

题目: QCD 简介

QCD 是基于 $SU(3)$ 对称性的规范相互作用理论

本讲座: 介绍 QCD 的基本元素 ($SU(N)$ 群基础)

从色荷角度来理解 韧致辐射和量子色动力学的微扰辐射修正。

* QCD 和 QED 的差异

	QED	QCD
对称性	$U(1)_{EM}$	$SU(3)_C$
charge	EM	color
媒介粒子	光子(γ)	胶子(g)
辐射粒子	带电物体	带色粒子和胶子

注意: 胶子的地位是特殊的, 既是载体又是辐射源。

1.1) 色代数基础

夸克: 具有 3 种不同的色态 (red, green, blue)

$$\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}, \quad \text{其中 } \psi_i: \text{Dirac Spinor}$$

Under the rotations in color space:

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$$\begin{cases} \psi'(x) = U(x) \psi(x) \\ \bar{\psi}'(x) = \bar{\psi}(x) U^\dagger(x) \end{cases} \quad \underline{U(x) \text{ 矩阵属于 } SU(3) \text{ 群}}$$

(*) $SU(N)$: special unitary $N \times N$ 矩阵

$$\begin{cases} U^\dagger U = 1 \\ \text{Det } U = e^{\text{Tr} \ln U} = 1 \end{cases}$$

$N \times N$ 矩阵的独立参数 $N^2 \times 2 \rightarrow \text{complex}$

$$U^\dagger U = 1 \quad N^2 \text{ 条件}$$

$$\text{Det } U = 1 \quad 1 \text{ 个条件}$$

\Rightarrow 独立实参数 $N^2 - 1$

1.1) 参数化. 无穷小变换. 和生成元

$$U(\omega) = e^{i\omega^a t^a} \quad a = 1, 2, \dots, N^2 - 1$$

\rightarrow 实参数 (color rotation angles)

t^a : $N \times N$ 矩阵

$$\begin{cases} U^\dagger U = 1 \\ \text{Det } U = 1 \end{cases} \Rightarrow \begin{cases} t^a = (t^a)^\dagger & \text{Hermitian} \\ \text{tr}(t^a) = 0 & \text{traceless} \end{cases}$$

无穷小变换.

$$U = 1 + i \delta \omega^a t^a + \dots$$

quark wave function.

$$\psi \rightarrow \psi' = U \psi \approx \psi + \delta \psi$$

$$(\delta \psi = i \delta \omega^a t^a \psi)$$

Anti-quark wave function

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} U^\dagger = \bar{\psi} + \delta \bar{\psi}$$

$$\delta \bar{\psi} = -i \delta \omega^a (\bar{\psi} t^a)$$

生成元: 在一个给定表示^R中 T^a 决定对象 R 的无穷小变换^{短符号法}
(T^a)

3: 夸克处于基础表示.

$$R^i = \psi^i, \quad T^a(3) = t^a.$$

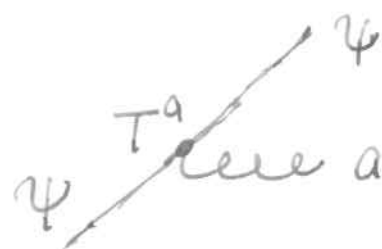
$$(T^a(3): R)^i = (t^a)^i_k R^k$$

$\bar{3}$: 反夸克处于共轭基础表示

$$R_i = \bar{\psi}_i, \quad T^a(\bar{3}) = -t^a$$

$$(T^a(\bar{3}): R)_i = -R_k (t^a)^k_i$$

在规范不变的QCD理论中 T^a 决定夸克辐射胶子的过程



我们可以将 T^a 视为“夸克色荷”

$$q: T^a(3) = t^a$$

$$\bar{q}: T^a(\bar{3}) = -t^a$$

夸克和反夸克的色荷相反，电荷相反

1.2 色荷中性化

我们首先考虑由夸克和反夸克组成的介子

色中性系统 R 满足

$$T^a: R = 0 \quad (\text{对所有的 } a)$$

可以验证如下定义的 R_0 是色中性 (或色单态)

$$\begin{aligned} R_0 \equiv R_{\text{white}} &= \bar{u}_1 u^1 + \bar{u}_2 u^2 + \bar{u}_3 u^3 \\ &= \sum_{i=1}^N \bar{u}_i u^i = (\bar{u} u) \end{aligned}$$

在色空间转动变化下

$$\delta R_0 = i \delta w^a \bar{u} (t^a u) + i \delta w^a (\bar{u} (-t^a) u) = 0$$

$\Rightarrow R_0$ 是 $SU(3)_c$ 的单态 (也叫不可约单态表示)

$$\text{生成元 } T^a(1) = 0$$

在实验上我们还观测到由3个夸克构成的色单态粒子。

\Rightarrow 应该如何构造?

1.3 双夸克组合 ($3 \times 3 = 6 + \bar{3}$)

将双夸克波函数 u^i 和 d^j 直乘 (为简便记述, 我们选 u 和 d 共有 $N \times N = 9$ 个分量。其在色空间转动变换下, 行为如下:

$$\begin{aligned}\delta\{u^i d^j\} &= (\delta u^i) d^j + u^i (\delta d^j) \\ &= i\delta\omega^a \left\{ (t^a)_k^i u^k d^j + u^i (t^a)_k^j d^k \right\}\end{aligned}$$

利用此9个分量, 我们可构造两个张量

$$u^i d^j = R^{\{ij\}} + R^{[ij]}$$

其中, 1) 对称开式

$$R^{\{ij\}} = \frac{1}{2}(u^i d^j + u^j d^i) \quad : \text{共有 } \frac{N(N+1)}{2} = 6 \text{ 分量}$$

2) 反对称开式

$$R^{[ij]} = \frac{1}{2}(u^i d^j - u^j d^i) \quad : \text{共有 } \frac{N(N-1)}{2} = 3 \text{ 分量}$$

利用 $\delta\{u^i d^j\}$ 的开式, 可以验证

$$\delta R^{\{ij\}} = \delta R^{\{ij\}} \quad \delta R^{[ij]} = -\delta R^{[ji]}$$

\Rightarrow 张量对称性在色转动操作下保持不变

即 $R^{\{ij\}}$ 和 $R^{[ij]}$ 是 $SU(3)_c$ 的不可约表示

• Sextet 6: R^{ij}

生成元: $T^a(6) = t^a \cdot \mathbb{1} + \mathbb{1} \cdot t^a$

作用在第1个指标 \rightarrow 作用在第2个指标

故有

$$(T^a(6):R)^{ij} = \left((t^a)_k^i \delta_l^j + \delta_k^i (t^a)_l^j \right) R^{kl}$$

• 反对称张量 $R^{[ij]}$

其维度和SU(3)的基础表示维度相同 $\frac{N(N-1)}{2} = 3$

下面我们验证:

张量 $R^{[ij]}$ 的变换性质和反夸克(3)相同。

首先,我们将 $R^{[ij]}$ 写作为矢量形式

$$R_m = \epsilon_{mij} u^i d^j = \epsilon_{mij} R^{[ij]} \quad (m=1,2,3)$$

$$\begin{cases} R_1 = u^2 d^3 - u^3 d^2 \\ R_2 = u^3 d^1 - u^1 d^3 \\ R_3 = u^1 d^2 - u^2 d^1 \end{cases} \quad \text{类似于3分量矢量}$$

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{132} = -\epsilon_{321} = +1$$

其次, 我们推导 R_m 的逆.

$$\begin{aligned}\varepsilon^{\alpha kl} R_\alpha &= \varepsilon^{\alpha kl} \varepsilon_{\alpha ij} R^{[ij]} \quad (\text{对 } \alpha=1,2,3 \text{ 求和}) \\&= (\delta_i^k \delta_j^l - \delta_j^k \delta_i^l) R^{[ij]} \\&= (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) R^{[ij]} \quad \left. \begin{array}{l} \text{利用 } R^{[ij]} \\ \text{反对称} \\ \text{性质} \end{array} \right\} \\&= 2R^{[kl]}\end{aligned}$$

$$\Rightarrow \underline{2R^{[ij]} = U^i d^j - U^j d^i = \varepsilon^{\alpha ij} R_\alpha} \quad (\star)$$

最后, 我们利用上面推导的各式来检验 δR_m .

$$\begin{aligned}\delta R_m &= \delta(\varepsilon_{mij} U^i d^j) \\&= i\delta\omega^a \varepsilon_{mij} \left((t^a U)^i d^j + U^i (t^a d)^j \right)\end{aligned}$$

$$\begin{aligned}\because \varepsilon_{mij} U^i (t^a d)^j &\stackrel{i \leftrightarrow j}{=} \varepsilon_{mji} U^j (t^a d)^i \\&= -\varepsilon_{mij} U^j (t^a d)^i\end{aligned}$$

$$\begin{aligned}\therefore \delta R_m &= i\delta\omega^a \varepsilon_{mij} \left((t^a U)^i d^j - U^j (t^a d)^i \right) \\&= i\delta\omega^a \varepsilon_{mij} (t^a)_k^i \underbrace{(U^k d^j - U^j d^k)}_{(\star)} \\&= i\delta\omega^a \varepsilon_{mij} (t^a)_k^i \varepsilon^{\alpha kj} R_\alpha \\&= i\delta\omega^a (t^a)_k^i (\delta_m^\alpha \delta_i^k - \delta_i^\alpha \delta_m^k) R_\alpha\end{aligned}$$

等式右方第1项为0, 因为

$$(t^a)_k^i \delta_m^\alpha \delta_i^k = (t^a)_i^i \delta_m^\alpha = \text{Tr}(t^a) \delta_m^\alpha = 0$$

因此,

$$\delta R_m = -i \delta \omega^a (t^a)_m^\alpha R_\alpha = i \delta \omega^a (-R t^a)_m$$

其变换行为和规范波函数一致

$\Rightarrow R^{[ij]}$ 按照 $T^a(\bar{3})$ 变换

- 总结, 我们将 $u^i d^j$ ~~按照~~ 波函数直乘作如下分解.

$$3 \times 3 = 6 + \bar{3}$$

1.4 (qqq) 重子中性化

在了解上面 (qq) 波函数性质后, 构造 (qqq) 自就非常容易了. 例如 (uds) 系统

$$R_0 = \epsilon_{mij} u^i d^j s^m$$

其变换形式如下

$$\begin{aligned} \delta(\underbrace{\epsilon_{mij} u^i d^j s^m}_{R_m}) &= (\delta R_m) s^m + R_m (\delta s^m) \\ &= i \delta \omega^a \{ -(R t^a) s + R (t^a s) \} = 0 \end{aligned}$$

注意, 值得指出介子和自重子的区别

介子: $(\mathbf{q}\bar{\mathbf{q}})$ 是任意 $SU(N)$ 群下的单态

重子: 依赖于 N . (要求重子中夸克数同 $=N$)

例如

① $SU(2)$ 同位旋 $N = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

构造同位旋单态

"介子" $\bar{N}_i N^i = \bar{n}_1 n_1 + \bar{n}_2 n_2$

"重子" $\epsilon_{ij} N^i N_j = n_1 n_2 - n_2 n_1$

② $SU(4)_c$: 假设夸克具有四种颜色

$$g^i g^j = 4 \otimes 4 = R^{\{ij\}} + R^{[ij]} = 10 + 6$$

$$g^i g^j g^k = 4 \otimes 4 \otimes 4 = \dots + \bar{4}$$

$SU(4)_c$ 的"自重子"

$$= \epsilon_{ijkl} g^i g^j g^k g^l$$

1.5 对易子和结构常数

考虑两个操作的对易子 (先后顺序的差异)

$$[U_2, U_1] \equiv U_2 U_1 - U_1 U_2$$

$$= (i\delta\omega_2^a)(i\delta\omega_1^b) \underline{[t^a, t^b]} + o(\delta\omega^3)$$

对易 = 0, 阿贝尔

不对易 $\neq 0$, 非阿贝尔

$$\text{因为 } \text{tr}(t^a t^b) = \text{tr}(t^b t^a)$$

$$\Rightarrow \text{tr}[t^a, t^b] = 0, \text{ 即 } [t^a, t^b] \text{ 无迹}$$

$$\text{所以 } [t^a, t^b] = i f_{abc} t^c, \quad a, b, c = 1, \dots, N^2-1$$

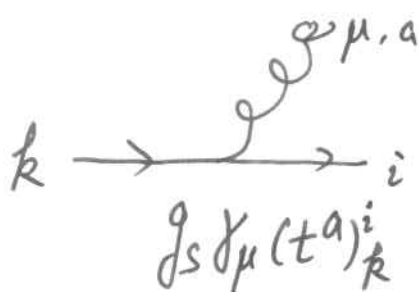
↓
反厄米

↙ 实数, 结构常数

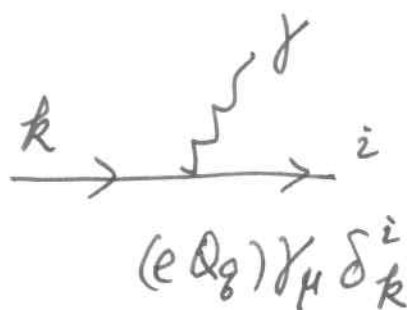
1.6 t -矩阵的归化常数

$$\text{Tr}(t^a t^b) \equiv \text{Tr} \delta^{ab} = \frac{1}{2} \delta^{ab}$$

选定 Tr 后就确定了 $N \times N$ 无迹矩阵空间中基矢
此时具有如下的费曼顶点。



$$g_s \gamma_\mu (t^a)^i_k$$



$$(e Q_g) \gamma_\mu \delta^i_k$$

1.7 $N \times N$ 矩阵按照 t^a 展开

$$M = n^0 \mathbb{1} + n^a t^a$$

$$\because \text{Tr}(M) = n^0 N \quad \text{且} \quad \text{Tr}(M t^b) = n^a \text{tr}(t^a t^b) \\ = \frac{1}{2} n^a \delta^{ab}$$

$$\therefore \boxed{M = \frac{1}{N} \text{Tr}(M) \mathbb{1} + 2 \text{Tr}(M t^a) t^a}$$

例1) 结构常数 f_{abc}

令 $M = [t^a, t^b]$, 则有

$$[t^a, t^b] = \frac{1}{N} \text{Tr}([t^a, t^b]) \mathbb{1} + 2 \text{Tr}([t^a, t^b] t^c) t^c$$

$$\Rightarrow i f_{abc} t^c = 2 \text{Tr}(t^c [t^a, t^b]) t^c$$

$$\Rightarrow i f_{abc} = 2 \text{Tr}(t^c [t^a, t^b])$$

可利用循环置换来证明

f_{abc} 对任意两个指标都是反对称的。

例2) $3 \times \bar{3} = 1 + 8$

考虑 $(q\bar{q})$ 系统, 其波函数直乘为

$$\psi^i \bar{\chi}_k = \frac{1}{N} \delta_k^i (\bar{\chi} \psi) + R_k^i$$

$$\text{其中 } R_k^i = \psi^i \bar{\chi}_k - \frac{1}{N} \delta_k^i (\bar{\chi} \psi)$$

是无迹张量, $R_i^i = 0$

在色旋转变换下,

$$\begin{aligned} \delta R_k^i &= i\delta\omega^a (t^a)_l^i R_k^l + i\delta\omega^a R_l^i (-t^a)_k^l \\ &= i\delta\omega^a ([t^a, R])_k^i \end{aligned}$$

$$\begin{aligned} \text{当 } i=k \text{ 时, } \delta R_i^i &= i\delta\omega^a ([t^a, R])_i^i = i\omega^a \text{Tr}([t^a, R]) \\ &= 0 = R_i^i \end{aligned}$$

所以, R_k^i 仍是无迹 $\Rightarrow R_k^i$ 构成 $SU(3)$ 八重态

• 矢量形式

我们可将 R_k^i 写作矢量形式 $\phi_a, a=1, 2, \dots, N^2-1$

$$R_k^i = \phi_b (t^b)_k^i$$

$$\phi_b = 2\text{Tr}(R t^b)$$

1.8) 伴随表示 — 生成元 $T^a(8)$

在前面我们已经看到 R_K^i 的无穷小变化行为是

$$\delta R_K^i = i \delta w^a ([t^a, R])_K^i$$

由生成元定义可知, 在八重态表示中生成元是

$$T^a(8): R = [t^a, R],$$

$$T^a(8) = [t^a, \dots] \text{ 生成元由对易子给出}$$

将 R_K^i 表示为矢量形式, 可得

$$R_K^i = \phi_c (t^c)_K^i, \quad \phi_c = 2 \text{Tr}(R t^c)$$

则有

$$T^a(8): R = T^a(8): (\phi_c t^c) \equiv t^b (T^a: \phi)_b$$

$$T^a(8): (\phi_c t^c) = \phi_c T^a(8): t^c$$

$$= \phi_c [t^a, t^c]$$

$$= i f_{acb} t^b \phi_c = -i f_{abc} t^b \phi_c$$

$$\text{也即 } (T^a: \phi)_b = -i f_{abc} \phi_c$$

矢量 ϕ 在色空间旋转变换下形式为

$$\delta\phi = i\delta W^a (T^a \cdot \phi)$$

$$\delta\phi_b = \delta W^a \underline{f_{abc}} \phi_c$$

生成元

生成元和结构常数相同的表示 —— 伴随表示

此时生成元为

$$(iT^a(8))_{bc} = f_{abc}$$

生成元
指标

b 和 c 为行列指标

1.9) Jacobi 恒等式

$$f_{abe} f_{cde} + f_{bce} f_{ade} + f_{cae} f_{bde} = 0$$

上式对 e 求和, $\{a, b, c\}$ 作循环置换

下面我们验证 Jacobi 恒等式可以保证

胶子场强张量满足正确的色变换.

胶子场强张量为

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c$$

其中包含胶子场的线性和双线性。在色旋转变换下，两者的变换行为应该保持一致。

首先, $iT^d(8): F^a = f_{dae} F^e$

下面我们考虑 上面的右方 的双线性项,

$$f_{dae} F^e \sim f_{dae} f_{ebc} A_\mu^b A_\nu^c$$

而等式左方

$$\begin{aligned} iT^d(8): \{ f_{abc} A_\mu^b A_\nu^c \} \\ &= f_{abc} \{ (iT^d(8): A_\mu)^b A_\nu^c + A_\mu^b (iT^d(8): A_\nu)^c \} \\ &= f_{abc} [f_{dbe} A_\mu^e A_\nu^c + f_{dce} A_\mu^b A_\nu^e] \\ &= (f_{aec} f_{deb} + f_{abe} f_{dec}) A_\mu^b A_\nu^c \end{aligned}$$

左右相等可得

$$f_{dae} f_{ebc} = f_{aec} f_{deb} + f_{abe} f_{dec}$$

整理后可得 Jacobi 恒等式

1.10 对易关系的普适性:

$$[T^a(R), T^b(R)] = if_{abc} T^c(R)$$

此普适性和色荷守恒关系联系紧密。

同时也保证带任意色荷的粒子 (处于 $SU(3)_c$ 群不同表示的粒子) 都遵从相同的相互作用。

2 Color charge

2.1 Casimir 标符和胶子辐射强度

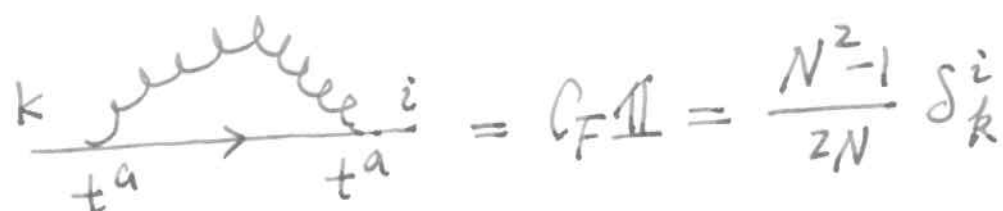
Casimir 标符 $T^2 \equiv T^a T^a$, $[T^2, T^a] = 0$

由 Schur's lemma 可知

T^2 正比于给定表示中的单位标符

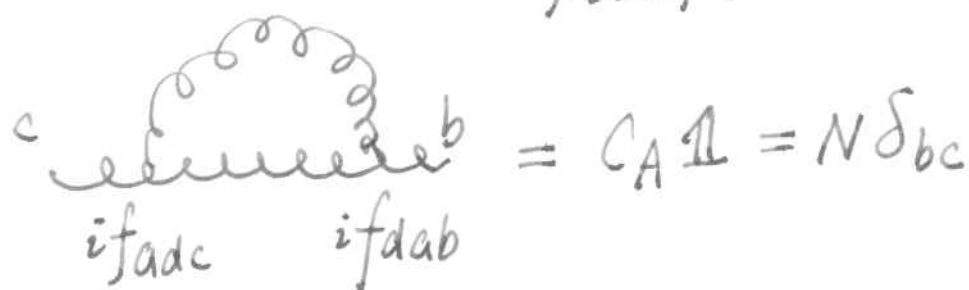
① 在基础表示中

$$(T^a(3))_j^i (T^a(3))_k^j = (t^a)_j^i (t^a)_k^j = C_F \delta_k^i \equiv C_F \mathbb{1}$$


$$= C_F \mathbb{1} = \frac{N^2 - 1}{2N} \delta_k^i$$

②

$$(T^a(8))_{bd} (T^a(8))_{dc} = (-if_{abd})(-if_{adc})$$
$$= f_{bad} f_{cad} = C_A \delta_{bc} \equiv C_A \mathbb{1}$$


$$= C_A \mathbb{1} = N \delta_{bc}$$

Schur's lemma 的物理图像:

辐射和吸收相同色态的胶子不改变辐射者的色态, 但量子效应反映在 Casimir 标符上。

下面我们推导这些“平方色荷” C_F 和 C_A

2.2 Fierz Identity

利用公式 $M = \frac{1}{N} \text{Tr}(M) \mathbb{1} + 2 \text{Tr}(M t^a) t^a$

取 $M_{jk}^i = \delta_{(j)}^i \delta_k^{(l)}$ (设 j 和 l 是固定的数)

则有 $\delta_j^i \delta_k^l = \frac{1}{N} \delta_k^i \delta_j^l + 2 (t^a)_k^i (t^a)_j^l$

上式可用图表示如下

$$\begin{array}{c} j \longrightarrow i \\ l \longleftarrow k \end{array} = \frac{1}{N} \begin{array}{c} j \text{---} \text{---} \text{---} i \\ l \text{---} \text{---} \text{---} k \end{array} + 2 \begin{array}{c} j \searrow \text{---} i \\ l \swarrow \text{---} k \end{array}$$

物理图像:

取 M_{jk}^i 中 i 为夸克指标, k 为反夸克指标

则上式可理解为将 $(3, \bar{3})$ 分解为色单态和色八重态

色投影标符:

$$\mathbb{1}(3) \cdot \mathbb{1}(3) = P(0) + P(8)$$

$$P(0) = \frac{1}{N} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array}$$

$$P(8) = 2 \begin{array}{c} \searrow \text{---} \swarrow \\ \swarrow \text{---} \searrow \end{array}$$

验证: $P_{(0)}$ 和 $P_{(8)}$ 满足投影符号的要求

$$1) \hat{P}_{(0)}^2 = \hat{P}_{(0)}$$

$$\left(\frac{1}{N} \text{diag}\right) \left(\frac{1}{N} \text{diag}\right) = \frac{1}{N^2} \text{diag} \square \text{diag} = \frac{1}{N} \text{diag}$$

(其中用到 $\square = N$, 每个封闭圈都给出 N)

$$2) \hat{P}_{(8)}^2 = \hat{P}_{(8)}$$

$$(2 \text{diag}) (2 \text{diag}) = 4 \text{diag} \diamond \text{diag} = 2 \text{diag}$$

(其中用到 $\text{diag} \bigcirc \text{diag} = \text{Tr} \delta^{ab} = \frac{1}{2} \delta^{ab}$)

$$3) \hat{P}_{(0)} \cdot \hat{P}_{(8)} = \hat{P}_{(8)} \hat{P}_{(0)} = 0$$

$$\left(\frac{1}{N} \text{diag}\right) (2 \text{diag}) = \frac{2}{N} \text{diag} \triangle \text{diag} = 0$$

(其中用到 $\text{diag} \bigcirc = \text{tr}(t^a) = 0$)

Fierz Identity 示例: color link

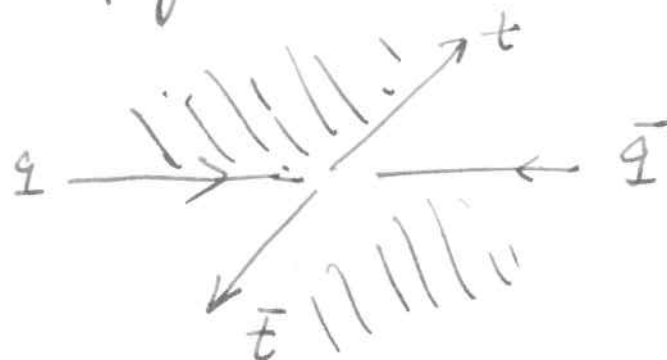
$$\textcircled{1} \begin{array}{c} q \rightarrow \text{---} q \\ \quad \quad \quad \updownarrow \\ \bar{q}' \leftarrow \text{---} \bar{q}' \end{array} = \frac{1}{2} \begin{array}{c} q \text{---} \downarrow \text{---} q \\ \bar{q}' \text{---} \uparrow \text{---} \bar{q}' \end{array} - \frac{1}{2N} \begin{array}{c} q \rightarrow \text{---} q \\ \bar{q} \leftarrow \text{---} \bar{q} \end{array}$$

$$\textcircled{2} \begin{array}{c} u \text{---} \updownarrow \text{---} u \\ d \text{---} \updownarrow \text{---} d \end{array} = \frac{1}{2} \begin{array}{c} u \rightarrow \text{---} u \\ d \rightarrow \text{---} d \end{array} - \frac{1}{2N} \begin{array}{c} u \rightarrow \text{---} u \\ d \rightarrow \text{---} d \end{array}$$

$$\textcircled{3} q\bar{q} \rightarrow t\bar{t} \quad \begin{array}{c} q \text{---} \text{---} t \\ \bar{q} \text{---} \text{---} \bar{t} \end{array} = \frac{1}{2} \begin{array}{c} q \text{---} \text{---} t \\ \bar{q} \text{---} \text{---} \bar{t} \end{array} - \frac{1}{2N} \begin{array}{c} q \text{---} \text{---} t \\ \bar{q} \text{---} \text{---} \bar{t} \end{array}$$

top夸克对通过交换u夸克的色荷(主要贡献)

Additional gluon radiation:



额外胶子主要出现在阴影区域

2.3) Quark "色荷" "Squared color charge" C_F

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$$\text{Diagram 1} = \frac{1}{N} \text{Diagram 2} + 2 \text{Diagram 3}$$

Diagram 1: A quark line with momentum l entering from the left and k exiting to the right. A dashed gluon loop is attached to the quark line.

Diagram 2: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

Diagram 3: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

$$\text{Diagram 4} = \frac{1}{N} \text{Diagram 5} + 2 \text{Diagram 6}$$

Diagram 4: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

Diagram 5: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

Diagram 6: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

$$N \delta_{lk}^l = \frac{1}{N} \delta_{lk}^l + 2 C_F \delta_{lk}^l \Rightarrow \underline{\underline{C_F = \frac{N^2 - 1}{2N} = \frac{4}{3}}}$$

2.4) QCD 顶点修正

$$\text{Diagram 1} = \frac{1}{N} \text{Diagram 2} + 2 \text{Diagram 3}$$

Diagram 1: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

Diagram 2: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

Diagram 3: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

$$0 = \frac{1}{N} \text{Diagram 4} + 2 \text{Diagram 5}$$

Diagram 4: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

Diagram 5: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

$$\Rightarrow \text{Diagram 6} = -\frac{1}{2N} \text{Diagram 7}$$

Diagram 6: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

Diagram 7: A quark line with momentum l entering from the left and k exiting to the right. A gluon line connects the quark line to itself, forming a loop.

代数语言: $\underline{\underline{t^a t^b t^a = -\frac{1}{2N} t^b}}$

注意: 我们画的仅是色空间中色因子贡献
等式左边是真正的费曼图, 但右边为“示意图”

QED
顶点:

$$\text{Feynman diagram: vertex with wavy line} : \text{Feynman diagram: loop with wavy line} C_F$$

我们也可用 Fierz identity 求之。

$$\text{Feynman diagram: vertex with wavy line} = \frac{1}{N} \text{Feynman diagram: vertex with wavy line and loop} + 2 \text{Feynman diagram: vertex with wavy line and loop}$$

$$N \delta_{\mathbf{k}}^l = \frac{1}{N} \text{Feynman diagram: vertex with wavy line and loop} + 2 \text{Feynman diagram: vertex with wavy line and loop}$$

$$\Rightarrow \text{Feynman diagram: vertex with wavy line} = \frac{1}{2} \left[\text{Feynman diagram: vertex with wavy line and loop} - \frac{1}{N} \text{Feynman diagram: vertex with wavy line and loop} \right]$$

$$= \frac{1}{2} \left(N - \frac{1}{N} \right) = \frac{N^2 - 1}{2N} \delta_{\mathbf{k}}^l$$

QCD Versus QED 修正

$$\text{Feynman diagram: vertex with wavy line} \left(-\frac{1}{2N} \right)$$

$$\text{Feynman diagram: vertex with wavy line} C_F = \frac{N^2 - 1}{2N}$$

胶子修正导致
排斥力

胶子修正导致吸引力

2.5 对 $8 \rightarrow 8$ 的 QCD 修正

首先考虑我们熟悉的 $8 \rightarrow 8$ 过程

$$\text{Diagram 1} \Rightarrow C_F \text{ Diagram 2}$$

$$\text{Diagram 3} \Rightarrow C_F \text{ Diagram 4}$$

由 Ward 恒式可知上面 QCD 修正中紫外发散抵消

(注意: 顶角修正和外线自能修正色因子相同)

而 QCD 中贡献不同.

$$\text{Diagram 5} = -\frac{1}{2N} \text{ Diagram 6}$$

δZ_{virt}
(vertex)

$$\text{Diagram 7} = \frac{N^2-1}{2N} \text{ Diagram 8}$$

δZ_{wf}
(wave function)

色因子不同, 紫外发散无法抵消

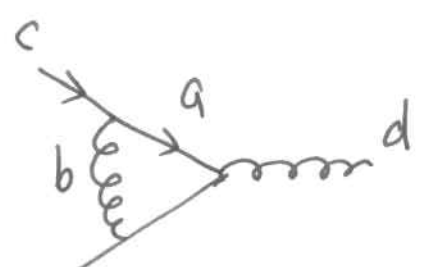
\Rightarrow 我们需要额外的贡献

$$\delta Z_{\text{wf}} + \delta Z_{\text{virt}} = \frac{N^2-1}{2N} A \ln \Lambda + (-\frac{1}{2N}) (-A \ln \Lambda)$$

(仅保留发散项, 方便讨论)

注意:  $\sim \frac{1}{N^2-1}$ 

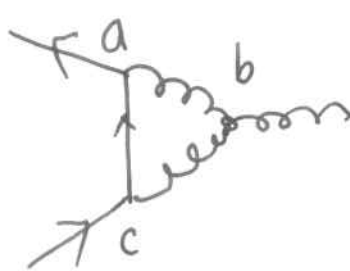

在大 N 极限下

 $\sim -\frac{1}{2N} \xrightarrow{dN} c$

这说明: $c \rightarrow a + g$ 后, a 夸克仅仅携带少量的色荷, 从而导致 a 和外部胶子(d)的相互作用减弱.

但因为 a 和 b 都是夸克, 并不会真正减少色荷。那么色荷到哪里去了?



\Rightarrow 还有1个费曼图

 $= \frac{C_A}{2}$ 

代数写法 $t^a + t^c f_{abc} = \frac{C_A}{2} t^b$

我们期待这个 Non-abelian 贡献可以抵消
顶点图和自能图.

$\Rightarrow C_A$ 一定不被大 N 压低

( + ) 的发散项的色因子是

$$\frac{N^2-1}{2N} t^b - (-\frac{1}{2N}) t^b = \frac{N}{2} t^b$$

$$\Rightarrow \text{要求 } C_A = N$$

下面我们验证 $C_A = N$ 。

将色矩阵显式写出如下

$$\begin{aligned} t^a t^a t^b - t^a t^b t^a &= t^a [t^a, t^b] = t^a i f_{abc} t^c \\ &= \left(\frac{1}{2} [t^a, t^c] + \frac{1}{2} \underbrace{\{t^a, t^c\}}_{\text{对称}} \right) i f_{abc} \\ &= \left(\frac{i}{2} f_{acd} t^d \right) i f_{abc} \\ &= \frac{1}{2} f_{acd} f_{acb} t^d \equiv \frac{C_A}{2} t^b \quad (\text{参见 } C_A \text{ 定义}) \\ &\Rightarrow \underline{C_A = N} \end{aligned}$$

注意: $\underline{t^a t^a t^b - t^a t^b t^a = C_F t^b - t^a t^b t^c = \frac{C_A}{2} t^b}$

在式中我们仅仅使用了反对易关系, 因为反对易关系与具体表示无关, 因此

$$T^a(R) T^b(R) T^a(R) = \left(C(R) - \frac{C_A}{2} \right) T^b(R)$$

故, $\left(\text{triangle with wavy line}^R + \text{triangle with wavy line}^R \right)$ 的发散项为

$$= C(R) A \ln \Lambda - \left(C(R) - \frac{C_A}{2} \right) A \ln \Lambda$$

$$= \frac{C_A}{2} A \ln \Lambda$$

两者之和消去和表示有关的发散。

$$\text{cloud diagram} = \frac{C_A}{2} (G - A) \ln \Lambda$$

$$\left(\text{box diagrams} \right) = \left\{ \frac{g_s^2}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} n_f \right) - \frac{C_A}{2} G \right\} \ln \Lambda$$

将所有图形求和之后可得

$$\delta g_s = g_s \left(\delta Z_{wf}^{(R)} + \delta Z_{virt} + \delta_{virt}^{(NA)} + \frac{1}{2} \delta Z_{w-f}^{(g/wf)} \right)$$

$$= \frac{g_s^3}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} n_f \right) \ln \Lambda$$

2.6 d-符号

$$d_{abc} = 2 \text{Tr}(t^a t^b t^c + t^b t^a t^c)$$

在具体记标中, 我们经常遇到 d-符号. 虽然它并不直接出现在费曼顶点中.

$$\text{令 } M = t^a t^b + t^b t^a - \frac{1}{N} \delta_{ab} \mathbb{1} \equiv d_{abc} t^c$$

① M 关于 a, b 是对称的

$$\text{② } \text{tr}(M) = \frac{1}{2} + \frac{1}{2} - \frac{1}{N} \times N = 0$$

将 M 按 $\mathbb{1}$ 和 t^c 展开可得

$$\begin{aligned} M &= \frac{1}{N} \text{Tr}(M) \mathbb{1} + 2 \text{Tr}(M t^c) t^c \\ &= 2 \text{Tr}\left((t^a t^b + t^b t^a - \frac{1}{N} \delta_{ab} \mathbb{1}) t^c\right) t^c \\ &= 2 \text{Tr}(t^a t^b t^c + t^b t^a t^c) t^c \\ &\equiv d_{abc} t^c \end{aligned}$$

$$\Rightarrow d_{abc} = 2 \text{Tr}(t^a t^b t^c + t^b t^a t^c)$$

图示:

The diagram shows a triangle loop with a cross inside, labeled d_{abc} below it. This is equal to 2 times the sum of two triangle loops. The first triangle loop has arrows on all three internal lines pointing in a clockwise direction. The second triangle loop has arrows on all three internal lines pointing in a counter-clockwise direction. The external lines are wavy and labeled with indices a, b, and c.

$$= 2 \left(\text{triangle with clockwise arrows} + \text{triangle with counter-clockwise arrows} \right)$$

d 符号性质

1) 令 M 中的 $a=b$, 则有

$$M = t^a t^a + t^a t^a - \frac{1}{N} \delta_{aa} \mathbb{1} = d_{aac} t^c$$

$$\Rightarrow 2C_F \mathbb{1} - \frac{N^2-1}{N} \mathbb{1} = d_{aac} t^c = 0$$

$$\Rightarrow d_{aac} = 0$$

这意味着 d_{abc} 是 $(N^2-1) \times (N^2-1) = 8 \times 8$ 无迹矩阵
共有 8 个.

2) $t^b M$:

$$\begin{aligned} \text{左方} &= t^b (t^a t^b + t^b t^a - \frac{1}{N} \delta_{ab} \mathbb{1}) = \left(-\frac{1}{2N} + \frac{N^2-1}{2N} - \frac{1}{N}\right) t^a \\ &= \frac{N^2-4}{2N} t^a \end{aligned}$$

$$\text{右方} = d_{abc} t^b t^c$$

$$\begin{aligned} &= d_{abc} \cdot \frac{1}{2} (t^b t^c + t^c t^b - \frac{1}{N} \delta_{bc} \mathbb{1}) \equiv \frac{1}{2} d_{abc} d_{bce} t^e \\ &\quad \text{从而有} \end{aligned}$$

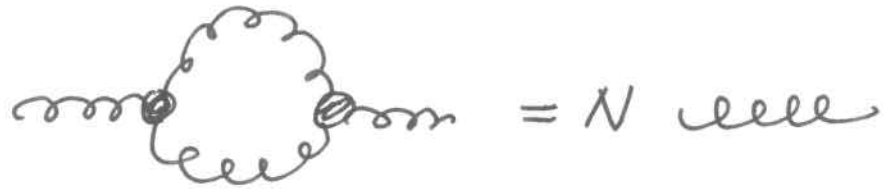
$$\frac{1}{2} d_{abc} d_{bce} t^e = \frac{N^2-4}{2N} t^a \Rightarrow \boxed{d_{abc} d_{bce} = \frac{N^2-4}{N} \delta_{ae}}$$

$$\text{Feynman diagram: a loop with two external wavy lines} = \frac{N^2-4}{N} \text{wavy line}$$

关于结构常数 f_{abc} , 我们有

$$f_{abc} f_{ebc} = N \delta_{ae}$$

即



$$= N \text{ (four-gluon vertex diagram)}$$

2.7 d符号应用示例 — 从夸克线上连续辐射双胶子

$$\text{令 } M = t^a t^b + t^b t^a - \frac{1}{N} \delta_{ab} \mathbb{1} \equiv d_{abc} t^c$$

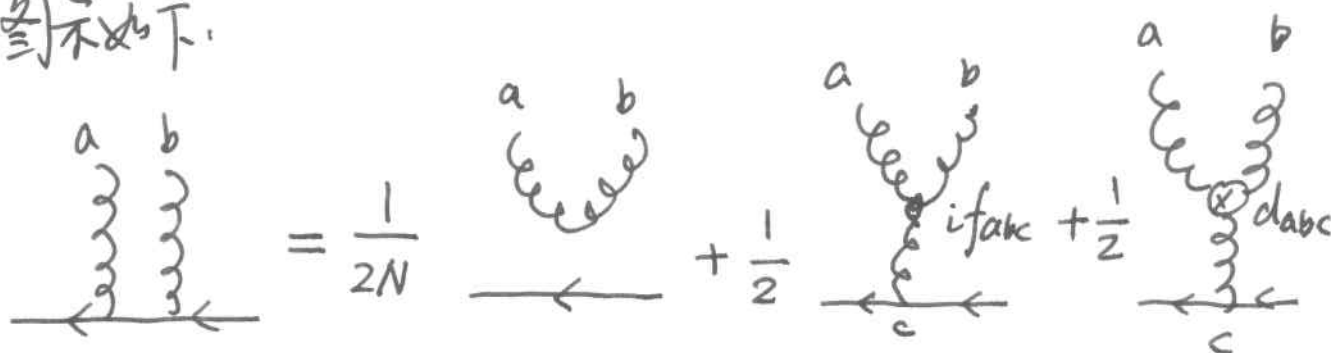
重新组合后可得

$$2t^a t^b - t^a t^b + t^b t^a - \frac{1}{N} \delta_{ab} \mathbb{1} = d_{abc} t^c$$

$$\Rightarrow 2t^a t^b = d_{abc} t^c + \frac{1}{N} \delta_{ab} \mathbb{1} + [t^a, t^b]$$

$$\Rightarrow t^a t^b = \frac{1}{2} d_{abc} t^c + \frac{1}{2N} \delta_{ab} \mathbb{1} + \frac{i}{2} f_{abc} t^c$$

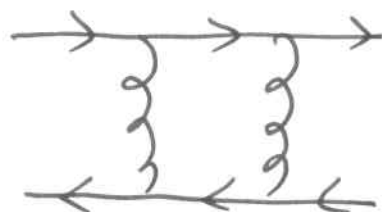
图示如下.



$$= \frac{1}{2N} \text{ (gluon loop)} + \frac{1}{2} \text{ (four-gluon vertex)} + \frac{1}{2} \text{ (triple-gluon vertex with } if_{abc} \text{)} + \frac{1}{2} \text{ (triple-gluon vertex with } d_{abc} \text{)}$$

其次, 我们考虑 $q\bar{q} \rightarrow q\bar{q}$ 的双胶子虚修正的色因子.

即



① 我们需把之前得到的双胶子连接到同一个夸克线上. 即

$$\begin{aligned}
 \text{Diagram} &= \frac{1}{2N} \text{Diagram 1} + \frac{1}{2} \text{Diagram 2} + \frac{1}{2} \text{Diagram 3} \\
 &= \frac{1}{2N} C_F \text{Diagram 4} + \frac{1}{2} \times \frac{C_A}{2} \text{Diagram 5} + \frac{1}{2} \text{Diagram 6}
 \end{aligned}$$

② 下面我们讨论右方第3个图形

$$\begin{aligned}
 \text{Diagram 3} &= \text{Diagram 7} \\
 &= \frac{1}{2N} \text{Diagram 8} + \frac{1}{2} \text{Diagram 9} + \frac{1}{2} \text{Diagram 10}
 \end{aligned}$$

$fabcdebc = 0$

即

$$\text{Diagram 9} = \frac{1}{2N} \text{Diagram 11} + \frac{1}{2} \text{Diagram 12}$$

$daac = 0$

③ 故而

$$\begin{array}{c} i \longrightarrow j \\ \downarrow \uparrow \\ k \longleftarrow l \end{array} = \frac{1}{2N} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} + \frac{1}{2} \times \frac{C_A}{2} \begin{array}{c} \longrightarrow \\ \downarrow \uparrow \\ \longrightarrow \end{array} \\
 + \frac{1}{2} \times \frac{1}{2} \begin{array}{c} \longrightarrow \\ \downarrow \uparrow \\ \downarrow \uparrow \\ \longrightarrow \end{array}$$

(利用 )

$$\begin{aligned}
 &= \frac{1}{2N} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} + \frac{1}{2} \times \frac{C_A}{2} \begin{array}{c} \longrightarrow \\ \downarrow \uparrow \\ \longrightarrow \end{array} + \frac{1}{2} \times \frac{1}{2} \times \frac{N^2-4}{N} \begin{array}{c} \longrightarrow \\ \downarrow \uparrow \\ \downarrow \uparrow \\ \longrightarrow \end{array} \\
 &= \frac{N^2-1}{4N^2} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} + \frac{N^2-2}{2N} \begin{array}{c} \longrightarrow \\ \downarrow \uparrow \\ \longrightarrow \end{array} \\
 &= \frac{N^2-1}{4N^2} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} + \frac{N^2-2}{2N} \left(\frac{1}{2} \begin{array}{c} \downarrow \uparrow \\ \downarrow \uparrow \end{array} - \frac{1}{2N} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \right) \\
 &= \frac{1}{4N^2} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} + \frac{N^2-2}{4N} \begin{array}{c} \downarrow \uparrow \\ \downarrow \uparrow \end{array}
 \end{aligned}$$

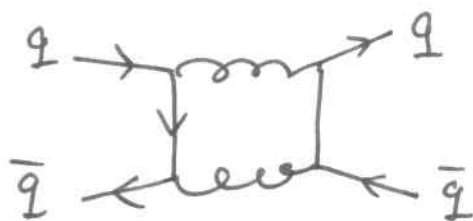
$$N=3 \quad \Rightarrow \quad \frac{1}{36} \delta_j^i \delta_k^l + \frac{7}{12} \delta_k^i \delta_l^j$$

类似地可得. ($qq \rightarrow qq$)

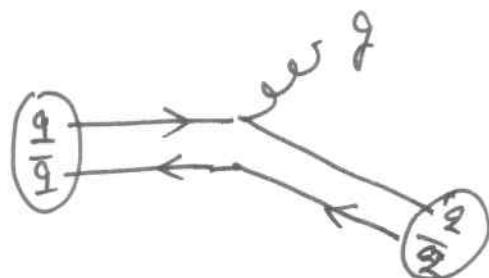
$$\begin{array}{c} \longrightarrow \longrightarrow \\ \downarrow \downarrow \\ \longrightarrow \longrightarrow \end{array} = \frac{N^2-1}{4N^2} \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} - \frac{1}{N} \begin{array}{c} \longrightarrow \longrightarrow \\ \downarrow \downarrow \\ \longrightarrow \longrightarrow \end{array}$$

作业:

- 1) 将 s-channel 散射的 $q\bar{q} \rightarrow q\bar{q}$ 过程分解为色单态和色八重态



- 2) 设一个处于 8 重态的 $(q\bar{q})$ 束缚态。当其辐射一个胶子后，其处于 8 重态或单态的几率比值如何

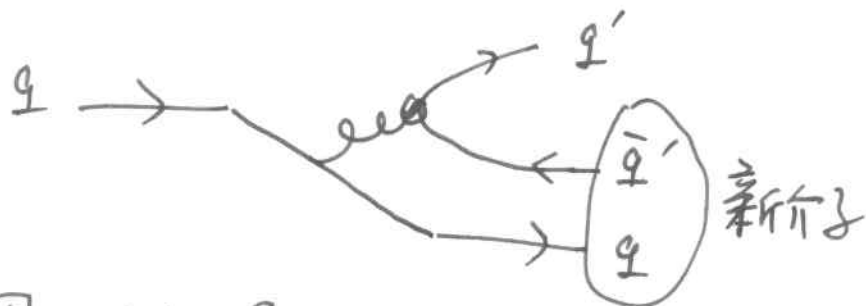


$$\omega_1 = ?$$

$$\omega_8 = ?$$

$$\frac{\omega_1}{\omega_8} = \frac{1}{7}$$

- 3) 初态夸克辐射一个胶子，此胶子逐步衰变成 $q'\bar{q}'$ 。
新反夸克 \bar{q}' 和辐射胶子后的夸克 q 形成束缚态介子。



请用图式法证明。

新介子处于 8 重态和单态的几率比值为

$$\frac{\omega_1}{\omega_8} = \frac{8}{1}$$

本讲座中使用的画示方法请参见 Cvitanovic 的书

基本性质:

$$\text{Tr} \mathbb{1} = N_c \quad \text{circle with arrow} = N_c$$

$$\text{Tr} t^a = 0 \quad \text{wavy line in a circle with arrow} = 0$$

$$\text{Tr}(t^a t^b) = \text{Tr} \delta^{ab} \quad \text{wavy line in a circle with wavy line} = \text{Tr} \text{wavy line}$$

示例 1. 胶子的色自由度的个数

$$\begin{aligned} N_g &= \text{cloud diagram} = \frac{1}{\text{Tr}} \text{cloud diagram with arrow} = \frac{1}{\text{Tr}} \text{circle with 3 wavy lines and arrow} \\ &= \text{circle with 2 wavy lines and arrow} - \frac{1}{N_c} \text{circle with arrow} = N_c^2 - 1 \end{aligned}$$

基本用到

$$\text{two wavy lines with arrows} = \text{Tr} \left[\text{square with arrows} - \frac{1}{N_c} \text{square with arrow} \right]$$

$$\Rightarrow \text{circle with 3 wavy lines} = \text{Tr} \left(\text{square with arrows} - \frac{1}{N_c} \text{square with arrow} \right)$$

示例2. 李克自能修正

$$\text{Diagram with a bubble} = \text{Tr} \left[\text{Diagram with a bubble and arrow} - \frac{1}{N_c} \rightarrow \right]$$

$$= \text{Tr} \left(N_c - \frac{1}{N_c} \right) \rightarrow$$

$$\Rightarrow \text{Diagram with a bubble} = C_F \rightarrow$$

代数: $t^a t^a = C_F$, $C_F = \text{Tr} \left(N_c - \frac{1}{N_c} \right)$

示例3. 三胶子相互作用顶点

由对易关系 $[t^a, t^b] = i f^{abc} t^c$

可知 $i f^{abc} = \frac{1}{\text{Tr}} \text{Tr} [t^a, t^b] t^c$

同时, $[t^a, t^b]_+ = 2 \frac{\text{Tr}}{N_c} \delta^{ab} + d^{abc} t^c$

$$\Rightarrow d^{abc} = \frac{1}{\text{Tr}} \text{Tr} [t^a, t^b]_+ t^c$$

$$= \frac{1}{\text{Tr}} \text{Tr} \{t^a, t^b\} t^c$$

(d^{abc} 是实常数)

上面的生成元对易关系可用下图表示

$$i f^{abc} = \text{Diagram 1} - \text{Diagram 2}$$

将费米子线封闭起来, 并连接到另一个胶子上.

$$\text{Diagram 1} = \text{Diagram 2} - \text{Diagram 3}$$

$$\Rightarrow \text{Tr} = \text{Diagram 1} - \text{Diagram 2}$$

$$\Rightarrow \text{Diagram 1} = \frac{1}{\text{Tr}} [\text{Diagram 2} - \text{Diagram 3}]$$

注意: 上面仅仅是色空间因子的示意图
并非费曼图!!!

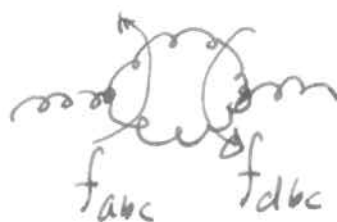
通过这种定义, 我们可以将三胶子相互作用顶点
从计标中去掉, 仅需计标 t^a 的迹.

同理可得: d^{abc}

$$\text{Diagram with a cross in a circle} = \frac{1}{\text{Tr}} \left[\text{Diagram 1} + \text{Diagram 2} \right]$$

注意: d^{abc} 对所有指标都对称

例 4:



$$= \frac{2}{\text{Tr}^2} \left[\text{Diagram 1} - \text{Diagram 2} \right]$$

$$= \frac{2}{\text{Tr}} \left[\text{Diagram 3} - \frac{1}{N_c} \text{Diagram 4} \right]$$

$$- \text{Diagram 5} + \frac{1}{N_c} \text{Diagram 6} \left] \right.$$

$$= \frac{2}{\text{Tr}} \left[\text{Diagram 7} - \text{Diagram 8} \right]$$

$$= 2 \left[\text{Diagram 9} - \frac{1}{N_c} \text{Diagram 10} - \text{Diagram 11} + \frac{1}{N_c} \text{Diagram 12} \right]$$

$$= 2 \text{Tr } N_c \text{Diagram 13}$$

此即  = C_A 

$$if^{acd}f^{bdc} = C_A \delta^{ab}$$

并且伴随表示中的 Casimir 标符是

$$C_A = 2T_R N_C = N_C$$

($T_R = 1/2$)

示例 5:

$$\begin{aligned} \text{Diagram 1} &= \frac{2}{T_R^2} \left[\text{Diagram 2} + \text{Diagram 3} \right] \\ &= \frac{2}{T_R} \left[\text{Diagram 4} - \frac{1}{N_C} \text{Diagram 5} \right. \\ &\quad \left. + \text{Diagram 6} - \frac{1}{N_C} \text{Diagram 7} \right] \\ &= \frac{2}{T_R} \left[\text{Diagram 8} + \text{Diagram 9} \right] - 4 \frac{T_R}{N_C} \text{Diagram 10} \\ &= 2 \left[\text{Diagram 11} - \frac{1}{N_C} \text{Diagram 12} + \text{Diagram 13} - \frac{1}{N_C} \text{Diagram 14} \right] \\ &\quad - 4 \frac{T_R}{N_C} \text{Diagram 10} \\ &= 2T_R \left(N_C - \frac{4}{N_C} \right) \text{Diagram 10} \end{aligned}$$

例6

$$\begin{aligned}
 \text{Diagram 1} &= T_R \left[\text{Diagram 2} - \frac{1}{N_c} \text{Diagram 3} \right] \\
 &= -\frac{T_R}{N_c} \text{Diagram 4}
 \end{aligned}$$

即: $t^a t^b t^c = -\frac{T_R}{N_c} t^b, \quad -\frac{T_R}{N_c} = C_F - \frac{C_A}{2}$

例7

$$\begin{aligned}
 \text{Diagram 1} &= \frac{1}{T_R} \left[\text{Diagram 2} - \text{Diagram 3} \right] \\
 &= \text{Diagram 4} - \frac{1}{N_c} \text{Diagram 5} - \text{Diagram 6} + \frac{1}{N_c} \text{Diagram 7} \\
 &= \text{Diagram 8} - \text{Diagram 9} \\
 &= T_R \left[\text{Diagram 10} - \frac{1}{N_c} \text{Diagram 11} - \text{Diagram 12} + \frac{1}{N_c} \text{Diagram 13} \right] \\
 &= T_R N_c \text{Diagram 14}
 \end{aligned}$$

即: $if^{abc} t^b t^a = \frac{C_A}{2} t^c$

另一种简单推导方法

$$\begin{aligned}
 \text{Diagram 1} &= \frac{1}{2} \left[\text{Diagram 2} - \text{Diagram 3} \right] \\
 &= \frac{1}{2} \text{Diagram 4} = \frac{C_A}{2} \text{Diagram 5}
 \end{aligned}$$

同样, 我们也会遇到如下图形.

$$\text{Diagram 6} = \frac{C_A}{2} \text{Diagram 7}$$

$$i f^{adf} i f^{bed} i f^{cfe} = \frac{C_A}{2} i f^{abc}$$