Exercises for "Resonances from LQCD"

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July 3, 2019

There are only exercises for Lectures 1-3, since the school ends after Lecture 4.

1 Lecture 1

1.1 *The K matrix.* Show that

$$\frac{1}{\mathcal{M}_2^{(\ell)}} = \frac{1}{\mathcal{K}_2^{(\ell)}} - i\rho\,, \tag{1}$$

with $\rho = q/(16\pi E^*)$ and $\mathcal{K}_2^{(\ell)}$ real, solves the unitarity condition

$$\mathrm{Im}\,\mathcal{M}_2^{(\ell)} = \rho\,|\mathcal{M}_2^{(\ell)}|^2\,. \tag{2}$$

This was discussed on slide 34 of Lecture 1.

1.2 *Scattering amplitudes in ϕ^4 theory.* Consider the theory of a real scalar field $\phi(x)$ with Minkowski Lagrange density (and mostly-minus metric)

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\,. \tag{3}$$

We will assume that λ is small and use perturbation theory.

(i) Draw all Feynman diagrams contributing to the two-particle scattering amplitude, $\mathcal{M}_2$, up to (and including) cubic order in λ.

(ii) Calculate $\mathcal{M}_2(s,t)$ at leading (linear) order in λ. (This is almost trivial.)

(iii) Draw all Feynman diagrams contributing to the three-particle scattering amplitude $\mathcal{M}_3$, up to (and including) cubic order in λ.

(iv) You should find that the leading order contribution to $\mathcal{M}_3$ is quadratic in λ. Calculate this contribution explicitly, and show that (a) it diverges for certain choices of the external momenta and (b) it diverges at threshold.

(Note that since we are only calculating tree-level diagrams we do not have to worry about renormalization.)

These divergences are "physical" in the sense that they are not removed by renormalization. One can show, however, that they do not lead to divergences in the scattering of three wavepackets, due to smearing over the delta-function-like divergences.

1.3 *Finite volume correlation functions (preparation for lecture 2).* Consider the two-point function

$$C_L(E, \vec{P}) = \int dt \int_L d^3x \,, e^{iEt - i\vec{P}\cdot\vec{x}} \langle 0| T\left\{\sigma^\dagger(t, \vec{x})\sigma(0)\right\}|0\rangle \,, \qquad (4)$$

where (Minkowski) time runs over the full range, $-\infty < t < \infty$, while $\vec{x}$ is restricted to a cubic box of side L, and σ is a local operator. The spatial boundary conditions are periodic, so that the total momentum is restricted to $\vec{P} = (2\pi/L)\vec{n}$, with $\vec{n}$ a vector of integers.

By inserting a complete set of states, show that, for fixed $\vec{P}$, C_L has poles in E at the position of the energies of the finite-volume states with the quantum numbers of σ and momentum $\vec{P}$. What are the residues of these poles?

2 Lecture 2

2.1 *Filling in steps in the derivation of the Lüscher zeta function.* In slide

20 of Lecture 2, we find the result

$$\frac{1}{2}\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m^2+i\epsilon}\frac{1}{b^2-m^2+i\epsilon}g(k) =$$

$$\frac{1}{2}\left(\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right)\frac{f(\vec{k}^*)g(\vec{k}^*)h(\vec{k})}{2\omega_k 2\omega_b(E-\omega_k-\omega_b+i\epsilon)} + \mathcal{O}(e^{-mL}), \quad (5)$$

where $b = P - k$, $h(\vec{k})$ is any UV regulator that equals unity when $E = \omega_k + \omega_b$, and $f(\vec{k}^*) \equiv f([\omega_k, \vec{k}])$ and similarly for $g(\vec{k}^*)$. Demonstrate this result, making sure to explain why there are exponentially suppressed corrections on the second line. For what conditions on E and $\vec{P}$ does it hold?

Comments: Recall that $\vec{k}^*$ is the three-momentum obtained when boosting $[\omega_k, \vec{k}]$ to the CM frame. We are simply choosing to express the function $f(k)$ when k is on shell, i.e. when $k = [\omega_k, \vec{k}]$, in terms of $\vec{k}^*$ rather than $\vec{k}$. This plays no role in the derivation. Note also that we are abusing notation by using the same name f for both a function of the four-vector k and the three-vector $\vec{k}^*$.

If you want to get more into the details, show also the next line on slide 20. For discussion of these steps, see Hanson & Sharpe, 1408.5933.

2.2 *Lüscher's formula in 1+1 dimensions.* (Adapted from Hansen & Sharpe, 1901.00483, with some typos corrected.) The limit of a single spatial dimension is nice because there are then no angular momentum indices and we can calculate the zeta function explicitly. We also simplify matters by considering the overall rest frame with $\vec{P} = 0$ (so $E^* = E$). Then we have

$$F_{\rm PV}(E,L) = \frac{1}{2}\left(\frac{1}{L}\sum_{k} - \mathrm{PV}\int \frac{dk}{2\pi}\right)\frac{1}{4\omega_k^2(E-2\omega_k)}\,. \quad (6)$$

Note that no UV regularization is needed, and that the quantization condition (see slide 36 of Lecture 2) is automatically algebraic:

$$F_{\rm PV}(E,L) = \frac{1}{\mathcal{K}_2(E)}\,. \quad (7)$$

Our aim is to manipulate this quantization condition into a more familiar and intuitive form.

(i) Show that, up to exponentially suppressed corrections, we can rewrite F_{PV} as

$$F_{\mathrm{PV}}(E,L) = \frac{L}{16E\pi^2}\left(\sum_n - \mathrm{PV}\int dn\right)\frac{1}{x^2-n^2}\,, \tag{8}$$

where $x = qL/(2\pi)$, with $q = \sqrt{E^2/4 - m^2}$ being the relative momentum were there two on shell particles having total CM energy E. Note that q is not a finite-volume momentum, i.e. is not quantized. It is just a proxy for the energy, E.

(ii) Show by explicit evaluation that

$$F_{\mathrm{PV}}(E,L) = \frac{L}{16E\pi^2}\frac{\pi\cot(\pi x)}{x}\,. \tag{9}$$

(iii) The phase space factor appearing in the unitarity relation in one spatial dimension is (being careful to note that there are two solutions to the delta function constraint)

$$\rho_1 = \frac{1}{2}\int\frac{dk}{2\pi}\frac{1}{4\omega_k^2}\pi\delta(E-2\omega_k) = \frac{1}{8qE}\,. \tag{10}$$

Thus the relation between $\mathcal{K}_2$ and the phase shift becomes

$$\mathcal{K}_2 = \rho_1\tan\delta(q)\,. \tag{11}$$

Using these results, manipulate the quantization condition into the form

$$\cot\delta(q) + \cot(qL/2) = 0\,. \tag{12}$$

(iv) Why is the final result expected? (Hint: think about scattering in QM.)

3 Lecture 3

3.1 *Comparing pole prescriptions.* Reproduce the results shown in the plots on slide 37 of Lecture 3. Note that, below threshold, the PV prescription is defined by analytic continuation of the above threshold result.

3.2 *Finite-volume shifts in ϕ^4 theory.* (Based on Hansen & Sharpe, 1509.07929.) The idea of this exercise is to gain more intuition for finite-volume energy shifts by explicitly calculating the two-point correlator C_L (see exercise 1.3 above) in perturbation theory in ϕ^4 theory. It turns out that it is not too hard to go to 3-loop order in such calculations (as I did in 1707.04279), and this allows one to provide a detailed test of the three-particle quantization condition, as mentioned in Lecture 4. But don't worry, here we will stick to one-loop order. Also, in order to keep this exercise within reasonable bounds, we will work out only the two-particle energy shift. You can read about the three-particle case in the above-noted references.

Specifically, we will consider the following modified Euclidean-space two-point correlation function in a time-momentum basis:

$$\widetilde{C}_L(\tau) = \frac{(2m)^2}{2L^6} e^{2m\tau} \langle \sigma(\tau)\sigma(0) \rangle \,, \tag{13}$$

$$\sigma(\tau) = \left(\int_L d^3x \, \phi(\vec{x}, \tau) \right)^2 . \tag{14}$$

Here ϕ is the same real scalar field we studied in Exercise 1.2, whose Euclidean Lagrange density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \,. \tag{15}$$

We have chosen the operator σ to consist of a product of zero-momentum wall sources, so that σ itself has zero momentum. To do the calculation we will need the scalar field propagator in time-momentum representation:

$$G_{\vec{p}}(\tau_1, \tau_2) \equiv \left\langle \left(\int_L d^3x \, e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{x}, \tau_1) \right) \phi(\tau_2) \right\rangle = \frac{e^{-\omega_p|\tau_1 - \tau_2|}}{2\omega_p} \,. \tag{16}$$

(If you are not familiar with this form for the free scalar propagator, you should derive it.)

(i) First we consider $\lambda = 0$, i.e. the noninteracting theory. Show that, in this case,

$$\widetilde{C}_L(\tau) = 1 \,. \tag{17}$$

In other words, the overall factors in Eq. (13) are chosen so that the correlator is independent of τ, and is normalized to unity.

(ii) Now we turn on λ to a nonzero (small) value. Show that

$$\widetilde{C}_L(\tau) = \sum_n Z_{2,n} e^{-\Delta E_n |\tau|}\,, \tag{18}$$

where the sum ranges over all states that couple to σ (and thus contain an even number of particles) and have zero total (spatial) momentum, and

$$\Delta E_n = E_n - 2m\,. \tag{19}$$

Give an interpretation for the overlap factors $Z_{2,n}$. (This problem is very similar to Exercise 1.3.)

(iii) Equation (18) means that we can use the τ dependence of $\widetilde{C}_L(\tau)$ to pick out the energies of the states in the interacting theory. Here we use this only for the two-particle state that lies closest to threshold, i.e. that goes over to two particles at rest as $\lambda \to 0$. We call this the threshold state. We will find for this state that, when we take the large L limit, $\Delta E = \mathcal{O}(\lambda/L^3)$. There will also be an infinite tower of excited states that become free-particle states with back-to-back nonzero momentum $\vec{p}$ as $\lambda \to 0$, and have $\Delta E \approx 2\omega_p - 2m \sim \mathcal{O}(1/L^2)$. The different L dependence allows us to separate these contributions by hand.

Our approach is thus to calculate $\widetilde{C}_L(\tau)$ order by order in perturbation theory, and discard any terms that correspond to excited states. The result we call $C_{\rm thr}(\tau)$, which has the simple form

$$C_{\rm thr}(\tau) = Z_{\rm thr} e^{-\Delta E_{\rm thr}|\tau|}\,. \tag{20}$$

In fact, we need consider only $\tau > 0$, so we can drop the absolute value. Given this form, show that

$$\Delta E_{\rm thr} = -\frac{\partial_\tau C_{\rm thr}(0)}{C_{\rm thr}(0)}\,. \tag{21}$$

Thus, after subtraction of excited-state contributions, we only need the derivative and value of the correlator at $\tau = 0$.

(iv) Next we expand the various quantities of interest in powers of λ:[1]

$$C_{\rm thr}(0) = 1 + \sum_{n=1}^{\infty} \lambda^n C_{\rm thr}^{(n)}(0)\,, \tag{22}$$

$$\partial_\tau C_{\rm thr}(0) = \sum_{n=1}^{\infty} \lambda^n \partial_\tau C_{\rm thr}^{(n)}(0)\,, \tag{23}$$

$$\Delta E_{\rm thr} = \sum_{n=1}^{\infty} \lambda^n \Delta E_{\rm thr}^{(n)}\,. \tag{24}$$

Show that

$$\Delta E_{\rm thr}^{(1)} = -\partial_\tau C_{\rm thr}^{(1)}(0)\,, \tag{25}$$

and derive the corresponding expression for $\Delta E_{\rm thr}^{(2)}$.

(v) Evaluate $C_{\rm thr}^{(1)}(0)$ and $\partial_\tau C_{\rm thr}^{(1)}(0)$ from the leading order (tree-level) diagram. (Hint: no subtraction of excited state contributions is needed.) Using your result, show that

$$\Delta E_{\rm thr}^{(1)} = \frac{1}{8m^2L^3}\,. \tag{26}$$

We thus find the result quoted in the Lectures (due to Lüscher) that the leading order threshold energy shift scales as $1/L^3$, which is an overlap factor. The coefficient is linear in λ and thus in the scattering amplitude.

(vi) If you have time and energy, consider the one-loop s-channel bubble diagram (the second diagram on Slide 32 of Lecture 1). Show that the contribution where the intermediate momentum is zero scales as λ^2/L^6, and is thus subleading as $L \to \infty$. More interesting is the contribution from nonzero momentum in the loop. Show that these contribute

$$\partial_\tau C_{\rm thr}^{(2)}(0) \supset \frac{1}{64m^2L^3}\frac{1}{L^3}\sum_{\vec p\neq 0}\frac{1}{\omega_p \vec p^{\,2}}\,, \tag{27}$$

which leads to a nonzero contribution to $\Delta E_{\rm thr}^{(2)}$.

[1] Strictly speaking we should use a bare coupling and counterterms to take care of renormalization. We will avoid this complication in this Exercise. See the references for discussion of this issue.

The sum is UV divergent, so we have to introduce a UV regulator and renormalize to do things properly. However, for large $|\vec{p}|$ the sum can be replaced by an integral as the summand is smooth. This then leads to a renormalization of the leading order $1/L^3$ contribution, which is not interesting. Indeed, one can choose a renormalization scheme in which it cancels exactly with a counterterm.

(vii) What is more interesting for us is the IR behavior, which is sensitive to the finiteness of the volume. We *cannot* replace the whole sum in Eq. (27) with an integral, since the summand is singular in the IR. Instead, we use the result

$$\left[\frac{1}{L^3}\sum_{\vec{p}=0}^{\Lambda} - \int^{\Lambda}\frac{d^3p}{(2\pi)^3}\right]\frac{f(p^2)}{p^2} = \frac{\mathcal{I}f(0)}{4\pi^2 L} - \frac{f'(0)}{L^3} + \mathcal{O}(e^{-mL})\,, \quad (28)$$

where $f(p^2)$ is a smooth, nonsingular function, and $\mathcal{I} = -8.914\ldots$ is a known constant (a particular value of a zeta function). To see a simple derivation of this result, see Appendix A of 1509.07929. It is an example of a general expression derived by Lüscher. The key point here is the appearance of the $1/L$ term, rather than the naive $1/L^3$ dependence. Using this result, and your earlier work, determine the corresponding $1/L^4$ contribution to $\Delta E^{(2)}_{\rm thr}$.

We will stop here. This has given an indication of how the power-law corrections to the leading $1/L^3$ behavior arise.