Chiral basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma^\mu_{\alpha\dot{\beta}} = (1, \vec{\sigma})_{\alpha\dot{\beta}}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (1, -\vec{\sigma})^{\dot{\alpha}\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\beta\alpha} \sigma^\mu_{\alpha\dot{\beta}}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \varepsilon^{12} &= +1 \\ \varepsilon_{12} &= -1 \end{aligned}}$$

☆ Spinor-helicity

Dixon: 1310.5353

Elvang & Huang: 1308.1697

massless fermion $\quad \not{p}\, u(p) = 0 = \not{p}\, v(p)$

can take $u(p) = v(p)$

$$\bar{u}(p)\not{p} = 0 = \bar{v}(p)\not{p}$$

$$\bar{u}(p) = \bar{v}(p)$$

$$\frac{1-\gamma_5}{2} u(p) \equiv \begin{pmatrix} |p]_\alpha \\ 0 \end{pmatrix} \equiv |p]$$

$$\frac{1+\gamma_5}{2} u(p) \equiv \begin{pmatrix} 0 \\ \underset{\sim}{|p\rangle^{\dot\alpha}} \end{pmatrix} \equiv |p\rangle$$

$$\bar{u}(p)\frac{1+\gamma_5}{2} \equiv \begin{pmatrix} 0 & \langle p|_{\dot\alpha} \end{pmatrix} \equiv \langle p|$$

$$\bar{u}(p)\frac{1-\gamma_5}{2} \equiv \begin{pmatrix} [p|^{\alpha} & 0 \end{pmatrix} \equiv [p|$$

$$\not{p} = p^\mu \gamma_\mu = \begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar\sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \underset{\sim}{p_{\alpha\dot\beta}} \\ p^{\dot\alpha\beta} & 0 \end{pmatrix}$$

$$p_\mu \sigma^\mu_{\alpha\dot\beta} \equiv p_{\alpha\dot\beta} \quad , \quad p^\mu \bar\sigma_\mu^{\dot\alpha\beta} \equiv \underset{\sim}{p^{\dot\alpha\beta}} = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix}$$

$$\det(p^{\dot\alpha\beta}) = p^2 = 0$$

Dirac eq. :

$$p_{\alpha\dot\beta}|p\rangle^{\dot\beta} = 0$$

$$p^{\dot\alpha\beta}|p]_\beta = 0$$

$$\langle p|_{\dot\alpha}\, p^{\dot\alpha\beta} = 0$$

$$[p|^{\alpha}\, p_{\alpha\dot\beta} = 0$$

7月16日

Spinor product

$$|p_i\rangle \equiv |i\rangle \quad , \quad |p_i] \equiv |i]$$

$$\langle i\, j\rangle \equiv \langle i|_{\dot{\alpha}}\, |j\rangle^{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\, |i\rangle^{\dot{\beta}}\, |j\rangle^{\dot{\alpha}} = -\langle j\, i\rangle$$

$$[i\, j] \equiv [i|^{\alpha}\, |j]_{\alpha} = -[j\, i]$$

$$\langle i|\gamma^{\mu}|j] = \langle i|_{\dot{\alpha}} (\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\, |j]_{\beta}$$

$$[i|\gamma^{\mu}|j\rangle = [i|^{\alpha}\, \sigma^{\mu}_{\alpha\dot{\beta}}\, |j\rangle^{\dot{\beta}}$$

* Momentum

$$\not{p} = \begin{pmatrix} 0 & \overbrace{p_{\alpha\dot\beta}}^{p\cdot\sigma} \\ \underbrace{p^{\dot\alpha\beta}}_{p\cdot\bar\sigma} & 0 \end{pmatrix}$$

$$\sigma^\mu_{\alpha\dot\beta}\,(\bar\sigma^\nu)^{\dot\beta\alpha} = \mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = 2g^{\mu\nu}$$

$$\Rightarrow p^{\dot\alpha\beta}\,\sigma^\mu_{\beta\dot\alpha} = \mathrm{Tr}(p\,\sigma^\mu) \overset{?}{=} 2p^\mu$$

$$= p_{\alpha\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha}$$

* $s_{ij} \equiv 2p_i \cdot p_j = \underline{\sigma^{\mu}_{\alpha\dot{\beta}}\, p_{i\mu}}\ \underline{\bar{\sigma}^{\nu\dot{\beta}\alpha}\, p_{j\nu}} = \underline{(p_i)_{\alpha\dot{\beta}}}\ \underline{(p_j)^{\dot{\beta}\alpha}}$

$$= |i]_{\alpha} \langle i|_{\dot{\beta}}\, |j\rangle^{\dot{\beta}} [j|^{\alpha} = \langle i\, j\rangle [j\, i]$$

* For real momenta

$$[p|^{\alpha} = \left(|p\rangle^{\dot{\alpha}}\right)^{*} \qquad (\alpha = \dot{\alpha})$$

$$\langle p|_{\dot{\alpha}} = \left(|p]_{\alpha}\right)^{*}$$

$$\Rightarrow \langle i\, j\rangle = [j\, i]^{*} = \sqrt{s_{ij}} \times e^{i\phi_{ij}}$$

☆ Back to $e^+e^- \to q\bar{q}$

$$iM_4(LRRL) = 2ie^2 Q_e Q_q \,\mathbb{I}\, A_4(LRRL)$$

$$A_4(LRRL) = \frac{1}{2S_{12}} \langle 2|\gamma^\mu|1] \, [3|\gamma_\mu|4\rangle$$

$$= \frac{1}{2S_{12}} 2\langle 2\,4\rangle[3\,1]$$

$$= \frac{\langle 2\,4\rangle[3\,1]}{\langle 1\,2\rangle[2\,1]}$$

* Other helicity configurations

$P: \; L \longleftrightarrow R, \quad |\rangle \longleftrightarrow |]$

$$\Longrightarrow A_4(RLLR) = \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 34\rangle} = \frac{[24]^2}{[12][34]}$$

C on 1-2 fermion-line : $1 \longleftrightarrow 2$

$$\Longrightarrow A_4(RLRL) = -\frac{\langle 14\rangle^2}{\langle 12\rangle\langle 34\rangle}$$

C on 3-4 line :

$$\Longrightarrow A_4(LRLR) = -\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle}$$

* Squared - amplitude

number of colors (=3)

$$\overline{|\mathcal{M}_4|^2} = e^4 Q_e^2 Q_q^2 N_c \sum_{hel} |A_4|^2$$

$$= 2 e^4 Q_e^2 Q_q^2 N_c \left[\left| \frac{\langle 24 \rangle^2}{\langle 12 \rangle \langle 34 \rangle} \right|^2 + \left| \frac{\langle 14 \rangle^2}{\langle 12 \rangle \langle 34 \rangle} \right|^2 \right]$$

$$= 2 e^4 Q_e^2 Q_q^2 N_c \underbrace{\frac{s_{24}^2 + s_{13}^2}{s_{12}^2}}$$

$$\hookrightarrow \frac{1}{4}(1-\cos\theta)^2 + \frac{1}{4}(1+\cos\theta)^2$$

$$= \frac{1}{2}(1+\cos^2\theta)$$

* Cross section $\qquad s = s_{12}$

$$\sigma = \frac{1}{2s} \int d\Phi_2 \, \overline{|\mathcal{M}_4|^2}$$

$\hookrightarrow$ 2-body phase space

$$d\Phi_2 = \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3\vec{p}_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

$$\frac{d^3\vec{p}_4}{2E_4} = d^4p_4 \, \delta(p_4^2) \, \Theta(E_4)$$

$\hookrightarrow |\vec{p}_4|$

$$\int d\Phi_2 = \frac{1}{(2\pi)^2}\int \frac{d^3\vec{p}_3}{2E_3}\,\underbrace{\delta\left((p_1+p_2-p_3)^2\right)}_{\delta(s-2\sqrt{s}E_3)}\,\underbrace{\theta(E_1+E_2-E_3)}_{\theta(\sqrt{s}-E_3)}$$

$|\vec{p}_3|$

c.o.m. frame

$$d^3\vec{p}_3 = |\vec{p}_3|^2\, d|\vec{p}_3|\, d\cos\theta\, d\phi$$

$$\frac{1}{16\pi}\int_{-1}^{+1} d\cos\theta$$

$$\alpha = \frac{e^2}{4\pi}$$

$$\sigma = \frac{e^4 Q_f^2 N_c}{32\pi s}\int_{-1}^{1} d\cos\theta\,(1+\cos^2\theta) = \frac{4\pi\alpha^2}{3s}\underline{N_c Q_f^2}$$

* R ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q Q_q^2$$

sum over "active" flavors

$\sqrt{s} > 2m_q$

Evidence for 3 colors

☆ Next-to-leading order

$$\sigma = \sigma_0 + \boxed{\alpha_s \bigcirc}$$