Expand by regions `box diagram`

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1 The box diagram

The box diagram:
The kinematic quantities:

$p_1 = \frac{q}{2} + p \qquad p_2 = \frac{q}{2} - p \qquad p_3 = \frac{q}{2} + p' \qquad p_4 = \frac{q}{2} - p'$

Define the variables $y = m^2 - \frac{q^2}{4} = p^2 = p'^2$ and $t = (p' - p)^2$

Since $p.q = p'.q = 0$
It's convinent to choose the frame in which $p = (0, \vec{\boldsymbol{p}}) \quad p' = (0, \vec{\boldsymbol{p'}}) \quad q = (q^0, \vec{\mathbf{0}})$

The threshold expansion is performed when $t \sim y << q^2$

The integral represented by the diagram can be written directly from the Feynman Rules:

$$\begin{aligned} I &= \int [dk] \frac{1}{((k+p_1)^2 - m^2)((p_2 - k)^2 - m^2)(k + p_1 - p_3)^2 k^2} \\ &= \int [dk] \frac{1}{((k+p)^2 + k.q - y)((k+p)^2 - k.q - y)(k + p - p')^2 k^2} \end{aligned} \tag{1}$$

where $[dk] = e^{\epsilon \gamma_E} \frac{d^D k}{i\pi^{\frac{D}{2}}}$

Near the threshold,we have four regions.In each region,perform the expansion in the small quantities of the integrand before the loop momentum integration.

1.1 Hard region

The loop momentum is of the order of the CMS energy,we say it's hard, ie. $k \sim q$. The integrand is expand in y,p and p'.

$$I^h = \int [dk] \frac{1}{(k^2 + k.q)(k^2 - k.q)(k^2)^2} \tag{2}$$

$$\frac{1}{(\ k^2 + k.q\ )(\ k^2 - k.q\ )(\ k^2)^2} \ = \frac{1}{2(k^2)^3} (\frac{1}{k^2 + k.q} + \frac{1}{k^2 - k.q})$$

Use Feynman parametrization

$$x(k^2 + k.q) + (1-x)k^2 = (k + \frac{x}{2} q)^2 - \frac{q^2}{4} x^2 \tag{3}$$

Integrate over k according to

$$\int [dl] \frac{1}{(l^2 - \Delta)^n} = (-1)^n \frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)} (\frac{1}{\Delta})^{n - \frac{D}{2}} \tag{4}$$

The integration left to be done is:

$$\frac{\Gamma(4)}{\Gamma(3)} \frac{\Gamma(2+\epsilon)}{\Gamma(4)} \int_0^1 dx \frac{(1-x)^2}{(\frac{q^2}{4} x^2)^{2+\epsilon}} = -\frac{8}{3} \tag{5}$$

1.2 Soft region

Whhe the loop momentum becomes soft,ie. $k \sim \sqrt{y}$. there is a contribution from the gluon poles. To the leading order expansion of the small quantities $y, (k+p)^2$,the integral is

$$\begin{aligned} I^s &= \int [dk] \frac{1}{(k.q + i0^+)(-k.q + i0^+)(k + p - p')^2 k^2} \\ &= \frac{1}{q^2} \int [dk] \frac{1}{k_0^2\ k^2\ (k + p - p')^2} \end{aligned} \tag{6}$$

Closing the upper complex plane,integrate over k^0.

$$\begin{aligned} I^s &= e^{\epsilon\gamma} \frac{2i\pi}{q^2} \int \frac{d^{D-1}k}{i\pi^{\frac{D}{2}}} \frac{1}{2(\vec{\boldsymbol{k}}^2)^{\frac{3}{2}}\ [\vec{\boldsymbol{k}}^2 - (\vec{\boldsymbol{k}} + \vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')^2]} + \frac{1}{2(\vec{\boldsymbol{k}} + \vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')^{\frac{3}{2}}\ [(\vec{\boldsymbol{k}} + \vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')^2 - \vec{\boldsymbol{k}}^2]} \\ &= e^{\epsilon\gamma} \frac{1}{q^2} \int \frac{d^{D-1}k}{\pi^{\frac{D}{2}-1}} \frac{1}{(\vec{\boldsymbol{k}}^2)^{\frac{3}{2}}} [\frac{1}{-2\vec{\boldsymbol{k}}.(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}') + t + i0^+} + \frac{1}{-2\vec{\boldsymbol{k}}.(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}') + t - i0^+}] \end{aligned}$$

According to $\dfrac{1}{(q^2)^n\ (qv)^m} = \dfrac{(n+m-1)!}{(n-1)!\ (m-1)!}\displaystyle\int_0^\infty \frac{2^m \lambda^{m-1} d\lambda}{(q^2+2\lambda qv)^{n+m}}$

$$\begin{aligned} I^s &= e^{\epsilon\gamma}\frac{3}{q^2}\int_0^\infty d\lambda \int \frac{d^{D-1}k}{\pi^{\frac{D}{2}-1}}\frac{1}{[\vec{\boldsymbol{k}}^2+(4\lambda^2+2\lambda)t+i0^+]^{\frac{5}{2}}} + (i0^+ \to -i0^+) \\ &= -e^{\epsilon\gamma}\frac{3\sqrt{\pi}}{q^2}\frac{\Gamma(1+\epsilon)}{\Gamma(\frac{5}{2})}\frac{1}{(-2t)^{1+\epsilon}}\int_0^\infty d\lambda\ [\frac{1}{(2\lambda^2+\lambda)t+i0^+}]^{1+\epsilon} + (i0^+ \to -i0^+) \\ &= \frac{1}{q^{2(2+\epsilon)}}\ [-\frac{4}{\hat{t}}\ (\ \frac{1}{\epsilon} - \log(-\hat{t})\ )] \end{aligned} \tag{7}$$

1.3 Potential region

When the loop momentum is potential, ie. $k^0 \sim \frac{y}{q}$ *and* $\vec{k} \sim \sqrt{y}$, expand in k_0^2.

$$I^p = \int [dk]\frac{1}{[-(\vec{k}+\vec{q})^2+k_0q_0-y+i0^+][-(\vec{k}+\vec{q})^2-k_0q_0-y+i0^+][-(\vec{k}+\vec{p}-\vec{p}')^2](-\vec{k}^2)} \tag{8}$$

$$= \frac{e^{\epsilon\gamma}}{q_0}\int\frac{d^{D-1}k}{\pi^{\frac{D}{2}-1}}\frac{1}{[(\vec{k}+\vec{p})^2+y-i0^+][(\vec{k}+\vec{p}-\vec{p}')^2-i0^+][\vec{k}^2-i0^+]} \tag{9}$$

$$\begin{aligned} &x_1[(\vec{k}+\vec{p}-\vec{p}')^2-i0^+]+x_2[(\vec{k}+\vec{p})^2+y-i0^+]+(1-x_1-x_2)[\vec{k}^2-i0^+] \\ &= [\vec{k}+x_1(\vec{p}-\vec{p}')+x_2\vec{p}]^2-[x_2^2\vec{p}^2+2x_1x_2\vec{p}(\vec{p}-\vec{p}')+x_1^2(\vec{p}-\vec{p}')^2+tx_1+i0^+] \end{aligned}$$

$$\begin{aligned} (\vec{p}-\vec{p}')^2 &= -t \\ \vec{p}^2 = \vec{p}'^2 &= -y \\ \Delta = x_2^2\vec{p}^2+2x_1x_2\vec{p}(\vec{p}-\vec{p}')+x_1^2(\vec{p}-\vec{p}')^2+tx_1+i0^+ &= -yx_2^2-tx_1^2+t(1-x_2)x_1 \end{aligned}$$

First do $x_1 -> 1-u_1$ and $x_2 -> u_2$;
and then do $u_1 -> x_1$ and $u_2 -> x_1x_2$

After Feynman parametrition the integral becomes:

$$I^p = \frac{i\sqrt{\pi}}{q_0}\Gamma(\frac{3}{2}+\epsilon)\int_0^1 dx_1\int_0^1 dx_2\frac{x_1}{\Delta^{\frac{3}{2}+\epsilon}} \tag{10}$$

$$\Delta = (-yx_2^2+tx_2-t)x_1^2+t(1-x_2)x_1-i0^+ \tag{11}$$

$$I^p = \frac{i\sqrt{\pi}}{q_0}\Gamma(\frac{3}{2}+\epsilon)\int_0^1 dx_1\int_0^1 dx_2\frac{x_1^{\frac{1}{2}}}{[x_1(-yx_2^2+tx_2-t)+t(1-x_2)+i0^+]^{\frac{3}{2}+\epsilon}} \tag{12}$$

First do integration over x_1 and we get the integrand for the integration over x_2 to be:

$$-\frac{2(t(1-x_2))^{-\epsilon-\frac{3}{2}}\,{}_2F_1\left(\frac{1}{2}-\epsilon,\epsilon+\frac{3}{2};\frac{3}{2}-\epsilon;\frac{x_2^2 y}{t(1-x_2)}+1\right)}{1-2\epsilon} \tag{13}$$

0 and 1 are singularity point for hypergeometric functions.Use an identity :

$${}_2F_1\left(\frac{1}{2}-\epsilon,\epsilon+\frac{3}{2};\frac{3}{2}-\epsilon;\frac{\mathrm{x2}^2 y}{t(1-\mathrm{x2})}+1\right)=\left(-\frac{\mathrm{x2}^2 y}{t(1-\mathrm{x2})}\right)^{\epsilon-\frac{1}{2}}\,{}_2F_1\left(\frac{1}{2}-\epsilon,-2\epsilon;\frac{3}{2}-\epsilon;\frac{t(1-\mathrm{x2})}{\mathrm{x2}^2 y}+1\right)$$

After using the identity,(10) becomes:

$$\frac{2t^{-2\epsilon-1}(1-\mathrm{x2})^{-2\epsilon-1}\mathrm{x2}^{2\epsilon-1}(-y)^{\epsilon-\frac{1}{2}}\,{}_2F_1\left(\frac{1}{2}-\epsilon,-2\epsilon;\frac{3}{2}-\epsilon;\frac{t(1-\mathrm{x2})}{\mathrm{x2}^2 y}+1\right)}{1-2\epsilon}$$

Divergences arise when x_2 is 0 or 1.Extract the singularities one by one. First use a trick to extract the singularity at $x_2=1$:

$$\int dx\frac{f(x,\epsilon)}{(1-x)^{2\epsilon+1}}=\int dx\frac{f(1,\epsilon)}{(1-x)^{2\epsilon+1}}+\int dx\frac{f(x,\epsilon)-f(1,\epsilon)}{(1-x)^{2\epsilon+1}}$$

The first integral in the RHS is divergent but here $f(1,\epsilon)$ is finite.The second integral in the RHS is finite,so we can set ϵ to be 0 there.

$$\frac{\pi}{2t\sqrt{y}}\left(\frac{1}{\epsilon}-2\log(t)+\log(-y)-\gamma+2-\psi^{(0)}\left(\frac{3}{2}\right)\right) \tag{14}$$

Then do the trick again to extract the singularity at $x_2=0$:

$$\int dx\frac{g(x,\epsilon)}{x^{-2\epsilon+1}}=\int dx\frac{g(0,\epsilon)}{x^{-2\epsilon+1}}+\int dx\frac{g(x,\epsilon)-g(0,\epsilon)}{(1-x)^{-2\epsilon+1}}$$

$$-\frac{\pi}{4t\sqrt{y}}\left(\frac{1}{\epsilon}-2\log(t)+\log(-y)+\gamma+2+\psi^{(0)}\left(\frac{3}{2}\right)\right) \tag{15}$$

Those two add up to obtain:

$$\frac{\pi}{4t\sqrt{y}}\left(\frac{1}{\epsilon}-2\log(t)+\log(-y)-\frac{3}{4}\gamma+2-\frac{3}{4}\psi^{(0)}\left(\frac{3}{2}\right)\right) \tag{16}$$

I found this can't get a finite answer for $\int dx\frac{g(x,\epsilon)-g(0,\epsilon)}{(1-x)^{-2\epsilon+1}}$,which should be finite.

The answer should be:

$$I^p=\frac{1}{(q^2)^{2+\epsilon}}\frac{\pi}{\hat{t}\sqrt{\hat{y}}}[\ \frac{1}{\epsilon}-\log(-\hat{t})\ ] \tag{17}$$

Sector Decomposition
(12) becomes:

$$I^p=\frac{i\sqrt{\pi}}{q}\Gamma[\frac{3}{2}+\epsilon][sectA+sectB] \tag{18}$$

$$sectA = \int_0^1 dx_1 \int_0^1 dw x_1^{-1-2\epsilon} w^{-\frac{1}{2}-\epsilon}[[tx_1 - y(1-x_1)^2]w + t]^{-\frac{3}{2}-\epsilon} \tag{19}$$

$$sectB = \int_0^1 dx_2 \int_0^1 du x_2^{-1-2\epsilon}[tux_2 - y(1-ux_2)^2 + tu]^{-\frac{3}{2}-\epsilon} \tag{20}$$

sectA and sectB is divergent at $x_1 -> 0$ or $x_2 -> 0$ respectively.Perform integration over x_1 in sectA and x_2 in sectB first and then do the w or u integration later. The divergent term is:
From sectA:

$$-\frac{1}{t\epsilon}\sqrt{\frac{1}{i\delta + t - y}}$$

From sectB:

$$\frac{1}{t\epsilon}\left(\frac{1}{\sqrt{i\delta + t - y}} - \frac{1}{\sqrt{-y + i\delta}}\right)$$

Add up to:

$$\frac{\pi}{2t\sqrt{y}\epsilon} \tag{21}$$

't Hooft & Veltaman's paper about one-loop scalar diagrams is a good reference. [1]

1.4 Ultrosoft region

When the loop momentum is ultrasoft, ie. $k \sim \frac{y}{q}$

$$\begin{aligned} I^{us} &= \frac{1}{t}\int [dk] \frac{1}{(q_0 k_0 + i0^+)(-q_0 k_0 + i0^+)k^2} \\ &= 0 \end{aligned} \tag{22}$$

2 Conclusion

We have done the leading term expansion in small quantities of each term of the denominators respectively,and we found there are 3 regions contributing to the box diagram. Adding them together is the leading term threshold expansion:

$$I = \frac{1}{q^{2(2+\epsilon)}}\left[\frac{\pi}{\hat{t}\sqrt{\hat{y}}} - \frac{4}{\hat{t}}\right]\left(\frac{1}{\epsilon} - \log(-\hat{t})\right) - \frac{8}{3} + \mathcal{O}(\hat{t}^{\frac{1}{2}}, \hat{y}^{\frac{1}{2}}) \tag{23}$$

References

[1] G. 't Hooft and M. J. G. Veltman, "Scalar One Loop Integrals," Nucl. Phys. B **153**, 365 (1979).