Note on Braaten's Paper

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1 Intro

Hamiltonian [Braaten and Platter(2008)]:
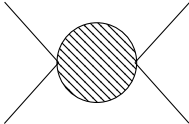
$$\mathcal{H} = \sum_\sigma \frac{1}{2m} \nabla \psi_\sigma^\dagger \cdot \nabla \psi_\sigma^{(\Lambda)} + \frac{g(\Lambda)}{m} \psi_1^\dagger \psi_2^\dagger \psi_3 \psi_4^{(\Lambda)} + \mathcal{V} \tag{1}$$

where the renormalized coupling

$$g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \tag{2}$$

2 Amplitude

Consider:

$$i\mathcal{A} = \langle 34 | \psi^\dagger \psi | 12 \rangle = \tag{3}$$

Define $P = p_1 + p_2 = (E, \mathbf{0})$, and $E = p^2/m$. The integral equation is

$$i\mathcal{A} = -\frac{ig(\Lambda)}{m} \left(1 + i\mathcal{A} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon} \frac{i}{k^0 - p^0 - \frac{|\mathbf{k}-\mathbf{p}|^2}{2m} + i\epsilon} \right) \tag{4}$$

The integral gives (redefine $\epsilon \to 2m\epsilon$)

$$\mathcal{I} = \frac{im}{2\pi^2} \left(-\Lambda + \sqrt{-mE - i\epsilon} \tan^{-1} \left(\frac{\Lambda}{\sqrt{-mE - i\epsilon}} \right) \right) = -\frac{i\Lambda m}{2\pi^2} + \frac{mp}{4\pi} \tag{5}$$

and

$$i\mathcal{A} = \frac{-1}{\mathcal{I} + \frac{m}{ig(\Lambda)}} = -\left[\frac{im\sqrt{-mE - i\epsilon} \tan^{-1} \left(\frac{\Lambda}{\sqrt{-mE - i\epsilon}} \right)}{2\pi^2} - \frac{im}{4\pi a} \right]^{-1} \tag{6}$$

$$\xrightarrow{\Lambda \to \infty} \frac{4i\pi/m}{-1/a + \sqrt{-mE - i\epsilon}} \tag{7}$$

Note that by definition, scattering length is the leading order momentum expansion of $1/\mathcal{A}$, which gives

$$\frac{1}{a} = \frac{4i\pi}{m}\left(\mathcal{I} + \frac{m}{ig(\Lambda)}\right)^{(0)} \tag{8}$$

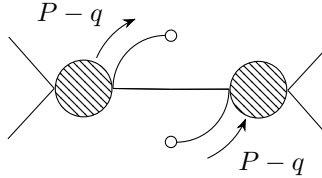$$= \frac{4\pi}{g(\Lambda)} + \frac{2\Lambda}{\pi} \tag{9}$$

$$\Rightarrow g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \tag{10}$$

and this is actually how we get the form of (2).

3 OPE

3.1 l.h.s.

Take what we got in the last section as a new non-perturbative vertex, we only need to deal with tree diagram this way. First we have Figure 2(a) in Braaten's paper:

$$= \langle 34|\psi^\dagger\left(-\frac{\mathbf{r}}{2}\right)\psi\left(\frac{\mathbf{r}}{2}\right)|12\rangle \tag{11}$$
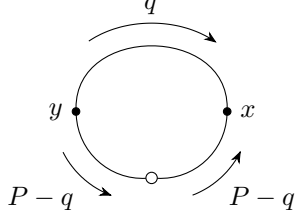
$$= (i\mathcal{A})^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{i}{q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon} \frac{i}{\left[E - q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon\right]^2} e^{i\mathbf{q}\cdot\mathbf{r}} \tag{12}$$

$$= \mathcal{A}^2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{m^2 e^{i\mathbf{q}\cdot\mathbf{r}}}{(\mathbf{q}^2 - p^2 - i\epsilon)^2} \tag{13}$$

$$= \frac{im^2 \mathcal{A}^2 e^{ipr}}{8\pi p} \tag{14}$$

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3.2 r.h.s.

For simplicity, we drop the external lines and focus on the internal subgraph. Consider Figure 2(b):

$$= \langle 34|\psi^\dagger \psi(0)|12\rangle_{amp} \tag{15}$$

$$= \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} e^{iP\cdot y} e^{-iP\cdot x} e^{-il_1\cdot y} e^{il_2\cdot x} e^{iq\cdot(x-y)} \tilde{D}(l_1)\tilde{D}(l_2)\tilde{D}(q) \tag{16}$$

$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \tilde{D}(P-q)\tilde{D}(P-q)\tilde{D}(q) \tag{17}$$

$$= -\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{m^2}{(\mathbf{q}^2 - p^2 - i\epsilon)^2} \tag{18}$$
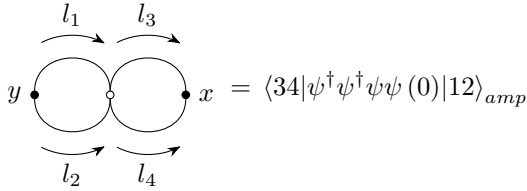
$$= -\frac{im^2}{8\pi p} \tag{19}$$

where $\tilde{D}$ marks momentum space propagator and two external vertexes give an $(i\mathcal{A})^2$ factor. The total contribution is

$$\frac{im^2\mathcal{A}^2}{8\pi p}, \tag{20}$$

the first order Fourier expansion of the l.h.s.

3.3 Higher dimensional operators

Figure 2(c) gives

$$= \langle 34|\psi^\dagger\psi^\dagger\psi\psi(0)|12\rangle_{amp} \tag{21}$$

$$= \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{\mathrm{d}^4 l_3}{(2\pi)^4} \frac{\mathrm{d}^4 l_4}{(2\pi)^4} e^{iP\cdot y} e^{-iP\cdot x} e^{-i(l_1+l_2)\cdot y} e^{i(l_3+l_4)\cdot x}$$
$$\tilde{D}(l_1)\tilde{D}(l_2)\tilde{D}(l_3)\tilde{D}(l_4) \tag{22}$$

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which becomes

 $\displaystyle = \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \tilde{D}(l_1) \tilde{D}(P - l_1) \tilde{D}(l_2) \tilde{D}(P - l_2)$ (23)

$$= -\int \frac{\mathrm{d}^3 \mathbf{l_1}}{(2\pi)^3} \frac{\mathrm{d}^3 \mathbf{l_2}}{(2\pi)^3} \frac{m^2}{\left(\mathbf{l_1}^2 - p^2 - i\epsilon\right)\left(\mathbf{l_2}^2 - p^2 - i\epsilon\right)}$$ (24)

$$= -\mathcal{I}^2$$ (25)

There're four diagrams in total:

 $= \mathcal{A}^2 \mathcal{I}^2$ (26)

 $= \mathcal{A}\mathcal{I}$ (27)

 $= \mathcal{A}\mathcal{I}$ (28)

 $= 1$ (29)

We have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm \mathbf{p}} = (\mathcal{A}\mathcal{I} + 1)^2$$ (30)

in total. Plug in

$$\mathcal{I} = -\frac{m}{ig(\Lambda)} - \frac{1}{\mathcal{A}}$$ (31)

we have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm \mathbf{p}} = m^2 g^{-2}(\Lambda) \mathcal{A}^2$$ (32)

The Wilson coefficient must be

$$-\frac{r}{8\pi} g^2(\Lambda)$$ (33)

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4 Contact

4.1 Definition

$$C = \int \mathrm{d}^3 R \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2(R) \right\rangle \tag{34}$$

4.2 Energy Relation

4.3 OPE for number density operators

We have a pair of number density operators

$$\psi_1^\dagger \psi_1(\mathbf{R} - \frac{1}{2}\mathbf{r}) \;\&\; \psi_2^\dagger \psi_2(\mathbf{R} + \frac{1}{2}\mathbf{r}) \tag{35}$$

and the diagram is

$$\tag{36}$$

$$= (i\mathcal{A})^2 \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4} \frac{\mathrm{d}^4 l_2}{(2\pi)^4} \frac{i}{l_1^0 - \frac{\mathbf{l_1}^2}{2m} + i\epsilon} \frac{i}{E - l_1^0 - \frac{\mathbf{l_1}^2}{2m} + i\epsilon} \frac{i}{l_2^0 - \frac{\mathbf{l_2}^2}{2m} + i\epsilon} \frac{i}{E - l_2^0 - \frac{\mathbf{l_2}^2}{2m} + i\epsilon} e^{i\mathbf{q}\cdot\mathbf{r}} \tag{37}$$

$$= \frac{\mathcal{A}^2 m^2}{16\pi^2 r^2} e^{2ipr} \tag{38}$$

Compare with the result of Figure 2(c) (32) we have the Wilson coefficient

$$\frac{g^2(\Lambda)}{16\pi^2 r^2} \tag{39}$$

References

[Braaten and Platter(2008)] E. Braaten and L. Platter, Phys. Rev. Lett. **100** (2008), 10.1103/physrevlett.100.205301.