Hydrogen

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1 Matching

QED Lagrangian is

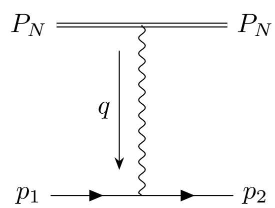
$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \tag{1}$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \tag{2}$$

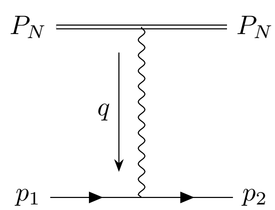
In tree level[1]

$$i\mathcal{M}^{(0)}_{QED} =$$



$$= -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \bar{u}_e(p_2)\gamma_0 u_e(p1)$$

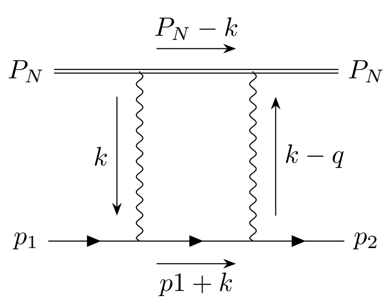
$$i\mathcal{M}^{(0)}_{NRQED} =$$



$$= -e^2 \bar{u}_N(P_N) v^0 u_N(P_N) \frac{i}{\mathbf{q}^2} \psi^\dagger(p_2)\psi(p1)$$

The box diagram for NRQED process is

$$i\mathcal{M}^{(1)}_{NRQED} =$$

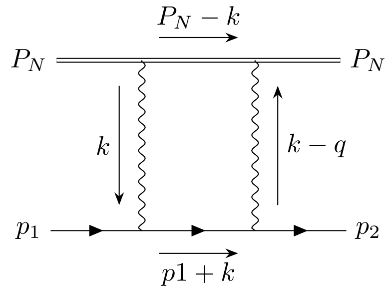


$$\begin{aligned}
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int [\mathrm{d}k] \frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(-k^0 + i\epsilon)(p_1^0 + k^0 - m - \frac{(\mathbf{p_1}+\mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(E_1 - \frac{(\mathbf{p_1}+\mathbf{k})^2}{2m})} \psi(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2(E_1 - \frac{\mathbf{k}^2}{2m})} \psi(p_1)
\end{aligned}$$

[1] Note that there's no Gamma matrice in the heavy particle side, they can only appear in the QED side.

The box and crossed box diagram for QED process is

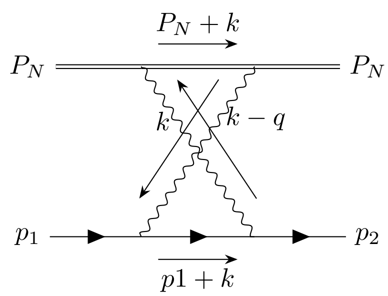
$$i\mathcal{M}_1^{(1)} =$$



$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0 + k_i\gamma^i\gamma^0 + \sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2 + m^2 - p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0 + (k_i - p_{1i})\gamma^i\gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 - p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.

$$i\mathcal{M}_2^{(1)} =$$



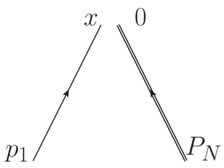
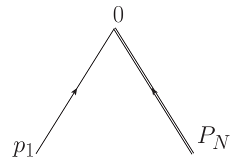
$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0 + k_i\gamma^i\gamma^0 - \sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2 + m^2 + p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0 + (k_i - p_{1i})\gamma^i\gamma^0 - \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 + p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

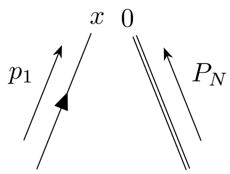
$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

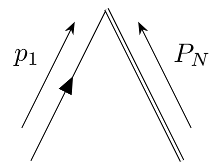
$$
\begin{aligned}
i\mathcal{M}_1^{(1)} + i\mathcal{M}_2^{(1)} &= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2 + k^2 + m^2 + (k_i - p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 - {p_1^0}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2 + k^2 + m^2 + (k_i - p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 - \mathbf{p_1}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)
\end{aligned}
$$

Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order. From this we can conclude that a contact interaction is presented at α^2 order with coefficient $\frac{4\pi e^4}{3m^2} = \frac{(4\pi)^3\alpha^2}{3m^2}$.

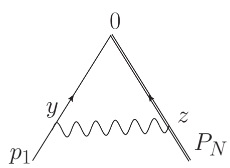
Now consider operator product expansion (all matrix elements below are under momentum space unless explicitly pointed out).

Tree level matching:  & 

$$\langle 0|T\psi(x)N(0)|pP_N\rangle =$$  $$= u_e(p)u_N(P_N)e^{-ip\cdot x}$$

$$\langle 0|T\psi_e(x)N(0)|pP_N\rangle =$$  $$= \psi_e(p)u_N(P_N)$$

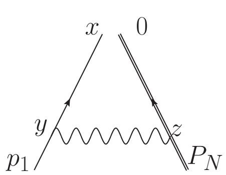
At leading order $u_e(p) = \begin{pmatrix}\psi_e(p)\\ 0\end{pmatrix}$. (If we're only interested in the hard region contribution, which is independent of states, the leading order is independent of any on-shell momentums.)

One loop scenario for NRQED case: 

$$\begin{aligned}
\langle 0|\psi_e(0)N(0)e\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle &= e^2u_N(v_N)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\
&= -ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)\\
&= -ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1)
\end{aligned}$$

drop p_1

$$= ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{2m}{\mathbf{k}^4}\psi(p_1)$$

For QED case: 

$$\begin{aligned}
\langle 0|\psi(x)N(0)e\int \mathrm{d}^4y\bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle\,[2] &= e^2u_N(v_N)\int[\mathrm{d}k]e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= e^2u_N(v_N)\int[\mathrm{d}k]e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= ie^2u_N(v_N)\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}$$

drop $\mathbf{p_1}$

$$
\begin{aligned}
&= ie^2 u_N(v_N) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{m+\sqrt{\mathbf{k}^2+m^2}}{2\mathbf{k}^2[\mathbf{k}^2+m^2-m\sqrt{\mathbf{k}^2+m^2}]} u_e(p_1) \\
&= ie^2 u_N(v_N) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{m+\sqrt{\mathbf{k}^2+m^2}}{2\mathbf{k}^2\sqrt{\mathbf{k}^2+m^2}[\sqrt{\mathbf{k}^2+m^2}-m]} u_e(p_1) \\
&= ie^2 u_N(v_N) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{2m^2+2m\sqrt{\mathbf{k}^2+m^2}+\mathbf{k}^2}{2\mathbf{k}^4\sqrt{\mathbf{k}^2+m^2}} u_e(p_1)
\end{aligned}
$$

we can see there's no UV divergence here. IR divergence is presented but it's exactly the same with the NRQED one so there's nothing that needs to concern about.

we can divide this integral into 3 parts. The easiest part is

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{m}{\mathbf{k}^4} = -m\pi x$$

and also

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{m^2}{\mathbf{k}^4\sqrt{\mathbf{k}^2+m^2}} = \frac{1}{mx} G^{2,1}_{1,3}\left(\frac{m^2x^2}{4} \middle| \begin{array}{c} 2 \\ \frac{1}{2}, \frac{3}{2}, 0 \end{array} \right)$$

and by applying contour integral around the branch cut we can do the transformation

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{1}{2\mathbf{k}^2\sqrt{\mathbf{k}^2+m^2}} = -\frac{i}{2x} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ikx}}{k\sqrt{k^2+m^2}}$$

push the contour

$$
\begin{aligned}
&= -\frac{i}{2x} 2\pi i \left(\int_{+i\infty-\epsilon}^{im} \mathrm{d}k \frac{e^{ikx}}{k\sqrt{k^2+m^2}} + \int_{im}^{+i\infty+\epsilon} \mathrm{d}k \frac{e^{ikx}}{k\sqrt{k^2+m^2}} \right) \\
&\overset{k\to -ik}{=\!=\!=} -\frac{-2\pi}{2x} \sqrt{2} e^{i3\pi/4} \int_m^{\infty} \mathrm{d}k \frac{e^{-kx}}{k\sqrt{k^2-m^2}} \\
&= \frac{\pi}{x} \sqrt{2} e^{i3\pi/4} \int \mathrm{d}x K_0(mx) \\
&= \frac{\pi^2\sqrt{2}}{2} e^{i3\pi/4} (\boldsymbol{L}_{-1}(mx)K_0(mx) + \boldsymbol{L}_0(mx)K_1(mx))
\end{aligned}
$$

If count the NRQED part in, the first part is

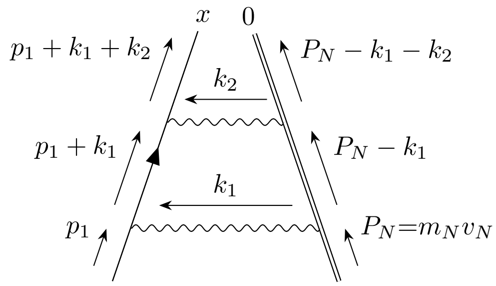
$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} m \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}-1}{\mathbf{k}^4} = -\frac{1}{2}\pi m x$$

the second part is

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}} m^2}{\mathbf{k}^4\sqrt{\mathbf{k}^2+m^2}} - \frac{m}{\mathbf{k}^4} = \frac{1}{4} mx G^{2,1}_{1,3}\left(\frac{m^2x^2}{4} \middle| \begin{array}{c} 1 \\ -\frac{1}{2}, \frac{1}{2}, -1 \end{array} \right)$$

[2] $\langle 0|\psi(x)N(0)e\int \mathrm{d}^4y\bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle = e^2\int \mathrm{d}^4y\int \mathrm{d}^4z\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{\mathbf{k^2}}e^{-ik\cdot(z-y)}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\tilde{S}_e(k_1)e^{-ik_1\cdot(y-x)}\int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\tilde{S}_N(k_2)u_N(v_N)u_e(p)e^{-ip_1\cdot y}$.

Two loop scenario for QED case $\langle 0|T\psi(x)N(0)e\int \mathrm{d}^4y_1\bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4z_1\bar{N}NA^0 e\int \mathrm{d}^4y_2\bar{\psi}\gamma^0\psi A^0 e\int \mathrm{d}^4z_2\bar{N}NA^0|eN\rangle$:

$-$ 

$$=e^4\int[dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{\not p_1+\not k_1+\not k_2+m}{(p_1+k_1+k_2)^2-m^2+i\epsilon}\gamma^0\frac{\not p_1+\not k_1+m}{(p_1+k_1)^2-m^2+i\epsilon}\gamma^0u_N(v_N)u_e(p_1)$$

$$=e^4\int[dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}\frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2}\frac{4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(p_1^0+k_1^0)(\not k_1+\not k_2)\gamma^0-\not k_2\not k_1}{[(p_1+k_1+k_2)^2-m^2+i\epsilon][(p_1+k_1)^2-m^2+i\epsilon][-k_1^0-k_2^0+i\epsilon][-k_1^0+i\epsilon]}u_N(v_N)u_e(p_1)$$ [3]

$$=ie^4\int[\mathrm{d}k_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2(k_1^0+p_1^0)[(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2}-k_1^0-p_1^0)+(k_2^i\gamma_i+\not k_1)\gamma^0]-[\gamma^0(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2}-k_1^0-p_1^0)-k_2^i\gamma_i]\not k_1}{2\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2}(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2}-p_1^0+\frac{2((\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2)+2\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2}-p_1^0}{2((\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2)}i\epsilon)}$$
$$\frac{1}{-k_1^0+i\epsilon}\frac{1}{(p_1+k_1)^2-m^2+i\epsilon}\frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2}u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

define $a=(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2$ and $b=\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2}-k_1^0-p_1^0+k_2^i\gamma_i\gamma^0=\sqrt{a}-k_1^0-p_1^0+k_2^i\gamma_i\gamma^0$ (pole location is $\sqrt{a}-k_1^0-p_1^0-\frac{i\epsilon}{2\sqrt{a}}$), and note that the long coefficient of the first ϵ above is positive

$$=ie^4\int[\mathrm{d}k_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(k_1^0+p_1^0)[b+\not k_1\gamma^0]-\gamma^0b\not k_1}{2\sqrt{a}(\sqrt{a}-p_1^0)[-k_1^0+i\epsilon][(p_1+k_1)^2-m^2+i\epsilon]}\frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2}u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

also define b' so that $b=b'-k_1^0$ ($b'=\sqrt{a}-p_1^0+k_2^i\gamma_i\gamma^0$) and $a'=(\mathbf{p_1}+\mathbf{k_1})^2+m^2$

$$=ie^4\int[\mathrm{d}k_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(k_1^0+p_1^0)[b'+k_1^i\gamma_i\gamma^0]-\gamma^0(b'-k_1^0)(k_1^0\gamma^0+k_1^i\gamma_i)}{2\sqrt{a}(\sqrt{a}-p_1^0)[-k_1^0+i\epsilon][(p_1+k_1)^2-m^2+i\epsilon]|\mathbf{k_1}|^2|\mathbf{k_2}|^2}u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

the pole location is $\sqrt{a'}-p_1^0-\frac{i\epsilon}{2\sqrt{a}}$

$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a'}(\sqrt{a}-p_1^0+(k_1+k_2)^i\gamma_i\gamma^0)-(\sqrt{a'}-\sqrt{a}+k_2^i\gamma_i\gamma^0)(\sqrt{a'}-p_1^0-k_1^i\gamma_i\gamma^0)+2{p_1^0}^2+2\mathbf{k_1}\cdot\mathbf{p_1}+2\sqrt{a'}p_1^0}{4\sqrt{a}\sqrt{a'}(\sqrt{a'}-p_1^0)(\sqrt{a}-p_1^0)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}$$
$$u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

shift both loop momentum[4] so that $a=|\mathbf{k_2}|^2+m^2$ and $a'=|\mathbf{k_1}|^2+m^2$, now $b=\sqrt{a}-k_1^0+(k_2-k_1)^i\gamma_i\gamma^0$ and $b'=\sqrt{a}+(k_2-k_1)^i\gamma_i\gamma^0$

$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$
$$\frac{2\sqrt{a'}(\sqrt{a}-p_1^0+(k_2-p_1)^i\gamma_i\gamma^0)-(\sqrt{a'}-\sqrt{a}+(k_2-k_1)^i\gamma_i\gamma^0)(\sqrt{a'}-p_1^0-(k_1-p_1)^i\gamma_i\gamma^0)+2m^2+2\mathbf{k_1}\cdot\mathbf{p_1}+2\sqrt{a'}p_1^0}{4\sqrt{a}\sqrt{a'}(\sqrt{a'}-p_1^0)(\sqrt{a}-p_1^0)|\mathbf{k_1}-\mathbf{p_1}|^2|\mathbf{k_2}-\mathbf{k_1}|^2}$$

drop $\mathbf{p_1}$

$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a'}(\sqrt{a}-m+k_2^i\gamma_i\gamma^0)-(\sqrt{a'}-\sqrt{a}+(k_2-k_1)^i\gamma_i\gamma^0)(\sqrt{a'}-m-k_1^i\gamma_i\gamma^0)+2m^2+2\sqrt{a'}m}{4\sqrt{a}\sqrt{a'}(\sqrt{a'}-m)(\sqrt{a}-m)|\mathbf{k_1}|^2|\mathbf{k_2}-\mathbf{k_1}|^2}$$
$$u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

rewrite it with $a_1=a'$ and $a_2=a$, now $a_1=\sqrt{|\mathbf{k_1}|^2+m^2}$ and $a_2=\sqrt{|\mathbf{k_2}|^2+m^2}$

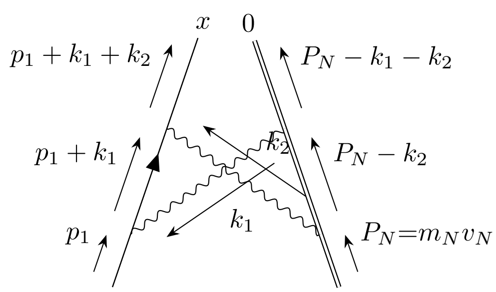
$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a_1}(\sqrt{a_2}-m+k_2^i\gamma_i\gamma^0)-(\sqrt{a_1}-\sqrt{a_2}+(k_2-k_1)^i\gamma_i\gamma^0)(\sqrt{a_1}-m-k_1^i\gamma_i\gamma^0)+2m^2+2\sqrt{a_1}m}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1}-m)(\sqrt{a_2}-m)|\mathbf{k_1}|^2|\mathbf{k_2}-\mathbf{k_1}|^2}$$
$$u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

to investigate the divergent behaviour of the integral, rewrite the integral before the shift ($a_1 = (\mathbf{p_1}+\mathbf{k_1})^2+m^2$ and $a_2 = (\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2+m^2$)

$$
\begin{aligned}
=&-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a_1}(\sqrt{a_2}-p_1^0+(k_1+k_2)^i\gamma_i\gamma^0)-(\sqrt{a_1}-\sqrt{a_2}+k_2^i\gamma_i\gamma^0)(\sqrt{a_1}-p_1^0-k_1^i\gamma_i\gamma^0)+2{p_1^0}^2+2\mathbf{k_1}\cdot\mathbf{p_1}+2\sqrt{a_1}p_1^0}{4\sqrt{a_2}\sqrt{a_1}(\sqrt{a_1}-p_1^0)(\sqrt{a_2}-p_1^0)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}\\
&u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}
\end{aligned}
$$

we can see it's UV logarithm divergent and IR logarithm divergent (only for the ${p_1^0}^2 \approx m^2$ term). Now we must regularize it to dimention $(d-1)$

$$
\begin{aligned}
=&-e^4\int\frac{\mathrm{d}^{d-1}\mathbf{k_1}}{(2\pi)^{d-1}}\frac{\mathrm{d}^{d-1}\mathbf{k_2}}{(2\pi)^{d-1}}\frac{2\sqrt{a_1}(\sqrt{a_2}-m+k_2^i\gamma_i\gamma^0)-(\sqrt{a_1}-\sqrt{a_2}+(k_2-k_1)^i\gamma_i\gamma^0)(\sqrt{a_1}-m-k_1^i\gamma_i\gamma^0)+2m^2+2\sqrt{a_1}m}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1}-m)(\sqrt{a_2}-m)|\mathbf{k_1}|^2|\mathbf{k_2}-\mathbf{k_1}|^2}\\
&u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}
\end{aligned}
$$

$-$ 

$$
\begin{aligned}
=&e^4\int[dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_2^0+i\epsilon}\frac{\not{p}_1+\not{k}_1+\not{k}_2+m}{(p_1+k_1+k_2)^2-m^2+i\epsilon}\gamma^0\frac{\not{p}_1+\not{k}_1+m}{(p_1+k_1)^2-m^2+i\epsilon}\gamma^0u_N(v_N)u_e(p_1)\\
=&e^4\int[dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}\frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2}\frac{4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(p_1^0+k_1^0)(\not{k}_1+\not{k}_2)\gamma^0-\not{k}_2\not{k}_1}{[(p_1+k_1+k_2)^2-m^2+i\epsilon][(p_1+k_1)^2-m^2+i\epsilon][-k_1^0-k_2^0+i\epsilon][-k_2^0+i\epsilon]}u_N(v_N)u_e(p_1)\\
=&ie^4\int[\mathrm{d}k_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(k_1^0+p_1^0)[b+\not{k}_1\gamma^0]-\gamma^0b\not{k}_1}{2\sqrt{a_2}(\sqrt{a_2}-k_1^0+i\epsilon)\left[k_1^0+p_1^0-\sqrt{a_2}+\frac{i\epsilon}{2\sqrt{a_2}}+i\epsilon\right][(p_1+k_1)^2-m^2+i\epsilon]}\frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2}u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}\\
=&-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a_1}(\sqrt{a_2}-p_1^0+(k_1+k_2)^i\gamma_i\gamma^0)+(\sqrt{a_1}+\sqrt{a_2}+k_2^i\gamma_i\gamma^0)(\sqrt{a_1}+p_1^0+k_1^i\gamma_i\gamma^0)-2{p_1^0}^2-2\mathbf{k_1}\cdot\mathbf{p_1}+2\sqrt{a_1}p_1^0}{4\sqrt{a_1}\sqrt{a_2}\left(\sqrt{a_1}+\sqrt{a_2}\right)\left(\sqrt{a_2}-p_1^0\right)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}\\
&u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}
\end{aligned}
$$

do the shift

$$
\begin{aligned}
=&-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a_1}(\sqrt{a_2}-p_1^0+(k_1+k_2)^i\gamma_i\gamma^0)+(\sqrt{a_1}+\sqrt{a_2}+k_2^i\gamma_i\gamma^0)(\sqrt{a_1}+p_1^0+k_1^i\gamma_i\gamma^0)-2{p_1^0}^2-2\mathbf{k_1}\cdot\mathbf{p_1}+2\sqrt{a_1}p_1^0}{4\sqrt{a_1}\sqrt{a_2}\left(\sqrt{a_1}+\sqrt{a_2}\right)\left(\sqrt{a_2}-p_1^0\right)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}\\
&u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}
\end{aligned}
$$

drop $\mathbf{p_1}$

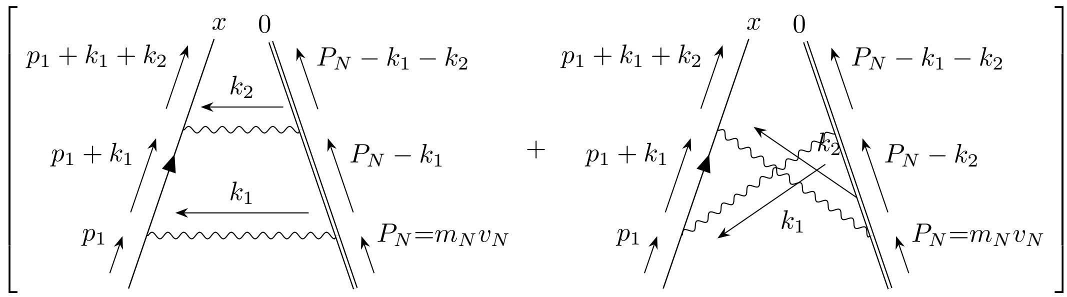
$$
\begin{aligned}
=&-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{2\sqrt{a_1}(\sqrt{a_2}-m+(k_1+k_2)^i\gamma_i\gamma^0)+(\sqrt{a_1}+\sqrt{a_2}+k_2^i\gamma_i\gamma^0)(\sqrt{a_1}+m+k_1^i\gamma_i\gamma^0)-2m^2+2\sqrt{a_1}m}{4\sqrt{a_1}\sqrt{a_2}\left(\sqrt{a_1}+\sqrt{a_2}\right)\left(\sqrt{a_2}-m\right)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}\\
&u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}
\end{aligned}
$$

---

[3] With a $u_e(p_1)$ on the right hand side, $(\not{p}_1+\not{k}_1+\not{k}_2+m)\gamma^0(\not{p}_1+\not{k}_1+m)\gamma^0 = (\not{p}_1+\not{k}_1+\not{k}_2+m)[2(p_1^0+k_1^0)\gamma^0-(\not{p}_1+\not{k}_1-m)] = 2(p_1^0+k_1^0)(\not{p}_1+\not{k}_1+\not{k}_2+m)\gamma^0-(\not{p}_1+\not{k}_1+\not{k}_2+m)(\not{p}_1+\not{k}_1-m) = 2(p_1^0+k_1^0)(\not{p}_1+\not{k}_1+\not{k}_2+m)\gamma^0-(\not{p}_1+\not{k}_1+\not{k}_2+m)\not{k}_1 = 2(p_1^0+k_1^0)(\not{p}_1+\not{k}_1+\not{k}_2+m)\gamma^0-(2p_1\cdot k_1-\not{k}_1\not{p}_1+\not{k}_2\not{k}_1+\not{k}_1m) = 2(p_1^0+k_1^0)(\not{p}_1+\not{k}_1+\not{k}_2+m)\gamma^0-(2p_1\cdot k_1+\not{k}_2\not{k}_1) = 2(p_1^0+k_1^0)[2(p_1^0+k_1^0+k_2^0)-\gamma^0(\not{p}_1+\not{k}_1+\not{k}_2-m)]-(2p_1\cdot k_1+\not{k}_2\not{k}_1) = 2(p_1^0+k_1^0)[2p_1^0+(\not{k}_1+\not{k}_2)\gamma^0]-(2p_1\cdot k_1+\not{k}_2\not{k}_1) = 4{p_1^0}^2+2p_1^0k_1^0+2\mathbf{p_1}\cdot\mathbf{k_1}+2(p_1^0+k_1^0)(\not{k}_1+\not{k}_2)\gamma^0-\not{k}_2\not{k}_1$.

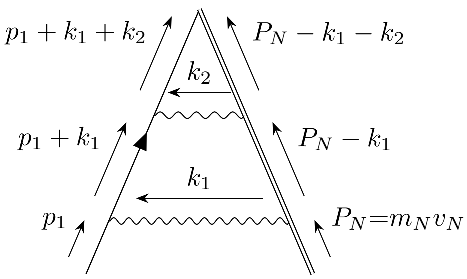
[4] $k_1\to k_1' = k_1+p_1$ and $k_2\to k_2' = k_1+k_2+p_1 = k_1'+k_2$.

The sum of QED diagram at NNLO is

$-$ 

$$=e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

For NRQED case ($\langle 0|\psi_e(0)N(0)e\int \mathrm{d}^4y_1\bar{\psi}_e\psi_eA^0e\int \mathrm{d}^4z_1\bar{N}NA^0e\int \mathrm{d}^4y_2\bar{\psi}_e\psi_eA^0e\int \mathrm{d}^4z_2\bar{N}NA^0|eN\rangle$)

$-$ 

$$=e^4\int[dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{1}{p_1^0+k_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{1}{p_1^0+k_1^0+k_2^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\psi_e(p_1)u_N(v_N)^{5}$$

$$=ie^4\int[dk_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0+i\epsilon}\frac{1}{p_1^0+k_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{1}{p_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+2i\epsilon}\psi_e(p_1)u_N(v_N)$$

$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{p_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1})^2}{2m}+2i\epsilon}\frac{1}{p_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+2i\epsilon}\psi_e(p_1)u_N(v_N)$$

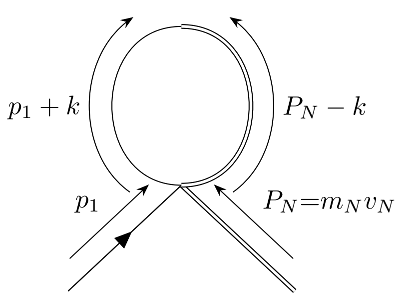
do the shift as above

$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{p_1^0-m-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{1}{p_1^0-m-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\psi_e(p_1)u_N(v_N)$$

drop $\mathbf{p_1}$

$$=-e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{1}{-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\psi_e(p_1)u_N(v_N)$$

There's also a contact term

$-$ 

$$=\frac{4\pi e^4}{3m}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon}\frac{1}{-k^0+i\epsilon}\psi_e(p_1)u_N(v_N)$$

$$=i\frac{4\pi e^4}{3m}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{p_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+2i\epsilon}\psi_e(p_1)u_N(v_N)$$

[5]Clearly in this line, if this NRQCD diagram is crossed, the second pole would become $-k_2^0+i\epsilon$ and the whole formula is zero (since both poles of k_1^0 would be in the same side).

drop $\mathbf{p_1}$

$$= i\frac{4\pi e^4}{3m}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{-\frac{\mathbf{k}^2}{2m}+2i\epsilon}\psi_e(p_1)u_N(v_N)$$

2 HSET

2.1 Lagrangian

For scalar QED

$$\mathcal{L} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi$$

where

$$D_\mu = \partial_\mu + ieA_\mu$$

In Schwartz's QFT (Chap. 35) he mentioned a choice of χ_v and $\tilde{\chi}_v$:

$$\phi(x) = e^{-imv\cdot x}\frac{1}{\sqrt{2m}}(\chi_v(x)+\tilde{\chi}_v(x)) \tag{3}$$

$$\chi_v(x) = e^{imv\cdot x}\frac{1}{\sqrt{2m}}(iv\cdot D+m)\phi(x),\ \tilde{\chi}_v(x) = e^{imv\cdot x}\frac{1}{\sqrt{2m}}(-iv\cdot D+m)\phi(x) \tag{4}$$

Put (3) into (4), a simple relation is derived:

$$(-iv\cdot D)\chi_v(x) = (2m+iv\cdot D)\tilde{\chi}_v(x)$$

It can also be writen as

$$2m\tilde{\chi}_v = (-iv\cdot D)(\chi_v+\tilde{\chi}_v)$$

Use this result

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2m}\Big\{\{[D^\mu(\chi_v+\tilde{\chi}_v)]^\dagger + imv^\mu(\chi_v+\tilde{\chi}_v)^\dagger\}\{[D_\mu(\chi_v+\tilde{\chi}_v)] - imv_\mu(\chi_v+\tilde{\chi}_v)\} - m^2(\chi_v+\tilde{\chi}_v)^\dagger(\chi_v+\tilde{\chi}_v)\Big\} \\
&= (\chi_v+\tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v+\tilde{\chi}_v) + \frac{1}{2m}[D^\mu(\chi_v+\tilde{\chi}_v)]^\dagger D_\mu(\chi_v+\tilde{\chi}_v) \qquad (5) \\
&= (\chi_v(x)+\tilde{\chi}_v(x))^\dagger(iv\cdot D)(\chi_v(x)+\tilde{\chi}_v(x)) + \mathcal{O}(\frac{1}{m}) \qquad (6)
\end{aligned}$$

(note that $D_\mu\phi = e^{-imv\cdot x}[D_\mu(\chi_v+\tilde{\chi}_v) - imv_\mu(\chi_v+\tilde{\chi}_v)]$ and $-imv^\mu[D_\mu(\chi_v+\tilde{\chi}_v)]^\dagger(\chi_v+\tilde{\chi}_v) = imv^\mu(\chi_v+\tilde{\chi}_v)^\dagger D_\mu(\chi_v+\tilde{\chi}_v) -$ *total derivative term*)

Use the leading order of (5)

$$\begin{aligned}
\mathcal{L}^{(0)} &= (\chi_v+\tilde{\chi}_v)^\dagger(iv\cdot D)(\chi_v+\tilde{\chi}_v) \\
&= \chi_v^\dagger iv\cdot D\chi_v + \tilde{\chi}_v^\dagger iv\cdot D(\chi_v+\tilde{\chi}_v) + \chi_v^\dagger iv\cdot D\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + (iv\cdot D\chi_v)^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + [(-2m-iv\cdot D)\tilde{\chi}_v]^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D+4m)\tilde{\chi}_v
\end{aligned}$$

We can have the final form[6]

$$\mathcal{L} = \chi_v^\dagger iv\cdot D\chi_v - \tilde{\chi}_v^\dagger(iv\cdot D+4m)\tilde{\chi}_v + \mathcal{O}(\frac{1}{m})$$

[6] With one problem: if we can tolerate coupled particle-anti particle pair, we can trade $iv\cdot D$ for mass term, so the leading part is the same but the anti-particle part could be different with the mixing?

2.2 Quantization

2.2.1 HQET as an example

The leading term of HQET Lagrangian is

$$\mathcal{L} = \bar{Q}_v (iv \cdot D) Q_v$$

$$Q_v(x) = e^{imv\cdot x}\frac{1+\not{v}}{2}\psi(x)$$

In free Dirac fermion theory we know that

$$\begin{aligned}\left\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\right\} &= \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab}\\ \{a_\mathbf{p}, a_{\mathbf{p}'}^\dagger\} &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\end{aligned}$$

also the plane wave expansion of ψ is

$$\begin{aligned}\psi(x) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a_\mathbf{p} u(p) e^{-ip\cdot x}\\ &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{\sqrt{2mv^0}} a_v \sqrt{m} u(v) e^{-imv\cdot x - ik\cdot x}\end{aligned}$$

using normalization of states $u(k) = \sqrt{m}u(v)$[7], $\langle p'|p\rangle = 2E_p(2\pi)^3\delta^{(3)}(\mathbf{p}'-\mathbf{p})$ and $\langle v', k'|v, k\rangle = 2v^0\delta_{vv'}(2\pi)^3\delta^{(3)}(\mathbf{k}'-\mathbf{k})$ we have $|p\rangle = \sqrt{m}\,|v\rangle$ ($|p\rangle = \sqrt{2E_p}a_\mathbf{p}^\dagger\,|0\rangle$ while $|v,k\rangle = \sqrt{2v^0}a_{v,\mathbf{k}}^\dagger\,|0\rangle$)

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}} a_v u(v) e^{-imv\cdot x - ik\cdot x}$$

Using the definition of $Q_v(x)$

$$\begin{aligned}Q_v(x) &= e^{imv\cdot x}\frac{1+\not{v}}{2}\psi(x)\\ &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}} a_v \frac{1+\not{v}}{2} u(v) e^{-ik\cdot x}\\ &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{\sqrt{2v^0}} a_v u(v) e^{-ik\cdot x}\end{aligned}$$

The commutation relation should be

$$\{Q_{va}(\mathbf{x}), Q_{v'b}(\mathbf{x}')\} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{\mathrm{d}^3 k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0 v'^0}}\{a_v, a_{v'}^\dagger\} u_a(v) u_b^\dagger(v') e^{-ik\cdot x + ik'\cdot x'}$$

using $\sum_s u_a(v)u_b^\dagger(v) = \frac{1}{m}\sum_s u_a(p)u_b^\dagger(p) = [(\not{v}+1)\gamma^0]_{ab}$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{\mathrm{d}^3 k'}{(2\pi)^3}\frac{1}{\sqrt{4v^0 v'^0}}\{a_v, a_{v'}^\dagger\}[(\not{v}+1)\gamma^0]_{ab} e^{-ik\cdot x + ik'\cdot x'}$$

assuming $\{a_v, a_{v'}\} = (2\pi)^3\delta_{vv'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$

$$\begin{aligned}&= \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{2v^0}[(\not{v}+1)\gamma^0]_{ab} e^{-ik\cdot(x-x')}\delta_{vv'}\\ &= [\frac{(\not{v}+1)\gamma^0}{2v^0}]_{ab}\delta_{vv'}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\end{aligned}$$

[7] The relation $\bar{u}^s(p)\gamma^\mu u^s(p) = 2p^\mu$ can be derived using Gordon identity, same for $\bar{u}^s(v)\gamma^\mu u^s(v) = 2v^\mu$, but it's actually $\bar{u}u$.

2.2.2 HSET

The anti-particle field is decoupled so we don't have to consider that for now. The equation-of-motion is

$$\begin{cases} v \cdot D\chi_v^\dagger = 0 \\ v \cdot D\chi_v = 0 \end{cases}$$

By definition

$$\begin{aligned} \chi_v(x) &= \frac{e^{imv\cdot x}}{\sqrt{2m}}(iv \cdot D + m)\phi(x) \\ &= \frac{1}{\sqrt{2m}}(iv \cdot D + 2m)e^{imv\cdot x}\phi(x) \end{aligned}$$

Obviously the plane wave expansion should be irrelevant of the heavy particle mass, which means the exponential part is $e^{-ik\cdot x}$ where k marks the offshellness.