Homework: Gauge Field Theory #3

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1. The standard model Lagrangian without fermion part:

$$\mathcal{L}=-\frac{1}{2}\operatorname{Tr}G^{\mu\nu}G_{\mu\nu}-\frac{1}{2}\operatorname{Tr}W^{\mu\nu}W_{\mu\nu}-\frac{1}{4}B^{\mu\nu}B_{\mu\nu}+|D^{\mu}\phi|^{2}-V(\phi)$$

note that G has 3 colors and W has 3 flavours, the covariant derivatives are

$$D^{\mu}=\partial^{\mu}-igW^{a,\mu}T^{a}+ig_{B}B^{\mu}$$

and the gauge transforms are:

SO(3):

$$G^{a,\mu}\to G^{a,\mu}+\frac{1}{g_G}\partial^{\mu}\beta^{a}-f^{abc}\beta^{b}G^{c,\mu}$$

U(1):

$$B^{\mu}\to B^{\mu}-\frac{1}{g_B}\partial^{\mu}\beta$$
$$\phi\to e^{i\beta(x)}\phi$$

SU(2):

$$W^{a,\mu}\to W^{a,\mu}+\frac{1}{g}\partial^{\mu}\alpha^{a}-f^{abc}\alpha^{b}W^{c,\mu}$$
$$\phi\to e^{i\alpha^{a}(x)T^{a}}\phi$$

Make $\phi=\frac{1}{\sqrt{2}}(v+h(x))$

$$\delta h=i\beta(x)(v+h)(U(1))$$
$$\delta h=i\alpha^{a}(x)T^{a}(v+h)(SU(2))$$

With R_ξ gauge, the gauge fixing term is

$$\mathcal{L}_{GF,gluon}=-\frac{1}{2\xi}(\partial_{\mu}G^{a,\mu})^{2}$$

$$\mathcal{L}_{GF}=-\frac{1}{2\xi}(\partial_{\mu}W^{a,\mu}-\xi gT^{a}_{ij}v_{j}h_{i})^{2}-\frac{1}{2\xi}(\partial_{\mu}B^{a,\mu}-\xi g_{B}v_{i}h_{i})^{2}$$

Then for gluon, the FP determinant is

$$\frac{\delta\partial_{\mu}G^{a,\mu}}{\delta\beta^{b}}=\frac{1}{g_G}\partial_{\mu}\partial^{\mu}\delta^{ab}-f^{abc}\partial_{\mu}G^{c,\mu}$$

so the ghost field part is

$$\mathcal{L}_{FP}=\bar{c}^{a}_{G}(\partial^{2}\delta^{ab}-g_{G}f^{abc}\partial_{\mu}G^{c,\mu})c^{b}_{G}$$

For the electro-weak part, the determinant is

$$\frac{\delta(\partial_{\mu}W^{a,\mu}-\xi gT^{a}_{ij}v_{j}h_{i})}{\delta\alpha^{b}}=\frac{1}{g}\partial^{2}\delta^{ab}-f^{abc}\partial_{\mu}W^{c,\mu}-i\xi gT^{a}_{ij}v_{j}T^{b}_{ik}(v+h)_{k}$$

$$\frac{\delta(\partial_{\mu}B^{a,\mu}-\xi g_{B}v_{i}h_{i})}{\delta\beta}=-\frac{1}{g_B}\partial^{2}-i\xi g_{B}v_{i}(v+h)_{i}$$

The ghost fields are

$$\mathcal{L}_{FP,W}=\bar{c}^{a}_{W}(\partial^{2}\delta^{ab}-gf^{abc}\partial_{\mu}W^{c,\mu}-i\xi gT^{a}_{ij}v_{j}T^{b}_{ik}(v+h)_{k})c^{b}_{W}$$
$$\mathcal{L}_{FP,B}=\bar{c}_{B}(\partial^{2}-i\xi g_{B}^{2}v_{i}(v+h)_{i})c_{B}$$

It should be easy to introduce Weinberg angle. (For real SM Higgs is a doublet, and for $W^{\pm}$ the higgs part should be real scalar fields phi^{+} and ϕ^{-} without breaking and for Z and A the higgs part should be complex scalar field.)

2. QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} + m)\psi$$

We can add gauge fixing term

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

and ignore the fermion part, the generating functional is then

$$Z[J] = \int D[A] e^{i\int \mathrm{d}^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - J^\mu A_\mu)}$$

Note that the kinetic term can be rewrite as

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A^\nu)^2 + \frac{1}{2}(\partial_\mu A^\mu)^2$$

so

$$\begin{aligned} Z[J] &= \int D[A] e^{i\int \mathrm{d}^4x(-\frac{1}{2}(\partial_\mu A^\nu)^2 + \frac{1-\xi^{-1}}{2}(\partial_\mu A^\mu)^2 - J^\mu A_\mu)} \\ &= \int D[A] e^{i\int \mathrm{d}^4x(A_\mu(\frac{1}{2}g^{\mu\nu}\partial^2 + \frac{1-\xi^{-1}}{2}\partial^\mu\partial^\nu)A_\nu - J^\mu A_\mu)} \end{aligned}$$

The propagator is then

$$(g^{\mu\nu}\partial^2 - (1-\xi^{-1})\partial^\mu\partial^\nu)\Delta_{\mu\nu}(x-y) = i\delta^4(x-y)$$

$$\Delta^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}(-i)(\frac{g^{\mu\nu}}{k^2+i\epsilon} - \frac{(1-\xi)k^\mu k^\nu/k^2}{k^2+i\epsilon})e^{ik\cdot(x-y)}$$

3. BRST symmetry.

We have

$$\begin{aligned} \delta_B\psi = -igc^aT^a\psi, \delta_B\bar{\psi} = \bar{\psi}(-igc^aT^a) \\ \delta_B G^{a,\mu} = (D^\mu)^{ab}c^b, \delta_B c^a = \frac{1}{2}gf^{abc}c^bc^c \\ \delta_B\bar{c}^a = B^a(x), \delta_B B^a = 0i, (D^\mu)^{ab} = \partial^\mu\delta^{ab} + gf^{cab}G^{c,\mu} \end{aligned}$$

so ($T^aT^b = if^{abc}T^c + T^bT^a = if^{abc}T^c + \frac{1}{2}\delta^{ab} - T^aT^b = \frac{i}{2}f^{abc}T^c$)

$$\begin{aligned}
&\delta_B(\delta_B\psi) = -ig(\delta_B c^a)T^a\psi + g^2c^aT^ac^bT^b\psi = -\frac{ig^2}{2}f^{abc}c^bc^cT^a\psi + g^2c^aT^ac^bT^b\psi = 0 \\
&\delta_B^2 c^a = \frac{1}{2}gf^{abc}(\frac{1}{2}gf^{bde}c^dc^ec^c - \frac{1}{2}gf^{cde}c^bc^dc^e) = \frac{g^2}{4}(f^{eac}f^{cbd}c^bc^dc^e - f^{abc}f^{cde}c^bc^dc^e) = -\frac{g^2}{4}f^{adc}f^{ceb}c^bc^dc^e = 0 \\
&\delta_B^2\bar{c}^a = 0 \\
&\delta_B^2\bar{\psi} = \bar{\psi}(-igc^aT^a)(-igc^bT^b) - \bar{\psi}(-ig\frac{1}{2}gf^{abc}c^bc^cT^a) = \bar{\psi}\frac{ig^2}{2}f^{abc}c^bc^cT^a - \bar{\psi}g^2c^aT^ac^bT^b = 0 \\
&\delta_B^2 G^{a,\mu} = \delta_B(\partial^\mu c^a + gf^{cab}G^{c,\mu}c^b) = \frac{1}{2}gf^{abc}\partial^\mu(c^bc^c) + gf^{cab}(\partial^\mu c^c + gf^{dce}G^{d,\mu}c^e)c^b + gf^{cab}G^{c,\mu}\frac{1}{2}gf^{bde}c^dc^e = \frac{1}{2}gf^{abc}\partial^\mu(c^bc^c) \\
&- gf^{abc}c^b(\partial^\mu c^c) + g^2f^{cab}f^{dce}G^{d,\mu}c^ec^b - \frac{g^2}{2}f^{cab}f^{bde}G^{c,\mu}c^dc^e = g^2f^{cab}f^{dce}G^{d,\mu}c^ec^b + \frac{g^2}{2}f^{cab}f^{bde}G^{c,\mu}c^dc^e \\
&= \frac{g^2}{2}(f^{aeb}f^{bdc} - f^{adb}f^{bec} + f^{cab}f^{bde})G^{c,\mu}c^dc^e = 0
\end{aligned}$$