Hydrogen

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1 Matching

QED Lagrangian is

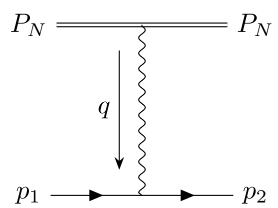
$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \tag{1}$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \tag{2}$$

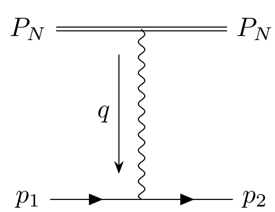
In tree level[1]

$i\mathcal{M}^{(0)}_{QED} =$



$$= -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \bar{u}_e(p_2)\gamma_0 u_e(p1)$$

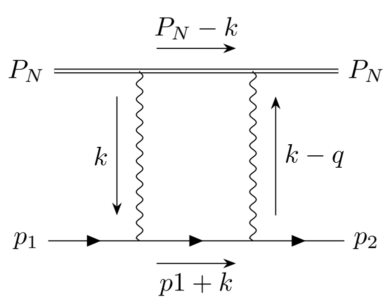
$i\mathcal{M}^{(0)}_{NRQED} =$



$$= -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \psi^\dagger(p_2)\psi(p1)$$

The box diagram for NRQED process is

$i\mathcal{M}^{(1)}_{NRQED} =$

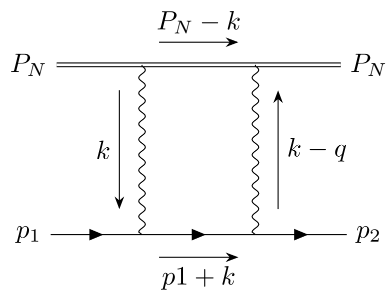


$$\begin{aligned}
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int [\mathrm{d}k] \frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(-k^0 + i\epsilon)(p_1^0 + k^0 - m - \frac{(\mathbf{p_1}+\mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(E_1 - \frac{(\mathbf{p_1}+\mathbf{k})^2}{2m})} \psi(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2(E_1 - \frac{\mathbf{k}^2}{2m})} \psi(p_1)
\end{aligned}$$

[1] Note that there's no Gamma matrice in the heavy particle side, they can only appear in the QED side.

The box and crossed box diagram for QED process is

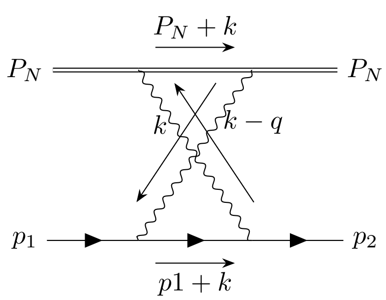
$$i\mathcal{M}_1^{(1)} =$$



$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.

$$i\mathcal{M}_2^{(1)} =$$



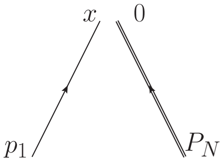
$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0-\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2+p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0-\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2+p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

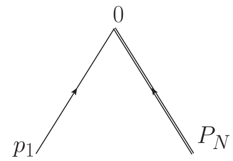
$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

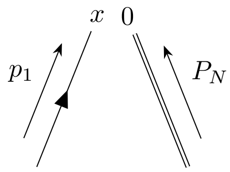
$$
\begin{aligned}
i\mathcal{M}_1^{(1)}+i\mathcal{M}_2^{(1)} &= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-{p_1^0}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2-\mathbf{p_1}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)
\end{aligned}
$$

Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order.

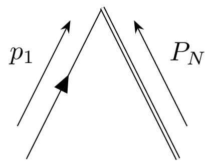
Now consider operator product expansion.

Tree level matching: 

& 

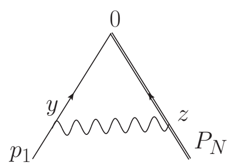
$\langle 0|T\psi(x)N(0)|pP_N\rangle =$ 

$= u_e(p)u_N(P_N)e^{-ip\cdot x}$

$\langle 0|T\psi_e(x)N(0)|pP_N\rangle =$ 

$= \psi_e(p)u_N(P_N)$

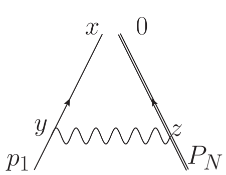
At leading order $u_e(p) = \begin{pmatrix}\psi_e(p)\\ 0\end{pmatrix}$. (If we're only interested in the hard region contribution, which is independent of states, the leading order is independent of any on-shell momentums.)

One loop scenario for NRQED case: 

$$\begin{aligned}\langle 0|\psi_e(0)N(0)e\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle &= e^2u_N(v_N)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\ &= -ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\ &= -ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1)\end{aligned}$$

drop p_1

$$= -ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1) = \pi ie^2\sqrt{\frac{2m}{E_1}}u_N(v_N)\psi(p_1)$$

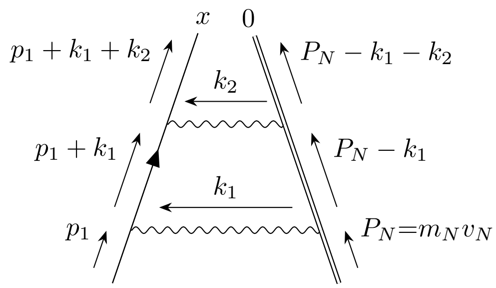
For QED case: 

$$\begin{aligned}\langle 0|\psi(x)N(0)e\int \mathrm{d}^4y\bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle\,{}^{[2]} &= e^2u_N(v_N)\int[\mathrm{d}k]e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\ &= e^2u_N(v_N)\int[\mathrm{d}k]e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\ &= ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\ &= ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)\end{aligned}$$

drop $\mathbf{p_1}$

$$= ie^2 u_N(v_N) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{p_1^0 + \sqrt{\mathbf{k}^2+m^2}}{2\mathbf{k}^2[\mathbf{k}^2 + m^2 - p_1^0\sqrt{\mathbf{k}^2+m^2}]} u_e(p_1)$$

Two loop scenario for QED case $\langle 0|T\psi(x)N(0)e\int \mathrm{d}^4 y_1 \bar{\psi}\gamma^0\psi A^0 e \int \mathrm{d}^4 z_1 \bar{N} N A^0 e \int \mathrm{d}^4 y_2 \bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4 z_2 \bar{N}NA^0|eN\rangle$:

$-$ 

$$=e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0-k_2^0+i\epsilon} \frac{1}{-k_1^0+i\epsilon} \frac{\not{p}_1 + \not{k}_1 + \not{k}_2 + m}{(p_1+k_1+k_2)^2 - m^2 + i\epsilon} \gamma^0 \frac{\not{p}_1+\not{k}_1+m}{(p_1+k_1)^2-m^2+i\epsilon} \gamma^0 u_N(v_N) u_e(p_1)$$

$$=e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}} \frac{1}{|\mathbf{k_1}|^2 |\mathbf{k_2}|^2} \frac{4{p_1^0}^2 + 2p_1^0 k_1^0 + 2\mathbf{p_1}\cdot\mathbf{k_1} + 2(p_1^0+k_1^0)(\not{k}_1+\not{k}_2)\gamma^0 - \not{k}_2\not{k}_1}{[(p_1+k_1+k_2)^2 - m^2 + i\epsilon][(p_1+k_1)^2 - m^2 + i\epsilon][-k_1^0-k_2^0+i\epsilon][-k_1^0+i\epsilon]} u_N(v_N) u_e(p_1)^3$$

$$=ie^4 \int [\mathrm{d}k_1] \frac{\mathrm{d}^3 \mathbf{k_2}}{(2\pi)^3} \frac{2(k_1^0+p_1^0)[(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2} - k_1^0 - p_1^0) + (k_2^i\gamma_i + \not{k}_1)\gamma^0] - [\gamma^0(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2} - k_1^0 - p_1^0) - k_2^i\gamma_i]\not{k}_1}{2\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2}(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2} - k_1^0 + \frac{2((\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2)+2\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2-p_1^0}}{2((\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2)} i\epsilon)}$$
$$\frac{1}{-k_1^0+i\epsilon} \frac{1}{(p_1+k_1)^2-m^2+i\epsilon} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

define $a = (\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2$ and $b = \sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2} - k_1^0 - p_1^0 + k_2^i\gamma_i\gamma^0 = \sqrt{a} - k_1^0 - p_1^0 + k_2^i\gamma_i\gamma^0$, and note that the long coefficient of the first ϵ above is positive

$$=ie^4 \int [\mathrm{d}k_1] \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2(k_1^0+p_1^0)[b+\not{k}_1\gamma^0] - \gamma^0 b\not{k}_1}{2\sqrt{a}(\sqrt{a}-k_1^0+i\epsilon)} \frac{1}{-k_1^0+i\epsilon} \frac{1}{(p_1+k_1)^2-m^2+i\epsilon} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

also define b' so that $b = b' - k_1^0$ ($b' = \sqrt{a} - p_1^0 + k_2^i\gamma_i\gamma^0$) and $a' = (\mathbf{p_1}+\mathbf{k_1})^2 + m^2$

$$=ie^4 \int [\mathrm{d}k_1] \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2(k_1^0+p_1^0)[b' + k_1^i\gamma_i\gamma^0] - \gamma^0(b'-k_1^0)(k_1^0\gamma^0 + k_1^i\gamma_i)}{2\sqrt{a}(\sqrt{a}-k_1^0+i\epsilon)[-k_1^0+i\epsilon]} \frac{1}{(p_1+k_1)^2-m^2+i\epsilon} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(b' + k_1^i\gamma_i\gamma^0) + \gamma^0(\sqrt{a'} - b' - p_1^0)(\sqrt{a'}\gamma^0 - p_1^0\gamma^0 + k_1^i\gamma_i)}{4\sqrt{a}\sqrt{a'}(\sqrt{a'}-p_1^0)(\sqrt{a}-\sqrt{a'}+p_1^0)} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

shift both loop momentum[4] so that $a = |\mathbf{k_2}|^2 + m^2$ and $a' = |\mathbf{k_1}|^2 + m^2$, now $b = \sqrt{a} - k_1^0 + (k_2-k_1)^i\gamma_i\gamma^0$ and $b' = \sqrt{a} + (k_2-k_1)^i\gamma_i\gamma^0$

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} + (k_2-p_1)^i\gamma_i\gamma^0) + (\sqrt{a'} - \sqrt{a} + (k_2-k_1)^i\gamma_i\gamma^0 - p_1^0)(\sqrt{a'} - p_1^0 - (k_1-p_1)^i\gamma_i\gamma^0)}{4\sqrt{a}\sqrt{a'}(\sqrt{a'}-p_1^0)(\sqrt{a}-\sqrt{a'}+p_1^0)|\mathbf{k_1}-\mathbf{p_1}|^2|\mathbf{k_2}-\mathbf{k_1}|^2} u_N(v_N) u_e(p_1) e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

drop $\mathbf{p_1}$

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} + k_2^i\gamma_i\gamma^0) + (\sqrt{a'} - \sqrt{a} + (k_2-k_1)^i\gamma_i\gamma^0 - p_1^0)(\sqrt{a'} - p_1^0 - k_1^i\gamma_i\gamma^0)}{4\sqrt{a}\sqrt{a'}(\sqrt{a'}-p_1^0)(\sqrt{a}-\sqrt{a'}+p_1^0)|\mathbf{k_1}|^2|\mathbf{k_2}-\mathbf{k_1}|^2} u_N(v_N) u_e(p_1) e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

rewrite it with $a_1 = a'$ and $a_2 = a$

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a_1}(\sqrt{a_2} + k_2^i\gamma_i\gamma^0) + (\sqrt{a_1} - \sqrt{a_2} + (k_2-k_1)^i\gamma_i\gamma^0 - p_1^0)(\sqrt{a_1} - p_1^0 - k_1^i\gamma_i\gamma^0)}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1}-p_1^0)(\sqrt{a_2}-\sqrt{a_1}+p_1^0)|\mathbf{k_1}|^2|\mathbf{k_2}-\mathbf{k_1}|^2} u_N(v_N) u_e(p_1) e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

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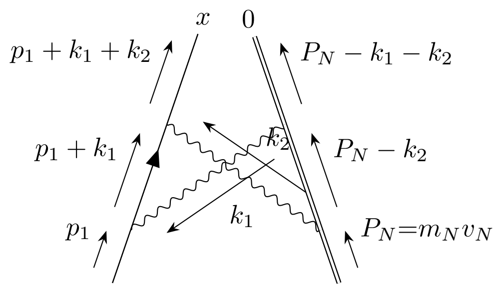
[2] $\langle 0|\psi(x)N(0)e\int \mathrm{d}^4 y \bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4 z \bar{N}NA^0|eN\rangle = e^2 \int \mathrm{d}^4 y \int \mathrm{d}^4 z \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{\mathbf{k^2}} e^{-ik\cdot(z-y)} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \tilde{S}_e(k_1) e^{-ik_1\cdot(y-x)} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \tilde{S}_N(k_2) u_N(v_N) u_e(p) e^{-ip_1\cdot y}$.

to investigate the divergent property of the integral, rewrite the integral before the shift ($a_1=(\mathbf{p_1}+\mathbf{k_1})^2+m^2$ and $a_2=(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2$)

$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a_1}(\sqrt{a_2}-p_1^0+(k_1+k_2)^i\gamma_i\gamma^0)+(\sqrt{a_1}-\sqrt{a_2}+k_2^i\gamma_i\gamma^0)(\sqrt{a_1}-p_1^0-k_1^i\gamma_i\gamma^0)}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1}-p_1^0)(\sqrt{a_2}-\sqrt{a_1}+p_1^0)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

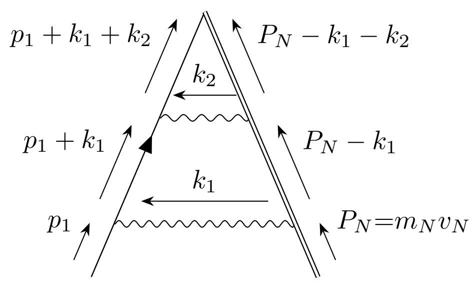
$-$ 

$$=e^4\int[dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_2^0+i\epsilon}\frac{\not p_1+\not k_1+\not k_2+m}{(p_1+k_1+k_2)^2-m^2+i\epsilon}\gamma^0\frac{\not p_1+\not k_1+m}{(p_1+k_1)^2-m^2+i\epsilon}\gamma^0u_N(v_N)u_e(p_1)$$

$$=e^4\int[dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}\frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2}\frac{4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(p_1^0+k_1^0)(\not k_1+\not k_2)\gamma^0-\not k_2\not k_1}{[(p_1+k_1+k_2)^2-m^2+i\epsilon][(p_1+k_1)^2-m^2+i\epsilon][-k_1^0-k_2^0+i\epsilon][-k_2^0+i\epsilon]}u_N(v_N)u_e(p_1)$$

$$=ie^4\int[\mathrm{d}k_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2(k_1^0+p_1^0)[b+\not k_1\gamma^0]-\gamma^0b\not k_1}{2\sqrt{a}(\sqrt{a}-k_1^0+i\epsilon)}\frac{1}{-\sqrt{a}+k_1^0+p_1^0+\frac{i\epsilon}{2\sqrt{a}}+i\epsilon}\frac{1}{(p_1+k_1)^2-m^2+i\epsilon}\frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2}u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

For NRQED case ($\langle 0|\psi_e(0)N(0)e\int\mathrm{d}^4y_1\bar\psi_e\psi_eA^0e\int\mathrm{d}^4z_1\bar NNA^0e\int\mathrm{d}^4y_2\bar\psi_e\psi_eA^0e\int\mathrm{d}^4z_2\bar NNA^0|eN\rangle$)

$-$ 

$$=e^4\int[dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_2^0+i\epsilon}\frac{1}{p_1^0+k_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{1}{p_1^0+k_1^0+k_2^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\psi_e(p_1)u_N(v_N)$$

2 HSET

2.1 Lagrangian

For scalar QED

$$\mathcal{L}=(D_\mu\phi)^\dagger D^\mu\phi-m^2\phi^\dagger\phi$$

where

$$D_\mu=\partial_\mu+ieA_\mu$$

In Schwartz's QFT (Chap. 35) he mentioned a choice of χ_v and $\tilde\chi_v$:

$$\phi(x)=e^{-imv\cdot x}\frac{1}{\sqrt{2m}}(\chi_v(x)+\tilde\chi_v(x))\tag{3}$$

[3] With a $u_e(p_1)$ on the right hand side, $(\not p_1+\not k_1+\not k_2+m)\gamma^0(\not p_1+\not k_1+m)\gamma^0=(\not p_1+\not k_1+\not k_2+m)[2(p_1^0+k_1^0)\gamma^0-(\not p_1+\not k_1-m)]=2(p_1^0+k_1^0)(\not p_1+\not k_1+\not k_2+m)\gamma^0-(\not p_1+\not k_1+\not k_2+m)(\not p_1+\not k_1-m)=2(p_1^0+k_1^0)(\not p_1+\not k_1+\not k_2+m)\gamma^0-(\not p_1+\not k_1+\not k_2+m)\not k_1=2(p_1^0+k_1^0)(\not p_1+\not k_1+\not k_2+m)\gamma^0-(2p_1\cdot k_1-\not k_1\not p_1+\not k_2\not k_1+\not k_1m)=2(p_1^0+k_1^0)(\not p_1+\not k_1+\not k_2+m)\gamma^0-(2p_1\cdot k_1+\not k_2\not k_1)=2(p_1^0+k_1^0)[2(p_1^0+k_1^0+k_2^0)-\gamma^0(\not p_1+\not k_1+\not k_2-m)]-(2p_1\cdot k_1+\not k_2\not k_1)=2(p_1^0+k_1^0)[2p_1^0+(\not k_1+\not k_2)\gamma^0]-(2p_1\cdot k_1+\not k_2\not k_1)=4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(p_1^0+k_1^0)(\not k_1+\not k_2)\gamma^0-\not k_2\not k_1$.

[4] $k_1\to k_1'=k_1+p_1$ and $k_2\to k_2'=k_1+k_2+p_1=k_1'+k_2$.

$$\chi_v(x) = e^{imv\cdot x}\frac{1}{\sqrt{2m}}(iv\cdot D+m)\phi(x),\ \tilde{\chi}_v(x) = e^{imv\cdot x}\frac{1}{\sqrt{2m}}(-iv\cdot D+m)\phi(x) \tag{4}$$

Put (3) into (4), a simple relation is derived:

$$(-iv\cdot D)\chi_v(x) = (2m+iv\cdot D)\tilde{\chi}_v(x)$$

It can also be writen as

$$2m\tilde{\chi}_v = (-iv\cdot D)(\chi_v+\tilde{\chi}_v)$$

Use this result

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2m}\Big\{\{[D^\mu(\chi_v+\tilde{\chi}_v)]^\dagger + imv^\mu(\chi_v+\tilde{\chi}_v)^\dagger\}\{[D_\mu(\chi_v+\tilde{\chi}_v)] - imv_\mu(\chi_v+\tilde{\chi}_v)\} - m^2(\chi_v+\tilde{\chi}_v)^\dagger(\chi_v+\tilde{\chi}_v)\Big\} \\
&= (\chi_v+\tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v+\tilde{\chi}_v) + \frac{1}{2m}[D^\mu(\chi_v+\tilde{\chi}_v)]^\dagger D_\mu(\chi_v+\tilde{\chi}_v) \qquad (5)\\
&= (\chi_v(x)+\tilde{\chi}_v(x))^\dagger(iv\cdot D)(\chi_v(x)+\tilde{\chi}_v(x)) + \mathcal{O}(\frac{1}{m}) \qquad (6)
\end{aligned}$$

(note that $D_\mu\phi = e^{-imv\cdot x}[D_\mu(\chi_v+\tilde{\chi}_v) - imv_\mu(\chi_v+\tilde{\chi}_v)]$ and $-imv^\mu[D_\mu(\chi_v+\tilde{\chi}_v)]^\dagger(\chi_v+\tilde{\chi}_v) = imv^\mu(\chi_v+\tilde{\chi}_v)^\dagger D_\mu(\chi_v+\tilde{\chi}_v) -$ *total derivative term*)

Use the leading order of (5)

$$\begin{aligned}
\mathcal{L}^{(0)} &= (\chi_v+\tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v+\tilde{\chi}_v) \\
&= \chi_v^\dagger iv\cdot D\chi_v + \tilde{\chi}_v^\dagger iv\cdot D(\chi_v+\tilde{\chi}_v) + \chi_v^\dagger iv\cdot D\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + (iv\cdot D\chi_v)^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + [(-2m-iv\cdot D)\tilde{\chi}_v]^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D+4m)\tilde{\chi}_v
\end{aligned}$$

We can have the final form[5]

$$\mathcal{L} = \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D+4m)\tilde{\chi}_v + \mathcal{O}(\frac{1}{m})$$

2.2 Quantization

2.2.1 HQET as an example

The leading term of HQET Lagrangian is

$$\mathcal{L} = \bar{Q}_v(iv\cdot D)Q_v$$

$$Q_v(x) = e^{imv\cdot x}\frac{1+\not{v}}{2}\psi(x)$$

In free Dirac fermion theory we know that

$$\begin{aligned}
\Big\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\Big\} &= \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab} \\
\{a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger\} &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')
\end{aligned}$$

also the plane wave expansion of ψ is

$$\begin{aligned}
\psi(x) &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_{\mathbf{p}}u(p)e^{-ip\cdot x} \\
&= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2mv^0}}a_v\sqrt{m}u(v)e^{-imv\cdot x-ik\cdot x}
\end{aligned}$$

[5]With one problem: if we can tolerate coupled particle-anti particle pair, we can trade $iv\cdot D$ for mass term, so the leading part is the same but the anti-particle part could be different with the mixing?

using normalization of states $u(k) = \sqrt{m}u(v)$[6], $\langle p'|p\rangle = 2E_p(2\pi)^3\delta^{(3)}(\mathbf{p'}-\mathbf{p})$ and $\langle v',k'|v,k\rangle = 2v^0\delta_{vv'}(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})$ we have $|p\rangle = \sqrt{m}\,|v\rangle$ ($|p\rangle = \sqrt{2E_p}a^\dagger_{\mathbf{p}}\,|0\rangle$ while $|v,k\rangle = \sqrt{2v^0}a^\dagger_{v,\mathbf{k}}\,|0\rangle$)

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v u(v)e^{-imv\cdot x - ik\cdot x}$$

Using the definition of $Q_v(x)$

$$\begin{aligned}
Q_v(x) &= e^{imv\cdot x}\frac{1+\not{v}}{2}\psi(x) \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v\frac{1+\not{v}}{2}u(v)e^{-ik\cdot x} \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v u(v)e^{-ik\cdot x}
\end{aligned}$$

The commutation relation should be

$$\{Q_{va}(\mathbf{x}), Q_{v'b}(\mathbf{x'})\} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0v'^0}}\{a_v, a^\dagger_{v'}\}u_a(v)u^\dagger_b(v')e^{-ik\cdot x + ik'\cdot x'}$$

using $\sum_s u_a(v)u^\dagger_b(v) = \frac{1}{m}\sum_s u_a(p)u^\dagger_b(p) = [(\not{v}+1)\gamma^0]_{ab}$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0v'^0}}\{a_v, a^\dagger_{v'}\}[(\not{v}+1)\gamma^0]_{ab}e^{-ik\cdot x + ik'\cdot x'}$$

assuming $\{a_v, a_{v'}\} = (2\pi)^3\delta_{vv'}\delta^{(3)}(\mathbf{k}-\mathbf{k'})$

$$\begin{aligned}
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2v^0}[(\not{v}+1)\gamma^0]_{ab}e^{-ik\cdot(x-x')}\delta_{vv'} \\
&= [\frac{(\not{v}+1)\gamma^0}{2v^0}]_{ab}\delta_{vv'}\delta^{(3)}(\mathbf{x}-\mathbf{x'})
\end{aligned}$$

2.2.2 HSET

The anti-particle field is decoupled so we don't have to consider that for now. The equation-of-motion is

$$\begin{cases} v\cdot D\chi_v^\dagger = 0 \\ v\cdot D\chi_v = 0 \end{cases}$$

By definition

$$\begin{aligned}
\chi_v(x) &= \frac{e^{imv\cdot x}}{\sqrt{2m}}(iv\cdot D + m)\phi(x) \\
&= \frac{1}{\sqrt{2m}}(iv\cdot D + 2m)e^{imv\cdot x}\phi(x)
\end{aligned}$$

[6] The relation $\bar{u}^s(p)\gamma^\mu u^s(p) = 2p^\mu$ can be derived using Gordon identity, same for $\bar{u}^s(v)\gamma^\mu u^s(v) = 2v^\mu$, but it's actually $\bar{u}u$.