

# Scalar QED

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## 1 Hydrogen Wavefunction Divergence in Klein-Gordon Equation and Schrödinger Equation

### 1.1 The Klein-Gordon part

The Klein-Gordon Hydrogen Equation is

$$((i\partial_0 + \frac{Z\alpha}{r})^2 + \nabla^2 - m^2)\Psi = 0 \quad (1)$$

For the bound state, the eigen value and the wave function are

$$E = m \frac{1}{\sqrt{1 + \frac{\alpha^2 Z^2}{(\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2 \alpha^2})^2}}} \quad (2)$$

$$\Psi = \frac{c}{\sqrt{4\pi}} e^{-kr} r^\lambda \quad (3)$$

where

$$\lambda = -\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2 \alpha^2} \quad c = \sqrt{\frac{(2k)^{2(1 + \sqrt{\frac{1}{4} - Z^2 \alpha^2})}}{\Gamma(2 + 2\sqrt{\frac{1}{4} - Z^2 \alpha^2})}} \quad k = \frac{m}{\sqrt{1 + \frac{(\frac{1}{2} + \sqrt{\frac{1}{4} - Z^2 \alpha^2})^2}{\alpha^2 Z^2}}} \quad (4)$$

c is the normalization factor for  $\int d^3r |\Psi|^2 = 1$ . For convenience, define

$$\Psi' = \frac{\Psi}{2(mZ\alpha)^{\frac{3}{2}}} \quad (5)$$

Now  $\Psi'$  is dimensionless and expand it in  $\alpha$ , we get the origin divergence comes from a term

$$-(Z\alpha)^2 \log(mr) \quad (6)$$

the  $m$  in log could be interpreted as a subtraction point  $\mu$ .

## 1.2 The Schrödinger part

The Hamiltonian is

$$H = H_0 + H_{int} \quad (7)$$

$$H_0 = -\frac{\nabla^2}{2m} - \frac{Z\alpha}{r}, \quad H_{int} = \frac{\nabla^4}{8m^3} + \frac{1}{32m^4}[-\nabla^2, [-\nabla^2, -\frac{Z\alpha}{r}]] \quad (8)$$

The first term of  $H_{int}$  is the relativistic kinematic  $v^2$  correction, the second one is the Darwin term. The  $H_0$  gives the radial wave functions as follows

$$R_{n0} = \frac{2(mZ\alpha)^{\frac{3}{2}}}{n^{\frac{3}{2}}} e^{-\frac{mZ\alpha}{n}r} F(1-n, 2, \frac{2mZ\alpha r}{n}), \quad E_n = -\frac{Z^2\alpha^2 m}{2n^2} \quad (9)$$

$$R_{k0} = \sqrt{\frac{2}{\pi}} (mZ\alpha)^{\frac{3}{2}} k e^{\frac{\pi}{2k}} |\Gamma(1 - \frac{i}{k})| e^{-imZ\alpha kr} F(1 + \frac{i}{k}, 2, 2imZ\alpha kr), \quad E_k = \frac{mZ^2\alpha^2 k^2}{2} \quad (10)$$

Within perturbation theory,  $E_1^{(1)} = \langle \phi | H_{int} | \phi \rangle$ , in quantum mechanics, the NLO energy correction is

$$E_1^{(1)} = E_1 Z^2 \alpha^2 \quad (11)$$

The NLO correction of the bound state wave function is

$$\sum_{n \neq 1} a_{n1} \phi_{n00} + \int dk a_{k1} \phi_{k00} \quad (12)$$

with

$$a_{n1} = \frac{\langle \phi_{n00} | H_{int} | \phi_{100} \rangle}{E_1 - E_n} \quad (13)$$

the discrete part of (12) is not divergent at  $r = 0$ . We now focus on the integration part and separate the relativistic kinematic term and the Darwin term. Since we are only interested in the divergent part, here we give a hard cutoff  $\frac{\Lambda}{m}$  as the up-limit of the integration and a also a down-limit  $\lambda$ , with  $\lambda \gg 1$  (note that the following wave function have been multiplied by  $2(mZ\alpha)^{\frac{3}{2}}$ )

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk \frac{2Z^2\alpha^2 k^{\frac{3}{2}}}{2\pi(\sqrt{1 - \exp(-\frac{2\pi}{k})})} (1 - \frac{2}{1+k^2} \exp(-\frac{2 \arctan(k)}{k})) e^{\frac{\pi}{2k}} |\Gamma(1 - \frac{i}{k})| \quad (14)$$

with the integral region we defined ( $k \gg 1$ ), it would be OK to expand the integrand in  $\frac{1}{k}$  (I haven't prove it yet), then the UV divergent term is

$$\Phi^{(1)}(0)_{kin} = \int_{\lambda}^{\frac{\Lambda}{m}} dk (Z\alpha)^2 (\frac{1}{\pi} + \frac{1}{k}) \quad (15)$$

$$\sim (\alpha Z)^2 (\frac{\Lambda}{\pi m} + \log(\frac{\Lambda}{m})) \quad (16)$$

The UV divergent part of Darwin term is

$$\Phi^{(1)}(0)_D = -\frac{(Z\alpha)^4}{8\pi} \int_{\lambda}^{\frac{\Lambda}{m}} dk k^2 e^{\frac{\pi}{k}} |\Gamma(1 - \frac{i}{k})|^2 \quad (17)$$

with the same trick as (15), the UV divergen part is

$$\Phi^{(1)}(0)_D = -(\alpha Z)^4 \int_{\lambda}^{\frac{\Lambda}{m}} dk \frac{k^2}{8\pi} + \frac{k}{8} + \frac{1}{24}\pi \quad (18)$$

$$\sim -\frac{(Z\alpha)^4}{8\pi} \left( \frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m} \right) \quad (19)$$

Now collect all the results we get as follow.

The K-G wave function's origin UV divergence is

$$K - G \text{ UV} : -(Z\alpha)^2 \log(mr) \quad (20)$$

The purterbative Schrödinger wave function's origin UV divergence, with a k cutoff  $\frac{\Lambda}{m}$ , is

$$Kin \text{ UV} : (\alpha Z)^2 \left( \frac{\Lambda}{\pi m} + \log\left(\frac{\Lambda}{m}\right) \right) \quad (21)$$

$$Darwin \text{ UV} : -\frac{(Z\alpha)^4}{8\pi} \left( \frac{\Lambda^3}{3m^3} + \frac{\pi\Lambda^2}{2m^2} + \frac{\pi^2\Lambda}{3m} \right) \quad (22)$$

All the  $m$ , under  $\Lambda$  or in a log, can be interpreted as a subtraction point  $\mu$ .

## 2 Non-relativistic Scalar QED (NRSQED) Matching

### 2.1 Feynman Rules

#### 2.1.1 Scalar QED (SQED)

Lagrangian

$$\mathcal{L}_{SQED} = |D_{\mu}\phi|^2 - m^2|\phi|^2 + \Phi_v^* i v \cdot D \Phi_v \quad (23)$$

with

$$D_{\mu}\phi = \partial_{\mu}\phi + ieA_{\mu}\phi$$

and

$$D_{\mu}\Phi_v = \partial_{\mu}\Phi - iZeA_{\mu}\Phi_v$$

But note that no  $\mathbf{A}$  can appear in actual calculation because here only static scalar potential exists. And the Feynman rules

### 2.1.2 NRSQED

Using the transformation  $\phi \rightarrow \frac{e^{-imt}}{\sqrt{2m}}\varphi$ , we can have the Lagrangian

$$\mathcal{L}_{NRSQED} = \varphi^* \left( iD_0 + \frac{\mathbf{D}^2}{2m} \right) \varphi + \delta\mathcal{L} + \Phi_v^* i v \cdot D \Phi_v \quad (24)$$

with the same notation above. Here  $\mathbf{D} = \nabla - ie\mathbf{A}$ .

Feynman rules are also the same except for the scalar electron side which becomes

We can ignore all interacting terms involving  $\mathbf{A}$ .

Since we need to match it to  $\mathcal{O}(v^2)$  order

$$\delta\mathcal{L} = \frac{(D_0\varphi)^*(D_0\varphi)}{2m} = \frac{\dot{\varphi}^*\dot{\varphi}}{2m} + \frac{e^2\varphi^*\varphi A_0^2}{2m} - \frac{ie}{2m}A_0(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) \quad (25)$$

and it changes the Feynman rules to<sup>1</sup>

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Since we rescaled  $\phi$  by  $\frac{1}{\sqrt{2m}}$  to get  $\varphi$ , the in/out states are also changed. We must multiply them by  $\sqrt{2p^0}$  to compensate that change.

Another way to achieve it is to use the transform rules of heavy scalar effective theory (HSET).

<sup>1</sup>In this note,  $p^0$  is the zero component of relativistic four momentum, and  $E = p^0 - m$ .

### 2.1.3 NRSQED-1

Let's focus on the lagrangian of only a single Scalar field

$$\mathcal{L}_{SQED} = |D_\mu \phi|^2 - m^2 |\phi|^2 \quad (26)$$

Substitute  $\phi$  for  $\phi = \frac{1}{\sqrt{2m}} e^{-imv \cdot x} (\varphi_v + \bar{\varphi}_v)$  with (Schwartz, Sec 35.2)

$$\varphi_v = \frac{1}{\sqrt{2m}} e^{imv \cdot x} (iv \cdot D + m) \phi \quad (27)$$

$$\bar{\varphi}_v = \frac{1}{\sqrt{2m}} e^{imv \cdot x} (-iv \cdot D + m) \phi \quad (28)$$

then

$$\mathcal{L}_{SQED} = \frac{1}{2m} (D_\mu (\varphi_v + \bar{\varphi}_v))^* D^\mu (\varphi_v + \bar{\varphi}_v) + (\varphi_v + \bar{\varphi}_v)^* iv \cdot D (\varphi_v + \bar{\varphi}_v) \quad (29)$$

Within our calculation, we set  $\vec{A} = 0, v = (1, \vec{0})$ . Then from (25), (26) and the motion equation derived from (27), we get two equation

$$\bar{\varphi}_v = \frac{-iD_0}{2m} (\varphi_v + \bar{\varphi}_v) \quad (30)$$

$$\frac{-iD_0}{2m} \varphi_v = \frac{\nabla^2}{4m^2} (\varphi_v + \bar{\varphi}_v) \quad (31)$$

And (27) could be transformed as

$$\mathcal{L}_{SQED} = (\varphi_v + \bar{\varphi}_v)^* (iD_0 + \frac{\nabla^2}{2m} - \frac{D_0^2}{2m}) (\varphi_v + \bar{\varphi}_v) \quad (32)$$

Substituting (28), (29) iteratively, then we get the  $\mathcal{L}$  expanding for  $v$  as

$$\mathcal{L}_{SQED} = \varphi_v^* (iD_0 + \frac{\nabla^2}{2m} + \frac{\nabla^4}{8m^3} + \frac{\nabla^6}{16m^5} + \frac{e}{32m^4} ([\nabla^2, [A_0, \nabla^2]] + [A_0, \nabla^4]) + \mathcal{O}(v^7)) \varphi_v \quad (33)$$

## 2.2 LO Matching

### 2.2.1 SQED

$$i\mathcal{M}_{SQED}^{(0)} = \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \text{---} \text{---} p_2 \end{array} = -e^2 v^0 \frac{i(p_1^0 + p_2^0)}{\mathbf{q}^2} = -e^2 v^0 \frac{i}{\mathbf{q}^2} (2m + 2E_1)$$

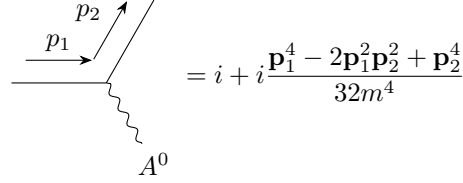
### 2.2.2 NRSQED

$$i\mathcal{M}_{NRSQED}^{(0)} = \begin{array}{c} P_N \text{---} \text{---} P_N \\ \downarrow q \\ p_1 \text{---} \text{---} p_2 \end{array} = -2\sqrt{p_1^0 p_2^0} e^2 v^0 \frac{i(1+\delta)}{\mathbf{q}^2} = -e^2 v^0 \times 2p_1^0 \frac{i(1+\delta)}{\mathbf{q}^2}$$

which gives

$$\delta = \frac{p_1^0 + p_2^0}{2\sqrt{p_1^0 p_2^0} i} - 1 \approx \frac{\mathbf{p}_1^4 - 2\mathbf{p}_1^2 \mathbf{p}_2^2 + \mathbf{p}_2^4}{32m^4}$$

The new electron-photon vertex is



$$= i + i \frac{\mathbf{p}_1^4 - 2\mathbf{p}_1^2 \mathbf{p}_2^2 + \mathbf{p}_2^4}{32m^4}$$

We can write the correction interacting term corresponding to  $\delta$  as

$$\begin{aligned} A_0 \varphi \nabla^4 \varphi^* - 2A_0 \nabla^2 \varphi^* \nabla^2 \varphi + \varphi^* A_0 \nabla^4 \varphi &= \varphi^* \nabla^4 (A_0 \varphi) - 2\varphi^* \nabla^2 (A_0 \nabla^2 \varphi) + \varphi^* A_0 \nabla^4 \varphi \\ &= \varphi^* (\nabla^2 [\nabla^2, A_0] - [\nabla^2, A_0] \nabla^2) \varphi \\ &= \varphi^* [\nabla^2, [\nabla^2, A_0]] \varphi \end{aligned}$$

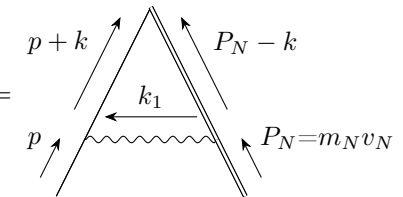
which is exactly the Darwin term in Holstein's *Advanced Topics in QM* with coefficient  $1/32m^4$ .

### 3 Local Operator and Matrix Element of NRSQED

To reproduce the singular behavior of “Klein-Gordon Hydrogen” wavefunction near origin, we can try OPE. But the dependence of  $x$  in OPE can be taken as a regularization scheme and thus the result should be the same as local one without renormalization. And the logarithmic terms of  $x$  in OPE can be reproduced by the logarithmic divergence of local operators. Since in the study of Klein-Gordon equation we know that the wavefunction only contains logarithmic divergence at the origin so that's the only type of divergence we're looking for.

#### 3.1 LO

#### 3.2 NLO



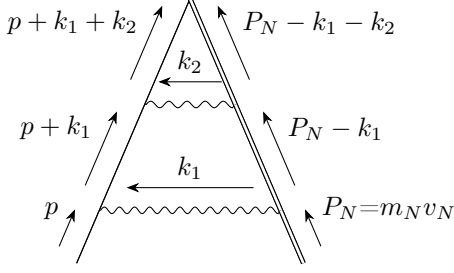
$$\langle 0 | \psi_e(0) N(0) (-ie\mu^{-\epsilon}) \int d^4 y \bar{\psi}_e \psi_e A^0 (-ie\mu^{-\epsilon}) \int d^4 z \bar{N} N A^0 | eN \rangle =$$

which doesn't have logarithm divergence<sup>2</sup>.

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<sup>2</sup>After dimensional regularization, the Gamma function in the numerator is something like  $\Gamma(n - d/2)$  and Gamma function doesn't have pole at half integer. We can rigorously prove this kind of diagrams do not have logarithmic divergence at one loop.

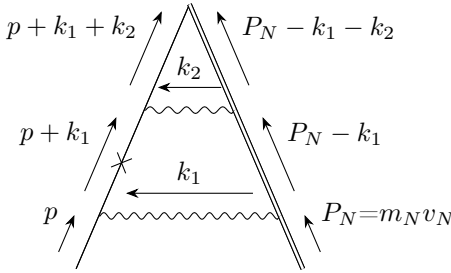
### 3.3 NNLO



$$= -\mu^{2\epsilon} Z^2 e^4 \left[ \int [dk_1][dk_2] \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{p^0 + k_1^0 - m - \frac{(\mathbf{p} + \mathbf{k}_1)^2}{2m} + i\epsilon} \frac{1}{p^0 + k_1^0 + k_2^0 - m - \frac{(\mathbf{p} + \mathbf{k}_1 + \mathbf{k}_2)^2}{2m} + i\epsilon} \right]$$

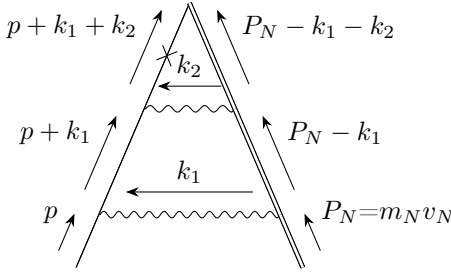
do the shift as above

$$= \mu^{2\epsilon} Z^2 e^4 \left[ \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{E - \frac{|\mathbf{k}_1|^2}{2m} + 2i\epsilon} \frac{1}{E - \frac{|\mathbf{k}_2|^2}{2m} + 2i\epsilon} \right] = 0$$



$$= 4m^2 \mu^{2\epsilon} Z^2 e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{|\mathbf{k}_1|^4 / 4m^2}{[|\mathbf{k}_1|^2 - 2mE]^2} \frac{1}{|\mathbf{k}_2|^2 - 2mE}$$

$$= Z^2 \alpha^2 \left( \frac{1}{2(3-d)} + \log \mu \right) + \text{finite terms}$$



$$= 4m^2 \mu^{2\epsilon} Z^2 e^4 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1 - \mathbf{p}|^2} \frac{1}{|\mathbf{k}_2 - \mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_1|^2 - 2mE} \frac{|\mathbf{k}_2|^4 / 4m^2}{[|\mathbf{k}_2|^2 - 2mE]^2}$$

$$= 0 + \text{finite terms}$$

# Appendices

## A Useful formulas in Feynman integrals

Feynman parametrization used here is

$$\frac{1}{\prod A_i^{d_i}} = \int \prod dx_i \delta(\sum x_i - 1) \frac{\prod x_i^{d_i-1}}{[\sum x_i A_i]^{\sum d_i}} \frac{\Gamma(\sum d_i)}{\prod \Gamma(d_i)} \quad (34)$$

Integral with the structure of the form

$$\int [dk_1][dk_2] \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{1}{[p^0 + k_1^0 - M - \frac{(\mathbf{p}+\mathbf{k}_1)^2}{2M} + i\epsilon]^m} \frac{1}{[p^0 + k_1^0 + k_2^0 - M - \frac{(\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)^2}{2M} + i\epsilon]^n}$$

will always produce

$$\int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \frac{1}{|\mathbf{k}_1|^2} \frac{1}{|\mathbf{k}_2|^2} \frac{1}{[p^0 - M - \frac{(\mathbf{p}+\mathbf{k}_1)^2}{2M}]^m} \frac{1}{[p^0 - M - \frac{(\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)^2}{2M} + i\epsilon]^n}$$

with  $k_1^0$  and  $k_2^0$  goes to zero.

For arbitrary one loop diagram of the following form, we have

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{2\beta}}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{n-\beta}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n - \beta - d/2)}{\Gamma(n)} \left(\frac{4\pi}{\Delta}\right)^{n-\beta-d/2} \quad (35a)$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\beta + d/2)}{\Gamma(d/2)} \frac{\Gamma(n - \beta - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\beta-d/2} \quad (35b)$$

where  $\frac{1}{(4\pi)^{d/2}}$  often appears as  $2^{-d}\pi^{-d/2}$ .

## B $\langle ++ \rangle$

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