Homework: Quantum Field Theory #8

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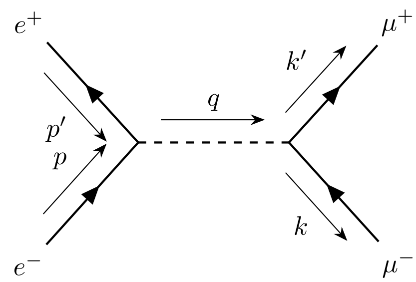
1. Yukawa theory.

(i) $e^+e^- \to \mu^+\mu^-$. $H_I = \int \mathrm{d}^3x[g_1\bar{\psi}_e\psi_e\phi + g_2\bar{\psi}_\mu\psi_\mu\phi]$, derive the $\mathcal{M}$ of the lowest order, calculate $\sum_{spins}|\mathcal{M}|^2$,$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ and σ.

$$\begin{aligned}
iT =& \langle kk'|Te^{-i\{g_1\int \mathrm{d}^4x\bar{\psi}_e\psi_e\phi + g_2\int \mathrm{d}^4x\bar{\psi}_\mu\psi_\mu\phi\}}|pp'\rangle \\
=& \langle kk'|\, g_2\int \mathrm{d}^4x\bar{\psi}_\mu\psi_\mu\phi g_1\int \mathrm{d}^4y\bar{\psi}_e\psi_e\phi\,|pp'\rangle \\
=& g_1g_2\int \mathrm{d}^4x\mathrm{d}^4y\bar{u}^s_\mu e^{ik\cdot x}v^{s'}_\mu e^{ik'\cdot x}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2-m^2}e^{-iq\cdot(x-y)}\bar{v}^re^{-ip'\cdot y}u^{r'}e^{-ip\cdot y} \\
=& g_1g_2\bar{u}^s_\mu v^{s'}_\mu\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2-m^2}\bar{v}^ru^{r'}(2\pi)^4\delta^4(k+k'-q)(2\pi)^4\delta^4(q-p-p') \\
=& \bar{u}^s(k)g_2v^{s'}(k')\frac{i}{(p+p')^2-m^2}\bar{v}^r(p')g_1u^{r'}(p)(2\pi)^4\delta^4(p+p'-k-k')
\end{aligned}$$

And the feynman diagram



The scattering amplitude

$$i\mathcal{M} = \bar{u}^s(k)g_2v^{s'}(k')\frac{i}{q^2-m^2}\bar{v}^r(p')g_1u^{r'}(p)$$

where $q = p + p'$. And

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|_{CM} = \frac{1}{2E_p2E_{p'}|v_p - v_{p'}|}\frac{|\mathbf{k}|}{(2\pi)^24E_{CM}}|\mathcal{M}|^2$$

The spin sum amplitude

$$\begin{aligned}
\sum_{spins}|\mathcal{M}|^2 =& \sum_{spins}g_1^2g_2^2[\bar{u}^s(k)v^{s'}(k')\frac{1}{q^2-m^2}\bar{v}^r(p')u^{r'}(p)][\bar{u}^s(k)v^{s'}(k')\frac{1}{q^2-m^2}\bar{v}^r(p')u^{r'}(p)]^* \\
=& \sum_{spins}g_1^2g_2^2[\bar{u}^s(k)v^{s'}(k')\frac{1}{q^2-m^2}\bar{v}^r(p')u^{r'}(p)][\bar{v}^{s'}(k')u^s(k)\frac{1}{q^2-m^2}\bar{u}^{r'}(p)v^r(p')] \\
=& \sum_{spins}\frac{g_1^2g_2^2}{(q^2-m^2)^2}\operatorname{tr}\Big\{u^s(k)\bar{u}^s(k)v^{s'}(k')\bar{v}^{s'}(k')\Big\}\operatorname{tr}\Big\{v^r(p')\bar{v}^r(p')u^{r'}(p)\bar{u}^{r'}(p)\Big\} \\
=& \frac{g_1^2g_2^2}{(q^2-m^2)^2}\operatorname{tr}\Big\{(\not{k}+m_\mu)(\not{k}'-m_\mu)\Big\}\operatorname{tr}\big\{(\not{p}'-m_e)(\not{p}+m_e)\big\} \\
=& \frac{g_1^2g_2^2}{(q^2-m^2)^2}(4k\cdot k'-m_\mu^2)(4p\cdot p'-m_e^2)
\end{aligned}$$

Now we know that in centre-of-mass frame

$$p^\mu = (E, 0, 0, p), p'^\mu = (E, 0, 0, -p), k^\mu = (E, k\sin\theta, 0, k\cos\theta), k'^\mu = (E, -k\sin\theta, 0, -k\cos\theta)$$

so

$$p \cdot p' = E^2 + p^2, k \cdot k' = E^2 + k^2, p^2 = E^2 - m_e^2, k^2 = E^2 - m_\mu^2, E_{CM} = 2E, q^2 = 4E^2 = s$$

Thus

$$\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{g_1^2 g_2^2}{4(q^2 - m^2)^2}(8E^2 - 5m_e^2)(8E^2 - 5m_\mu^2)$$

The differential cross section is (the mass of electron is neglected)

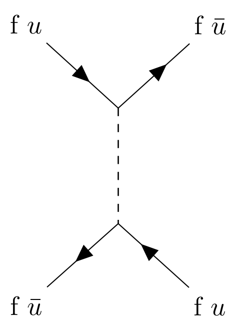
$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|_{CM} &= \frac{1}{2E_p 2E_{p'}|v_p - v_{p'}|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{CM}}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 \\
&= \frac{1}{2E_{CM}^2}\frac{k}{16\pi^2 E_{CM}}\frac{g_1^2 g_2^2}{4(q^2 - m^2)^2}8E^2(8E^2 - 5m_\mu^2) \\
&= \frac{\sqrt{E^2 - m_\mu^2}}{16\pi^2 s}\frac{g_1^2 g_2^2}{4(s - m^2)^2}(8E^2 - 5m_\mu^2) \\
&= \frac{\sqrt{s - 4m_\mu^2}}{128\pi^2 s}\frac{g_1^2 g_2^2}{(s - m^2)^2}(2s - 5m_\mu^2)
\end{aligned}$$

and total cross section

$$\sigma = \frac{\sqrt{s - 4m_\mu^2}}{32\pi s}\frac{g_1^2 g_2^2}{(s - m^2)^2}(2s - 5m_\mu^2)$$

(ii) NR scattering.

(a) $ff \to ff$.



$$= (-g^2)[\bar{u}(k)u(p)\frac{i}{q^2 - M^2}\bar{u}(k')u(p') - \bar{u}(k')u(p)\frac{i}{q^2 - M^2}\bar{u}(k)u(p')]$$

Use Yukawa potential

$$i\mathcal{M} = (-g^2)[\bar{u}(k)u(p)\frac{i}{q^2 - M^2}\bar{u}(k')u(p') - \bar{u}(k')u(p)\frac{i}{q^2 - M^2}\bar{u}(k)u(p')]$$

For non-relativistic limit ($p = (m, \mathbf{p})$ and so on)

$$\bar{u}^s(k)u^r(p) = 2m\delta^{sr}$$

and so on. So

$$i\mathcal{M} = (-g^2)[\frac{i}{(\mathbf{k} - \mathbf{p})^2 - M^2}2m\delta^{rs}2m\delta^{r's'} - \frac{i}{(\mathbf{k'} - \mathbf{p})^2 - M^2}2m\delta^{r's}2m\delta^{rs'}]$$

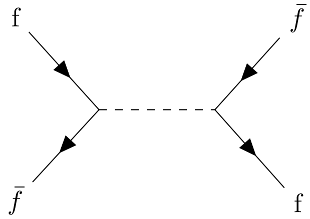
And in Born approximation

$$\langle p'|iT|p\rangle = -i\tilde{V}(\mathbf{q})(2\pi)\delta^4(E_{\mathbf{p'}} - E_{\mathbf{p}})$$

Comparing with the former one we have for each $\mathbf{q}$

$$\tilde{V}(\mathbf{q}) = (-g^2)[\frac{i}{\mathbf{q}^2 - M^2}$$

(b) $f\bar{f} \to f\bar{f}$.



$$= (-g^2)\bar{u}(k)v(k')\frac{i}{q^2 - M^2}\bar{v}(p')u(p)$$

Similarly to the discussion we made before, the sign of the potential in Born approximation is entirely dependent on the vertex. So it's again attractive.

2. Trace technology.

(i) $\mathrm{tr}\{\gamma^5\}$

$$\begin{aligned}\mathrm{tr}\{\gamma^5\} &= i\,\mathrm{tr}\{\gamma^0\gamma^1\gamma^2\gamma^3\} \\ &= 4i(g^{01}g^{23} - g^{02}g^{13} + g^{03}g^{12}) \\ &= 0\end{aligned}$$

(ii) $\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\}$

$$\begin{aligned}\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\} &= \mathrm{tr}\{\gamma^0\gamma^0\gamma^5\gamma^\mu\gamma^\nu\} \\ &= \mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^0\}\end{aligned}$$

note that $\mathrm{tr}\{\gamma^5\gamma^\nu\gamma^0\} = \mathrm{tr}\{\gamma^1\gamma^2\gamma^3\gamma^\nu\} = 0$

$$\begin{aligned}&= (-1)^3\,\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^0\gamma^0\} \\ &= -\,\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\} \\ &= 0\end{aligned}$$

(iii) $\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\}$

For a proof of identity 6, the same trick still works unless $(\mu\nu\rho\sigma)$ is some permutation of (0123), so that all 4 gammas appear. The anticommutation rules imply that interchanging two of the indices changes the sign of the trace, so $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5)$ must be proportional to $\epsilon^{\mu\nu\rho\sigma}$ $(\epsilon^{0123} = \eta^{0\mu}\eta^{1\nu}\eta^{2\rho}\eta^{3\sigma}\epsilon_{\mu\nu\rho\sigma} = \eta^{00}\eta^{11}\eta^{22}\eta^{33}\epsilon_{0123} = -1)$. The proportionality constant is $4i$, as can be checked by plugging in $(\mu\nu\rho\sigma) = (0123)$, writing out γ^5, and remembering that the trace of the identity is 4.

$$\begin{aligned}\mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= \mathrm{tr}\{\gamma^0\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} \\ &= -\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^0\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} \\ &= -2g^{0\mu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + \mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma\} \\ &= -2g^{0\mu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + 2g^{0\nu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\} - \mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^0\gamma^\rho\gamma^\sigma\} \\ &= -2g^{0\mu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + 2g^{0\nu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\} - 2g^{0\rho}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\} + \mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^0\gamma^\sigma\} \\ &= -2g^{0\mu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + 2g^{0\nu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\} - 2g^{0\rho}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\} + 2g^{0\sigma}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\} \\ &\quad - \mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^0\} \\ &= -2g^{0\mu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + 2g^{0\nu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\} - 2g^{0\rho}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\} + 2g^{0\sigma}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\} \\ &\quad - \mathrm{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} \\ &= -g^{0\mu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + g^{0\nu}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\} - g^{0\rho}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\} + g^{0\sigma}\,\mathrm{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\}\end{aligned}$$

For $\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\}$

$$\begin{aligned}\mathrm{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} &= -\,\mathrm{tr}\{g^1\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} \\ &= -2g^{1\nu}\,\mathrm{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} + 2g^{1\rho}\,\mathrm{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\sigma\} - 2g^{1\sigma}\,\mathrm{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\rho\} + \mathrm{tr}\{\gamma^1\gamma^1\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} \\ &= -g^{1\nu}\,\mathrm{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} + g^{1\rho}\,\mathrm{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\sigma\} - g^{1\sigma}\,\mathrm{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\rho\}\end{aligned}$$

For $\operatorname{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\}$

$$\begin{aligned}
\operatorname{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} &= -\operatorname{tr}\{\gamma^2\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} \\
&= \operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^2\gamma^\rho\gamma^\sigma\} \\
&= 2g^{2\rho}\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} - 2g^{2\sigma}\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\} + \operatorname{tr}\{\gamma^2\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} \\
&= g^{2\rho}\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} - g^{2\sigma}\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\}
\end{aligned}$$

For $\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\}$

$$\begin{aligned}
\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} &= -\operatorname{tr}\{\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} \\
&= -\operatorname{tr}\{\gamma^3\gamma^2\gamma^1\gamma^0\gamma^5\gamma^3\gamma^\sigma\} \\
&= -g^{3\sigma}\operatorname{tr}\{\gamma^3\gamma^2\gamma^1\gamma^0\gamma^5\} \\
&= -ig^{3\sigma}
\end{aligned}$$

Put these back and

$$\begin{aligned}
\operatorname{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} &= g^{2\rho}\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} - g^{2\sigma}\operatorname{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\} \\
&= -ig^{2\rho}g^{3\sigma} + ig^{2\sigma}g^{3\rho}
\end{aligned}$$

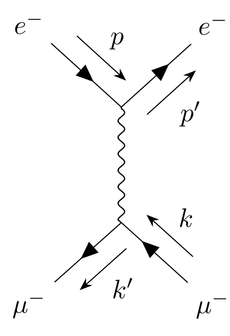
$$\begin{aligned}
\operatorname{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} &= -g^{1\nu}\operatorname{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} + g^{1\rho}\operatorname{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\sigma\} - g^{1\sigma}\operatorname{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\rho\} \\
&= -g^{1\nu}(-g^{2\rho}g^{3\sigma} + g^{2\sigma}g^{3\rho}) + g^{1\rho}(-g^{2\nu}g^{3\sigma} + g^{2\sigma}g^{3\nu}) - g^{1\sigma}(-g^{2\nu}g^{3\rho} + g^{2\rho}g^{3\nu}) \\
&= ig^{1\nu}g^{2\rho}g^{3\sigma} - ig^{1\nu}g^{2\sigma}g^{3\rho} - ig^{1\rho}g^{2\nu}g^{3\sigma} + ig^{1\rho}g^{2\sigma}g^{3\nu} + ig^{1\sigma}g^{2\nu}g^{3\rho} - ig^{1\sigma}g^{2\rho}g^{3\nu}
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= -g^{0\mu}\operatorname{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + g^{0\nu}\operatorname{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\} - g^{0\rho}\operatorname{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\} + g^{0\sigma}\operatorname{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\} \\
&= -ig^{0\mu}(g^{1\nu}g^{2\rho}g^{3\sigma} - g^{1\nu}g^{2\sigma}g^{3\rho} - g^{1\rho}g^{2\nu}g^{3\sigma} + g^{1\rho}g^{2\sigma}g^{3\nu} + g^{1\sigma}g^{2\nu}g^{3\rho} - g^{1\sigma}g^{2\rho}g^{3\nu}) \\
&\quad + ig^{0\nu}(g^{1\mu}g^{2\rho}g^{3\sigma} - g^{1\mu}g^{2\sigma}g^{3\rho} - g^{1\rho}g^{2\mu}g^{3\sigma} + g^{1\rho}g^{2\sigma}g^{3\mu} + g^{1\sigma}g^{2\mu}g^{3\rho} - g^{1\sigma}g^{2\rho}g^{3\mu}) \\
&\quad - ig^{0\rho}(g^{1\mu}g^{2\nu}g^{3\sigma} - g^{1\mu}g^{2\sigma}g^{3\nu} - g^{1\nu}g^{2\mu}g^{3\sigma} + g^{1\nu}g^{2\sigma}g^{3\mu} + g^{1\sigma}g^{2\mu}g^{3\nu} - g^{1\sigma}g^{2\nu}g^{3\mu}) \\
&\quad + ig^{0\sigma}(g^{1\mu}g^{2\nu}g^{3\rho} - g^{1\mu}g^{2\rho}g^{3\nu} - g^{1\nu}g^{2\mu}g^{3\rho} + g^{1\nu}g^{2\rho}g^{3\mu} + g^{1\rho}g^{2\mu}g^{3\nu} - g^{1\rho}g^{2\nu}g^{3\mu}) \\
&= -4i\epsilon^{\mu\nu\rho\sigma}
\end{aligned}$$

(Use the identity mentioned before.)

3. The electron muon scattering process $e^-\mu^- \to e^-\mu^-$.

We can plot the feynman diagram and give the invariant amplitude



$$i\mathcal{M} = (ie^2)\bar{u}^s(p)\gamma^\mu u^{s'}(p')\frac{g_{\mu\nu}}{q^2}\bar{u}^r(k)\gamma^\nu u^r(k')$$

The spin sum amplitude

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\operatorname{tr}\{(\not{p} + m_e)\gamma^\mu(\not{p}' + m_e)\gamma^\nu\}\operatorname{tr}\Big\{(\not{k} + m_\mu)\gamma_\mu(\not{k}' + m_\mu)\gamma_\nu\Big\} \\
&= \frac{e^4}{4q^4}[4(p^\mu p'^\nu + p'^\mu p^\nu - p\cdot p' g^{\mu\nu}) + 4m_e^2 g^{\mu\nu}][4(k^\mu k'^\nu + k'^\mu k^\nu - k\cdot k' g_{\mu\nu}) + 4m_\mu^2 g_{\mu\nu}] \\
&= \frac{4e^4}{q^4}[p^\mu p'^\nu + p'^\mu p^\nu - p\cdot p' g^{\mu\nu} + m_e^2 g^{\mu\nu}][k^\mu k'^\nu + k'^\mu k^\nu - k\cdot k' g_{\mu\nu} + m_\mu^2 g_{\mu\nu}]
\end{aligned}$$

ignore the electron mass m_e

$$= \frac{4e^4}{q^4}[p^\mu p'^\nu + p'^\mu p^\nu - p\cdot p' g^{\mu\nu}][k^\mu k'^\nu + k'^\mu k^\nu - k\cdot k' g_{\mu\nu} + m_\mu^2 g_{\mu\nu}]$$
$$= \frac{4e^4}{q^4}[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) - (p\cdot p')(k\cdot k') + m_\mu^2(p\cdot p') + (p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) - (p'\cdot p)(k\cdot k')$$
$$+ m_\mu^2(p'\cdot p) - (p\cdot p')(k\cdot k') - (p\cdot p')(k\cdot k') + 4(p\cdot p')(k\cdot k') - 4m_\mu^2(p\cdot p')]$$
$$= \frac{8e^4}{q^4}[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) - m_\mu^2(p\cdot p')]$$

where $q = p' - p$.

The differential cross section (θ is the angle between the initial muon and final muon, and in centre-of-mass frame)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|_{CM} = \frac{1}{2E_p 2E_k|v_p - v_k|}\frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$

for our problem we set $|v_p - v_k| = 1$ first and we'll plug it back later, $E_p = \omega$

$$= \frac{1}{2\omega 2E_k}\frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}}\frac{8e^4}{q^4}[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) - m_\mu^2(p\cdot p')]$$

and

$$p = (\omega, \omega\hat{z}), k = (E_k, -\omega\hat{z}), p' = (\omega_{p'}, \omega_{\mathbf{p}'}), k' = (E_{k'}, -\omega_{\mathbf{p}'}), \omega_{\mathbf{p}'} = (-\omega_{p'}\sin\theta, 0, -\omega_{p'}\cos\theta)$$

which explicitly is

$$p = (\omega, \omega\hat{z}), k = (E_k, -\omega\hat{z}), p' = (\omega_{p'}, -\omega_{p'}\sin\theta, 0, -\omega_{p'}\cos\theta), k' = (E_{k'}, \omega_{p'}\sin\theta, 0, \omega_{p'}\cos\theta)$$

so

$$p\cdot k = \omega E_k + \omega^2, p'\cdot k' = \omega_{p'}(\omega + E_k), E_k^2 = \omega^2 + m_\mu^2, p\cdot k' = \omega^2 + \omega E_k - \omega\omega_{p'} - \omega\omega_{p'}\cos\theta, p'\cdot k = \omega_{p'}E_k - \omega\omega_{p'}\cos\theta$$
$$p\cdot p' = \omega\omega_{p'}(1+\cos\theta), q^2 = (p'-p)^2 = 2\omega_{p'}\omega(1-\cos\theta)$$

and gives

$$= \frac{1}{2\omega 2E_k}\frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}}\frac{8e^4}{q^4}[(\omega E_k + \omega^2)(\omega_{p'}(\omega + E_k)) + (\omega^2 + \omega E_k - \omega\omega_{p'} - \omega\omega_{p'}\cos\theta)(\omega_{p'}E_k - \omega\omega_{p'}\cos\theta)$$
$$- m_\mu^2(\omega\omega_{p'}(1+\cos\theta))]$$
$$= \frac{1}{2\omega 2E_k}\frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}}\frac{8e^4}{q^4}[\omega E_k\omega_{p'}\omega + \omega E_k\omega_{p'}E_k + \omega^2\omega_{p'}\omega + \omega^2\omega_{p'}E_k + \omega^2\omega_{p'}E_k - \omega^2\omega\omega_{p'}\cos\theta + \omega E_k\omega_{p'}E_k$$
$$- \omega E_k\omega\omega_{p'}\cos\theta - \omega\omega_{p'}\omega_{p'}E_k + \omega\omega_{p'}\omega\omega_{p'}\cos\theta - \omega\omega_{p'}\cos\theta\omega_{p'}E_k + \omega\omega_{p'}\cos\theta\omega\omega_{p'}\cos\theta - m_\mu^2\omega\omega_{p'}(1+\cos\theta)$$
$$= \frac{1}{2\omega 2E_k}\frac{\omega_{p'}}{(2\pi)^2 4(\omega + E_k)}\frac{8e^4}{q^4}[3\omega^2 E_k\omega_{p'} + 2\omega E_k^2\omega_{p'} + \omega^3\omega_{p'} - \omega^3\omega_{p'}\cos\theta - \omega^2 E_k\omega_{p'}\cos\theta - \omega\omega_{p'}^2 E_k(1+\cos\theta) + \omega^2\omega_{p'}^2\cos\theta$$
$$+ \omega^2\omega_{p'}^2\cos^2\theta - m_\mu^2\omega\omega_{p'}(1+\cos\theta)]$$

assuming no large change in energy and gives $\omega_{p'} = \omega$, so

$$= \frac{1}{2\omega 2E_k}\frac{\omega}{(2\pi)^2 4(\omega + E_k)}\frac{8e^4}{q^4}[2\omega^3 E_k + 2\omega^2 E_k^2 + \omega^4 - 2\omega^3 E_k\cos\theta + \omega^4\cos^2\theta - m_\mu^2\omega^2(1+\cos\theta)]$$
$$= \frac{1}{2\omega 2E_k}\frac{\omega}{(2\pi)^2 4(\omega + E_k)}\frac{2e^4}{\omega^4(1-\cos\theta)^2}[2\omega^3 E_k + 2\omega^2 E_k^2 + \omega^4 - 2\omega^3 E_k\cos\theta + \omega^4\cos^2\theta - m_\mu^2\omega^2(1+\cos\theta)]$$
$$= \frac{1}{2\omega 2E_k}\frac{\omega}{(2\pi)^2 4(\omega + E_k)}\frac{2e^4}{\omega^2(1-\cos\theta)^2}[2\omega E_k + 2E_k^2 + \omega^2 - 2\omega E_k\cos\theta + \omega^2\cos^2\theta - m_\mu^2(1+\cos\theta)]$$
$$= \frac{1}{2\omega 2E_k}\frac{\omega}{(2\pi)^2 4(\omega + E_k)}\frac{2e^4}{\omega^2(1-\cos\theta)^2}[(E_k + \omega)^2 + (E_k - \omega\cos\theta)^2 - m_\mu^2(1+\cos\theta)]$$

plug in $|v_p - v_k| = \left|1 + \frac{\omega}{E_k}\right|$

$$
\begin{aligned}
&= \frac{1}{2\omega 2E_k\left|1+\frac{\omega}{E_k}\right|} \frac{\omega}{(2\pi)^2 4(\omega+E_k)} \frac{2e^4}{\omega^2(1-\cos\theta)^2}[(E_k+\omega)^2 + (E_k - \omega\cos\theta)^2 - m_\mu^2(1+\cos\theta)] \\
&= \frac{1}{4|E_k+\omega|} \frac{1}{(2\pi)^2 4(\omega+E_k)} \frac{2e^4}{\omega^2(1-\cos\theta)^2}[(E_k+\omega)^2 + (E_k - \omega\cos\theta)^2 - m_\mu^2(1+\cos\theta)] \\
&= \frac{1}{64\pi^2(\omega+E_k)^2} \frac{2e^4}{\omega^2(1-\cos\theta)^2}[(E_k+\omega)^2 + (E_k - \omega\cos\theta)^2 - m_\mu^2(1+\cos\theta)] \\
&= \frac{1}{2(\omega+E_k)^2} \frac{\alpha^2}{\omega^2(1-\cos\theta)^2}[(E_k+\omega)^2 + (E_k - \omega\cos\theta)^2 - m_\mu^2(1+\cos\theta)]
\end{aligned}
$$

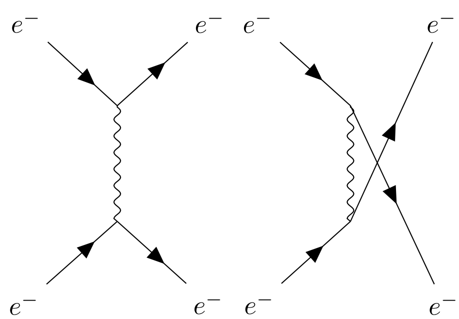
where $E_k = \sqrt{\omega^2 + m_\mu^2}$.

In high energy limit $m_\mu \to 0$

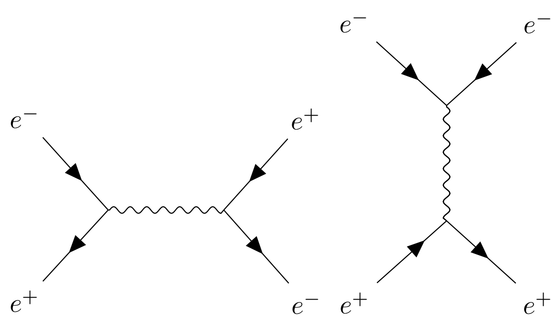
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|_{CM} = \frac{1}{2E_{CM}^2} \frac{\alpha^2}{(1-\cos\theta)^2}[4 + (1-\cos\theta)^2]$$

4. Feynman diagrams.

(i) $e^- e^- \to e^- e^-$



(ii) $e^- e^+ \to e^- e^+$

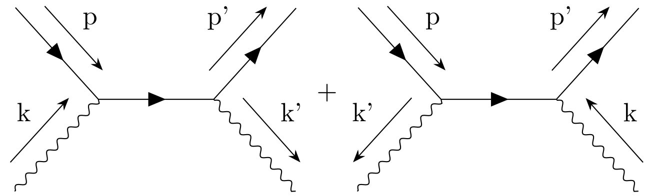


5. Verify Ward identity. (Use condition (i) and (ii) and abtain the result $\mathcal{M} = 0$ which verified Ward identity.)

We already have the invariant amplitude as follows

$$i\mathcal{M} = -ie^2 \epsilon_\mu^*(k')\epsilon_\nu(k)\bar{u}(p')[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}]u(p)$$

and from the feynman diagram



we have

$$p + k = p' + k', k^2 = k'^2 = 0$$

And note that from Dirac equation

$$\not{p}u(p) = mu(p)$$

$$\bar{u}(p)\not{p} = m\bar{u}(p)$$

(i) $\epsilon^*_\mu(k') \to k'_\mu$

$$\begin{aligned}
i\mathcal{M} &= -ie^2 k'_\mu \epsilon_\nu(k)\bar{u}(p')[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}]u(p) \\
&= -ie^2 \epsilon_\nu(k)\bar{u}(p')[\frac{\not{k}'\not{k}\gamma^\nu + 2\not{k}'p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}'\not{k}' - 2\gamma^\nu (p\cdot k')}{2p\cdot k'}]u(p)
\end{aligned}$$

$$\frac{\gamma^\nu \not{k}'\not{k}' - 2\gamma^\nu(p\cdot k')}{2p\cdot k'} = \frac{\gamma^\nu(k'\cdot k') - 2\gamma^\nu(p\cdot k')}{2p\cdot k'} = -\gamma^\nu$$

$$\begin{aligned}
\bar{u}(p')[\frac{\not{k}'\not{k}\gamma^\nu + 2\not{k}'p^\nu}{2p\cdot k}]u(p) &= \bar{u}(p')[\frac{(\not{p}+\not{k}-\not{p}')\not{k}\gamma^\nu + 2(\not{p}+\not{k}-\not{p}')p^\nu}{2p\cdot k}]u(p) \\
&= \bar{u}(p')[\frac{\not{p}\not{k}\gamma^\nu - \not{p}'\not{k}\gamma^\nu + 2(\not{p}+\not{k}-\not{p}')p^\nu}{2p\cdot k}]u(p) \\
&= \bar{u}(p')[\frac{2(p\cdot k)\gamma^\nu - 2\not{k}p^\nu + \not{k}\gamma^\nu\not{p} - \not{p}'\not{k}\gamma^\nu + 2(\not{p}+\not{k}-\not{p}')p^\nu}{2p\cdot k}]u(p) \\
&= \bar{u}(p')[\frac{2(p\cdot k)\gamma^\nu + \not{k}\gamma^\nu\not{p} - \not{p}'\not{k}\gamma^\nu + 2(\not{p}-\not{p}')p^\nu}{2p\cdot k}]u(p)
\end{aligned}$$

use Dirac equation

$$\begin{aligned}
&= \bar{u}(p')[\frac{2(p\cdot k)\gamma^\nu}{2p\cdot k}]u(p) \\
&= \gamma^\nu
\end{aligned}$$

So $i\mathcal{M} = 0$.

(ii) $\epsilon_\nu(k) \to k_\nu$

$$\begin{aligned}
i\mathcal{M} &= -ie^2 \epsilon^*_\mu(k')k_\nu\bar{u}(p')[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}]u(p) \\
&= -ie^2 \epsilon^*_\mu(k')\bar{u}(p')[\frac{\gamma^\mu \not{k}\not{k} + 2\gamma^\mu (k\cdot p)}{2p\cdot k} + \frac{\not{k}\not{k}'\gamma^\mu - 2\not{k}p^\mu}{2p\cdot k'}]u(p)
\end{aligned}$$

$$\frac{\gamma^\mu \not{k}\not{k} + 2\gamma^\mu(k\cdot p)}{2p\cdot k} = \frac{\gamma^\mu(k\cdot k) + 2\gamma^\mu(k\cdot p)}{2p\cdot k} = \gamma^\mu$$

$$
\begin{aligned}
\bar{u}(p')[\frac{\not{k}\not{k}'\gamma^\mu - 2\not{k}p^\mu}{2p\cdot k'}]u(p) &= \bar{u}(p')[\frac{(\not{p}'+\not{k}'-\not{p})\not{k}'\gamma^\mu - 2(\not{p}'+\not{k}'-\not{p})p^\mu}{2p\cdot k'}]u(p) \\
&= \bar{u}(p')[\frac{\not{p}'\not{k}'\gamma^\mu - \not{p}\not{k}'\gamma^\mu - 2(\not{p}'+\not{k}'-\not{p})p^\mu}{2p\cdot k'}]u(p) \\
&= \bar{u}(p')[\frac{\not{p}'\not{k}'\gamma^\mu - 2\gamma^\mu(p\cdot k') + 2\not{k}'p^\mu - \not{k}'\gamma^\mu\not{p} - 2(\not{p}'+\not{k}'-\not{p})p^\mu}{2p\cdot k'}]u(p) \\
&= \bar{u}(p')[\frac{\not{p}'\not{k}'\gamma^\mu - 2\gamma^\mu(p\cdot k') - \not{k}'\gamma^\mu\not{p} - 2(\not{p}'-\not{p})p^\mu}{2p\cdot k'}]u(p)
\end{aligned}
$$

use Dirac equation

$$
\begin{aligned}
&= \bar{u}(p')[\frac{-2\gamma^\mu(p\cdot k')}{2p\cdot k'}]u(p) \\
&= -\gamma^\mu
\end{aligned}
$$

So $i\mathcal{M} = 0$.

6. Feynman parametres.

$$
\begin{aligned}
\frac{1}{AB} &= -\frac{1}{A-B}(\frac{1}{A}-\frac{1}{B}) \\
&= \frac{1}{A-B}\int_B^A \mathrm{d}x' \frac{1}{x'^2}
\end{aligned}
$$

substitute x' with x which satisfies $x' = (A-B)x + B$

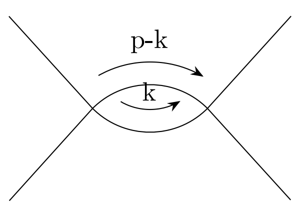
$$
\begin{aligned}
&= \int_0^1 \mathrm{d}x \frac{1}{[(A-B)x+B]^2} \\
&= \int_0^1 \mathrm{d}x \frac{1}{[xA+(1-x)B]^2} \\
&= \int_0^1 \mathrm{d}x\mathrm{d}y\delta(x+y-1)\frac{1}{[xA+yB]^2}
\end{aligned}
$$

7. ϕ^4 theory.

The Lagrangian

$$
\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4
$$

Calculate



In the second order (all external lines are 1)

$$
i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2(p-k)^2}
$$

(i) Cutoff.

Apply Feynman parameters

$$
\begin{aligned}
i\mathcal{M}_2 &= \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{1}{[x(p-k)^2+(1-x)k^2]^2} \\
&= \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{1}{[xp^2-2xp\cdot k+k^2]^2}
\end{aligned}
$$

$k \to k + xp$

$$
\begin{aligned}
&= \frac{(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{[xp^2 - 2xp \cdot (k + xp) + (k + xp)^2]^2} \\
&= \frac{(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{[k^2 + x(1-x)p^2 + i\epsilon]^2}
\end{aligned}
$$

Now apply Wick rotation (The nature of Wick rotation is to rotate k into Euclidean space, so the metric of k_E is of Euclidean space, and $k_E^2 > 0$.)

$$k^0 \to ik_E^0, \mathbf{k} = \mathbf{k_E}, k^2 = -k_E^2$$

$$
\begin{aligned}
i\mathcal{M}_2 &= \frac{(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{[k^2 + x(1-x)p^2 + i\epsilon]^2} \\
&= \frac{i(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{[k_E^2 - (x(1-x)p^2 + i\epsilon)]^2}
\end{aligned}
$$

$\Delta \equiv -x(1-x)p^2 - i\epsilon$

$$
\begin{aligned}
&= \frac{i(-i\lambda)^2}{2} \int \frac{\mathrm{d}^4 k_E}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{[k_E^2 + \Delta]^2} \\
&= \frac{i(-i\lambda)^2}{2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_4}{(2\pi)^4} \int_0^\infty \mathrm{d}k_E \frac{k_E^3}{[k_E^2 + \Delta]^2}
\end{aligned}
$$

Variables substitution

$$
\begin{aligned}
&= \frac{i(-i\lambda)^2}{4} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_4}{(2\pi)^4} \int_0^\infty \mathrm{d}k_E \frac{k_E}{[k_E + \Delta]^2} \\
&= \frac{i(-i\lambda)^2}{4} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_4}{(2\pi)^4} \int_\Delta^\infty \mathrm{d}k_E \frac{k_E - \Delta}{k_E^2} \\
&= \frac{i(-i\lambda)^2}{4} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_4}{(2\pi)^4} (\ln k_E + \frac{\Delta}{k_E})|_{k_E=\Delta}^{\infty}
\end{aligned}
$$

Ultraviolet divergence appears, use cutoff regularization

$$
\begin{aligned}
&= \frac{i(-i\lambda)^2}{4} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_4}{(2\pi)^4} (\ln k_E + \frac{\Delta}{k_E})|_{k_E=\Delta}^{\Lambda} \\
&= \frac{i(-i\lambda)^2}{4} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_4}{(2\pi)^4} (\ln \Lambda + \frac{\Delta}{\Lambda} - \ln \Delta - \frac{\Delta}{\Delta}) \\
&= \frac{i(-i\lambda)^2}{4} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}\Omega_4}{(2\pi)^4} (\ln \Lambda + \frac{\Delta}{\Lambda} - \ln \Delta - 1)
\end{aligned}
$$

$\mathrm{d}\Omega_4 = \sin^2\theta \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\omega, \int \Omega_4 = 2\pi^2$

$$= \frac{-i\lambda^2}{32\pi^2} \int_0^1 \mathrm{d}x (\ln \Lambda + \frac{\Delta}{\Lambda} - \ln \Delta - 1)$$

$\int_0^1 \mathrm{d}x \frac{-x(1-x)p^2 - i\epsilon}{\Lambda} = \frac{p^2}{3\Lambda} - \frac{p^2}{2\Lambda} - \frac{i\epsilon}{\Lambda}, \int_0^1 \mathrm{d}x \ln(-x(1-x)p^2 - i\epsilon) = -2 + \ln(-p^2)$

$$= \frac{-i\lambda^2}{32\pi^2} (\ln \Lambda + \frac{p^2}{3\Lambda} - \frac{p^2}{2\Lambda} + 2 - \ln(-p^2) - 1)$$

if we insert Λ before the variable substitution, we'll have $\Lambda \to \Lambda^2$

$$= \frac{-i\lambda^2}{32\pi^2} (\ln \Lambda^2 + \frac{p^2}{3\Lambda^2} - \frac{p^2}{2\Lambda^2} + 2 - \ln(-p^2) - 1)$$

if $\Lambda \to \Lambda^2 + \Delta$

$$= \frac{-i\lambda^2}{32\pi^2}(2\ln(\Lambda) + \frac{2\sqrt{4\Lambda^2 - p^2}\tan^{-1}\left(\frac{p}{\sqrt{4\Lambda^2-p^2}}\right)}{p} - \frac{p^2}{6\Lambda} - 2\ln(p) - i\pi - 1)$$

So

$$\mathcal{M}_2 = \frac{\lambda^2}{32\pi^2}(\ln\frac{s}{\Lambda^2} + i - 1)$$

$$\mathcal{M}(s) = -\lambda + \frac{\lambda^2}{32\pi^2}\ln\frac{s}{\Lambda^2} - \mathcal{O}(\lambda^3)$$

Now we perform the renormalization

$$\mathcal{M}(s_1) - \mathcal{M}(s_2) = \frac{\lambda^2}{32\pi^2}\ln\frac{s_1}{s_2}$$

$$\lambda_R \equiv -\mathcal{M}(s_0) = \lambda - \frac{\lambda^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2} + \mathcal{O}(\lambda^3)$$

Now assuming λ is large so

$$\lambda = \lambda_R + a\lambda_R^2 + \cdots$$

$$\lambda_R = (\lambda_R + a\lambda_R^2 + \cdots) - \frac{(\lambda_R + a\lambda_R^2 + \cdots)^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2} + \mathcal{O}(\lambda^3)$$

For second order λ_R

$$a = \frac{\ln\frac{s_0}{\Lambda^2}}{32\pi^2}$$

So

$$\lambda = \lambda_R + \frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2} + \mathcal{O}(\lambda_R^2)$$

$$\mathcal{M}(s) = -\lambda + \frac{\lambda^2}{32\pi^2}\ln\frac{s}{\Lambda^2} = -(\lambda_R + \frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2}) + \frac{(\lambda_R + \frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2})^2}{32\pi^2}\ln\frac{s}{\Lambda^2} = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{s} + \cdots$$

to the second order.

(ii) Dimensional regularization.

Use the Wick rotated integration

$$\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{1}{[k_E^2 + \Delta]^2}$$

replace the dimension with d

$$\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{1}{[k_E^2 + \Delta]^2}$$

Integration involving k_E

$$\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{1}{[k_E^2 + \Delta]^2} = \int\frac{\mathrm{d}\Omega_d}{(2\pi)^d}\mathrm{d}k_E\frac{k_E^{d-1}}{[k_E^2 + \Delta]^2}$$

$\int\mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$

$$= \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty\mathrm{d}k_E\frac{k_E^{d-1}}{[k_E^2 + \Delta]^2}$$

$$= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty\mathrm{d}k_E\frac{k_E^{d/2-1}}{[k_E + \Delta]^2}$$

$l = \Delta/(k_E + \Delta)$

$$= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_1^0\mathrm{d}l\frac{-\Delta}{l^2}\frac{l^2}{\Delta^2}(\Delta\frac{1-l}{l})^{d/2-1}$$

$$= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\Delta^{d/2-2}\int_0^1\mathrm{d}l(l)^{1-d/2}(1-l)^{d/2-1}$$

use the definition of beta function $\int_0^1 \mathrm{d}x x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$\begin{aligned}
&= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\Delta^{d/2-2}B(2-d/2,d/2) \\
&= \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\Gamma(2)}\Delta^{d/2-2}
\end{aligned}$$

use the approximation $\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$ where $\epsilon = 4-d$ and γ is the Euler-Mascheroni constant, note that this approximation takes effect near $d=4$

$$\begin{aligned}
&\xrightarrow{d\to 4} \frac{\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)}{(4\pi)^2}(1 - \frac{1}{2}\ln\frac{\Delta}{4\pi}\epsilon + \mathcal{O}(\epsilon^2)) \\
&= \frac{1}{(4\pi)^2}(\frac{2}{\epsilon} - \gamma - \ln\Delta + \ln 4\pi + \mathcal{O}(\epsilon))s
\end{aligned}$$

Now

$$\begin{aligned}
i\mathcal{M}_2(p^2) &= \frac{-i\lambda^2}{2}\int_0^1 \mathrm{d}x \frac{1}{(4\pi)^2}(\frac{2}{\epsilon} - \gamma - \ln\Delta + \ln 4\pi + \mathcal{O}(\epsilon)) \\
&= \frac{-i\lambda^2}{2}\frac{1}{(4\pi)^2}(\frac{2}{\epsilon} - \gamma + 2 - \ln(-p^2) + \ln 4\pi + \mathcal{O}(\epsilon))
\end{aligned}$$

$$i\mathcal{M}(s) = -i\lambda - \frac{i\lambda^2}{2}\frac{1}{(4\pi)^2}(\frac{2}{\epsilon} - \gamma + 2 - \ln(-p^2) + \ln 4\pi + \mathcal{O}(\epsilon))$$

Isolate the divergent term

$$\frac{-i\lambda^2}{(4\pi)^2\epsilon}$$

$$\mathcal{M}(s) = -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s))$$

Follow the same procedure

$$\lambda_R = \lambda + \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0))$$

$$\lambda = \lambda_R + a\lambda_R^2$$

$$\lambda_R = \lambda_R + a\lambda_R^2 + \frac{(\lambda_R + a\lambda_R^2)^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0)) = \lambda_R + a\lambda_R^2 + \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0))$$

$$a = -\frac{1}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0))$$

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0))$$

$$\mathcal{M}(s) = -\lambda - \frac{\lambda^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s)) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s_0)) - \frac{\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} - \ln(-s)) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\ln\frac{s}{s_0}$$

which is exactly the same as cutoff regularization.

The field strength renormalization factor follows, from LSZ formula:

$$\begin{aligned}
&\int \frac{\mathrm{d}^4 p_1}{(2\pi)^4}e^{ip_1\cdot x_1}\frac{\mathrm{d}^4 p_2}{(2\pi)^4}e^{ip_2\cdot x_2}\frac{\mathrm{d}^4 k_1}{(2\pi)^4}e^{-ik_1\cdot y_1}\frac{\mathrm{d}^4 k_2}{(2\pi)^4}e^{-ik_2\cdot y_2}\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}|\Omega\rangle \\
=&Z^2\frac{1}{p_1^2-m^2}\frac{1}{p_2^2-m^2}\frac{1}{k_1^2-m^2}\frac{1}{k_2^2-m^2}\langle p_1 p_2|S|k_1 k_2\rangle
\end{aligned}$$

$$Z^2\mathcal{M} \sim Z\lambda = (1+\delta Z)(\lambda_R + a\lambda_R^2) = \lambda_R + \lambda_R\delta Z + a\lambda_R^2 = \lambda_R$$

$$\delta Z = -a\lambda_R \Longrightarrow \lambda_R\delta Z = -a\lambda_R^2$$