Homework: Quantum Field Theory

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3.2. Derive the *Gordon identity*

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{1}$$

where $q = (p' - p)$.

From the standard covariant form of Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

and can be written as

$$\gamma^\mu p_\mu u(p) = mu(p) \tag{2}$$

From previous definition

$$\bar{u}(p) \equiv u^\dagger(p)\gamma^0$$

and

$$u^\dagger(p)p_\mu^\dagger(\gamma^\mu)^\dagger = mu^\dagger(p)$$

So we have

$$\bar{u}(p)\gamma^0 p_\mu^\dagger(\gamma^\mu)^\dagger\gamma^0 = m\bar{u}(p)$$

Then

$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \frac{\bar{u}(p')\gamma^0 {p'_{\mu'}}^\dagger(\gamma^{\mu'})^\dagger\gamma^0}{m}\gamma^\mu\frac{\gamma^{\mu''}p_{\mu''}u(p)}{m} \\
&= \bar{u}(p')\frac{\gamma^0 {p'_{\mu'}}^\dagger(\gamma^{\mu'})^\dagger\gamma^0\gamma^\mu\gamma^{\mu''}p_{\mu''}}{m^2}u(p)
\end{aligned}$$

Note that p_μ and γ commute, and

$$\begin{aligned}
\gamma^0(\gamma^\mu)^\dagger\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} \\
&= \gamma^\mu
\end{aligned}$$

which means

$$\bar{u}(p)\gamma^\mu p_\mu = m\bar{u}(p)$$

and

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\frac{\gamma^\nu p'_\nu \gamma^\mu \gamma^\nu p_\nu}{m^2}u(p)$$

Now we observe

$$\begin{aligned}
i\sigma^{\mu\nu}q_\nu &= -\frac{1}{2}[\gamma^\mu,\gamma^\nu](p'_\nu - p_\nu) \\
&= -\frac{1}{2}(\gamma^\mu\gamma^\nu p'_\nu - \gamma^\nu\gamma^\mu p'_\nu - \gamma^\mu\gamma^\nu p_\nu + \gamma^\nu\gamma^\mu p_\nu)
\end{aligned}$$

and

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$$

We have

$$\begin{aligned}
i\sigma^{\mu\nu}q_\nu &= -\frac{1}{2}(2\gamma^\mu\gamma^\nu p'_\nu - 2g^{\mu\nu}p'_\nu - 2\gamma^\mu\gamma^\nu p_\nu + 2g^{\mu\nu}p_\nu) \\
&= (p'^\mu - p^\mu) - \gamma^\mu\gamma^\nu(p'_\nu - p_\nu)
\end{aligned}$$

With this (1) becomes

$$\begin{aligned}
\bar{u}(p')\gamma^\mu u(p) &= \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{(p'^\mu - p^\mu) - \gamma^\mu\gamma^\nu(p'_\nu - p_\nu)}{2m}\right]u(p) \\
&= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{\gamma^\mu\gamma^\nu(p'_\nu - p_\nu)}{2m}\right]u(p) \\
&= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{\gamma^\mu\gamma^\nu(p'_\nu - p_\nu)}{2m}\right]u(p)
\end{aligned}$$

We know that

$$\begin{aligned}
\bar{u}(p')\frac{\gamma^\nu p'_\nu\gamma^\mu\gamma^\nu p_\nu}{m^2}u(p) &= \frac{1}{2}\left\{\bar{u}(p')\frac{-\gamma^\nu p'_\nu\gamma^\nu\gamma^\mu p_\nu + 2\gamma^\nu p'_\nu g^{\mu\nu}p_\nu - \gamma^\mu p'_\nu\gamma^\nu\gamma^\nu p_\nu + 2p'_\nu g^{\mu\nu}\gamma^\nu p_\nu}{m^2}u(p)\right\} \\
&= \frac{1}{2}\left\{\bar{u}(p')\frac{-m\gamma^\nu\gamma^\mu p_\nu + 2\gamma^\nu p'_\nu g^{\mu\nu}p_\nu - \gamma^\mu p'_\nu\gamma^\nu m + 2p'_\nu g^{\mu\nu}\gamma^\nu p_\nu}{m^2}u(p)\right\} \\
&= \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{m} - \frac{\gamma^\nu\gamma^\mu p_\nu + \gamma^\mu p'_\nu\gamma^\nu}{2m}\right]u(p) \\
&= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{-\gamma^\mu\gamma^\nu p'_\nu + \gamma^\mu p'_\nu\gamma^\nu}{2m}\right]u(p) \\
&= \bar{u}(p')\left[\frac{p'^\mu}{m} - \frac{\gamma^\mu\gamma^\nu(p'_\nu - p_\nu)}{2m}\right]u(p)
\end{aligned}$$

And it consists with the former one.