

# Note on Braaten's Paper

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## 1 Intro

Hamiltonian [Braaten and Platter(2008)]:

$$\mathcal{H} = \sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)} + \frac{g(\Lambda)}{m} \psi_1^{\dagger} \psi_2^{\dagger} \psi_3 \psi_4^{(\Lambda)} + \mathcal{V} \quad (1)$$

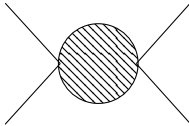
where the renormalized coupling

$$g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \quad (2)$$

has dimension of  $[M]^{-1}$ .

## 2 Amplitude

Consider:

$$i\mathcal{A} = \langle 34 | \psi^{\dagger} \psi | 12 \rangle = \text{diagram} \quad (3)$$


Define  $P = p_1 + p_2 = (E, \mathbf{0})$ , and  $E = p^2/m$ . The integral equation is

$$i\mathcal{A} = -\frac{ig(\Lambda)}{m} \left( 1 + i\mathcal{A} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\epsilon} \frac{i}{P^0 - k^0 - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + i\epsilon} \right) \quad (4)$$

The integral gives (redefine  $\epsilon \rightarrow 2m\epsilon$ )

$$\mathcal{I} = \frac{im}{2\pi^2} \left( -\Lambda + \sqrt{-mE - i\epsilon} \tan^{-1} \left( \frac{\Lambda}{\sqrt{-mE - i\epsilon}} \right) \right) = -\frac{i\Lambda m}{2\pi^2} + \frac{mp}{4\pi} \quad (5)$$

(the last step isn't carefully checked and might have sign error concerning the contour on the complex plane) and

$$i\mathcal{A} = \frac{-1}{\mathcal{I} + \frac{m}{ig(\Lambda)}} = - \left[ \frac{im\sqrt{-mE - i\epsilon} \tan^{-1} \left( \frac{\Lambda}{\sqrt{-mE - i\epsilon}} \right)}{2\pi^2} - \frac{im}{4\pi a} \right]^{-1} \quad (6)$$

$$\xrightarrow{\Lambda \rightarrow \infty} \frac{4i\pi/m}{-1/a + \sqrt{-mE - i\epsilon}} \quad (7)$$

Note that by definition, scattering length is the leading order momentum expansion of  $1/\mathcal{A}$ , which gives

$$\frac{1}{a} = \frac{4i\pi}{m} \left( \mathcal{I} + \frac{m}{ig(\Lambda)} \right)^{(0)} \quad (8)$$

$$= \frac{4\pi}{g(\Lambda)} + \frac{2\Lambda}{\pi} \quad (9)$$

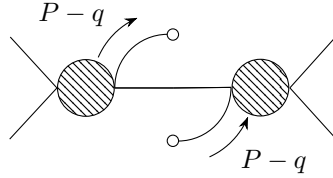
$$\Rightarrow g(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi} \quad (10)$$

and this is actually how we get the form of (2).

### 3 OPE

#### 3.1 l.h.s.

Take what we got in the last section as a new non-perturbative vertex, we only need to deal with tree diagram this way. First we have Figure 2(a) in Braaten's paper:



$$= \langle 34 | \psi^\dagger \left( -\frac{\mathbf{r}}{2} \right) \psi \left( \frac{\mathbf{r}}{2} \right) | 12 \rangle \quad (11)$$

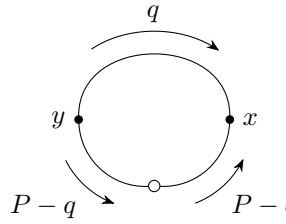
$$= (i\mathcal{A})^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon} \frac{i}{\left[ E - q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon \right]^2} e^{i\mathbf{q} \cdot \mathbf{r}} \quad (12)$$

$$= \mathcal{A}^2 \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{m^2 e^{i\mathbf{q} \cdot \mathbf{r}}}{(\mathbf{q}^2 - p^2 - i\epsilon)^2} \quad (13)$$

$$= \frac{im^2 \mathcal{A}^2 e^{ipr}}{8\pi p} \quad (14)$$

#### 3.2 r.h.s.

For simplicity, we drop the external lines and focus on the internal subgraph. Consider Figure 2(b):



$$= \langle 34 | \psi^\dagger \psi(0) | 12 \rangle_{amp} \quad (15)$$

$$= \int d^4 x \int d^4 y \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} e^{iP \cdot y} e^{-iP \cdot x} e^{-il_1 \cdot y} e^{il_2 \cdot x} e^{iq \cdot (x-y)} \tilde{D}(l_1) \tilde{D}(l_2) \tilde{D}(q) \quad (16)$$

$$= \int \frac{d^4 q}{(2\pi)^4} \tilde{D}(P-q) \tilde{D}(P-q) \tilde{D}(q) \quad (17)$$

$$= - \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{m^2}{(\mathbf{q}^2 - p^2 - i\epsilon)^2} \quad (18)$$

$$= - \frac{im^2}{8\pi p} \quad (19)$$

where  $\tilde{D}$  marks momentum space propagator and two external vertexes give an  $(i\mathcal{A})^2$  factor. The total contribution is

$$\frac{im^2 \mathcal{A}^2}{8\pi p}, \quad (20)$$

the first order Fourier expansion of the l.h.s.

### 3.3 Higher dimensional operators

Figure 2(c) gives

$$\begin{array}{c} \begin{array}{c} \xrightarrow{l_1} \quad \xrightarrow{l_3} \\ \bullet \quad \bullet \\ \xleftarrow{l_2} \quad \xleftarrow{l_4} \end{array} \\ y \quad x \end{array} = \langle 34 | \psi^\dagger \psi^\dagger \psi \psi (0) | 12 \rangle_{amp} \quad (21)$$

$$= \int d^4 x \int d^4 y \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{d^4 l_3}{(2\pi)^4} \frac{d^4 l_4}{(2\pi)^4} e^{iP \cdot y} e^{-iP \cdot x} e^{-i(l_1+l_2) \cdot y} e^{i(l_3+l_4) \cdot x} \tilde{D}(l_1) \tilde{D}(l_2) \tilde{D}(l_3) \tilde{D}(l_4) \quad (22)$$

which becomes

$$\begin{array}{c} \begin{array}{c} \xrightarrow{l_1} \quad \xrightarrow{l_2} \\ \bullet \quad \bullet \\ \xleftarrow{P-l_1} \quad \xleftarrow{P-l_2} \end{array} \\ y \quad x \end{array} = \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \tilde{D}(l_1) \tilde{D}(P-l_1) \tilde{D}(l_2) \tilde{D}(P-l_2) \quad (23)$$

$$= - \int \frac{d^3 \mathbf{l}_1}{(2\pi)^3} \frac{d^3 \mathbf{l}_2}{(2\pi)^3} \frac{m^2}{(\mathbf{l}_1^2 - p^2 - i\epsilon) (\mathbf{l}_2^2 - p^2 - i\epsilon)} \quad (24)$$

$$= -\mathcal{I}^2 \quad (25)$$

There're four diagrams in total:

$$\begin{array}{c} \text{Diagram 1: Two shaded circles connected by two unshaded circles, with external lines crossing.} \\ = \mathcal{A}^2 \mathcal{I}^2 \end{array} \quad (26)$$

$$\begin{array}{c} \text{Diagram 2: One shaded circle connected by two unshaded circles, with external lines crossing.} \\ = \mathcal{A} \mathcal{I} \end{array} \quad (27)$$



$$= \mathcal{AI} \quad (28)$$



$$= 1 \quad (29)$$

We have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm \mathbf{p}} = (\mathcal{AI} + 1)^2 \quad (30)$$

in total. Plug in

$$\mathcal{I} = -\frac{m}{ig(\Lambda)} - \frac{1}{\mathcal{A}} \quad (31)$$

we have

$$\left\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2^{(\Lambda)}(0) \right\rangle_{\pm \mathbf{p}} = m^2 g^{-2}(\Lambda) \mathcal{A}^2 \quad (32)$$

The Wilson coefficient must be

$$-\frac{r}{8\pi} g^2(\Lambda) \quad (33)$$

## 4 Contact

### 4.1 Definition

$$C = \int d^3 R \left\langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2(R) \right\rangle \quad (34)$$

### 4.2 Energy Relation

According to the Hamiltonian:

$$\mathcal{H} = \left( \sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)} - \frac{\Lambda}{2\pi^2 m} g^2(\Lambda) \psi_1^{\dagger} \psi_2^{\dagger} \psi_1 \psi_2 \right) + \frac{1}{4\pi m a} g^2(\Lambda) \psi_1^{\dagger} \psi_2^{\dagger} \psi_1 \psi_2 + \mathcal{V} \quad (35)$$

where the matrix elements of those three operators are finite. The  $\nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)}$  part gives a linear divergence  $2 \times \frac{\Lambda m \mathcal{A}^2}{4\pi^2}$  for two spin states in total while the other one gives  $-\frac{\Lambda m \mathcal{A}^2}{2\pi^2}$ , we can see that the linear divergence is cancelled. Integrating over positions, we obtain

$$\int d^3 R \left\langle \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma} \right\rangle = \int d^3 R d^3 r \delta^{(3)}(\mathbf{r}) \left\langle \nabla \psi_{\sigma}^{\dagger}(\mathbf{R} - \frac{\mathbf{r}}{2}) \cdot \nabla \psi_{\sigma}(\mathbf{R} + \frac{\mathbf{r}}{2}) \right\rangle = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} k^2 \rho_{\sigma}(k) \quad (36)$$

$$\frac{1}{4\pi m a} \int d^3 R \left\langle g^2 \psi_1^{\dagger} \psi_2^{\dagger} \psi_1 \psi_2 \right\rangle = \frac{1}{4\pi m a} C \quad (37)$$

also notice

$$\int^{\Lambda} \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{\mathbf{k}^2} = \frac{\Lambda}{2\pi^2} \quad (38)$$

we have

$$\int d^3R \frac{\Lambda}{2\pi^2 m} \langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 \rangle = \sum_\sigma \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{k^2}{2m} \frac{C}{\mathbf{k}^4} \quad (39)$$

we achieve

$$E = \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m} \left( \rho_\sigma(k) - \frac{C}{k^4} \right) + \frac{C}{4\pi m a} + \int d^3R \langle V \rangle \quad (40)$$

### 4.3 Adiabatic relation

Using F-H theorem

$$dE/da = \int d^3R \langle \partial \mathcal{H} / \partial a \rangle \quad (41)$$

it's straightforward that

$$\partial \mathcal{H} / \partial a = g^2 \psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2 / (4\pi m a^2) \quad (42)$$

We then have

$$\frac{dE}{d(1/a)} = -\frac{1}{4\pi m} C \quad (43)$$

using (34).

### 4.4 Viral Theorem

Dimensional analysis requires

$$\left[ \omega \frac{\partial}{\partial \omega} - \frac{1}{2} a \frac{\partial}{\partial a} \right] \int d^3R \langle \mathcal{H} \rangle = 1 \quad (44)$$

Together with F-H theorem

$$\frac{a}{2} \frac{\partial}{\partial a} \int d^3R \langle \mathcal{H} \rangle = \frac{C}{8\pi m a} + \frac{a}{2} \frac{\partial}{\partial a} \langle \mathcal{V} \rangle = \frac{a}{2} \frac{dE}{da} \quad (45)$$

$$\frac{\partial}{\partial \omega} \int d^3R \langle \mathcal{H} \rangle = \frac{\partial}{\partial \omega} \langle \mathcal{V} \rangle = \frac{dE}{d\omega} \quad (46)$$

### 4.5 OPE for number density operators

We have a pair of number density operators

$$\psi_1^\dagger \psi_1(\mathbf{R} - \frac{1}{2}\mathbf{r}) \text{ \& } \psi_2^\dagger \psi_2(\mathbf{R} + \frac{1}{2}\mathbf{r}) \quad (47)$$

and the diagram is

$$= (i\mathcal{A})^2 \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{i}{l_1^0 - \frac{\mathbf{l}_1^2}{2m} + i\epsilon} \frac{i}{E - l_1^0 - \frac{\mathbf{l}_1^2}{2m} + i\epsilon} \frac{i}{l_2^0 - \frac{\mathbf{l}_2^2}{2m} + i\epsilon} \frac{i}{E - l_2^0 - \frac{\mathbf{l}_2^2}{2m} + i\epsilon} e^{i\mathbf{q}\cdot\mathbf{r}} \quad (49)$$

$$= \frac{\mathcal{A}^2 m^2}{16\pi^2 r^2} e^{2ipr} \quad (50)$$

Compare with the result of Figure 2(c) (32) we have the Wilson coefficient

$$\frac{g^2(\Lambda)}{16\pi^2 r^2} \quad (51)$$

## 5 Reformulate in Dimensional Regularization

The renormalized Hamiltonian reads

$$\mathcal{H} = \sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma} + Z_g \frac{g}{m} \psi_1^{\dagger} \psi_2^{\dagger} \psi_3 \psi_4 \quad (52)$$

as well as

$$S = \int d^{d+1}x \left[ \sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma} + \mu^{3-d} \frac{g}{m} \psi_1^{\dagger} \psi_2^{\dagger} \psi_3 \psi_4 + \delta_g \frac{g}{m} \psi_1^{\dagger} \psi_2^{\dagger} \psi_3 \psi_4 \right] \quad (53)$$

We have

$$i\mathcal{A} = \text{diagram} \quad (54)$$

$$\mathcal{I} \equiv \text{diagram} \quad (55)$$

The integral equation is

$$i\mathcal{A} = -\frac{iZ_g(\mu)g}{m} (1 + i\mathcal{A}\mathcal{I}) \quad (56)$$

$$\mathcal{I} \equiv \mu^{3-d} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\eta} \frac{i}{P^0 - k^0 - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + i\eta} \quad (57)$$

Graphically we have

$$\text{diagram} = \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}$$

The integral equals to

$$\mathcal{I} = \mu^{3-d} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{i}{k^0 - \frac{\mathbf{k}^2}{2m} + i\eta} \frac{i}{k^0 - P^0 - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + i\eta} \quad (58)$$

$$= \mu^{3-d} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{i}{P^0 - \frac{\mathbf{k}^2}{2m} - \frac{|\mathbf{k}-\mathbf{P}|^2}{2m} + 2i\eta} \quad (59)$$

$$= \mu^{3-d} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{im}{p^2 - \mathbf{k}^2 + i\eta} \quad (60)$$

$$= -\mu^{3-d} \frac{1}{2\pi^2} \frac{im}{2} \pi (-p^2 - i\eta)^{\frac{d-2}{2}} \csc\left(\frac{\pi d}{2}\right) \quad (61)$$

$$\xrightarrow{d \rightarrow 4} \frac{im(p^2 + i\eta)}{2\pi^2(d-4)\mu} + \frac{im(p^2 + i\eta) \log\left(-\frac{p^2 + i\eta}{\mu^2}\right)}{4\pi^2\mu} + \mathcal{O}(d-4) \quad (62)$$

When we set  $\mu = p$ , the logarithm disappears and the finite term goes to our result with cutoff regularization

$$\frac{im(p^2 + i\eta) \log\left(-\frac{p^2 + i\eta}{\mu^2}\right)}{4\pi^2\mu} \xrightarrow{\mu \rightarrow p, \eta \rightarrow 0} -\frac{mp}{4\pi} \quad (63)$$

Similar to our previous calculation, we have

$$\mathcal{I} = -\frac{imp^2}{2\pi^2\epsilon\mu} + \frac{im(p^2 + i\eta) \log\left(-\frac{p^2 + i\eta}{\mu^2}\right)}{4\pi^2\mu} \quad (64)$$

with  $\epsilon \equiv 4 - d$ . We put it back to the integral equation

$$i\mathcal{A} = \frac{-1}{\mathcal{I} + \frac{m}{iZ_g(\mu)g}} \equiv \frac{4i\pi/m}{-1/a + \sqrt{-p^2 - i\eta}} \quad (65)$$

With this definition we have

$$\frac{1}{a} = \frac{4i\pi}{m} \left( \mathcal{I} + \frac{m}{iZ_g(p)g} \right)^{(0)} \quad (66)$$

$$= \frac{2p}{\pi\epsilon} + \frac{4\pi}{Z_g(p)g} \quad (67)$$

$$\Rightarrow Z_g(p)g = \frac{4\pi a}{1 - 2ap/(\pi\epsilon)} \quad (68)$$

We could see that  $p/\epsilon$  is effectively  $\Lambda$ .

Now without pre-determined scale  $\mu$  we have

$$\frac{1}{a} = \frac{2p^2}{\pi\mu\epsilon} + \left( \frac{4\pi}{g} + \sqrt{-p^2 - i\eta} - \frac{p^2 \log\left(-\frac{p^2 + i\eta}{\mu^2}\right)}{\pi\mu} - \frac{i\eta \log\left(-\frac{p^2 + i\eta}{\mu^2}\right)}{\pi\mu} \right) \quad (69)$$

$$\Rightarrow Z_g(\mu)g = \frac{4\pi^2\mu}{p^2 \log\left(-\frac{p^2 + i\eta}{\mu^2}\right) + i\eta \log\left(-\frac{p^2 + i\eta}{\mu^2}\right) - \pi\mu\sqrt{-p^2 - i\eta} - 2p^2/\epsilon + \pi\mu/a} \quad (70)$$

## A Perturbative view of integral equation

If we're to have the integral equation only with perturbative counterterms, for a start we can write

$$\begin{aligned}
 & \text{Shaded circle with 4 external lines} = \text{Cross} + \text{Circle with cross} + \text{Circle} \\
 & + \text{Two circles with shaded circle} + \text{Circle with cross and shaded circle}
 \end{aligned}$$

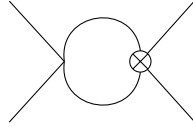
with one-loop counterterm.

The iterative form is

$$\text{Shaded circle with 4 external lines} = \text{Cross} + \text{Circle with cross} + \text{Circle} \tag{71}$$

$$\begin{aligned}
 & + \text{Two circles} + \text{Two circles with cross} + \text{Three circles} \\
 & + \text{Circle with cross} + \text{Circle with cross and cross} + \text{Circle with cross and circle}
 \end{aligned} \tag{72}$$

It'd appeared that we missed one diagram during the iteration:



If we add this one, the r.h.s. of the integral equation won't be finite again. Thus, only the counterterm from one loop is not sufficient enough to cancel all the divergences.

## References

[Braaten and Platter(2008)] E. Braaten and L. Platter, *Phys. Rev. Lett.* **100** (2008), 10.1103/physrevlett.100.205301.