Homework: Gauge Field Theory #1

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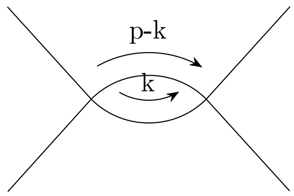
1. ϕ^4 theory ($\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$). Verify optical theorem in the lowest order.



$$= \frac{(-i\lambda)^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p-k)^2-m^2+i\epsilon}$$

For simplicity, we ignore the mass term.

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2(p-k)^2}$$

Apply feynman parameterization

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{[x(p-k)^2+(1-x)k^2]^2}$$

$k \to k + xp$

$$= \frac{(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{[k^2+x(1-x)p^2+i\epsilon]^2}$$

Set $\Delta \equiv -x(1-x)p^2 + i\epsilon$, and apply wick rotation

$$i\mathcal{M}_2 = \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2}$$

Dimensional regularization

$$\begin{aligned}
i\mathcal{M}_2 &= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{1}{[k_E^2+\Delta]^2} \\
&= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\mathrm{d}k_E\frac{k_E^{d-1}}{[k_E^2+\Delta]^2} \\
&= \frac{i(-i\lambda)^2}{2}\int_0^1 \mathrm{d}x \frac{\pi^{d/2}\Gamma(2-d/2)}{\Gamma(2)(2\pi)^d}\Delta^{d/2-2} \\
&\xrightarrow{d\to 4} -i\lambda^2\frac{\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)}{32\pi^2}\int_0^1 \mathrm{d}x(\frac{\Delta}{4\pi})^{-\epsilon/2} \\
&= -i\lambda^2\frac{\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)}{32\pi^2}\int_0^1 \mathrm{d}x(1-\frac{\epsilon}{2}\ln\frac{\Delta}{4\pi}) \\
&= \frac{-i\lambda^2}{32\pi^2}(\frac{2}{\epsilon}-\gamma+2-\ln\left(-p^2\right)+\ln(4\pi)+\mathcal{O}(\epsilon))
\end{aligned}$$

where $\epsilon = 4-d$.

So

$$i\mathcal{M}(s) = -i\lambda + \frac{-i\lambda^2}{32\pi^2}(\frac{2}{\epsilon}-\gamma+2-\ln(-s)+\ln(4\pi))$$

$$\mathcal{M}(s) = -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \gamma + 2 - \ln(-s) + \ln(4\pi)) = -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s) + finite\ terms)$$

where $finite\ terms = \ln(4\pi) + 2 - \gamma$.

$$\lambda_R = \lambda + \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms)$$

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms)$$

$$\begin{aligned}
\mathcal{M}(s) &= -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s) + finite\ terms) \\
&= -\lambda_R + \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0) + finite\ terms) - \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s) + finite\ terms) \\
&= -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{s}
\end{aligned}$$

As the lowest order, the results are always $-\lambda$.

Optical theorem concludes that

$$\frac{\lambda^2}{16\pi} = \int \mathrm{d}\Pi\lambda^2$$

where

$$\begin{aligned}
\int \mathrm{d}\Pi\lambda^2 &= \int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 4E_1E_2}(2\pi)^4\delta^4(p - p_1 - p_2)\lambda^2 \\
&= \frac{1}{16\pi}\lambda^2
\end{aligned}$$

2. Proca field, QED with massive photon. Calculate the leading order of $e^-e^- \to e^-e^-$.

The propagator

$$\langle 0|T\{A^\mu_{in}(x)A^\nu_{in}(y)\}|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i(-g^{\mu\nu} + \frac{k^\mu k^\nu}{\mu^2})}{k^2 - \mu^2 + i\epsilon} + \frac{i}{\mu^2}\delta^4(x-y)\delta^{\mu 0}\delta^{\nu 0}$$

The Lagarangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu + \bar{\psi}(i\not{D} - m)\psi$$

and the interaction part

$$\mathcal{L}_I = e\bar{\psi}\gamma^\mu\psi A_\mu$$

$(\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu + \bar{\psi}(i\not{\partial} - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu)$. The corresponding Hamiltonian is

$$\mathcal{H}_I = A^\mu J_\mu + \frac{1}{2\mu^2}J_0^2 = -e\bar{\psi}\gamma^\mu\psi A_\mu + \frac{e^2}{2\mu^2}\bar{\psi}\gamma^0\psi\bar{\psi}\gamma_0\psi$$

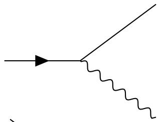
and we have the propagator

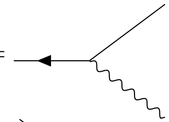
$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{i(-g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2})}{k^2 - \mu^2 + i\epsilon} + \frac{i}{\mu^2}\delta^4(x-y)\delta^0_\mu\delta^0_\nu$$

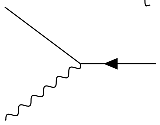
and

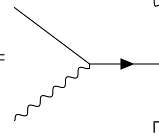
$$\langle k_1k_2|T\{-i\mathcal{H}_I\}|p_1p_2\rangle = i\langle k_1k_2|T\{e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{e^2}{2\mu^2}\bar{\psi}\gamma^0\psi\bar{\psi}\gamma_0\psi\}|p_1p_2\rangle$$

The feynman rules are pretty straight forward now:

$\overbrace{\psi\,|p,s\rangle}$ =  $= u_s(p)$

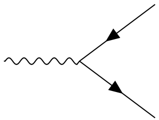
$\overbrace{\bar\psi\,|p,s\rangle}$ =  $= \bar v_s(p)$

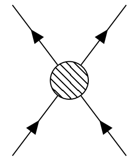
$\langle k,s|\,\overbrace{\psi}$ =  $= v_s(p)$

$\langle k,s|\,\overbrace{\bar\psi}$ =  $= \bar u_s(p)$

Photon propagator: $\overbrace{A^\mu A^\nu}$ = $= \frac{i(-g^{\mu\nu}+\frac{k^\mu k^\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}+\frac{i}{\mu^2}\delta^4(x-y)\delta^0_\mu\delta^0_\nu$

Fermion propagator: $\overbrace{\bar\psi(x)\psi(y)}$ =  $= \frac{i}{\not k-m+i\epsilon}$

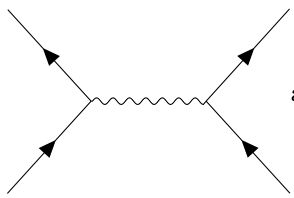
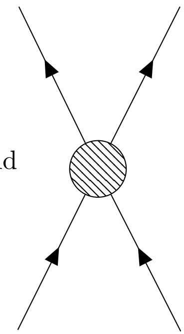
3-particle vertex:  $= -ie\gamma^\mu$

4-particle vertex:  $= i\frac{e^2}{2\mu^2}\gamma^0\times\gamma^0$

At tree level(to e^2 order), the first part must be

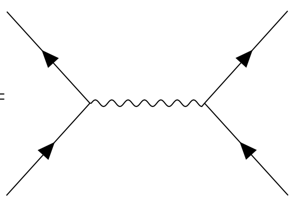
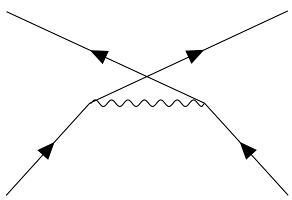
$$-e^2\,\langle k_1k_2|T\{\bar\psi\gamma^\mu\psi A_\mu\bar\psi\gamma^\nu\psi A_\nu\}|p_1p_2\rangle$$

so generally we have two diagrams

 and 

with some exchange in external legs.

The contribution of the first one is

$-\frac{1}{e^2}i\mathcal{M}_1 =$  + 

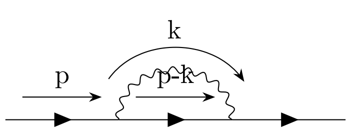
$$\begin{aligned}
=&\bar u(k_1)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}+\frac{i}{\mu^2}\delta^0_\mu\delta^0_\nu]\bar u(k_2)\gamma^\nu u(p_2)-\bar u(k_2)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}+\frac{i}{\mu^2}\delta^0_\mu\delta^0_\nu]\bar u(k_1)\gamma^\nu u(p_2)\\
=&\bar u(k_1)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}]\bar u(k_2)\gamma^\nu u(p_2)-\bar u(k_2)\gamma^\mu u(p_1)[\frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2+i\epsilon}]\bar u(k_1)\gamma^\nu u(p_2)\\
&+\frac{i}{\mu^2}\bar u(k_1)\gamma^0 u(p_1)\bar u(k_2)\gamma^0 u(p_2)-\frac{i}{\mu^2}\bar u(k_2)\gamma^0 u(p_1)\bar u(k_1)\gamma^0 u(p_2)
\end{aligned}$$

and the second one is

$$i\mathcal{M}_2 = \frac{ie^2}{\mu^2}(\bar u(k_1)\gamma^0 u(p_1)\bar u(k_2)\gamma^0 u(p_2)-\bar u(k_2)\gamma^0 u(p_1)\bar u(k_1)\gamma^0 u(p_2))$$

Combine these two and the incovariant terms are automatically canceled.

3. One loop self energy.



$= \frac{i}{\not p-m}(-i\Sigma_2(p))\frac{i}{\not p-m}$

where

$$-i\Sigma_2(p) = (-ie^2)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\gamma^\mu \frac{i}{\not{k}-m}\gamma^\nu \frac{i(-g_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2}$$

Devide $-i\Sigma_2(p)$ into two parts (first part is exactly the same as massless QED, just without infrared divergence), we write the second part as

$$\begin{aligned}
-i\Sigma_{2s}(p) &= -e^2\int \frac{\mathrm{d}^4k}{(2\pi)^4}\gamma^\mu \frac{i}{\not{p}-\not{k}-m}\gamma^\nu \frac{ik_\mu k_\nu}{\mu^2(k^2-\mu^2)} \\
&= \frac{e^2}{\mu^2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\not{k}^2}{(\not{p}-\not{k}-m)(k^2-\mu^2)} \\
&= \frac{e^2}{\mu^2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2}{(\not{p}-\not{k}-m)(k^2-\mu^2)}
\end{aligned}$$

Use Feynman parameters, the integral becomes

$$\begin{aligned}
\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2}{(\not{p}-\not{k}-m)(k^2-\mu^2)} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2(\not{p}-\not{k}+m)}{((p-k)^2-m^2)(k^2-\mu^2)} \\
&= \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2(\not{p}-\not{k}+m)}{[k^2-2xk\cdot p+xp^2-xm^2-(1-x)\mu^2+i\epsilon]^2} \\
&\underline{\underline{l\equiv k-xp}} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4l}{(2\pi)^4}\frac{(l+xp)^2(\not{p}-\not{l}-x\not{p}+m)}{[l^2-\Delta+i\epsilon]^2}
\end{aligned}$$

where $\Delta = -x(1-x)p^2+xm^2+(1-x)\mu^2$.

The numerator is (dropping terms with l to odd orders)

$$(l+xp)^2[(1-x)\not{p}+m-\not{l}] = l^2[(1-x)\not{p}+m]-2xl\cdot p\not{l}+x^2p^2[(1-x)\not{p}+m]$$

note that the second term becomes $-\frac{1}{2}x\not{p}l^2$ under Lorentz invariance, so eventually the numerator is

$$l^2[(1-\frac{3}{2})\not{p}+m]+x^2p^2[(1-x)\not{p}+m]$$

Put these together

$$-i\Sigma_{2s}(p) = \frac{ie^2}{\mu^2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^4l_E}{(2\pi)^4}\frac{-[(1-\frac{3}{2}x)\not{p}+m]l_E^2+x^2p^2[(1-x)\not{p}+m]}{[l_E^2+\Delta]^2}$$

Again we use dimensional regularization, the second term is given

$$\int \frac{\mathrm{d}^4l_E}{(2\pi)^4}\frac{1}{[l_E^2+\Delta]^2} = \frac{1}{(4\pi)^2}(\frac{2}{\epsilon}-\log\Delta-\gamma+\log(4\pi)+\mathcal{O}(\epsilon))$$

where $\epsilon = 4-d$, and the first term is $(d\to 4)$

$$\begin{aligned}
\int \frac{\mathrm{d}^dl_E}{(2\pi)^d}\frac{l_E^2}{[l_E^2+\Delta]^2} &= \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(1-d/2)}{\Gamma(2)}(\frac{1}{\Delta})^{1-d/2} \\
&= -\frac{1}{(4\pi)^{d/2}}\frac{d}{2}(\frac{2}{\epsilon}-\gamma+1+\mathcal{O}(\epsilon))(\frac{1}{\Delta})^{1-d/2} \\
&= -\frac{\Delta}{8\pi^2}(\frac{2}{\epsilon}-\log\Delta+\frac{1}{2}-\gamma+\log(4\pi))
\end{aligned}$$

Now we are able to collect the divergent terms:

$$\frac{i\alpha}{\pi\mu^2}\int_0^1 \mathrm{d}x[(1-\frac{3}{2}x)\not{p}+m]\frac{\Delta}{\epsilon}+x^2p^2[(1-x)\not{p}+m]\frac{1}{2\epsilon}$$

and there's also contribution from $g^{\mu\nu}$ term of the propagator

$$\frac{-i\alpha}{2\pi}\int_0^1 \mathrm{d}x(2\mu-x\not{p})(\frac{2}{\epsilon}-\log\Delta-\gamma+\log(4\pi)+\mathcal{O}(\epsilon)) \Longrightarrow \frac{-i\alpha}{\pi}\int_0^1 \mathrm{d}x(2\mu-x\not{p})\frac{1}{\epsilon}$$

The divergent type in this topic are logarithmic divergence and square divergence, but latter can't be seen in dimensional regularization.