

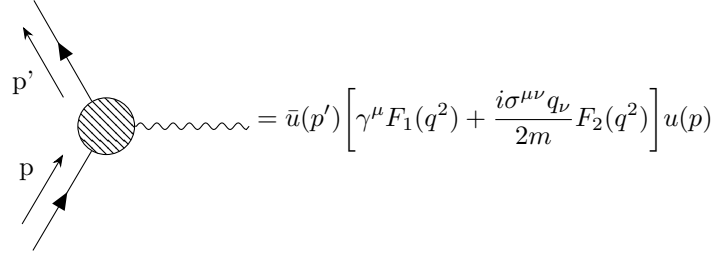
# Homework: Quantum Field Theory #9

Yingsheng Huang

February 19, 2017

## 6.1. Rosenbluth Formula.

The QED vertex:



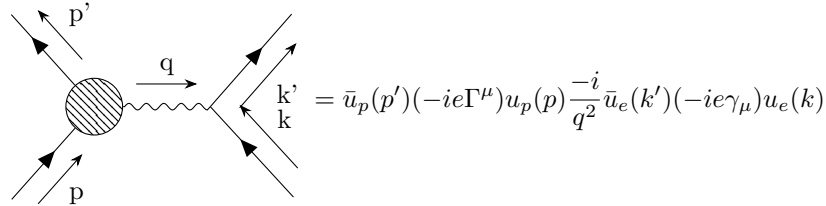
$$= \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p)$$

where  $q = p' - p$  and  $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ . Consider the fermions strongly interacting ones (such as protons), the form factor reflects the structure of strong interaction and therefore can only be determined by experiment. Consider a electron ( $E \ll m_e$ ) scattering from a proton initially at rest. Show that this leads to the Rosenbluth formula

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 \left[ (F_1^2 - \frac{q^2}{4m^2} F_2^2) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right]}{2E^2 \left[ 1 + \frac{2E}{m} \sin^2 \frac{\theta}{2} \right] \sin^4 \frac{\theta}{2}}$$

for elastic scattering cross section, computed to leading order in  $\alpha$ .

The whole diagram



$$= \bar{u}_p(p') (-ie\Gamma^\mu) u_p(p) \frac{-i}{q^2} \bar{u}_e(k') (-ie\gamma_\mu) u_e(k)$$

Use spin sum

$$\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \sum_{spins} \{ \bar{u}_p(p') \Gamma^\mu u_p(p) \bar{u}_e(k') \gamma_\mu u_e(k) [\bar{u}_p(p') \Gamma^\nu u_p(p) \bar{u}_e(k') \gamma_\nu u_e(k)]^\dagger \}$$

while

$$\begin{aligned} \sum_{spins} \bar{u}_e(k') \gamma_\mu u_e(k) [\bar{u}_e(k') \gamma_\nu u_e(k)]^\dagger &= \sum_{spins} \bar{u}_e(k') \gamma_\mu \bar{u}_e(k) [u_e(k) \gamma_\nu u_e(k')] \\ &= \sum_{spins} \text{tr} \{ u_e(k') \bar{u}_e(k') \gamma_\mu u_e(k) \bar{u}_e(k) \gamma_\nu \} \\ &= \text{tr} \{ (\not{k}' + m_e) \gamma_\mu (\not{k} + m_e) \gamma_\nu \} \end{aligned}$$

and

$$\begin{aligned}
& \sum_{spins} \bar{u}_p(p') \Gamma^\mu u_p(p) \bar{u}_p(p') \Gamma^\nu u_p(p) \\
&= \sum_{spins} \bar{u}_p(p') \left\{ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\rho} q_\rho}{2m} F_2(q^2) \right\} u_p(p) \bar{u}_p(p) \left\{ \gamma^\nu F_1(q^2) - \frac{i\sigma^{\nu\sigma} q_\sigma}{2m} F_2(q^2) \right\} u_p(p') \\
&= \text{tr} \{ (\not{p}' + m) \gamma^\mu (\not{p} + m) \gamma^\nu \} F_1^2 + (\text{tr} \{ (\not{p}' + m) \sigma^{\mu\rho} (\not{p} + m) \gamma^\nu \} - \text{tr} \{ (\not{p}' + m) \gamma^\mu (\not{p} + m) \sigma^{\nu\rho} \}) \frac{i q_\rho}{2m} F_1 F_2 \\
&\quad + \text{tr} \left\{ (\not{p}' + m) \frac{\sigma^{\mu\rho} q_\rho}{2m} (\not{p} + m) \frac{\sigma^{\nu\sigma} q_\sigma}{2m} \right\} F_2^2
\end{aligned}$$

(Still need to prove  $[\bar{u}(p') \sigma^{\mu\nu} u(p)]^\dagger = \bar{u}(p) \sigma^{\mu\nu} u(p')$ , which only requires  $[\bar{u}(p') \gamma^\mu \gamma^\nu u(p)]^\dagger = \bar{u}(p) \gamma^\mu \gamma^\nu u(p')$ , so

$$[\bar{u}(p') \gamma^\mu \gamma^\nu u(p)]^\dagger = (u(p))^\dagger \gamma^{\nu\dagger} \gamma^{\mu\dagger} \gamma^{0\dagger} u(p') = (u(p))^\dagger \gamma^\nu \gamma^\mu u(p') = (u(p))^\dagger \gamma^0 \gamma^\nu \gamma^\mu u(p') = \bar{u}(p) \gamma^\nu \gamma^\mu u(p')$$

note that there's a extra minus sign.)

Now we calculate the traces term by term.

$$\begin{aligned}
& \text{tr} \{ (\not{k}' + m_e) \gamma_\mu (\not{k} + m_e) \gamma_\nu \} = 4(k'_\mu k_\nu + k_\mu k'_\nu - k' \cdot k g_{\mu\nu}) + 4m_e^2 g_{\mu\nu} \\
& m_e \text{ is small comparing to } m \text{ so we ignore it: } = 4(k'_\mu k_\nu + k_\mu k'_\nu - k' \cdot k g_{\mu\nu})
\end{aligned}$$

$$\begin{aligned}
& \text{tr} \{ (\not{p}' + m) \gamma^\mu (\not{p} + m) \gamma^\nu \} = 4(p'^\mu p^\nu + p^\mu p'^\nu - p' \cdot p g^{\mu\nu}) + 4m^2 g^{\mu\nu} \\
& \text{tr} \{ (\not{p}' + m) \gamma^\mu \gamma^\rho (\not{p} + m) \gamma^\nu \} = 4m(p'^\mu g^{\rho\nu} + p'^\nu g^{\rho\mu} - p'^\rho g^{\mu\nu}) + 4m(p^\nu g^{\mu\rho} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu})
\end{aligned}$$

$$\begin{aligned}
& \text{tr} \{ (\not{p}' + m) \sigma^{\mu\rho} (\not{p} + m) \gamma^\nu \} = \frac{i}{2} 4m \{ [p'^\mu g^{\rho\nu} + p'^\nu g^{\rho\mu} - p'^\rho g^{\mu\nu} + p^\nu g^{\mu\rho} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu}] \\
& \quad - [p'^\rho g^{\mu\nu} + p'^\nu g^{\mu\rho} - p'^\mu g^{\rho\nu} + p^\nu g^{\rho\mu} + p^\mu g^{\rho\nu} - p^\rho g^{\mu\nu}] \} \\
& = 4m \frac{i}{2} \{ 2p'^\mu g^{\rho\nu} - 2p'^\rho g^{\mu\nu} + 2p^\rho g^{\mu\nu} - 2p^\mu g^{\rho\nu} \} \\
& = 4mi \{ p'^\mu g^{\rho\nu} - p'^\rho g^{\mu\nu} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu} \}
\end{aligned}$$

$$\begin{aligned}
& \text{tr} \{ (\not{p}' + m) \sigma^{\mu\rho} (\not{p} + m) \gamma^\nu \} - \text{tr} \{ (\not{p}' + m) \gamma^\mu (\not{p} + m) \sigma^{\nu\rho} \} \\
&= \text{tr} \{ (\not{p}' + m) \sigma^{\mu\rho} (\not{p} + m) \gamma^\nu \} - \text{tr} \{ (\not{p} + m) \sigma^{\nu\rho} (\not{p}' + m) \gamma^\mu \} \\
&= 4mi \{ [p'^\mu g^{\rho\nu} - p'^\rho g^{\mu\nu} + p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu}] - [p^\nu g^{\rho\mu} - p^\rho g^{\mu\nu} + p'^\rho g^{\mu\nu} - p'^\nu g^{\rho\mu}] \} \\
&= 4mi \{ p'^\mu g^{\rho\nu} - 2p'^\rho g^{\mu\nu} + 2p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu} - p^\nu g^{\rho\mu} + p'^\nu g^{\rho\mu} \}
\end{aligned}$$

$$\begin{aligned}
& \text{tr} \left\{ (\not{p}' + m) \frac{\sigma^{\mu\rho} q_\rho}{2m} (\not{p} + m) \frac{\sigma^{\nu\sigma} q_\sigma}{2m} \right\} = -4(p'^\mu p^\nu g^{\rho\sigma} - p'^\mu p^\sigma g^{\rho\nu} + p'^\rho p^\sigma g^{\mu\nu} - p'^\rho p^\nu g^{\mu\sigma} + p' \cdot p g^{\mu\sigma} g^{\rho\nu} - p' \cdot p g^{\mu\nu} g^{\rho\sigma} + p'^\nu p^\mu g^{\rho\sigma} \\
& \quad - p'^\nu p^\rho g^{\mu\sigma} + p'^\sigma p^\rho g^{\mu\nu} - p'^\sigma p^\mu g^{\rho\nu}) \frac{q_\rho}{2m} \frac{q_\sigma}{2m} + 4m^2 (g^{\mu\nu} g^{\rho\sigma} - g^{\rho\nu} g^{\mu\sigma}) \frac{q_\rho}{2m} \frac{q_\sigma}{2m} \\
&= -\frac{1}{m^2} (p'^\mu p^\nu q^2 - p'^\mu (p \cdot q) q^\nu + (p' \cdot q) (p \cdot q) g^{\mu\nu} - (p' \cdot q) p^\nu q^\mu + (p' \cdot p) q^\mu q^\nu - (p' \cdot p) g^{\mu\nu} q^2 \\
& \quad + p'^\nu p^\mu q^2 - p'^\nu (p \cdot q) q^\mu + (p' \cdot q) (p \cdot q) g^{\mu\nu} - (p' \cdot q) p^\mu q^\nu) + (g^{\mu\nu} q^2 - q^\nu q^\mu) \\
&= -\frac{1}{m^2} (p'^\mu p^\nu q^2 - p'^\mu (p \cdot q) q^\nu + 2(p' \cdot q) (p \cdot q) g^{\mu\nu} - (p' \cdot q) p^\nu q^\mu + (p' \cdot p) q^\mu q^\nu - (p' \cdot p) g^{\mu\nu} q^2 \\
& \quad + p'^\nu p^\mu q^2 - p'^\nu (p \cdot q) q^\mu - (p' \cdot q) p^\mu q^\nu) + (g^{\mu\nu} q^2 - q^\nu q^\mu)
\end{aligned}$$

(Trace of six gamma matrices: (the Latin letters are just for convenience)

$$\begin{aligned}
& \text{tr} \{ \gamma^a \gamma^b \gamma^c \gamma^d \gamma^e \gamma^f \} = 2g^{ab} \text{tr} \{ \gamma^c \gamma^d \gamma^e \gamma^f \} - 2g^{ac} \text{tr} \{ \gamma^b \gamma^d \gamma^e \gamma^f \} + 2g^{ad} \text{tr} \{ \gamma^b \gamma^c \gamma^e \gamma^f \} \\
& \quad - 2g^{ae} \text{tr} \{ \gamma^b \gamma^c \gamma^d \gamma^f \} + 2g^{af} \text{tr} \{ \gamma^b \gamma^c \gamma^d \gamma^e \} - \text{tr} \{ \gamma^a \gamma^b \gamma^c \gamma^d \gamma^e \gamma^f \} \\
&= 4 \{ g^{ab} (g^{cd} g^{ef} - g^{ce} g^{df} + g^{cf} g^{de}) - g^{ac} (g^{bd} g^{ef} - g^{be} g^{df} + g^{bf} g^{de}) + g^{ad} (g^{bc} g^{ef} - g^{be} g^{cf} + g^{bf} g^{ce}) \\
& \quad - g^{ae} (g^{bc} g^{df} - g^{bd} g^{cf} + g^{bf} g^{cd}) + g^{af} (g^{bc} g^{de} - g^{bd} g^{ce} + g^{be} g^{cd}) \}
\end{aligned}$$

and what we need is  $a$  and  $d$  to contract with momentum and  $b, c$  &  $e, f$  reversed and cancelled, so

$$\begin{aligned}
\frac{1}{4} \text{tr}\{\not{p}'[\gamma^b, \gamma^c]\not{p}[\gamma^e, \gamma^f]\} &= p'_a p_d \{ [g^{ab}(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}) - g^{ac}(g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}) + g^{ad}(g^{bc}g^{ef} - g^{be}g^{cf} + g^{bf}g^{ce}) \\
&\quad - g^{ae}(g^{bc}g^{df} - g^{bd}g^{cf} + g^{bf}g^{cd}) + g^{af}(g^{bc}g^{de} - g^{bd}g^{ce} + g^{be}g^{cd})] \\
&\quad + [-g^{ac}(g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}) + g^{ab}(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}) - g^{ad}(g^{cb}g^{ef} - g^{ce}g^{bf} + g^{cf}g^{be}) \\
&\quad + g^{ae}(g^{cb}g^{df} - g^{cd}g^{bf} + g^{cf}g^{bd}) - g^{af}(g^{cb}g^{de} - g^{cd}g^{be} + g^{ce}g^{bd})] \\
&\quad + [-g^{ab}(g^{cd}g^{fe} - g^{cf}g^{de} + g^{ce}g^{df}) + g^{ac}(g^{bd}g^{fe} - g^{bf}g^{de} + g^{be}g^{df}) - g^{ad}(g^{bc}g^{fe} - g^{bf}g^{ce} + g^{be}g^{cf}) \\
&\quad + g^{af}(g^{bc}g^{de} - g^{bd}g^{ce} + g^{be}g^{cd}) - g^{ae}(g^{bc}g^{df} - g^{bd}g^{cf} + g^{bf}g^{cd})] \\
&\quad + [g^{ac}(g^{bd}g^{fe} - g^{bf}g^{de} + g^{be}g^{df}) - g^{ab}(g^{cd}g^{fe} - g^{cf}g^{de} + g^{ce}g^{df}) + g^{ad}(g^{cb}g^{fe} - g^{cf}g^{be} + g^{ce}g^{bf}) \\
&\quad - g^{af}(g^{cb}g^{de} - g^{cd}g^{be} + g^{ce}g^{bd}) + g^{ae}(g^{cb}g^{df} - g^{cd}g^{bf} + g^{cf}g^{bd})] \} \\
&= p'_a p_d \{ 4g^{ab}g^{cf}g^{de} - 4g^{ab}g^{ce}g^{df} + 4g^{ac}g^{be}g^{df} - 4g^{ac}g^{bf}g^{de} + 4g^{ad}g^{bf}g^{ce} - 4g^{ad}g^{be}g^{cf} \\
&\quad + 4g^{ae}g^{bd}g^{cf} - 4g^{ae}g^{bf}g^{cd} + 4g^{af}g^{be}g^{cd} - 4g^{af}g^{bd}g^{ce} \} \\
&= 4(p'^b p^e g^{cf} - p'^b p^f g^{ce} + p'^c p^f g^{be} - p'^c p^e g^{bf} + p' \cdot p g^{bf} g^{ce} - p' \cdot p g^{be} g^{cf} + p'^e p^b g^{cf} \\
&\quad - p'^e p^c g^{bf} + p'^f p^c g^{be} - p'^f p^b g^{ce})
\end{aligned}$$

Trace of two sigma bilinear:

$$\begin{aligned}
\text{tr}\{\sigma^{\mu\rho}\sigma^{\nu\sigma}\} &= -\frac{1}{4}\{\text{tr}\{\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma\} - \text{tr}\{\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma\} - \text{tr}\{\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu\} + \text{tr}\{\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\}\} \\
&= -\{(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu}) - (g^{\mu\rho}g^{\nu\sigma} - g^{\rho\nu}g^{\mu\sigma} + g^{\rho\sigma}g^{\mu\nu}) - (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\rho\nu} + g^{\mu\nu}g^{\rho\sigma}) \\
&\quad + (g^{\mu\rho}g^{\nu\sigma} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma})\} \\
&= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\rho\nu}g^{\mu\sigma})
\end{aligned}$$

)

The invariant amplitude

$$\begin{aligned}
\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 &= \frac{e^4}{q^4} (k'_\mu k_\nu + k_\mu k'_\nu - k' \cdot k g_{\mu\nu}) \left\{ (p'^\mu p^\nu + p^\mu p'^\nu - p' \cdot p g^{\mu\nu} + m^2 g^{\mu\nu}) F_1^2 \right. \\
&\quad - m(p'^\mu g^{\rho\nu} - 2p'^\rho g^{\mu\nu} + 2p^\rho g^{\mu\nu} - p^\mu g^{\rho\nu} - p^\nu g^{\rho\mu} + p'^\nu g^{\rho\mu}) \frac{q_\rho}{2m} F_1 F_2 \\
&\quad + [-\frac{1}{m^2} (p'^\mu p^\nu q^2 - p'^\mu (p \cdot q) q^\nu + 2(p' \cdot q)(p \cdot q) g^{\mu\nu} - (p' \cdot q) p^\nu q^\mu + (p' \cdot p) q^\mu q^\nu - (p' \cdot p) g^{\mu\nu} q^2 \\
&\quad + p'^\nu p^\mu q^2 - p'^\nu (p \cdot q) q^\mu - (p' \cdot q) p^\mu q^\nu) + (g^{\mu\nu} q^2 - q^\nu q^\mu)] F_2^2 \} \\
&= \frac{e^4}{q^4} \left\{ [(k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') - (p' \cdot p)(k' \cdot k) + m^2(k' \cdot k) + (k \cdot p')(k' \cdot p) + (k \cdot p)(k' \cdot p') - (p \cdot p')(k \cdot k')] \right. \\
&\quad + m^2(k \cdot k') - 2(k \cdot k')(p \cdot p') + 4(k \cdot k')(p \cdot p') - 4m^2(k \cdot k')] F_1^2 - \} \\
&= \frac{e^4}{q^4} \left\{ [2(k' \cdot p')(k \cdot p) + 2(k' \cdot p)(k \cdot p') - 2m^2(k' \cdot k)] F_1^2 - \right\}
\end{aligned}$$