Chiral basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma^\mu_{\alpha\dot{\beta}} = (1, \vec{\sigma})_{\alpha\dot{\beta}}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (1, -\vec{\sigma})^{\dot{\alpha}\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\beta\alpha} \sigma^\mu_{\alpha\dot{\beta}}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \varepsilon^{12} &= +1 \\ \varepsilon_{12} &= -1 \end{aligned}}$$

☆ Spinor-helicity

Dixon: 1310.5353

Elvang & Huang: 1308.1697

massless fermion

$$\not{p}\, u(p) = 0 = \not{p}\, v(p)$$

can take $u(p) = v(p)$

$$\bar{u}(p)\not{p} = 0 = \bar{v}(p)\not{p}$$

$$\bar{u}(p) = \bar{v}(p)$$

$$\frac{1-\gamma_5}{2} u(p) \equiv \begin{pmatrix} |p]_\alpha \\ 0 \end{pmatrix} \equiv |p]$$

$$\frac{1+\gamma_5}{2} u(p) \equiv \begin{pmatrix} 0 \\ \underset{\sim}{|p\rangle^{\dot\alpha}} \end{pmatrix} \equiv |p\rangle$$

$$\bar{u}(p)\frac{1+\gamma_5}{2} \equiv \begin{pmatrix} 0 & \langle p|_{\dot\alpha} \end{pmatrix} \equiv \langle p|$$

$$\bar{u}(p)\frac{1-\gamma_5}{2} \equiv \begin{pmatrix} [p|^{\alpha} & 0 \end{pmatrix} \equiv [p|$$

$$\not{p} = p^{\mu}\gamma_{\mu} = \begin{pmatrix} 0 & p\cdot\sigma \\ p\cdot\bar\sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \underset{\sim}{p_{\alpha\dot\beta}} \\ p^{\dot\alpha\beta} & 0 \end{pmatrix}$$

$$p_{\mu}\sigma^{\mu}_{\alpha\dot\beta} \equiv p_{\alpha\dot\beta}\;, \quad p^{\mu}\bar\sigma_{\mu}^{\dot\alpha\beta} \equiv \underset{\sim}{p^{\dot\alpha\beta}} = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix}$$

$$\det(p^{\dot\alpha\beta}) = p^2 = 0$$

Dirac eq.:
$$p_{\alpha\dot\beta}|p\rangle^{\dot\beta}=0$$
$$p^{\dot\alpha\beta}|p]_{\beta}=0$$
$$\langle p|_{\dot\alpha}\,p^{\dot\alpha\beta}=0$$
$$[p|^{\alpha}\,p_{\alpha\dot\beta}=0$$

7月16日

Spinor product

$$|p_i\rangle \equiv |i\rangle \quad , \quad |p_i] \equiv |i]$$

$$\langle i\, j\rangle \equiv \langle i|_{\dot\alpha}\, |j\rangle^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\, |i\rangle^{\dot\beta}\, |j\rangle^{\dot\alpha} = -\langle j\, i\rangle$$

$$[i\, j] \equiv [i|^{\alpha}\, |j]_{\alpha} = -[j\, i]$$

$$\langle i|\gamma^{\mu}|j] = \langle i|_{\dot\alpha} (\bar\sigma^{\mu})^{\dot\alpha\beta}\, |j]_{\beta}$$

$$[i|\gamma^{\mu}|j\rangle = [i|^{\alpha}\, \sigma^{\mu}_{\alpha\dot\beta}\, |j\rangle^{\dot\beta}$$

* Momentum

$$\not{p} = \begin{pmatrix} 0 & \overbrace{p_{\alpha\dot\beta}}^{p\cdot\sigma} \\ \underbrace{p^{\dot\alpha\beta}}_{p\cdot\bar\sigma} & 0 \end{pmatrix}$$

$$\sigma^\mu_{\alpha\dot\beta}(\bar\sigma^\nu)^{\dot\beta\alpha} = \mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = 2g^{\mu\nu}$$

$$\Rightarrow p^{\dot\alpha\beta}\sigma^\mu_{\beta\dot\alpha} = \mathrm{Tr}(p\sigma^\mu) \overset{?}{=} 2p^\mu$$

$$p_{\alpha\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha} =$$

* $S_{ij} \equiv 2p_i \cdot p_j = \underline{\sigma^{\mu}_{\alpha\dot\beta}\, p_{i\mu}}\; \underline{\bar\sigma^{\nu\dot\beta\alpha}\, p_{j\nu}} = \underline{(p_i)_{\alpha\dot\beta}}\; \underline{(p_j)^{\dot\beta\alpha}}$

$$= |i]_\alpha \langle i|_{\dot\beta}\, |j\rangle^{\dot\beta} [j|^\alpha = \langle i\, j\rangle [j\, i]$$

* For real momenta $\quad [p|^\alpha = \left(|p\rangle^{\dot\alpha}\right)^* \quad (\alpha = \dot\alpha)$

$$\langle p|_{\dot\alpha} = \left(|p]_\alpha\right)^*$$

$$\Longrightarrow \langle i\, j\rangle = [j\, i]^* = \sqrt{S_{ij}} \times e^{i\phi_{ij}}$$

★ Back to $e^+e^- \to q\bar{q}$

$$iM_4(LRRL) = 2ie^2 Q_e Q_q \mathbb{1}\, A_4(LRRL)$$

$$A_4(LRRL) = \frac{1}{2S_{12}} \langle 2|\gamma^\mu|1] [3|\gamma_\mu|4\rangle$$

$$= \frac{1}{2S_{12}}\, 2\langle 2\, 4\rangle [3\, 1]$$

$$= \frac{\langle 2\, 4\rangle [3\, 1]}{\langle 1\, 2\rangle [2\, 1]}$$

* Other helicity configurations

$$P: \; L \longleftrightarrow R, \quad |\rangle \longleftrightarrow |]$$

$$\Longrightarrow A_4(RLLR) = \frac{\langle 1\,3\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle} = \frac{[2\,4]^2}{[1\,2][3\,4]}$$

C on 1-2 fermion-line : $1 \longleftrightarrow 2$

$$\Longrightarrow A_4(RLRL) = -\frac{\langle 1 4\rangle^2}{\langle 1\,2\rangle\langle 3 4\rangle}$$

C on 3-4 line :

$$\Longrightarrow A_4(LRLR) = -\frac{\langle 2\,3\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

* Squared - amplitude

number of colors (=3)

$$\overline{|\mathcal{M}_4|^2} = e^4 Q_e^2 Q_q^2 N_c \sum_{hel} |A_4|^2$$

$$= 2 e^4 Q_e^2 Q_q^2 N_c \left[\left| \frac{\langle 24 \rangle^2}{\langle 12 \rangle \langle 34 \rangle} \right|^2 + \left| \frac{\langle 14 \rangle^2}{\langle 12 \rangle \langle 34 \rangle} \right|^2 \right]$$

$$= 2 e^4 Q_e^2 Q_q^2 N_c \underbrace{\frac{s_{24}^2 + s_{13}^2}{s_{12}^2}}$$

$$\hookrightarrow \frac{1}{4}(1-\cos\theta)^2 + \frac{1}{4}(1+\cos\theta)^2$$

$$= \frac{1}{2}(1+\cos^2\theta)$$

* Cross section $\qquad s = s_{12}$

$$\sigma = \frac{1}{2s} \int d\Phi_2 \, \overline{|\mathcal{M}_4|^2}$$

$d\Phi_2$ → 2-body phase space

$$d\Phi_2 = \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3\vec{p}_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)$$

$$\frac{d^3\vec{p}_4}{\underbrace{2E_4}_{\sim|\vec{p}_4|}} = d^4p_4 \, \delta(p_4^2)\, \theta(E_4)$$

$$\int d\Phi_2 = \frac{1}{(2\pi)^2}\int \frac{d^3\vec{p}_3}{2E_3}\,\underbrace{\delta\left((p_1+p_2-p_3)^2\right)}_{\delta(s-2\sqrt{s}E_3)}\,\underbrace{\theta(E_1+E_2-E_3)}_{\theta(\sqrt{s}-E_3)}$$

$$2E_3 \to |\vec{p}_3|$$

c.o.m. frame

$$d^3\vec{p}_3 = |\vec{p}_3|^2\, d|\vec{p}_3|\, d\cos\theta\, d\phi$$

$$\alpha = \frac{e^2}{4\pi}$$

$$\frac{1}{16\pi}\int_{-1}^{+1} d\cos\theta$$

$$\sigma = \frac{e^4 Q_q^2 N_c}{32\pi s}\int_{-1}^{1} d\cos\theta\,(1+\cos^2\theta) = \underline{\frac{4\pi\alpha^2}{3s} N_c Q_q^2}$$

* R ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q Q_q^2$$

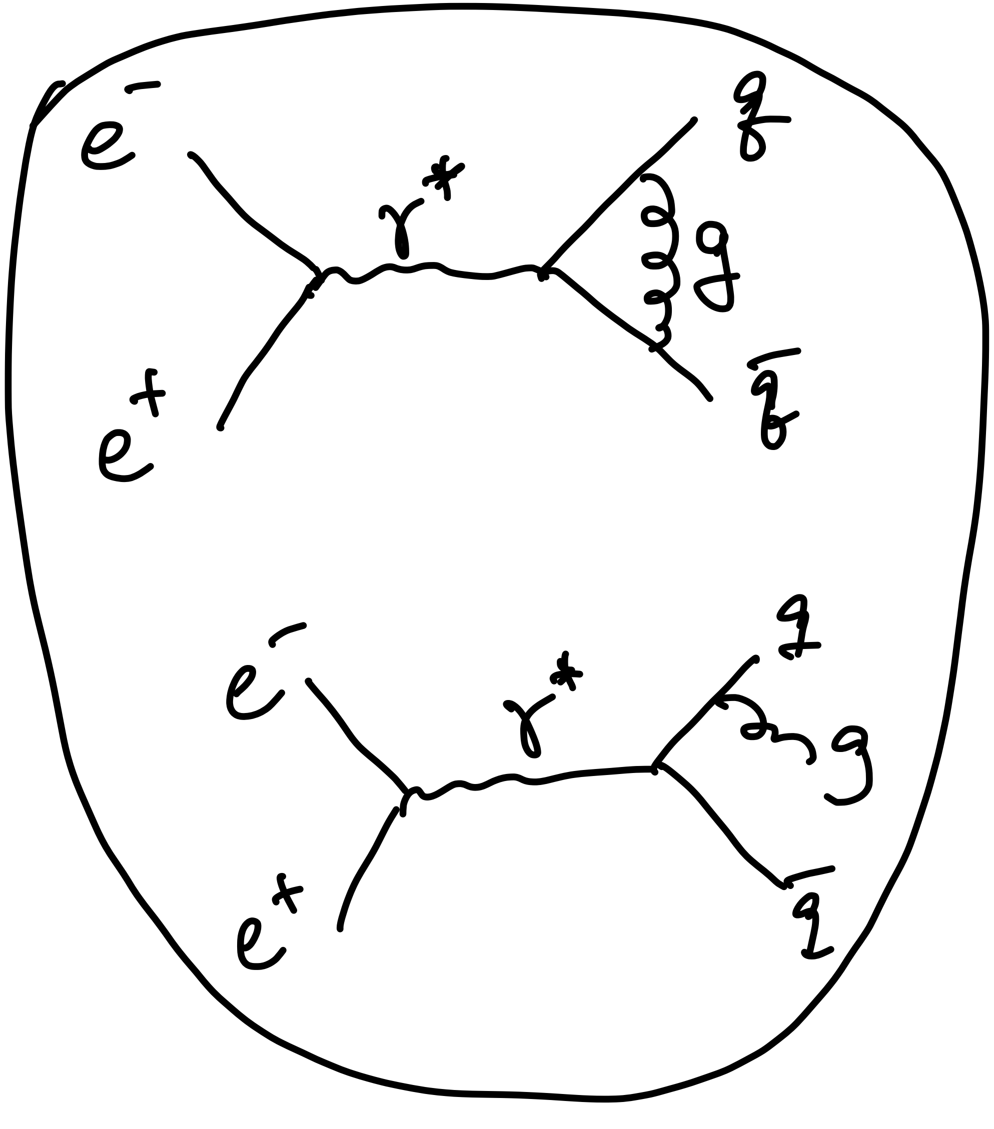
sum over "active" flavors

$\sqrt{s} > 2m_q$

Evidence for 3 colors

☆ Next-to-leading order

$$\sigma = \sigma_0 + \boxed{\alpha_s \bigcirc}$$



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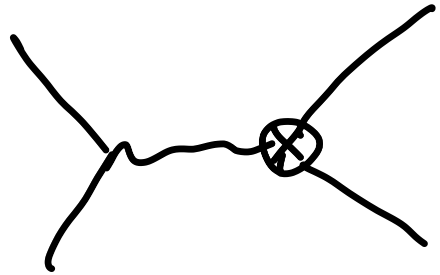
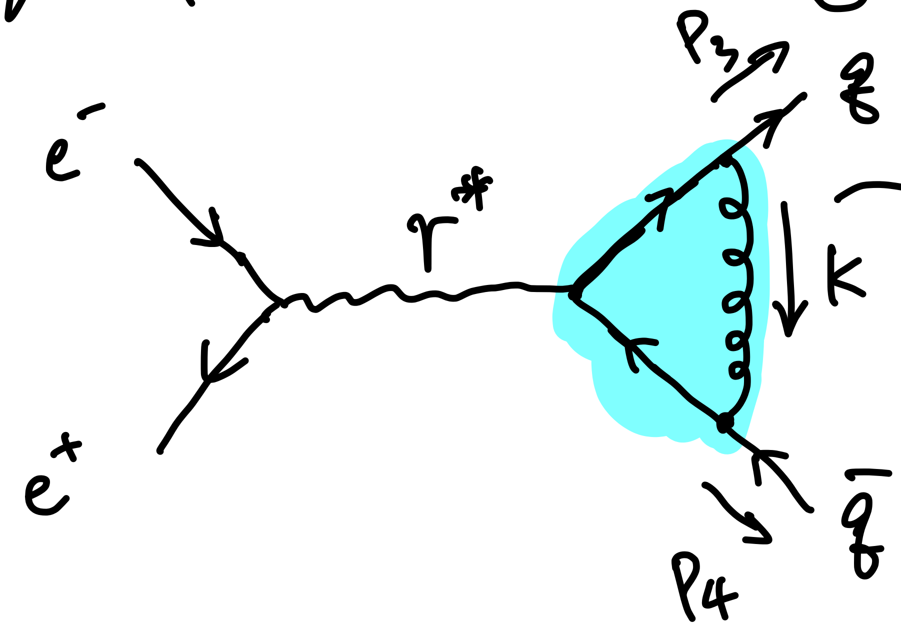


7.18 $e^+e^- \longrightarrow$ hadrons

☆ Next-to-leading order (NLO) $\quad \underline{\alpha_s}$



$$t^a t^a = C_F \mathbb{1}$$

$$i\mathcal{M}_4^V = \frac{i e^2 Q_e Q_q}{s} \bar{v}(p_2)\gamma^\mu u(p_1)\, \bar{u}(p_3)\Gamma_\mu v(p_4)$$

$$\Gamma_\mu = \int \frac{d^4k}{(2\pi)^4}\, i g_s \gamma^\nu t^a \, \frac{i(\cancel{k}+\cancel{p}_3)}{(k+p_3)^2}\, \gamma_\mu \, \frac{i(\cancel{k}-\cancel{p}_4)}{(k-p_4)^2}\, i g_s \gamma_\nu t^a \, \frac{-i}{k^2}$$

$$= -i g_s^2 C_F \mathbb{I} \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu (\cancel{k}+\cancel{p}_3)\gamma_\mu (\cancel{k}-\cancel{p}_4)\gamma_\nu}{k^2 (k+p_3)^2 (k-p_4)^2}}_{\text{Both UV and IR divergent}}$$

* UV div. : $k \to \infty$

$$\sim \int d^4k \, \frac{k^2}{k^6} \sim \int \frac{d|k_E|}{|k_E|} \Rightarrow \text{log. div.}$$

* IR div.

$$I = \int d^4k \frac{1}{k^2 (k+p_3)^2 (k-p_4)^2}$$

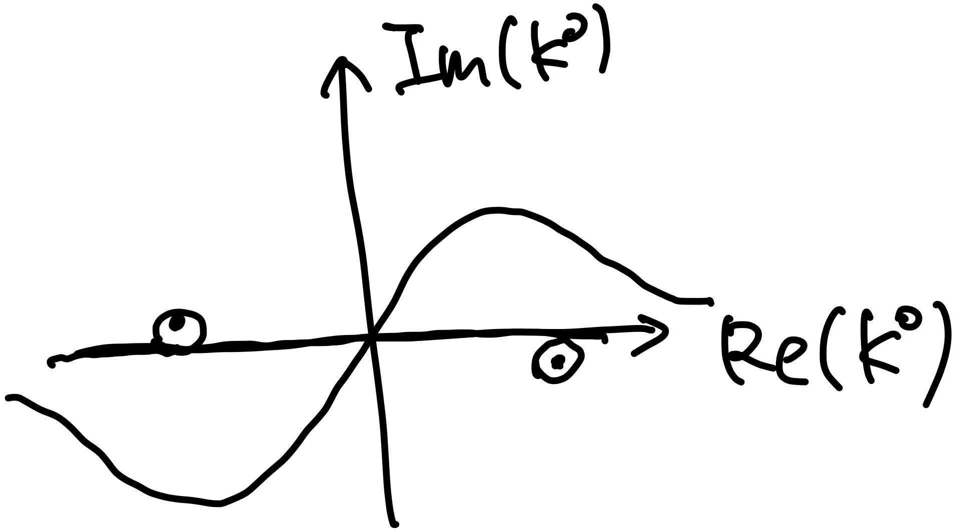
Feynman parametrization

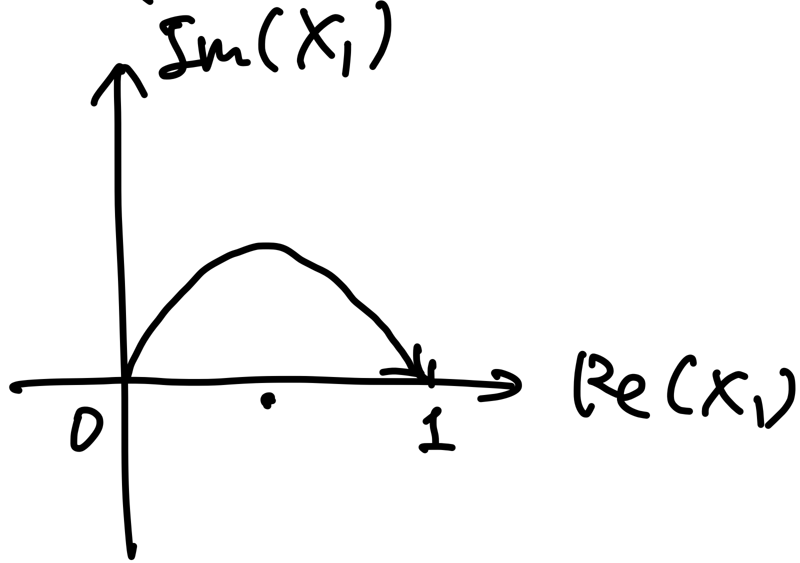
$$I = \int d^4k \int_0^1 dx_1 dx_2 dx_3 \; \delta(x_1 + x_2 + x_3 - 1) \frac{\Gamma(3)}{D^3}$$

$$D = \underline{x_1} k^2 + x_2 \underline{(k+p_3)}^2 + x_3 (k-p_4)^2 + i\delta$$

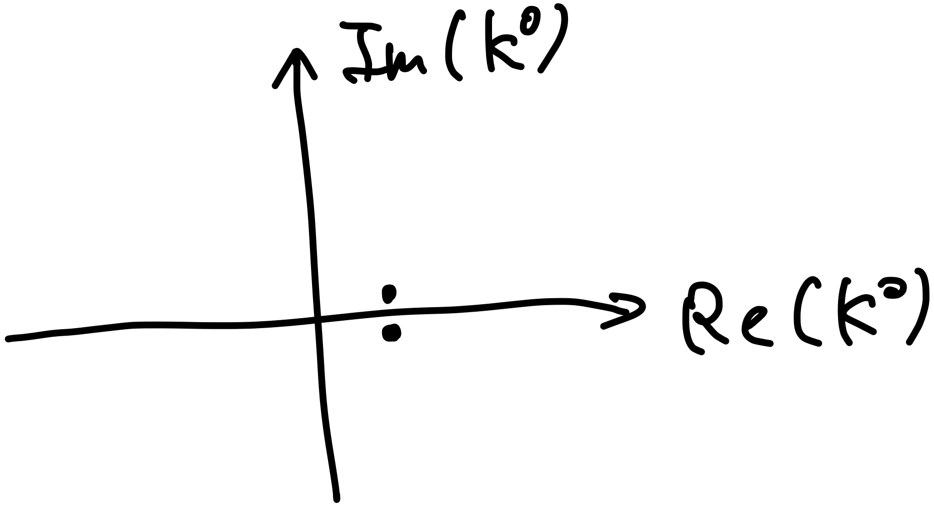
→ Integration over 7 complex variables

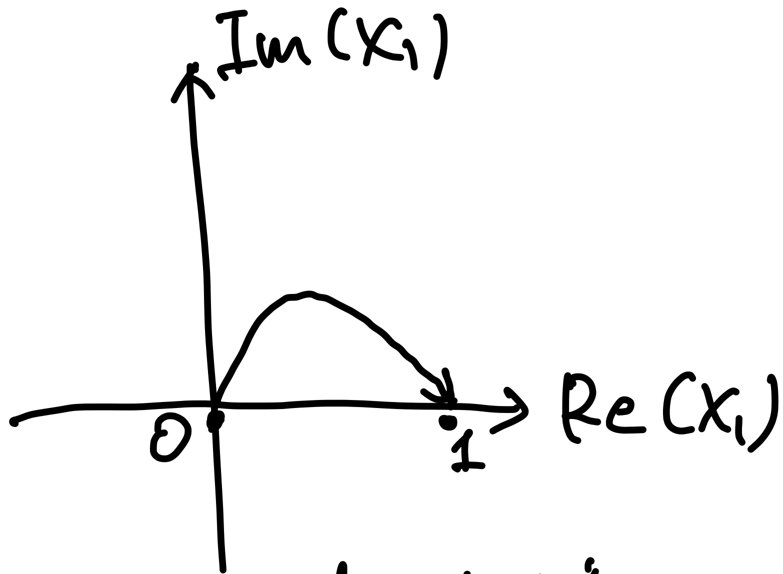
$D = 0 \Longrightarrow$ possible singular points





"Pinched singularities"





"Endpoint"

Pinched :

$$\begin{cases} D = 0 \\ \dfrac{\partial}{\partial k^\mu} D = 0 \end{cases}$$

Landau equations

(double pole)

$D \sim (k^\mu - \Lambda)^2$

Sterman: hep-ph/9606312

$$\begin{cases} D = x_1 k^2 + x_2 (k+p_3)^2 + x_3 (k-p_4)^2 + i\delta = 0 \\ \dfrac{\partial}{\partial k^\mu} D = 0 \implies x_1 k^\mu + x_2 (k+p_3)^\mu + x_3 (k-p_4)^\mu = 0 \end{cases}$$

$p_3^2 = p_4^2 = 0$

Two kinds of solutions

- Soft: $k^\mu = 0$, $\frac{x_2}{x_1} = \frac{x_3}{x_1} = 0$

- Collinear: $k \parallel p_3$: $x_3 = 0$, $k^\mu = -z p_3^\mu$

  $x_1 z = x_2(1-z)$

  $k \parallel p_4$: $x_2 = 0$, $k^\mu = z p_4^\mu$

  $x_1 z = x_3(1-z)$

* Soft limit: $k^\mu \sim \sqrt{s}(\lambda, \lambda, \lambda, \lambda)$, $\lambda \ll 1$

$k^2 \sim s\lambda^2$

$$I \approx \int d^4k \frac{1}{k^2 (2k\cdot p_3)(-2k\cdot p_4)} \sim \int d^4\lambda \frac{1}{\lambda^4}$$ log. div.

* Collinear limit: $k^\mu // p_3^\mu$

$$k\cdot p_3 \sim \lambda^2 s\,, \quad k^2 \sim \lambda^2 s\,, \quad k\cdot p_4 \sim s$$

$$I \approx \int d^4k \, \frac{1}{k^2 (k^2 + 2k\cdot p_3)(-2k\cdot p_4)} \sim \int d^4\lambda \, \frac{1}{\lambda^4}$$

☆ Regularization

$\underbrace{\text{dimensional regularization}}$ (DREG)

↓

amplitude as a function of $d = 4 - 2\epsilon$

dimension

complex variable

* One-loop scalar integrals in DREG

('t Hooft & Veltman)

$$B_0(p^2,0,0) \equiv \frac{\mu^{2\epsilon}}{i\pi^{2-\epsilon} r_\Gamma} \int \underline{d^dk} \, \frac{1}{k^2(k+p)^2} \qquad (p^2 \neq 0)$$

$$r_\Gamma = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \epsilon \gamma_E + O(\epsilon^2)$$

Euler constant

UV divergent for $d \geq 4$

$\boxed{\text{Convergent for } d<4}$ analytic continuation

$$B_0(p^2,0,0) = \frac{\mu^{2\epsilon}}{i\pi^{2-\epsilon} r_\Gamma} \int d^d k \int dx_1 dx_2 \, \delta(x_1+x_2-1)$$

$$\times \frac{\Gamma(2)}{[x_1 k^2 + x_2 (k+p)^2 + i\delta]^2}$$

$$= \frac{\Gamma(\epsilon)}{r_\Gamma} \underbrace{\left(\frac{\mu^2}{-p^2-i\delta}\right)^\epsilon}_{1+\epsilon \ln \frac{\mu^2}{-p^2-i\delta}} \int_0^1 dx_2 \, x_2^{-\epsilon} (1-x_2)^{-\epsilon}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} + \cdots$$

$$= \left(\frac{\mu^2}{-p^2-i\delta}\right)^\epsilon \frac{1}{1-2\epsilon} \frac{1}{\epsilon}$$

$$C_0(0,0,S_{34},0,0,0)$$

$$p_3^2=0$$
$$p_4^2=0$$
$$2p_3\cdot p_4=S_{34}=S$$

$$=\frac{\mu^{2\epsilon}}{i\pi^{2-\epsilon}r_\Gamma}\int d^dk\,\frac{1}{k^2(k+p_3)^2(k-p_4)^2}$$

IR div. for $d\leq 4$

$\boxed{\text{Convergent for } d>4}$

$$\frac{1}{S_{34}}\frac{2}{r_\Gamma\Gamma(2-\epsilon)}\left(\frac{\mu^2}{-S_{34}-i\delta}\right)^{\epsilon}\underline{\int_0^\infty dt\,t^{1-\epsilon}(1+t)^{-3}}$$

$$\times\int dx_1dx_2dx_3\,\delta(x_1+x_2+x_3-1)\,\underline{x_2^{-1-\epsilon}\,x_3^{-1-\epsilon}}$$

Convergent for $\epsilon < 0$

$$= \frac{1}{S_{34}} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-S_{34} - i\delta} \right)^{\epsilon}$$

Finally, a special integral

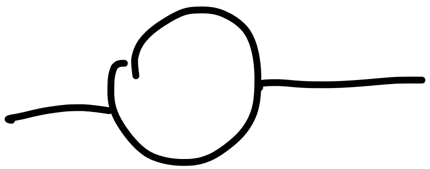
$$B_0(0,0,0) = \frac{\mu^{2\epsilon}}{i\pi^{2-\epsilon} r_\Gamma} \underbrace{\int d^d k \frac{1}{k^2 (k+p)^2}}_{\text{UV AND IR div.}} \quad (p^2 = 0)$$

In DREG: $\quad B_0(0,0,0) = \underbrace{\frac{1}{\epsilon}}_{\text{UV}} - \underbrace{\frac{1}{\epsilon}}_{\text{IR}}$

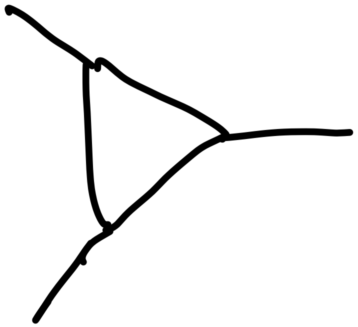
* Tensor integrals (one-loop)

Passarino
Veltman

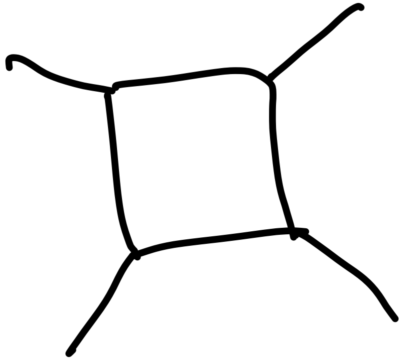
$\Downarrow$



B_0



C_0



D_0

"integral reduction"

- PV tensor reduction
- Generalized unitarity

· Integration-By-Parts (IBP)

$$* \quad \Gamma_\mu = \gamma_\mu \mathbb{1} \times \frac{\alpha_s}{4\pi} C_F (4\pi)^\epsilon r_\Gamma \times \Big[4B_0(0,0,0) - 3B_0(s,0,0) - 2s\, C_0(0,0,s,0,0,0) - 2 \Big]$$

$$= \gamma_\mu \mathbb{1} \times \frac{\alpha_s}{4\pi} C_F \underline{(4\pi)^\epsilon e^{-\epsilon \gamma_E}} \left(\frac{\mu^2}{-s-i\delta} \right)^\epsilon \times \Big[\underbrace{\frac{1}{\epsilon}}_{\text{UV}} \underbrace{- \frac{2}{\epsilon^2} - \frac{4}{\epsilon}}_{\text{IR}} - 8 + \frac{\pi^2}{6} \Big]$$

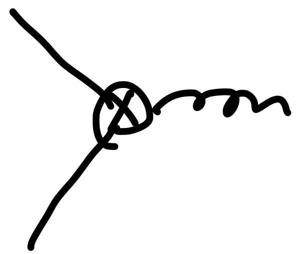
* Renormalization

$$\mathcal{L} = -\frac{1}{4} Z_A F^a_{\mu\nu} F^{a\mu\nu} + Z_\psi \bar{\psi} i \not{D} \psi$$

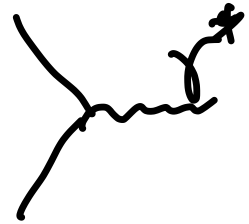
$$D_\mu = \partial_\mu - \underline{i Z_g g_s \mu^{\epsilon} Z_A^{\frac{1}{2}} A^a_\mu t^a}$$

$\downarrow$

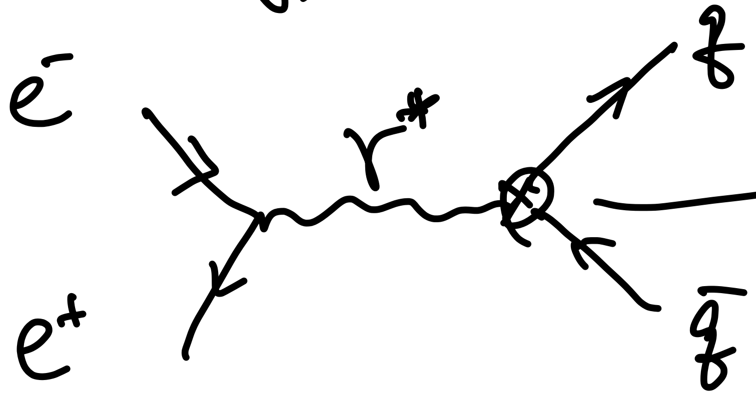
$i g_s \gamma^\mu t^a$



$$\left(Z_g Z_A^{\frac{1}{2}} Z_\psi - 1 \right) i g_s \gamma^\mu t^a$$

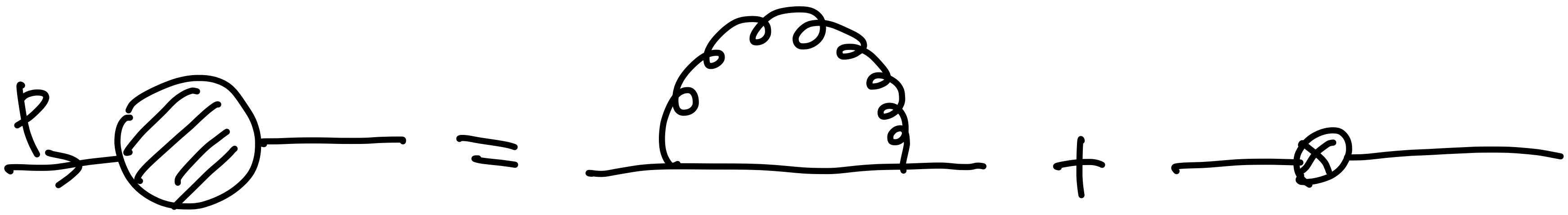


$$\left(Z_e Z_\gamma^{\frac{1}{2}} Z_\psi - 1 \right)$$
$$\times\, i e Q_q \gamma^\mu$$



$\longrightarrow$ $\underline{\left(Z_\psi - 1 \right)}$

Z_ψ : make quark self-energy UV finite



$-i\Sigma(\not p)$ $\qquad\qquad$ $\underline{i\not p(Z_\psi-1)}$

$$i\not p\,\frac{\alpha_s}{4\pi}C_F(4\pi)^\epsilon\, r_\Gamma(1-\epsilon)\,B_0(p^2,0,0)$$

$$= i\not p\,\frac{\alpha_s}{4\pi}C_F(4\pi)^\epsilon e^{-\epsilon\gamma_E}\left(\underbrace{\frac{1}{\epsilon}}_{\text{UV pole}}+\cdots\right)$$

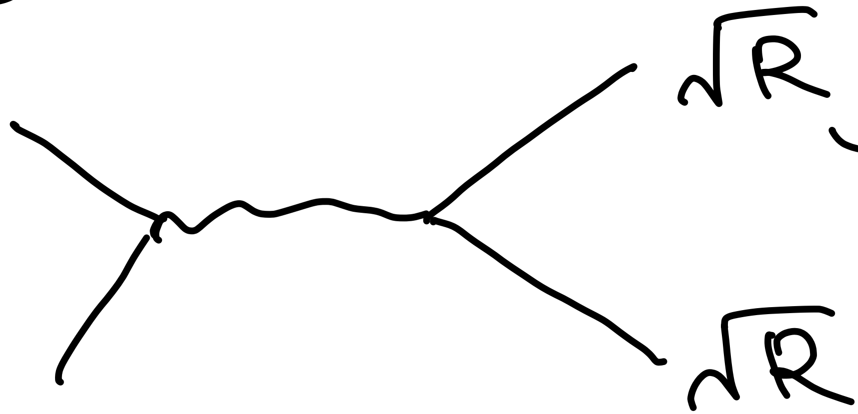
$$\Longrightarrow Z_\psi = 1 - \frac{\alpha_s}{4\pi} C_F (4\pi)^\epsilon e^{-\epsilon\gamma_E} \left(\frac{1}{\epsilon} + \cdots \right)$$

"renormalization scheme"

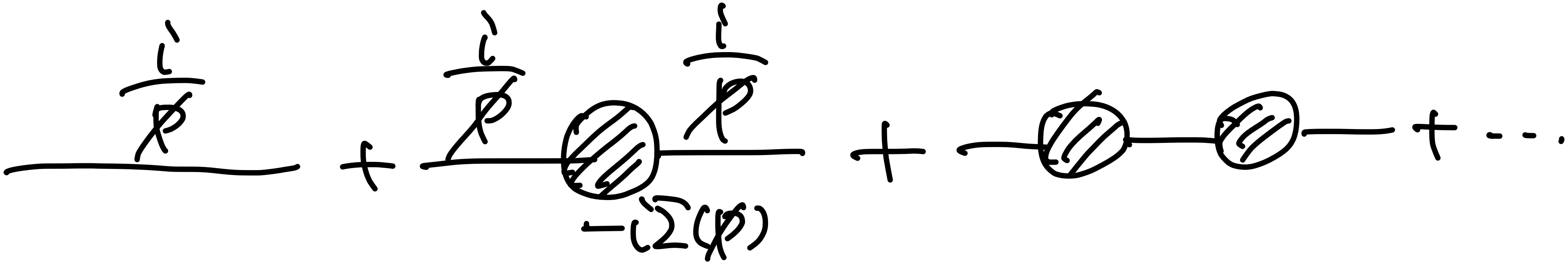
In QCD, $\overline{\text{MS}}$ scheme

$$Z_\psi = 1 - \frac{\alpha_s}{4\pi} C_F (4\pi)^\epsilon e^{-\epsilon\gamma_E} \frac{1}{\epsilon}$$

LSZ reduction



$\rightarrow$ residue of propagators

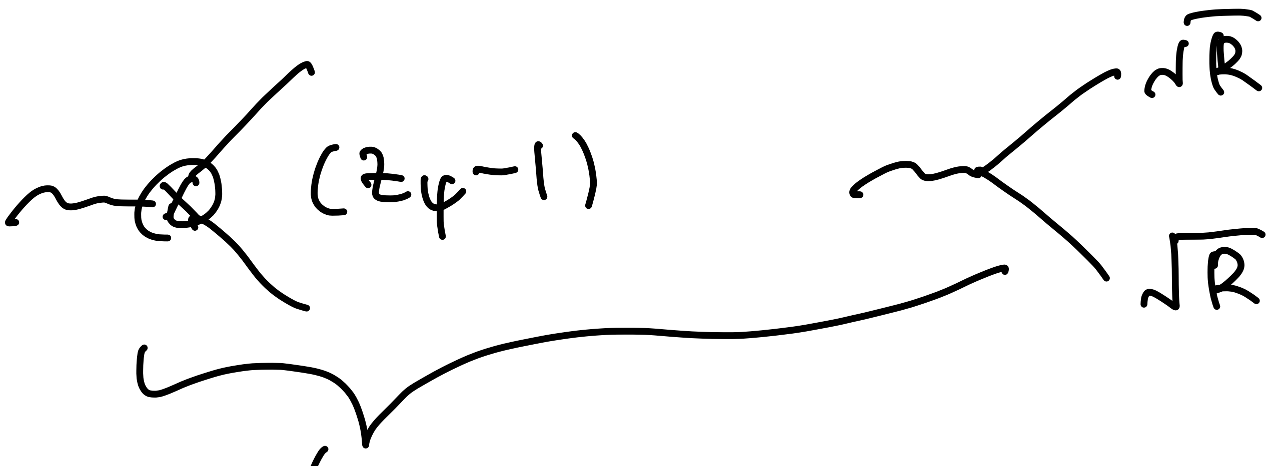


$$= \frac{i}{\not{p}-\Sigma(\not{p})} \underset{\not{p}\to 0}{\sim} \frac{iR}{\not{p}}$$

$$R = 1 - \frac{\alpha_s}{4\pi}(4\pi)^{\epsilon} e^{-\epsilon\gamma_E}\left[\underbrace{\lim_{p\to 0} B_0(p^2,0,0)}_{\underbrace{\frac{1}{\epsilon}}_{UV} - \underbrace{\frac{1}{\epsilon}}_{IR}} - \underbrace{\frac{1}{\epsilon}}_{UV}\right]$$

$$R = 1 + \frac{\alpha_s}{4\pi}(4\pi)^{\epsilon} e^{-\epsilon\gamma_E}\,\frac{1}{\epsilon}$$

IR pole



$$\delta\Gamma_\mu = \gamma_\mu \mathbb{1} \, \frac{\alpha_s}{4\pi} C_F (4\pi)^\epsilon e^{-\epsilon\gamma_E} \left(\underbrace{\frac{1}{\epsilon}}_{\text{IR}} - \underbrace{\frac{1}{\epsilon}}_{\text{UV}} \right)$$

$$\Gamma_\mu = \gamma_\mu \mathbb{1} \, \frac{\alpha_s}{4\pi} C_F (4\pi)^\epsilon e^{-\epsilon\gamma_E} \left(\frac{\mu^2}{-s-i\delta} \right)^\epsilon$$

$$\times \left[\underbrace{-\frac{2}{\epsilon^2} - \frac{3}{\epsilon}}_{\text{IR}} - 8 + \frac{\pi^2}{6} \right]$$