Scalar QED

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1 Hydrogen Wavefunction Divergence in Klein-Gordon Equation and Schrödinger Equation

1.1 The Klein-Gordon part

The Klein-Gordon Hydrogen Equation is

$$((i\partial_0 + \frac{Z\alpha}{r})^2 + \nabla^2 - m^2)\Psi = 0 \tag{1}$$

For the bound state, the eigen value and the wave function are

$$E = m\frac{1}{\sqrt{1+\frac{\alpha^2 Z^2}{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}}} \tag{2}$$

$$\Psi = \frac{c}{\sqrt{4\pi}}e^{-kr}r^{\lambda} \tag{3}$$

where

$$\lambda = -\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2\alpha^2} \qquad c = \sqrt{\frac{(2k)^{2(1+\sqrt{\frac{1}{4}-Z^2\alpha^2})}}{\Gamma(2+2\sqrt{\frac{1}{4}-Z^2\alpha^2})}} \qquad k = \frac{m}{\sqrt{1+\frac{(\frac{1}{2}+\sqrt{\frac{1}{4}-Z^2\alpha^2})^2}{\alpha^2 Z^2}}} \tag{4}$$

c is the normaliztion factor for $\int d^3r|\Psi|^2 = 1$. For convenience, define

$$\Psi' = \frac{\Psi}{2(mZ\alpha)^{\frac{3}{2}}} \tag{5}$$

Now Ψ' is dimensionless and expand it in α, we get the origin divergence comes from a term

$$-(Z\alpha)^2\log(mr) \tag{6}$$

the m in log could be interpreted as a subtraction point μ.

1.2 The Schrödinger part

The Hamiltonian is

$$H = H_0 + H_{int} \tag{7}$$
$$H_0 = -\frac{\nabla^2}{2m} - \frac{Z\alpha}{r}, \quad H_{int} = \frac{\nabla^4}{8m^3} + \frac{1}{32m^4}[-\nabla^2, [-\nabla^2, -\frac{Z\alpha}{r}]] \tag{8}$$

The first term of H_{int} is the relativistic kinematic v^2 correction, the second one is the Darwin term.
The H_0 gives the radial wave functions as follows

$$R_{n0} = \frac{2(mZ\alpha)^{\frac{3}{2}}}{n^{\frac{3}{2}}} e^{-\frac{mZ\alpha}{n}r} F(1-n, 2, \frac{2mZ\alpha r}{n}), \quad E_n = -\frac{Z^2\alpha^2 m}{2n^2} \tag{9}$$
$$R_{k0} = \sqrt{\frac{2}{\pi}}(mZ\alpha)^{\frac{3}{2}} k e^{\frac{\pi}{2k}} |\Gamma(1-\frac{i}{k})| e^{-imz\alpha kr} F(1+\frac{i}{k}, 2, 2imZ\alpha kr), \quad E_k = \frac{mZ^2\alpha^2 k^2}{2} \tag{10}$$

Within perturbation theory,$E_1^{(1)} =< \phi|H_{int}|\phi >$, in quantum mechanics, the NLO energy correction is

$$E_1^{(1)} = E_1 Z^2 \alpha^2 \tag{11}$$

The NLO corretion of the bound state wave function is

$$\sum_{n\neq 1} a_{n1}\phi_{n00} + \int dk a_{k1}\phi_{k00} \tag{12}$$

with

$$a_{n1} = \frac{< \phi_{n00}|H_{int}|\phi_{100} >}{E_1 - E_n} \tag{13}$$

the discrete part of (12) is not divergent at $r = 0$. We now focus on the integration part and seperate the relativstic kinematic term and the Darwin term. Since we are only interested in the divergent part, here we give a hard cutoff $\frac{\Lambda}{m}$ as the up-limit of the integration and a also a down-limit λ, with $\lambda >> 1$ (note that the following wave function have been multiplied by $2(mZ\alpha)^{\frac{3}{2}}$)

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk \frac{2Z^2\alpha^2 k^{\frac{3}{2}}}{2\pi(\sqrt{1-\exp(-\frac{2\pi}{k})})}(1 - \frac{2}{1+k^2}\exp\left(-\frac{2\arctan(k)}{k}\right))e^{\frac{\pi}{2k}}|\Gamma(1-\frac{i}{k})| \tag{14}$$

with the integral region we defined($k >> 1$), it would be OK to expand the integrand in $\frac{1}{k}$ (I havn't prove it yet), then the UV divergent term is

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk (Z\alpha)^2 (\frac{1}{\pi} + \frac{1}{k}) \tag{15}$$
$$\sim (\alpha Z)^2 (\frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right)) \tag{16}$$

The UV divergent part of Darwin term is

$$\Phi^{(1)}(0)_D = -\frac{(Z\alpha)^4}{8\pi}\int_\lambda^{\frac{\Lambda}{m}} dk k^2 e^{\frac{\pi}{k}} |\Gamma(1-\frac{i}{k})|^2 \tag{17}$$

with the same trick as (15), the UV divergen part is

$$\Phi^{(1)}(0)_D = -(\alpha Z)^4 \int_\lambda^{\frac{\Lambda}{m}} dk \frac{k^2}{8\pi} + \frac{k}{8} + \frac{1}{24}\pi \tag{18}$$

$$\sim -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m}) \tag{19}$$

Now collect all the results we get as follow.
The K-G wave function's origin UV divergence is

$$K-G \;\; UV : -(Z\alpha)^2 \log(mr) \tag{20}$$

The purterbative Schrödinger wave function's origin UV divergence, with a k cutoff $\frac{\Lambda}{m}$, is

$$Kin \;\; UV : (\alpha Z)^2(\frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right)) \tag{21}$$

$$Darwin \;\; UV : -\frac{(Z\alpha)^4}{8\pi}(\frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m}) \tag{22}$$

All the m, under Λ or in a log, can be interpreted as a subtraction point μ.

2 Non-relativistic Scalar QED (NRSQED) Matching

2.1 Feynman Rules

2.1.1 Scalar QED (SQED)

Lagrangian

$$\mathcal{L}_{SQED} = |D_\mu\phi|^2 - m^2|\phi|^2 + \Phi_v^* iv \cdot D\Phi_v \tag{23}$$

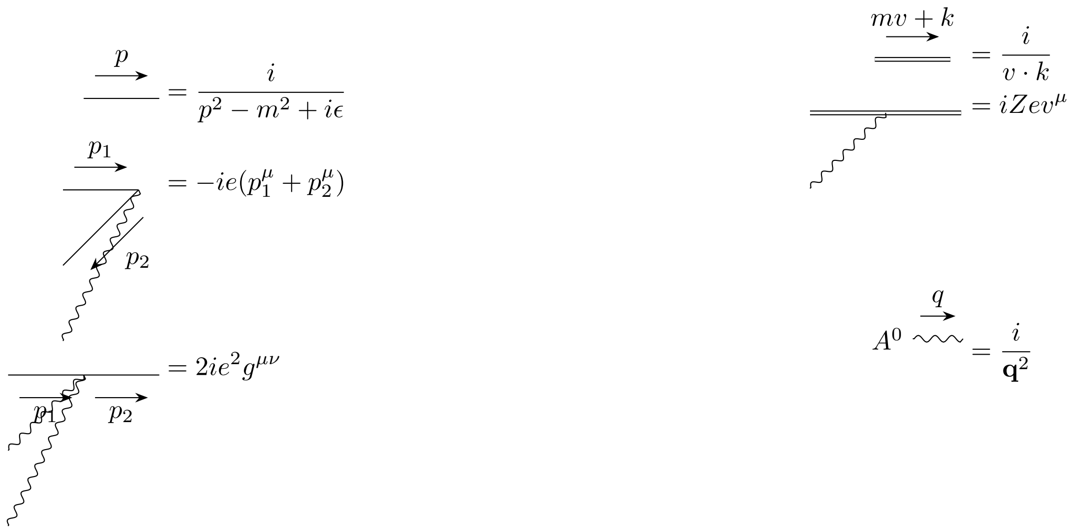
with

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$$

and

$$D_\mu\Phi_v = \partial_\mu\Phi - iZeA_\mu\Phi_v$$

But note that no $\mathbf{A}$ can appear in actual calculation because here only static scalar potential exists. And the Feynman rules



2.1.2 NRSQED-1

Let's focus on the lagrangian of only a single Scalar field

$$\mathcal{L}_{SQED} = |D_\mu \phi|^2 - m^2 |\phi|^2 \tag{24}$$

Substitute ϕ for $\phi = \frac{1}{\sqrt{2m}} e^{-imv\cdot x}(\chi_v + \bar{\chi}_v)$ with (Schwartz, Sec 35.2)

$$\chi_v = \frac{1}{\sqrt{2m}} e^{imv\cdot x}(iv \cdot D + m)\phi \tag{25}$$

$$\bar{\chi}_v = \frac{1}{\sqrt{2m}} e^{imv\cdot x}(-iv \cdot D + m)\phi \tag{26}$$

then

$$\mathcal{L}_{SQED} = \frac{1}{2m}(D_\mu(\chi_v + \bar{\chi}_v))^* D^\mu(\chi_v + \bar{\chi}_v) + (\chi_v + \bar{\chi}_v)^* iv \cdot D(\chi + \bar{\chi}_v) \tag{27}$$

Within our calculation, we set $\vec{A} = 0, v = (1, \vec{0})$. Then from (25), (26) and the motion equation derived from (27), we get two equation

$$\bar{\chi}_v = \frac{-iD_0}{2m}(\chi_v + \bar{\chi}_v) \tag{28}$$

$$\frac{-iD_0}{2m}\chi_v = \frac{\nabla^2}{4m^2}(\chi_v + \bar{\chi}_v) \tag{29}$$

And (27) could be transformed as

$$\mathcal{L}_{SQED} = (\chi_v + \bar{\chi}_v)^*(iD_0 + \frac{\nabla^2}{2m} - \frac{D_0^2}{2m})(\chi + \bar{\chi}_v) \tag{30}$$

Substituting (28), (29) iteratively, then we get the $\mathcal{L}$ expanding for v as

$$\mathcal{L}_{SQED} = \chi_v^*(iD_0 + \frac{\nabla^2}{2m} + \frac{\nabla^4}{8m^3} + \frac{\nabla^6}{16m^5} + \frac{e}{32m^4}([\nabla^2, [A_0, \nabla^2]] + [A_0, \nabla^4]) + \mathcal{O}(v^7))\chi_v \tag{31}$$

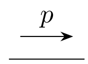
2.1.3 NRSQED

Using the transformation $\phi \to \frac{e^{-imt}}{\sqrt{2m}}\varphi$, we can have the Lagrangian

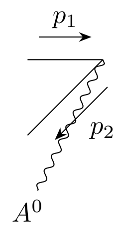
$$\mathcal{L}_{NRSQED} = \varphi^* \left(iD_0 + \frac{\mathbf{D}^2}{2m} \right) \varphi + \delta\mathcal{L} + \Phi_v^* iv \cdot D \Phi_v \tag{32}$$

with the same notation above. Here $\mathbf{D}=\nabla - ie\mathbf{A}$.

Feynman rules are also the same except for the scalar electron side which becomes



$$= \frac{i}{E-\frac{\mathbf{p}^2}{2m}+i\epsilon}$$



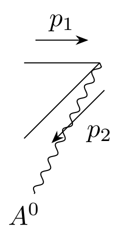
$$= -ie$$

We can ignore all interacting terms involving $\mathbf{A}$.

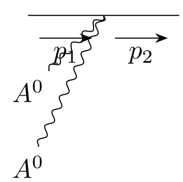
Since we need to match it to $\mathcal{O}(v^2)$ order

$$\delta\mathcal{L} = \frac{(D_0\varphi)^*(D_0\varphi)}{2m} = \frac{\dot{\varphi}^*\dot{\varphi}}{2m} + \frac{e^2\varphi^*\varphi A_0^2}{2m} - \frac{ie}{2m}A_0(\varphi^*\dot{\varphi}-\dot{\varphi}^*\varphi) \tag{33}$$

and it changes the Feynman rules to[1]



$$= -ie\left(1+\frac{E_1+E_2}{2m}\right)$$



$$= \frac{ie^2}{2m}$$

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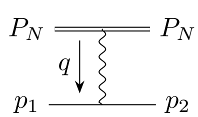
Since we rescaled ϕ by $\frac{1}{\sqrt{2m}}$ to get φ, the in/out states are also changed. We must multiply them by $\sqrt{2p^0}$ to compensate that change.

Another way to achieve it is to use the transform rules of heavy scalar effective theory (HSET).

2.2 LO

2.2.1 SQED

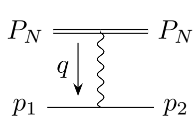
$$i\mathcal{M}^{(0)}_{SQED} =$$



$$= -e^2v^0\frac{i(p_1^0+p_2^0)}{\mathbf{q}^2} = -e^2v^0\frac{i}{\mathbf{q}^2}(2m+2E_1)$$

2.2.2 NRSQED

$$i\mathcal{M}^{(0)}_{NRSQED} =$$



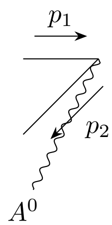
$$= -2\sqrt{p_1^0p_2^0}e^2v^0\frac{i(1+\delta)}{\mathbf{q}^2} = -e^2v^0\times 2p_1^0\frac{i(1+\delta)}{\mathbf{q}^2}$$

which gives

$$\delta = \frac{p_1^0+p_2^0}{2\sqrt{p_1^0p_2^0}i} - 1 \approx \frac{\mathbf{p}_1^4-2\mathbf{p}_1^2\mathbf{p}_2^2+\mathbf{p}_2^4}{32m^4}$$

[1] In this note, p^0 is the zero component of relativistic four momentum, and $E=p^0-m$.

The new electron-photon vertex is



$$= i + i\frac{\mathbf{p}_1^4 - 2\mathbf{p}_1^2\mathbf{p}_2^2 + \mathbf{p}_2^4}{32m^4}$$

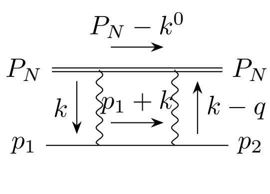
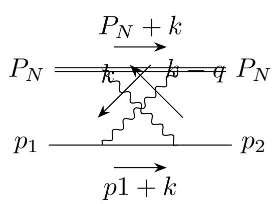
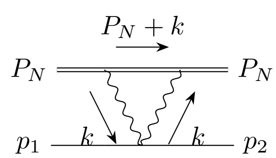
We can write the correction interacting term corresponding to δ as

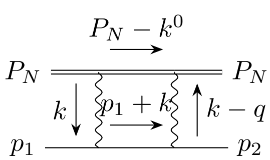
$$\begin{aligned} A_0\varphi\nabla^4\varphi^* - 2A_0\nabla^2\varphi^*\nabla^2\varphi + \varphi^* A_0\nabla^4\varphi &= \varphi^*\nabla^4(A_0\varphi) - 2\varphi^*\nabla^2(A_0\nabla^2\varphi) + \varphi^* A_0\nabla^4\varphi \\ &= \varphi^*(\nabla^2[\nabla^2, A_0] - [\nabla^2, A_0]\nabla^2)\varphi \\ &= \varphi^*[\nabla^2, [\nabla^2, A_0]]\varphi \end{aligned}$$

which is exactly the Darwin term in Holstein's *Advanced Topics in QM* with coefficient $1/32m^4$.

2.3 NLO

2.3.1 SQED

$i\mathcal{M}^{(1)}_{SQED} =$  $+$  $+$ 



$$= -e^2 v^0 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{\mathbf{k}^2} \quad < +content+ >$$

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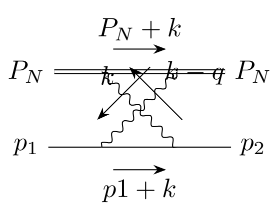

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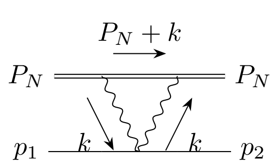

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2.3.2 NRSQED

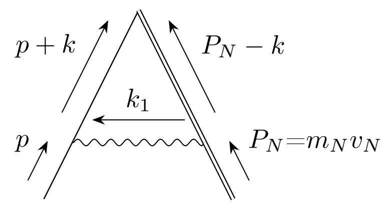
3 Local Operator and Matrix Element of NRSQED

To reproduce the singular behavior of "Klein-Gordon Hydrogen" wavefunction near origin, we can try OPE. But the dependence of x in OPE can be taken as a regularization scheme and thus the result should be the same as local one without renormalization. And the logarithmic terms of x in OPE can be reproduced by the logarithmic divergence of local operators. Since in the study of Klein-Gordon equation we know that the wavefunction only contains logarithmic divergence at the origin so that's the only type of divergence we're looking for.

3.1 LO

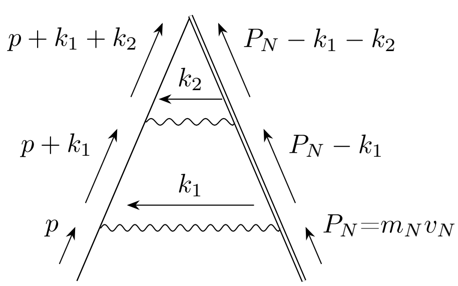
3.2 NLO

$$\langle 0|\psi_e(0)N(0)(-ie\mu^{-\epsilon})\int \mathrm{d}^4y\bar{\psi}_e\psi_e A^0(-ie\mu^{-\epsilon})\int \mathrm{d}^4z\bar{N}NA^0|eN\rangle =$$



which doesn't have logarithm divergence[2].

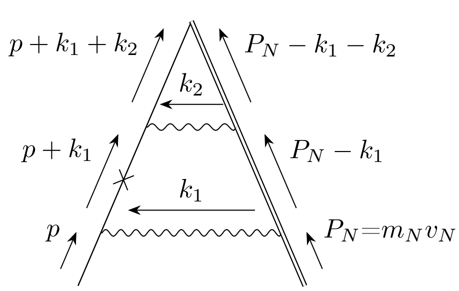
3.3 NNLO



$$= -\mu^{2\epsilon}Z^2e^4\left[\int [dk_1][dk_2]\frac{1}{|\mathbf{k_1}|^2}\frac{1}{|\mathbf{k_2}|^2}\frac{1}{-k_1^0-k_2^0+i\epsilon}\frac{1}{-k_1^0+i\epsilon}\frac{1}{p^0+k_1^0-m-\frac{(\mathbf{p}+\mathbf{k_1})^2}{2m}+i\epsilon}\frac{1}{p^0+k_1^0+k_2^0-m-\frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon}\right]$$

do the shift as above

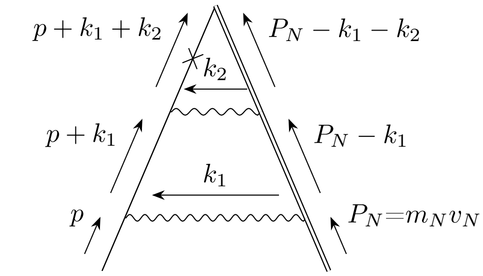
$$=\mu^{2\epsilon}Z^2e^4\left[\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{1}{E-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon}\frac{1}{E-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon}\right]=0$$



$$= 4m^2\mu^{2\epsilon}Z^2e^4\int\frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{|\mathbf{k_1}-\mathbf{p}|^2}\frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2}\frac{|\mathbf{k_1}|^4/4m^2}{[|\mathbf{k_1}|^2-2mE]^2}\frac{1}{|\mathbf{k_2}|^2-2mE}$$

$$=Z^2\alpha^2\left(\frac{1}{2(3-d)}+\log\mu\right)+\text{finite terms}$$

[2] After dimensional regularization, the Gamma function in the numerator is something like $\Gamma(n-d/2)$ and Gamma function doesn't have pole at half integer. We can rigorously prove this kind of diagrams do not have logarithmic divergence at one loop.



$$= 4m^2 \mu^{2\epsilon} Z^2 e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1} - \mathbf{p}|^2} \frac{1}{|\mathbf{k_2} - \mathbf{k_1}|^2} \frac{1}{|\mathbf{k_1}|^2 - 2mE} \frac{|\mathbf{k_2}|^4/4m^2}{[|\mathbf{k_2}|^2 - 2mE]^2}$$

$$= 0 + \text{finite terms}$$

Appendices

A Useful formulas in Feynman integrals

Feynman parametrization used here is

$$\frac{1}{\prod A_i^{d_i}} = \int \prod \mathrm{d}x_i \delta(\sum x_i - 1) \frac{\prod x_i^{d_i - 1}}{[\sum x_i A_i]^{\sum d_i}} \frac{\Gamma(\sum d_i)}{\prod \Gamma(d_i)} \tag{34}$$

Integral with the structure of the form

$$\int [dk_1][dk_2] \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{[p^0 + k_1^0 - M - \frac{(\mathbf{p}+\mathbf{k_1})^2}{2M} + i\epsilon]^m} \frac{1}{[p^0 + k_1^0 + k_2^0 - M - \frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2M} + i\epsilon]^n}$$

will always produce

$$\int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{[p^0 - M - \frac{(\mathbf{p}+\mathbf{k_1})^2}{2M}]^m} \frac{1}{[p^0 - M - \frac{(\mathbf{p}+\mathbf{k_1}+\mathbf{k_2})^2}{2M} + i\epsilon]^n}$$

with k_1^0 and k_2^0 goes to zero.

For arbitary one loop diagram of the following form, we have

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{2\beta}}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{n-\beta}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n - \beta - d/2)}{\Gamma(n)} \left(\frac{4\pi}{\Delta}\right)^{n-\beta-d/2} \tag{35a}$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n - \beta - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\beta-d/2} \tag{35b}$$

where $\frac{1}{(4\pi)^{d/2}}$ often appears as $2^{-d}\pi^{-d/2}$.

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