Homework: Particle Physics

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1. The force range and characteristic interaction time for all three types of interaction but gravity are

	Force Range	Characteristic interaction time
Strong interaction	1 fm	1×10^{-23} s
Electromagnetic interaction	∞	1×10^{-16} s
Weak interaction	1/400 fm	1×10^{-10} s

2. For electron in an EM field, Dirac equation can be written as [1]

$$[i\frac{\partial}{\partial t} + e\phi - \boldsymbol{\alpha} \cdot (\mathbf{P} + e\mathbf{A}) - m\beta]\psi = 0 \tag{1}$$

where $\mathbf{P} = -i\nabla$. Set

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-imt}$$

so that the certain part of electron rest mass can be removed. Then we have

$$\begin{aligned} i\frac{\partial}{\partial t}\varphi &= \boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})\chi - e\phi\varphi \\ i\frac{\partial}{\partial t}\chi &= \boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})\varphi - e\phi\chi - 2m\chi \end{aligned}$$

At nonrelativistic limit, we have

$$\chi \approx \frac{1}{2m}\boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})\varphi \tag{2}$$

where $\chi/\varphi \ll 1$. Then

$$i\frac{\partial}{\partial t}\varphi = \frac{1}{2m}[\boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})]^2\varphi - e\phi\varphi$$

Using

$$\begin{aligned} [\boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})]^2 &= (\mathbf{P} + e\mathbf{A})^2 + i\boldsymbol{\sigma} \cdot [(\mathbf{P} + e\mathbf{A}) \times (\mathbf{P} + e\mathbf{A})] \\ &= (\mathbf{P} + e\mathbf{A})^2 + ie\boldsymbol{\sigma} \cdot [\mathbf{P} \times \mathbf{A} + \mathbf{A} \times \mathbf{P}] \\ &= (\mathbf{P} + e\mathbf{A})^2 + e\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A}) \\ &= (\mathbf{P} + e\mathbf{A})^2 + e\boldsymbol{\sigma} \cdot \end{aligned}$$

Then (2) becomes

$$
\begin{aligned}
i\frac{\partial}{\partial t}\varphi &= \left[\frac{1}{2m}(\mathbf{P}+e\mathbf{A})^2 + \frac{e}{2m}\boldsymbol{\sigma}\cdot\ - e\phi\right]\varphi \\
&= \left[\frac{1}{2m}(\mathbf{P}+e\mathbf{A})^2 - \boldsymbol{\mu}\cdot\ - e\phi\right]\varphi
\end{aligned}
$$

where $\boldsymbol{\mu} = -\frac{e\hbar}{2mc}\boldsymbol{\sigma}$ (in our previous calculation $\hbar = c = 1$) is the intrinsic magnetic moment of electron.

3. From the definition of Breit-Wigner formula, we have the distribution function for decay to a specific quantum state

$$
P(E) = \frac{1}{2\pi}\frac{\Gamma}{(E-M)^2 + \Gamma^2/4} \tag{3}
$$

and also

$$
\Gamma = \frac{1}{\tau}
$$

(3) can be proved by (using some inturtive fourier transformations)

$$
\psi(M) = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sqrt{2\pi}}e^{iMt}\left|t\right\rangle = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sqrt{2\pi}}e^{iMt}e^{-i(m-i\frac{\Gamma}{2})t}\left|0\right\rangle = \frac{1}{\sqrt{2\pi}}\frac{1}{(M-m)+i\frac{\Gamma}{2}}
$$

and the mass distribution function would be

$$
\rho(M) \equiv \Gamma\psi(M)\psi^*(M) = \frac{1}{2\pi}\frac{\Gamma}{(E-M)^2 + \Gamma^2/4}
$$

More strict prove can be found in B.R.Martin's *Particle Physics* which I briefly listed in the end of this problem.

With the rest mass M and the lifetime τ of the particle, the distribution function would be

$$
P(E) = \frac{1}{2\pi}\frac{\Gamma}{(E-M)^2 + 1/(4\tau^2)} \tag{4}
$$

Prove [2] of (3): We choose the wavefunctions of possible final states to be orthonormal, the wavefunction describes state of any timestamp can be expanded by

$$
\mathbf{\Psi}(\mathbf{r}, t) = \sum_{n=0}^{\infty} a_n(t)e^{-iE_nt}\psi_n(\mathbf{r})
$$

where

$$
a_0(0) = 1, \qquad a_n(0) = 0 (n \geq 1)
$$

and

$$
E_n = H_{nn} = \int \psi_n^* H \psi_n \mathrm{d}x
$$

To determine the time dependency of $a_n(t)$, we use the Schrödinger equation

$$
i\frac{\partial \mathbf{\Psi}}{\partial t} = H\mathbf{\Psi}
$$

and

$$\sum_m \left\{ i\frac{\mathrm{d}a_m}{\mathrm{d}t}e^{-iE_m t}\psi_m + E_m a_m e^{-iE_m t}\psi_m \right\} = \sum_m a_m e^{-iE_m t} H\psi_m$$

which gives (assuming terms when $n \neq m$ are small except for the first)

$$i\frac{\mathrm{d}a_n}{\mathrm{d}t} = H_{n0}e^{-i(E_0-E_n)t}a_0$$

and also we evaluate this problem in the rest frame for the decaying particle, and asuume $a_0(t) = e^{-\Gamma t/2}$ which consist with the exponential decay law. Now we have

$$ia_n = -iH_{n0}\left\{ \frac{e^{-i[(M-E_n)-i\Gamma/2]t} - 1}{(E_n - M) + i\Gamma/2} \right\} \xlongequal{t\gg 1/\Gamma} \frac{iH_{n0}}{(E_n - M) + i\Gamma/2}$$

Now the probability would be

$$P_n = a_n^2 = \frac{|H_{n0}|^2}{(E_n - M)^2 + \Gamma^2/4} = \frac{2\pi}{\Gamma}|H_{n0}|^2 P(E_n - M)$$

where

$$P(E_n - M) = \frac{\Gamma/2\pi}{(E-M)^2 + \Gamma^2/4}$$

We can verify it's normalized.

4. For unstable particles with long enough lifetime to be tracked, their lifetime can be calculated using simple Lorentz invariance property

$$m\tau = Et - pL = E\frac{L}{v} - pL = E^2\frac{L}{p} - pL = \frac{m^2 L}{p}$$

which means we can estimate the particle lifetime by $\tau = \frac{mL}{p}$.

5. The mass of J/ψ particle is 3.1GeV, and which of electron is 0.5MeV. To produce J/ψ particle in e^+e collision, we have

$$M = 3.1\text{GeV}, \ \ m = 0.5\text{MeV}$$

and in center-of-mass frame

$$4E^2 \geq M^2$$

where E is the energy of one electron/positron. Note that

$$E = \gamma m = \frac{1}{\sqrt{1-\beta^2}}m$$

So we have

$$\frac{1}{\sqrt{1-\beta^2}}m \geq 2M$$

which gives

$$\beta \geq \sqrt{1 - \frac{m^2}{4M^2}} \approx 0.99999999024453693539961460987023$$

And that's the minimum velocity electron/positron must go.

6. The Mandelstam variables can be expressed by

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{5}$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{6}$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{7}$$

From which we have (in CM frame)

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1p_2 - 2p_1p_3 - 2p_1p_4 \end{aligned}$$

Assuming

$$p_1 = (E_1, \mathbf{p_1}), \quad p_2 = (E_2, \mathbf{p_2}), \quad p_3 = (E_3, \mathbf{p_3}), \quad p_4 = (E_4, \mathbf{p_4})$$
$$E_1 + E_2 = E_3 + E_4, \quad \mathbf{p_1} + \mathbf{p_2} = \mathbf{p_3} + \mathbf{p_4}$$

and

$$\begin{aligned} s + t + u &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1p_2 - 2p_1p_3 - 2p_1p_4 \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2E_1(E_2 - E_3 - E_4) - 2\mathbf{p_1}(\mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4}) \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 - 2E_1^2 + 2|\mathbf{p_1}|^2 \\ &= p_1^2 + p_2^2 + p_3^2 + p_4^2 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned}$$

7. For branch $A \to BC$, the partial width is [2]

$$\Gamma(A \to BC) = \frac{g_{ABC}^2}{8\pi} \frac{|p_B|}{M_A^2}$$

If A is an unstable particle with width Γ, from Breit-Wigner formula, we have

$$P_f(E) = \frac{1}{2\pi} \frac{\Gamma_f}{(E - M)^2 + \Gamma^2/4} \tag{8}$$

which satisfies normalization $\int_{-\infty}^{\infty} \mathrm{d}E P_f(E) = 1$. Considering the momentum of particle B is (using the fact that B and C have the same mass, which means they have opposite momentums, and we choose the centre-of-mass frame of A)

$$p_B = \sqrt{\left(\frac{E}{2}\right)^2 - M_B^2}$$

Then the decay width becomes

$$\begin{aligned} \Gamma &= \frac{g_{ABC}^2}{8\pi} \int_{-\infty}^{\infty} \mathrm{d}E P_f(E) \frac{p_B}{E^2} \\ &= \frac{g_{ABC}^2}{8\pi} \int_{-\infty}^{\infty} \mathrm{d}E \frac{1}{2\pi} \frac{\Gamma_f}{(E - M_A)^2 + \Gamma^2/4} \frac{p_B}{E^2} \\ &= \frac{g_{ABC}^2}{16\pi^2} \int_{-\infty}^{\infty} \mathrm{d}E \frac{\Gamma_f}{(E - M_A)^2 + \Gamma^2/4} \frac{\sqrt{\left(\frac{E}{2}\right)^2 - M_B^2}}{E^2} \end{aligned}$$

8. 3-particle decay phase space integrals can be derived as follows [3]:

First we know that the gengeral non-relativistic expression for N-body phase space is

$$\mathrm{d}n = (2\pi)^3 \prod_{i=1}^{N} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3} \delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i\right) \tag{9}$$

where $\mathbf{p}_a$ is the momentum of the decaying particle. (This expression can be derived easily from the phase space volume of each particle.) According to Fermi's golden rule (and notice the $(2E_i)^{1/2}$ ratio difference between $\mathcal{M}_{fi}$ and T_{fi}), the decay rate can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - \sum_{i=1}^{N} E_i) \delta^3(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i) \prod_{i=1}^{N} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i} \tag{10}$$

So for 3-particle decay the decay rate can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - E_1 - E_2 - E_3) \delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3}$$

Now we consider it in the centre-of-mass frame of the decaying particle A, which means $E_a = m_a$ and $\mathbf{p}_a = 0$. Through the integration of delta function and $\mathrm{d}^3\mathbf{p}_3$, we have

$$\begin{aligned}
\Gamma_{fi} &= \frac{(2\pi)^4}{2m_a} \int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2 - E_3) \delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3} \\
&= \frac{(2\pi)^4}{2m_a} \int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2 - E_3) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{1}{(2\pi)^3 2E_3} \\
&= \frac{1}{2^4 m_a (2\pi)^5} \int \frac{|\mathcal{M}_{fi}|^2}{E_1 E_2 E_3} \delta(m_a - E_1 - E_2 - E_3) \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \\
&= \frac{1}{2^4 m_a (2\pi)^5} \int \frac{|\mathcal{M}_{fi}|^2}{E_1 E_2 E_3} \delta(m_a - E_1 - E_2 - E_3) \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \\
&= \frac{1}{2^4 m_a (2\pi)^5} \int \frac{|\mathcal{M}_{fi}|^2 |\mathbf{p}_1|^2 |\mathbf{p}_2|^2 \mathrm{d}p_1 \, \mathrm{d}(\cos\theta_1) \, \mathrm{d}\phi_1 \mathrm{d}p_2 \, \mathrm{d}(\cos\theta_2) \, \mathrm{d}\phi_2}{E_1 E_2 \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}} \\
&\qquad\qquad \delta(m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2})
\end{aligned}$$

where θ_1, ϕ_1 and ϕ_2 are independent of the integral and therefore can be integrated first. Note that

$$\frac{\mathrm{d}|\mathbf{p}_i|}{\mathrm{d}E_i} = \frac{E_i}{|\mathbf{p}_i|}$$

which means

$$\mathrm{d}|\mathbf{p}_i| = \frac{E_i}{|\mathbf{p}_i|} \mathrm{d}E_i$$

and mark the kernel of δ function as

$$f(\cos\theta_2) \equiv m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}$$

we have

$$f'(\cos\theta_2) = -\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}}$$

and the real root of $f(\cos\theta_2)=0$ is

$$\cos\theta_2' = \frac{(m_a-E_1-E_2)^2-m_3^2-|\mathbf{p}_1|^2-|\mathbf{p}_2|^2}{2|\mathbf{p}_1||\mathbf{p}_2|}$$

So we have

$$\begin{aligned}
&\delta(m_a-E_1-E_2-\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_2|^2+2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2+m_3^2})\\
=&\frac{\delta(\cos\theta_2-\cos\theta_2')}{\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_2|^2+2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2'+m_3^2}}}
\end{aligned}$$

And the original formula becomes

$$\begin{aligned}
\Gamma_{fi} &= \frac{8\pi^2}{2^4 m_a(2\pi)^5}\int\frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1|^2|\mathbf{p}_2|^2\mathrm{d}p_1\mathrm{d}p_2\,\mathrm{d}(\cos\theta_2)}{E_1E_2\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_2|^2+2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2+m_3^2}}\frac{\delta(\cos\theta_2-\cos\theta_2')}{\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_2|^2+2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2'+m_3^2}}}\\
&= \frac{1}{2^3 m_a(2\pi)^3}\int\frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1|^2|\mathbf{p}_2|^2\mathrm{d}p_1\mathrm{d}p_2}{E_1E_2\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_2|^2+2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2'+m_3^2}}\frac{2\sqrt{|\mathbf{p}_1|^2+|\mathbf{p}_2|^2+2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2'+m_3^2}}{2|\mathbf{p}_1||\mathbf{p}_2|}\\
&= \frac{1}{2^3 m_a(2\pi)^3}\int\frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1||\mathbf{p}_2|\mathrm{d}p_1\mathrm{d}p_2}{E_1E_2}\\
&= \frac{1}{2^3 m_a(2\pi)^3}\int\frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1||\mathbf{p}_2|\frac{E_1}{|\mathbf{p}_1|}\mathrm{d}E_1\frac{E_2}{|\mathbf{p}_2|}\mathrm{d}E_2}{E_1E_2}\\
&= \frac{1}{8m_a(2\pi)^3}\int|\mathcal{M}_{fi}|^2\mathrm{d}E_1\mathrm{d}E_2
\end{aligned}$$

which is exactly the form of square Dalitz plot. Transform it a little bit and we have the standard form of Dalitz plot (note that $s_2=(p_2+p_3)^2=(p_a-p_1)^2\to\mathrm{d}s_2=-2m_a\mathrm{d}E_1$ and similar for s_3)

$$\Gamma_{fi}=\frac{1}{32m_a(2\pi)^3}\int|\mathcal{M}_{fi}|^2\mathrm{d}s_2\mathrm{d}s_3$$

Now let's review another form of the standard Dalitz form

$$\frac{\mathrm{d}\Gamma_{fi}}{\mathrm{d}s_2\mathrm{d}s_3}=\frac{1}{32m_a(2\pi)^3}|\mathcal{M}_{fi}|^2$$

and its physical meaning is obvious: the density of data points on a Dalitz plot is proportional to the decay matrix element.

References

[1] 曾谨言. 量子力学 (卷 II), 第 5 版. 北京：科学出版社, 2013.

[2] Martin, Brian R., and Graham Shaw. Particle physics. John Wiley & Sons, 2013.

[3] Thomson, Mark. Modern particle physics. Cambridge University Press, 2013.