

Homework: General Relativity #3

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1. Turtle coordinate

$$\tilde{t} = t + 2GM \ln \left| \frac{r}{2GM} - 1 \right|, \tilde{r} = r$$

so

$$\begin{aligned} d\tilde{t} &= dt + 2GM \frac{\frac{1}{r}}{\frac{r}{2GM} - 1} dr \\ &= dt + \left(\frac{r}{2GM} - 1 \right)^{-1} dr \end{aligned}$$

The original Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

After coordinate transformation (without explicit 'tilde')

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right)\left(dt - \left(\frac{r}{2GM} - 1\right)^{-1}dr\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\left(1 - \frac{2GM}{r}\right)(dt^2 - 2\left(\frac{r}{2GM} - 1\right)^{-1}dtdr + \left(\frac{r}{2GM} - 1\right)^{-2}dr^2) + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\left(1 - \frac{2GM}{r}\right)dt^2 + 2\left(1 - \frac{2GM}{r}\right)\left(\frac{r}{2GM} - 1\right)^{-1}dtdr - \left(1 - \frac{2GM}{r}\right)\left(\frac{r}{2GM} - 1\right)^{-2}dr^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{4GM}{r}dtdr - \frac{(2GM)^2}{r(r-2GM)}dr^2 + \frac{r}{r-2GM}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{4GM}{r}dtdr - \frac{(2GM)^2 - r^2}{r(r-2GM)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{4GM}{r}dtdr + \left(1 + \frac{2GM}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{2GM}{r}dt^2 + \frac{4GM}{r}dtdr + \frac{2GM}{r}dr^2 \\ &= -dt^2 + dr^2 + r^2d\Omega^2 + \frac{2GM}{r}(dt + dr)^2 \end{aligned}$$

2. The reversed Eddington metric.

$$1. ds^2 = -dt^2 + dr^2 + r^2d\Omega^2 + \frac{2GM}{r}(dt + dr)^2.$$

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & \frac{2GM}{r} & 0 & 0 \\ \frac{2GM}{r} & \left(1 + \frac{2GM}{r}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 - \frac{2GM}{r} & \frac{2GM}{r} & 0 & 0 \\ \frac{2GM}{r} & 1 - \frac{2GM}{r} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} \sin^{-2}\theta \end{pmatrix}$$

$$2. \quad ds^2 = -(1 - \frac{2GM}{r})d\tilde{t}^2 + 2d\tilde{t}dr + r^2d\Omega^2.$$

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2GM}{r}) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - \frac{2GM}{r} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} \sin^{-2} \theta \end{pmatrix}$$

3. Under the conformally flat coordinate condition

$$r = \rho(1 + \frac{GM}{2\rho})^2$$

and

$$dr = d\left(\rho + GM + \frac{(GM)^2}{4\rho}\right) = (1 - \frac{(GM)^2}{4\rho^2})d\rho$$

the Schwarzschild metric becomes

$$\begin{aligned} ds^2 &= -(1 - \frac{2GM}{\rho(1 + \frac{GM}{2\rho})^2})dt^2 + (1 - \frac{2GM}{\rho(1 + \frac{GM}{2\rho})^2})^{-1}d\left(\rho(1 + \frac{GM}{2\rho})^2\right)^2 + (\rho(1 + \frac{GM}{2\rho})^2)^2(d\theta^2 + \sin^2 \theta d\phi^2) \\ &= -(\frac{4\rho^2 - 4GM\rho + G^2M^2}{4\rho^2 + 4GM\rho + G^2M^2})dt^2 + (\frac{4\rho^2 - 4GM\rho + G^2M^2}{4\rho^2 + 4GM\rho + G^2M^2})^{-1}(1 - \frac{(GM)^2}{4\rho^2})^2d\rho^2 + (1 + \frac{GM}{2\rho})^4\rho^2(d\theta^2 + \sin^2 \theta d\phi^2) \\ &= -\frac{(1 - \frac{GM}{2\rho})^2}{(1 + \frac{GM}{2\rho})^2}dt^2 + \frac{(1 + \frac{GM}{2\rho})^2}{(1 - \frac{GM}{2\rho})^2}(1 - \frac{GM}{2\rho})^2(1 + \frac{GM}{2\rho})^2d\rho^2 + (1 + \frac{GM}{2\rho})^4\rho^2(d\theta^2 + \sin^2 \theta d\phi^2) \\ &= -\frac{(1 - \frac{GM}{2\rho})^2}{(1 + \frac{GM}{2\rho})^2}dt^2 + (1 + \frac{GM}{2\rho})^4d\rho^2 + (1 + \frac{GM}{2\rho})^4\rho^2(d\theta^2 + \sin^2 \theta d\phi^2) \\ &= -\frac{(1 - \frac{GM}{2\rho})^2}{(1 + \frac{GM}{2\rho})^2}dt^2 + (1 + \frac{GM}{2\rho})^4[d\rho^2 + \rho^2(d\theta^2 + \sin^2 \theta d\phi^2)] \end{aligned}$$

4. From

$$(-1 - \frac{2Mr(r^2 + a^2)}{\Delta\rho^2})E^2 + \frac{4Mar}{\Delta\rho^2}EL + \frac{\rho^2 - 2Mr}{\Delta\rho^2 \sin^2 \theta}L^2 + \frac{\rho^2}{\Delta}(\frac{dr}{d\tau})^2 + \rho^2(\frac{d\theta}{d\tau})^2 = -1$$

where $\Delta = r^2 - 2Mr + a^2$, $\rho^2 = r^2 + a^2 \cos^2 \theta$, derive the radial equation if $\theta = \frac{\pi}{2}$.

If $\theta = \frac{\pi}{2}$, $\rho^2 = r^2$

$$\begin{aligned} &(-1 - \frac{2Mr(r^2 + a^2)}{\Delta r^2})E^2 + \frac{4Mar}{\Delta r^2}EL + \frac{r^2 - 2Mr}{\Delta r^2}L^2 + \frac{r^2}{\Delta}(\frac{dr}{d\tau})^2 = -1 \\ &-2Mr(r^2 + a^2)E^2 + 4MarEL + (r^2 - 2Mr)L^2 + r^4(\frac{dr}{d\tau})^2 = \Delta r^2(E^2 - 1) \\ &-2Mr(r^2 + a^2)E^2 + 4MarEL + (\Delta - a^2)L^2 + r^4(\frac{dr}{d\tau})^2 = \Delta r^2(E^2 - 1) \end{aligned}$$

Q.E.D.

5. The action in EM field

$$I = \int (-m\sqrt{-g_{\alpha\beta}(x)}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + qA_\mu(x)\frac{dx^\mu}{d\lambda})d\lambda$$

$$\delta I = \int \left\{ \frac{m}{2}(-g_{\alpha\beta}(x))\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right\}^{-\frac{1}{2}}(g_{\mu\nu,\rho}\delta x^\rho\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}) + qA_{\mu,\rho}\delta x^\rho\frac{dx^\mu}{d\lambda} + qA_\mu\frac{d\delta x^\mu}{d\lambda}\right\}d\lambda$$

The first term

$$\begin{aligned}
& \frac{m}{2} \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \left\{ g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + 2g_{\mu\nu} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right\} \\
&= \frac{m}{2} \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \left\{ g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + 2g_{\mu\nu} \frac{d}{d\lambda} \left(\delta x^\mu \frac{dx^\mu}{d\lambda} \right) - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\lambda^2} \right\} \\
&= \frac{m}{2} \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \left\{ g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + 2 \frac{d}{d\lambda} \left(g_{\mu\nu} \delta x^\mu \frac{dx^\mu}{d\lambda} \right) - 2g_{\mu\nu,\rho} \frac{dx^\rho}{d\lambda} \delta x^\mu \frac{dx^\mu}{d\lambda} - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\lambda^2} \right\} \\
&= \frac{m}{2} \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \left\{ g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu,\rho} \frac{dx^\rho}{d\lambda} \delta x^\mu \frac{dx^\mu}{d\lambda} - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\lambda^2} \right\} + m \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \frac{d}{d\lambda} \left(g_{\mu\nu} \delta x^\mu \frac{dx^\mu}{d\lambda} \right)
\end{aligned}$$

and

$$\begin{aligned}
& m \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \frac{d}{d\lambda} \left(g_{\mu\nu} \delta x^\mu \frac{dx^\mu}{d\lambda} \right) \\
&= m \frac{d}{d\lambda} \left\{ \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} g_{\mu\nu} \delta x^\mu \frac{dx^\mu}{d\lambda} \right\} - m \left\{ \frac{d}{d\lambda} \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \right\} g_{\mu\nu} \delta x^\mu \frac{dx^\mu}{d\lambda}
\end{aligned}$$

Now the total derivative term can be ignored so the first term becomes

$$\begin{aligned}
& \int \frac{m}{2} \left(-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{-\frac{1}{2}} (g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + 2g_{\mu\nu} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}) d\lambda \\
&= \frac{m}{2} \int \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \left\{ g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu,\rho} \frac{dx^\rho}{d\lambda} \delta x^\mu \frac{dx^\nu}{d\lambda} - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\lambda^2} \right\} d\lambda \\
&\quad - m \int \left\{ \frac{d}{d\lambda} \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{-\frac{1}{2}} \right\} g_{\mu\nu} \delta x^\mu \frac{dx^\nu}{d\lambda} d\lambda
\end{aligned}$$

take $\lambda = \tau$

$$\begin{aligned}
&= \frac{m}{2} \int \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right]^{-\frac{1}{2}} \left\{ g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 2g_{\mu\nu,\rho} \frac{dx^\rho}{d\tau} \delta x^\mu \frac{dx^\nu}{d\tau} - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\tau^2} \right\} d\tau \\
&\quad - m \int \left\{ \frac{d}{d\tau} \left[-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right]^{-\frac{1}{2}} \right\} g_{\mu\nu} \delta x^\mu \frac{dx^\nu}{d\tau} d\tau \\
&= \frac{m}{2} \int \left\{ g_{\mu\nu,\rho} \delta x^\rho \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 2g_{\mu\nu,\rho} \frac{dx^\rho}{d\tau} \delta x^\mu \frac{dx^\nu}{d\tau} - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\tau^2} \right\} d\tau \\
&= \frac{m}{2} \int \left\{ g_{\rho\nu,\mu} \delta x^\mu \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} - 2g_{\mu\nu,\rho} \frac{dx^\rho}{d\tau} \delta x^\mu \frac{dx^\nu}{d\tau} - 2g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\tau^2} \right\} d\tau \\
&= \frac{m}{2} \int \left\{ g_{\rho\nu,\mu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} - 2g_{\mu\nu,\rho} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} - 2g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} \right\} \delta x^\mu d\tau
\end{aligned}$$

The rest terms

$$\begin{aligned}
& qA_{\mu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} + qA_\mu \frac{d\delta x^\mu}{d\lambda} \\
&= qA_{\mu,\rho} \delta x^\rho \frac{dx^\mu}{d\lambda} + q \frac{d}{d\lambda} (A_\mu \delta x^\mu) - q \frac{dA_\mu}{d\lambda} \delta x^\mu
\end{aligned}$$

drop total derivative term, and take $\lambda = \tau$

$$\begin{aligned}
&= qA_{\mu,\rho} \delta x^\rho \frac{dx^\mu}{d\tau} - q \frac{dA_\mu}{d\tau} \delta x^\mu \\
&= [qA_{\rho,\mu} \frac{dx^\rho}{d\tau} - q \frac{dA_\mu}{d\tau}] \delta x^\mu
\end{aligned}$$

So the equation of motion

$$m[g_{\mu\nu,\rho}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} - \frac{1}{2}g_{\rho\nu,\mu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}] - qA_{\rho,\mu}\frac{dx^\rho}{d\tau} + q\frac{dA_\mu}{d\tau} = 0$$

6. Prove $\delta(g^{\alpha\beta}g_{\beta\gamma}) = 0 \implies \delta(g^{\alpha\beta}) = -(\delta g^{\alpha\beta} =: g^{\alpha\mu}(\delta g_{\mu\nu})g^{\nu\beta})$.

$$\begin{aligned}\delta(g^{\alpha\beta}) &= \delta(g^{\alpha\mu}g_{\mu\nu}g^{\nu\beta}) = (\delta g^{\alpha\mu})g_{\mu\nu}g^{\nu\beta} + g^{\alpha\mu}(\delta g_{\mu\nu})g^{\nu\beta} + g^{\alpha\mu}g_{\mu\nu}\delta g^{\nu\beta} \\ &= \delta(g^{\alpha\mu}g_{\mu\nu})g^{\nu\beta} + g^{\alpha\mu}g_{\mu\nu}\delta g^{\nu\beta} \\ &= g^{\alpha\mu}g_{\mu\nu}\delta g^{\nu\beta} \\ &= (\delta g^{\alpha\mu})g_{\mu\nu}g^{\nu\beta} \\ &= -g^{\alpha\mu}(\delta g_{\mu\nu})g^{\nu\beta} = -\delta g^{\alpha\beta}\end{aligned}$$