

Hydrogen

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QED Lagrangian is

$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD_0)N - \mathcal{L}_\gamma \quad (1)$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \quad (2)$$

In tree level

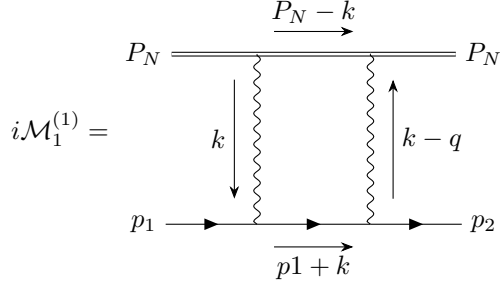
$$i\mathcal{M}_{QED}^{(0)} = \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \longrightarrow p_2 \end{array} = -e^2 \bar{u}_N(P_N) v^\mu u_N(P_N) \frac{i}{\mathbf{q}^2} \bar{u}_e(p_2) \gamma_\mu u_e(p_1)$$

$$i\mathcal{M}_{NRQED}^{(0)} = \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \longrightarrow p_2 \end{array} = -e^2 \bar{u}_N(P_N) v^\mu u_N(P_N) \frac{i}{\mathbf{q}^2} \psi^\dagger(p_2) \gamma_\mu \psi(p_1)$$

The box diagram for NRQED process is

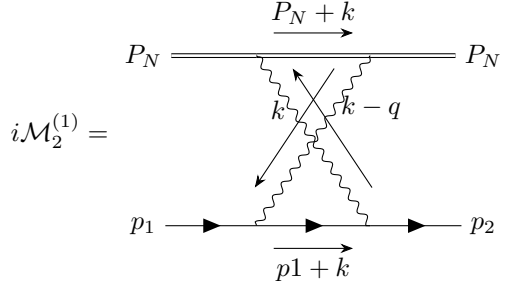
$$\begin{aligned} i\mathcal{M}_{NRQED}^{(1)} &= \begin{array}{c} P_N \text{---} \xrightarrow{P_N - k} P_N \\ \downarrow k \quad \uparrow k - q \\ p_1 \longrightarrow p_2 \\ \xrightarrow{p_1 + k} \end{array} \\ &= e^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int [dk] \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (-k^0 + i\epsilon) (p_1^0 + k^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\ &= -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\mathbf{k}^2 (\mathbf{k} - \mathbf{q})^2 (E_1 - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m})} \psi(p_1) \\ &= -ie^4 \bar{u}_N(P_N) \frac{1 + \gamma^0}{2} u_N(P_N) \psi^\dagger(p_2) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p}_1)^2 (\mathbf{k} - \mathbf{p}_2)^2 (E_1 - \frac{\mathbf{k}^2}{2m})} \psi(p_1) \end{aligned}$$

The box and crossed box diagram for QED process is



$$\begin{aligned}
i\mathcal{M}_1^{(1)} &= \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 - p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.



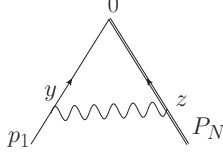
$$\begin{aligned}
i\mathcal{M}_2^{(1)} &= \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{(\not{p}_1 + \not{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [dk] \frac{2p_1^0 + \not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 - \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 + p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
&= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 - \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 + p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

$$\begin{aligned}
i\mathcal{M}_1^{(1)} + i\mathcal{M}_2^{(1)} &= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^{0^2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 + m^2 - p_1^{0^2}]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1) \\
&= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{d^3k}{(2\pi)^3} \frac{p_1^{0^2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k} - \mathbf{p}_1)^2(\mathbf{k} - \mathbf{p}_2)^2[\mathbf{k}^2 - \mathbf{p}_1^2]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1)
\end{aligned}$$

Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order.

Now consider operator product expansion.

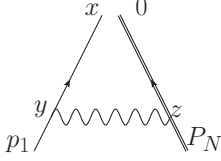


One loop scenario for NRQED case:

$$\begin{aligned}
\langle 0 | \psi_e(0) N(0) e \int d^4 y \bar{\psi}_e \psi_e A^0 e \int d^4 z \bar{N} N A^0 | e N \rangle &= e^2 u_N(P_N) \int [dk] \frac{1}{\mathbf{k}^2(-k^0 + i\epsilon)(p_1^0 + k^0 - m - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\
&= -ie^2 u_N(P_N) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(E_1 - \frac{(\mathbf{p}_1 + \mathbf{k})^2}{2m} + i\epsilon)} \psi(p_1) \\
&= -ie^2 u_N(P_N) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(\mathbf{k} - \mathbf{p}_1)^2(E_1 - \frac{\mathbf{k}^2}{2m} + i\epsilon)} \psi(p_1)
\end{aligned}$$

drop p_1

$$= ie^2 u_N(P_N) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\mathbf{k}^2(E_1 - \frac{\mathbf{k}^2}{2m} + i\epsilon)} \psi(p_1) = \pi i e^2 \sqrt{\frac{2m}{E_1}} u_N(P_N) \psi(p_1)$$



For QED case:

$$\begin{aligned}
\langle 0 | \psi(x) N(0) e \int d^4 y \bar{\psi} \gamma^0 \psi A^0 e \int d^4 z \bar{N} N A^0 | e N \rangle &= e^2 u_N(P_N) \int [dk] e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{(\not{p}_1 + \not{k} + m) \gamma^0}{\mathbf{k}^2[(p_1 + k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= e^2 u_N(P_N) \int [dk] e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{2p_1^0 + \not{k} \gamma^0}{\mathbf{k}^2[(p_1 + k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1) \\
&= ie^2 u_N(P_N) \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}}{2\mathbf{k}^2[(\mathbf{k} + \mathbf{p}_1)^2 + m^2 - p_1^0 \sqrt{(\mathbf{k} + \mathbf{p}_1)^2 + m^2}]} u_e(p_1) \\
&= ie^2 u_N(P_N) \int \frac{d^3 k}{(2\pi)^3} e^{-i(\mathbf{k} - \mathbf{p}_1) \cdot \mathbf{x}} \frac{p_1^0 + (k_i - p_{1i}) \gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k} - \mathbf{p}_1)^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)
\end{aligned}$$

drop p_1

$$= ie^2 u_N(P_N) \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{p_1^0 + \sqrt{\mathbf{k}^2 + m^2}}{2\mathbf{k}^2[\mathbf{k}^2 + m^2 - p_1^0 \sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)$$