Scalar QED

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1 Hydrogen Wavefunction Divergence in Klein-Gordon Equation and Schrödinger Equation

2 Non-relativistic Scalar QED (NRSQED) Matching

2.1 Feynman Rules

2.1.1 Scalar QED (SQED)

Lagrangian

$$\mathcal{L}_{SQED} = |D_\mu \phi|^2 - m^2 |\phi|^2 + \Phi_v^* iv \cdot D\Phi_v \tag{1}$$

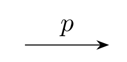
with

$$D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi$$

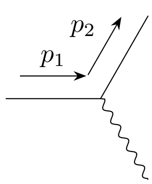
and

$$D_\mu \Phi_v = \partial_\mu \Phi - iZeA_\mu \Phi_v$$

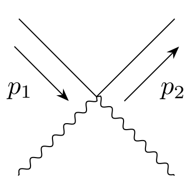
But note that no $\mathbf{A}$ can appear in actual calculation because here only static scalar potential exists. And the Feynman rules



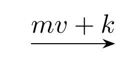
$$= \frac{i}{p^2 - m^2 + i\epsilon}$$



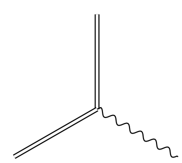
$$= -ie(p_1^\mu + p_2^\mu)$$



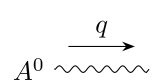
$$= 2ie^2 g^{\mu\nu}$$



$$= \frac{i}{v \cdot k}$$



$$= iZev^\mu$$



$$= \frac{i}{\mathbf{q}^2}$$

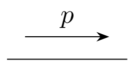
2.1.2 NRSQED

Lagrangian

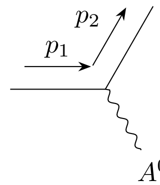
$$\mathcal{L}_{NRSQED} = \varphi^* \left(iD_0 + \frac{\mathbf{D}^2}{2m} \right) \varphi + \delta\mathcal{L} + \Phi_v^* iv \cdot D\Phi_v \tag{2}$$

with the same notation above. Here $\mathbf{D} = \nabla - ie\mathbf{A}$.

Feynman rules are also the same except for the scalar electron side which becomes



$$= \frac{i}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon}$$



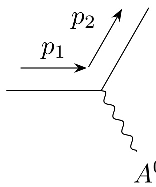
$$= -ie$$

We can ignore all interacting terms involving $\mathbf{A}$.

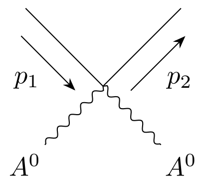
Since we need to match it to $\mathcal{O}(v^2)$ order

$$\delta\mathcal{L} = (D_0\varphi)^*(D_0\varphi) = \frac{\dot{\varphi}^*\dot{\varphi}}{2m} + \frac{e^2\varphi^*\varphi A_0^2}{2m} - \frac{ie}{2m}A_0(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) \tag{3}$$

and it changes the Feynman rules to[1]



$$= -ie(1 + \frac{E_1 + E_2}{2m})$$



$$= \frac{ie^2}{2m}$$

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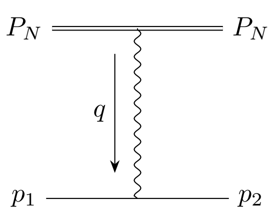
Since we rescaled ϕ by $\frac{1}{\sqrt{2m}}$ to get φ, the in/out states are also changed. We must multiply them by $\sqrt{2m}$ to compensate that change.

2.2 LO

2.2.1 SQED

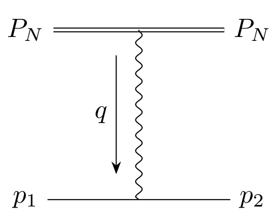
$$i\mathcal{M}^{(0)}_{SQED} =$$



$$= -e^2 v^0 \frac{i(p_1^0 + p_2^0)}{\mathbf{q}^2} = -e^2 v^0 \frac{i}{\mathbf{q}^2}(2m + E_1 + E_2)$$

2.2.2 NRSQED

$$i\mathcal{M}^{(0)}_{NRSQED} =$$



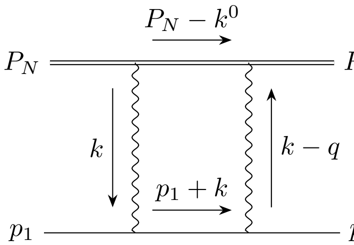
$$= -2me^2 v^0 \frac{i(1 + \frac{E_1 + E_2}{2m})}{\mathbf{q}^2}$$

[1] In this note, p^0 is the zero component of relativistic four momentum, and $E = p^0 - m$.

2.3 NLO

2.3.1 SQED

$i\mathcal{M}^{(1)}_{SQED} =$



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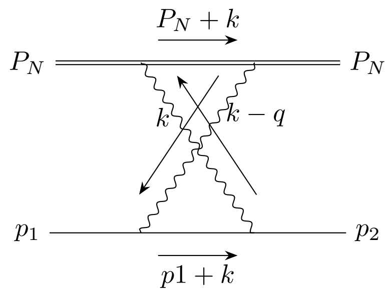

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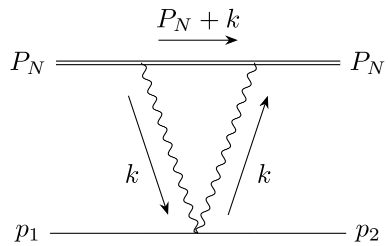

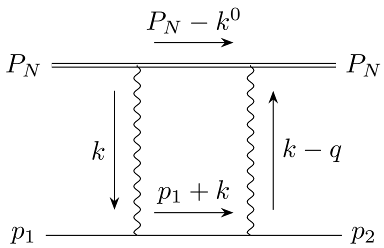

$$= -e^2 v^0 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{\mathbf{k}^2} \quad < +content+ >$$

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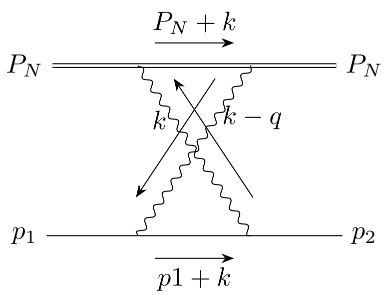

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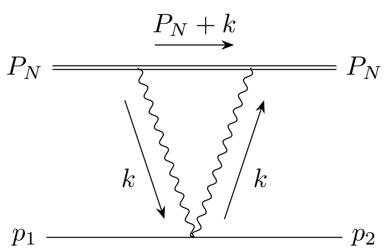

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2.3.2 NRSQED

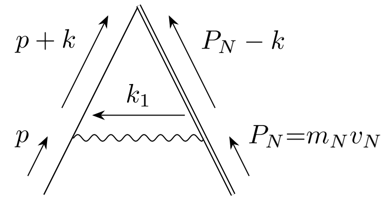
3 Local Operator and Matrix Element of NRSQED

To reproduce the singular behavior of "Klein-Gordon Hydrogen" wavefunction near origin, we can try OPE. But the dependence of x in OPE can be taken as a regularization scheme and thus the result should be the same as local one without renormalization. And the logarithmic terms of x in OPE can be reproduced by the logarithmic divergence of local operators. Since in the study of Klein-Gordon equation we know that the wavefunction only contains logarithmic divergence at the origin so that's the only type of divergence we're looking for.

3.1 LO

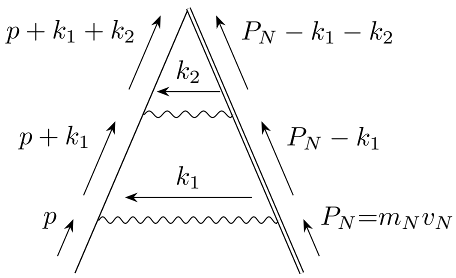
3.2 NLO

$$\langle 0|\psi_e(0)N(0)(-ie\mu^{-\epsilon})\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0(-ie\mu^{-\epsilon})\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle =$$



which doesn't have logarithm divergence[2].

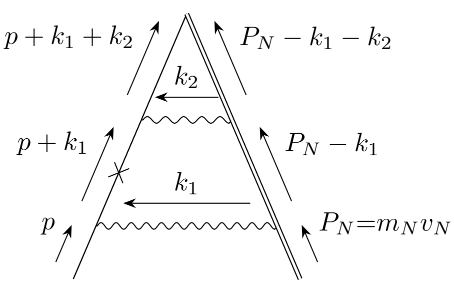
3.3 NNLO



$$=-\mu^{-4\epsilon}e^4\left[\int [dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{2m+2E+k_1^0}{p^0+k_1^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{2m+2E+2k_1^0+k_2^0}{p^0+k_1^0+k_2^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\right]$$

do the shift as above

$$=e^4\left[\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{2m+2E}{E-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{2m+2E}{E-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]$$



$$=16m^2(m+E)\mu^{-4\epsilon}e^4\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{1}{|\mathbf{k_2}|^2-2mE}$$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{\left(\frac{4\pi}{\Delta_2}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^2\Gamma(2)}$$

where $\Delta_2=(1-x)\left(|\mathbf{k_1}|^2x-2Em\right)$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{1}{(|\mathbf{k_1}|^2-2mE/x)^{2-d/2}}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}\Gamma(2-d/2)$$

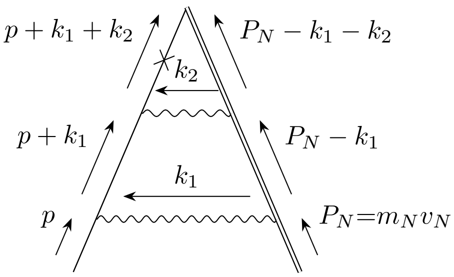
$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\mathrm{d}z\mathrm{d}t\delta(y+z+t-1)\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{zt^{1-d/2}|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2+\Delta_1]^{5-d/2}}\frac{\Gamma(5-d/2)}{\Gamma(2-d/2)}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}\Gamma(2-d/2)$$

[2]After dimensional regularization, the Gamma function in the numerator is something like $\Gamma(n-d/2)$ and Gamma function doesn't have pole at half integer.

where $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mE(z+t/x)$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2(4\pi)^2}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\mathrm{d}z\mathrm{d}t\delta(y+z+t-1)zt^{1-d/2}\frac{d(d+2)}{4}\frac{\Gamma\left(3-d\right)}{(4\pi)^{3-d/2}}\left(\frac{4\pi}{\Delta_1}\right)^{3-d}\left(\frac{4\pi}{x(1-x)}\right)^{2-d/2}$$

$$=-16m^2(m+E)\frac{1}{128\pi^2m^2}(\frac{1}{d-3}+4\log\mu)+\text{finite terms}$$



$$= 16m^2(m+E)\mu^{-4\epsilon}e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_1}|^2-2mE}\frac{|\mathbf{k_2}|^4/4m^2}{[|\mathbf{k_2}|^2-2mE]^2}$$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_1}|^2-2mE}\frac{(1-x)\Gamma(1-d/2)}{8\pi}\left(\frac{4\pi}{\Delta_2}\right)^{1-d/2}\frac{d(d+2)}{4}$$

where $\Delta_2 = x(1-x)|\mathbf{k_1}|^2 - 2mE(1-x)$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\mathrm{d}z\mathrm{d}t\frac{t^{-d/2}}{[|\mathbf{k_1}|^2+\Delta_1]^{3-d/2}}\frac{\Gamma(3-d/2)}{\Gamma(1-d/2)}\delta(y+z+t-1)\frac{\Gamma(1-d/2)}{8\pi}\left(\frac{4\pi}{x(1-x)}\right)^{1-d/2}\frac{d(d+2)x}{4}$$

where $\Delta_1 = y(1-y)\mathbf{p}^2 - 2mEz - 2mE\frac{t}{x}$

$$=16m^2(m+E)\mu^{-4\epsilon}e^4\frac{1}{4m^2}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\mathrm{d}z\mathrm{d}t\delta(y+z+t-1)\frac{1}{(4\pi)^{3-d/2}}\left(\frac{4\pi}{\Delta_1}\right)^{3-d}\frac{\Gamma\left(3-d\right)}{8\pi}\left(\frac{4\pi}{x(1-x)}\right)^{1-d/2}\frac{d(d+2)x}{4}t^{-d/2}$$

$$=16m^2(m+E)\frac{15}{8192\pi^2m^2}(\frac{1}{d-3}+4\log\mu)+\text{finite terms}$$

Appendices

Integral with the structure of the form

$$\int [dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{1}{[p^0+k_1^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}+i\epsilon]^m}\frac{1}{[p^0+k_1^0+k_2^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon]^n}$$

will always produce

$$\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{[p^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}]^m}\frac{1}{[p^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon]^n}$$

with k_1^0 and k_2^0 goes to zero.

For arbitary one loop diagram of the following form, we have

$$\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{k^{2\beta}}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{n-\beta}}\frac{\Gamma(\beta+d/2)}{\Gamma(d/2)}\frac{\Gamma(n-\beta-d/2)}{\Gamma(n)}\left(\frac{4\pi}{\Delta}\right)^{n-\beta-d/2} \tag{4a}$$

$$= \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\beta+d/2)}{\Gamma(d/2)}\frac{\Gamma(n-\beta-d/2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\beta-d/2} \tag{4b}$$

For two loop diagrams of this form ($\epsilon = 3-d$)

$$\mu^{-4\epsilon}\int \frac{\mathrm{d}^d\mathbf{k}_1}{(2\pi)^d}\frac{\mathrm{d}^d\mathbf{k}_2}{(2\pi)^d}\frac{1}{(\mathbf{k_1}-\mathbf{a})^2}\frac{1}{(\mathbf{k_2}-\mathbf{k_1})^2}\frac{\mathbf{k}_1^{2\alpha}}{(\mathbf{k}_1^2-c)^m}\frac{\mathbf{k}_2^{2\beta}}{(\mathbf{k}_2^2-d)^n} \tag{5}$$

The integral is evaluated to

$$\mu^{-4\epsilon}\int_0^1\prod_{i=1}^{2}\mathrm{d}x_i\delta(\sum x_i-1)\prod x_i^{d_i-1}\frac{\Gamma(n+1)}{\Gamma(n)}\frac{1}{(4\pi)^{n+1-\beta}}\frac{\Gamma(\beta+d/2)}{\Gamma(d/2)}\frac{\Gamma(n+1-\beta-d/2)}{\Gamma(n+1)}\left(\frac{4\pi}{\alpha(x_i)}\right)^{n+1-\beta-d/2}$$
$$\int\frac{\mathrm{d}^d\mathbf{k}_1}{(2\pi)^d}\frac{1}{(\mathbf{k_1}-\mathbf{a})^2}\frac{\mathbf{k}_1^{2\alpha}}{(\mathbf{k}_1^2-c)^m}\frac{1}{(\mathbf{k}_1-\Delta_2)^{n+1-\beta-d/2}}$$
$$=\mu^{-4\epsilon}\int_0^1\prod_{i=1}^{2}\mathrm{d}x_i\delta(\sum x_i-1)\prod x_i^{d_i-1}\frac{1}{(4\pi)^{n+1-\beta}}\frac{\Gamma(\beta+d/2)}{\Gamma(d/2)}\frac{\Gamma(n+1-\beta-d/2)}{\Gamma(n)}\left(\frac{4\pi}{\alpha(x_i)}\right)^{n+1-\beta-d/2}\int_0^1\prod_{i=1}^{3}\mathrm{d}y_j\delta(\sum y_j-1)\prod y_j^{d_j-1}$$
$$\frac{\Gamma(m+n+2-\beta-d/2)}{\Gamma(m)\Gamma(n+1-\beta-d/2)}\frac{1}{(4\pi)^{m+n+2-\alpha-\beta-d/2}}\frac{\Gamma(\alpha+d/2)}{\Gamma(d/2)}\frac{\Gamma(m+n+2-\alpha-\beta-d)}{\Gamma(m+n+2-\beta-d/2)}\left(\frac{4\pi}{\Delta_1}\right)^{m+n+2-\alpha-\beta-d}$$
$$=\mu^{-4\epsilon}\int_0^1\prod_{i=1}^{2}\mathrm{d}x_i\delta(\sum x_i-1)\prod x_i^{d_i-1}\frac{1}{(4\pi)^d}\frac{\Gamma(\beta+d/2)}{\Gamma(d/2)}\frac{1}{\Gamma(n)}\left(\frac{1}{\alpha(x_i)}\right)^{n+1-\beta-d/2}$$
$$\int_0^1\prod_{i=1}^{3}\mathrm{d}y_j\delta(\sum y_j-1)\prod y_j^{d_j-1}\frac{\Gamma(\alpha+d/2)}{\Gamma(d/2)}\frac{\Gamma(m+n+2-\alpha-\beta-d)}{\Gamma(m)}\left(\frac{1}{\Delta_1}\right)^{m+n+2-\alpha-\beta-d}$$

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