

# Homework: Quantum Field Theory #8

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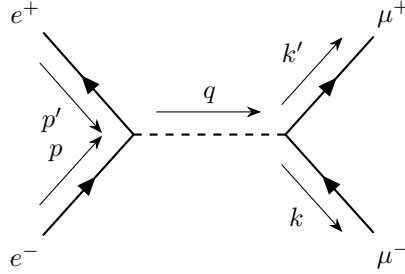
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## 1. Yukawa theory.

(i)  $e^+e^- \rightarrow \mu^+\mu^-$ .  $H_I = \int d^3x [g_1 \bar{\psi}_e \psi_e \phi + g_2 \bar{\psi}_\mu \psi_\mu \phi]$ , derive the  $\mathcal{M}$  of the lowest order, calculate  $\sum_{spins} |\mathcal{M}|^2$ ,  $\frac{d\sigma}{d\Omega}$  and  $\sigma$ .

$$\begin{aligned}
 iT &= \langle kk' | T e^{-i \int d^4x [\bar{\psi}_e \psi_e \phi + g_2 \bar{\psi}_\mu \psi_\mu \phi]} | pp' \rangle \\
 &= \langle kk' | g_2 \int d^4x \bar{\psi}_\mu \psi_\mu \phi g_1 \int d^4y \bar{\psi}_e \psi_e \phi | pp' \rangle \\
 &= g_1 g_2 \int d^4x d^4y \bar{u}_\mu^s e^{ik \cdot x} v_\mu^{s'} e^{ik' \cdot x} \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2} e^{-iq \cdot (x-y)} \bar{v}^r e^{-ip' \cdot y} u^{r'} e^{-ip \cdot y} \\
 &= g_1 g_2 \bar{u}_\mu^s v_\mu^{s'} \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2} \bar{v}^r u^{r'} (2\pi)^4 \delta^4(k + k' - q) (2\pi)^4 \delta^4(q - p - p') \\
 &= \bar{u}^s(k) g_2 v^{s'}(k') \frac{i}{(p + p')^2 - m^2} \bar{v}^r(p') g_1 u^{r'}(p) (2\pi)^4 \delta^4(p + p' - k - k')
 \end{aligned}$$

And the feynman diagram



The scattering amplitude

$$i\mathcal{M} = \bar{u}^s(k) g_2 v^{s'}(k') \frac{i}{q^2 - m^2} \bar{v}^r(p') g_1 u^{r'}(p)$$

where  $q = p + p'$ . And

$$\frac{d\sigma}{d\Omega} |_{CM} = \frac{1}{2E_p 2E_{p'} |v_p - v_{p'}|} \frac{|\mathbf{k}|}{(2\pi)^2 4E_{CM}} |\mathcal{M}|^2$$

The spin sum amplitude

$$\begin{aligned}
 \sum_{spins} |\mathcal{M}|^2 &= \sum_{spins} g_1^2 g_2^2 [\bar{u}^s(k) v^{s'}(k') \frac{1}{q^2 - m^2} \bar{v}^r(p') u^{r'}(p)] [\bar{u}^s(k) v^{s'}(k') \frac{1}{q^2 - m^2} \bar{v}^r(p') u^{r'}(p)]^* \\
 &= \sum_{spins} g_1^2 g_2^2 [\bar{u}^s(k) v^{s'}(k') \frac{1}{q^2 - m^2} \bar{v}^r(p') u^{r'}(p)] [\bar{v}^{s'}(k') u^s(k) \frac{1}{q^2 - m^2} \bar{u}^{r'}(p) v^r(p')] \\
 &= \sum_{spins} \frac{g_1^2 g_2^2}{(q^2 - m^2)^2} \text{tr} \left\{ u^s(k) \bar{u}^s(k) v^{s'}(k') \bar{v}^{s'}(k') \right\} \text{tr} \left\{ v^r(p') \bar{v}^r(p') u^{r'}(p) \bar{u}^{r'}(p) \right\} \\
 &= \frac{g_1^2 g_2^2}{(q^2 - m^2)^2} \text{tr} \left\{ (\not{k} + m_\mu)(\not{k}' - m_\mu) \right\} \text{tr} \left\{ (\not{p}' - m_e)(\not{p} + m_e) \right\} \\
 &= \frac{g_1^2 g_2^2}{(q^2 - m^2)^2} (4k \cdot k' - m_\mu^2)(4p \cdot p' - m_e^2)
 \end{aligned}$$

Now we know that in centre-of-mass frame

$$p^\mu = (E, 0, 0, p), p'^\mu = (E, 0, 0, -p), k^\mu = (E, k \sin \theta, 0, k \cos \theta), k'^\mu = (E, -k \sin \theta, 0, -k \cos \theta)$$

so

$$p \cdot p' = E^2 + p^2, k \cdot k' = E^2 + k^2, p^2 = E^2 - m_e^2, k^2 = E^2 - m_\mu^2, E_{CM} = 2E, q^2 = 4E^2 = s$$

Thus

$$\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = \frac{g_1^2 g_2^2}{4(q^2 - m^2)^2} (8E^2 - 5m_e^2)(8E^2 - 5m_\mu^2)$$

The differential cross section is (the mass of electron is neglected)

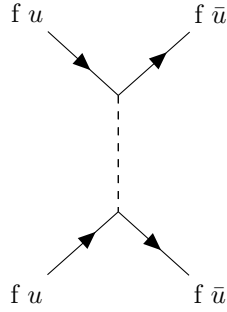
$$\begin{aligned} \frac{d\sigma}{d\Omega}|_{CM} &= \frac{1}{2E_p 2E_{p'} |v_p - v_{p'}|} \frac{|\mathbf{k}|}{(2\pi)^2 4E_{CM}} \frac{1}{4} \sum_{spins} |\mathcal{M}|^2 \\ &= \frac{1}{2E_{CM}^2} \frac{k}{16\pi^2 E_{CM}} \frac{g_1^2 g_2^2}{4(q^2 - m^2)^2} 8E^2 (8E^2 - 5m_\mu^2) \\ &= \frac{\sqrt{E^2 - m_\mu^2}}{16\pi^2 s} \frac{g_1^2 g_2^2}{4(s - m^2)^2} (8E^2 - 5m_\mu^2) \\ &= \frac{\sqrt{s - 4m_\mu^2}}{128\pi^2 s} \frac{g_1^2 g_2^2}{(s - m^2)^2} (2s - 5m_\mu^2) \end{aligned}$$

and total cross section

$$\sigma = \frac{\sqrt{s - 4m_\mu^2}}{32\pi s} \frac{g_1^2 g_2^2}{(s - m^2)^2} (2s - 5m_\mu^2)$$

(ii) NR scattering.

(a)  $ff \rightarrow ff$ .



$$= (-g^2) [\bar{u}(k)u(p) \frac{i}{q^2 - M^2} \bar{u}(k')u(p') - \bar{u}(k')u(p) \frac{i}{q^2 - M^2} \bar{u}(k)u(p')] ]$$

Use Yukawa potential

$$i\mathcal{M} = (-g^2) [\bar{u}(k)u(p) \frac{i}{q^2 - M^2} \bar{u}(k')u(p') - \bar{u}(k')u(p) \frac{i}{q^2 - M^2} \bar{u}(k)u(p')] ]$$

For non-relativistic limit ( $p = (m, \mathbf{p})$  and so on)

$$\bar{u}^s(k)u^r(p) = 2m\delta^{sr}$$

and so on. So

$$i\mathcal{M} = (-g^2) \left[ \frac{i}{(\mathbf{k} - \mathbf{p})^2 - M^2} 2m\delta^{rs} 2m\delta^{r's'} - \frac{i}{(\mathbf{k}' - \mathbf{p})^2 - M^2} 2m\delta^{r's} 2m\delta^{rs'} \right]$$

And in Born approximation

$$\langle p' | iT | p \rangle = -i\tilde{V}(\mathbf{q})(2\pi)\delta^4(E_{\mathbf{p}'} - E_{\mathbf{p}})$$

Comparing with the former one we have for each  $\mathbf{q}$

$$\tilde{V}(\mathbf{q}) = (-g^2) \left[ \frac{i}{\mathbf{q}^2 - M^2} \right]$$

(b)  $f\bar{f} \rightarrow f\bar{f}$ .

$$= (-g^2) \bar{u}(k) v(k') \frac{i}{q^2 - M^2} \bar{v}(p') u(p)$$

Similarly to the discussion we made before, the sign of the potential in Born approximation is entirely dependent on the vertex. So it's again attractive.

## 2. Trace technology.

(i)  $\text{tr}\{\gamma^5\}$

$$\begin{aligned} \text{tr}\{\gamma^5\} &= i \text{tr}\{\gamma^0 \gamma^1 \gamma^2 \gamma^3\} \\ &= 4i(g^{01}g^{23} - g^{02}g^{13} + g^{03}g^{12}) \\ &= 0 \end{aligned}$$

(ii)  $\text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu\}$

$$\begin{aligned} \text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu\} &= \text{tr}\{\gamma^0 \gamma^0 \gamma^5 \gamma^\mu \gamma^\nu\} \\ &= \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^0\} \end{aligned}$$

note that  $\text{tr}\{\gamma^5 \gamma^\nu \gamma^0\} = \text{tr}\{\gamma^1 \gamma^2 \gamma^3 \gamma^\nu\} = 0$

$$\begin{aligned} &= (-1)^3 \text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu \gamma^0 \gamma^0\} \\ &= -\text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu\} \\ &= 0 \end{aligned}$$

(iii)  $\text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\}$

For a proof of identity 6, the same trick still works unless  $(\mu\nu\rho\sigma)$  is some permutation of (0123), so that all 4 gammas appear. The anticommutation rules imply that interchanging two of the indices changes the sign of the trace, so  $\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5)$  must be proportional to  $\epsilon^{\mu\nu\rho\sigma}$  ( $\epsilon^{0123} = \eta^{0\mu} \eta^{1\nu} \eta^{2\rho} \eta^{3\sigma} \epsilon_{\mu\nu\rho\sigma} = \eta^{00} \eta^{11} \eta^{22} \eta^{33} \epsilon_{0123} = -1$ ). The proportionality constant is  $4i$ , as can be checked by plugging in  $(\mu\nu\rho\sigma) = (0123)$ , writing out  $\gamma^5$ , and remembering that the trace of the identity is 4.

$$\begin{aligned} \text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} &= \text{tr}\{\gamma^0 \gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} \\ &= -\text{tr}\{\gamma^0 \gamma^5 \gamma^0 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} \\ &= -2g^{0\mu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} + \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^0 \gamma^\nu \gamma^\rho \gamma^\sigma\} \\ &= -2g^{0\mu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} + 2g^{0\nu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\rho \gamma^\sigma\} - \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^0 \gamma^\rho \gamma^\sigma\} \\ &= -2g^{0\mu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} + 2g^{0\nu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\rho \gamma^\sigma\} - 2g^{0\rho} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma\} + \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^0 \gamma^\sigma\} \\ &= -2g^{0\mu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} + 2g^{0\nu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\rho \gamma^\sigma\} - 2g^{0\rho} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma\} + 2g^{0\sigma} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho\} \\ &\quad - \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^0\} \\ &= -2g^{0\mu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} + 2g^{0\nu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\rho \gamma^\sigma\} - 2g^{0\rho} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma\} + 2g^{0\sigma} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho\} \\ &\quad - \text{tr}\{\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} \\ &= -g^{0\mu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} + g^{0\nu} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\rho \gamma^\sigma\} - g^{0\rho} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma\} + g^{0\sigma} \text{tr}\{\gamma^0 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho\} \end{aligned}$$

For  $\text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\}$

$$\begin{aligned} \text{tr}\{\gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} &= -\text{tr}\{g^1 \gamma^1 \gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} \\ &= -2g^{1\nu} \text{tr}\{\gamma^1 \gamma^0 \gamma^5 \gamma^\rho \gamma^\sigma\} + 2g^{1\rho} \text{tr}\{\gamma^1 \gamma^0 \gamma^5 \gamma^\nu \gamma^\sigma\} - 2g^{1\sigma} \text{tr}\{\gamma^1 \gamma^0 \gamma^5 \gamma^\nu \gamma^\rho\} + \text{tr}\{\gamma^1 \gamma^1 \gamma^0 \gamma^5 \gamma^\nu \gamma^\rho \gamma^\sigma\} \\ &= -g^{1\nu} \text{tr}\{\gamma^1 \gamma^0 \gamma^5 \gamma^\rho \gamma^\sigma\} + g^{1\rho} \text{tr}\{\gamma^1 \gamma^0 \gamma^5 \gamma^\nu \gamma^\sigma\} - g^{1\sigma} \text{tr}\{\gamma^1 \gamma^0 \gamma^5 \gamma^\nu \gamma^\rho\} \end{aligned}$$

For  $\text{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\}$

$$\begin{aligned}\text{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} &= -\text{tr}\{\gamma^2\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} \\ &= \text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^2\gamma^\rho\gamma^\sigma\} \\ &= 2g^{2\rho}\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} - 2g^{2\sigma}\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\} + \text{tr}\{\gamma^2\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} \\ &= g^{2\rho}\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} - g^{2\sigma}\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\}\end{aligned}$$

For  $\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\}$

$$\begin{aligned}\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} &= -\text{tr}\{\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} \\ &= -\text{tr}\{\gamma^3\gamma^2\gamma^1\gamma^0\gamma^5\gamma^3\gamma^\sigma\} \\ &= -g^{3\sigma}\text{tr}\{\gamma^3\gamma^2\gamma^1\gamma^0\gamma^5\} \\ &= -ig^{3\sigma}\end{aligned}$$

Put these back and

$$\begin{aligned}\text{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} &= g^{2\rho}\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\sigma\} - g^{2\sigma}\text{tr}\{\gamma^2\gamma^1\gamma^0\gamma^5\gamma^\rho\} \\ &= -ig^{2\rho}g^{3\sigma} + ig^{2\sigma}g^{3\rho}\end{aligned}$$

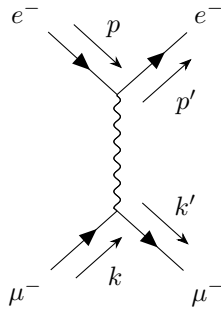
$$\begin{aligned}\text{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} &= -g^{1\nu}\text{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\rho\gamma^\sigma\} + g^{1\rho}\text{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\sigma\} - g^{1\sigma}\text{tr}\{\gamma^1\gamma^0\gamma^5\gamma^\nu\gamma^\rho\} \\ &= -g^{1\nu}(-g^{2\rho}g^{3\sigma} + g^{2\sigma}g^{3\rho}) + g^{1\rho}(-g^{2\nu}g^{3\sigma} + g^{2\sigma}g^{3\nu}) - g^{1\sigma}(-g^{2\nu}g^{3\rho} + g^{2\rho}g^{3\nu}) \\ &= ig^{1\nu}g^{2\rho}g^{3\sigma} - ig^{1\nu}g^{2\sigma}g^{3\rho} - ig^{1\rho}g^{2\nu}g^{3\sigma} + ig^{1\rho}g^{2\sigma}g^{3\nu} + ig^{1\sigma}g^{2\nu}g^{3\rho} - ig^{1\sigma}g^{2\rho}g^{3\nu}\end{aligned}$$

$$\begin{aligned}\text{tr}\{\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} &= -g^{0\mu}\text{tr}\{\gamma^0\gamma^5\gamma^\nu\gamma^\rho\gamma^\sigma\} + g^{0\nu}\text{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\} - g^{0\rho}\text{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\} + g^{0\sigma}\text{tr}\{\gamma^0\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\} \\ &= -ig^{0\mu}(g^{1\nu}g^{2\rho}g^{3\sigma} - g^{1\nu}g^{2\sigma}g^{3\rho} - g^{1\rho}g^{2\nu}g^{3\sigma} + g^{1\rho}g^{2\sigma}g^{3\nu} + g^{1\sigma}g^{2\nu}g^{3\rho} - g^{1\sigma}g^{2\rho}g^{3\nu}) \\ &\quad + ig^{0\nu}(g^{1\mu}g^{2\rho}g^{3\sigma} - g^{1\mu}g^{2\sigma}g^{3\rho} - g^{1\rho}g^{2\mu}g^{3\sigma} + g^{1\rho}g^{2\sigma}g^{3\mu} + g^{1\sigma}g^{2\mu}g^{3\rho} - g^{1\sigma}g^{2\rho}g^{3\mu}) \\ &\quad - ig^{0\rho}(g^{1\mu}g^{2\nu}g^{3\sigma} - g^{1\mu}g^{2\sigma}g^{3\nu} - g^{1\nu}g^{2\mu}g^{3\sigma} + g^{1\nu}g^{2\sigma}g^{3\mu} + g^{1\sigma}g^{2\mu}g^{3\nu} - g^{1\sigma}g^{2\nu}g^{3\mu}) \\ &\quad + ig^{0\sigma}(g^{1\mu}g^{2\nu}g^{3\rho} - g^{1\mu}g^{2\rho}g^{3\nu} - g^{1\nu}g^{2\mu}g^{3\rho} + g^{1\nu}g^{2\rho}g^{3\mu} + g^{1\rho}g^{2\mu}g^{3\nu} - g^{1\rho}g^{2\nu}g^{3\mu}) \\ &= -4i\epsilon^{\mu\nu\rho\sigma}\end{aligned}$$

(Use the identity mentioned before.)

**3.** The electron muon scattering process  $e^-\mu^- \rightarrow e^-\mu^-$ .

We can plot the feynman diagram and give the invariant amplitude



$$i\mathcal{M} = (ie^2)\bar{u}^s(p)\gamma^\mu u^{s'}(p')\frac{g_{\mu\nu}}{q^2}\bar{u}^r(k)\gamma^\nu u^r(k')$$

The spin sum amplitude

$$\begin{aligned}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\text{tr}\{(\not{p} + m_e)\gamma^\mu(\not{p}' + m_e)\gamma^\nu\}\text{tr}\{(\not{k} + m_\mu)\gamma_\mu(\not{k}' + m_\mu)\gamma_\nu\} \\ &= \frac{e^4}{4q^4}[4(p^\mu p'^\nu + p'^\mu p^\nu - p \cdot p' g^{\mu\nu}) + 4m_e^2 g^{\mu\nu}][4(k^\mu k'^\nu + k'^\mu k^\nu - k \cdot k' g_{\mu\nu}) + 4m_\mu^2 g_{\mu\nu}] \\ &= \frac{4e^4}{q^4}[p^\mu p'^\nu + p'^\mu p^\nu - p \cdot p' g^{\mu\nu} + m_e^2 g^{\mu\nu}][k^\mu k'^\nu + k'^\mu k^\nu - k \cdot k' g_{\mu\nu} + m_\mu^2 g_{\mu\nu}]\end{aligned}$$

ignore the electron mass  $m_e$

$$\begin{aligned}
&= \frac{4e^4}{q^4} [p^\mu p'^\nu + p'^\mu p^\nu - p \cdot p' g^{\mu\nu}] [k^\mu k'^\nu + k'^\mu k^\nu - k \cdot k' g_{\mu\nu} + m_\mu^2 g_{\mu\nu}] \\
&= \frac{4e^4}{q^4} [(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) - (p \cdot p')(k \cdot k') + m_\mu^2(p \cdot p') + (p' \cdot k)(p \cdot k') + (p' \cdot k')(p \cdot k) - (p' \cdot p)(k \cdot k') \\
&\quad + m_\mu^2(p' \cdot p) - (p \cdot p')(k \cdot k') - (p \cdot p')(k \cdot k') + 4(p \cdot p')(k \cdot k') - 4m_\mu^2(p \cdot p')] \\
&= \frac{8e^4}{q^4} [(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) - m_\mu^2(p \cdot p')]
\end{aligned}$$

where  $q = p' - p$ .

The differential cross section ( $\theta$  is the angle between the initial muon and final muon, and in centre-of-mass frame)

$$\frac{d\sigma}{d\Omega}|_{CM} = \frac{1}{2E_p 2E_k |v_p - v_k|} \frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}} \frac{1}{4} \sum_{spins} |\mathcal{M}|^2$$

for our problem we set  $|v_p - v_k| = 1$  first and we'll plug it back later,  $E_p = \omega$

$$= \frac{1}{2\omega 2E_k} \frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}} \frac{8e^4}{q^4} [(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) - m_\mu^2(p \cdot p')]$$

and

$$p = (\omega, \omega \hat{z}), k = (E_k, -\omega \hat{z}), p' = (\omega_{p'}, \omega_{\mathbf{p}'}), k' = (E_{k'}, -\omega_{\mathbf{p}'}), \omega_{\mathbf{p}'} = (-\omega_{p'} \sin \theta, 0, -\omega_{p'} \cos \theta)$$

which explicitly is

$$p = (\omega, \omega \hat{z}), k = (E_k, -\omega \hat{z}), p' = (\omega_{p'}, -\omega_{p'} \sin \theta, 0, -\omega_{p'} \cos \theta), k' = (E_{k'}, \omega_{p'} \sin \theta, 0, \omega_{p'} \cos \theta)$$

so

$$\begin{aligned}
p \cdot k &= \omega E_k + \omega^2, p' \cdot k' = \omega_{p'}(\omega + E_k), E_k^2 = \omega^2 + m_\mu^2, p \cdot k' = \omega^2 + \omega E_k - \omega \omega_{p'} - \omega \omega_{p'} \cos \theta, p' \cdot k = \omega_{p'} E_k - \omega \omega_{p'} \cos \theta \\
p \cdot p' &= \omega \omega_{p'}(1 + \cos \theta), q^2 = (p' - p)^2 = 2\omega_{p'} \omega(1 - \cos \theta)
\end{aligned}$$

and gives

$$\begin{aligned}
&= \frac{1}{2\omega 2E_k} \frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}} \frac{8e^4}{q^4} [(\omega E_k + \omega^2)(\omega_{p'}(\omega + E_k)) + (\omega^2 + \omega E_k - \omega \omega_{p'} - \omega \omega_{p'} \cos \theta)(\omega_{p'} E_k - \omega \omega_{p'} \cos \theta) \\
&\quad - m_\mu^2(\omega \omega_{p'}(1 + \cos \theta))] \\
&= \frac{1}{2\omega 2E_k} \frac{|\mathbf{p}'|}{(2\pi)^2 4E_{CM}} \frac{8e^4}{q^4} [\omega E_k \omega_{p'} \omega + \omega E_k \omega_{p'} E_k + \omega^2 \omega_{p'} \omega + \omega^2 \omega_{p'} E_k + \omega^2 \omega_{p'} E_k - \omega^2 \omega \omega_{p'} \cos \theta + \omega E_k \omega_{p'} E_k \\
&\quad - \omega E_k \omega \omega_{p'} \cos \theta - \omega \omega_{p'} \omega_{p'} E_k + \omega \omega_{p'} \omega \omega_{p'} \cos \theta - \omega \omega_{p'} \cos \theta \omega_{p'} E_k + \omega \omega_{p'} \cos \theta \omega \omega_{p'} \cos \theta - m_\mu^2 \omega \omega_{p'}(1 + \cos \theta)] \\
&= \frac{1}{2\omega 2E_k} \frac{\omega_{p'}}{(2\pi)^2 4(\omega + E_k)} \frac{8e^4}{q^4} [3\omega^2 E_k \omega_{p'} + 2\omega E_k^2 \omega_{p'} + \omega^3 \omega_{p'} - \omega^3 \omega_{p'} \cos \theta - \omega^2 E_k \omega_{p'} \cos \theta - \omega \omega_{p'}^2 E_k(1 + \cos \theta) + \omega^2 \omega_{p'}^2 \cos \theta \\
&\quad + \omega^2 \omega_{p'}^2 \cos^2 \theta - m_\mu^2 \omega \omega_{p'}(1 + \cos \theta)]
\end{aligned}$$

assuming no large change in energy and gives  $\omega_{p'} = \omega$ , so

$$\begin{aligned}
&= \frac{1}{2\omega 2E_k} \frac{\omega}{(2\pi)^2 4(\omega + E_k)} \frac{8e^4}{q^4} [2\omega^3 E_k + 2\omega^2 E_k^2 + \omega^4 - 2\omega^3 E_k \cos \theta + \omega^4 \cos^2 \theta - m_\mu^2 \omega^2(1 + \cos \theta)] \\
&= \frac{1}{2\omega 2E_k} \frac{\omega}{(2\pi)^2 4(\omega + E_k)} \frac{2e^4}{\omega^4(1 - \cos \theta)^2} [2\omega^3 E_k + 2\omega^2 E_k^2 + \omega^4 - 2\omega^3 E_k \cos \theta + \omega^4 \cos^2 \theta - m_\mu^2 \omega^2(1 + \cos \theta)] \\
&= \frac{1}{2\omega 2E_k} \frac{\omega}{(2\pi)^2 4(\omega + E_k)} \frac{2e^4}{\omega^2(1 - \cos \theta)^2} [2\omega E_k + 2E_k^2 + \omega^2 - 2\omega E_k \cos \theta + \omega^2 \cos^2 \theta - m_\mu^2(1 + \cos \theta)] \\
&= \frac{1}{2\omega 2E_k} \frac{\omega}{(2\pi)^2 4(\omega + E_k)} \frac{2e^4}{\omega^2(1 - \cos \theta)^2} [(E_k + \omega)^2 + (E_k - \omega \cos \theta)^2 - m_\mu^2(1 + \cos \theta)]
\end{aligned}$$

plug in  $|v_p - v_k| = \left|1 + \frac{\omega}{E_k}\right|$

$$\begin{aligned}
&= \frac{1}{2\omega 2E_k \left|1 + \frac{\omega}{E_k}\right|} \frac{\omega}{(2\pi)^2 4(\omega + E_k)} \frac{2e^4}{\omega^2(1 - \cos\theta)^2} [(E_k + \omega)^2 + (E_k - \omega \cos\theta)^2 - m_\mu^2(1 + \cos\theta)] \\
&= \frac{1}{4|E_k + \omega|} \frac{1}{(2\pi)^2 4(\omega + E_k)} \frac{2e^4}{\omega^2(1 - \cos\theta)^2} [(E_k + \omega)^2 + (E_k - \omega \cos\theta)^2 - m_\mu^2(1 + \cos\theta)] \\
&= \frac{1}{64\pi^2(\omega + E_k)^2} \frac{2e^4}{\omega^2(1 - \cos\theta)^2} [(E_k + \omega)^2 + (E_k - \omega \cos\theta)^2 - m_\mu^2(1 + \cos\theta)] \\
&= \frac{1}{2(\omega + E_k)^2} \frac{\alpha^2}{\omega^2(1 - \cos\theta)^2} [(E_k + \omega)^2 + (E_k - \omega \cos\theta)^2 - m_\mu^2(1 + \cos\theta)]
\end{aligned}$$

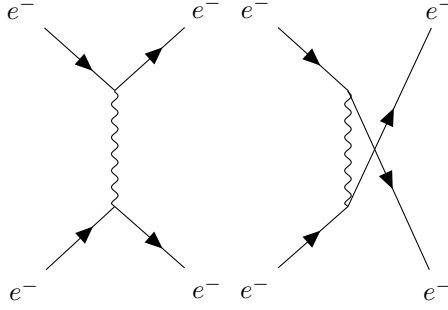
where  $E_k = \sqrt{\omega^2 + m_\mu^2}$ .

In high energy limit  $m_\mu \rightarrow 0$

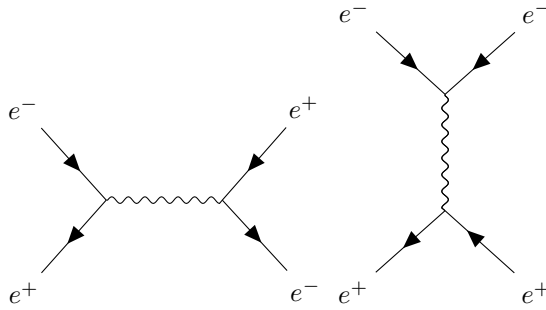
$$\frac{d\sigma}{d\Omega}|_{CM} = \frac{1}{2E_{CM}^2} \frac{\alpha^2}{(1 - \cos\theta)^2} [4 + (1 - \cos\theta)^2]$$

#### 4. Feynman diagrams.

(i)  $e^-e^- \rightarrow e^-e^-$



(ii)  $e^-e^+ \rightarrow e^-e^+$

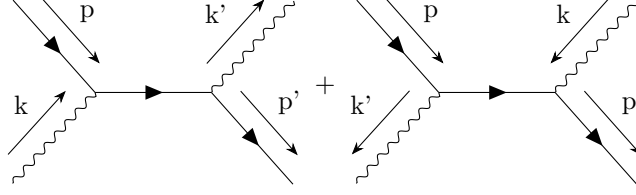


#### 5. Verify Ward identity. (Use condition (i) and (ii) and obtain the result $\mathcal{M} = 0$ which verified Ward identity.)

We already have the invariant amplitude as follows

$$i\mathcal{M} = -ie^2 \epsilon_\mu^*(k') \epsilon_\nu(k) \bar{u}(p') \left[ \frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2p \cdot k'} \right] u(p)$$

and from the feynman diagram



we have

$$p + k = p' + k', k^2 = k'^2 = 0$$

And note that from Dirac equation

$$\not{p}u(p) = mu(p)$$

$$\bar{u}(p)\not{p} = m\bar{u}(p)$$

(i)  $\epsilon_\mu^*(k') \rightarrow k'_\mu$

$$\begin{aligned} i\mathcal{M} &= -ie^2 k'_\mu \epsilon_\nu(k) \bar{u}(p') \left[ \frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2p \cdot k'} \right] u(p) \\ &= -ie^2 \epsilon_\nu(k) \bar{u}(p') \left[ \frac{\not{k}' \not{k} \gamma^\nu + 2\not{k}' p^\nu}{2p \cdot k} + \frac{\gamma^\nu \not{k}' \not{k} - 2\gamma^\nu (p \cdot k')}{2p \cdot k'} \right] u(p) \end{aligned}$$

$$\frac{\gamma^\nu \not{k}' \not{k} - 2\gamma^\nu (p \cdot k')}{2p \cdot k'} = \frac{\gamma^\nu (k' \cdot k') - 2\gamma^\nu (p \cdot k')}{2p \cdot k'} = -\gamma^\nu$$

$$\begin{aligned} \bar{u}(p') \left[ \frac{\not{k}' \not{k} \gamma^\nu + 2\not{k}' p^\nu}{2p \cdot k} \right] u(p) &= \bar{u}(p') \left[ \frac{(\not{p} + \not{k} - \not{p}') \not{k} \gamma^\nu + 2(\not{p} + \not{k} - \not{p}') p^\nu}{2p \cdot k} \right] u(p) \\ &= \bar{u}(p') \left[ \frac{\not{p} \not{k} \gamma^\nu - \not{p}' \not{k} \gamma^\nu + 2(\not{p} + \not{k} - \not{p}') p^\nu}{2p \cdot k} \right] u(p) \\ &= \bar{u}(p') \left[ \frac{2(p \cdot k) \gamma^\nu - 2\not{k} p^\nu + \not{k} \gamma^\nu \not{p} - \not{p}' \not{k} \gamma^\nu + 2(\not{p} + \not{k} - \not{p}') p^\nu}{2p \cdot k} \right] u(p) \\ &= \bar{u}(p') \left[ \frac{2(p \cdot k) \gamma^\nu + \not{k} \gamma^\nu \not{p} - \not{p}' \not{k} \gamma^\nu + 2(\not{p} - \not{p}') p^\nu}{2p \cdot k} \right] u(p) \end{aligned}$$

use Dirac equation

$$\begin{aligned} &= \bar{u}(p') \left[ \frac{2(p \cdot k) \gamma^\nu}{2p \cdot k} \right] u(p) \\ &= \gamma^\nu \end{aligned}$$

So  $i\mathcal{M} = 0$ .

(ii)  $\epsilon_\nu(k) \rightarrow k_\nu$

$$\begin{aligned} i\mathcal{M} &= -ie^2 \epsilon_\mu^*(k') k_\nu \bar{u}(p') \left[ \frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2p \cdot k'} \right] u(p) \\ &= -ie^2 \epsilon_\mu^*(k') \bar{u}(p') \left[ \frac{\gamma^\mu \not{k} \not{k} + 2\gamma^\mu (k \cdot p)}{2p \cdot k} + \frac{\not{k} \not{k}' \gamma^\mu - 2\not{k} p^\mu}{2p \cdot k'} \right] u(p) \end{aligned}$$

$$\frac{\gamma^\mu \not{k} \not{k} + 2\gamma^\mu (k \cdot p)}{2p \cdot k} = \frac{\gamma^\mu (k \cdot k) + 2\gamma^\mu (k \cdot p)}{2p \cdot k} = \gamma^\mu$$

$$\begin{aligned}
\bar{u}(p') \left[ \frac{\not{k} \not{k}' \gamma^\mu - 2 \not{k} p^\mu}{2p \cdot k'} \right] u(p) &= \bar{u}(p') \left[ \frac{(\not{p}' + \not{k}' - \not{p}) \not{k}' \gamma^\mu - 2(\not{p}' + \not{k}' - \not{p}) p^\mu}{2p \cdot k'} \right] u(p) \\
&= \bar{u}(p') \left[ \frac{\not{p}' \not{k}' \gamma^\mu - \not{p} \not{k}' \gamma^\mu - 2(\not{p}' + \not{k}' - \not{p}) p^\mu}{2p \cdot k'} \right] u(p) \\
&= \bar{u}(p') \left[ \frac{\not{p}' \not{k}' \gamma^\mu - 2\gamma^\mu (p \cdot k') + 2 \not{k}' p^\mu - \not{k}' \gamma^\mu \not{p} - 2(\not{p}' + \not{k}' - \not{p}) p^\mu}{2p \cdot k'} \right] u(p) \\
&= \bar{u}(p') \left[ \frac{\not{p}' \not{k}' \gamma^\mu - 2\gamma^\mu (p \cdot k') - \not{k}' \gamma^\mu \not{p} - 2(\not{p}' - \not{p}) p^\mu}{2p \cdot k'} \right] u(p)
\end{aligned}$$

use Dirac equation

$$\begin{aligned}
&= \bar{u}(p') \left[ \frac{-2\gamma^\mu (p \cdot k')}{2p \cdot k'} \right] u(p) \\
&= -\gamma^\mu
\end{aligned}$$

So  $i\mathcal{M} = 0$ .

6. Feynman parametres.

$$\begin{aligned}
\frac{1}{AB} &= -\frac{1}{A-B} \left( \frac{1}{A} - \frac{1}{B} \right) \\
&= \frac{1}{A-B} \int_B^A dx' \frac{1}{x'^2}
\end{aligned}$$

substitute  $x'$  with  $x$  which satisfies  $x' = (A-B)x + B$

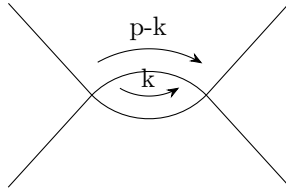
$$\begin{aligned}
&= \int_0^1 dx \frac{1}{[(A-B)x + B]^2} \\
&= \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \\
&= \int_0^1 dx dy \delta(x+y-1) \frac{1}{[xA + yB]^2}
\end{aligned}$$

7.  $\phi^4$  theory.

The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!} \phi^4$$

Calculate



In the second order (all external lines are 1)

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 (p-k)^2}$$

(i) Cutoff.

Apply Feynman parameters

$$\begin{aligned}
i\mathcal{M}_2 &= \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx \frac{1}{[x(p-k)^2 + (1-x)k^2]^2} \\
&= \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx \frac{1}{[xp^2 - 2xp \cdot k + k^2]^2}
\end{aligned}$$



$$k \rightarrow k + xp$$

$$\begin{aligned} &= \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx \frac{1}{[xp^2 - 2xp \cdot (k + xp) + (k + xp)^2]^2} \\ &= \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx \frac{1}{[k^2 + x(1-x)p^2 + i\epsilon]^2} \end{aligned}$$

Now apply Wick rotation (The nature of Wick rotation is to rotate  $k$  into Euclidean space, so the metric of  $k_E$  is of Euclidean space, and  $k_E^2 > 0$ .)

$$k^0 \rightarrow ik_E^0, \mathbf{k} = \mathbf{k}_E, k^2 = -k_E^2$$

$$\begin{aligned} i\mathcal{M}_2 &= \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \int_0^1 dx \frac{1}{[k^2 + x(1-x)p^2 + i\epsilon]^2} \\ &= \frac{i(-i\lambda)^2}{2} \int \frac{d^4 k_E}{(2\pi)^4} \int_0^1 dx \frac{1}{[k_E^2 - (x(1-x)p^2 + i\epsilon)]^2} \end{aligned}$$

$$\Delta \equiv -x(1-x)p^2 - i\epsilon$$

$$\begin{aligned} &= \frac{i(-i\lambda)^2}{2} \int \frac{d^4 k_E}{(2\pi)^4} \int_0^1 dx \frac{1}{[k_E^2 + \Delta]^2} \\ &= \frac{i(-i\lambda)^2}{2} \int_0^1 dx \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\infty dk_E \frac{k_E^3}{[k_E^2 + \Delta]^2} \end{aligned}$$

Variables substitution

$$\begin{aligned} &= \frac{i(-i\lambda)^2}{4} \int_0^1 dx \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\infty dk_E \frac{k_E}{[k_E^2 + \Delta]^2} \\ &= \frac{i(-i\lambda)^2}{4} \int_0^1 dx \int \frac{d\Omega_4}{(2\pi)^4} \int_\Delta^\infty dk_E \frac{k_E - \Delta}{k_E^2} \\ &= \frac{i(-i\lambda)^2}{4} \int_0^1 dx \int \frac{d\Omega_4}{(2\pi)^4} (\ln k_E + \frac{\Delta}{k_E})|_{k_E=\Delta}^\infty \end{aligned}$$

Ultraviolet divergence appears, use cutoff regularization

$$\begin{aligned} &= \frac{i(-i\lambda)^2}{4} \int_0^1 dx \int \frac{d\Omega_4}{(2\pi)^4} (\ln k_E + \frac{\Delta}{k_E})|_{k_E=\Delta}^\Lambda \\ &= \frac{i(-i\lambda)^2}{4} \int_0^1 dx \int \frac{d\Omega_4}{(2\pi)^4} (\ln \Lambda + \frac{\Delta}{\Lambda} - \ln \Delta - \frac{\Delta}{\Delta}) \\ &= \frac{i(-i\lambda)^2}{4} \int_0^1 dx \int \frac{d\Omega_4}{(2\pi)^4} (\ln \Lambda + \frac{\Delta}{\Lambda} - \ln \Delta - 1) \end{aligned}$$

$$d\Omega_4 = \sin^2 \theta \sin \phi d\theta d\phi d\omega, \int \Omega_4 = 2\pi^2$$

$$= \frac{-i\lambda^2}{32\pi^2} \int_0^1 dx (\ln \Lambda + \frac{\Delta}{\Lambda} - \ln \Delta - 1)$$

$$\begin{aligned} \int_0^1 dx \frac{-x(1-x)p^2 - i\epsilon}{\Lambda} &= \frac{p^2}{3\Lambda} - \frac{p^2}{2\Lambda} - \frac{i\epsilon}{\Lambda}, \int_0^1 dx \ln(-x(1-x)p^2 - i\epsilon) = -2 + \ln(-p^2) \\ &= \frac{-i\lambda^2}{32\pi^2} (\ln \Lambda + \frac{p^2}{3\Lambda} - \frac{p^2}{2\Lambda} + 2 - \ln(-p^2) - 1) \end{aligned}$$

if we insert  $\Lambda$  before the variable substitution, we'll have  $\Lambda \rightarrow \Lambda^2$

$$= \frac{-i\lambda^2}{32\pi^2} (\ln \Lambda^2 + \frac{p^2}{3\Lambda^2} - \frac{p^2}{2\Lambda^2} + 2 - \ln(-p^2) - 1)$$

if  $\Lambda \rightarrow \Lambda^2 + \Delta$

$$= \frac{-i\lambda^2}{32\pi^2} (2\ln(\Lambda) + \frac{2\sqrt{4\Lambda^2 - p^2} \tan^{-1}\left(\frac{p}{\sqrt{4\Lambda^2 - p^2}}\right)}{p} - \frac{p^2}{6\Lambda} - 2\ln(p) - i\pi - 1)$$

So

$$\mathcal{M}_2 = \frac{\lambda^2}{32\pi^2} (\ln \frac{s}{\Lambda^2} + i - 1)$$

$$\mathcal{M}(s) = -\lambda + \frac{\lambda^2}{32\pi^2} \ln \frac{s}{\Lambda^2} - \mathcal{O}(\lambda^3)$$

Now we perform the renormalization

$$\mathcal{M}(s_1) - \mathcal{M}(s_2) = \frac{\lambda^2}{32\pi^2} \ln \frac{s_1}{s_2}$$

$$\lambda_R \equiv -\mathcal{M}(s_0) = \lambda - \frac{\lambda^2}{32\pi^2} \ln \frac{s_0}{\Lambda^2} + \mathcal{O}(\lambda^3)$$

Now assuming  $\lambda$  is large so

$$\lambda = \lambda_R + a\lambda_R^2 + \dots$$

$$\lambda_R = (\lambda_R + a\lambda_R^2 + \dots) - \frac{(\lambda_R + a\lambda_R^2 + \dots)^2}{32\pi^2} \ln \frac{s_0}{\Lambda^2} + \mathcal{O}(\lambda^3)$$

For second order  $\lambda_R$

$$a = \frac{\ln \frac{s_0}{\Lambda^2}}{32\pi^2}$$

So

$$\lambda = \lambda_R + \frac{\lambda_R^2}{32\pi^2} \ln \frac{s_0}{\Lambda^2} + \mathcal{O}(\lambda_R^2)$$

$$\mathcal{M}(s) = -\lambda + \frac{\lambda^2}{32\pi^2} \ln \frac{s}{\Lambda^2} = -(\lambda_R + \frac{\lambda_R^2}{32\pi^2} \ln \frac{s_0}{\Lambda^2}) + \frac{(\lambda_R + \frac{\lambda_R^2}{32\pi^2} \ln \frac{s_0}{\Lambda^2})^2}{32\pi^2} \ln \frac{s}{\Lambda^2} = -\lambda_R - \frac{\lambda_R^2}{32\pi^2} \ln \frac{s_0}{s} + \dots$$

to the second order.

(ii) Dimensional regularization.

Use the Wick rotated integration

$$\int \frac{d^4 k_E}{(2\pi)^4} \int_0^1 dx \frac{1}{[k_E^2 + \Delta]^2}$$

replace the dimension with  $d$

$$\int \frac{d^d k_E}{(2\pi)^d} \int_0^1 dx \frac{1}{[k_E^2 + \Delta]^2}$$

Integration involving  $k_E$

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{[k_E^2 + \Delta]^2} = \int \frac{d\Omega_d}{(2\pi)^d} dk_E \frac{k_E^{d-1}}{[k_E^2 + \Delta]^2}$$

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$= \frac{2}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty dk_E \frac{k_E^{d-1}}{[k_E^2 + \Delta]^2}$$

$$= \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty dk_E \frac{k_E^{d/2-1}}{[k_E + \Delta]^2}$$

$$l = \Delta/(k_E + \Delta)$$

$$= \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \int_1^0 dl \frac{-\Delta}{l^2} \frac{l^2}{\Delta^2} (\Delta \frac{1-l}{l})^{d/2-1}$$

$$= \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \Delta^{d/2-2} \int_0^1 dl (l)^{1-d/2} (1-l)^{d/2-1}$$

use the definition of beta function  $\int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$\begin{aligned} &= \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \Delta^{d/2-2} B(2-d/2, d/2) \\ &= \frac{\Gamma(2-d/2)}{(4\pi)^{d/2} \Gamma(2)} \Delta^{d/2-2} \end{aligned}$$

use the approximation  $\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$  where  $\epsilon = 4-d$  and  $\gamma$  is the Euler-Mascheroni constant, note that this approximation takes effect near  $d=4$

$$\begin{aligned} &\xrightarrow{d \rightarrow 4} \frac{\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)}{(4\pi)^2} \left(1 - \frac{1}{2} \ln \frac{\Delta}{4\pi} \epsilon + \mathcal{O}(\epsilon^2)\right) \\ &= \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma - \ln \Delta + \ln 4\pi + \mathcal{O}(\epsilon)\right) s \end{aligned}$$

Now

$$\begin{aligned} i\mathcal{M}_2(p^2) &= \frac{-i\lambda^2}{2} \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma - \ln \Delta + \ln 4\pi + \mathcal{O}(\epsilon)\right) \\ &= \frac{-i\lambda^2}{2} \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + 2 - \ln(-p^2) + \ln 4\pi + \mathcal{O}(\epsilon)\right) \\ i\mathcal{M}(s) &= -i\lambda - \frac{i\lambda^2}{2} \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + 2 - \ln(-p^2) + \ln 4\pi + \mathcal{O}(\epsilon)\right) \end{aligned}$$

Isolate the divergent term

$$\begin{aligned} &\frac{-i\lambda^2}{(4\pi)^2 \epsilon} \\ \mathcal{M}(s) &= -\lambda - \frac{\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s)\right) \end{aligned}$$

Follow the same procedure

$$\begin{aligned} \lambda_R &= \lambda + \frac{\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s_0)\right) \\ \lambda &= \lambda_R + a\lambda_R^2 \\ \lambda_R &= \lambda_R + a\lambda_R^2 + \frac{(\lambda_R + a\lambda_R^2)^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s_0)\right) = \lambda_R + a\lambda_R^2 + \frac{\lambda_R^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s_0)\right) \\ a &= -\frac{1}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s_0)\right) \\ \lambda &= \lambda_R - \frac{\lambda_R^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s_0)\right) \\ \mathcal{M}(s) &= -\lambda - \frac{\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s)\right) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s_0)\right) - \frac{\lambda_R^2}{32\pi^2} \left(\frac{2}{\epsilon} - \ln(-s)\right) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2} \ln \frac{s}{s_0} \end{aligned}$$

which is exactly the same as cutoff regularization.

The field strength renormalization factor follows, from LSZ formula:

$$\begin{aligned} &\int \frac{d^4 p_1}{(2\pi)^4} e^{ip_1 \cdot x_1} \frac{d^4 p_2}{(2\pi)^4} e^{ip_2 \cdot x_2} \frac{d^4 k_1}{(2\pi)^4} e^{-ik_1 \cdot y_1} \frac{d^4 k_2}{(2\pi)^4} e^{-ik_2 \cdot y_2} \langle \Omega | T \{ \phi(x_1) \phi(x_2) \phi(y_1) \phi(y_2) \} | \Omega \rangle \\ &= Z^2 \frac{1}{p_1^2 - m^2} \frac{1}{p_2^2 - m^2} \frac{1}{k_1^2 - m^2} \frac{1}{k_2^2 - m^2} \langle p_1 p_2 | S | k_1 k_2 \rangle \\ Z^2 \mathcal{M} &\sim Z \lambda = (1 + \delta Z) (\lambda_R + a\lambda_R^2) = \lambda_R + \lambda_R \delta Z + a\lambda_R^2 = \lambda_R \\ \delta Z &= -a\lambda_R \implies \lambda_R \delta Z = -a\lambda_R^2 \end{aligned}$$