Homework: General Relativity #3

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1. Turtle coordinate

$$\tilde{t} = t + 2GM \ln\left|\frac{r}{2GM} - 1\right|, \tilde{r} = r$$

so

$$\begin{aligned}
\mathrm{d}\tilde{t} &= \mathrm{d}t + 2GM\frac{\frac{1}{2GM}}{\frac{r}{2GM} - 1}\mathrm{d}r \\
&= \mathrm{d}t + (\frac{r}{2GM} - 1)^{-1}\mathrm{d}r
\end{aligned}$$

The original Schwarzschild metric is

$$\mathrm{d}s^2 = -(1 - \frac{2GM}{r})\mathrm{d}t^2 + (1 - \frac{2GM}{r})^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

After coordinate transformation (without explicit 'tilde')

$$\begin{aligned}
\mathrm{d}s^2 &= -(1 - \frac{2GM}{r})(\mathrm{d}t - (\frac{r}{2GM} - 1)^{-1}\mathrm{d}r)^2 + (1 - \frac{2GM}{r})^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
&= -(1 - \frac{2GM}{r})(\mathrm{d}t^2 - 2(\frac{r}{2GM} - 1)^{-1}\mathrm{d}t\mathrm{d}r + (\frac{r}{2GM} - 1)^{-2}\mathrm{d}r^2) + (1 - \frac{2GM}{r})^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
&= -(1 - \frac{2GM}{r})\mathrm{d}t^2 + 2(1 - \frac{2GM}{r})(\frac{r}{2GM} - 1)^{-1}\mathrm{d}t\mathrm{d}r - (1 - \frac{2GM}{r})(\frac{r}{2GM} - 1)^{-2}\mathrm{d}r^2 + (1 - \frac{2GM}{r})^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
&= -(1 - \frac{2GM}{r})\mathrm{d}t^2 + \frac{4GM}{r}\mathrm{d}t\mathrm{d}r - \frac{(2GM)^2}{r(r - 2GM)}\mathrm{d}r^2 + \frac{r}{r - 2GM}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
&= -(1 - \frac{2GM}{r})\mathrm{d}t^2 + \frac{4GM}{r}\mathrm{d}t\mathrm{d}r - \frac{(2GM)^2 - r^2}{r(r - 2GM)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
&= -(1 - \frac{2GM}{r})\mathrm{d}t^2 + \frac{4GM}{r}\mathrm{d}t\mathrm{d}r + (1 + \frac{2GM}{r})\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
&= -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{2GM}{r}\mathrm{d}t^2 + \frac{4GM}{r}\mathrm{d}t\mathrm{d}r + \frac{2GM}{r}\mathrm{d}r^2 \\
&= -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + \frac{2GM}{r}(\mathrm{d}t + \mathrm{d}r)^2
\end{aligned}$$

2. The reversed Eddington metric.

1. $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + \frac{2GM}{r}(\mathrm{d}t + \mathrm{d}r)^2$.

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2GM}{r}) & \frac{2GM}{r} & 0 & 0 \\ \frac{2GM}{r} & (1 + \frac{2GM}{r}) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 - \frac{2GM}{r} & \frac{2GM}{r} & 0 & 0 \\ \frac{2GM}{r} & 1 - \frac{2GM}{r} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

2. $\mathrm{d}s^2 = -(1-\frac{2GM}{r})\mathrm{d}\tilde{t}^2 + 2\mathrm{d}\tilde{t}\mathrm{d}r + r^2\mathrm{d}\Omega^2$.

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{2GM}{r}) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1-\frac{2GM}{r} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

3. Under the conformally flat coordinate condition

$$r = \rho(1+\frac{GM}{2\rho})^2$$

and

$$\mathrm{d}r = \mathrm{d}\left(\rho + GM + \frac{(GM)^2}{4\rho}\right) = (1-\frac{(GM)^2}{4\rho^2})\mathrm{d}\rho$$

the Schwarzschild metric becomes

$$\begin{aligned}
\mathrm{d}s^2 &= -(1-\frac{2GM}{\rho(1+\frac{GM}{2\rho})^2})\mathrm{d}t^2 + (1-\frac{2GM}{\rho(1+\frac{GM}{2\rho})^2})^{-1}\,\mathrm{d}\left(\rho(1+\frac{GM}{2\rho})^2\right)^2 + (\rho(1+\frac{GM}{2\rho})^2)^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \\
&= -(\frac{4\rho^2-4GM\rho+G^2M^2}{4\rho^2+4GM\rho+G^2M^2})\mathrm{d}t^2 + (\frac{4\rho^2-4GM\rho+G^2M^2}{4\rho^2+4GM\rho+G^2M^2})^{-1}(1-\frac{(GM)^2}{4\rho^2})^2\mathrm{d}\rho^2 + (1+\frac{GM}{2\rho})^4\rho^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \\
&= -\frac{(1-\frac{GM}{2\rho})^2}{(1+\frac{GM}{2\rho})^2}\mathrm{d}t^2 + \frac{(1+\frac{GM}{2\rho})^2}{(1-\frac{GM}{2\rho})^2}(1-\frac{GM}{2\rho})^2(1+\frac{GM}{2\rho})^2\mathrm{d}\rho^2 + (1+\frac{GM}{2\rho})^4\rho^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \\
&= -\frac{(1-\frac{GM}{2\rho})^2}{(1+\frac{GM}{2\rho})^2}\mathrm{d}t^2 + (1+\frac{GM}{2\rho})^4\mathrm{d}\rho^2 + (1+\frac{GM}{2\rho})^4\rho^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \\
&= -\frac{(1-\frac{GM}{2\rho})^2}{(1+\frac{GM}{2\rho})^2}\mathrm{d}t^2 + (1+\frac{GM}{2\rho})^4[\mathrm{d}\rho^2 + \rho^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)]
\end{aligned}$$

4. From

$$(-1-\frac{2Mr(r^2+a^2)}{\Delta\rho^2})E^2 + \frac{4Mar}{\Delta\rho^2}EL + \frac{\rho^2-2Mr}{\Delta\rho^2\sin^2\theta}L^2 + \frac{\rho^2}{\Delta}(\frac{\mathrm{d}r}{\mathrm{d}\tau})^2 + \rho^2(\frac{\mathrm{d}\theta}{\mathrm{d}\tau})^2 = -1$$

where $\Delta = r^2 - 2Mr + a^2$, $\rho^2 = r^2 + a^2\cos^2\theta$, derive the radial equation if $\theta = \frac{\pi}{2}$.

If $\theta = \frac{\pi}{2}$, $\rho^2 = r^2$

$$(-1-\frac{2Mr(r^2+a^2)}{\Delta r^2})E^2 + \frac{4Mar}{\Delta r^2}EL + \frac{r^2-2Mr}{\Delta r^2}L^2 + \frac{r^2}{\Delta}(\frac{\mathrm{d}r}{\mathrm{d}\tau})^2 = -1$$

$$-2Mr(r^2+a^2)E^2 + 4MarEL + (r^2-2Mr)L^2 + r^4(\frac{\mathrm{d}r}{\mathrm{d}\tau})^2 = \Delta r^2(E^2-1)$$

$$-2Mr(r^2+a^2)E^2 + 4MarEL + (\Delta-a^2)L^2 + r^4(\frac{\mathrm{d}r}{\mathrm{d}\tau})^2 = \Delta r^2(E^2-1)$$

Q.E.D.

5. The action in EM field

$$I = \int(-m\sqrt{-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}} + qA_\mu(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda})\mathrm{d}\lambda$$

$$\delta I = \int\left\{\frac{m}{2}(-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda})^{-\frac{1}{2}}(g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + 2g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}) + qA_{\mu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} + qA_\mu\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda$$

The first term

$$
\begin{aligned}
&\frac{m}{2}\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\left\{g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}+2g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right\}\\
=&\frac{m}{2}\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\left\{g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}+2g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)-2g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2}\right\}\\
=&\frac{m}{2}\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\left\{g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}+2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)-2g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}-2g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2}\right\}\\
=&\frac{m}{2}\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\left\{g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}-2g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}-2g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2}\right\}+m\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&m\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)\\
=&m\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right\}-m\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\right\}g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}
\end{aligned}
$$

Now the total derivative term can be ignored so the first term becomes

$$
\begin{aligned}
&\int\frac{m}{2}(-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda})^{-\frac{1}{2}}(g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}+2g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda})\mathrm{d}\lambda\\
=&\frac{m}{2}\int\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\left\{g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}-2g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\delta x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}-2g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2}\right\}\mathrm{d}\lambda\\
&\qquad-m\int\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\right\}g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda
\end{aligned}
$$

take $\lambda=\tau$

$$
\begin{aligned}
=&\frac{m}{2}\int\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right]^{-\frac{1}{2}}\left\{g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\delta x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right\}\mathrm{d}\tau\\
&\qquad-m\int\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[-g_{\alpha\beta}(x)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right]^{-\frac{1}{2}}\right\}g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\mathrm{d}\tau\\
=&\frac{m}{2}\int\left\{g_{\mu\nu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\delta x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right\}\mathrm{d}\tau\\
=&\frac{m}{2}\int\left\{g_{\rho\nu,\mu}\delta x^\mu\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\delta x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu}\delta x^\mu\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right\}\mathrm{d}\tau\\
=&\frac{m}{2}\int\left\{g_{\rho\nu,\mu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}-2g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}\right\}\delta x^\mu\mathrm{d}\tau
\end{aligned}
$$

The rest terms

$$
\begin{aligned}
&qA_{\mu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}+qA_\mu\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\lambda}\\
=&qA_{\mu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}+q\frac{\mathrm{d}}{\mathrm{d}\lambda}(A_\mu\delta x^\mu)-q\frac{\mathrm{d}A_\mu}{\mathrm{d}\lambda}\delta x^\mu
\end{aligned}
$$

drop total derivative term, and take $\lambda=\tau$

$$
\begin{aligned}
=&qA_{\mu,\rho}\delta x^\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}-q\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}\delta x^\mu\\
=&[qA_{\rho,\mu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}-q\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}]\delta x^\mu
\end{aligned}
$$

So the equation of motion

$$m[g_{\mu\nu,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}+g_{\mu\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}-\frac{1}{2}g_{\rho\nu,\mu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}]-qA_{\rho,\mu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}+q\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}=0$$

6. Prove $\delta(g^{\alpha\beta}g_{\beta\gamma})=0\Longrightarrow\delta(g^{\alpha\beta})=-(\delta g^{\alpha\beta}=:g^{\alpha\mu}(\delta g_{\mu\nu})g^{\nu\beta})$.

$$\begin{aligned}\delta(g^{\alpha\beta})&=\delta(g^{\alpha\mu}g_{\mu\nu}g^{\nu\beta})=(\delta g^{\alpha\mu})g_{\mu\nu}g^{\nu\beta}+g^{\alpha\mu}(\delta g_{\mu\nu})g^{\nu\beta}+g^{\alpha\mu}g_{\mu\nu}\delta g^{\nu\beta}\\&=\delta(g^{\alpha\mu}g_{\mu\nu})g^{\nu\beta}+g^{\alpha\mu}g_{\mu\nu}\delta g^{\nu\beta}\\&=g^{\alpha\mu}g_{\mu\nu}\delta g^{\nu\beta}\\&=(\delta g^{\alpha\mu})g_{\mu\nu}g^{\nu\beta}\\&=-g^{\alpha\mu}(\delta g_{\mu\nu})g^{\nu\beta}=-\delta g^{\alpha\beta}\end{aligned}$$