Quantum Field Theory(Spring)

Yingsheng Huang

March 3, 2017

1 Noether's Theorem

- Symmetry
- Lorentz/Poincare Group
- Tensor analysis

Poincare group:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$$

investigate the inverse, closure character:

$$x''^{\mu} = \bar{\Lambda}^{\mu}_{\rho} x'^{\rho} + \bar{a}^{\mu} = (\bar{\Lambda}\Lambda)^{\mu}_{\nu} x^{\nu} + (\bar{\Lambda}^{\mu}_{\rho} a^{\rho} + \bar{a}^{\mu})$$

$$(\bar{\Lambda}, \bar{a}) \times (\Lambda, a) = (\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a})$$

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}$$

translation:

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}$$

$$\begin{aligned} g_{\rho\sigma} &= g_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma} = g_{\mu\nu}(\delta^{\mu}_{\nu''} + \omega^{\mu}_{\nu''})(\delta^{\nu}_{\nu'} + \omega^{\nu}_{\nu'}) \\ \delta x^{\mu} &= \omega^{\mu}_{\nu} x^{\nu} \\ &= \omega^{\mu\nu} x_{\nu} = \omega^{\rho\sigma} x_{\sigma} \delta^{\mu}_{\rho} = \omega^{\rho\sigma} x_{\sigma} \partial_{\rho} x^{\mu} \\ &= \frac{i}{2} \omega^{\rho\sigma} \hat{L}_{\rho\sigma} x^{\mu} \end{aligned}$$

$$\hat{L}_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$$

Poincare algebra: commutation relations.

Local field transform under Poincare group:

$$f(x) \to f'(x')$$

$$\begin{aligned} \delta f \equiv f'(x') - f(x) &= f'(x + \delta x) - f(x) \\ f'(x) - f(x) + \delta x^{\mu}\partial_{\mu} f(x) + (O)(\delta x) & \\ &= \delta_0 f + \delta x^{\mu}\partial_{\mu} \end{aligned}$$

$$\delta \equiv \delta_0(\text{functional change}) + \delta x^{\mu}\partial_{\mu}$$

translation:

$$\delta f = 0 = \delta_0 f + \epsilon^{\mu}\partial_{\mu} f \Longrightarrow \delta_0 f = -\epsilon^{\mu}\partial_{\mu} f = -i\epsilon \cdot \hat{p} f$$

Lorentz:

$$\delta_0 \phi = -\delta x^{\mu}\partial_{\mu}\phi = \omega^{\mu\rho} x_{\rho}\partial_{\mu}\phi = \frac{1}{2}\omega^{\rho\sigma}(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho})$$

$$\delta_0 \phi = -\frac{i}{2}\omega^{\rho\sigma} L_{\rho\sigma}\phi$$

$$\delta_0(\partial_\mu\phi) = -\frac{i}{2}\omega^{\rho\sigma}L_{\rho\sigma}(\partial_{\mu\phi}) = \frac{i}{2}\omega^{\rho\sigma}S_{\rho\sigma}$$
$$\implies (S_{\mu\nu})^\rho_\sigma = \dots$$

Noether's theorem:

$$\delta S = \delta\int \mathrm{d}^4x\mathcal{L} = \int(\delta\mathrm{d}^4x)\mathcal{L} + \int \mathrm{d}^4x\delta\mathcal{L}$$
$$\delta\mathrm{d}^4x = \mathrm{d}^4x(\partial_\mu\delta x^\mu)$$

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x[\partial_\mu(\mathcal{L}\delta x) + \delta_0\mathcal{L}] \\
&= \int \mathrm{d}^4x[\partial_\mu(\mathcal{L}\delta x^\mu) + \partial_\mu\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}(\partial_\mu\phi)} + \text{E-L eq terms}] \\
&= \int_R \partial_\mu[\mathcal{L}\delta^\mu_\rho - \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}(\partial_\mu\phi)}\partial_\rho\phi)\delta x^\rho + \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}(\partial_\mu\phi)}\delta\phi]
\end{aligned}$$