Tan Relations within QFT Framework

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Outlines

What’re Tan Relations?

Ultracold Atomic Fermi Gas

- ☐ dilute gases
- ☐ only has short range interaction
- ☐ when scattering length is much larger than the range of the force, short range details become insignificant, we can mimic the effect with a delta potential
- ☐ within a small range of scattering length a, similar to nuclear system

Tan Relations

A set of relations representing physical quantities with a universal function C, the *integrated contact density* or simply *contact*, in large momentum limit (Tan, 2008a, 2008b, 2008c).

Contact

$$C \equiv \int \mathrm{d}^3 R \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2(R) \right\rangle$$

Energy relation

$$E = \sum_\sigma \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{2m} \left(\rho_\sigma(k) - \frac{C}{k^4} \right) + \frac{C}{4\pi m a} + \int d^3 R \langle V \rangle$$

Adiabatic relation

$$\frac{\mathrm{d}E}{\mathrm{d}(1/a)} = -\frac{1}{4\pi m} C$$

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Virial theorem

$$E = 2 \int d^3 R \langle \mathcal{V} \rangle - C/(8\pi m a)$$

Momentum distribution

$$\rho_\sigma(\boldsymbol{k}) \longrightarrow C/k^4$$

Number density of fermion pairs

$$\left\langle N_{\text{pair}}\left(\boldsymbol{R}, \boldsymbol{s}\right)\right\rangle \longrightarrow s^1$$

Pionless EFT

- ☐ Successful in describing nuclei.
- ☐ Introduced for few body atomic systems (see review by Platter, 2009).
  The use of contact interaction in an EFT for dilute gases can be dated to the ’90s.

Effective Range Expansion

At zero energy limit

$$u(r \gg R) \approx 1 - r/a$$

ERE

$$k \cot \delta_0 = -\frac{1}{a} + \frac{r_s}{2} k^2 + \cdots \tag{1}$$

The amplitude is

$$\begin{aligned} T &= \frac{4\pi}{m} \frac{1}{k \cot \delta - ik} \\ &= \frac{4\pi}{m} \frac{1}{-\frac{1}{a_s} + \frac{r_{0s}}{2} k^2 + \ldots - ik} \end{aligned} \tag{2}$$

Pionless EFT Lagrangian

$$\mathcal{L}_{\text{eft}} = \psi^\dagger \left[i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} \right] \psi - \frac{C_0}{2} \left(\psi^\dagger \psi \right)^2 + \frac{C_2}{16} \left[(\psi\psi)^\dagger \left(\psi \overleftrightarrow{\nabla}^2 \psi \right) + \text{h.c.} \right] \\ + \frac{C_2'}{8} (\psi\vec{\nabla}\psi)^\dagger \cdot (\psi\vec{\nabla}\psi) - \frac{D_0}{6} \left(\psi^\dagger \psi \right)^3 + \dots \tag{3}$$

We only consider the leading order, which contains only C_0 term. This leaves us (using a dimensionless coupling constant $g(\Lambda)$)

Leading order Pionless EFT Lagrangian

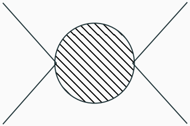
$$\mathcal{L} = \psi^\dagger \left[i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} \right] \psi - \frac{g(\Lambda)}{m} \left(\psi^\dagger \psi \right)^2 \tag{4}$$

There's no gauge symmetry present. However, one could introduce new fields and currents that has gauge symmetry, i.e. include photons to account for isospin breaking effect. So far these're all fixed by NN scattering data, there're also terms need extra data.

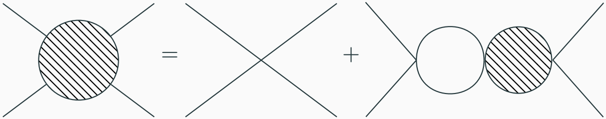
Bubble resum

Consider:

$$i\mathcal{A} = \langle 34|\psi^\dagger\psi|12\rangle = \tag{5}$$



Define $P = p_1 + p_2 = (E, \mathbf{0})$, and $E = p^2/m$. The integral equation is



(6)

$$i\mathcal{A} = -\frac{ig(\Lambda)}{m}\left(1 + i\mathcal{A}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon}\frac{i}{P^0 - k^0 - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + i\epsilon}\right) \tag{7}$$

The integral gives (rescale $\epsilon \to 2m\epsilon$)

$$\mathcal{I} = \frac{im}{2\pi^2}\left(-\Lambda + \sqrt{-mE - i\epsilon}\tan^{-1}\left(\frac{\Lambda}{\sqrt{-mE - i\epsilon}}\right)\right) = -\frac{i\Lambda m}{2\pi^2} + \frac{mp}{4\pi} \tag{8}$$

The amplitude, expressed with g and $\mathcal{I}$, is

$$i\mathcal{A} = \frac{-1}{\mathcal{I} + \frac{m}{ig(\Lambda)}} = \frac{i}{-\frac{\Lambda m}{2\pi^2} - i\frac{mp}{4\pi} - i\frac{m}{ig(\Lambda)}} = \frac{i\frac{4\pi}{m}}{-\frac{2\Lambda}{\pi} - \frac{4\pi}{g(\Lambda)} - ip} \tag{9}$$

Compare to the one we got from ERE

$$\frac{4i\pi/m}{-1/a + \sqrt{-mE - i\epsilon}} = \frac{4i\pi/m}{-1/a - ip} \tag{10}$$

we can express $g(\Lambda)$ with a:

$$g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \tag{11}$$

Some remarks

- Large scattering length is essentially fine-tuned.
- Natural case: Given force range R, $a \sim r_0 \sim R$. With DR and MS, a perturbative expansion in $C_0 = 4\pi a/M$ is achieved.

$$T = \frac{4\pi}{M}\left(-a + ika^2 + \left(\frac{a^2 r_0}{2} + a^3\right)k^2 + \dots\right)$$

  If use cutoff regulator, by choosing $\Lambda \sim 1/R \sim 1/a$, all loops contain divergence and must be resumed.

- Unnatural case: $a \gg r_0 \sim R$ (shallow bound states). For deuteron, $1/a_t \simeq 36$ MeV $\ll m_\pi \simeq 140$ MeV. For ^{4}He atoms, $a \sim 18 R_{vW}$. For a singlet NN scattering,

$$T = -\frac{4\pi}{M}\left(\frac{a_s}{1 + ika_s} + \frac{k^2 a_s^2 r_{0s}}{2}\frac{1}{(1 + ika_s)^2} + \dots\right)$$

- ”PDS” scheme

Braaten’s Approach with OPE

Hamiltonian for fermions with two spin states

Braaten's Hamiltonian

$$\mathcal{H} = \sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(A)} + \frac{g(\Lambda)}{m} \psi_1^{\dagger} \psi_2^{\dagger} \psi_1 \psi_2^{(\Lambda)} + \mathcal{V} \tag{12}$$

where

$$\mathcal{V} = V(\boldsymbol{R}) \sum_{\sigma} \psi_{\sigma}^{\dagger} \psi_{\sigma}$$

$$g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi}$$

A $2 \to 2$ amplitude is

$$\mathcal{A}(E) = \frac{4\pi/m}{-1/a + \sqrt{-mE - i\epsilon}} \tag{13}$$

Our first goal here, is to understand the power-law behavior of the momentum distribution $\rho_\sigma(\mathbf{k})$. The asymptotic behavior of $1/k^4$, in coordinate space, is proportional to $|\mathbf{r}|$.

Given the definition of momentum distribution:

$$\rho_\sigma(\boldsymbol{k}) = \int d^3R \int d^3r e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \left\langle \psi_\sigma^\dagger \left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}\right) \psi_\sigma \left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}\right) \right\rangle \tag{14}$$

we deploy the following OPE in Braaten and Platter (2008):

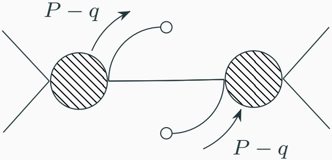
$$\psi_\sigma^\dagger \left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}\right) \psi_\sigma \left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}\right) = \sum_n C_{\sigma,n}(\boldsymbol{r}) \mathcal{O}_n(\boldsymbol{R}) \tag{15}$$

$$= 1 \times \psi_\sigma^\dagger \psi_\sigma(\mathbf{R}) - \frac{g^2(\Lambda) r}{8\pi} \times \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(\mathbf{R}) + \cdots \tag{16}$$

where the second term reproduces exactly what we expect: a coefficient linear in r.

In the following, we'll prove this equation.

First we have Figure 2(a) in Braaten's paper:



(17)

$$= \langle 34|\psi^\dagger\left(-\frac{\mathbf{r}}{2}\right)\psi\left(\frac{\mathbf{r}}{2}\right)|12\rangle \tag{18}$$

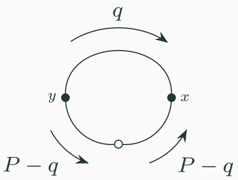
$$= (i\mathcal{A})^2\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{i}{q^0-\frac{\mathbf{q}^2}{2m}+i\epsilon}\frac{i}{\left[E-q^0-\frac{\mathbf{q}^2}{2m}+i\epsilon\right]^2}e^{i\mathbf{q}\cdot\mathbf{r}} \tag{19}$$

$$= \mathcal{A}^2\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{m^2 e^{i\mathbf{q}\cdot\mathbf{r}}}{\left(\mathbf{q}^2-p^2-i\epsilon\right)^2} \tag{20}$$

$$= \frac{im^2\mathcal{A}^2 e^{ipr}}{8\pi p} \tag{21}$$

2-pt correlator: r.h.s. (1)

For simplicity, we drop the external lines and focus on the internal subgraph. Consider Figure 2(b):

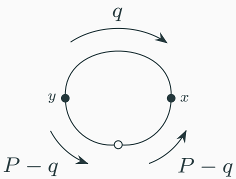


$$= \langle 34|\psi^\dagger \psi\,(0)|12\rangle_{amp} \tag{22}$$

$$\begin{aligned}
&= \int \mathrm{d}^4x \int \mathrm{d}^4y \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\frac{\mathrm{d}^4 l_2}{(2\pi)^4}\frac{\mathrm{d}^4 q}{(2\pi)^4} e^{iP\cdot y} e^{-iP\cdot x} e^{-il_1\cdot y} e^{il_2\cdot x} e^{iq\cdot(x-y)} \tilde{D}(l_1)\tilde{D}(l_2)\tilde{D}(q) \\
&= \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \tilde{D}(P-q)\tilde{D}(P-q)\tilde{D}(q) \\
&= -\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{m^2}{\left(\mathbf{q}^2 - p^2 - i\epsilon\right)^2} = -\frac{im^2}{8\pi p}
\end{aligned}$$

where $\tilde{D}$ marks momentum space propagator and two external vertexes give an $(i\mathcal{A})^2$ factor.

For simplicity, we drop the external lines and focus on the internal subgraph. Consider Figure 2(b):



$$= \langle 34|\psi^{\dagger}\psi\,(0)|12\rangle_{amp} \tag{22}$$

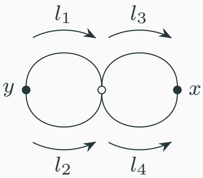
The total contribution is

$$\frac{im^2\mathcal{A}^2}{8\pi p}, \tag{23}$$

the first order Fourier expansion of the l.h.s.. The Wilson coefficient of this order is 1.

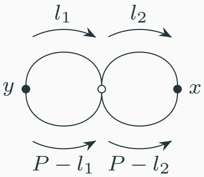
2-pt correlator: r.h.s. (2)

Figure 2(c) gives



$$= \langle 34|\psi^\dagger\psi^\dagger\psi\psi\,(0)|12\rangle_{amp} \tag{24}$$

which becomes



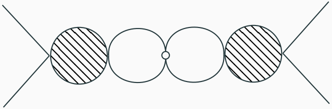
$$= \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\frac{\mathrm{d}^4 l_2}{(2\pi)^4}\tilde{D}(l_1)\tilde{D}(P-l_1)\tilde{D}(l_2)\tilde{D}(P-l_2) \tag{25}$$

$$= -\int \frac{\mathrm{d}^3\mathbf{l_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{l_2}}{(2\pi)^3}\frac{m^2}{\left(\mathbf{l_1}^2 - p^2 - i\epsilon\right)\left(\mathbf{l_2}^2 - p^2 - i\epsilon\right)} \tag{26}$$

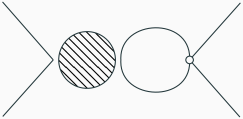
$$= -\mathcal{I}^2 \tag{27}$$

2-pt correlator: r.h.s. (2)

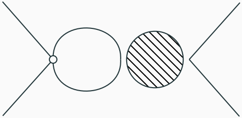
There're four diagrams in total:



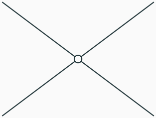
$$= \mathcal{A}^2\mathcal{I}^2 \tag{28}$$



$$= \mathcal{A}\mathcal{I} \tag{29}$$



$$= \mathcal{A}\mathcal{I} \tag{30}$$



$$= 1 \tag{31}$$

We have in total

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm\mathbf{p}} = (\mathcal{A}\mathcal{I} + 1)^2 \tag{32}$$

2-pt correlator: r.h.s. (2)

Plug in

$$\mathcal{I} = -\frac{m}{ig(\Lambda)} - \frac{1}{\mathcal{A}} \tag{33}$$

we have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm\mathbf{p}} = m^2 g^{-2}(\Lambda) \mathcal{A}^2 \tag{34}$$

The Wilson coefficient must be

$$-\frac{r}{8\pi} g^2(\Lambda) \tag{35}$$

We have proven the OPE relation.

Momentum tail and contact

Let's recall some previous results:

$$C \equiv \int \mathrm{d}^3 R \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2(R) \right\rangle$$

$$\rho_\sigma(\boldsymbol{k}) = \int d^3 R \int d^3 r e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \left\langle \psi_\sigma^\dagger \left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}\right) \psi_\sigma \left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}\right) \right\rangle$$

$$\psi_\sigma^\dagger \left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}\right) \psi_\sigma \left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}\right) = 1 \times \psi_\sigma^\dagger \psi_\sigma(\mathbf{R}) - \frac{g^2(\Lambda) r}{8\pi} \times \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(\mathbf{R}) + \cdots$$

Put them together (the 1st term in OPE is 0 in large-k limit):

$$\rho_\sigma(\boldsymbol{k}) \xlongequal{k\to\infty} \frac{C}{k^4} + \cdots \tag{36}$$

Note that the 3-dimensional Fourier transform of $\frac{r}{8\pi}$ is exactly $1/k^4$.

Applications of the Contact

Energy relation

According to the Hamiltonian:

$$\mathcal{H} = \left(\sum_{\sigma} \frac{1}{2m} \nabla\psi_{\sigma}^{\dagger} \cdot \nabla\psi_{\sigma}^{(\Lambda)} - \frac{\Lambda}{2\pi^2 m} g^2(\Lambda) \psi_1^{\dagger}\psi_2^{\dagger}\psi_1\psi_2 \right) + \frac{1}{4\pi m a} g^2(\Lambda) \psi_1^{\dagger}\psi_2^{\dagger}\psi_1\psi_2 + \mathcal{V} \tag{37}$$

where the matrix elements of those three operators are finite. The $\nabla\psi_{\sigma}^{\dagger} \cdot \nabla\psi_{\sigma}^{(\Lambda)}$ part gives a linear divergence $2 \times \frac{\Lambda m \mathcal{A}^2}{4\pi^2}$ for two spin states in total while the other one gives $-\frac{\Lambda m \mathcal{A}^2}{2\pi^2}$, we can see that the linear divergence is cancelled. Integrating over positions, we obtain

$$\begin{aligned} \int \mathrm{d}^3 R \left\langle \nabla\psi_{\sigma}^{\dagger} \cdot \nabla\psi_{\sigma} \right\rangle &= \int \mathrm{d}^3 R \mathrm{d}^3 r \delta^{(3)}(\mathbf{r}) \left\langle \nabla\psi_{\sigma}^{\dagger}(\mathbf{R} - \frac{\mathbf{r}}{\mathbf{2}}) \cdot \nabla\psi_{\sigma}(\mathbf{R} + \frac{\mathbf{r}}{\mathbf{2}}) \right\rangle \\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} k^2 \rho_{\sigma}(k) \end{aligned} \tag{38}$$

$$\frac{1}{4\pi m a} \int \mathrm{d}^3 R \left\langle g^2 \psi_1^{\dagger}\psi_2^{\dagger}\psi_1\psi_2 \right\rangle = \frac{1}{4\pi m a} C \tag{39}$$

Also notice

$$\int^{\Lambda} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{\mathbf{k}^2} = \frac{\Lambda}{2\pi^2} \tag{40}$$

we have

$$\int \mathrm{d}^3 R \frac{\Lambda}{2\pi^2 m} \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 \right\rangle = \sum_\sigma \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{k^2}{2m} \frac{C}{\mathbf{k}^4} \tag{41}$$

we achieve

$$E = \sum_\sigma \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{2m} \left(\rho_\sigma(k) - \frac{C}{k^4} \right) + \frac{C}{4\pi m a} + \int d^3 R \langle V \rangle \tag{42}$$

Adiabatic relation

Using F-H theorem

$$\mathrm{d}E/\mathrm{d}a = \int \mathrm{d}^3 R \langle \partial \mathcal{H} / \partial a \rangle \tag{43}$$

it's straightforward that

$$\partial \mathcal{H} / \partial a = g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 / (4\pi m a^2) \tag{44}$$

We then have

$$\frac{\mathrm{d}E}{\mathrm{d}(1/a)} = -\frac{1}{4\pi m} C \tag{45}$$

Virial theorem

Given a harmonic trapping potential:

$$V(\mathbf{R}) = \frac{m}{2}\omega^2 R^2 \tag{46}$$

Dimensional analysis requires

$$\left[\omega \frac{\partial}{\partial \omega} - \frac{1}{2} a \frac{\partial}{\partial a}\right] \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle = \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle \tag{47}$$

Together with F-H theorem

$$\frac{a}{2} \frac{\partial}{\partial a} \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle = \frac{C}{8\pi m a} \tag{48}$$

$$\frac{\partial}{\partial \omega} \int \mathrm{d}^3 R \, \langle \mathcal{H} \rangle = \frac{\partial}{\partial \omega} \int \mathrm{d}^3 R \, \langle \mathcal{V} \rangle = 2 \int \mathrm{d}^3 R \, \langle \mathcal{V} \rangle \tag{49}$$

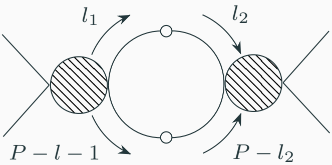
$$E = 2 \int d^3 R \langle \mathcal{V} \rangle - C/(8\pi m a) \tag{50}$$

Number density operator of fermion pairs

We have number density operator for fermion pairs

$$\psi_1^\dagger\psi_1(\mathbf{R}-\frac{1}{2}\mathbf{r})\psi_2^\dagger\psi_2(\mathbf{R}+\frac{1}{2}\mathbf{r}) \tag{51}$$

and the diagram is



$$= (i\mathcal{A})^2\int\frac{\mathrm{d}^4l_1}{(2\pi)^4}\frac{\mathrm{d}^4l_2}{(2\pi)^4}\frac{i}{l_1^0-\frac{\mathbf{l_1}^2}{2m}+i\epsilon}\frac{i}{E-l_1^0-\frac{\mathbf{l_1}^2}{2m}+i\epsilon}\frac{i}{l_2^0-\frac{\mathbf{l_2}^2}{2m}+i\epsilon}\frac{ie^{i\mathbf{q}\cdot\mathbf{r}}}{E-l_2^0-\frac{\mathbf{l_2}^2}{2m}+i\epsilon}$$

$$= \frac{\mathcal{A}^2m^2}{16\pi^2r^2}e^{2ipr}$$

Compare with the result of Figure 2(c) (34) we have

$$\psi_1^\dagger\psi_1\left(\boldsymbol{R}-\frac{1}{2}\boldsymbol{r}\right)\psi_2^\dagger\psi_2\left(\boldsymbol{R}+\frac{1}{2}\boldsymbol{r}\right)\to\frac{1}{16\pi^2r^2}g^2\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2(\boldsymbol{R}) \tag{52}$$

Contact?

Define $N_{\text{pair}}(\boldsymbol{R}, s)$ to describe the number of fermion pairs within a sphere of radius s

$$N_{\text{pair}}(\boldsymbol{R}, s) \equiv \int_{|\mathbf{r}|<s} \mathrm{d}^3\mathbf{r}\psi_1^\dagger\psi_1\left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}\right)\psi_2^\dagger\psi_2\left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}\right)$$

In the absence of interactions, $\left\langle N_{\text{pair}}\left(\boldsymbol{R}, s\right)\right\rangle$ scales as s^3 as $s \to 0$.

In the case of a large scattering length $\left\langle N_{\text{pair}}\left(\boldsymbol{R}, s\right)\right\rangle$ scales as s^1. We can interpret the contact density operator $g^2\psi_1^\dagger\psi_2^\dagger\psi_1\psi_2$ as the limit as $s \to 0$ of $(4\pi/s)N_{\text{pair}}\left(R, s\right)$.

Consider inelastic scattering into other spin states that has much lower energy. The effect on a state with definite energy E is to change its time-dependence from $\exp(-iEt)$ to $\exp(-i(E - i\Gamma/2)t)$. The probability in that state decreases with time at the rate Γ. The adiabatic relation can be used to derive an expression for Γ to leading order in the imaginary part of a

$$\Gamma \approx \frac{(-\operatorname{Im} a)}{2\pi m|a|^2}C$$

Thus C determines the rate at which low-energy fermions are depleted by inelastic collisions.

Questions?

Backup

PDS scheme (Kaplan et al., 1998)

We start with the one loop bubble in $d \to 3$ dimension, for latter convenience we rescale the usual μ by $1/2$

$$\mathcal{I} = -i\left(\frac{\mu}{2}\right)^{3-d} \int \frac{\mathrm{d}^d \mathbf{l}}{(2\pi)^d} \frac{m}{\mathbf{l}^2 - p^2} = -i \frac{\pi^{1-\frac{d}{2}} m \mu^{3-d} \left(-\frac{1}{p^2}\right)^{1-\frac{d}{2}} \csc\left(\frac{\pi d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \tag{53}$$

For MS scheme, we only subtract poles at $d = 3$. For PDS scheme, we also subtract poles at $d = 2$. The counterterm is

$$\delta\mathcal{I} = \frac{-i\mu m}{4\pi(d-2)} \tag{54}$$

Now we add $\delta\mathcal{I}$ to $\mathcal{I}$, then expand the expression in $d = 3$:

$$\mathcal{I} + \delta\mathcal{I} = \frac{-im(\mu + ip)}{4\pi}$$

The total amplitude is

$$i\mathcal{A}_{\mathrm{PDS}} = \frac{-1}{\mathcal{I} + \frac{m}{ig(\mu)}} = \frac{-1}{\frac{-im(\mu+ip)}{4\pi} + \frac{m}{ig(\mu)}} \tag{55}$$

PDS scheme

With a little algebra

$$i\mathcal{A}_{\mathrm{PDS}} = \frac{-1}{\frac{-im(\mu+ip)}{4\pi} + \frac{m}{ig(\mu)}} = \frac{4i\pi/m}{-\mu - \frac{4\pi}{ig(\mu)} - ip}$$

We have

$$g(\mu) = \frac{-4\pi i}{1/a - \mu} \tag{56}$$

In hard cutoff scheme, by simple power-counting, the tree diagram is leading. But that's not true, because there're cancellations between orders. In the end, one needs to resum all orders.

In PDS scheme, we can see all orders are of the same magnitude, thus the need for resummation is explicit.

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