$$

6 Matching to PDF

The matching coefficient Z is defined as

$$\tilde{q}(x) = \int_0^1 \frac{dy}{y} Z\left(\frac{x}{y}, \frac{P^z}{\mu}\right) q(y) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{P_z^2}, \frac{M^2}{P_z^2}\right) \tag{70}$$

For leading order, it's

$$\delta(1-x) = \int_0^1 \frac{\mathrm{d}y}{y} Z\Big(\frac{x}{y}, \frac{P^z}{\mu}\Big)\delta(1-y) \tag{71}$$

It's straightforward to get (under the assumption that the heaviside theta function $\theta(0) = 0$)

$$Z\Big(\frac{x}{y}, \frac{P^z}{\mu}\Big) = \delta\Big(1 - \frac{x}{y}\Big) \tag{72}$$

The one loop matching factor is

$$\left(1 + \delta\tilde{Z}_F\right)\delta(1-x) + \tilde{q}^{(1)}(x) = \int_0^1 \frac{dy}{y}\left[\delta\left(\frac{x}{y} - 1\right) + Z^{(1)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right)\right]\left[\left(1 + \delta Z_F\right)\delta(1-y) + q^{(1)}(y)\right] \tag{73}$$

where Z_F is independent of x and y. In axial gauge, Z_F is the fermion self energy, while in Feynman gauge, it's the combination of fermion self energy, gauge link self energy and loops between quark line and gauge link line involving only one side of the uncut diagram. Up to one loop the equation becomes

$$\begin{aligned}\left(1 + \delta\tilde{Z}_F\right)\delta(1-x) + \tilde{q}^{(1)}(x) &= \int_0^1 \frac{dy}{y}\left[\delta\left(\frac{x}{y} - 1\right)\left(1 + \delta Z_F\right)\delta(1-y) + \delta\left(\frac{x}{y} - 1\right)q^{(1)}(y) + Z^{(1)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right)\delta(1-y)\right] \\ &= \left(1 + \delta Z_F\right)\delta(1-x) + q^{(1)}(x)\theta(x)\theta(1-x) + Z^{(1)}\left(x, \frac{p^z}{\mu}\right)\end{aligned}$$

We then have (note that the momentum fraction x of PDF can't exceed $(0, 1)$)

$$\delta\tilde{Z}_F\delta(1-x) + \tilde{q}^{(1)}(x) = \delta Z_F\delta(1-x) + q^{(1)}(x) + Z^{(1)}\left(x, \frac{p^z}{\mu}\right) \tag{74}$$

$$Z^{(1)}\left(x, \frac{p^z}{\mu}\right) = \left[\delta\tilde{Z}_F - \delta Z_F\right]\delta(1-x) + \left[\tilde{q}^{(1)}(x) - q^{(1)}(x)\right] \tag{75}$$

δZ_F and $\delta\tilde{Z}_F$ combined with delta functions are to be merged into $q^{(1)}$ and $\tilde{q}^{(1)}$ and becomes plus distribution.

At two loop, we're looking at

$$\begin{aligned}&\left(1 + \delta\tilde{Z}_F^{(1)} + \delta\tilde{Z}_F^{(2)}\right)\delta(1-x) + \tilde{q}^{(1)}(x) + \tilde{q}^{(2)}(x) \\ &= \int_0^1 \frac{dy}{y}\left[\delta\left(\frac{x}{y} - 1\right) + Z^{(1)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right) + Z^{(2)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right)\right]\left[\left(1 + \delta Z_F^{(1)} + \delta Z_F^{(2)}\right)\delta(1-y) + q^{(1)}(y) + q^{(2)}(y)\right]\end{aligned} \tag{76}$$

Then

$$\begin{aligned}&\left(1 + \delta\tilde{Z}_F^{(1)} + \delta\tilde{Z}_F^{(2)}\right)\delta(1-x) + \tilde{q}^{(1)}(x) + \tilde{q}^{(2)}(x) \\ &= \int_0^1 \frac{dy}{y}\Big\{\delta\left(\frac{x}{y} - 1\right)\left[\left(1 + \delta Z_F^{(1)} + \delta Z_F^{(2)}\right)\delta(1-y) + q^{(1)}(y) + q^{(2)}(y)\right] \\ &\qquad + \left[\left(1 + \delta Z_F^{(1)}\right)\delta(1-y) + q^{(1)}(y)\right]Z^{(1)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right) + Z^{(2)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right)\delta(1-y)\Big\}\delta(1-y) \\ &= \left(1 + \delta Z_F^{(1)} + \delta Z_F^{(2)}\right)\delta(1-x) + q^{(1)}(x) + q^{(2)}(x) + \left(1 + \delta Z_F^{(1)}\right)Z^{(1)}\left(x, \frac{p^z}{\mu}\right) + \int_0^1 \frac{dy}{y}q^{(1)}(y)Z^{(1)}\left(\frac{x}{y}, \frac{p^z}{\mu}\right) + Z^{(2)}\left(x, \frac{p^z}{\mu}\right)\end{aligned} \tag{77}$$

$$
\begin{aligned}
&= \left(1+\delta Z_F^{(1)}+\delta Z_F^{(2)}\right)\delta(1-x)+q^{(1)}(x)+q^{(2)}(x)+\left(1+\delta Z_F^{(1)}\right)\left\{\left[\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}\right]\delta(1-x)+\left[\tilde{q}^{(1)}(x)-q^{(1)}(x)\right]\right\} \\
&\qquad + \int_0^1 \frac{dy}{y} q^{(1)}(y)\left\{\left[\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}\right]\delta(1-\frac{x}{y})+\left[\tilde{q}^{(1)}\left(\frac{x}{y}\right)-q^{(1)}\left(\frac{x}{y}\right)\right]\right\}+Z^{(2)}\left(x,\frac{p^z}{\mu}\right) \\
&= \left(1+\delta Z_F^{(1)}+\delta Z_F^{(2)}\right)\delta(1-x)+q^{(1)}(x)+q^{(2)}(x)+\left(1+\delta Z_F^{(1)}\right)\left\{\left[\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}\right]\delta(1-x)+\left[\tilde{q}^{(1)}(x)-q^{(1)}(x)\right]\right\} \\
&\qquad + q^{(1)}(x)\left[\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}\right]+\int_0^1 \frac{dy}{y} q^{(1)}(y)\left[\tilde{q}^{(1)}\left(\frac{x}{y}\right)-q^{(1)}\left(\frac{x}{y}\right)\right]+Z^{(2)}\left(x,\frac{p^z}{\mu}\right) \\
&= \left(1+\delta Z_F^{(1)}+\delta Z_F^{(2)}\right)\delta(1-x)+\tilde{q}^{(1)}(x)+q^{(2)}(x)+\left(1+\delta Z_F^{(1)}\right)\left[\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}\right]\delta(1-x)+\delta Z_F^{(1)}\tilde{q}^{(1)}(x) \\
&\qquad -\left(2\delta Z_F^{(1)}-\delta\tilde{Z}_F^{(1)}\right)q^{(1)}(x)+\int_0^1 \frac{dy}{y} q^{(1)}(y)\left[\tilde{q}^{(1)}\left(\frac{x}{y}\right)-q^{(1)}\left(\frac{x}{y}\right)\right]+Z^{(2)}\left(x,\frac{p^z}{\mu}\right) \qquad (78)
\end{aligned}
$$

and this concludes to

$$
\begin{aligned}
Z^{(2)}\left(x,\frac{p^z}{\mu}\right) &= \left(1+\delta\tilde{Z}_F^{(1)}+\delta\tilde{Z}_F^{(2)}\right)\delta(1-x)+\tilde{q}^{(1)}(x)+\tilde{q}^{(2)}(x)-\left(1+\delta Z_F^{(1)}+\delta Z_F^{(2)}\right)\delta(1-x)-q^{(1)}(x)-q^{(2)}(x) \\
&\qquad -\left(1+\delta Z_F^{(1)}\right)Z^{(1)}\left(x,\frac{p^z}{\mu}\right)-\int_0^1 \frac{dy}{y} q^{(1)}(y)Z^{(1)}\left(\frac{x}{y},\frac{p^z}{\mu}\right) \\
&= \left(\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}+\delta\tilde{Z}_F^{(2)}-\delta Z_F^{(2)}\right)\delta(1-x)+\tilde{q}^{(1)}(x)-q^{(1)}(x)+\tilde{q}^{(2)}(x)-q^{(2)}(x) \qquad (79) \\
&\qquad -\left(1+\delta Z_F^{(1)}\right)Z^{(1)}\left(x,\frac{p^z}{\mu}\right)-\int_0^1 \frac{dy}{y} q^{(1)}(y)Z^{(1)}\left(\frac{x}{y},\frac{p^z}{\mu}\right) \\
&= \left(\delta\tilde{Z}_F^{(1)}+\delta\tilde{Z}_F^{(2)}-\delta Z_F^{(1)}-\delta Z_F^{(2)}\right)\delta(1-x)+\tilde{q}^{(1)}(x)-q^{(1)}(x)+\tilde{q}^{(2)}(x)-q^{(2)}(x)+\delta Z_F^{(1)}\tilde{q}^{(1)}(x) \\
&\qquad -\left(2\delta Z_F^{(1)}-\delta\tilde{Z}_F^{(1)}\right)q^{(1)}(x)+\left(1+\delta Z_F^{(1)}\right)\left[\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}\right]\delta(1-x)+\int_0^1 \frac{dy}{y} q^{(1)}(y)\left[\tilde{q}^{(1)}\left(\frac{x}{y}\right)-q^{(1)}\left(\frac{x}{y}\right)\right] \\
&= \left(\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}+\delta\tilde{Z}_F^{(2)}-\delta Z_F^{(2)}\right)\delta(1-x)+\tilde{q}^{(1)}(x)-q^{(1)}(x)+\tilde{q}^{(2)}(x)-q^{(2)}(x)+\delta Z_F^{(1)}\tilde{q}^{(1)}(x) \\
&\qquad -\left(2\delta Z_F^{(1)}-\delta\tilde{Z}_F^{(1)}\right)q^{(1)}(x)+\left(1+\delta Z_F^{(1)}\right)\left[\delta\tilde{Z}_F^{(1)}-\delta Z_F^{(1)}\right]\delta(1-x)+\int_0^1 \frac{dy}{y} q^{(1)}(y)\left[\tilde{q}^{(1)}\left(\frac{x}{y}\right)-q^{(1)}\left(\frac{x}{y}\right)\right] \\
&\qquad (80)
\end{aligned}
$$

A Conventions

A.1 QFT basics

The quark field ψ reads

$$\psi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{2E_k} \left[u(k) e^{-ik\cdot x} b_k + v(k) e^{ik\cdot x} d_k^\dagger \right] \tag{81}$$

and the projection of single particle state is

$$|p\rangle = b_p^\dagger |0\rangle \tag{82}$$

$$\left\{ b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger} \right\} = (2\pi)^3 2E \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs} \tag{83}$$

The Dirac spinor is normalized to

$$\bar{u}^s(p) u(p) = 2m\delta^{rs} \tag{84}$$

With Gordon identity, one can derive [Srednicki(2007)]

$$\bar{u}(P)\gamma^\mu u(P) = 2P^\mu \tag{85}$$

The axial gauge propagator is

$$\tilde{D}_G^{A\mu\nu}(p) = -i\delta_{ab} \left(g^{\mu\nu} - \frac{n^\mu p^\nu + n^\nu p^\mu}{n\cdot p} + n\cdot n \frac{p^\mu p^\nu}{(n\cdot p)^2} \right) \frac{1}{p^2} \tag{86}$$

The Feynman gauge propagator is

$$\tilde{D}_G^{F\mu\nu}(p) = \frac{-ig^{\mu\nu}\delta_{ab}}{p^2 + i\epsilon} \tag{87}$$

State contract with field:

$$\begin{aligned}
\overbrace{\psi(x)|P\rangle} &= \int \frac{\mathrm{d}^3\mathbf{l}}{(2\pi)^3} \frac{1}{2E_{\mathbf{l}}} \left[b_{\mathbf{l}} u(l) e^{-il\cdot x} + d_{\mathbf{l}}^\dagger v(l) e^{il\cdot x} \right] b_{\mathbf{P}}^\dagger |0\rangle \\
&= \int \frac{\mathrm{d}^3\mathbf{l}}{(2\pi)^3} \frac{1}{2E_{\mathbf{l}}} u(l) e^{-il\cdot x} (2\pi)^3 2E\delta^{(3)}(\mathbf{l} - \mathbf{P}) |0\rangle \\
&= u(P) e^{-iP\cdot x}
\end{aligned} \tag{88}$$

and correspondingly

$$\langle \overbrace{P|\bar{\psi}}(x) = \bar{u}(P) e^{iP\cdot x} \tag{89}$$

A.2 Plus function and Heaviside Theta function

Plus function is defined as [Collins(2009)]

$$\int_0^1 \mathrm{d}x \left(\frac{1}{1-x} \right)_+ T(x) \equiv \int_0^1 \mathrm{d}x \frac{T(x) - T(1)}{1-x} \tag{90}$$

$$\int_0^1 \mathrm{d}x \frac{A(x)}{(1-x)_+} T(x) \equiv \int_0^1 \mathrm{d}x \frac{A(x)T(x) - A(1)T(1)}{1-x} \tag{91}$$

The plus function could also be defined in a different fashion:

$$F_+(x) := \lim_{\beta\to 0}\left(F(x)\theta(1-\beta-x) - \delta(1-\beta-x)\int_0^{1-\beta} \mathrm{d}yF(y)\right) \tag{92}$$

and with a smooth test function it's defined the same way as above:

$$\int_0^1 \mathrm{d}xF_+(x)G(x) = \int_0^1 \mathrm{d}xF(x)[G(x)-G(1)] \tag{93}$$

with

$$\int_0^1 \mathrm{d}xF(x)_+ = 0 \tag{94}$$

For a flexible lower boundary:

$$\int_a^1 \mathrm{d}xF_+(x)G(x) = \int_a^1 \mathrm{d}xF(x)[G(x)-G(1)] + G(1)\int_0^a \mathrm{d}xF(x) \tag{95}$$

Also if rule out some smoothness problem

$$[f(x)]_+ = f(x) - \delta(1-x)\int_0^1 f(z)\mathrm{d}z \tag{96}$$

Here we use a modified version of plus function (to not confuse this with those used in [Stewart and Zhao(2018)], we use boxplus for a plus function with integration limit from minus infinity to plus infinity)

$$f_{\boxplus}(x) = f(x) - \delta(1-x)\int_{-\infty}^{\infty} \mathrm{d}yf(y) \tag{97}$$

$$= f(x) - \delta(1-x)\int_{-\infty}^{\infty} \mathrm{d}l^z f(y)/P^z \tag{98}$$

For functions as

$$\left(\frac{1}{1-x}\right)^{1-\epsilon} \tag{99}$$

instead of expanding them in $\epsilon- > 0$ limit, we can treat them as distributions and expand them into the combinations of delta functions and plus functions.

$$\left(\frac{1}{1-x}\right)^{1-\epsilon} = \left(\frac{1}{1-x}\right)_+^{1-\epsilon} + \delta(1-x)\int_0^1 \left(\frac{1}{1-z}\right)^{1-\epsilon}\mathrm{d}z \tag{100}$$

$$= \left(\frac{1}{1-x}\right)_+^{1-\epsilon} + \frac{\delta(1-x)}{\epsilon} \tag{101}$$

$$= \left(\frac{1}{1-x}\right)_+ + \frac{\delta(1-x)}{\epsilon} \tag{102}$$

and

$$\left(\frac{1}{1-x}\right)^{1-\epsilon} = \left(\frac{1}{1-x}\right)_{\boxplus}^{1-\epsilon} + \delta(1-x)\int_{-\infty}^{\infty} \left(\frac{1}{1-z}\right)^{1-\epsilon}\mathrm{d}z \tag{103}$$

$$= \left(\frac{1}{1-x}\right)_{\boxplus}^{1-\epsilon} + 0\frac{\delta(1-x)}{\epsilon} \tag{104}$$

$$\tag{105}$$

The convention for path ordering is that the field with higher value of the integration variable s goes to the left.

The definition of Heaviside theta function follows

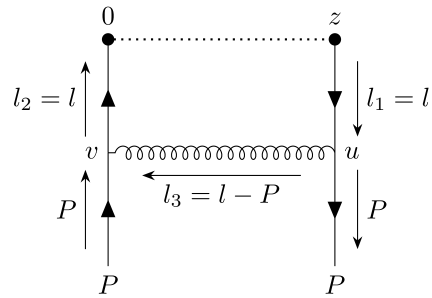
$$\theta(z) = \int \frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz}}{w+i\epsilon} \tag{106}$$

and here we choose

$$\theta(0) = 1 \tag{107}$$

B Wick Contraction: Axial Gauge

Take diagram 1b as an example



This corresponds to

$$\frac{1}{4\pi}\int \mathrm{d}z e^{ixP^z z}\langle P|\int \mathrm{d}^4u\bar{\psi}_u\psi_u A_u\bar{\psi}(z)\gamma^z\psi(0)\int \mathrm{d}^4v\bar{\psi}_v\psi_v A_v|P\rangle \tag{108}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z e^{ixP^z z}\int \mathrm{d}^4u\mathrm{d}^4v\bar{u}(P)e^{iP\cdot u}\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\tilde{D}_F(l_1)e^{-il_1\cdot(u-z)}\gamma^z\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\tilde{D}_F(l_2)e^{-il_2\cdot(-v)}$$
$$\int\frac{\mathrm{d}^4l_3}{(2\pi)^4}\tilde{D}_G(l_3)e^{-il_3\cdot(v-u)}u(P)e^{-iP\cdot v} \tag{109}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z\int \mathrm{d}^4u\mathrm{d}^4v\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\int\frac{\mathrm{d}^4l_3}{(2\pi)^4}e^{ixP^z z+il_1\cdot z}e^{i(P-l_1+l_3)\cdot u}e^{i(l_2-l_3-P)\cdot v}\bar{u}(P)\tilde{D}_F(l_1)\gamma^z\tilde{D}_F(l_2)$$
$$\tilde{D}_G(l_3)u(P) \tag{110}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z\int\frac{\mathrm{d}^4l}{(2\pi)^4}e^{ixP^z z+il\cdot z}\bar{u}(P)\tilde{D}_F(l)\gamma^z\tilde{D}_F(l)\tilde{D}_G(l-P)u(P) \tag{111}$$

$$=\frac{1}{4\pi}\int \mathrm{d}z\int\frac{\mathrm{d}^4l}{(2\pi)^4}e^{i(xP^z-l^z)z}\bar{u}(P)\tilde{D}_F(l)\gamma^z\tilde{D}_F(l)\tilde{D}_G(l-P)u(P) \tag{112}$$

$$=\frac{1}{4\pi}\int\frac{\mathrm{d}l^0}{2\pi}\int\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\ \bar{u}(P)\tilde{D}_F(l)\gamma^z\tilde{D}_F(l)\tilde{D}_G(l-P)u(P)\Big|_{l^z=xP^z} \tag{113}$$

where

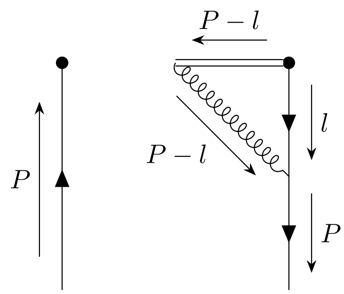
$$\int \mathrm{d}z e^{i(xP^z-l^z)z} = 2\pi\delta(l^z - xP^z) \tag{114}$$

This indicates what the Feynman diagram actually means: a normal Feynman diagram with only 3-momentum integration, where the axial momentum is fixed by $l^z = xP^z$, and with an extra $1/4\pi$ factor.

C Wick Contraction: Feynman Gauge

C.1 Normal Diagrams

Let's take diagram 2f as an example:



The one loop quasi PDF is

$$\tilde{q}_1^{(1)}(x) = \frac{1}{2}\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z}\langle P,S|\int \mathrm{d}^4u(-ig_s t^a\gamma_\mu)\bar{\psi}_u\psi_u A_u^\mu\bar{\psi}(z)\gamma^z\tilde{\mathcal{W}}[z,0]\psi(0)|P,S\rangle \tag{115}$$

where

$$\tilde{\mathcal{W}}[z,0] = \mathcal{P}\exp\left[-ig_s\int_0^z \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right] \tag{116}$$

We should rewrite the gauge link to the product of two gauge links connect to infinity

$$\tilde{\mathcal{W}}[z,0] = \tilde{\mathcal{W}}[z,+\infty]\tilde{\mathcal{W}}[\infty,0] \tag{117}$$

and in one loop level it equals to

$$\mathcal{P}\exp\left[ig_s\int_\infty^z \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right]\mathcal{P}\exp\left[ig_s\int_0^\infty \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right] = \left[ig_s\mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z'+z)\,\mathrm{t}^a\right] - \left[ig_s\mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right]$$

The path ordering gives

$$\mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z') = \int \mathrm{d}z' A^{a,z}(z')\,\theta(z') = \int \mathrm{d}z' A^{a,z}(z')\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \tag{118}$$

and

$$\mathcal{P}\left[\int_0^\infty \mathrm{d}z' A^{a,z}(z')\right]^2 = \int \mathrm{d}z' A^{a,z}(z')\,\theta(z')\int \mathrm{d}z'' A^{a,z}(z'')\,\theta(z''-z') \tag{119}$$

with all momenta involved with z' must be in the z-direction (the exponent is actually $z'n\cdot w$ if a four-vector w actually exists). Consider the second gauge link first, the matrix element is then (discarding all couplings)

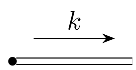
$$\begin{aligned}
&\langle P,S|\bar{\psi}_u\psi_u A_u^\mu\bar{\psi}(z)\gamma^z\int \mathrm{d}z' A^{a,z}(z'+z)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\psi(0)|P,S\rangle \\
&= \int \mathrm{d}^4u\langle P,S|\bar{\psi}_u\psi_u A_u\bar{\psi}(z)\gamma^z\int \mathrm{d}z' A^{a,z}(z'+z)\,\psi(0)|P,S\rangle\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \\
&= \int \mathrm{d}^4u\langle P,S|\bar{\psi}_u\psi_u A_u\bar{\psi}(z)\gamma^z\int \mathrm{d}z' A^{a,z}(z'+z)\,\psi(0)|P,S\rangle\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \\
&= \int \mathrm{d}z'\int \mathrm{d}^4u\bar{u}(P)e^{iP\cdot u}\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\tilde{D}_F(l_1)e^{-il_1\cdot(u-z)}\gamma^z\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\tilde{D}_G(l_2)e^{-il_2\cdot(u-z'-z)}u(P)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon} \\
&= \int \mathrm{d}z'\int \mathrm{d}^4u\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\int\frac{\mathrm{d}^4l_2}{(2\pi)^4}\int\frac{\mathrm{d}w}{2\pi}\bar{u}(P)e^{i(P-l_1-l_2)\cdot u}e^{i(l_1+l_2)\cdot z}e^{-iwz'-il_2\cdot z'}\tilde{D}_F(l_1)\gamma^z\tilde{D}_G(l_2)u(P)\frac{i}{w+i\epsilon}
\end{aligned}$$

$$=\int \mathrm{d}z' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \int \frac{\mathrm{d}w}{2\pi} \bar{u}(P) e^{iP\cdot z} e^{-iwz' + i(P-l)^z z'} \tilde{D}_F(l)\gamma^z \tilde{D}_G(P-l) u(P) \frac{i}{w+i\epsilon}$$
$$= \bar{u}(P) e^{iP\cdot z} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_F(l)\gamma^z \tilde{D}_G^{\mu z}(P-l) \frac{i}{P^z - l^z + i\epsilon} u(P)$$

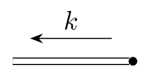
The complete quasi PDF at one loop is

$$\frac{g_s^2 C_F}{2} \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \bar{u}(P)\gamma_\mu e^{iP\cdot z} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_F(l)\gamma^z \tilde{D}_G^{\mu z}(P-l) \frac{i}{P^z - l^z + i\epsilon} u(P)$$
$$= \frac{g_s^2 C_F}{2P^z} \bar{u}(P)\gamma_\mu \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_F(l)\gamma^z \tilde{D}_G^{\mu z}(P-l) \frac{i}{P^z - l^z + i\epsilon} u(P)\delta(1-x)$$

multiplied by those couplings. This basically established that the momentum of a gluon equals to the momentum of the eikonal line it attaches to. We then have the Feynman rule:



$$= \frac{i}{n\cdot k + i\epsilon};$$

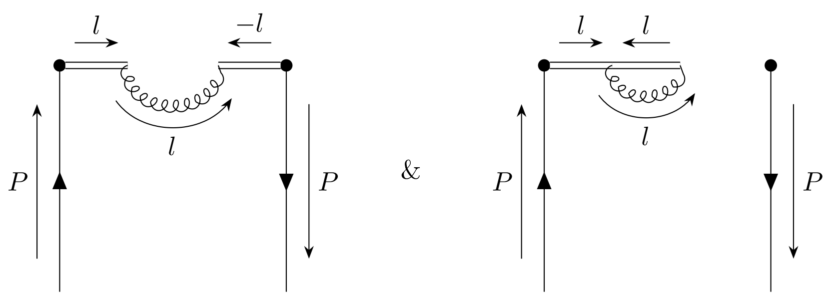


$$= \frac{i}{n\cdot k + i\epsilon} \tag{120}$$

and for the gluon-eikonal vertex on the r.h.s., an extra minus sign is added for the normal $(-ig_s t^a)$.

C.2 Gauge Link Self Energy Related Diagrams

The next job is to determine the Feynman rule for



The first one is

$$\frac{1}{2}\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \left[ig_s \mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z'+z)\,\mathrm{t}^a \right] \left[-ig_s \mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a \right] \psi(0)|P,S\rangle \tag{121}$$

Let's look at the coupling-free form:

$$\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \left[\mathcal{P}\int_0^\infty \mathrm{d}z' A^{a,z}(z'+z) \right] \left[\mathcal{P}\int_0^\infty \mathrm{d}z'' A^{a,z}(z'') \right] \psi(0)|P,S\rangle$$
$$= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \int \mathrm{d}z' A^{a,z}(z'+z) \int \mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ihz''}}{h+i\epsilon}$$
$$= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P,S|\bar{\psi}(z)\gamma^z \int \mathrm{d}z' A^{a,z}(z'+z) \int \mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ihz''}}{h+i\epsilon}$$
$$= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \bar{u}(P) e^{iP\cdot z}\gamma^z \int \mathrm{d}z'\mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) e^{-il\cdot(z''-z'-z)} u(P) \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ihz''}}{h+i\epsilon}$$
$$= \bar{u}(P) \int \frac{\mathrm{d}z}{2\pi} e^{i(x-1)P^z z + il^z z}\gamma^z \int \mathrm{d}z'\mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \int \frac{\mathrm{d}w}{2\pi} \frac{i}{w+i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{i}{h+i\epsilon} e^{-i(w-l)\cdot z'} e^{-i(l+h)\cdot z''} u(P)$$
$$= \bar{u}(P) \int \frac{\mathrm{d}z}{2\pi} e^{-i(1-x)P^z z + il^z z}\gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{l^z+i\epsilon} \frac{i}{-l^z+i\epsilon} u(P)$$
$$= \bar{u}(P)\gamma^z \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{l^z+i\epsilon} \frac{i}{-l^z+i\epsilon} \delta(l^z - (1-x)P^z) u(P)$$

One can start with a different route:

$$\bar{u}(P)\int\frac{\mathrm{d}z}{2\pi}e^{-i(1-x)P^z z+il^z z}\gamma^z\int\mathrm{d}z'\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\int\frac{\mathrm{d}w}{2\pi}\frac{i}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{i}{h+i\epsilon}e^{-i(w-l)\cdot z'}e^{-i(l+h)\cdot z''}u(P) \tag{122}$$

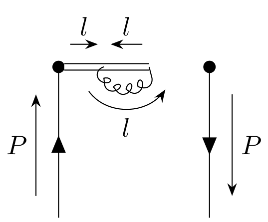
The second one is

$$\frac{1}{2}\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z\frac{\mathcal{P}\left[-ig_s\int_0^\infty\mathrm{d}z' A^{a,z}(z')\,\mathrm{t}^a\right]\left[-ig_s\int_0^\infty\mathrm{d}z'' A^{a,z}(z'')\,\mathrm{t}^a\right]}{2}\psi(0)|P,S\rangle \tag{123}$$

The coupling-free form is[4]

$$\begin{aligned}
&\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z\mathcal{P}\left[\int_0^\infty\mathrm{d}z' A^{a,z}(z')\int_0^\infty\mathrm{d}z'' A^{a,z}(z'')\right]\psi(0)|P,S\rangle\\
=&\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z\int_0^\infty\mathrm{d}z' A^{a,z}(z')\int_0^\infty\mathrm{d}z'' A^{a,z}(z'')\left[\theta(z'-z'')+\theta(z''-z')\right]\psi(0)|P,S\rangle\\
=&\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z\int_0^\infty\mathrm{d}z' A^{a,z}(z')\int_0^\infty\mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle\\
=&\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z\int\mathrm{d}z' A^{a,z}(z')\int\mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
=&\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\langle P,S|\bar{\psi}(z)\gamma^z\int\mathrm{d}z' A^{a,z}(z')\int\mathrm{d}z'' A^{a,z}(z'')\,\psi(0)|P,S\rangle\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
=&\int\frac{\mathrm{d}z}{2\pi}e^{ixP^z z}\bar{u}(P)e^{iP\cdot z}\gamma^z\int\mathrm{d}z'\int\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)e^{-il\cdot(z''-z')}u(P)\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iwz'}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}\\
=&\bar{u}(P)\int\frac{\mathrm{d}z}{2\pi}e^{-i(1-x)P^z z}\gamma^z\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\int\mathrm{d}z'\int\mathrm{d}z''\int\frac{\mathrm{d}w}{2\pi}\frac{i}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{i}{h+i\epsilon}e^{-i(w-l)\cdot z'}e^{-i(h+l)\cdot z''}u(P)\\
=&\bar{u}(P)\gamma^z u(P)\int\frac{\mathrm{d}z}{2\pi}e^{-i(1-x)P^z z}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}\\
=&2\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon}
\end{aligned}$$

This gives



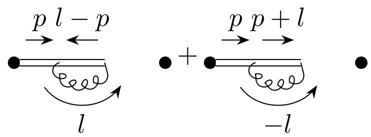
$$= -\frac{g_s^2 C_F}{2}\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G^{zz}(l)\frac{i}{l^z+i\epsilon}\frac{i}{-l^z+i\epsilon} \tag{124}$$

Let's start from the second line of above derivation and add a small momentum p

$$\begin{aligned}
&\int_0^\infty\mathrm{d}z' A^{a,z}(z')\int_0^\infty\mathrm{d}z'' A^{a,z}(z'')\left[\theta(z'-z'')+\theta(z''-z')\right]\\
=&\int_0^\infty\mathrm{d}z'\int_0^\infty\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G(l)e^{il^z(z''-z')}\left[\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iw(z'-z'')}}{w+i\epsilon}+\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iw(z''-z')}}{w+i\epsilon}\right]\\
=&\int_0^\infty\mathrm{d}z'\int_0^\infty\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G(l)\left[\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iw(z'-z'')}e^{il^z(z''-z')}}{w+i\epsilon}+\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iw(z''-z')}e^{-il^z(z''-z')}}{w+i\epsilon}\right]\\
=&\int\mathrm{d}z'\int\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G(l)\left[\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iw(z'-z'')}e^{il^z(z''-z')}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz''}}{h+i\epsilon}+\int\frac{\mathrm{d}w}{2\pi}\frac{ie^{-iw(z''-z')}e^{-il^z(z''-z')}}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{ie^{-ihz'}}{h+i\epsilon}\right]\\
=&\int\mathrm{d}z'\int\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G(l)\int\frac{\mathrm{d}w}{2\pi}\frac{i}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{i}{h+i\epsilon}\left[e^{-iz'(w+l^z)-iz''(h-w-l^z)}+e^{-iz'(w-l^z)-iz''(h-w+l^z)}\right]\lim_{p\to0}e^{ipz'}\\
=&\int\mathrm{d}z'\int\mathrm{d}z''\int\frac{\mathrm{d}^4l}{(2\pi)^4}\tilde{D}_G(l)\int\frac{\mathrm{d}w}{2\pi}\frac{i}{w+i\epsilon}\int\frac{\mathrm{d}h}{2\pi}\frac{i}{h+i\epsilon}\lim_{p\to0}\left[e^{-iz'(w+l^z-p)-iz''(h-w-l^z)}+e^{-iz'(w-l^z-p)-iz''(h-w+l^z)}\right]
\end{aligned}$$

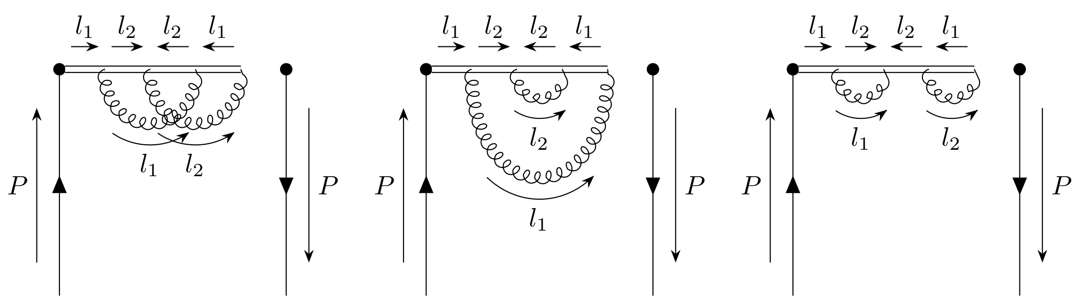
$$= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G(l) \lim_{p\to 0} \left[\frac{i}{p - l^z + i\epsilon} \frac{i}{p + i\epsilon} + \frac{i}{p + l^z + i\epsilon} \frac{i}{p + i\epsilon} \right]$$
$$= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G(l) \frac{i}{l^z + i\epsilon} \frac{i}{-l^z + i\epsilon} \times 2$$

We can reverse the loop momentum of one of the path and add a small inflowing momentum p (in the following diagrams each diagram only represents one specific path, that means the sum of both diagram is the value of the original diagram):



$$= -\frac{g_s^2 C_F}{2} \delta(1-x) \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \left[\frac{i}{p + i\epsilon} \frac{i}{p - l^z + i\epsilon} + \frac{i}{p + i\epsilon} \frac{i}{p + l^z + i\epsilon} \right]$$
$$= -\frac{g_s^2 C_F}{2} \delta(1-x) \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{p + i\epsilon} \frac{2ip}{(p - l^z + i\epsilon)(p + l^z + i\epsilon)}$$
$$= -\frac{g_s^2 C_F}{2} \delta(1-x) \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{p - l^z + i\epsilon} \frac{2i}{p + l^z + i\epsilon}$$

At two loop, we hope one can still try this. Let's discuss the following diagram first



(a) (b) (c)

Take the first diagram as an example: the gauge link part of the amplitude of a single possible path is

$$\int_0^\infty \mathrm{d}z_1 \int_0^\infty \mathrm{d}z_2 \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 A^{a,z}(z_1) A^{b,z}(z_2) A^{c,z}(z_3) A^{d,z}(z_4) \theta(z_1 - z_2)\theta(z_2 - z_3)\theta(z_3 - z_4) \tag{125}$$

where the overall factor $1/4!$ is cancelled by the sum of all possible path. [5] For convenience we now label the coordinate with subscript. First we check if the equivalence stands:

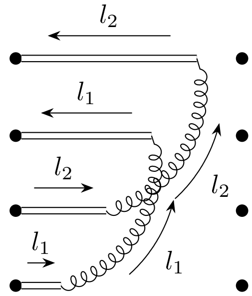
$$A_1^a A_2^b A_3^c A_4^d \theta_{12}\theta_{23}\theta_{34} \stackrel{?}{=} A_1^a A_2^b A_4^d A_3^c \theta_{12}\theta_{24}\theta_{43}$$

The integrand becomes

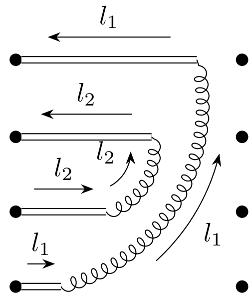
$$\tilde{D}_G(l_1)\tilde{D}_G(l_2) \prod_i^4 \frac{i}{w_i + i\epsilon} \quad \Big\{ e^{-il_1(z_3 - z_1)} e^{-il_2(z_4 - z_2)} e^{-iw_1(z_1 - z_2)} e^{-iw_2(z_2 - z_3)} e^{-iw_3(z_3 - z_4)} e^{-iw_4 z_4}$$
$$\stackrel{?}{=} e^{-il_1(z_4 - z_1)} e^{-il_2(z_3 - z_2)} e^{-iw_1(z_1 - z_2)} e^{-iw_2(z_2 - z_4)} e^{-iw_3(z_4 - z_3)} e^{-iw_4 z_3} \Big\}$$
$$= \Big\{ e^{-i(w_1 - l_1)z_1} e^{-i(w_2 - w_1 - l_2)z_2} e^{-i(w_3 + l_1 - w_2)z_3} e^{-i(l_2 + w_4 - w_3)z_4}$$
$$\stackrel{?}{=} e^{-i(w_1 - l_1)z_1} e^{-i(w_2 - w_1 - l_2)z_2} e^{-i(w_4 + l_1 - w_2)z_3} e^{-i(l_2 + w_3 - w_4)z_4} \Big\}$$
$$= \left\{ \frac{i}{l_1 + i\epsilon} \frac{i}{l_1 + l_2 + i\epsilon} \frac{i}{l_2 + i\epsilon} \frac{i}{0 + i\epsilon} \stackrel{?}{=} \frac{i}{l_1 + i\epsilon} \frac{i}{l_1 + l_2 + i\epsilon} \frac{i}{0 + i\epsilon} \frac{i}{l_2 + i\epsilon} \right\}$$

These stand for two different path order, but should be equivalent with the same contraction choice. So by summing up all possible path, one collects all regions in the whole $z_i > 0$ space. Then all theta functions except for $\theta(z_i)$ vanish.

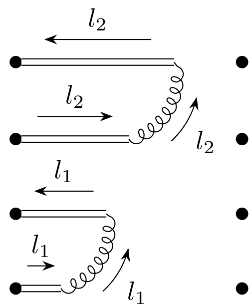
It's actually



$$= \frac{i}{l_1^z + i\epsilon}\frac{i}{l_2^z + i\epsilon}\frac{i}{-l_1^z + i\epsilon}\frac{i}{-l_2^z + i\epsilon} = \frac{1}{{l_1^z}^2 {l_2^z}^2}$$



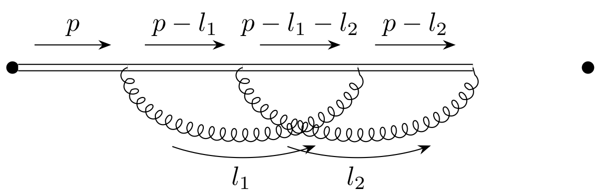
$$= \frac{i}{l_1^z + i\epsilon}\frac{i}{l_2^z + i\epsilon}\frac{i}{-l_2^z + i\epsilon}\frac{i}{-l_1^z + i\epsilon} = \frac{1}{{l_1^z}^2 {l_2^z}^2}$$

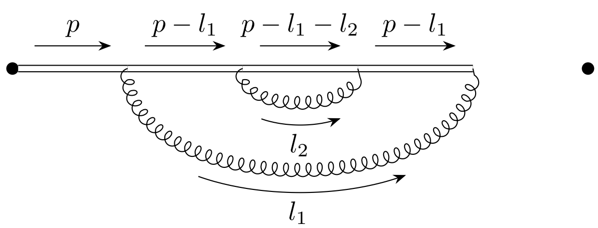


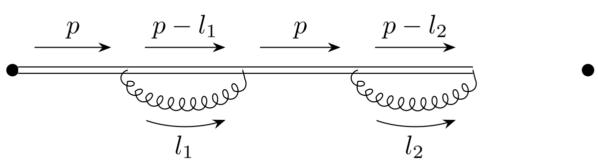
$$= \frac{i}{l_1^z + i\epsilon}\frac{i}{-l_1^z + i\epsilon}\frac{i}{l_2^z + i\epsilon}\frac{i}{-l_2^z + i\epsilon} = \frac{1}{{l_1^z}^2 {l_2^z}^2}$$

with all theta function being $\prod \theta(z_i)$.

But to directly use the Feynman rules, we can impose a small momentum inflow p as in the following diagram







Define

$$\mathcal{I}_1 = \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \tilde{D}_G(l_1) \tilde{D}_G(l_2) \frac{i}{p} \left\{ \frac{i}{p - l_1^z} \frac{i}{p - l_1^z - l_2^z} \frac{i}{p - l_2^z} \right\} \tag{126}$$

$$\mathcal{I}_2 = \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \tilde{D}_G(l_1) \tilde{D}_G(l_2) \frac{i}{p} \left\{ \frac{i}{p - l_1^z} \frac{i}{p - l_1^z - l_2^z} \frac{i}{p - l_1^z} \right\} \tag{127}$$

$$\mathcal{I}_3 = \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \tilde{D}_G(l_1) \tilde{D}_G(l_2) \frac{i}{p} \left\{ \frac{i}{p - l_1^z} \frac{i}{p} \frac{i}{p - l_2^z} \right\} \tag{128}$$

There seems to be no way to remove the $1/p$ divergence.

According to Tong, one can manually add a regulating momentum and then take the derivative to eliminate the effect. [6]

Diagram 2h is

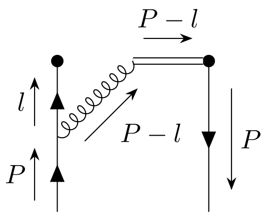
$$\frac{1}{2} \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \frac{\mathcal{P}\left[ig_s \int_0^\infty \mathrm{d}z' A^{a,z}\left(z' + z\right) \mathrm{t}^a\right] \left[ig_s \int_0^\infty \mathrm{d}z'' A^{a,z}\left(z'' + z\right) \mathrm{t}^a\right]}{2} \psi(0) | P, S \rangle \tag{129}$$

and it behaves exactly like 2g, since those extra $(+z)$s will be cancelled in the Wick contraction. These three diagrams are combined to form a plus function of diagram 2d. [7]

D Comparing two different prescription of Feynman integrals

Take diagram 2b as an example (we use a boldface font with subscript 3 to symbolize the first three components of a four vector: $\tilde{\mathbf{l}}^\mu = (l^0, \mathbf{l}_T)$)

$$\frac{1}{2}\sum_s$$



$$
\begin{aligned}
&= \frac{ig_s^2 C_F}{4}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\mathrm{Tr}\big\{(\not{P}+m)\gamma^z(\not{l}+m)\gamma^z\big\}}{l^2-m^2+i\epsilon}\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(P-l)^2+i\epsilon}\delta(l^z-xP^z)\\
&= ig_s^2 C_F\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+l^zP^z-m^2}{\tilde{\mathbf{l}}^2-(l^z)^2-m^2+i\epsilon}\frac{1}{l^z-P^z+i\epsilon}\frac{1}{(\tilde{\mathbf{P}}-\tilde{\mathbf{l}})^2-(P^z-l^z)^2+i\epsilon}\delta(l^z-xP^z)\\
&= \frac{ig_s^2 C_F}{2\pi P^z(x-1)}\int \frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+x(P^z)^2-m^2}{\tilde{\mathbf{l}}^2-x^2(P^z)^2-m^2+i\epsilon}\frac{1}{(\tilde{\mathbf{l}}-\tilde{\mathbf{P}})^2-(P^z)^2(1-x)^2+i\epsilon}
\end{aligned}
\tag{130}
$$

$$
\begin{aligned}
&y\Big(\tilde{\mathbf{l}}^2-x^2(P^z)^2-m^2+i\epsilon\Big)+(1-y)\Big((\tilde{\mathbf{l}}-\tilde{\mathbf{P}})^2-(P^z)^2(1-x)^2+i\epsilon\Big)\\
=&\tilde{\mathbf{l}}^2-x^2y(P^z)^2-ym^2-2(1-y)\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+(1-y)\tilde{\mathbf{P}}^2-(1-y)(P^z)^2(1-x)^2+i\epsilon\\
=&(\tilde{\mathbf{l}}-(1-y)\tilde{\mathbf{P}})^2+y(1-y)\tilde{\mathbf{P}}^2-x^2y(P^z)^2-ym^2-(1-y)(P^z)^2(1-x)^2+i\epsilon
\end{aligned}
\tag{131}
$$

$$\Delta = -y(1-y)\tilde{\mathbf{P}}^2+x^2y(P^z)^2+ym^2+(1-y)(P^z)^2(1-x)^2 \tag{132}$$

The integral is

$$
\begin{aligned}
&\int_0^1\mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+x(P^z)^2-m^2}{\tilde{\mathbf{l}}^2-x^2(P^z)^2-m^2+i\epsilon}\frac{1}{(\tilde{\mathbf{l}}-\tilde{\mathbf{P}})^2-(P^z)^2(1-x)^2+i\epsilon}\\
=&\int_0^1\mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{(\tilde{\mathbf{l}}+(1-y)\tilde{\mathbf{P}})\cdot\tilde{\mathbf{P}}+x(P^z)^2-m^2}{\Big[\tilde{\mathbf{l}}^2-\Delta+i\epsilon\Big]^2}\\
=&\int_0^1\mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{\tilde{\mathbf{l}}\cdot\tilde{\mathbf{P}}+(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{\Big[\tilde{\mathbf{l}}^2-\Delta+i\epsilon\Big]^2}
\end{aligned}
\tag{133}
$$

The first term in the numerator vanishes

$$=\int_0^1\mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{\Big[\tilde{\mathbf{l}}^2-\Delta+i\epsilon\Big]^2}$$

after Wick rotation

$$
\begin{aligned}
=&\frac{i}{(-1)^2}\int_0^1\mathrm{d}y\int\frac{\mathrm{d}^3\tilde{\mathbf{l}}}{(2\pi)^3}\frac{(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{\Big[\tilde{\mathbf{l}}^2+\Delta-i\epsilon\Big]^2}\\
=&i\int_0^1\mathrm{d}y\frac{(1-y)\tilde{\mathbf{P}}^2+x(P^z)^2-m^2}{8\pi\sqrt{\Delta}}
\end{aligned}
\tag{134}
$$

The final result agrees with what we got from integrating l^0 first:

$$
\begin{aligned}
&\frac{C_F g_s^2}{32\pi^2 P^z(x-1)\sqrt{m^2+{P^z}^2}}\Bigg\{{P^z}^2x\Bigg\{3\log\left(|x-1|\sqrt{m^2+{P^z}^2}+P^z(x-1)\right)-\log\left(|x-1|\sqrt{m^2+{P^z}^2}+P^z(-x)+P^z\right)\\
&-3\log\left(\sqrt{\left(m^2+{P^z}^2\right)\left(m^2+{P^z}^2x^2\right)}+m^2+{P^z}^2x\right)+\log\left(\sqrt{\left(m^2+{P^z}^2\right)\left(m^2+{P^z}^2x^2\right)}-m^2-{P^z}^2x\right)+2\log(P^z)\Bigg\}\\
&-2P^z\,|x-1|\sqrt{m^2+{P^z}^2}+2\sqrt{\left(m^2+{P^z}^2\right)\left(m^2+{P^z}^2x^2\right)}\Bigg\}
\end{aligned}
$$

E Diagram 1b Comparing (Defuncted)

Let's start with

$$\begin{aligned}
&\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\left(-ig_s t^a\gamma^\mu\right)\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}\left(-ig_s t^a\gamma^\nu\right)\tilde{D}^A_{G\mu\nu}(P-l)u(P)\Big|_{l^z=xP^z}\\
=&-g_s^2C_F\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\gamma^\mu\frac{i(\not{l}+m)}{l^2-m^2}\gamma^z\frac{i(\not{l}+m)}{l^2-m^2}\gamma^\nu\tilde{D}^A_{G\mu\nu}(P-l)u(P)\Big|_{l^z=xP^z}\\
=&-ig_s^2C_F\bar{u}(P)\int\frac{\mathrm{d}l^0}{2\pi}\frac{\mathrm{d}^2\mathbf{l}_T}{(2\pi)^2}\gamma^\mu\frac{\not{l}+m}{l^2-m^2}\gamma^z\frac{\not{l}+m}{l^2-m^2}\gamma^\nu\frac{1}{(P-l)^2+i\epsilon}u(P)\\
&\left[\bar{g}^{\mu\nu}-\frac{n^\nu\left(P^\mu-l^\mu\right)+n^\mu\left(P^\nu-l^\nu\right)}{n\cdot(P-l)}+\frac{n^2\left(P^\mu-l^\mu\right)\left(P^\nu-l^\nu\right)}{\left(n\cdot P-n\cdot l\right)^2}\right]\Bigg|_{l^z=xP^z}
\end{aligned}\tag{135}$$

We consider the numerator as a first step

$$\bar{u}(P)\gamma^\mu(\not{l}+m)\gamma^z(\not{l}+m)\gamma^\nu\left[(P-l)^2\tilde{D}^A_{G\mu\nu}(P-l)\right]u(P)\tag{136}$$

We can separate the gluon propagator into there parts. The first one gives a metric tensor and the final result

$$4l^3\left(m\bar{u}(P)u(P)-\bar{u}(P)\not{l}u(P)\right)-2\left(m^2-l^2\right)\bar{u}(P)\gamma^3u(P)\tag{137}$$

The combined result can be further separated with respect to the structure of gamma matrices. The first one is for $\bar{u}(P)\not{l}u(P)$:

$$\begin{aligned}
&\frac{2l^z\left(2l^z\left(P^z-l^z\right)-l^2+m^2\right)}{\left(l^2-m^2\right)^2(P-l)^2\left(l^z-P^z\right)^2}\\
=&-\frac{4(l^z)^2}{\left(l^2-m^2\right)^2(P-l)^2\left(l^z-P^z\right)}-\frac{2l^z}{\left(l^2-m^2\right)(P-l)^2\left(l^z-P^z\right)^2}
\end{aligned}$$

for $\bar{u}(P)u(p)$:

$$\begin{aligned}
&\frac{2ml^z\left(-6l^zP^z+4\left(l^z\right)^2+2\left(P^z\right)^2+l^2-m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{2ml^z\left(-4l^zP^z+4\left(l^z\right)^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2ml^z\left(-2l^zP^z+2\left(P^z\right)^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2ml^z\left(l^2-m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{8m\left(l^z\right)^2\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}-\frac{4ml^zP^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2ml^z\left(l^2-m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{8m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{4ml^zP^z}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4m\left(l^z\right)^2-4ml^zP^z}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4ml^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4ml^z}{\left(l^2-m^2\right)^2(l-P)^2}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}
\end{aligned}$$

for $\bar{u}(P)\gamma^zu(p)$:

$$\frac{(l-P)^2\left(2l^z\left(P^z-l^z\right)-l^2+m^2\right)+2\left(m^2-l^2\right)P^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}$$

$$
\begin{aligned}
=&\frac{(l-P)^2\left(2l^z\left(P^z-l^z\right)-l^2+m^2\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}+\frac{2\left(m^2-l^2\right)P^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)^2}\\
=&\frac{2l^z\left(P^z-l^z\right)-l^2+m^2}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)^2}-\frac{2P^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\\
=&\frac{2l^z\left(P^z-l^z\right)}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)^2}-\frac{l^2-m^2}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)^2}-\frac{2P^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\\
=&-\frac{2l^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{1}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}-\frac{2P^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}
\end{aligned}
$$

The total result is

$$
\begin{aligned}
&\bar{u}(P)\Bigg\{-\frac{4(l^z)^2\not{l}}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\not{l}}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
&\quad+\frac{4m\left(l^z\right)^2}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}+\frac{4ml^z}{\left(l^2-m^2\right)^2(l-P)^2}+\frac{2ml^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
&\quad-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}-\frac{2P^z\gamma^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\Bigg\}u(P)
\end{aligned}
\tag{138}
$$

$$
\begin{aligned}
=\bar{u}(P)\Bigg\{&\frac{-4\left(l^z\right)^2\left(\not{l}-m\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\left(\not{l}-m\right)}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}+\frac{4ml^z}{\left(l^2-m^2\right)^2(l-P)^2}\\
&-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}-\frac{2P^z\gamma^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)}\Bigg\}u(P)
\end{aligned}
\tag{139}
$$

$$
\begin{aligned}
=\bar{u}(P)\Bigg\{&\frac{-4\left(l^z\right)^2\left(\not{l}-m\right)+4ml^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\left(\not{l}-m\right)+2P^z\gamma^z\left(l^z-P^z\right)}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
&-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}\Bigg\}u(P)
\end{aligned}
\tag{140}
$$

$$
\begin{aligned}
=\bar{u}(P)\Bigg\{&-\frac{4\left(l^z\right)^2\left(\not{l}-2m\right)+4ml^zP^z}{\left(l^2-m^2\right)^2(l-P)^2\left(l^z-P^z\right)}-\frac{2l^z\left(\not{l}-m+P^z\gamma^z\right)-2(P^z)^2\gamma^z}{\left(l^2-m^2\right)(l-P)^2\left(l^z-P^z\right)^2}\\
&-\frac{2l^z\gamma^z}{\left(l^2-m^2\right)^2\left(l^z-P^z\right)}-\frac{\gamma^z}{\left(l^2-m^2\right)\left(l^z-P^z\right)^2}\Bigg\}u(P)
\end{aligned}
\tag{141}
$$

Xiong's result is

$$
\begin{aligned}
&-ig_s^2C_F\int\frac{d^4k}{(2\pi)^4}\bar{u}(P)\left[\frac{2\gamma^z}{\left(k^2-m^2\right)(P-k)^2}+\frac{4\left(2m-k^z\right)\not{k}}{\left(k^2-m^2\right)^2(P-k)^2}\right.\\
&\left.+\frac{2\left(k^z\gamma^z+\not{k}-m\right)}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)}-\frac{\gamma^z}{(P-k)^2\left(P^z-k^z\right)^2}\right]P^z\delta\left(k^z-xP^z\right)u(P)
\end{aligned}
\tag{142}
$$

As we discussed earlier, it can be dissected into

$$
\frac{4\left(2m-k^z\right)\not{k}}{\left(k^2-m^2\right)^2(P-k)^2}+\frac{2\not{k}}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)}\qquad\bar{u}(P)\not{k}u(P)\tag{143}
$$

$$
\frac{-2m}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)}\qquad\bar{u}(P)u(P)\tag{144}
$$

$$
\frac{2\gamma^z}{\left(k^2-m^2\right)(P-k)^2}+\frac{2k^z\gamma^z}{\left(k^2-m^2\right)(P-k)^2\left(P^z-k^z\right)}-\frac{\gamma^z}{(P-k)^2\left(P^z-k^z\right)^2}\qquad\bar{u}(P)\gamma^zu(P)\tag{145}
$$

Notes

1. The constant mentioned above is

$$-\frac{C_F g_s^2\left(-\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}-\frac{2P_3^2\left(\log\left(2\left(m^2+P_3^2\right)\right)\right)}{\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}+m^2+P_3^2}+\Lambda\sqrt{m^2+P_3^2}+m^2+P_3^2\right)}{16\pi^2\left(P_3-l_3\right)\sqrt{m^2+P_3^2}}-\frac{C_F g_s^2}{16\pi^2\mathrm{sgn}\left(l_3-P_3\right)}$$

$$\frac{P_3 C_F g_s^2\left(\log\left(l_3-P_3\right)^2-\frac{2\left(\log\left(2\left(\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}-m^2-P_3^2\right)\right)\right)}{\Lambda}\right)}{16\pi^2\sqrt{m^2+P_3^2}}+\frac{m^4 P_3 C_F g_s^2\left(\Lambda-\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}\right)}{16\pi^2\Lambda\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}$$

$$-\frac{m^2 P_3^3 C_F g_s^2\left(\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}+\sqrt{m^2+P_3^2}\right)}{8\pi^2\Lambda\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}-\frac{P_3^5 C_F g_s^2\left(\Lambda+\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}+2\sqrt{m^2+P_3^2}\right)}{16\pi^2\Lambda\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}$$

$$-\frac{\Lambda^2 m^2 P_3 C_F g_s^2\left(\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}-1\right)}{16\pi^2\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}-\frac{P_3^3 C_F g_s^2\left(\Lambda\left(\Lambda+\Lambda\sqrt{\frac{m^2+P_3^2}{\Lambda^2+m^2+P_3^2}}+2\sqrt{m^2+P_3^2}\right)-2\sqrt{\left(m^2+P_3^2\right)\left(\Lambda^2+m^2+P_3^2\right)}\right)}{16\pi^2\left(m^2+P_3^2\right)^{3/2}\left(\Lambda^2+m^2+P_3^2\right)}$$

multiplied by the delta function and integration.

2. Wrong prescription: Having $\left(\int_0^\infty-\int_1^\infty\right)\mathrm{d}xF(x)[G(x)-G(1)]=\int_0^1\mathrm{d}xF_+(x)G(x)$

$$\int\mathrm{d}y\frac{\delta(y-x)-\delta(1-x)}{|y-1|}=\frac{\theta(1-x)\theta(x)}{(1-x)_+}+\frac{\theta(-x)}{2(x-1)}+\frac{3\theta(x-1)}{2(x-1)}+\frac{\theta(1-x)\theta(x)}{x-1} \tag{146}$$

$$\int\mathrm{d}y\left(\frac{\log\frac{y-1}{\Lambda/P^z}}{y-1}\right)[\delta(y-x)-\delta(1-x)]=\left(\frac{\log\frac{y-1}{\Lambda/P^z}}{y-1}\right)_++\frac{\log\left(\frac{x-1}{\Lambda/P^z}\right)\theta(x-1)}{x-1}+\frac{\log(1-x)\theta(1-x)}{x-1} \tag{147}$$

3. We can also parametrize this as

$$4ig_s^2C_F\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\int_0^\infty\mathrm{d}\lambda\frac{\lambda}{\left(l^2+2\lambda n\cdot l+i\epsilon\right)^3}=4ig_s^2C_F\delta(1-x)\int\frac{\mathrm{d}^4l}{(2\pi)^4}\int_0^\infty\mathrm{d}\lambda\frac{1}{\left(l^2-\lambda^2+i\epsilon\right)^3} \tag{148}$$

4. To check if the sum of theta function can actually leads to unity

$$\begin{aligned}&\int\mathrm{d}zf(z)[\theta(z-a)+\theta(a-z)]\\ =&\int\mathrm{d}zf(z)\left[\int\frac{\mathrm{d}w_1}{2\pi}\frac{ie^{-iw_1(z-a)}}{w_1+i\epsilon}+\int\frac{\mathrm{d}w_1}{2\pi}\frac{ie^{-iw_1(a-z)}}{w_1+i\epsilon}\right]\\ =&\int\mathrm{d}zf(z)e^{\frac{\epsilon(z-a)}{\mathrm{sgn}(a-z)}}\\ \overset{\epsilon\to0}{=\!=}&\int\mathrm{d}zf(z)\end{aligned}$$

5. A step-by-step analysis should give all possible path summed up to be unity:

$$\begin{aligned}&\int\prod_i\mathrm{d}z_if(z_i)[\theta(z_3-z_2)\theta(z_2-z_1)+\theta(z_3-z_1)\theta(z_1-z_2)]\\ =&\int\mathrm{d}zf(z)\left[\int\frac{\mathrm{d}w_1}{2\pi}\frac{ie^{-iw_1(z_3-z_2)}}{w_1+i\epsilon}\int\frac{\mathrm{d}w_2}{2\pi}\frac{ie^{-iw_2(z_2-z_1)}}{w_2+i\epsilon}+\int\frac{\mathrm{d}w_1}{2\pi}\frac{ie^{-iw_1(z_3-z_1)}}{w_1+i\epsilon}\int\frac{\mathrm{d}w_2}{2\pi}\frac{ie^{-iw_2(z_1-z_2)}}{w_2+i\epsilon}\right]\\ \overset{\epsilon\to0}{=\!=}&\int\mathrm{d}zf(z)\end{aligned}$$

where $f(x)$ contains the gluon part and $\theta(z_4-z_3)$. *Mathematica* will give 1/2 for $z_1=z_3$ or $z_2=z_3$ scenario but this should come from the convention *Mathematica* take for $\theta(0)$.

In order to discuss the first diagram, we must perform the Wick contraction in specific order. This indicates it might be more reasonable to use $\theta_{43}\theta_{32}\theta_{21}+\theta_{41}\theta_{12}\theta_{23}$.

$$\begin{aligned}&\int\prod_i\mathrm{d}z_if(z_i)[\theta(z_4-z_3)\theta(z_3-z_2)\theta(z_2-z_1)+\theta(z_4-z_1)\theta(z_1-z_2)\theta(z_2-z_3)]\\ =&\int\prod_i\mathrm{d}z_if(z_i)\left[\int\frac{\mathrm{d}w_1}{2\pi}\frac{ie^{-iw_1(z_4-z_3)}}{w_1+i\epsilon}\int\frac{\mathrm{d}w_2}{2\pi}\frac{ie^{-iw_2(z_3-z_2)}}{w_2+i\epsilon}\int\frac{\mathrm{d}w_3}{2\pi}\frac{ie^{-iw_3(z_2-z_1)}}{w_3+i\epsilon}+\int\frac{\mathrm{d}w_1}{2\pi}\frac{ie^{-iw_1(z_4-z_1)}}{w_1+i\epsilon}\int\frac{\mathrm{d}w_2}{2\pi}\frac{ie^{-iw_2(z_1-z_2)}}{w_2+i\epsilon}\int\frac{\mathrm{d}w_3}{2\pi}\frac{ie^{-iw_3(z_2-z_3)}}{w_3+i\epsilon}\right.\end{aligned}$$

$$
\begin{aligned}
&= \int \prod_i \mathrm{d}z_i f(z_i) \int \frac{i\mathrm{d}w_1}{2\pi} \int \frac{i\mathrm{d}w_2}{2\pi} \int \frac{i\mathrm{d}w_3}{2\pi} e^{-iw_1 z_4} e^{iw_2 z_2 - iw_3 z_2} \frac{e^{iw_1 z_3} e^{-iw_2 z_3} e^{iw_3 z_1} + e^{iw_1 z_1} e^{-iw_2 z_1} e^{iw_3 z_3}}{(w_1 + i\epsilon)(w_2 + i\epsilon)(w_3 + i\epsilon)} \\
&= \int \prod_i \mathrm{d}z_i f(z_i) \int \frac{i\mathrm{d}w_1}{2\pi} \int \frac{i\mathrm{d}w_2}{2\pi} \int \frac{i\mathrm{d}w_3}{2\pi} e^{-iw_1 z_4} e^{i(w_2 - w_3) z_2} \frac{e^{i(w_1 - w_2) z_3} e^{iw_3 z_1} + e^{i(w_1 - w_2) z_1} e^{iw_3 z_3}}{(w_1 + i\epsilon)(w_2 + i\epsilon)(w_3 + i\epsilon)} \\
&= \int \prod_i \mathrm{d}z_i f(z_i) \int \frac{i\mathrm{d}w_1}{2\pi} \int \frac{i\mathrm{d}w_2}{2\pi} \int \frac{i\mathrm{d}w_3}{2\pi} \frac{4e^{-i(w_1 + w_2) z_4/2} e^{i((w_1 - w_2)/2 - w_3) z_2}}{w_1 + w_2 + i\epsilon} \frac{e^{iw_2 z_3} e^{iw_3 z_1} + e^{iw_2 z_1} e^{iw_3 z_3}}{(w_1 - w_2 + i\epsilon)(w_3 + i\epsilon)} \\
&\xlongequal{\epsilon \to 0} \int \prod_i \mathrm{d}z_i f(z_i)
\end{aligned}
$$

6. According to Tong, one can choose

$$
\begin{aligned}
\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{0 + i\epsilon} \frac{i}{-l^z + i\epsilon} &= \lim_{p \to 0} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \frac{i}{p + i\epsilon} \frac{i}{p - l^z + i\epsilon} && (149) \\
&= \lim_{p \to 0} \frac{i}{p + i\epsilon} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{\tilde{\mathbf{l}}^2 - l^{z2} + i\epsilon} \frac{i}{p - l^z + i\epsilon} && (150)
\end{aligned}
$$

where

$$
\mathcal{I} \equiv \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{\tilde{\mathbf{l}}^2 - l^{z2} + i\epsilon} \frac{i}{p - l^z + i\epsilon} \tag{151}
$$

With partial derivative operator

$$
\frac{\partial}{\partial p} \mathcal{I} = -\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{\tilde{\mathbf{l}}^2 - l^{z2} + i\epsilon} \frac{i}{[p - l^z + i\epsilon]^2} \tag{152}
$$

$$
\mathcal{I} = \frac{\partial}{\partial p} \mathcal{I}(l^z - p) \tag{153}
$$

We can evaluate the value of $\frac{\partial}{\partial p} \mathcal{I} l^z$

$$
\begin{aligned}
\frac{i}{p} \mathcal{I} l^z &= \frac{i}{p} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{\tilde{\mathbf{l}}^2 - l^{z2} + i\epsilon} \frac{i l^z}{p - l^z + i\epsilon} \\
&= \frac{i}{p} \int \frac{\mathrm{d}^3 \tilde{\mathbf{l}}}{(2\pi)^3} \frac{-i/2}{\sqrt{\tilde{\mathbf{l}}^2} + p + i\epsilon} \\
&= \frac{2p^2(\log(p) - \log(p + i\Lambda)) + \Lambda(\Lambda + 2ip)}{8\pi^2 p}
\end{aligned}
$$

$$
-\frac{i}{p} \frac{\partial \mathcal{I}}{\partial p} p = -i \frac{\partial \mathcal{I}}{\partial p}
$$

With Dim-Reg $\mathcal{I} l^z / p \to 0$. The original diagram gives

$$
-i \frac{\partial \mathcal{I}}{\partial p} = i \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{i}{\tilde{\mathbf{l}}^2 - l^{z2} + i\epsilon} \frac{i}{[l^z - i\epsilon]^2} \tag{154}
$$

7. Take only one combination of the theta function/one possible path

$$
\begin{aligned}
&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \int_0^\infty \mathrm{d}z' A^{a,z}\left(z'\right) \int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''\right) \theta(z' - z'') \psi(0) | P, S \rangle \\
&= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \int \mathrm{d}z' A^{a,z}\left(z'\right) \int \mathrm{d}z'' A^{a,z}\left(z''\right) \psi(0) | P, S \rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz''}}{w + i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z' - z'')}}{h + i\epsilon} \\
&= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \psi(0) | P, S \rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) e^{-il\cdot(z'' - z')} \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz''}}{w + i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z' - z'')}}{h + i\epsilon} \\
&= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \psi(0) | P, S \rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \int \frac{\mathrm{d}w}{2\pi} \frac{i}{w + i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{i}{h + i\epsilon} e^{-i(l + h) z'} e^{-i(w - h - l) z''}
\end{aligned}
$$

The other one is

$$
\begin{aligned}
&\int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \int_0^\infty \mathrm{d}z'' A^{a,z}\left(z''\right) \int_0^\infty \mathrm{d}z' A^{a,z}\left(z'\right) \theta(z'' - z') \psi(0) | P, S \rangle \\
&= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \int \mathrm{d}z'' A^{a,z}\left(z''\right) \int \mathrm{d}z' A^{a,z}\left(z'\right) \psi(0) | P, S \rangle \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w + i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z'' - z')}}{h + i\epsilon} \\
&= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \psi(0) | P, S \rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) e^{-il\cdot(z' - z'')} \int \frac{\mathrm{d}w}{2\pi} \frac{ie^{-iwz'}}{w + i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{ie^{-ih(z'' - z')}}{h + i\epsilon} \\
&= \int \frac{\mathrm{d}z}{2\pi} e^{ixP^z z} \langle P, S | \bar{\psi}(z) \gamma^z \psi(0) | P, S \rangle \int \mathrm{d}z' \int \mathrm{d}z'' \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \tilde{D}_G^{zz}(l) \int \frac{\mathrm{d}w}{2\pi} \frac{i}{w + i\epsilon} \int \frac{\mathrm{d}h}{2\pi} \frac{i}{h + i\epsilon} e^{-i(l + h) z''} e^{-i(w - h - l) z'}
\end{aligned}
$$

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