Hydrogen

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QED Lagrangian is

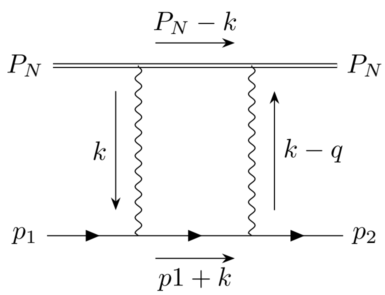
$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \tag{1}$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \tag{2}$$

The box diagram for NRQED process is

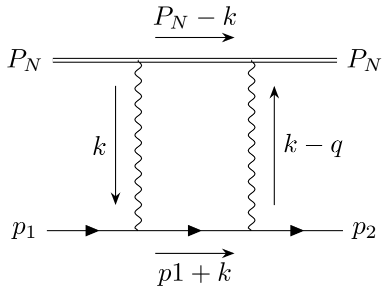
$$i\mathcal{M}_{NRQED} =$$



$$\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{\mathbf{p_1}+\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1) \\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(E_1-\frac{\mathbf{p_1}+\mathbf{k}^2}{2m})}\psi(p_1) \\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2(E_1-\frac{\mathbf{k}^2}{2m})}\psi(p_1)
\end{aligned}$$

The box and crossed box diagram for QED process is

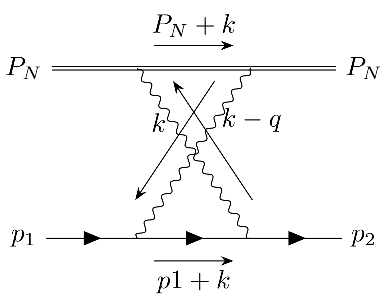
$i\mathcal{M}_1 =$



$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

$i\mathcal{M}_1$ has infrared log divergence and no ultraviolet divergence.

$i\mathcal{M}_2 =$



$$
\begin{aligned}
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)\\
&= e^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int[\mathrm{d}k]\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2-m^2+i\epsilon](k^0+i\epsilon)}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+k_i\gamma^i\gamma^0-\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2+m^2+p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)\\
&= -ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0-\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2+p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)
\end{aligned}
$$

$i\mathcal{M}_2$ has no infrared or ultraviolet divergence.

$$
\begin{aligned}
i\mathcal{M}_1 + i\mathcal{M}_2 &= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2+m^2-{p_1^0}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)\\
&= ie^4 \bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)u_e^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{{p_1^0}^2+k^2+m^2+(k_i-p_{1i})p_1^0\gamma^i\gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2-\mathbf{p_1}^2]\sqrt{\mathbf{k}^2+m^2}}u_e(p_1)
\end{aligned}
$$