Hydrogen

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1 Matching

QED Lagrangian is

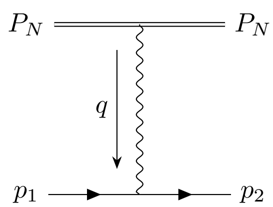
$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \tag{1}$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \tag{2}$$

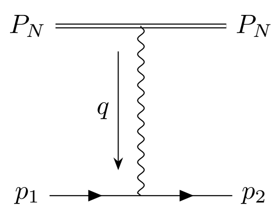
In tree level

$$i\mathcal{M}^{(0)}_{QED} =$$



$$= -e^2\bar{u}_N(P_N)v^\mu u_N(P_N)\frac{i}{\mathbf{q}^2}\bar{u}_e(p_2)\gamma_\mu u_e(p1)$$

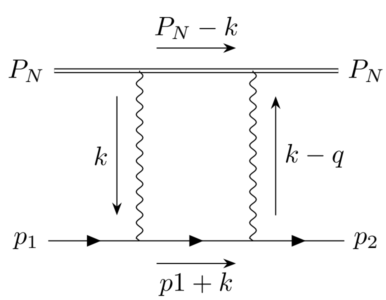
$$i\mathcal{M}^{(0)}_{NRQED} =$$



$$= -e^2\bar{u}_N(P_N)v^\mu u_N(P_N)\frac{i}{\mathbf{q}^2}\psi^\dagger(p_2)\gamma_\mu\psi(p1)$$

The box diagram for NRQED process is

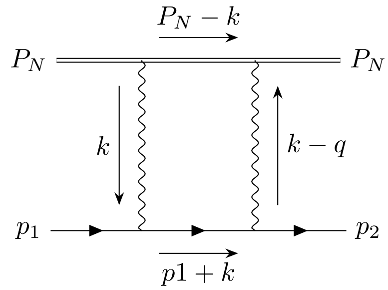
$$i\mathcal{M}^{(1)}_{NRQED} =$$



$$\begin{aligned}
&= e^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\
&= -ie^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m})}\psi(p_1)\\
&= -ie^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2(E_1-\frac{\mathbf{k}^2}{2m})}\psi(p_1)
\end{aligned}$$

The box and crossed box diagram for QED process is

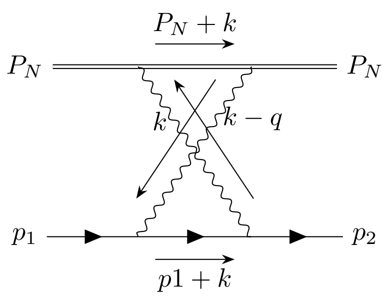
$i\mathcal{M}_1^{(1)} =$



$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{p_1^0 + k_i\gamma^i\gamma^0 + \sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2 + m^2 - p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}]} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i\gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 - p_1^0\sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}
$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.

$i\mathcal{M}_2^{(1)} =$



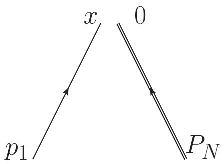
$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{p_1^0 + k_i\gamma^i\gamma^0 - \sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2 + m^2 + p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}]} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i\gamma^0 - \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 + p_1^0\sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}
$$

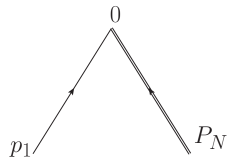
$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

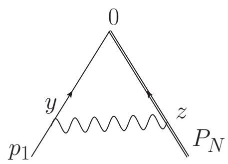
$$
\begin{aligned}
i\mathcal{M}_1^{(1)} + i\mathcal{M}_2^{(1)} &= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{{p_1^0}^2 + k^2 + m^2 + (k_i - p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 - {p_1^0}^2]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{{p_1^0}^2 + k^2 + m^2 + (k_i - p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 - \mathbf{p_1}^2]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1)
\end{aligned}
$$

Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order.

Now consider operator product expansion.

Tree level matching: 

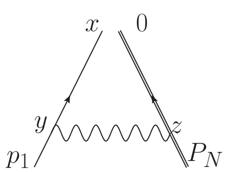
& 

One loop scenario for NRQED case: 

$$
\begin{aligned}
\langle 0|\psi_e(0)N(0)e\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle &= e^2u_N(P_N)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\
&= -ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\
&= -ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1)
\end{aligned}
$$

drop p_1

$$
= -ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1) = \pi ie^2\sqrt{\frac{2m}{E_1}}u_N(P_N)\psi(p_1)
$$

For QED case: 

$$
\begin{aligned}
\langle 0|\psi(x)N(0)e\int \mathrm{d}^4y\bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle &= e^2u_N(P_N)\int[\mathrm{d}k]e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= e^2u_N(P_N)\int[\mathrm{d}k]e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{p_1})\cdot\mathbf{x}}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

drop p_1

$$
= ie^2u_N(P_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{p_1^0+\sqrt{\mathbf{k}^2+m^2}}{2\mathbf{k}^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
$$

2 HSET

2.1 Lagrangian

For scalar QED

$$
\mathcal{L} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi
$$

where

$$
D_\mu = \partial_\mu + ieA_\mu
$$

In Schwartz's QFT (Chap. 35) he mentioned a choice of χ_v and $\tilde{\chi}_v$:

$$\phi(x) = e^{-imv\cdot x}\frac{1}{\sqrt{2m}}(\chi_v(x) + \tilde{\chi}_v(x)) \tag{3}$$

$$\chi_v(x) = e^{imv\cdot x}\frac{1}{\sqrt{2m}}(iv\cdot D + m)\phi(x),\ \tilde{\chi}_v(x) = e^{imv\cdot x}\frac{1}{\sqrt{2m}}(-iv\cdot D + m)\phi(x) \tag{4}$$

Put (3) into (4), a simple relation is derived:

$$(-iv\cdot D)\chi_v(x) = (2m + iv\cdot D)\tilde{\chi}_v(x)$$

It can also be writen as

$$2m\tilde{\chi}_v = (-iv\cdot D)(\chi_v + \tilde{\chi}_v)$$

Use this result

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2m}\Big\{\{[D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger + imv^\mu(\chi_v + \tilde{\chi}_v)^\dagger\}\{[D_\mu(\chi_v + \tilde{\chi}_v)] - imv_\mu(\chi_v + \tilde{\chi}_v)\} - m^2(\chi_v + \tilde{\chi}_v)^\dagger(\chi_v + \tilde{\chi}_v)\Big\} \\
&= (\chi_v + \tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v + \tilde{\chi}_v) + \frac{1}{2m}[D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger D_\mu(\chi_v + \tilde{\chi}_v) \qquad (5) \\
&= (\chi_v(x) + \tilde{\chi}_v(x))^\dagger(iv\cdot D)(\chi_v(x) + \tilde{\chi}_v(x)) + \mathcal{O}(\frac{1}{m}) \qquad (6)
\end{aligned}$$

(note that $D_\mu\phi = e^{-imv\cdot x}[D_\mu(\chi_v + \tilde{\chi}_v) - imv_\mu(\chi_v + \tilde{\chi}_v)]$ and $-imv^\mu[D_\mu(\chi_v + \tilde{\chi}_v)]^\dagger(\chi_v + \tilde{\chi}_v) = imv^\mu(\chi_v + \tilde{\chi}_v)^\dagger D_\mu(\chi_v + \tilde{\chi}_v) -$ *total derivative term*)

Use the leading order of (5)

$$\begin{aligned}
\mathcal{L}^{(0)} &= (\chi_v + \tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v + \tilde{\chi}_v) \\
&= \chi_v^\dagger iv\cdot D\chi_v + \tilde{\chi}_v^\dagger iv\cdot D(\chi_v + \tilde{\chi}_v) + \chi_v^\dagger iv\cdot D\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + (iv\cdot D\chi_v)^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + [(-2m - iv\cdot D)\tilde{\chi}_v]^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D + 4m)\tilde{\chi}_v
\end{aligned}$$

We can have the final form

$$\mathcal{L} = \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D + 4m)\tilde{\chi}_v + \mathcal{O}(\frac{1}{m})$$

2.2 Quantization

2.2.1 HQET as an example

The leading term of HQET Lagrangian is

$$\mathcal{L} = \bar{Q}_v(iv\cdot D)Q_v$$

$$Q_v(x) = e^{imv\cdot x}\frac{1 + \not{v}}{2}\psi(x)$$

In free Dirac fermion theory we know that

$$\Big\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\Big\} = \delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{ab}$$
$$\{a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger\} = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

also the plane wave expansion of ψ is

$$\begin{aligned}
\psi(x) &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_{\mathbf{p}}u(p)e^{-ip\cdot x} \\
&= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2mv^0}}a_v\sqrt{m}u(v)e^{-imv\cdot x - ik\cdot x}
\end{aligned}$$

using normalization of states $u(k) = \sqrt{m}u(v)$[1], $\langle p'|p\rangle = 2E_p(2\pi)^3\delta^{(3)}(\mathbf{p'}-\mathbf{p})$ and $\langle v', k'|v, k\rangle = 2v^0\delta_{vv'}(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})$ we have $|p\rangle = \sqrt{m}\,|v\rangle$ ($|p\rangle = \sqrt{2E_p}a^\dagger_{\mathbf{p}}\,|0\rangle$ while $|v,k\rangle = \sqrt{2v^0}a^\dagger_{v,\mathbf{k}}\,|0\rangle$)

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v u(v)e^{-imv\cdot x - ik\cdot x}$$

Using the definition of $Q_v(x)$

$$\begin{aligned}
Q_v(x) &= e^{imv\cdot x}\frac{1+\not{v}}{2}\psi(x)\\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v\frac{1+\not{v}}{2}u(v)e^{-ik\cdot x}\\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}}a_v u(v)e^{-ik\cdot x}
\end{aligned}$$

The commutation relation should be

$$\{Q_{va}(\mathbf{x}), Q_{v'b}(\mathbf{x}')\} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0v'^0}}\{a_v, a^\dagger_{v'}\}u_a(v)u^\dagger_b(v')e^{-ik\cdot x + ik'\cdot x'}$$

using $\sum_s u_a(v)u^\dagger_b(v) = \frac{1}{m}\sum_s u_a(p)u^\dagger_b(p) = [(\not{v}+1)\gamma^0]_{ab}$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0v'^0}}\{a_v, a^\dagger_{v'}\}[(\not{v}+1)\gamma^0]_{ab}e^{-ik\cdot x + ik'\cdot x'}$$

assuming $\{a_v, a_{v'}\} = (2\pi)^3\delta_{vv'}\delta^{(3)}(\mathbf{k}-\mathbf{k'})$

$$\begin{aligned}
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2v^0}[(\not{v}+1)\gamma^0]_{ab}e^{-ik\cdot(x-x')}\delta_{vv'}\\
&= [\frac{(\not{v}+1)\gamma^0}{2v^0}]_{ab}\delta_{vv'}\delta^{(3)}(\mathbf{x}-\mathbf{x'})
\end{aligned}$$

2.2.2 HSET

The anti-particle field is decoupled so we don't have to consider that for now. The equation-of-motion is

$$\begin{cases} v\cdot D\chi^\dagger_v = 0\\ v\cdot D\chi_v = 0\end{cases}$$

By definition

$$\begin{aligned}
\chi_v(x) &= \frac{e^{imv\cdot x}}{\sqrt{2m}}(iv\cdot D + m)\phi(x)\\
&= \frac{1}{\sqrt{2m}}(iv\cdot D + 2m)e^{imv\cdot x}\phi(x)
\end{aligned}$$

[1]The relation $\bar{u}^s(p)\gamma^\mu u^s(p) = 2p^\mu$ can be derived using Gordon identity, same for $\bar{u}^s(v)\gamma^\mu u^s(v) = 2v^\mu$, but it's actually $\bar{u}u$.