

# Local Operator Divergence

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NRQED matrix element at NLO

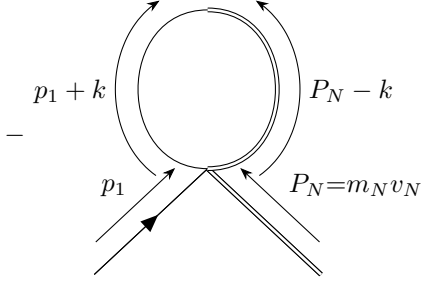
$$\begin{aligned}
 \langle 0 | \psi_e(0) N(0) (-ie) \int d^4 y \bar{\psi}_e \psi_e A^0 (-ie) \int d^4 z \bar{N} N A^0 | e N \rangle &= \\
 \text{Diagram: A triangle loop with a wavy line. Left side: incoming $p$, outgoing $p+k$. Top: incoming $P_N$, outgoing $P_N-k$. Bottom: incoming $P_N=m_N v_N$, outgoing $P_N$. Internal lines: $k_1$ (top), $k_1$ (bottom), $k_1$ (wavy).} & \\
 = ie^2 u_N(v_N) \int [dk] \frac{1}{\mathbf{k}^2 (-k^0 + i\epsilon) (0 + k^0 - m - \frac{(\mathbf{p}+\mathbf{k})^2}{2m} + \frac{(\mathbf{p}+\mathbf{k})^4}{8m^3} + i\epsilon)} \psi(p) & \\
 = e^2 u_N(v_N) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{p})^2 (E - \frac{\mathbf{k}^2}{2m} + \frac{\mathbf{k}^4}{8m^3})} \psi(p) & \\
 = e^2 u_N(v_N) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{p})^2 (E - \frac{\mathbf{k}^2}{2m})} (1 - \frac{\mathbf{k}^4}{8m^3 (E - \frac{\mathbf{k}^2}{2m})}) \psi(p) & \\
 = e^2 u_N(v_N) \psi(p) [\frac{\pi}{2p}] = e^2 u_N(v_N) \psi(p) [\frac{\pi}{2mv}] &
 \end{aligned}$$

At NNLO (where we're only interested in divergent parts)

$$\begin{aligned}
 \text{Diagram: A triangle loop with two wavy lines. Left side: incoming $p$, outgoing $p+k_1+k_2$. Top: incoming $P_N$, outgoing $P_N-k_1-k_2$. Bottom: incoming $P_N=m_N v_N$, outgoing $P_N-k_1$. Internal lines: $k_1$ (top), $k_2$ (bottom), $k_1$ (wavy), $k_2$ (wavy).} & \\
 = e^4 \int [dk_1][dk_2] \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{p^0 + k_1^0 - m - \frac{(\mathbf{p}+\mathbf{k}_1)^2}{2m} + i\epsilon} \frac{1}{p^0 + k_1^0 + k_2^0 - m - \frac{(\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)^2}{2m} + i\epsilon} \psi_e(p) u_N(v_N) & \\
 \text{do the shift as above} & \\
 = -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{E - \frac{|\mathbf{k}_1|^2}{2m} + 2i\epsilon} \frac{1}{E - \frac{|\mathbf{k}_2|^2}{2m} + 2i\epsilon} \psi_e(p) u_N(v_N) & \\
 \text{drop } \mathbf{p} & \\
 = -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_1|^2 - \frac{|\mathbf{k}_1|^4}{4m^2} + 2i\epsilon} \frac{1}{|\mathbf{k}_2|^2 - \frac{|\mathbf{k}_2|^4}{4m^2} + 2i\epsilon} \psi_e(p) u_N(v_N) & \\
 \text{if we add higher relativistic correction} & \\
 = -e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{2m}{|\mathbf{k}_1|^2 - \frac{|\mathbf{k}_1|^4}{4m^2}} \frac{2m}{|\mathbf{k}_2|^2 - \frac{|\mathbf{k}_2|^4}{4m^2}} \psi_e(p) u_N(v_N) & \\
 = -4m^2 e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_1|^2} (1 + \frac{|\mathbf{k}_1|^2}{4m^2}) \frac{1}{|\mathbf{k}_2|^2} (1 + \frac{|\mathbf{k}_2|^2}{4m^2}) \psi_e(p) u_N(v_N) &
 \end{aligned}$$

The integral

$$\begin{aligned} & \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_1|^2} \left(1 + \frac{|\mathbf{k}_1|^2}{4m^2}\right) \frac{1}{|\mathbf{k}_2|^2} \left(1 + \frac{|\mathbf{k}_2|^2}{4m^2}\right) \\ &= \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_1|^2} \left(1 + \frac{|\mathbf{k}_1|^2}{4m^2}\right) \frac{1}{|\mathbf{k}_2|^2} \left(1 + \frac{|\mathbf{k}_2|^2}{4m^2}\right) \end{aligned}$$



$$= \frac{4\pi e^4}{3m} \int \frac{d^4k}{(2\pi)^4} \frac{1}{p_1^0 + k^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon} \frac{1}{-k^0 + i\epsilon} \psi_e(p_1) u_N(v_N)$$

$$= i \frac{4\pi e^4}{3m} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{p_1^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + 2i\epsilon} \psi_e(p_1) u_N(v_N)$$

drop  $\mathbf{p}_1$

$$= i \frac{4\pi e^4}{3m} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{-\frac{\mathbf{k}^2}{2m} + 2i\epsilon} \psi_e(p_1) u_N(v_N)$$

if the dispersion relation is up to  $\mathbf{k}^4$  then

$$= \frac{4e^4 m}{3}$$