# Elements of Group Theory

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#### Abstract

- 1. Generalities
- 2. Lie groups and Lie algebras
- 3. The unitary groups
- 4. Representations of the SU(n) groups (and of their algebras)
- 5. The tensor method for unitary groups, and the permutation group
- 6. Relativistic invariance. The Lorentz group
- 7. General representation of relativistic states

#### Foreword

The following notes are the basis for a graduate course in the Universidad Autónoma de Madrid. They are oriented towards the application of group theory to particle physics, although some of it can be used for general quantum mechanics. They have no pretense of mathematical rigour; but I hope no gross mathematical inaccuracy has got into them.

The notes can be broadly split into three parts: from Sect. 1 to sect 3, they deal with abstract mathematical concepts. Generally speaking, I have not attempted to give proofs of the statements made. These sections I have mostly taken from some lectures I gave at the Menendez Pelayo University, in the summer of 1965. In Sects. 3 through 5, we consider specific groups, particularly the so-called classical groups, which are the ones that have wider application in particle physics. We then describe practical methods to study their representations, which is the way that most applications of groups appear in high energy physics. Finally, the last two sections 6 and 7 deal with properties and representations of the Lorentz group. It is really a shame that so many physicists, who show an astounding familiarity with p-dimensional noncommutative membranes, have only a vague idea of why the photon has two polarization states (although its spin is 1) or how to transform a particle to a moving reference frame.

There are few people with whom I have discussed about the contents of these notes, besides A. Galindo in what respects the first sections, long time ago; but I would like to record here my gratefulness to Maria Herrero, whose enthusiasm decided me to give the lectures, and produce the text (besides providing a useful reference for some of the matters treated in Sects. 3, 4).

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#### §1. Generalities

#### 1.1. Groups and subgroups. Homomorphisms

A set of elements, G, is said to form a group if there exists an associative operation, that we will call multiplication, and an element,  $e \in G$ , called the identity or unity, with the following properties:

- 1. For every  $f, g \in G$  there exists the element h in G such that fg = h;
- 2. For all  $g \in G$ , eg = ge = g.
- 3. For every element  $g \in G$  there exists an element  $g^{-1}$ , also in G, called the *inverse*, such that  $g^{-1}g = gg^{-1} = e$ .

In general,  $fg \neq gf$ . If one has fg = gf for all  $f, g \in G$ , we say that the group is abelian, or commutative. For abelian groups, the operation is at times called sum and denoted by f + g.

A subgroup, H of G, is a subset of G which is itself a group. Given a subgroup H of G we say that it is invariant if, for every  $h \in H$ , and all  $g \in G$ , the element  $ghg^{-1}$  is in H. The element e by itself, and the whole group G, are invariant subgroups; they are called the *trivial* subgroups. If a group has no invariant subgroup other than the trivial ones, then we say that the group is *simple*. If a group has no abelian invariant subgroup (apart from the identity) we say that the group is *semisimple*.

Examples: The *n*-dimensional Euclidean space,  $\mathbb{R}^n = \{\mathbf{v}\}\$ , with

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

the  $v_i$  real numbers, is an abelian group with the vector law of composition: if

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}.$$

The same is true for the complex euclidean space,  $\mathbb{C}^n$ , where the vector components are complex numbers. The set  $\mathbb{R}_+$  of *positive* real numbers is an abelian group with the operation of ordinary multiplication. The set  $\mathcal{T}_n$  of translations in  $\mathbb{R}^n$  is an abelian group.

The set of rotations defined by a three-dimensional vector,  $\boldsymbol{\theta}$ , by angle  $\theta = |\boldsymbol{\theta}|$ , around the (fixed) direction of  $\boldsymbol{\theta}$  in the sense of a corkscrew that advances with  $\boldsymbol{\theta}$  is an abelian group. If we do not fix the direction, then we get the group of three-dimensional rotations, which is *not* abelian.

Let G, G' be groups. Let f be an application of G in G'. We say that it is a homomorphism if it preserves the group operations, i.e., if for all  $a, b \in G$ ,

$$f(a) = a', \ f(b) = b' \text{ implies } f(ab^{-1}) = a'b'^{-1}.$$

If the image of G is all of G', and the inverse application also exists and is a homomorphism, we say that we have an *isomorphism*. If G = G', and the image of G is the whole of G, we say that the homomorphism is an *automorphism*.

The various groups G', G'',... isomorphic to a group G, and the group G itself, may be thought of as realizations of a single abstract group, G.

The set  $K_f \subset G$  of elements such that  $k \in K_f$  implies f(k) = e' (e' is the unit of G') is called the kernel of the homomorphism. If  $K_f = G$ , we say that f is trivial; if  $K_f = \{e\}$  and the image of G is all of G', then f is an isomorphism.

THEOREM.

 $K_f$  is an invariant subgroup of G. Hence, if G is simple, every homomorphism of G is an isomorphism. If an automorphism f of G is induced by the formula

$$f(a) = aaa^{-1}$$
, with  $a \in G$ 

we say that the automorphism is *internal*; if no such a exists, we say that it is external.

EXAMPLE: The application

$$\exp: \xi \in \mathbb{R}^1 \to e^{i\alpha\xi} \quad \alpha \neq 0 \text{ fixed}$$

is a homomorphism; its kernel is  $K_{\text{exp}} = \{\xi : \xi = 2n\pi/\alpha\}, n \text{ an arbitrary integer.}$ 

EXAMPLE: Consider the group SL(n,C) consisting of  $n \times n$  matrices,  $n \ge 2$ , with complex elements, and unit determinant. The transformation  $g \to g^*$ , where the star means the complex conjugate, is an external automorphism.

EXAMPLE: Let us characterize a rotation of angle  $\theta$  around the origin in two (real) dimensions by  $R(\theta)$ . The set of all  $R(\theta)$  forms a group, that we may call SO(2). The application D of SO(2) on  $2 \times 2$  matrices

$$D(R(\theta)) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

is an isomorphism.

Given two groups,  $G_1$ ,  $G_2$ , we define their direct product,  $G = G_1 \times G_2$  as the set of elements  $(g_1, g_2)$  with  $g_i \in G_i$ ,  $g \in G$ , that we will write in the form  $g = g_1g_2$  when there is no danger of confusion, with the product law

$$gh \equiv (g_1g_2)(h_1h_2) = (g_1h_1)(g_2h_2).$$

Let G be a group with I and H subgroups of it, I being invariant. If every element  $g \in G$  may be written as

$$q = hi, \quad h \in H, \ i \in I,$$

then we say that G is the semidirect product of H and I, written as

$$G = H \widetilde{\times} I$$
.

EXAMPLE: Consider the euclidean group in n dimensions,  $\mathcal{E}_n$ , consisting of the rotations (SO(n)) and translations  $\mathcal{T}_n$  in  $\mathbb{R}^n$ . Then,  $\mathcal{E}_n = \mathrm{SO}(n) \times \mathcal{T}_n$ . If R is a general element in SO(n) and  $\mathbf{a}$  one in  $\mathcal{T}_n$ , a general element g in  $\mathcal{E}_n$  can be written as  $g = (\mathbf{a}, R)$ ; it acts on an arbitrary vector  $\mathbf{r}$  in  $\mathbb{R}^n$  by

$$(\mathbf{a}, R): \mathbf{r} \to R\mathbf{r} + \mathbf{a}.$$

The unit element is e = (0, 1) and the product law is

$$(\mathbf{a}, R)(\mathbf{b}, S) = (\mathbf{a} + R\mathbf{b}, RS).$$

EXERCISES: Verify that  $\mathcal{T}_n$  is invariant. Evaluate the inverse of  $(\mathbf{a}, R)$ .

#### 1.2. Representations

A representation D of the group G is a homomorphism

$$D: q \in G \to D(q) \in \mathcal{O}(\mathfrak{H}),$$

where  $\mathcal{O}(\mathfrak{H})$  is the set of linear operators in the Hilbert space  $\mathfrak{H}$ , over the complex numbers. To avoid inessential complications we will assume that, as happens in physical applications, both D,  $D^{-1}$  are bounded operators. We will generally write the scalar product in  $\mathfrak{H}$  as  $\langle \phi | \psi \rangle$  for any pair  $\phi$ ,  $\psi \in \mathfrak{H}$ .

We say that D is finite if the Hilbert space has finite dimension; hence, it is equivalent to the space  $\mathbb{C}^n$  and the D(g) are equivalent to  $n \times n$  complex matrices.

If we have two representations  $D_1$ ,  $D_2$  acting into the same  $\mathcal{O}(\mathfrak{H})$ , and there exists the (bounded) linear operator S in  $\mathcal{O}(\mathfrak{H})$  such that, for all g,

$$D_1(g) = SD_2(g)S^{-1}$$

then we say that  $D_1$  and  $D_2$  are equivalent; indeed, they can be deduced one from the other by the change of basis in  $\mathfrak{H}$  induced by S.

If all the D(g) are unitary,  $D(g)^{\dagger} = D(g)^{-1}$ , we say that D is a unitary representation; if D is an isomorphism, we say that D is faithful; if, for all g, D(g) = 1, we say that D is trivial.

If the (nontrivial<sup>1</sup>) subspace  $\mathfrak{K}$  of  $\mathfrak{H}$  is invariant under all the D(g), then we say that D is partially reducible. If also the complementary<sup>2</sup>  $\mathfrak{H} \ominus \mathfrak{K}$  is invariant, we say that the representation is (fully) reducible.

As an example of a representation which is reducible, but not fully reducible, consider the euclidean group in two dimensions, with rotations  $R(\theta)$  and translations by the vectors  $\mathbf{a} = (a_1, a_2)$ ; we write its elements as  $(\mathbf{a}, R(\theta))$ . The group can be represented by the matrices

$$D(\mathbf{a}, R(\theta)) \to \begin{pmatrix} e^{\mathrm{i}\theta/2} & e^{-\mathrm{i}\theta/2}(a+\mathrm{i}b) \\ 0 & e^{-\mathrm{i}\theta/2} \end{pmatrix}.$$

These leave invariant the subspace of vectors of the form  $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ , but not its orthogonal,  $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$ .

EXERCISE: Prove that a unitary representation that is partially reducible is always fully reducible.

Given two representations,  $D_1$  and  $D_2$ , acting on  $\mathcal{O}(\mathfrak{H}_1)$  and  $\mathcal{O}(\mathfrak{H}_2)$ , we can form two new representations  $D_1 \oplus D_2$  and  $D_1 \otimes D_2$  called, respectively, their direct sum and direct product as follows. First we define the direct sum of Hilbert spaces  $\mathfrak{H}_1$ ,  $\mathfrak{H}_2$ , denoted by  $\mathfrak{H} \equiv \mathfrak{H}_1 \oplus \mathfrak{H}_2$  as the set of pairs

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \text{ with } \phi_i \in \mathfrak{H}_i,$$

with the natural definitions of linear combinations and scalar products; e.g.,  $\langle \phi | \psi \rangle = \langle \phi_1 | \psi_1 \rangle + \langle \phi_2 | \psi_2 \rangle$ . We then define  $D \equiv D_1 \oplus D_2$ , acting on  $\mathfrak{H}$ , by

$$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}.$$

Clearly, D is reducible; its invariant subspaces  $\mathfrak{K}_i$  are formed by vectors of the form

$$\mathfrak{K}_1 = \left\{ \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{K}_2 = \left\{ \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right\}.$$

As for the direct product, we start by defining the direct product of two Hilbert spaces,  $\mathcal{O}(\mathfrak{H}_1)$  and  $\mathcal{O}(\mathfrak{H}_2)$ , assumed to be separable. Hence, they have numerable orthonormal bases, that we denote by  $\{\epsilon_n^{(1)}\}, \{\epsilon_n^{(2)}\}$  respectively. We now form a new Hilbert space,  $\mathfrak{H} \equiv \mathfrak{H}_1 \otimes \mathfrak{H}_2$ , as that generated by the basis  $(\{\epsilon_i^{(1)}, \epsilon_j^{(2)}\})$ , that we will simply write  $(\{\epsilon_i^{(1)}, \epsilon_j^{(2)}\}) \rightarrow \{\epsilon_i^{(1)} \epsilon_j^{(2)}\}$ . Its vectors are thus of the form

$$\phi = \sum_{ij} \alpha_{ij} \epsilon_i^{(1)} \epsilon_j^{(2)}$$

 $<sup>^1</sup>$  The trivial subspaces are  $\mathfrak H$  itself, and that subspace formed by just the zero vector.

<sup>&</sup>lt;sup>2</sup> The complementary,  $\mathfrak{H} \ominus \mathfrak{K}$ , is defined as the set of vectors orthogonal to  $\mathfrak{K}$ .

and the operations of linear combination and scalar product are defined in the natural manner; for e.g. the second, if we have

$$\phi = \sum_{ij} \alpha_{ij} \epsilon_i^{(1)} \epsilon_j^{(2)}, \quad \psi = \sum_{ij} \beta_{ij} \epsilon_i^{(1)} \epsilon_j^{(2)}$$

then

$$\langle \phi | \psi \rangle \equiv \sum_{ij} \alpha_{ij}^* \beta_{ij}.$$

The direct product  $D \equiv D_1 \otimes D_2$  is then defined as follows: if  $\phi = \sum_{ij} \alpha_{ij} \epsilon_i^{(1)} \epsilon_j^{(2)}$ ; and if  $D_1 \epsilon_i^{(1)} = \sum_{i'} d_{ii'}^{(1)} \epsilon_{i'}^{(1)}$ ,  $D_2 \epsilon_j^{(2)} = \sum_{j'} d_{jj'}^{(2)} \epsilon_{j'}^{(2)}$ , then

$$D\phi = \sum_{ij} \sum_{i'j'} \alpha_{ij} d_{ii'}^{(1)} d_{jj'}^{(2)} \epsilon_{i'}^{(1)} \epsilon_{j'}^{(2)}.$$

EXERCISES: Check that direct sum and product are commutative. Check that, for the finite dimensional case, direct sum and product agree with the ordinary direct sum and product of matrices. Check that the dimension of the direct sum is the sum of the dimensions, and the dimension of the direct product is the product of the dimensions.

In the finite dimensional case, with dimensions  $\mu$ ,  $\nu$ , if  $D_1(g) = (a_{nm})$  and  $D_2(g) = (b_{nm})$ , then  $D \equiv D_1 \otimes D_2$  is the matrix

$$D \equiv \begin{pmatrix} a_{11} \begin{pmatrix} b_{11} & \cdots & b_{1\nu} \\ & \cdots & & \\ b_{\nu 1} & \cdots & b_{\nu \nu} \end{pmatrix} & \cdots & & \\ & & \cdots & & \\ & & & a_{\mu \mu} \begin{pmatrix} b_{11} & \cdots & b_{1\nu} \\ & \cdots & & \\ b_{\nu 1} & \cdots & b_{\nu \nu} \end{pmatrix} \end{pmatrix}.$$

A representation that cannot be split in the sum of two or more representations is called *irreducible*. A useful criterion for reducibility is the following:

Lemma (Schur).

If an operator F commutes with all the representatives of a group representation,

$$[F, D(g)] = 0,$$

then either the representation is reducible, or F is a multiple of the identity operator.

A second related lemma, also due to Schur, is the following:

LEMMA.

If the representations D, D' are irreducible; and if the operator A verifies AD(g) = D'(g)A, for all g (if the dimensions of D, D' are different, A would be a square matrix) then either D, D' are equivalent, or A = 0.

#### 1.3. Finite groups. The permutation group. Cayley's theorem

If the number of elements in a group is finite, it is said to be a finite group. Important finite groups (that, however, we will not study here; see e.g. Lyubarskii, 1960; Hamermesh, 1963) are the crystallographic groups. Another important group is the group  $\Pi_n$  of permutations of n elements, called the *permutation* or *symmetric* group. It is defined as follows. Let the n elements be labeled  $v_i$ , i = 1, ... n. Let us consider two arrays of these elements,

$$v_{i_1}, \ldots v_{i_n}; \quad v_{j_1}, \ldots v_{j_n}.$$

A permutation P is the application of the first array over the second; we will denote it by

$$P \equiv P(\{v_{i_1}, \dots v_{i_n}\} \to \{v_{i_1}, \dots v_{i_n}\}).$$

We will denote permutations by the letters P, Q, R... We have the product law

$$P(\{v_{i_1}, \dots, v_{i_n}\}) \to \{v_{j_1}, \dots, v_{j_n}\})Q(\{v_{j_1}, \dots, v_{j_n}\}) \to \{v_{k_1}, \dots, v_{k_n}\}) = R(\{v_{i_1}, \dots, v_{i_n}\}) \to \{v_{k_1}, \dots, v_{k_n}\}).$$

The inverse  $P^{-1}$  of P is given by

$$P^{-1} \equiv [P(\{v_{i_1}, \dots v_{i_n}\} \to \{v_{i_1}, \dots v_{i_n}\})]^{-1} = Q(\{v_{i_1}, \dots v_{i_n}\} \to \{v_{i_1}, \dots v_{i_n}\}).$$

Clearly, the permutation group is not abelian.

A transposition,  $T(v_i \leftrightarrow v_j)$  is a permutation that only changes  $v_i$  into  $v_j$ , and  $v_j$  into  $v_i$ . Any permutation may be written as a product of transpositions. The quantity  $\delta_P \equiv (-1)^{\nu_P}$ , where  $\nu_P$  is the number of such transpositions, is called the *parity* of P. Although the decomposition in transpositions is not unique, and hence neither is  $\nu_P$ , the parity only depends on the permutation P and not on how it was decomposed in transpositions.

The permutation group is also important because it exhausts the set of all finite groups, in the following sense:

THEOREM (CAYLEY).

Any finite subgroup is isomorphic with a subgroup of the permutation group. That is to say, given a finite group G, there exists an n, and a subgroup  $G_n$  of  $\Pi_n$ , such that  $G_n$  is isomorphic to G.

For more details, see Hamermesh (1963).

## 1.4. The classical groups

Among the more important groups are those defined in terms of matrices, often called *classical groups*. We here describe a number of these; several among them will be studied in more detail later on.

GL(n,C). (General complex linear group). This is the group of complex  $n \times n$  matrices with nonzero determinant.

GL(n,R). (General real linear group). This is the group of real  $n \times n$  matrices with determinant  $\neq 0$ .

O(n,C). (Complex orthogonal group). This is the group of complex orthogonal  $n \times n$  matrices, i.e., such that if  $M \in O(n,C)$ , then  $MM^T = 1$  where  $M^T$  is the transpose of M.

O(n). (Orthogonal group). This is the group of real orthogonal  $n \times n$  matrices, i.e., such that if  $M \in O(n)$ , then  $MM^{T} = 1$  where  $M^{T}$  is the transpose of M.

U(n). (Unitary group). The group of unitary complex  $n \times n$  matrices.

Sp(2k). (Simplectic group). The group that leaves invariant the simplectic form in the 2k-dimensional euclidean space.

EXERCISE: Which of these groups is not simple? Find abelian invariant subgroups.

The definitions of these groups are all well known and elementary except, perhaps, that of the simplectic group. It is the group of real transformations in the 2k-dimensional space that leave invariant the skew-symmetric quadratic form  $[\mathbf{x}\mathbf{y}]$  defined by

$$[\mathbf{xy}] \equiv x_1 y_1 - x_2 y_2 + \dots + x_{2k-1} y_{2k-1} - x_{2k} y_{2k}.$$

Important subgroups of these groups are those obtained requiring unit determinant; the corresponding matrices are called *unimodular*. They are denoted by adding the letter S (and the calificative special) to the name of the group, except for the first two which are called SL(n,C) and SL(n,R). Thus, SO(n) is the special orthogonal group consisting of real orthogonal matrices in  $n \times n$  dimensions, and with unit determinant.

EXERCISE: Prove that SO(n) coincides with the group of rotations in  $\mathbb{R}^n$ .

The standard text on the classical groups is that of Weyl (1946); that of Hamermesh (1963) is more oriented towards physical applications.

# §2. Lie groups and Lie algebras

#### 2.1. Definitions

Many of the groups of interest in physics are Lie groups.<sup>3</sup> A group G is a Lie group, of dimension d (d finite) if every element  $g \in G$  is specified by d real parameters:  $g \equiv g(\alpha_1, \ldots, \alpha_d)$  in such a way that, if  $\alpha_1, \ldots, \alpha_d$  are the parameters of g,  $\beta_1, \ldots, \beta_d$  those of h and  $\gamma_1, \ldots, \gamma_d$  those of  $gh^{-1}$ , then the  $\gamma_n = \gamma_n(\alpha_1, \ldots, \alpha_d; \beta_1, \ldots, \beta_d)$  are analytic functions of the  $\alpha_i$  and  $\beta_j$ . We will assume that the parameters are essential; that is to say,  $g(\alpha_1, \ldots, \alpha_d) = h(\beta_1, \ldots, \beta_d)$  only if  $\alpha_1 = \beta_1, \ldots, \alpha_d = \beta_d$ .

For Lie groups we will narrow the definition of simple and semisimple groups as follows: we say that a Lie group is simple if it has no invariant subgroups that are also Lie groups; and we say that it is semisimple if it has no abelian invariant subgroups that are also Lie groups. (However, *simple* or *semisimple* Lie groups may have invariant *discrete* abelian subgroups.)

EXAMPLE: The "special" groups SU(n), SL(n,C) and SL(n,R) are all simple as Lie groups but, for n =even, the discrete subgroup  $\{1, -1\}$  of SU(n) is invariant.

#### THEOREM.

It is possible to reparametrize a Lie group in such a way that the parameters are normal, that is to say, they verify  $g(0,\ldots,0)=e$  (e being the unity) and, if the vectors  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are parallel, then

$$g(\alpha_1, \ldots, \alpha_d)h(\beta_1, \ldots, \beta_d) = f(\alpha_1 + \beta_1, \ldots, \alpha_d + \beta_d).$$

The interest of normal parameters is that one can reduce a finite transformation to powers of infinitesimal ones:

$$g(\boldsymbol{\alpha}) = \left[g(\boldsymbol{\alpha}/N)\right]^N$$
.

For groups whose elements are matrices (or, more generally, operators) this allows us to get finite group elements by exponentiation:

$$g(\boldsymbol{\alpha}) = \lim_{N \to \infty} [g(\boldsymbol{\alpha}/N)]^N = \exp \boldsymbol{\alpha} \mathbf{L}, \quad L_i \equiv \partial g(\boldsymbol{\alpha})/\partial \alpha_i|_{\boldsymbol{\alpha}=0}.$$

Let G be a Lie group, in normal coordinates. Let  $g = g(\alpha_1, \ldots, \alpha_d)$ ,  $h = h(\beta_1, \ldots, \beta_d)$  and define the Weyl commutator  $c = g^{-1}h^{-1}gh \equiv c(\gamma_1, \ldots, \gamma_d)$ . Then, the quantities  $C_{ik\nu}$  given by

$$C_{ik\nu} \equiv \left. \frac{\partial^2 \gamma_{\nu}(\alpha_1, \dots, \alpha_d; \beta_1, \dots, \beta_d)}{\partial \alpha_i \partial \beta_k} \right|_{\alpha = \beta = 0}$$

are called the structure constants of the group.

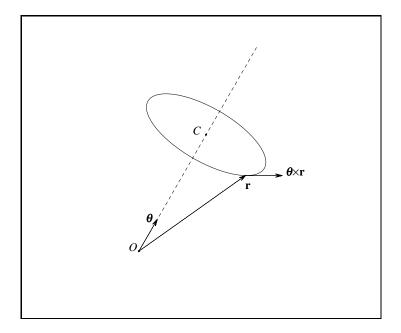
A fundamental theorem is the following:

### THEOREM.

If the group G is simple, the structure constants calculated for the group G, or for any nontrivial representation of G, are identical.

It follows that we can evaluate the  $C_{ik\nu}$  in whatever representation is convenient.

<sup>&</sup>lt;sup>3</sup> The proof of the majority of result we will give on Lie groups, as well as a wealth of supplementary information on them, may be found in the classic treatise of Chevalley (1946).



The action of the rotation  $R(\boldsymbol{\theta})$ .

We say that the Lie group G is *compact* if the subset of  $\mathbb{R}^n$  over which the parameters  $\alpha_1, \ldots, \alpha_d$  vary when  $g(\alpha_1, \ldots, \alpha_d)$  ranges over the whole group is compact; for normal parameters, this essentially means that it is bounded. SO(n) and SU(n) are compact Lie groups; SL(n,C) and SL(n,R) are also Lie groups, but they are not compact.

A simple and important example of Lie group is the rotation group, SO(3). We can parametrize the elements R of SO(3) by three parameters,  $\theta_i$ , so that, on any vector  $\mathbf{r}$  in three-dimensional space,  $R(\boldsymbol{\theta})$  acts as follows:

$$\mathbf{r} \to \mathbf{r}' = R(\boldsymbol{\theta})\mathbf{r} = (\cos\theta)\mathbf{r} + (1-\cos\theta)\frac{\boldsymbol{\theta}\mathbf{r}}{\theta^2}\boldsymbol{\theta} + \frac{\sin\theta}{\theta}\boldsymbol{\theta} \times \mathbf{r};$$

see the figure. For  $\boldsymbol{\theta}$  infinitesimal,

$$R(\boldsymbol{\theta})\mathbf{r} = \mathbf{r} + \boldsymbol{\theta} \times \mathbf{r} + O(\theta^2).$$

A subtle point is that we must restrict  $\boldsymbol{\theta}$  to  $|\boldsymbol{\theta}| \leq 2\pi$ , and we have to identify the rotations  $R(\boldsymbol{\theta})$  for  $|\boldsymbol{\theta}| = 2\pi$  with the unity.

EXERCISES: Check that the matrix  $R_{ij}$  is orthogonal and that  $det(R_{ij}) = 1$ . Check that SO(3) is compact. Try to draw the parameter space for SO(3).

We finish this subsection with two important theorems:

# THEOREM.

If the group G is compact, then all its irreducible, finite dimensional representations, are equivalent to unitary representations (i.e., representations in which the matrices D(q) are all unitary).

#### THEOREM.

If the group G is not compact, then it does not have unitary finite dimensional representations.

# 2.2. Functions over the group; group integration; the regular representation. Character of a representation

Let G be an arbitrary Lie group. We consider the space  $\mathcal{F}(G)$  of functions, with complex values, and defined over the group,

$$\phi: g = g(\alpha_1, \dots, \alpha_d) \to \phi(g) \in \mathbb{C}.$$

Because g is given by the parameters  $\alpha_1, \ldots, \alpha_d$ , we can consider  $\phi$  as an ordinary function of d variables,  $\phi(g) = \phi(\alpha_1, \ldots, \alpha_d)$ .

THEOREM (HAAR INTEGRAL).

If G is compact there exists a nonegative function  $\mu(g) = \mu(\alpha_1, \ldots, \alpha_d)$ , unique up to normalization, called the Haar measure, such that the integral

$$\int_{G} d\mu(g)\phi(g) \equiv \int_{\{\alpha\}} d\mu(\alpha_1, \dots, \alpha_d)\phi(\alpha_1, \dots, \alpha_d)$$

exists provided  $\phi$  is bounded in all G. Moreover,  $\mu$  is left and right invariant:  $d\mu(hg) = d\mu(gh) = d\mu(g)$ .

If the group is not compact, but is semisimple, the result is still true but we have to restrict the function  $\phi$  to decrease at infinity in parameter space. The proof of this theorem may be found in Naimark (1956); cf. also Chevalley (1946). An intuitive discussion may be seen in Wigner (1959).

We may define a scalar product in the subset  $\mathcal{C}(G) \subset \mathcal{F}(G)$  of continuous functions on G (of fast decrease in parameter space, if the group is not compact); we write

$$\langle \phi | \psi \rangle \equiv \int_G \mathrm{d}\mu(g) \phi(g)^* \psi(g).$$

Then,  $\mathcal{C}(G)$  can be extended to a Hilbert space,  $L^2(G)$ .

For compact groups, the integral  $\int_G d\mu(g)$  is finite. In this case one can, if so wished, normalize the Haar measure so that  $\int_G d\mu(g) = 1$ .

The Haar measure can be reduced to an ordinary integral by writing

$$d\mu(\alpha_1,\ldots,\alpha_d)=j(\alpha_1,\ldots,\alpha_d)d\alpha_1\cdots d\alpha_d.$$

The functions j can be found, for several important groups, in Hamermesh (1963).

EXERCISE: Prove that, for SO(3), characterizing its elements as before by  $R(\boldsymbol{\theta})$ , one simply has  $d\mu = d\theta_1 d\theta_2 d\theta_3$ .

The notion of Haar integral can be extended to finite groups. If G is a finite group with elements  $g_i$ , i = 1, ..., n then the "Haar integral" is simply the sum over all group elements:

$$\int \mathrm{d}\mu \, \phi \equiv \sum_{i=1}^{n} \phi(g_i).$$

It is possible to construct a representation of the group G over the set of functions  $L^2(G)$ , which is at times called the *regular* representation. For an element  $a \in G$ , it is defined by

$$reg(a): \phi(q) \to \phi(aq).$$

More on the important properties of the regular representation may be found in Naimark (1959).

EXERCISE: Prove that the regular representation is unitary.

An important group function is what is called the *character* of a (finite-dimensional) representation, D(g). It is defined by  $\chi_D(g) = \text{Tr } D(g)$ . An important property of the character is that it is intrinsic to the representation, in the sense that, if D, D' are equivalent, then  $\chi_D(g) = \chi_{D'}(g)$ . Moreover, if D, D' are not equivalent, their characters are orthogonal:

$$\int d\mu(g) \, \chi_D^*(g) \chi_{D'}(g) = 0.$$

This is a consequence of the Peter-Weyl theorem, that we will consider later.

The theory of characters is very important in the study of representations of *finite* groups, in particular the permutation group or chrystalographic groups; see Lyubarskii (1960) or Hamermesh (1963).

#### 2.3. Lie algebras

Consider a linear space, L, with elements L that verify the following conditions:<sup>4</sup>

- 1. Any linear combination with real constants,  $aL_1 + bL_2$ ,  $L_i \in \mathbf{L}$ , is also in  $\mathbf{L}$ ;
- 2. There exists a composition law, called the *commutator*,  $[L_1, L_2] = -[L_2, L_1] \in \mathbf{L}$  such that it is linear in both arguments;
- 3. For any three  $L_i$ , i = 1, 2, 3 in **L** one has the Jacobi identity

$$\sum_{\text{cyclic}} [L_1, [L_2, L_3]] = 0.$$

Then we say that L is a Lie algebra. If all commutators vanish we say that L is abelian.

If **H** is a linear subspace in **L**, which is in itself a Lie algebra, we say that it is *invariant* if, for all  $H \in \mathbf{H}$ ,  $L \in \mathbf{L}$ , the commutator [H, L] belongs to **H**. We say that **L** is *simple* if it has no invariant subalgebra (except the trivial ones). We say that **L** is *semisimple* if it has no abelian (nontrivial) invariant subalgebra.

If **L** is a Lie algebra and it has a basis  $L_i$ , i = 1, ..., d, then we can write

$$[L_i, L_j] = \sum_{\nu} C_{ij\nu} L_{\nu}.$$

The  $C_{ij\nu}$  are called the structure constants of the Lie algebra.

Given a Lie group, G, we can construct a corresponding Lie algebra as follows: consider the regular representation. Then the set G of operators L of the form

$$L = \sum_{i} a_{i} \left. \frac{\partial \operatorname{reg}(g(\alpha_{1}, \dots, \alpha_{d}))}{\partial \alpha_{i}} \right|_{\boldsymbol{\alpha} = 0}, \quad a_{i} \text{ real},$$

is a Lie algebra. We say that G is the Lie algebra of G.

EXERCISE: Check that the structure constants of the group G are the same as those of its corresponding Lie algebra, G.

One has the following fundamental theorem:

THEOREM (LIE AND E. CARTAN).

To every (finite dimensional) Lie algebra  $\mathbf{L}$  there corresponds at least a group, G, whose Lie algebra  $\mathbf{G}$  is identical with  $\mathbf{L}$ ,  $\mathbf{G} = \mathbf{L}$ .

<sup>&</sup>lt;sup>4</sup> A very comprehensive (and comprehensible) book on Lie algebras is Jacobson (1962). In the present notes, we will only consider *finite* Lie algebras, i.e., such that the linear space  $\mathcal{L}$  has finite dimension.

EXAMPLE: The set  $\mathbf{M}_n$ ,  $n \geq 2$ , of real  $n \times n$  matrices M, with zero trace,  $\operatorname{Tr} M = 0$ , is a Lie algebra. A basis of this algebra is formed by the matrices

$$L_{ij} = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & \dots & 1 & (ij) & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \text{ for } i \neq j; \qquad L_k = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 1 & (k) & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & -1 & (k+1) & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}.$$

The corresponding Lie group is SL(n,R).

EXERCISE: Evaluate the structure constants of  $\mathbf{M}_n$  for n=2 and n=3. What is the dimension of  $\mathbf{M}_n$ ?

EXERCISE: Consider the set  $\mathbf{A}_{n-1}$  of complex  $n \times n$  matrices A anti-hermitean (i.e.,  $A^{\dagger} = -A$ ) and of zero trace,  $\operatorname{Tr} A = 0$ . Prove that it is a Lie algebra. Find a basis and the structure constants for  $\mathbf{A}_{n-1}$ . What is the dimension of  $\mathbf{A}_{n-1}$ ?

Given a Lie algebra  $\mathbf{L}$ , with generators  $L_n$ , we can form a new Lie algebra, over the complex numbers, that we call the *complexification* of  $\mathbf{L}$  and we denote by  $\mathbf{L}^{\mathbb{C}}$  (or by the same letter,  $\mathbf{L}$ , if there is no danger of confusion), by admitting linear combinations with *complex* coefficients,

$$\sum_{n} \alpha_n L_n, \quad \alpha_n \in \mathbb{C}.$$

From any complex Lie algebra,  $\mathbf{L}^{\mathbb{C}}$ , we can generate a new *real* Lie algebra,  $(\mathbf{L}^{\mathbb{C}})^{\mathbb{R}}$  whose basis is formed by the set  $\{L_n, \sqrt{-1}L_m\}$ .

EXERCISE: Prove that the complexification of  $\mathbf{A}_{n-1}$  coincides with that of  $\mathbf{M}_n$ , and both with the Lie algebra of  $\mathrm{SL}(\mathrm{n,C})$ .

The definitions of representations, direct product and direct sum for Lie algebras are similar to those for groups. Thus, a representation of  $\mathbf{L}$  is an application into the set of operators in a Hilbert space, D(L), such that

$$D(\alpha L + \beta L') = \alpha D(L) + \beta D(L'); \quad D([L, L']) = [D(L), D(L')].$$

Likewise, we define *reducible* representations of Lie algebras to be those that can be written as direct sum of nontrivial representations.

#### 2.4. The universal covering group

Consider two closed, oriented curves,  $\ell$ ,  $\ell'$ , in a group G, such that both  $\ell$ ,  $\ell'$  run through the identity e. We will say that  $\ell$  is homotopic to  $\ell'$  if  $\ell$  can be continuously deformed into  $\ell'$  (without going out of G). Let us define the product  $\ell\ell'$  as the curve obtained joining  $\ell$  and  $\ell'$ , and call a null curve to one that can be continuously deformed into the point e. If, moreover, we identify homotopic curves, we obtain a set  $\mathcal{P}$  with a structure of abelian group, called the homotopy or Poincaré group.

#### THEOREM.

Given a Lie group, G, there exists a unique group  $\hat{G}$ , called the universal covering group of G such that

- i)  $\dim G = \dim G$ ;
- ii)  $\hat{G}/\mathcal{P}=G;$
- iii) The Lie algebras of G and  $\hat{G}$  are identical.

If the number of elements of  $\mathcal{P}$  is N, we say that  $\hat{G}$  covers G N times.

EXAMPLES: The homotopy group of SO(3) is isomorphic to the group  $\{1, -1\}$  (with the ordinary multiplication law). The Lie algebra of SO(3) is  $A_1$ . The covering group of SO(3) is SU(2). The homotopy groups of SO(4), SO(6) or the (orthocronous, proper) Lorentz group,  $\mathcal{L} \equiv \mathcal{L}_+^{\uparrow}$  are also isomorphic to  $\{1, -1\}$ . The covering group of SO(6) is SU(4). The covering group of  $\mathcal{L}$  is SL(2,C).

EXERCISE: Consider the rotation group in two dimensions, SO(2), with elements characterized by the angle  $\theta$ ,  $0 \le \theta < 2\pi$ . It can be mapped into the group of complex numbers of the form  $e^{i\theta}$ . One can extend the group to include the rotation by  $2\pi$  by identifying  $e^{2\pi i} \equiv 1$ . Use this to find the homotopy group of SO(2) (it is isomorphic to the integers) and the covering group of SO(2) (it is isomorphic to the set of real numbers).

Because in quantum mechanics the vectors  $|\phi\rangle$  and  $e^{i\lambda}|\phi\rangle$  represent the same state, covering groups play an important role there, as we will see later.

We next establish the correspondence  $SO(3) \rightarrow SU(2)$ . We let  $\sigma_i$  be the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

EXERCISE: Check that

$$\sigma_a \sigma_b = i \sum_c \epsilon_{abc} \sigma_c + 2\delta_{ab}.$$

To every three-vector,  $\mathbf{v}$  we make correspond a hermitean, traceless  $2\times 2$  matrix  $\hat{v}$ ,

$$\hat{v} \equiv \mathbf{v}\boldsymbol{\sigma}: \quad \hat{v}^{\dagger} = \hat{v}, \quad \operatorname{Tr} \hat{v} = 0; \quad \det \hat{v} = -\mathbf{v}^2.$$

If R is an element of SO(3) (a rotation), and  $\mathbf{v}_R$  the image of  $\mathbf{v}$  under R,  $v_i = \sum_j R_{ij}v_j$ , then the matrix

$$\hat{v}_R \equiv \mathbf{v}_R \boldsymbol{\sigma}$$

is still hermitean and traceless. It can be written as

$$\hat{v}_R = U \,\hat{v} \, U^{\dagger}$$

with U unitary and of unit determinant. In fact, the explicit form of U is obtained as follows. Let  $\boldsymbol{\theta}$  be the parameters that determine R,  $R = R(\boldsymbol{\theta})$ . Then,

$$U = \pm \exp(-i\boldsymbol{\sigma}\boldsymbol{\theta}/2).$$

The correspondence  $SO(3) \rightarrow SU(2)$  is bi-valued; that of  $SU(2) \rightarrow SO(3)$  is single-valued.

EXERCISE: Prove all this. Hint: calculate for infinitesimal parameters  $\boldsymbol{\theta}$  and exponentiate.

EXERCISE: Calculate the  $R(\boldsymbol{\theta})$  that corresponds to a given  $U(\boldsymbol{\theta})$ . Hint: consider the quantity  $\operatorname{Tr} \sigma_a \hat{n}_R$ , where **n** is a unitary vector along the *n*-th axis.

If a Lie group is a matrix group, we may consider its Lie algebra to be a matrix algebra. The restriction to matrix groups is really no restriction as it can be proved that any Lie group has a faithful matrix representation. We have,

THEOREM.

If G is a matrix Lie group, and **G** its matrix Lie algebra, with basis  $\{L_n\}_1^d$ , then the set of elements of the form  $\exp \sum_{1}^{d} \alpha_n L_n$ ,  $\alpha_n$  real, generates the group  $\hat{G}$ .

For this reason, the elements  $L_n$  are also called the generators of the group (or of the Lie algebra).

THEOREM.

If  $\hat{G}$  is abelian, simple, semisimple then G is also abelian, simple, semisimple; and conversely.

The proof of the last theorem is based on the relation, valid for small L, L',

$$\mathbf{e}^L \mathbf{e}^{L'} \mathbf{e}^{-L} \mathbf{e}^{-L'} = [L, L'] + \text{third order terms}.$$

 $e^{L}e^{L'}e^{-L}e^{-L'}$  is called the Weyl commutator.

There are two generalizations of the concept of (unitary) group representations which are important in physics. One are the representations up to a phase, which are applications such that

$$D(g)D(h) = e^{i\lambda(g,h)}D(gh).$$

The other are multivalued representations,

$$g \in G \to e^{i\lambda}D(g)$$

where the phase  $\lambda$  may take several values; for example, one may have  $g \to \pm D(g)$  as in the correspondence  $SO(3) \to SU(2)$  above.

With respect to the first, Wigner has shown that (for the groups of interest in physics) one can choose the phases of the vectors in the Hilbert spaces in which the D(g) act so that  $\phi(g,h) \equiv 0$ : that is to say, they can be reduced to ordinary representations. With respect to multivalued representations, one can show (see Chevalley 1946) that they correspond to single valued representations of the covering group,  $\hat{G}$ .

In the particular case of the rotation group, it follows that multiple-valued representations of SO(3) become single valued representations of SU(2). Likewise, multiple-valued representations of the Lorentz group,  $\mathcal{L}$  (that we will discuss later) become single-valued representations of its covering group,  $SL(2,\mathbb{C})$ . Because  $SL(2,\mathbb{C})$  doubly covers  $\mathcal{L}$ , and SU(2) doubly covers SO(3), this implies that representations of SO(3) or  $\mathcal{L}$  can be at most double-valued. Hence, in particular, spin can only be integer or half integer. For massive particles this follows also from the commutation relations of the generators of SO(3); for massless particles, the proof based on the covering group is the only one known to the author.

EXERCISE: From the fact that that the covering group of the rotation group in two dimensions, SO(2), is isomorphic to the group of the real line deduce that, in two dimensions, one can have any real value for the angular momentum; i.e., in two dimensions the angular momentum can vary continuously.

#### 2.5. The adjoint representation. Cartan's tensor and Cartan's basis

An important representation of Lie groups and Lie algebras is the so-called adjoint representation. It represents the element  $L_n$  in a Lie algebra G of dimension d by the matrix  $\mathrm{ad}_{G}(L_n)$  with components

$$(\operatorname{ad}_{\mathbf{G}}(L_n))_{ij} = C_{ijn};$$

the  $C_{ijn}$  are the structure constants. The dimension of this representation is, clearly, that of the Lie algebra, d. This representation generates, by exponentiation, a representation of the covering group  $\hat{G}$ . In turn, this representation induces a metric tensor  $g_{ik}$ , called the Cartan tensor (or also *Killing form*), as follows:

$$g_{ik} = \operatorname{Tr} L_i L_k = \sum_{nm} C_{nmi} C_{mnk}.$$

If  $g_{ik}$  is negative-definite, we say that **G** is compact.

THEOREM (E. CARTAN).

The tensor  $g_{ik}$  is non-degenerate if, and only if, G is semisimple.

THEOREM (H. WEYL).

**G** is compact if, and only if, G is compact.

Given a semisimple, complex Lie algebra,  $\mathbf{G}$ , consider all its abelian subalgebras (which cannot be invariant). Among these, that of maximum dimension,<sup>5</sup>  $\mathbf{H}$ , is called the maximal abelian subalgebra; if l is its dimension, we also say that l is its rank. Consider now the maximal abelian subalgebra  $\mathbf{H}$ , and let us denote by  $H_i$  to a basis of  $\mathbf{H}$ . We let the  $E_{\alpha}$  be the remaining elements, obviously in  $\mathbf{G} \ominus \mathbf{H}$ , that complete a basis of  $\mathbf{G}$ . One has:

THEOREM (KILLING AND E. CARTAN).

There exists a basis of  $\mathbf{G}^{\mathbb{C}}$  (we will simply denote  $\mathbf{G}^{\mathbb{C}}$  by  $\mathbf{G}$ ) such that all the  $\mathrm{ad}_{\mathbf{G}}(H_i)$  are self-adjoint. Moreover, we can choose the  $E_{\alpha}$  such that they are eigenvectors of the  $H_i$ ,

$$[H_i, E_{\alpha}] = r_i(\alpha) E_{\alpha};$$

for every  $E_{\alpha}$  there exists  $E_{-\alpha}$  with

$$[H_i, E_{-\alpha}] = -r_i(\alpha)E_{-\alpha}$$

and

$$[E_{\alpha}, E_{-\alpha}] = r^{i}(\alpha)H_{i}, \quad r^{i}(\alpha) = \sum_{j} g_{ij}r_{j}(\alpha)$$

and, finally,

$$[E_{\alpha}, E_{\beta}] = n_{\alpha\beta} E_{\alpha+\beta}.$$

Here  $n_{\alpha\beta} = C_{\alpha+\beta,\alpha\beta}$  if  $E_{\alpha+\beta}$  exists; otherwise,  $n_{\alpha\beta} = 0$ .

The l-dimensional vectors  $\boldsymbol{\alpha}$  with components  $r_i(\alpha)$  are called roots of  $\mathbf{G}$ .

THEOREM (KILLING AND E. CARTAN).

Apart from the so-called exceptional algebras, which we will not study here,<sup>6</sup> the only possible compact algebras are those of the following table, where we also give the corresponding classical groups:

 $egin{array}{ll} {\bf A}_l: & {
m SU}(l+1) \\ {\bf B}_l: & {
m O}(2l+1) \\ {\bf C}_l: & {
m Sp}(2l) \\ {\bf D}_l: & {
m O}(2l). \\ \end{array}$ 

We note that some of the lower dimensionality algebras are in fact isomorphic:  $\mathbf{B}_1$  and  $\mathbf{A}_1$ ,  $\mathbf{D}_2$  and  $\mathbf{A}_1 \times \mathbf{A}_1$  and  $\mathbf{D}_3$  and  $\mathbf{A}_3$ .

It is possible to give a concise characterization of all the compact Lie algebras in terms of the root diagrams; we will give these in a few simple cases. An even more concise characterization is in terms of the so-called *Dynkin diagrams*, which we will not discuss here. We refer the reader to the text of Jacobson (1962), where one can also find the proofs of many of the statements of this section, as well as the description of the so-called exceptional groups (and algebras) of E. Cartan.

<sup>&</sup>lt;sup>5</sup> There may exist several abelian subalgebras with the same maximum dimension; the results are independent of which one we choose as maximal abelian subalgebra.

<sup>&</sup>lt;sup>6</sup> There are five such algebras, denoted by  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$  and  $E_8$ ; the index is the rank. They may be found in Jacobson (1962).

# §3. The unitary groups

The study of the unitary groups, SU(n), is equivalent to the study of the corresponding Lie algebras,  $A_{n-1}$ . Because the groups SU(n) are their own covering groups, one can be obtained from the other by exponentiation or differentiation with respect to the parameters. We will in this, and the following sections, study in some detail the simplest groups corresponding to n=2, 3, as well as their representations.

EXERCISE: Prove that the automorphism  $U \to U^*$  in SU(n) is external for  $N \ge 3$ . Prove that it is internal for n = 2. Hint: for the second, write  $U = \exp i\theta\sigma/2$  and consider the transformation  $U \to CUC^{-1}$  with  $C = i\sigma_2$  ( $\sigma_2$  the Pauli matrix) in SU(2).

#### 3.1. The group SU(2) and the Lie algebra $A_1$

By far the more important Lie groups are the unitary ones, SU(n). We will now construct explicitly their corresponding Lie algebras for n = 2, 3.

 $\mathbf{A}_1$ . The (real)  $\mathbf{A}_1$  algebra consists of traceless, antihermitean  $2 \times 2$  matrices. A convenient basis for it are the  $L_a = (-\mathrm{i}/2)\sigma_a$ , with  $\sigma_a$  the Pauli matrices. The commutation relations are

$$[L_a, L_b] = \sum_c \epsilon_{abc} L_c,$$

and  $\epsilon_{abc}$  is the antisymmetric Levi-Civita tensor. Thus, the structure constants are  $C_{abc} = \epsilon_{abc}$ . The adjoint representation is three-dimensional and has as basis the matrices with components

$$(\operatorname{ad}(L_a))_{ij} = \epsilon_{aij}.$$

The Cartan tensor is  $g_{ij} = -2\delta_{ij}$ .

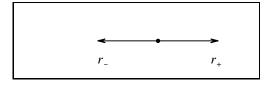
The maximal abelian subalgebra consists of the multiples of a single generator, that we may take  $T_3 = iL_3$ ; we change somewhat the names and definitions to be in agreement with what is usual in physical applications. We will also work with the complexified algebra,  $\mathbf{A}_1^{\mathbb{C}}$ , that we will go on calling simply  $\mathbf{A}_1$ . The Cartan basis of this (complex) algebra is completed with the elements

$$T_{\pm 1} = \mathrm{i} \left( L_1 \pm \mathrm{i} L_2 \right),\,$$

and one can easily check that

$$[T_3, T_{+1}] = \pm T_{+1}, \quad [T_{+1}, T_{-1}] = 2H.$$

The root diagram of  $A_1$  is one dimensional, as shown in the figure.



The root diagram for  $A_1$ .

# 3.2. The groups SO(4) and $SU(2) \times SU(2)$

We will here establish a correspondence between the groups SO(4) and SU(2)×SU(2) (in fact, between the corresponding Lie algebras; we will work infinitesimally). For this, consider the set of matrices  $\sigma_A$ , A = 1, 2, 3, 4 with  $\sigma_4 = i$ , and  $\sigma_i$  the Pauli matrices for i = 1, 2, 3.

For any real four-dimensional vector, v we will designate its components by  $(\mathbf{v}, v_4)$ . The scalar product in  $\mathbb{R}^4$  we then write as

$$v \cdot w = \mathbf{v}\mathbf{w} + v_4 w_4.$$

For any vector v, we form the  $2 \times 2$  matrix

$$\hat{v} = v \cdot \sigma = \mathbf{v}\boldsymbol{\sigma} + \mathrm{i}v_4,$$

and we note that

$$\det \hat{v} = -v \cdot v.$$

We now consider the transformation

$$\hat{v} \to \hat{v'} = v' \cdot \sigma = V \hat{v} U^{\dagger}, \quad U, V \in SU(2).$$
 (1)

The set of such transformations builds the product group  $SU(2)\times SU(2)$ . One can therefore write U, V in all generality as

$$U = e^{-i\boldsymbol{\alpha}\boldsymbol{\sigma}}, \quad V = e^{-i\boldsymbol{\beta}\boldsymbol{\sigma}}.$$

Eq. (1) establishes a correspondence between vectors in  $\mathbb{R}^4$ ,

$$v \rightarrow v'$$

which it is easy to check that it is linear and such that  $v \cdot v = v' \cdot v'$ . It only remains to verify that v' is real to conclude that we can write

$$v_A' = \sum_B R_{AB} v_B, \quad R \in SO(4).$$

We do this for infinitesimal  $\boldsymbol{\alpha}$ ,  $\boldsymbol{\beta}$ , that is to say, we take

$$U = 1 - i\boldsymbol{\alpha}\boldsymbol{\sigma} + O(\alpha^2), \quad V = 1 - i\boldsymbol{\beta}\boldsymbol{\sigma} + O(\beta^2);$$

we will then neglect quadratic terms systematically. It follows that, if we write

$$v' \cdot \sigma = V(v \cdot \sigma)U; \quad v'_A = \sum_B R_{AB} v_B$$

then, for infinitesimal transformations, the matrix elements  $R_{AB}$  are given by

$$\mathbf{v}' = \mathbf{v} - (\boldsymbol{\alpha} + \boldsymbol{\beta}) \times \mathbf{v} + v_4(\boldsymbol{\alpha} - \boldsymbol{\beta}),$$
  

$$v_4' = v_4 - (\boldsymbol{\alpha} - \boldsymbol{\beta})\mathbf{v}.$$
(2)

This is clearly real, and therefore Eq. (2) sets up the mapping

$$(\pm V, \pm U) \in SU(2) \times SU(2) \rightarrow (R_{AB}) \in SO(4)$$

for infinitesimal transformations.

EXERCISE: Extend this to finite transformations.

### 3.3. The group SU(3) and the Lie algebra $A_2$

We now have  $3 \times 3$  traceless, antihermitean matrices. For physical applications it is convenient to start with the basis  $L_a = -(i/2)\lambda_a$ ,  $a = 1, \dots, 8$ ;  $\lambda_a$  are the Gell-Mann matrices

$$\lambda_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The commutation relations are now

$$[L_a, L_b] = \sum_c f_{abc} L_c,$$

so the structure constants are  $C_{ikn} = f_{ikn}$ , and only nonzero elements of the f, up to permutations, are as follows:

$$1 = f_{123} = 2f_{147} = 2f_{246} = 2f_{257} = 2f_{345}$$
$$= -2f_{156} = -2f_{367} = \frac{2}{\sqrt{3}}f_{458} = \frac{2}{\sqrt{3}}f_{678}.$$

For physical applications it is interesting to note that the  $\lambda_a$  verify the anticommutation relations

$$\{\lambda_a, \lambda_b\} = 2\sum d_{abc}\lambda_c + \frac{4}{3}\delta_{ab}$$

with the d fully symmetric and all of them zero except for the following (and their permutations):

$$\frac{1}{\sqrt{3}} = d_{118} = d_{228} = d_{338} = -d_{888}, \quad -\frac{1}{2\sqrt{3}} = d_{448} = d_{558} = d_{668} = d_{778},$$

$$\frac{1}{2} = d_{146} = d_{157} = d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377}.$$

EXERCISE: Evaluate the Cartan tensor for SU(3).

The maximal abelian subalgebra of SU(3) has now dimension 2; we may take as its basis the elements

$$T_3 = iL_3, \quad Y = \frac{2}{\sqrt{3}}iL_8;$$

again here we use these names (instead of  $H_1$ ,  $H_2$ ) and definitions because they are the conventional ones in applications to particle physics. With them the  $T_3$ , Y are hermitean (instead of antihermitean). Likewise, we will use names other than  $E_{\alpha}$  for the remaining terms in a Cartan basis. To be precise, we define

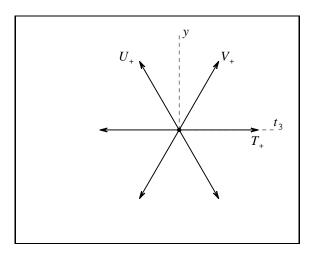
$$T_{+} = i(L_{1} \pm iL_{2}); \quad U_{+} = i(L_{6} \pm iL_{7}); \quad V_{+} = i(L_{4} \pm iL_{5}).$$

In terms of these operators, the commutation relations are

$$\begin{split} [T_3,Y] &= 0, \quad [T_3,T_\pm] = \pm T_\pm, \quad [T_+,T_-] = 2T_3, \quad [Y,T_\pm] = 0; \\ [T_3,U_\pm] &= \mp \frac{1}{2}U_\pm, \quad [T_3,V_\pm] = \pm \frac{1}{2}V_\pm, \quad [Y,U_\pm] = \pm \frac{1}{2}U_\pm, \quad [Y,V_\pm] = \pm \frac{1}{2}V_\pm; \\ [U_+,U_-] &= \frac{3}{2}Y - T_3 \equiv 2U_3, \quad [V_+,V_-] = \frac{3}{2}Y + T_3 \equiv 2V_3; \\ [T_+,U_+] &= V_+, \quad [T_+,V_-] = -U_-, \quad [U_+,V_-] = T_-; \\ [T_+,V_+] &= [T_+,U_-] = [U_+,V_+] = 0. \end{split}$$

EXERCISE: Prove that the three  $T_{\pm}$ ,  $T_3$  form the basis of a  $\mathbf{A}_1$  subalgebra of  $\mathbf{A}_2$ . Check that, with the  $U_3$ ,  $V_3$  just defined, the same is true for the three  $U_5$ ,  $V_5$ .

EXERCISE: Verify that the root diagram of  $A_2$  is as in the figure.



The root diagram for  $A_2$ .

# §4. Representations of the SU(n) groups (and of their Lie algebras)

Because the groups SU(n) are their own covering groups, it follows that their representations may be obtained from the representations of their (complex) Lie algebras,  $\mathbf{A}_{n-1}$ : a much simpler task. This task is further simplified because a representation of a real Lie algebra,  $\mathbf{L}$ , can be extended to a representation of its complexification,  $\mathbf{L}^{\mathbb{C}}$ , by the simple expedient of allowing multiplication by complex numbers. We will use this trick systematically.

In the present section we will construct explicitly the representations of these Lie algebras for l = n - 1 = 1, 2; and, later on, of the groups for all n. There is a particularly important representation of the groups SU(n), namely that acting in a complex n-dimensional space in which the representatives of the elements in SU(n) are the very unimodular, unitary  $n \times n$  matrices in SU(n). It is called the fundamental representation. One has the important result that all the representations of SU(n) can be generated by multiplying the fundamental representation by itself (Weyl, 1946).

A very understandable treatise on representations of Lie groups, in particular of SU(n) and SL(n,C), is that of Hamermesh (1963); for the rotation group, see Wigner (1959).

# 4.1. The representations of $A_1$

The representations of the  $A_1$  Lie algebra are well known from elementary quantum mechanics, but we will review them here because of their importance for more complicated cases. We work with the Cartan basis given above and look for irreducible, finite dimensional representations. Hence, in these representations the operators representing the  $T_a$ , a=1,2,3 [which we denote with the same letters,  $D(T_a) \to T_a$ ] can be taken to be hermitean operators. Because of this, one has  $T_+^{\dagger} = T_-$ . We construct an orthonormal basis of vectors  $|t,t_3\rangle$  which are eigenvalues of  $T_3$ :

$$T_3|t,t_3\rangle = t_3|t,t_3\rangle;$$

the quantity t, that (as we will see) fully characterizes the representation is defined as the maximum of  $t_3$ ; hence, there exists a state (that we assume to be *unique*; see below)  $|t,t\rangle$  with this maximum value of  $t_3$ . Because the transformation  $T_3 \to -T_3$  is a symmetry, it follows that, for each state  $|t,t_3\rangle$ , there exists the state  $|t,t_3\rangle$ . It thus follows that the state with *minimum* value of  $t_3$  is  $|t,t\rangle$ .

The commutation relations of the  $T_3$ ,  $T_{\pm}$  can be used to verify that the last act as rising/lowering operators for  $t_3$ . Hence the state

$$T_{-}^{n}|t,t\rangle \equiv C_{t,t-n}|t,t-n\rangle$$

is such that

$$T_3|t, t-n\rangle = (t-n)|t, t-n\rangle.$$

The  $C_{t,t-n}$  are constants introduced to make the states  $|t,t-n\rangle$  normalized to unity; see below. A first consequence of this is that one must necessarily have

$$T_{+}|t,t\rangle = T_{-}|t,-t\rangle = 0.$$

It is easy to check that the operator  $\sum_a T_a^2$  commutes with all the generators; hence, by virtue of the Schur Lemma, it has to be a multiple of the identity,  $\sum_a T_a^2 = \lambda$ . The number  $\lambda$  is evaluated as follows. First, we note the identity

$$T_{+}T_{-} = \sum_{a} T_{a}^{2} - T_{3}^{2} + T_{3}; \tag{1}$$

then we apply it to  $|t, -t\rangle$ . We find

$$0 = T_{+}T_{-}|t, -t\rangle = \left(\sum_{a} T_{a} - T_{3}^{2} + T_{3}\right)|t, -t\rangle = (\lambda - t^{2} - t)|t, -t\rangle$$

and hence

$$\sum_{a} T_a^2 = t(t+1). (2)$$

An operator like  $\sum_a T_a^2$  that commutes with all the generators is called a Casimir operator.

Let us continue with the construction of the basis  $|t, t_3\rangle$ . When we apply  $T_-^n$  to  $|t, t\rangle$  with n > 2t we must find zero. Hence we have the 2t + 1 basis vectors

$$|t,t\rangle, |t,t-1\rangle, \ldots, |t,-t\rangle.$$

EXERCISE: Prove that this implies that t and the  $t_3$  must be either integer or half-integer.

We next have to find the coefficients  $C_{tt_3}$ . This is done by establishing a recursion relation as follows:

$$1 = \langle t, t_3 | t, t_3 \rangle = \frac{1}{|C_{tt_3}|^2} \langle t, t | T_+^n T_-^n | t, t \rangle = \frac{|C_{t,t_3+1}|^2}{|C_{tt_3}|^2} \langle t, t_3 + 1 | T_+ T_- | t, t_3 + 1 \rangle$$

$$= \frac{|C_{t,t_3+1}|^2}{|C_{tt_3}|^2} \langle t, t_3 + 1 | \left( \sum_a T_a - T_3^2 + T_3 \right) | t, t_3 + 1 \rangle = \frac{|C_{t,t_3+1}|^2}{|C_{tt_3}|^2} \left[ t(t+1) - t_3(t_3+1) \right].$$

This implies the recursion formula

$$|C_{t,t_3+1}| = |C_{t,t_3}|/\sqrt{t(t+1)-t_3(t_3+1)}$$

which, together with the requirement that  $C_{tt} = 1$  and that the  $C_{tt_3}$  be positive gives all these coefficients. In particular, we find the action of the  $T_{\pm}$  on our basis,

$$T_{\pm}|t,t_3\rangle = \sqrt{t(t+1) - t_3(t_3 \pm 1)}|t,t_3 \pm 1\rangle,$$
 (3)

which completely solves the problem.

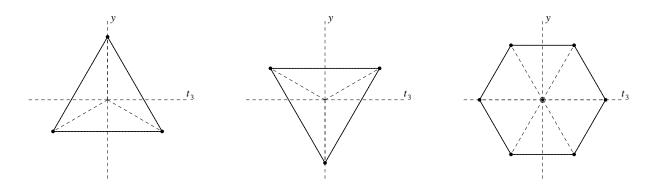
EXERCISE: Prove that, if there existed more than one state with maximum value of  $t_3$ , say, if one had  $|t,t; I\rangle$  and  $|t,t; II\rangle$ , not proportional, then the representation would be reducible.

$$t_3$$
 $-3/2$   $-1/2$  0  $1/2$   $3/2$ 

The representation of  $\mathbf{A}_1$  for t = 3/2.

#### 4.2. The representations of $A_2$

We have now two independent commuting operators,  $T_3$  and Y. So, we have to specify two eigenvalues,  $t_3$  and y, and the diagrams for the representations of  $\mathbf{A}_2$  are two-dimensional. Another thing in that the representations of  $\mathrm{SU}(3)$  differ from those of  $\mathrm{SU}(2)$  is that, if D(g) is a representation of  $\mathrm{SU}(3)$ , the representation  $D(g)^*$  may not be equivalent to it. When  $D(g)^*$  is equivalent to D(g), we say that the representation is real. Thus, the 8-dimensional representation of  $\mathrm{SU}(3)$  is real, but the 3-, 6- or 10-dimensional representations are not: the representations  $3^*$ ,  $6^*$  or  $10^*$  (with self-explanatory notation) are not equivalent to them. In the following figures we show the  $t_3$ , y diagrams of the lowest dimensional representations of  $\mathbf{A}_2$  (the representations  $6^*$ , which is the up-down mirror image of the 6, and  $10^*$ , the mirror image of 10, are not shown).



The representations 3,  $3^*$  and 8.



The representations 6 and 10.

EXERCISE: Prove that the representations of SU(2) (that we deduced in the previous section) are all real. Hint: the matrix that does the trick is the representative of  $i\sigma_2$ .

To describe the irreducible representations of  $A_2$  we consider the plane  $t_3 y$  and put a dot for each state of said representation at the corresponding location on this plane. We then have a diagram that, as we shall see, fully characterizes the representation. On can move among the dots of the diagram with the operators<sup>7</sup>  $T_{\pm}$ ,  $U_{\pm}$  and  $V_{\pm}$ ; in fact, using the commutation relations we can easily verify the following properties:

 $T_+$  raises  $t_3$  by 1 unit, and leaves y unchanged;

 $U_+$  lowers  $t_3$  by  $\frac{1}{2}$  unit and raises y by 1 unit (we note that the units of y have a length  $\sqrt{3}/2$  those of  $t_3$ ).

 $V_{+}$  raises  $t_{3}$  by 1 unit and raises y by 1 unit.

The  $T_-$ ,  $U_-$  and  $V_-$  have the opposite effect. In view of this, it follows that by applying the  $T_{\pm}$ ,  $U_{\pm}$  and  $V_{\pm}$  we move in the diagram along lines forming angles multiple of 60°, including 0°.

Another important property of the diagram of a representation is that its boundary forms a hexagon, in general irregular, symmetric around the y axis, and where the length of the sides, equal to the number of states in such side minus 1, is given by just two integers, p and q. Thus, the representation 8 (see figure) has p = 1, q = 1; the representations 3, 6 and 10 are degenerate hexagons, with q = 0 and p = 1, 2, 3 respectively. For p = q = 0 we have a single point, the trivial representation.

<sup>&</sup>lt;sup>7</sup> We also here denote with the same letters the elements of the Lie algebra and their representatives.

To construct all the points in a diagram, we start from the site with largest value of  $t_3 = t = (p+q)/2$  (it can be proved that there is a single one),  $|y,t\rangle$ , and apply all operators  $T_{\pm}$ ,  $U_{\pm}$  and  $V_{\pm}$  to  $|y,t\rangle$ , thereby generating the diagram. We note that some of the points are multiple; thus, in diagrams 3, 6, 8, 10 all points are simple, except for the central point in 8 which is double. We can separate the two points there by the value of the operator  $\sum_{a} T_a^2$ .

EXERCISE: Reconstruct, from a single point with maximum  $t_3$ , the diagrams for the representations 3, 6, 8, 10;  $3^*$ ,  $6^*$ ,  $10^*$  shown in previous figures.

EXERCISE: Arrange the baryons with spin 1/2, n, p,  $\Sigma$ s,  $\Lambda$  and  $\Xi$ s into an SU(3) octet; and the spin 3/2 resonances ( $\Delta$ s, etc.) into a decuplet.

# 4.3. Products of representations. The Peter-Weyl theorem and the Clebsch-Gordan coefficients. Product of representations of SU(2)

Let us label the irreducible unitary representations of a compact group G as  $D^{(l)}(g)$ . We then have:

THEOREM (PETER-WEYL).

The set of functions  $D^{(l)}_{ik}(g)$  forms a complete orthonormal basis in the space  $L^2(G)$  with respect to the Haar measure  $\mu$ , normalized to  $\int_G d\mu(g) = 1$ . That is to say, one has

$$\int_{G} d\mu(g) D_{ik}^{(l)}(g)^* D_{i'k'}^{(l')}(g) = \delta_{ll'} \delta_{ii'} \delta_{kk'}$$

and any function  $\phi(g)$  may be expanded in this basis.

For the proof, see Naimark (1959) or Chevalley (1946).

If we consider now the tensor product of two unitary, finite dimensional representations of  $\mathbf{A}_1$ ,  $D^{(l_1)} \otimes D^{(l_2)}$ , it will be reducible in general. The Peter–Weyl theorem guarantees that we can expand it as a direct sum of irreducible representations

$$D^{(l_1)} \otimes D^{(l_2)} = \bigoplus_{l} D^{(l)}.$$

For the individual states we then find

$$|\psi^{(l_1)}\rangle \otimes |\psi^{(l_2)}\rangle = \sum_{l,\phi^{(l)}} C(\phi^{(l)};\psi^{(l_1)},\psi^{(l_2)}) \, |\phi^{(l)}\rangle.$$

The coefficients  $C(\phi^{(l)}; \psi^{(l_1)}, \psi^{(l_2)})$  are called *Clebsch–Gordan* coefficients and we will show how to calculate them in simple cases; here we start with SU(2) (actually, with  $\mathbf{A}_1$ ).

We consider two representations D', D'', corresponding to the numbers t', t'', and denote by  $T'_a$ ,  $T''_a$  to the operators that represent the Lie algebra in each of the two spaces. We will label the corresponding states as

$$|t',t_3'\rangle\otimes|t'',t_3''\rangle.$$

The operator  $T_3$  corresponding to the product representation is obviously

$$T_3 = T_3' + T_3''$$

hence its possible eigenvalues are  $t_3' + t_3''$ . It is also clear that there is only one state with maximum value of  $T_3$ , viz.,  $|t',t'\rangle \otimes |t'',t''\rangle$ , for which  $t_3 = t' + t''$ .

Instead of considering the product  $D' \otimes D''$ , we could project it on the possible irreducible representations that it contains,  $D^{(t)}$ . We would than have a basis

$$|t,t_3\rangle$$
.

By using the commutation relations one can verify the relations

$$\mathbf{T}'\mathbf{T}'' = \left\{ t(t+1) - t'(t'+1) - t''(t''+1) \right\}$$

$$\mathbf{T}''\mathbf{T}' = \left\{ t(t+1) + t'(t'+1) - t''(t''+1) \right\}.$$
(1)

Let us now find the possible values of t, and the Clebsch–Gordan coefficients. First of all, we have that the maximum possible value of  $t_3$  is t' + t''; hence the product  $D' \otimes D''$  contains the representation characterized by such t. Then, we start with the state

$$|t'+t'',t'+t''\rangle=|t',t'\rangle\otimes|t'',t''\rangle.$$

We then apply  $T_{-}$  to this state. On one hand,

$$T_{-}|t'+t'',t'+t''\rangle = \sqrt{t'+t''}|t'+t'',t'+t''-1\rangle,$$

and, on the other,

$$T_{-}|t'+t'',t'+t''\rangle = T'_{-}|t',t'\rangle \otimes |t'',t''\rangle + |t',t'\rangle \otimes T''_{-}|t'',t''\rangle$$
$$= \sqrt{t'}|t',t'-1\rangle \otimes |t'',t''\rangle + \sqrt{t''}|t',t'\rangle \otimes |t'',t''-1\rangle$$

and we have used Eq. (3) in Sect. 4.1. Equating,

$$|t'+t'',t'+t''-1\rangle = \sqrt{\frac{t'}{t'+t''}}|t',t'-1\rangle \otimes |t'',t''\rangle + \sqrt{\frac{t''}{t'+t''}}|t',t'\rangle \otimes |t'',t''-1\rangle$$
(2)

and, iterating the procedure, we would find all the states

$$|t'+t'',t_3\rangle$$
,  $t_3=t'+t'',t'+t''-1,\ldots,-(t'+t'')$ .

The vector  $|t'+t'',t'+t''-1\rangle$  is not the only one with  $t_3=t'+t''-1$ . In fact, this value of  $t_3$  may be obtained adding t' and t''-1 or t'-1 and t'': we also have the combination

$$|t'+t'',t'+t''-1\rangle_{\perp} = \sqrt{\frac{t''}{t'+t''}}|t',t'-1\rangle \otimes |t'',t''\rangle - \sqrt{\frac{t'}{t'+t''}}|t',t'\rangle \otimes |t'',t''-1\rangle.$$

which is orthogonal to the one above. [We have fixed the phases so that the corresponding Clebsch-Gordan is real and, for the rest, followed the standard conventions of Condon and Shortley (1951).]

If we applied  $T_+$  to this state we would get zero: which means that it corresponds to a representation with t = t' + t'': we can write above equality as

$$|t'+t'',t'+t''-1\rangle_{\perp} \equiv |t'+t'',t'+t''-1\rangle = \sqrt{\frac{t''}{t'+t''}}|t',t'-1\rangle \otimes |t'',t''\rangle - \sqrt{\frac{t'}{t'+t''}}|t',t'\rangle \otimes |t'',t''-1\rangle.$$

Applying repeatedly  $T_{-}$  to this state, we would generate all the states

$$|t' + t'' - 1, t_3\rangle$$

in terms of the  $|t', t_3'\rangle \otimes |t'', t_3''\rangle$ .

We may then go to the states with  $t_3 = t' + t'' - 2$ . They can be obtained in three ways; two correspond to states already constructed. The third is obtained by taking a combination orthogonal to the other two. We can then continue the process (in which we evaluate all the Clebsch–Gordan coefficients) and find that

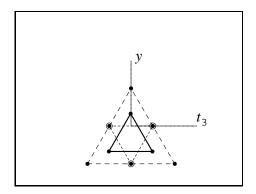
$$D' \otimes D'' = \bigoplus_{t=|t'-t''|}^{t=t'+t''} D^{(t)}.$$

The lower limit is obtained by remarking that, in the direct product basis we have (2t'+1)(2t''+1), states while in the direct sum basis we have  $\sum_{t_{\min}}^{t'+t''}(2t+1)$ : equality is only possible if  $t_{\min}=|t'-t''|$ .

Explicit expressions for the representations of SU(2) and for their Clebsch–Gordan coefficients may be found in Wigner (1959); the book of Condon and Shortley (1951) contains a large number of properties and applications of products of representations of SU(2).

# 4.4. Products of representations of $A_2$

The most powerful method for multiplying (and, indeed, constructing) representations of the unitary groups is the tensor method; we will describe it below. Here we will follow a method similar to that used for SU(2). If we have two irreducible representations of  $\mathbf{A}_2$ , D', D'', with diagrams  $\mathcal{D}'$ ,  $\mathcal{D}''$ , the  $t_3$  and y quantum numbers of  $D = D' \times D''$  must be such that they are obtained by adding the corresponding quantum numbers of D', D'':  $t_3 = t_3' + t_3''$ , y = y' + y''. Hence, the diagrams contained in the product representation must be contained in the diagram obtained by putting the center of the diagram  $\mathcal{D}'$  on each of the points of  $\mathcal{D}''$ . The array of points so obtained may be resolved into the different diagrams for the irreducible representations that we have generated in a previous section. Thus, for example, multiplying  $3 \times 3^*$  one recognizes the superposition of the diagrams for 8 and 1; and multiplying  $3 \times 3$  we get an array that can be resolved into the superposition of the diagrams for 6 and  $3^*$  (see figure).



$$3 \times 3 = 3^* + 6.$$

EXERCISE: Verify that  $3 \times 3 \times 3 = 1 + 8 + 8 + 10$ . What is the result of  $8 \times 8$ ?

The values of the Clebsch–Gordan coefficients can be obtained as for products of representations of  $\mathbf{A}_1$ , starting with the state in  $D' \times D''$  with largest  $t_3$  and generating all the other states by applying the  $T_{\pm}$ ,  $U_{\pm}$ ,  $V_{\pm}$ . This is a very cumbersome procedure; we will not give more details.

EXERCISE: Assume that the particles in the 3 representation of SU(3) are the quarks u, d, s. Identify the mesons contained in the product  $3 \times 3^*$  depending on the spin being 0 or 1; consider that the quarks are in a relative S-wave.

A detailed description of the representations of  $A_2$ , and their Clebsch–Gordan coefficients, may be found in the treatise of Hamermesh (1963) and, especially, in the review of de Swart (1963).

#### §5. The tensor method for unitary groups, and the permutation group

#### 5.1. SU(n) tensors

SU(n) tensors are the obvious generalization of ordinary tensors. A SU(n) tensor of rank r is a set of complex numbers, with r indices:  $\psi_{a_1,...,a_r}$ , and the  $a_i$  vary from 1 to n. They are assumed to transform,

<sup>&</sup>lt;sup>8</sup> We will henceforth simplify the notation by using simple multiplication sign,  $\times$ , instead of the  $\otimes$  one, for tensor products, and simple sum signs, + instead of  $\oplus$ , when there is no danger of confusion.

<sup>&</sup>lt;sup>9</sup> All the algebraic developments that we will give for SU(n) can be extended to SL(n,C) tensors in a straightforward manner. The tensor analysis of SL(n,C) [indeed, of GL(n,C)] may be found in Hamermesh (1963).

under unimodular unitary matrices U, as

$$U: \psi_{a_1,\dots,a_r} \to \psi_{U;a_1,\dots,a_r} \equiv \sum_{a'_1,\dots,a'_r} U_{a_1,a'_1} \cdots U_{a_r,a'_r} \psi_{a'_1,\dots,a'_r}. \tag{1}$$

We say that this is a *covariant* tensor. If instead we had an object  $\psi^{a_1,...,a_r}$  with the transformation law

$$U: \psi^{a_1,\dots,a_r} \to \psi^{U;a_1,\dots,a_r} \equiv \sum_{a'_1,\dots,a'_r} U^*_{a_1,a'_1} \cdots U^*_{a_r,a'_r} \psi^{a'_1,\dots,a'_r}$$
(2)

we would say that the tensor is *contravariant*. We will write contravariant tensors with superindices. Another common notation is to put dots on contravariant indices, so we would have  $\psi^{a_1,...,a_r} \equiv \psi_{\dot{a}_1,...,\dot{a}_r}$ . We will here use the upper indices notation. It is also clear that tensors provide a representation of the group SU(n), in general reducible.

Because the U are unitary, we obviously have

$$\sum_{a_1,\dots,a_r} \psi^{a_1,\dots,a_r} \psi_{a_1,\dots,a_r} = \text{scalar invariant}.$$

More generally, we may define an invariant scalar product of tensors  $\psi$ ,  $\phi$  with the same rank by

$$\langle \psi, \phi \rangle \equiv \sum_{a_1, \dots, a_r} \psi_{a_1, \dots, a_r}^* \phi_{a_1, \dots, a_r}.$$

It is also easy to verify that the Levi-Cività tensor in n dimensions,  $\epsilon_{a_1,...,a_n}$  is an invariant tensor (of rank n). It can also be considered a contravariant tensor, writing

$$\epsilon^{a_1,\dots,a_n} \equiv \epsilon_{a_1,\dots,a_n}.$$

It and the Kronecker delta  $\delta_a^b$  (or products thereof) are the only invariant numerical tensors. The proof is left as an exercise.

EXERCISE: Prove that, for any nonsingular matrix S,

$$\sum_{a'_1, \dots, a'_n} S_{a_1 a'_1} \dots S_{a_n a'_n} \epsilon_{a'_1, \dots, a'_n} = (\det S) \epsilon_{a_1, \dots, a_n}.$$

The unitarity of the U can be used to prove the following result: if  $\psi_{a_1,...,a_r}$  is a covariant tensor of rank r, then

$$\psi^{a_{r+1},\dots,a_n} = \sum_{a_1,\dots,a_r} \epsilon^{a_1,\dots,a_n} \psi_{a_1,\dots,a_r}$$
 (3)

is a contravariant tensor of rank n-r.

We could also construct mixed tensors (the Kronecker delta is one example) with r subindices and s superindices,  $\psi_{a_1,\dots,a_r}^{a_{r+1},\dots,a_{r+s}}$ ; but this is not more general in the sense that we can use (3) to reduce them to e.g. covariant tensors, which are the ones that we will (mostly) consider henceforth.

An important property of the tensor representations is that the permutations of the indices commute with the SU(n) transformations. This occurs because all the U in Eq. (1) are the same. We can thus classify tensors according to their symmetric properties under the permutation group, and this classification will be SU(n) invariant: this will allow us to explicitly construct all the irreducible representations of SU(n). For example, consider a tensor of rank 2,  $\psi_{ab}$ . We may split it as

$$\psi_{ab} = \frac{1}{2} \left\{ \psi_{ab}^{\mathrm{S}} + \psi_{ab}^{\mathrm{A}} \right\}$$

where the symmetrized (S) or antisymmetrized (A) combinations are

$$\psi_{ab}^{S} = \psi_{ab} + \psi_{ba}, \quad \psi_{ab}^{A} = \psi_{ab} - \psi_{ba}.$$

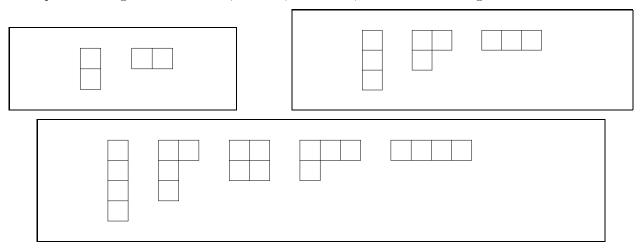
Both  $\psi_{ab}^{{\rm S,A}}$  are invariant under SU(n) transformations.

Because of this, the problem of constructing and multiplying tensor representations is related to that of constructing the irreducible representations of the permutation group, which we will discuss below.

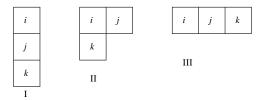
#### 5.2. The tensor representations of the SU(n) group. Young tableaux and patterns

The classification and product of representations of the SU(n) groups with the tensor method uses the technique of the so-called *Young tableaux*. This technique was first developed for the permutation group; it may be found applied to it in Hamermesh (1963). Here we will develop it directly for representations of SU(n). The results found are valid *tels quels* for SL(n,C).

Let us consider a tensor  $\psi_{i_1,...,i_r}$ , where some of the indices may be repeated, and we assume that there are n different indices. This is what we would have if  $\psi_{i_1,...,i_r}$  was a general tensor under SU(n). We first define the Young frames as arrays of r equal squares (that we take of unit length) into rows, left justified. If there are  $\rho$  rows and their lengths are  $l_1, \ldots, l_{\rho}$ , then we require  $l_1 \geq l_2 \geq \ldots \geq l_{\rho}$ . Examples of Young frames for r = 2, 3 and 4, and  $n \geq 4$ , are shown in the figures below.



Once we have a Young frame, we define a Young tableau by putting an index among the  $i_1, \ldots, i_r$  into each frame. Thus, from the frames in the second figure above we obtain the following tableaux:



EXERCISE: Fill in the other two sets of frames to get the corresponding Young tableaux.

When putting actual numbers (in lieu of the abstract indices ijk) in a Young tableau, we have a number of possibilities depending on which numbers we choose. We say that a tableau with actual numbers is a  $standard\ tableau$  if the value of the indices does not decrease as we go to the right along a row, for all rows, and it does increase as we go downwards along a column, for all columns.

For typographical reasons, as well as for ease when making hand drawings, one can replace the Young frames and tableaux by *Young patterns*, as follows. Instead of the boxes of a Young frame, we put an array of dots. And, instead of the indices inside boxes in a tableau, we merely put the indices instead of the dots in the corresponding array. Thus, the pattern corresponding to the frame



is the array

Likewise, to the tableau



corresponds the pattern  $\begin{pmatrix} i & j \\ k \end{pmatrix}$ .

With each Young tableau we associate the following operation on a tensor,  $\psi_{i_1,...,i_r}$ :

- 1.- Indices appearing in the same column of the tableau are antisymmetrized. This gives a tensor, sum of the several tensors that are generated by the symmetrization.
- 2.- Subsequently, in the sum just obtained, indices appearing in the same row (of the tableau) are symmetrized.

Thus, from the three Young tableaux above we find the following tensors:

$$\mathcal{Y}^{\mathrm{I}}\psi_{ijk} \equiv \psi_{ijk}^{\mathrm{I}} = \psi_{ijk} - \psi_{ikj} - \psi_{jik} + \psi_{jki} - \psi_{kij} + \psi_{kji};$$

$$\mathcal{Y}^{\mathrm{II}}\psi_{ijk} \equiv \psi_{ijk}^{\mathrm{II}} = \psi_{ijk} + \psi_{jik} - \psi_{kji} - \psi_{kij};$$

$$\mathcal{Y}^{\mathrm{III}}\psi_{ijk} \equiv \psi_{ijk}^{\mathrm{III}} = \psi_{ijk} + \psi_{ikj} + \psi_{jik} + \psi_{jki} + \psi_{kij} + \psi_{kji}.$$

$$(1)$$

Exercise: Show that, for A, B = I, II, III,

$$\mathcal{Y}^{A}\left(\mathcal{Y}^{B}\psi_{ijk}\right) = (Const.) \times \delta_{AB}\left(\mathcal{Y}^{B}\psi_{ijk}\right),$$

i.e., the operations  $\mathcal{Y}^{I}$ ,  $\mathcal{Y}^{II}$ ,  $\mathcal{Y}^{III}$  are mutually orthogonal. Evaluate the constants above.

| $i_1$ | $i_2$ |
|-------|-------|
| $i_3$ | $i_4$ |

The tableau  $\mathcal{Y}$ .

As a second example of Young tableaux we apply the tableau of the figure above, that we denote by  $\mathcal{Y}$ , to the tensor  $\psi_{i_1i_2i_3i_4}$ .

First we antisymmetrize  $i_1$ ,  $i_3$ , and  $i_2$ ,  $i_4$ , and  $i_1$ ,  $i_3$  plus  $i_2$ ,  $i_4$  getting

$$\psi_{i_1 i_2 i_3 i_4} - \psi_{i_3 i_2 i_1 i_4} - \psi_{i_1 i_4 i_3 i_2} + \psi_{i_3 i_4 i_1 i_2}.$$

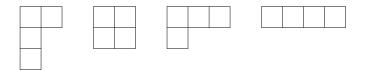
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Then, we symmetrize the result in  $i_1$ ,  $i_2$ , and  $i_3$ ,  $i_4$  and  $i_1$ ,  $i_2$  plus  $i_3$ ,  $i_4$ . The final result is then

$$\begin{split} \mathcal{Y}\psi_{i_1i_2i_3i_4} &= \psi_{i_1i_2i_3i_4} - \psi_{i_3i_2i_1i_4} - \psi_{i_1i_4i_3i_2} + \psi_{i_3i_4i_1i_2} \\ &+ \psi_{i_2i_1i_3i_4} - \psi_{i_3i_1i_2i_4} - \psi_{i_2i_4i_3i_1} + \psi_{i_3i_4i_2i_1} \\ &+ \psi_{i_1i_2i_4i_3} - \psi_{i_4i_2i_1i_3} - \psi_{i_1i_3i_4i_2} + \psi_{i_4i_3i_1i_2} \\ &+ \psi_{i_2i_1i_4i_3} - \psi_{i_4i_1i_2i_3} - \psi_{i_2i_3i_4i_1} + \psi_{i_4i_3i_2i_1}. \end{split}$$

EXERCISE: Show that, if  $n \geq 3$ , the three tensors above are irreducible under SU(n).

EXERCISE: Show that, for SU(3), the only rank four Young tableaux have the frames shown in the figure:



There is no vertical tableau with 4 or more rows for SU(3).

Let us return to the example (1). When substituting actual numbers in lieu of the ijk, we need only do so with numbers that would lead to a standard tableau. If they formed a nonstandard tableau, the result would be (after appropriate symmetrization) either zero or a combination of the  $\psi^{I,II,III}$ . We then find the following standard tableaux: for the case (I), there is only one, that of the figure.

The only standard tableau corresponding to the tensor  $\psi^{\rm I}_{ijk}$ 

For the case (II), we have 8 standard tableaux, as shown below.



The eight standard tableaux corresponding to the tensor  $\psi_{ijk}^{\text{II}}$ .

EXERCISE: Construct the 10 standard tableaux corresponding to  $\psi_{ijk}^{\text{III}}$ .

In view of these results, it follows that the tensor corresponding to (I) has a single component, i.e., it is an invariant singlet; that corresponding to (II) has 8 components (and thus the tensor is a realization of the adjoint representation) and the tensor corresponding to (III) is a decuplet. The (rather cumbersome) general formula for the dimension of the representation associated to a Young tableau may be found in Hamermesh (1963), pp. 384 ff. It is obtained by calculating how many standard tableaux exist for a given Young frame.

# 5.3. Product of representations in terms of Young tableaux

Consider two representations of SU(n), corresponding to the Young tableaux  $\mathcal{Y}$  and  $\mathcal{Y}'$ . The product of the two representations may be decomposed into irreducible representations, with corresponding Young tableaux  $\mathcal{Y}^{(l)}$ ,  $l=1,2,\ldots$ ; we remind the reader that the product is commutative. We will write this symbolically as

$$\mathcal{Y} \times \mathcal{Y}' = \mathcal{Y}^{(1)} + \mathcal{Y}^{(2)} + \cdots \tag{1}$$

We now give a procedure to find the tableaux  $\mathcal{Y}^{(l)}$ . We do this in steps.

Step 1. Label the boxes of tableau  $\mathcal{Y}'$  by putting the same index, a in all the boxes in the first row; the same index, b, in all the boxes in the second row; the same index c in all the boxes of the third row, etc. Note that we assume the tableau  $\mathcal{Y}'$  to be standard, so we must have  $a < b < c, \cdots$ 

Step 2. Glue all boxes labeled a to the tableau  $\mathcal{Y}$ , in all possible combinations, in such a way that you form Young tableaux, but so that two identical letters do *not* appear in the same column. In this way one finds a set of tableaux,

$$\mathcal{Y}_1, \ \mathcal{Y}_2, \ \dots, \mathcal{Y}_{J_1}. \tag{2}$$

Step 3. Glue the boxes labeled b to the tableaux in (2), with the same conditions as in Step 2, to get a second set of tableaux,

$$\mathcal{Y}_{1,1}, \mathcal{Y}_{1,2}, \dots, \mathcal{Y}_{1,J_2}$$
 $\dots \dots$ 
 $\mathcal{Y}_{J_1,1}, \mathcal{Y}_{J_1,2}, \dots, \mathcal{Y}_{J_1,J_2}.$ 
(3)

Step 4. Do the same with the boxes labeled c, etc.

Step 5. Once finished the process, consider each of the ensuing tableaux. For a given one, form the sequence of symbols  $a, b, \ldots$  by starting, from right to left, from the upper row, then continuing along the second row, etc. This will give a sequence aabcc... If the sequence is such that, to the left of any of its symbols, there are more a than b, of b than c, etc., then the tableau is to be rejected.

Step 6. Remove the symbols  $a, b, c, \ldots$  from the remaining tableaux (keeping the boxes). These form the set

$$\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{J_1}$$
.

The whole procedure is best seen with an example. Consider the product of the tableau of the figure by itself.



According to the rules laid before, we must form the tableaux of the figure below:

<sup>&</sup>lt;sup>10</sup>For reasons that escape the present author, such a sequence is said not to form a lattice permutation; cf. Hamermesh (1963), p. 198.



Instead, we will use the pattern representation and thus have the two following patterns:

• • a a a b

By glueing the "boxes" with a to the first pattern, we get the equivalent of (2),

Note that the array

• • • a a

need not be considered, as it vanishes under antisymmetrization.

We then glue the box containing b to [1] in all (consistent) possible manners, finding

Likewise, we glue the box containing b to [2] and get the patterns

 $[2,1]: \begin{array}{cccc} \bullet & \bullet & a \\ \bullet & a & b \end{array} \qquad [2,2]: \begin{array}{cccc} \bullet & a \\ b \end{array} \qquad (4ii)$ 

With [3], we have

Finally, from [4],

Among the patterns so obtained, there appear some that we rejected because they do not form a "lattice permutation"; they are, for example, the patterns

• • a a b • a b In both cases, the procedure of Step 5 gives the sequence baa, which has too many as to the right of b. The set of tableaux obtained by replacing the letters in Eqs. (4) by dots gives the full set of

The set of tableaux obtained by replacing the letters in Eqs. (4) by dots gives the full set of tableaux that appear in the decomposition (1). Note that the pattern

• • •

appears twice, as it can be reached by two independent paths, [2,1] and [3,1]. This indicates that the corresponding representation will also appear twice in the reduction of the product.

#### 5.4. Product of representations in the tensor formalism

We will consider in detail the case SU(3); this will indicate the generalization to higher groups.

First of all, we will construct all representations by composing the fundamental representation with itself. We consider tensors made up of products of vectors  $u_i^{(\alpha)}$  (the index i denotes the components) in the 3-dimensional complex space,  $u^{(\alpha)} \in \mathbb{C}^3$ : thus, we have a rank 1 tensor,  $u_i$ ; rank two tensors,  $u_iv_j$ ; rank three tensors,  $u_iv_jw_k$ ; rank four tensors,  $u_iu_jv_kw_l$ ; ...; rank r tensors  $u_{i_1}^{(1)}u_{i_2}^{(2)}\dots u_{i_r}^{(r)}$ . It is not difficult to prove that forming linear combinations of these tensors we generate all the tensors, i.e., the tensors  $u_{i_1}^{(1)}u_{i_2}^{(2)}\dots u_{i_r}^{(r)}$  form a complete basis. In particular, putting them in Young tableaux we generate all the irreducible tensors. Thus we have:

$$\begin{aligned} & \text{Rank 1: } T_i^{(3)} = u_i \quad [3]. \\ & \text{Rank 2: } T_{ij}^{(3^*)} = \frac{1}{\sqrt{2}} \left( u_i v_j - u_j v_i \right) \quad [3^*]; \quad T_{ij}^{(6)} = \frac{1}{\sqrt{2}} \left( u_i v_j + u_j v_i \right) \quad [6]. \\ & \text{Rank 3: } \\ & T_{ijk}^{(1)} = \frac{1}{\sqrt{6}} \left( u_i v_j w_k - u_j v_i w_k - u_i v_k w_j + u_k v_i w_j - u_k v_j w_i + u_j v_k w_i \right) \quad [1]; \\ & T_{ijk}^{(8)} = \frac{1}{\sqrt{4}} \left( u_i v_k w_j - u_k v_i w_j + u_k v_j w_i - u_j v_k w_i \right) \quad [8]; \\ & T_{ijk}^{(10)} = \frac{1}{\sqrt{6}} \left( u_i v_j w_k + u_j v_i w_k + u_i v_k w_j + u_k v_i w_j + u_k v_j w_i + u_j v_k w_i \right) \quad [10]. \end{aligned}$$

etc. We have arranged the numerical factors so that, if the  $u, v, \ldots$  are of unit length, so are the higher rank tensors. In brackets we have put the dimensionality of each representation.

EXERCISES: Identify these tensors with the corresponding Young tableaux. Check that, if we assume the u, v, w to be an orthonormal set, so are the tensors  $T^{(I)}$  above.

Instead of multiplying abstract representations, it is much simpler to multiply these explicit representations and merely project them in the ones we have. We show this with an explicit example. We start by multiplying  $3 \times 3$  and find the tensor  $u_i v_i$ ; it can be expanded into rank 2 tensors trivially,

$$u_i v_j = \frac{1}{\sqrt{2}} T_{ij}^{(3^*)} + \frac{1}{\sqrt{2}} T_{ij}^{(6)},$$

hence we recover (with Clebsch–Gordan coefficients included!) the result  $3 \times 3 = 3^* + 6$ . If we multiply again by a vector we find

$$T_{ij}^{(3^*)}w_k = \frac{1}{\sqrt{2}}(u_iv_j - u_jv_i)w_k$$

and it is easy to see that one has

$$T_{ij}^{(3^*)} w_k = \frac{1}{\sqrt{2}} \left( \sqrt{6} \, T_{ijk}^{(1)} + \sqrt{4} \, T_{ijk}^{(8)} \right) :$$

thus, we find  $3^* \times 3 = 1 + 8$ , again including the Clebsch–Gordan coefficients. This expansion can be done in a systematic manner by applying the Young tableaux of rank 3 to the tensor  $T_{ij}^{(3^*)}w_k = \frac{1}{\sqrt{2}}(u_iv_j - u_jv_i)w_k$ .

EXERCISES: i) Decompose the product,  $T_{ij}^{(6)}w_k$ . ii) Form baryons from the u, d, s quarks, taking into account the colour quantum number (which generates a SU(3) invariance), including the requirement of colour singlet for "physical" hadrons.

The book of Cheng and Li (1984) contains a readable elementary description of the SU(n) groups, their representations and their multiplication, which the reader may find sufficient for most physical applications (although, of course, the basic reference is the text of Hamermesh, 1963).

EXERCISE: By going to Lie algebras, and then to the complexified Lie algebras, show that everything that has been said for the Young tableaux-tensor formalism of SU(n) holds also for GL(n,C).

#### 5.5. Representations of the permutation group

The method of Young tableaux allows us also to find the representations of the permutation group. We will here only give a few results, without proofs; a detailed treatment may be found in the books of Weyl (1946) and Hamermesh (1963).

Consider the permutation group of n elements,  $\Pi_n$ , and take all the Young tableaux of rank n. We may interpret the permutations as acting on the indices in the Young tableaux. For each Young tableau,  $\mathcal{Y}$ , we assign a representation of  $\Pi_n$  as follows. Denote by p to the subgroup of all permutations that leave each box in the same row (but not necessarily in the same column) that it occupied before applying the permutation; and denote by q to the subgroup of permutations which move the boxes only inside the same column. It is evident that the sets p, q will be different for different tableaux. We then introduce the function  $\phi(P)$ ,  $P \in \Pi_n$  by requiring

$$\phi(P) = \begin{cases} 0, \text{ when } P \text{ is not contained in the product } pq; \\ \delta_P \text{ if } P = P_p Q_q \text{ with } P_p \in p, \ Q_q \in q. \end{cases}$$

Here  $\delta_P$  is the parity of the permutation P. The functions of the form

$$f(Q) = \sum_{P} a_{P} \phi(QP)$$

with  $a_P$  real numbers generate a linear space, that we may call  $\mathfrak{H}(\mathcal{Y})$ , associated with the given Young tableau. We finally define the operator D(S) that represents the permutation P on the functions  $\mathfrak{H}(\mathcal{Y})$  by

$$D(S): f(P) \to f(SP).$$

It is easy to verify that these operators form a representation of  $\Pi_n$ . Although it is more difficult, it can also be shown that the representation is irreducible, that the representations corresponding to different tableaux are inequivalent, and that they exhaust the set of all representations of  $\Pi_n$ .

A more detailed discussion of representations of the permutation group may be found in the treatises of Weyl (1946), Hammermesh (1963) or Lyubarskii (1960).

# 6. Relativistic invariance. The Lorentz group

# 6.1. Lorentz transformations. Normal parameters

In relativity theory the passage from one inertial system to another one, moving with respect to it with speed  $\mathbf{v}$ , is given by the *Lorentz boosts* (or accelerations). Starting with the case where  $\mathbf{v}$  is parallel to the OZ axis, these boosts are given by<sup>11</sup>

$$\begin{split} x &\to x, \quad y \to y, \\ z &\to \frac{1}{\sqrt{1-v^2/c^2}}(z+vt), \\ t &\to \frac{1}{\sqrt{1-v^2/c^2}}(t+\frac{v}{c^2}z). \end{split}$$

Here and henceforth c will denote the speed of light.

We also write this with shorthand notation

$$\mathbf{r} \to L(\mathbf{v}_z)\mathbf{r}, \quad t \to L(\mathbf{v}_z)t.$$

(This really is shorthand:  $L(\mathbf{v})\mathbf{r}$  depends also on t, and not only on  $\mathbf{v}$ ,  $\mathbf{r}$ ; likewise,  $L(\mathbf{v})t$  depends also on  $\mathbf{r}$ .) For  $\mathbf{v}$  directed in an arbitrary way, we use the following trick. Let  $R(\mathbf{z} \to \mathbf{v})$  be a rotation carrying the OZ axis over  $\mathbf{v}$ . For example, we may choose

$$R(\mathbf{z} \to \mathbf{v}) = R(\boldsymbol{\alpha}), \ R(\boldsymbol{\alpha})\mathbf{z} = \mathbf{v}/|\mathbf{v}|,$$

with  $\mathbf{z}$  the unit vector along OZ and

$$\cos \alpha = v_3/v$$
,  $\boldsymbol{\alpha} = (\alpha/v)(\sin \alpha)\mathbf{z} \times \mathbf{v}$ .

Denoting by  $L(\mathbf{v})$  the Lorentz boost with velocity  $\mathbf{v}$ , we define

$$L(\mathbf{v}) = R(\mathbf{z} \to \mathbf{v})L(\mathbf{v}_z)R^{-1}(\mathbf{z} \to \mathbf{v}),$$

where  $\mathbf{v}_z$  is a vector of length v along OZ. Using the explicit formulas for  $L(\mathbf{v}_z)$  and R we find that

$$\mathbf{r} \to L(\mathbf{v})\mathbf{r} = \mathbf{r} - \frac{\mathbf{v}\mathbf{r}}{v^2}\mathbf{v} + \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \left(\frac{1}{v^2}\mathbf{r}\mathbf{v} + t\right)\mathbf{v},$$

$$t \to L(\mathbf{v})t = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \left(t + \frac{\mathbf{v}\mathbf{r}}{c^2}\right).$$

EXERCISE: Verify that, for  $t, t', \mathbf{r}, \mathbf{r}', \mathbf{v}$  arbitrary,

$$c^{2}(L(\mathbf{v})t)(L(\mathbf{v})t') - (L(\mathbf{v})\mathbf{r})(L(\mathbf{v})\mathbf{r}') = c^{2}tt' - \mathbf{r}\mathbf{r}',$$

i.e., that under Lorentz boosts one has

$$c^2 t t' - \mathbf{r} \mathbf{r}' = \text{invariant.}$$

<sup>&</sup>lt;sup>11</sup>The contents of this and the following sections is adapted from the author's textbook on relativistic quantum mechanics, Ynduráin (1996).

The parameters  $\mathbf{v}$  are now *not* normal; it is not true that the product of boosts by  $\mathbf{v}$ ,  $\mathbf{v}'$  is the boost by  $\mathbf{v} + \mathbf{v}'$  (which does not even exist if  $|\mathbf{v} + \mathbf{v}'| \ge c$ ). It is then convenient to use other parameters, which will be denoted by  $\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$  such that, whenever  $\boldsymbol{\xi}$  and  $\boldsymbol{\eta}$  are parallel,

$$L(\boldsymbol{\xi})L(\boldsymbol{\eta}) = L(\boldsymbol{\xi} + \boldsymbol{\eta}).$$

Note that we use the same notation for  $L(\mathbf{v})$  and  $L(\boldsymbol{\xi})$ ; the context, and the latin/greek characters should be enough to indicate whether we are using velocities or the new normal parameters.

Let us choose  $\xi$  along OZ. If we write

$$L(\boldsymbol{\xi})z = A(\xi)z + B(\xi)ct, \quad L(\boldsymbol{\xi})t = \frac{1}{c}C(\xi)z + D(\xi)t,$$

where A, B, C, D are functions to be determined, we get the consistency conditions

$$AB = CD$$
,  $A^2 - C^2 = D^2 - B^2 = 1$ ,

so that we can find  $\varphi(\boldsymbol{\xi})$  verifying

$$A = D = \cosh \varphi(\xi), \quad B = C = \sinh \varphi(\xi).$$

This relation implies that

$$\cosh(\varphi(\boldsymbol{\xi}) + \varphi(\boldsymbol{\eta})) = \cosh\varphi(\boldsymbol{\xi})\cosh\varphi(\boldsymbol{\eta}) + \sinh\varphi(\boldsymbol{\xi})\sinh\varphi(\boldsymbol{\eta})$$

$$\sinh(\varphi(\boldsymbol{\xi}) + \varphi(\boldsymbol{\eta})) = \cosh\varphi(\boldsymbol{\xi}) \sinh\varphi(\boldsymbol{\eta}) + \sinh\varphi(\boldsymbol{\xi}) \cosh\varphi(\boldsymbol{\eta}),$$

and we can thus choose  $\varphi(\xi) = \xi \equiv |\xi|$ . Finally

$$x \to x, \quad y \to y,$$
  
 $z \to (\cosh \xi)z + (\sinh \xi)ct,$   
 $t \to \frac{1}{c}(\sinh \xi)z + (\cosh \xi)t, \quad \boldsymbol{\xi} \parallel OZ.$ 

The relation between the  $\xi$  and  $\mathbf{v}$  is found by comparison of these relations:

$$\cos \xi = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}, \quad \sinh \xi = \frac{|\mathbf{v}|}{c} \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}, \quad \boldsymbol{\xi} \parallel \mathbf{v}.$$

 $\pmb{\xi}$  is sometimes called the *rapidity*. For a boost along an arbitrary  $\pmb{\xi}$ , we find

$$\mathbf{r} \to L(\boldsymbol{\xi})\mathbf{r} = \mathbf{r} - \frac{\boldsymbol{\xi}\mathbf{r}}{\xi^2}\boldsymbol{\xi} + \frac{1}{\xi} \left\{ (\cosh \xi) \frac{\boldsymbol{\xi}\mathbf{r}}{\xi} \boldsymbol{\xi} + c(\sinh \xi) t \boldsymbol{\xi} \right\},$$
$$t \to L(\boldsymbol{\xi})t = (\cosh \xi)t + \frac{1}{c} \frac{\sinh \xi}{\xi} \boldsymbol{\xi}\mathbf{r}.$$

For speeds small compared with c,

$$\boldsymbol{\xi} \simeq \mathbf{v}/c$$
,

and a Lorentz boost coincides with a Galilean boost.

The transformations  $\Lambda$  of the set  $(\mathbf{r},t)$  obtained by applying rotations and Lorentz boosts as a product,

$$\Lambda = LR$$

are called *Lorentz transformations*. As we will see in the next sections, they form a group, called the *Lorentz group*, or, sometimes, and for reasons that will be apparent presently, the *orthochronous*, proper *Lorentz group*.

If we include possible products by space,  $I_s$ , and time,  $I_t$ , reversals,

$$I_s: \mathbf{r} \to -\mathbf{r}, t \to t; I_t: \mathbf{r} \to \mathbf{r}, t \to -t,$$

we obtain a set (which is also a group) called the *full Lorentz group*. Its elements are of one of the following forms:

$$LR$$
,  $I_sLR$ ,  $I_tLR$ ,  $I_sI_tLR$ .

# 6.2. Minkowski Space. The Full Lorentz Group

As we saw in the previous section, Lorentz boosts mix space and time. A unified treatment of relativistic transformations demands that we work in a set that contains both. This is *Minkowskian spacetime* (or just *Minkowski space*). Its elements, or points, which will be denoted<sup>12</sup> by letters  $x, y, \ldots$ , are called four-vectors, and are determined by four coordinates,  $x_{\mu}, \mu = 0, 1, 2, 3$ ,

$$x \sim \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where  $x_0 = ct$  corresponds to a time coordinate and  $x_j = r_j$ , j = 1, 2, 3 are purely spatial coordinates.<sup>13</sup> We will consistently tag Minkowskian coordinates with Greek indices  $\mu, \nu, \ldots$  varying from 0 to 3; latin indices  $i, j, \ldots$  will be restricted to varying from 1 to 3. We will also denote by  $\mathbf{r}$  the spatial part of x, and x may thus also be written as

$$x \sim \begin{pmatrix} ct \\ \mathbf{r} \end{pmatrix}$$
.

At times a horizontal notation is convenient, and we write  $x \sim (ct, \mathbf{r})$ .

Lorentz boosts may be represented by  $4 \times 4$  matrices  $L, x \to Lx$ , with elements  $L_{\mu\nu}$ , so that

$$(Lx)_{\mu} = \sum_{\nu=0}^{3} L_{\mu\nu} x_{\nu};$$

explicitly, we have

$$(Lx)_0 = (\cosh \xi)x_0 + \frac{\sinh \xi}{\xi} \sum_{j=1}^3 \xi_j x_j,$$

$$1 \left( \cosh \xi \sum_{j=1}^n \frac{1}{\xi} \right) \left($$

$$(Lx)_i = x_i - \frac{1}{\xi^2} \left( \sum_j \xi_j x_j \right) \xi_i + \frac{1}{\xi} \left( \frac{\cosh \xi}{\xi} \sum_j \xi_j x_j + x_0 \sinh \xi \right) \xi_i.$$

Rotations can also be defined as transformations in Minkowski space:  $x \to Rx$ , with

$$(Rx)_{\mu} = \sum_{\nu} R_{\mu\nu} x_{\nu},$$

and

$$(Rx)_0 = x_0,$$

$$(Rx)_i = (\cos \theta)x_i + \frac{1 - \cos \theta}{\theta^2} \left(\sum_j \theta_j x_j\right) \theta_i + \frac{\sin \theta}{\theta} \sum_{kl} \epsilon_{ikl} \theta_k x_l.$$

Here  $\epsilon_{ikl}$  is the Levi–Cività symbol.

 $<sup>^{12}\</sup>mathrm{Our}$  conventions are not universal, although they are certainly quite common.

<sup>&</sup>lt;sup>13</sup>For the sake of definiteness, we work here with the space-time Minkowski space; the considerations are of course also valid for the energy-momentum Minkowski space of vectors p, with  $\mathbf{p}$  the momentum and  $p_0 = E/c$ , E the energy.

The transformations L, R leave invariant the quadratic form  $x \cdot y$  defined by

$$x \cdot y \equiv x_0 y_0 - \sum_{j=1}^3 x_j y_j.$$

This form is known as the *Minkowski* (pseudo) scalar product, and can be also written in terms of the (pseudo) metric tensor G, with components  $g_{\mu\nu}$ ,

$$g_{\mu\nu} = 0, \mu \neq \nu, \quad g_{\mu\nu} = 1, \mu = \nu = 0, \quad g_{\mu\nu} = -1, \mu = \nu \neq 0.$$

Indeed,

$$x \cdot y = \sum_{\mu\nu} g_{\mu\nu} x_{\mu} y_{\nu} = \sum_{\mu} g_{\mu\mu} x_{\mu} y_{\mu} = x^{\mathrm{T}} G y.$$

In the last expression, x, y are taken to be matrices. The *Minkowski square*, denoted by  $x^2$  if there is no danger of confusion, is defined as  $x^2 \equiv x \cdot x$ .

As stated above, one can verify, by direct computation, that, when  $\Lambda = LR$  for any L, R, then, for every pair x, y,

$$(\Lambda x) \cdot (\Lambda y) = x \cdot y.$$

In terms of the metric tensor,

$$\Lambda^{\mathrm{T}}G\Lambda = G.$$

These relations suggest that we define a group, called the *full Lorentz group*, and denoted by  $\overline{\mathcal{L}}$ , to be the set of all matrices  $\overline{\Lambda}$  such that

$$\overline{\Lambda}^{\mathrm{T}} G \overline{\Lambda} = G.$$

It is obvious that such  $\overline{\Lambda}$  form a group, and it is easy to verify that one also has

$$\overline{\Lambda}G\overline{\Lambda}^{\mathrm{T}} = G.$$

Let us take determinants in  $\Lambda^{\mathrm{T}}G\Lambda = G$ . We find that  $(\det \overline{\Lambda})^2 = 1$ , and hence  $\det \overline{\Lambda} = \pm 1$ . Consider space reversal, acting in Minkowski space by  $(I_s x)_0 = x_0$ ,  $(I_s x)_i = -x_i$ . Clearly,  $I_s$  is in  $\overline{\mathcal{L}}$  and moreover  $\det I_s = -1$ . If  $\overline{\Lambda}$  belongs to  $\overline{\mathcal{L}}$  and  $\det \overline{\Lambda} = -1$ , then we can write identically

$$\overline{\Lambda} = I_s(I_s\overline{\Lambda}).$$

and now  $\det(I_s\overline{\Lambda}) = +1$ . If we denote by  $\mathcal{L}_+$  to the subgroup of  $\overline{\mathcal{L}}$  consisting of matrices with determinant unity, we have just shown that  $\overline{\mathcal{L}}$  consists of matrices either in  $\mathcal{L}_+$  or products of  $I_s$  time matrices in  $\mathcal{L}_+$ .

Consider next the four-vector  $n_t$ , a unit vector along the time axis, with components  $n_{t\mu} = \delta_{\mu 0}$ . Given  $\overline{\Lambda}$  in  $\overline{\mathcal{L}}$ , we may have either  $(\overline{\Lambda}n_t)_0 > 0$  or  $(\overline{\Lambda}n_t)_0 < 0$ ; it is not possible to have  $(\overline{\Lambda}n_t)_0 = 0$ . Moreover, if  $(\overline{\Lambda}n_t)_0 > 0$  and  $(\overline{\Lambda}'n_t)_0 > 0$ , then  $(\overline{\Lambda}^{-1}n_t)_0 > 0$  and  $(\overline{\Lambda}\overline{\Lambda}'n_t)_0 > 0$ . (The proofs of these statements are left as exercises.) It then follows that the subset of  $\overline{\mathcal{L}}$  consisting of transformations  $\overline{\Lambda}$  with  $(\overline{\Lambda}n_t)_0 > 0$  forms a group, called the *orthochronous Lorentz group*, and denoted by  $\mathcal{L}^{\uparrow}$ ; the corresponding transformations preserve the arrow of time. If the matrix  $\overline{\Lambda}$  in  $\overline{\mathcal{L}}$  is such that  $(\overline{\Lambda}n_t)_0 < 0$ , then we can write identically

$$\overline{\Lambda} = I(I\overline{\Lambda}),$$

where I is the total reversal,  $I = I_t I_s$ :  $Ix \equiv -x$ . Clearly,  $(I\overline{\Lambda}n_t)_0$  is now positive. We have proved that any element of  $\overline{\mathcal{L}}$  is either an element of  $\mathcal{L}^{\uparrow}$  or a product  $I\Lambda$  with  $\Lambda$  in  $\mathcal{L}^{\uparrow}$ .

Finally, the proper, orthochronous Lorentz group  $\mathcal{L}_{+}^{\uparrow}$  (which we simply call, if there is no danger of confusion, the Lorentz group,  $\mathcal{L}$ ) is the group of matrices  $\Lambda$  such that

$$\Lambda^{\mathrm{T}}G\Lambda = G$$
,  $\det \Lambda = 1$ ,  $\Lambda_{00} > 0$ .

As we have just shown, we have that any element in  $\overline{\mathcal{L}}$ ,  $\overline{\Lambda}$  is of one of the forms

$$I_s\Lambda$$
,  $I_t\Lambda$ ,  $I_sI_t\Lambda$ ,  $\Lambda$ 

with  $\Lambda$  in  $\mathcal{L}_{+}^{\uparrow}$ .

The transformations  $I_s$ ,  $I_t$ , I are at times called improper transformations.

EXERCISE: Prove that  $\Lambda^{\mathrm{T}}G\Lambda=G$  implies that  $\Lambda n_t\neq 0$ . Solution: Consider the 00 components of  $\Lambda^{\mathrm{T}}G\Lambda=G$ , and  $\Lambda G\Lambda^{\mathrm{T}}=G$ ; then,

$$\varLambda_{00}^2 - \sum_i \varLambda_{i0}^2 = 1; \quad \varLambda_{00}^2 - \sum_i \varLambda_{0i}^2 = 1.$$

From any of these,  $|\Lambda_{00}| \ge 1$  so  $|(\Lambda n_t)_0| \ge 1$ .

EXERCISE: Show that  $\Lambda_{00} > 0$ ,  $\Lambda'_{00} > 0$  imply that  $(\Lambda \Lambda')_{00} > 0$ . Solution: Using the evaluations of the previous problem and Schwartz's inequality,

$$\left| \sum_{i} \Lambda_{0i} \Lambda'_{i0} \right| \leq \sqrt{\sum \Lambda_{0i} \Lambda_{0i}} \sqrt{\sum \Lambda'_{i0} \Lambda'_{i0}} < \Lambda_{00} \Lambda'_{00}.$$

Hence.

$$(\Lambda \Lambda')_{00} = \Lambda_{00} \Lambda'_{00} + \sum_{i} \Lambda_{0i} \Lambda'_{i0} > \Lambda_{00} \Lambda'_{00} - \left| \sum_{i} \Lambda_{0i} \Lambda'_{i0} \right| > 0.$$

EXERCISE: Show that  $\Lambda_{00} > 0$  implies that  $(\Lambda^{-1})_{00} > 0$ .

# 6.3. More on the Lorentz Group

In this section we further characterize the (orthochronous, proper) Lorentz group. We start by proving a simple, but basic, theorem.

THEOREM 1.

If R is in  $\mathcal{L}$  and  $Rn_t = n_t$ , then R is a rotation.

To prove this, we note that the condition  $Rn_t = n_t$  implies that R is of the form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \hat{R} & & \\ 0 & & & \end{pmatrix},$$

with  $\hat{R}$  a  $3 \times 3$  matrix. The condition  $R^{\mathrm{T}}GR = G$  implies that  $\hat{R}^{\mathrm{T}}\hat{R} = 1$ ; and  $\det R = +1$  implies that also  $\det \hat{R} = +1$ . Therefore,  $\hat{R} \in \mathrm{SO}(3)$ , i.e., it is a three-dimensional rotation. From now on we will denote by the same symbol R the Minkowski space transformation and the restriction  $(\hat{R})$  to ordinary three-space.

Now let  $\Lambda$  be an arbitrary transformation in  $\mathcal{L}$ , and let  $u \equiv \Lambda n_t$ . We have  $u_0 > 0$  and  $u \cdot u = 1$ . Consider the vector  $\boldsymbol{\xi}$  such that  $u_0 = \cosh |\boldsymbol{\xi}|$ ,  $|\mathbf{u}| = \sinh |\boldsymbol{\xi}|$ ; this is possible because

$$1 = u \cdot u = (u_0)^2 - |\mathbf{u}|^2 = \cosh^2 \xi - \sinh^2 \xi.$$

We choose  $\boldsymbol{\xi}$  directed along  $\mathbf{u}$ ,

$$\boldsymbol{\xi}/|\boldsymbol{\xi}| = \mathbf{u}/|\mathbf{u}|,$$

so that

$$u_0 = \cosh \xi, \quad u_i = \frac{1}{\xi} (\sinh \xi) \xi_i.$$

Using the explicit expressions for  $L(\boldsymbol{\xi})$ , we see that  $L(\boldsymbol{\xi})n_t = u$ . It follows that the transformation  $L^{-1}(\boldsymbol{\xi})\Lambda$  is such that

$$L^{-1}(\boldsymbol{\xi})\Lambda n_t = n_t,$$

so by Theorem 1,  $L^{-1}(\boldsymbol{\xi})\Lambda \equiv R$  has to be a rotation, characterized by some  $\boldsymbol{\theta}$ . We have therefore proved the following theorem:

Theorem 2.

Any (proper, orthochronous) Lorentz transformation,  $\Lambda$ , can be written as

$$\Lambda = L(\boldsymbol{\xi})R(\boldsymbol{\theta}),$$

where R is a rotation and L a Lorentz boost (the decomposition is not unique).

In particular it follows from this that the Lorentz group is a six-dimensional Lie group (three parameters from  $\boldsymbol{\theta}$  and three from  $\boldsymbol{\xi}$ ). It is clearly non-compact (the parameters  $\boldsymbol{\xi}$  can take arbitrarily large values) and it is also simple and doubly connected; later we will find its covering group, which coincides with SL(2,C).

We may recall that the Lorentz boost  $L(\xi)$  can be written as

$$R'L(\boldsymbol{\xi}_z)R'',$$

with  $R', R'' = R'^{-1}$  rotations and  $L(\boldsymbol{\xi}_z)$  an acceleration along the OZ axis. Thus, the general study of Lorentz transformations is reduced to that of rotations and pure accelerations, that may be taken to be along the OZ axis.

EXERCISE: Given two pure boosts  $L(\boldsymbol{\xi})$ ,  $L(\boldsymbol{\eta})$ , find  $L(\boldsymbol{\zeta})$ ,  $R(\boldsymbol{\theta})$  such that

$$L(\boldsymbol{\xi})L(\boldsymbol{\eta}) = L(\boldsymbol{\zeta})R(\boldsymbol{\theta}).$$

Note that in general (unless  $\xi, \eta$  are parallel) the product of two boosts is not a pure boost

We finish the characterization by presenting two more theorems, and a covariant parametrization of the Lorentz transformation  $\Lambda$ .

THEOREM.

A Lorentz transformation  $\Lambda$  such that  $\Lambda n_t = u$  is a pure boost, times a rotation around  $\boldsymbol{\xi}$  (where  $\boldsymbol{\xi}$  is given in terms of u by  $\cosh \xi = u_0$ ,  $\boldsymbol{\xi}/\xi = \mathbf{u}/|\mathbf{u}|$ ) if, and only if,  $\Lambda$  commutes with all rotations around  $\boldsymbol{\xi}$ .

To prove this, we use that a rotation around  $\boldsymbol{\xi}$ , which we denote by  $R_{\boldsymbol{\xi}}$ , leaves  $\boldsymbol{\xi}$  invariant; hence, it follows that  $L(\boldsymbol{\xi})$  and  $R_{\boldsymbol{\xi}}$  commute. [Use that  $\boldsymbol{\xi}(R_{\boldsymbol{\xi}}\mathbf{r}) = (R_{\boldsymbol{\xi}}^{-1}\boldsymbol{\xi})\mathbf{r} = \boldsymbol{\xi}\mathbf{r}$  for any  $\mathbf{r}$ ]. The reciprocal is also easy. Given that  $u = \Lambda n_t$ , we construct  $\boldsymbol{\xi}$  as before, and then  $L(\boldsymbol{\xi})$ . Now,  $L^{-1}(\boldsymbol{\xi})\Lambda = R$  is a rotation. As we have just seen,  $L(\boldsymbol{\xi})$  commutes with rotations  $R_{\boldsymbol{\xi}}$ ; so does  $\Lambda$ , and hence R. But a rotation that commutes with all rotations around an axis  $\boldsymbol{\xi}$  is itself a rotation around that axis, so  $\Lambda = L(\boldsymbol{\xi})R_{\boldsymbol{\xi}}$ , finishing the proof.

THEOREM.

We have, for any  $\xi$  and any rotation R,

$$RL(\boldsymbol{\xi})R^{-1} = L(R\boldsymbol{\xi}),$$

where  $L(R\xi)$  is the boost characterized by  $R\xi$ .

The proof is straightforward and is left as an exercise.

Instead of parametrizing a Lorentz transformation  $\Lambda = L(\boldsymbol{\xi})R(\boldsymbol{\theta})$  by the parameters  $\boldsymbol{\xi}$ ,  $\boldsymbol{\theta}$ , it is at times convenient to use what is called a *covariant parametrization*. We define the set of parameters  $\omega_{\mu\nu}$  in terms of  $\boldsymbol{\xi}$ ,  $\boldsymbol{\theta}$  by

$$\sum_{jk} \epsilon_{jkl} \omega_{jk} = \theta_l, \quad \omega_{j0} = \frac{1}{2} \xi_j; \quad \omega_{\alpha\beta} = -\omega_{\beta\alpha}.$$

For  $\omega$  infinitesimal we write a Lorentz transformation as

$$\Lambda = 1 - \sum \omega_{\alpha\beta} X^{(\alpha\beta)} + O(\omega^2).$$

Then, the matrices  $X^{(\alpha\beta)}$  have components

$$X_{\mu\nu}^{(\alpha\beta)} = -(\delta_{\mu\alpha}g_{\nu\beta} - \delta_{\mu\beta}g_{\nu\alpha}).$$

To prove this, we note that, on the one hand, and from the definition of X,

$$(\Lambda(\omega)x)_{\mu} \simeq x_{\mu} - \sum_{\alpha\beta} \sum_{\nu} \omega_{\alpha\beta} X_{\mu\nu}^{(\alpha\beta)} x_{\nu};$$

on the other, from the explicit formulas for R, L,

$$(R(\boldsymbol{\theta})x)_0 = x_0, \quad (R(\boldsymbol{\theta})x)_i = x_i - \sum 2\omega_{ik}x_k;$$

$$(L(\xi)x)_0 \simeq x_0 + \sum 2\omega_{j0}x_j, \ (L(\xi)x)_i \simeq x_i + 2\omega_{i0}x_0,$$

so that letting  $\Lambda = LR$ , we get

$$(\Lambda x)_0 \simeq x_0 - \sum 2\omega_{0j}x_j, \ (\Lambda x)_i \simeq x_i + 2\omega_{i0}x_0 - \sum 2\omega_{ik}x_k$$

from which the desired result follows.

Beyond  $\mathcal{L}_{+}^{\uparrow}$ , the invariance group of relativity also includes space translations,

$$\mathbf{r} \rightarrow \mathbf{r} + \mathbf{a}$$

and time translations,

$$ct \rightarrow ct + a_0$$
;

in four-vector notation,

$$x_{\mu} \rightarrow x_{\mu} + a_{\mu}$$
.

The group obtained by adjoining to  $\mathcal{L}$  the translations will be called the *Poincaré*, or *inhomogeneous Lorentz group*, written  $\mathcal{JL}$ . Its elements are pairs  $(a, \Lambda)$  with a a four-vector and  $\Lambda$  in  $\mathcal{L}$ . They act on an arbitrary vector x by

$$(a, \Lambda)x = a + \Lambda x,$$

and satisfy the ensuing product and inverse law:

$$(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda \Lambda'),$$
  
 $(a, \Lambda)^{-1} = (-\Lambda^{-1}a, \Lambda^{-1}).$ 

The unit element of the group is the transformation (0,1). At times we will simplify the notation writing a instead of (a,1) and  $\Lambda$  instead of  $(0,\Lambda)$ . The mathematical structure of  $\mathcal{IL}$  is

$$\mathcal{IL} = \mathcal{L} \widetilde{\times} \mathcal{T}_4.$$

# 6.4. Geometry of Minkowski Space

The geometrical properties of spacetime present some peculiarities owing to the indefinite character of the metric. A first peculiarity is that we can classify vectors v of a Minkowskian space, in a relativistically invariant way, in the following classes: timelike, lightlike, and spacelike vectors. Timelike vectors v are such that  $v \cdot v > 0$ . If  $v_0 > 0$ , we say they are positive timelike; if  $v_0 < 0$ , negative ( $v_0 = 0$  is impossible). Lightlike vectors v, which satisfy  $v \cdot v = 0$ , are positive lightlike if  $v_0 > 0$ , negative if  $v_0 < 0$ .  $v_0 = 0$  is only possible for the null vector, v = 0. Finally, we say that v is spacelike if  $v \cdot v < 0$ ; the sign of  $v_0$  is not invariant now.

EXERCISES: i) Prove that this classification is invariant under transformations in  $\mathcal{L}_{+}^{\uparrow}$ ; in particular check invariance of sign  $v_0$  if  $v^2 \geq 0$ . ii) Show that the trajectory of a particle with mass is given by a positive timelike vector, and that of a light ray by a positive lightlike vector. Hint: Let  $\mathbf{r}$  be the location of a particle (or signal) at time t. Form the four-vector  $x, x_0 = ct, \mathbf{x} = \mathbf{r}$ . The velocity of the particle (assuming uniform motion) is  $\mathbf{v} = \mathbf{r}/t$ 

The following lemma is very useful:

# LEMMA.

(i) If v is positive (negative) timelike, then there exists a vector  $v^{(0)}$  and a Lorentz transformation  $\Lambda$  such that  $v = \Lambda v^{(0)}$ , and  $v_0^{(0)} = \pm m$ ,  $\mathbf{v}^{(0)} = 0$ , m > 0. (ii) If v is positive (negative) lightlike there exists a  $\overline{v}$  and  $\Lambda$  with  $v = \Lambda \overline{v}$  and  $\overline{v}_0 = \pm 1$ ,  $\overline{v}_1 = \overline{v}_2 = 0$ ,  $\overline{v}_3 = 1$ . (Here and before the signs  $(\pm)$  are correlated to positive-negative.) (iii) If v is spacelike, there exist a  $v^{(3)}$  and  $\Lambda$  with  $v = \Lambda v^{(3)}$ ,  $v_{\mu}^{(3)} = \delta_{\mu 3} v_3^{(3)}$ ,  $v_3^{(3)} > 0$ .

This means that, in an appropriate reference system, a positive lightlike vector (e.g.) can be chosen to be of the form  $\overline{v}$ ,

$$\overline{v} = (1, 0, 0, 1).$$

The clumsy but simple proof of this lemma uses the explicit expression for the Lorentz transformations to build explicit constructions.

The difference between an Euclidean space and Minkowski space is also apparent in the two following results:

## THEOREM.

If both v and v' are lightlike and they are orthogonal, i.e.,  $v \cdot v' = 0$ , then they are parallel:  $v' = \alpha v$ .

The proof is left as a simple exercise, using the previous Lemma.

#### THEOREM.

If  $v \cdot v > 0$  and  $v \cdot u = 0$ , either v and u are proportional or necessarily u is spacelike.

The proof is again left as an exercise, using the Lemma.

# THEOREM.

The only invariant numerical tensors in Minkowski space are combinations of the metric tensor,  $g_{\mu\nu}$ , and the Levi-Cività tensor  $\epsilon_{\mu\nu\rho\sigma}$ ,

 $\epsilon_{\mu\nu\rho\sigma} = 1$ , if  $\mu\nu\rho\sigma$  is an even permutation of 1230,

 $\epsilon_{\mu\nu\rho\sigma}$  – 1, if  $\mu\nu\rho\sigma$  is an odd permutation of 1230,

 $\epsilon_{\mu\nu\rho\sigma} = 0$ , if two indices are equal.

Note that  $\epsilon_{ijk0} = \epsilon_{ijk}$ , where  $\epsilon_{ijk}$  is the Levi–Cività tensor in ordinary three-space.

#### THEOREM.

Given a set of Minkowski vectors  $v^{(a)}$ , the only invariants that are continuous and that can be formed with them are functions of the scalar products  $v^{(a)} \cdot v^{(b)}$  and, if there are four or more vectors, of the quantities

$$\sum \epsilon^{\mu\nu\rho\sigma} v_{\mu}^{(a)} v_{\nu}^{(b)} v_{\rho}^{(c)} v_{\sigma}^{(d)}.$$

In spite of the fact that these theorems are similar to their analogues in Euclidean space and also in spite of their apparent simplicity, proofs are very complicated. For example, the later Theorem fails if we remove the requisite of continuity: the functions (sign  $v_0$ ) $\theta(v^2)$  or  $\delta_4(v) \equiv \delta(v_0)\delta(\mathbf{v})$  are invariant: yet they cannot be written in terms of invariants. Proofs of the two Theorems can be found in, for example, the treatise of Bogoliubov, Logunov and Todorov (1975).

Given a Minkowski vector, v, the set of Lorentz transformations  $\Gamma$  that leave it invariant is called its little group<sup>14</sup> (or stabilizer), W(v). The little group of a vector v depends only upon the sign of  $v \cdot v$ , in the sense that if, for example,  $v \cdot v > 0$  and  $u \cdot u > 0$ , then the little groups W(v), W(u) are isomorphic. To prove this, we first note that W(v) and  $W(\Lambda v)$  are isomorphic for any  $\Lambda$ . Indeed, if  $\Gamma v = v$ , then  $\Lambda \Gamma \Lambda^{-1}$  is in  $W(\Lambda v)$ , and vice versa. Moreover, W(v) is identical with  $W(\alpha v)$  for any number  $\alpha \neq 0$ . Using this in conjunction with Lemma 1, we find that there are essentially only three little groups. To be precise, we have that, if  $v \cdot v > 0$ , the little group is isomorphic to  $W(n_t)$ ; if  $v \cdot v = 0$ , the little group is isomorphic to W(v), v = 0, the little group is isomorphic to W(v).

#### THEOREM

One has, (A)  $W(n_t) = SO(3)$ , where by SO(3) we denote the group of ordinary rotations. (B)  $W(\overline{v}) = SO(2) \times \mathcal{T}_2$ , where  $SO_z(2)$  is the group of rotations around OZ, and  $\mathcal{T}_2$  is defined below. (C)  $W(n^{(3)}) = \mathcal{L}_+^{\uparrow}(3)$ , where  $\mathcal{L}_+^{\uparrow}(3)$  is identical to a Lorentz-like group (in three dimensions) that acts only on time and the spatial plane XOY, but leaves OZ invariant.

The result (A) is already known to us. Result (C) is left as a simple exercise. We turn to the lightlike case (B). Let  $\Gamma$  be an element of  $\mathcal{W}(\overline{v})$ , and let N be the subspace of Minkowski space orthogonal to  $\overline{v}$ , that is, if u is in N, then  $u \cdot \overline{v} = 0$ .

Clearly, the subspace N is also invariant under  $\Gamma$ . A basis of N is formed by the three vectors  $v^{(a)}, a = 1, 2, 3$  with  $v^{(1)} = n^{(1)}, \ v^{(2)} = n^{(2)}, \ n_{\mu}^{(a)} = \delta_{a\mu}$ , and  $v^{(3)} = \overline{v}$ : because  $\overline{v}$  is lightlike the subspace orthogonal to  $\overline{v}$  contains  $\overline{v}$  itself. If u is in N, we write  $u = \sum_a \alpha_a v^{(a)}$ . Because  $\Gamma u$  is also in N, we can write

$$\Gamma u = \sum_{ab} \Gamma_{ab} \alpha_b v^{(a)};$$

thus the matrix elements  $\Gamma_{ab}$  determine  $\Gamma$ , and vice versa. The conditions  $\Gamma u \cdot \Gamma u' = u \cdot u'$  and  $\Gamma \overline{v} = \overline{v}$  imply that

$$(\Gamma_{ab}) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ \Gamma_{31} & \Gamma_{32} & 1 \end{pmatrix},$$

with  $\Gamma_{31}$ ,  $\Gamma_{32}$  arbitrary. This set of matrices has a mathematical structure like that of the Euclidean group of the plane,  $SO_z(2) \times \mathcal{T}_2$  where  $SO_z(2)$  are rotations around OZ,

$$\begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

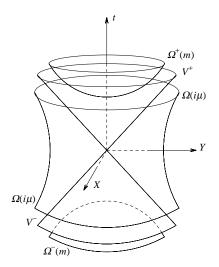
and the "translations"  $\mathcal{T}_2$  are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \Gamma_{31} & \Gamma_{32} & 1 \end{pmatrix}.$$

To finish this section we present a few more definitions (see the figure). The light cone is the set of vectors v with  $v^2=0$ . If, moreover,  $v_0>0$  ( $v_0<0$ ), we speak of the future, forward or positive (past, backward or negative) light cone, denoted by  $V^+$  ( $V^-$ ). The set of vectors u with  $u^2=m^2>0$  is denoted by  $\Omega^{\pm}(m)$ , ( $\pm$ ) according to the sign of  $u_0$ , and is called the future, forward or positive (past, backward or negative) mass hyperboloid, for  $u_0>0$  ( $u_0<0$ ). This name derives from (momentum) Minkowski space. The set of w with  $w\cdot w=-\mu^2$ ,  $\mu^2>0$  is called the imaginary mass hyperboloid,  $\Omega(i\mu)$ .

EXERCISE: Verify that the sets  $V^+$ ,  $V^-$ ,  $\Omega^+(m)$ ,  $\Omega^-(m)$ ,  $\Omega(i\mu)$  are invariant under  $\mathcal{L}_+^{\uparrow}$ , and that each vector in one of them can be reached by an appropriate transformation from any other one in the same set.

<sup>&</sup>lt;sup>14</sup>Little groups, first introduced by Wigner (1939), play a key role in the study of relativistic particle states.



Various regions in Minkowski space.

# 6.5. Finite dimensional representations of the Lorentz group

# i. The correspondence $\mathcal{L} \to \mathrm{SL}(2,\mathbb{C})$

To every Minkowski vector v with components  $v_{\mu}$  we associate the  $2 \times 2$  complex matrix

$$\tilde{v} = v_0 + \boldsymbol{\sigma} \mathbf{v} = \sum_{\mu\nu} g_{\mu\nu} \tilde{\sigma}_{\mu} v_{\nu} = \begin{pmatrix} v_0 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & v_0 - v_3 \end{pmatrix},$$
$$\tilde{\sigma}_0 = \sigma_0 = 1, \ \tilde{\sigma}_i = -\sigma_i.$$

We have

$$\tilde{\sigma}_{\mu} = \sum_{\nu} g_{\mu\nu} \sigma_{\nu}; \quad \text{Tr } \tilde{\sigma}_{\mu} \sigma_{\nu} = 2g_{\mu\nu};$$

$$\det \tilde{v} = v \cdot v, \quad v_{\mu} = \frac{1}{2} \operatorname{Tr} \sigma_{\mu} \tilde{v}; \quad \tilde{v}^{\dagger} = \tilde{v},$$

the last relation holding if the  $v_{\mu}$  are real.

For every Lorentz transformation,

$$\Lambda : v \to \Lambda v \equiv v_{\Lambda},$$

we have a corresponding matrix A, A in  $SL(2,\mathbb{C})$ . We define A by

$$A\tilde{v}A^{\dagger} = \tilde{v}_{\Lambda} = \tilde{\sigma} \cdot \Lambda v. \tag{1}$$

Actually, both  $\pm A$  correspond to the same  $\Lambda$ . An explicit formula for the correspondence is obtained as follows. Choose the vectors  $v^{(\alpha)}$  with  $v^{(\alpha)}_{\mu} = \delta_{\alpha\mu}$ . Applying (1) to these, we get immediately

$$\Lambda_{\beta\alpha} = \frac{1}{2} \operatorname{Tr} \sigma_{\beta} A \sigma_{\alpha} A^{\dagger}.$$

The inverse is slightly more difficult to obtain. We will consider separately accelerations L(v) such that

$$L(v)n_t = v; \ n_{t\mu} = \delta_{\mu 0},$$

and rotations, R. For the first, and because  $\tilde{n}_t = 1$ , (1) gives

$$A(L(v))A^{\dagger}(L(v)) = \tilde{v},$$

with solution

$$A(L(v)) = +\tilde{v}^{1/2}.$$

Note that  $\tilde{v} = L(v)n_t$  is positive definite. We choose the sign (+) for the square root for continuity. For a pure boost,  $A(L(v))^{\dagger} = A(L(v))$ .

EXERCISE: Prove this.

For rotations, R, we have  $Rn_t = n_t$ ; hence (1) gives

$$A(R)A^{\dagger}(R) = 1,$$

i.e., A is unitary. Let  $\boldsymbol{\theta}$  be the parameters of R. For  $\boldsymbol{\theta}$  infinitesimal, and  $v_0 = 0$ ,

$$\tilde{v} \equiv \boldsymbol{\sigma} \mathbf{v} \rightarrow \boldsymbol{\sigma} \mathbf{v} + \sum \sigma_j \theta_k v_l \epsilon_{jkl}.$$

If we write

$$A(R) = \exp i\theta \lambda \simeq 1 + i\theta \lambda$$

we then get, from (1),

$$(1+\mathrm{i}\boldsymbol{ heta}\boldsymbol{\lambda})\boldsymbol{\sigma}\mathbf{v}(1-i\boldsymbol{ heta}\boldsymbol{\lambda})\simeq\boldsymbol{\sigma}\mathbf{v}+\sum\epsilon_{jkl}\sigma_j\theta_kv_l,$$

from which

$$[\lambda_j, \sigma_k] = -i \sum \epsilon_{jkl} \sigma_l,$$

and hence  $\lambda = -\sigma/2$ :

$$A(R(\boldsymbol{\theta})) = \exp \frac{-\mathrm{i}}{2} \boldsymbol{\theta} \boldsymbol{\sigma}. \tag{2}$$

If the four-vector v is such that  $v^2 = 1$ ,  $v_0 > 0$ , we define  $\xi$  by

$$\cosh \xi = v_0, \sinh \xi = |\mathbf{v}|, \boldsymbol{\xi}/|\boldsymbol{\xi}| = \mathbf{v}/|\mathbf{v}|.$$

Then,

$$\tilde{v}^{1/2} = \cosh \frac{\xi}{2} + \frac{1}{\xi} \boldsymbol{\xi} \boldsymbol{\sigma} \sinh \frac{\xi}{2} = \exp \frac{1}{2} \boldsymbol{\xi} \boldsymbol{\sigma},$$

so that

$$A(L(v)) = \exp \frac{1}{2} \boldsymbol{\xi} \boldsymbol{\sigma}. \tag{3}$$

EXERCISE: Prove that  $\det A(L(v)) = \det A(R(\boldsymbol{\theta})) = 1$ . Prove that the set  $A(L(v))A(R(\boldsymbol{\theta}))$  exhausts the group SL (2,C). Hint. Use the polar decomposition: any matrix A may be written as

$$A = HU$$

with H positive definite and U unitary. If  $\det A = 1$ ,  $\det H$ ,  $\det U$  can also be taken to be so. Check that any such H may be written as (3), and any such U as in (2).

We next find the images of the little groups in  $SL(2,\mathbb{C})$ . For the timelike case, this is accomplished by choosing the vector  $n_t$ , with  $n_{t\mu} = \delta_{\mu 0}$ . Then,  $\tilde{n}_t = 1$  and the image U of a rotation R has to verify  $UU^{\dagger} = 1$ , i.e., the image of the SO(3) subgroup of  $\mathcal{L}$  is the SU(2) subgroup of  $SL(2,\mathbb{C})$ .

For the case of lightlike vectors, we choose  $n = n_t + n^{(3)}$  with  $n_t$  as before and  $n_{\mu}^{(3)} = \delta_{\mu 3}$ . Then

$$\widetilde{n} = 1 + \sigma_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

If N is the image in  $SL(2,\mathbb{C})$  of the little group transformation  $\Gamma$ ,  $\Gamma n = n$ , then it must satisfy the conditions

$$N\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} N^{\dagger} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \det N = 1$$

from which it follows that one can write

$$N = \begin{pmatrix} e^{i\theta/2} & e^{-i\theta/2}(a+ib) \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

EXERCISE: Find the image in SL(2,C) of the little group of a spacelike vector.

ii. Connection with the Dirac formalism

Let us use the notation

$$D_{\alpha\beta}^{(1/2)}(\Lambda) \equiv A_{\alpha\beta}(\Lambda),$$
  
$$\tilde{D}_{\dot{\alpha}\dot{\beta}}^{(1/2)}(\Lambda) \equiv (A^{-1+}(\Lambda))_{\dot{\alpha}\dot{\beta}}.$$

We also define

$$\hat{v} \equiv v_0 - \boldsymbol{\sigma} \mathbf{v} = \sigma \cdot v,$$
  
 $\hat{v}_{\Lambda} \equiv \sigma \cdot \Lambda v.$ 

One may check by explicit verification that

$$A^{-1+}\hat{v}A^{-1} = \hat{v}_A,\tag{4}$$

a formula which is the counterpart of (1) and which indeed provides another representation of  $\mathcal{L}$  into  $SL(2,\mathbb{C})$ , inequivalent to that given by (1). (It is actually equivalent to the representation  $\Lambda \to A^*$ .)

EXERCISE: prove that the representations  $\Lambda \to A$  and  $\Lambda \to (A^{\mathrm{T}})^{-1}$  are equivalent. Hint: the matrix that does it is  $C = \mathrm{i}\sigma_2$ .

We link this to the standard Dirac formalism by noting that, in the Weyl realization of the gamma matrices,

 $\gamma_{\mu} = \begin{pmatrix} 0 & \tilde{\sigma}_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix}, \quad \sigma_{0} = 1$ 

one has

$$\gamma \cdot v = \begin{pmatrix} 0 & \tilde{v} \\ \hat{v} & 0 \end{pmatrix}.$$

We then define

$$D(\Lambda) = \begin{pmatrix} D^{(1/2)}(\Lambda) & 0\\ 0 & \tilde{D}^{(1/2)}(\Lambda) \end{pmatrix}$$
$$= \begin{pmatrix} A_{\alpha\beta}(\Lambda) & 0\\ 0 & (A^{-1+}(\Lambda))_{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

As an application we prove the transformation properties of the Dirac  $\gamma$  matrices. In the Weyl realization, and for an arbitrary four-vector v,

$$\begin{split} D^{-1}(\varLambda)\gamma \cdot vD(\varLambda) &= \begin{pmatrix} A^{-1} & 0 \\ 0 & A^{\dagger} \end{pmatrix} \begin{pmatrix} 0 & \tilde{v} \\ \hat{v} & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1}^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} 0 & A^{-1}\hat{v}A^{-1}^{\dagger} \\ A^{\dagger}\hat{v}A & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\sigma} \cdot \varLambda^{-1}v \\ \sigma \cdot \varLambda^{-1}v & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (\varLambda\tilde{\sigma}) \cdot v \\ (\varLambda\sigma) \cdot v & 0 \end{pmatrix} = (\varLambda\gamma) \cdot \sigma, \end{split}$$

and we have used (1), (4). Because v is arbitrary, this gives

$$D^{-1}(\Lambda)\gamma_{\mu}D(\Lambda) = \sum \Lambda_{\mu\nu}\gamma_{\nu}.$$

The similitude with the treatment of the group SO(4) in Sect. 3.2 will be noted. In fact, the groups SO(4) and  $\mathcal{L}$  can be related one to the other through analytical continuation on the variable  $v_0$  and the complexification of their Lie algebras coincide. We will not delve into this question further.

# iii. The finite-dimensional representations of SL(2,C)

The finite dimensional representations of SL(2,C) are very easy to construct. Denoting by  $\mathbf{M}_2$  to the Lie algebra of SL(2,C), it is easily seen to consist of  $2 \times 2$  complex traceless matrices. It is obvious that, if we complexify the  $\mathbf{A}_1$  algebra corresponding to the SU(2) subgroup of SL(2,C), it generates all of  $\mathbf{M}_2$ :  $\mathbf{A}_1^{\mathbb{C}} = \mathbf{M}_2$ . Therefore, we may generate in this way the representations of the Lorentz group from those of the rotation group. In particular, it follows that the Clebsch–Gordan coefficients of SU(2) and SL(2,C) are the same. Thus, we may, by simple tensor product

$$A_{\alpha_1\beta_1}A_{\alpha_2\beta_2}\cdots A_{\alpha_i\beta_i}$$

construct a representation of  $SL(2,\mathbb{C})$  which, when restricted to the rotation subgroup, corresponds to spin j/2.

More on the matters treated in this section may be found in Bogoliubov, Logunov and Todorov (1975) or Wightman (1960).

# §7. General Description of Relativistic States

#### 7.1. Preliminaries

It is in many applications convenient to introduce an abstract characterization of relativistic states, freeing it from the problems encountered in explicit realizations. We will thus describe the states by "safe" observables: momentum  $\mathbf{p}$  and another one that we label  $\zeta$  and that will be related to a spin component: our task will then be to construct the states,  $|\mathbf{p}, \zeta\rangle$ , and study their transformation properties under relativistic transformations. This we will do from the next section onwards; in what remains of the present section we will introduce some standard theorems on group representations, without proofs, and, at the end, describe the group of relativistic transformations, the Poincaré group.

The invariance group of relativity is the Poincaré group, also called the inhomogeneous Lorentz group. Its elements are pairs  $(a, \Lambda)$  with a a four-translation consisting of a spatial translation by  $\mathbf{a}$ , and a time translation by  $a_0/c$ ; and a (proper, orthochronous) Lorentz transformation,  $\Lambda$ . The generators of the Poincaré group may be described as generators of rotations, boosts and translations. Let us consider any representation,  $U(a, \Lambda)$  of the Poincaré group; then, for infinitesimal transformations we write

$$U(0, R(\boldsymbol{\theta})) \simeq 1 - \frac{\mathrm{i}}{\hbar} \boldsymbol{\theta} \mathbf{L},$$
  
 $U(0, L(\boldsymbol{\xi})) \simeq 1 - \frac{\mathrm{i}}{\hbar} \boldsymbol{\xi} \mathbf{N},$   
 $U(a, 1) \simeq 1 + \frac{\mathrm{i}}{\hbar} a \cdot P.$ 

The commutation relations may be evaluated in any (faithful) representation; indeed, since these respect product and inverse rules, commutators will also be respected. We may then choose the regular representation with the U acting on scalar functions of  $a, \Lambda$ . We can then take

$$L_{j} = i\hbar \sum_{k} \epsilon_{jkl} x_{k} \partial_{l},$$

$$N_{j} = i\hbar (x_{0}\partial_{j} - x_{j}\partial_{0}),$$

$$P_{j} = i\hbar \partial_{j}, P_{0} = i\hbar \partial_{0}$$

and evaluate the commutators with these explicit expressions. That way we find the relations, valid in

any representation,

$$[L_k, L_j] = i\hbar \sum_{k} \epsilon_{kjl} L_l,$$

$$[L_k, N_j] = i\hbar \sum_{k} \epsilon_{kjl} N_l,$$

$$[L_k, P_j] = i\hbar \sum_{k} \epsilon_{kjl} P_l;$$

$$[L_k, P_0] = 0, \quad [P_\mu, P_\nu] = 0;$$

$$[N_k, N_j] = -i\hbar \sum_{k} \epsilon_{kjl} L_l,$$

$$[N_k, P_j] = -i\hbar \delta_{kj} P_0,$$

$$[N_k, P_0] = -i\hbar P_k.$$

We may also write them in covariant form. If we let

$$U(\Lambda) \simeq 1 - \frac{i}{\hbar} \omega^{\mu\nu} M_{\mu\nu},$$

then a simple calculation, making use of the fact that

$$[\partial_{\mu}, x_{\nu}] = g_{\mu\nu}$$

allows us to write the commutation relations in the form

$$[M_{\mu\nu}, P_{\alpha}] = i\hbar (g_{\nu\alpha}P_{\mu} - g_{\mu\alpha}P_{\nu}),$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = i\hbar (g_{\mu\alpha}M_{\beta\nu} + g_{\mu\beta}M_{\nu\alpha} + g_{\nu\alpha}M_{\mu\beta} + g_{\nu\beta}M_{\alpha\mu}),$$

$$[P_{\mu}, P_{\nu}] = 0.$$

Consider now a quantum system represented by the state  $|\Psi\rangle$ . A Poincaré transformation g will carry it over a new state,  $|\Psi_g\rangle$ . According to the rules of quantum mechanics, we expect that this will be implemented by a linear unitary operator,

$$U(g) = U(a, \Lambda) :$$
  
 $|\Psi_g\rangle = U(a, \Lambda)|\Psi\rangle.$ 

We will require that this be a representation of the Poincaré group. Actually, this is asking for too much; in principle, one could have, more generally, a representation up to a phase:

$$U(a, \Lambda)U(a', \Lambda') = e^{i\varphi}U(a + \Lambda a', \Lambda \Lambda').$$

In the following sections we will give an explicit construction with  $\varphi = 0$ ; the proof that the result is general is fairly complicated and will not be given here (see Wigner, 1939).

We will then consider unitary representations of the Poincaré group. Since a reducible representation can be decomposed into orthogonal irreducible ones, we need only consider the latter, which may be identified as those describing elementary systems that we will call particles. Note that here "elementarity" is not used in a dynamical sense; it only means that the corresponding isolated system cannot be described as two or more systems, also isolated <sup>15</sup>.

<sup>&</sup>lt;sup>15</sup>Our treatment will not be mathematically rigorous. Mathematical rigour can be provided by consulting the treatises of Bogoliubov, Logunov and Todorov (1975) or Wightman (1960). The problem of giving the general description of relativistically invariant systems was first fully solved by Wigner (1939), whose paper we will essentially follow.

## 7.2. Relativistic one-particle states: general description

Let us denote by  $\mathfrak{H}$  the Hilbert space for free one-particle states. We will construct a basis of  $\mathfrak{H}$ , working in the Heisenberg picture, the simplest one to use for our analysis.

Consider the operators that represent translations,  $U(a,1) \equiv U(a)$ . If we write them in exponential form,

$$U(a) = \exp ia \cdot P,$$

then unitarity of U implies Hermiticity of the  $P_{\mu}$ . We will identify  $P_0$  with the energy<sup>16</sup> operator (the Hamiltonian), and  $\mathbf{P}$  the ordinary momentum operator; the four  $P_{\mu}$  form the four-momentum operator.

From the commutation relations, it follows that the operator  $P^2 = P \cdot P$  commutes with all the generators of the Poincaré group, and hence also with all the  $U(a, \Lambda)$ . Schur's lemma then implies that it is a constant, which we identify with the square of the mass (which can be zero):

$$m^2 = P \cdot P$$
.

Because of this, it follows that, for free particles, the operator  $P_0$  is actually a function of the **P**:

$$P_0 = +(m^2 + \mathbf{P}^2)^{1/2},$$

where we have chosen the positive square root to get positive energies. If  $\mathbf{p}$  are the eigenvalues of the  $\mathbf{P}$ , and  $p_0$  those of  $P_0$ , we thus have

$$p_0 = +\sqrt{m^2 + \mathbf{p}^2},$$

as was to be expected for a relativistic particle.

As we know, the  $P_{\mu}$  commute among themselves. We can then diagonalize them simultaneously, and consider the corresponding eigenvectors as the desired base of  $\mathfrak{H}$ , which we denote by  $|p,\zeta\rangle$ , with  $\zeta$  being whatever extra quantum numbers necessary to specify the states; as we will see, the  $\zeta$  will be essentially a spin component. Note that the notation  $|p,\zeta\rangle$ , although convenient, is redundant; we could also write  $|p,\zeta\rangle = |\mathbf{p},\zeta\rangle$ , since  $p_0$  is fixed once  $\mathbf{p}$  is given.

Because  $|p,\zeta\rangle$  are eigensates of the  $P_{\mu}$ , we have

$$P_{\mu}|p,\zeta\rangle = p_{\mu}|p,\zeta\rangle,$$

and, exponentiating, and writing U(a) for U(a, 1),

$$U(a)|p,\zeta\rangle = e^{ia \cdot P}|p,\zeta\rangle = e^{ia \cdot p}|p,\zeta\rangle.$$

Let us select a fixed momentum,  $\overline{p}$ , with  $\overline{p} \cdot \overline{p} = m^2$ ,  $\overline{p}_0 > 0$ . This means that we are choosing a fixed reference system. Any admissible four-vector for the particle, p, may be written as

$$p = \Lambda(p)\overline{p},$$

where  $\Lambda(p)$  is a (not unique) Lorentz transformation. We then *choose* a family of such Lorentz transformations,  $\Lambda(p)$ , one for each p. The basis we will find will depend on the family of  $\Lambda(p)$  we choose; but the choice will be left unspecified for the moment. Then, we define the basis  $|\Lambda(p), \zeta\rangle$  by<sup>17</sup>

$$|\Lambda(p),\zeta\rangle \equiv U(\Lambda(p))|\overline{p},\zeta\rangle,$$

i.e., by accelerating via  $\Lambda(p)$  to momentum p; to simplify the notation, we write  $U(\Lambda)$  for  $U(0,\Lambda)$ .

Let us first prove that the state  $|\Lambda(p),\zeta\rangle$  corresponds to four-momentum p. To see this, we evaluate

$$U(a)|\Lambda(p),\zeta\rangle = U(a)U(\Lambda(p))|\overline{p},\zeta\rangle.$$

<sup>&</sup>lt;sup>16</sup>Unless otherwise explicitly stated, we will use natural units with  $\hbar = c = 1$ .

<sup>&</sup>lt;sup>17</sup>The notation  $|\Lambda(p), \zeta\rangle$  is shorthand. A more precise notation for this state would be  $|p, \zeta; \Lambda(p)\rangle$ , i.e., a state with momentum p, other quantum number  $\zeta$ , and obtained with the Lorentz transformation  $\Lambda(p)$ . Our notation is simpler and, hopefully, transparent enough.

Using the identity

$$U(a)U(\Lambda(p)) = U(a, \Lambda(p)) = U(\Lambda(p))U(\Lambda(p)^{-1}a),$$

we obtain

$$U(a)|\Lambda(p),\zeta\rangle = U(\Lambda(p))U(\Lambda(p)^{-1}a)|\overline{p},\zeta\rangle.$$

Taking into account that

$$(\Lambda(p)^{-1}a) \cdot \overline{p} = a \cdot \Lambda(p)\overline{p} = a \cdot p,$$

we get

$$U(\Lambda(p))U(\Lambda(p)^{-1}a)|\overline{p},\zeta\rangle$$

$$=U(\Lambda(p))e^{i(\Lambda(p)^{-1}a)\cdot\overline{p}}|\overline{p},\zeta\rangle$$

$$=e^{ip\cdot a}U(\Lambda(p))|\overline{p},\zeta\rangle$$

$$=e^{ip\cdot a}|\Lambda(p),\zeta\rangle.$$

We have thus shown that

$$U(a)|\Lambda(p),\zeta\rangle = e^{ia\cdot p}|\Lambda(p),\zeta\rangle,$$

and (for example, by differentiating with respect to  $a_{\mu}$  at a=0) that  $|\Lambda(p),\zeta\rangle$  is a state with momentum p, as claimed above:

$$P_{\mu}|\Lambda(p),\zeta\rangle = p_{\mu}|\Lambda(p),\zeta\rangle.$$

These equation tell us how the translations act upon our basis of state vectors,  $|\Lambda(p),\zeta\rangle$ . We will now deduce corresponding formulas for Lorentz transformations. To do so, we start by considering transformations, which we will denote by  $\Gamma,\Gamma',\ldots$ , contained in the little group of  $\overline{p}$ ,  $\mathcal{W}(\overline{p})$ ; and we will let these transformations act on  $|\overline{p},\zeta\rangle\equiv |\Lambda(\overline{p}),\zeta\rangle$  itself. Because the  $\Gamma$  leave  $\overline{p}$  invariant, it follows that the state vector  $U(\Gamma)|\overline{p},\zeta\rangle$  still corresponds to momentum  $\overline{p}$ . Therefore, it will have to be a linear combination of vectors  $|\overline{p},\zeta'\rangle$ :

$$U(\Gamma)|\overline{p},\zeta\rangle = \sum_{\zeta'} D_{\zeta'\zeta}(\Gamma)|\overline{p},\zeta'\rangle,$$

where the  $D_{\zeta'\zeta}$  are certain coefficients. So, in the case of massive particles of spin 1/2, the parameter  $\zeta$  will, for example, represent the third component of spin. Thus, we can have  $\zeta = \pm 1/2$ . It is easy to verify that the conditions

$$U(\Gamma)U(\Gamma') = U(\Gamma\Gamma'), \ U(\Gamma^{-1}) = U^{-1}(\Gamma), \ U^{\dagger}(\Gamma) = U^{-1}(\Gamma)$$

imply that

$$D(\Gamma)D(\Gamma') = D(\Gamma\Gamma'),$$
  

$$D(\Gamma^{-1}) = D(\Gamma)^{-1},$$
  

$$D^{\dagger}(\Gamma) = D(\Gamma)^{-1};$$

$$\begin{pmatrix} D_{1/2,1/2} & D_{1/2,-1/2} \\ D_{-1/2,1/2} & D_{-1/2,-1/2} \end{pmatrix} \equiv \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

that we may take to be the components of a matrix D:

$$D(\Gamma)^{\mathrm{T}} = (D_{\zeta'\zeta}(\Gamma)), \text{ i.e., } D(\Gamma) = (D_{\zeta\zeta'}(\Gamma)).$$

<sup>18</sup> In some cases it may be convenient to label the matrix elements not with the indices  $\pm 1/2$ , but with indices 1, 2. We thus identify

it follows that the matrices D build up a unitary representation of the little group,  $\mathcal{W}(\overline{p})$ . From the "elementarity" of the system, that is to say, from the fact that  $U(a, \Lambda)$  is irreducible, we can deduce that the representation D must also be irreducible.

Exercise: Prove this.

The specific form of the D will be given in the next two sections. For the moment we will assume that we have such a representation, so that we know the values of the coefficients  $D_{\zeta'\zeta}(\Gamma)$ ; with their help we will be able to solve in full generality the problem of finding how arbitrary Lorentz transformations act. In fact, we have,

$$U(\Lambda)|\Lambda(p),\zeta\rangle = U(\Lambda)U(\Lambda(p))|\overline{p},\zeta\rangle$$
  
=  $U(\Lambda(\Lambda p))U(\Lambda(\Lambda p))^{-1}U(\Lambda\Lambda(p))|\overline{p},\zeta\rangle$   
=  $U(\Lambda(\Lambda p))U(\Lambda(\Lambda p)^{-1}\Lambda\Lambda(p))|\overline{p},\zeta\rangle$ ,

where  $\Lambda(\Lambda p)\overline{p} = \Lambda p$ , and we have introduced a term  $U(\Lambda(\Lambda p))U(\Lambda(\Lambda p))^{-1} = 1$  and used the group properties of the U. Now,

$$(\Lambda(\Lambda p))^{-1}\Lambda\Lambda(p)\overline{p} = (\Lambda(\Lambda p))^{-1}\Lambda p = \overline{p}.$$

so that the transformation  $(\Lambda(\Lambda p))^{-1}\Lambda\Lambda(p)$ , which we will write as  $\Gamma(p,\Lambda)$ , is in  $\mathcal{W}(\overline{p})$ , since it leaves  $\overline{p}$  invariant. We thus find

$$U(\Gamma(p,\Lambda))|\overline{p},\zeta\rangle = \sum_{\zeta'} D_{\zeta'\zeta}(\Gamma(p,\Lambda))|\overline{p},\zeta'\rangle;$$

substituting this we get the explicit formula

$$U(\Lambda)|\Lambda(p),\zeta\rangle = \sum_{\zeta'} D_{\zeta'\zeta}(\Gamma(p,\Lambda))|\Lambda(\Lambda p),\zeta'\rangle,$$
  
$$\Gamma(p,\Lambda) \equiv (\Lambda(\Lambda p))^{-1}\Lambda\Lambda(p).$$

Besides choosing the family of  $\Lambda(p)$ , and finding the explicit values of the  $D_{\zeta'\zeta}$ , the only thing that we need to have the problem totally solved is to find the normalization of the states  $|\Lambda(p),\zeta\rangle$  such that relativistic transformations leave it invariant, i.e., such that the  $U(a,\Lambda)$  are unitary.

The U(a) are unitary by construction. If we assume the  $\zeta$  to be eigenvalues of an observable, we will have

$$\langle \Lambda(p), \zeta | \Lambda(p'), \zeta' \rangle = N(p)\delta(\mathbf{p} - \mathbf{p}')\delta_{\zeta\zeta'},$$

where N is a factor to be determined by the requirement that, for any  $\Lambda$ ,

$$\langle U(\Lambda)(\Lambda(p),\zeta)|U(\Lambda)(\Lambda(p'),\zeta')\rangle$$
  
=  $\langle \Lambda(p),\zeta|\Lambda(p'),\zeta'\rangle$ 

(unitarity). Substituting and recalling that the matrix  $D = (D_{\zeta'\zeta})$  is unitary, we find the condition

$$N(\Lambda p)\delta(\Lambda \mathbf{p} - \Lambda \mathbf{p}') = N(p)\delta(\mathbf{p} - \mathbf{p}').$$

If  $\Lambda$  is a rotation R, and since  $\delta(R\mathbf{p}) = \delta(\mathbf{p})$ , it follows that N can only depend on  $|\mathbf{p}|$ , or, equivalently, on  $p_0$ ,  $N = N(p_0)$ . Considering next a boost along OZ,  $L_z$ , with parameter  $\xi$ ,

$$L_z: p_0 \to (\cosh \xi) p_0 + (\sinh \xi) p_3,$$
  

$$p_3 \to (\cosh \xi) p_3 + (\sinh \xi) p_0,$$
  

$$p_1 \to p_1, \ p_2 \to p_2:$$

we find

$$N((\cosh \xi)p_0)\frac{1}{(\cosh \xi)p_0}\delta(\mathbf{p} - \mathbf{p}') = N(p_0)\delta(\mathbf{p} - \mathbf{p}'),$$

for any  $\xi$ , so that we get  $N(p_0) = \text{constant} \times p_0$ . We will follow custom in choosing this constant equal to 2, so the invariant form of the scalar product is finally

$$\langle \Lambda(p), \zeta | \Lambda(p'), \zeta' \rangle = 2p_0 \delta(\mathbf{p} - \mathbf{p}') \delta_{\zeta\zeta'}, \quad p_0 = +\sqrt{m^2 + \mathbf{p}^2}.$$

Before moving on to the detailed analysis of the various different cases, a few more words on general matters are in order. First of all we again remark that the analysis of this section is valid for massive as well as massless particles; for the latter it is sufficient to set m=0 in the appropriate formulas. Secondly, it may appear that our analysis is dependent on the fixed vector (or reference system)  $\overline{p}$ , from which we build the basis. This is not so; because the little groups of two  $\overline{p}$ ,  $\overline{p}$  ' are isomorphic, it follows that substituting  $\overline{p}$  ' for  $\overline{p}$  merely result in a change of basis in  $\mathfrak{H}$ . The same is true if we replace the family  $\Lambda(p)$  by another family,  $\Lambda'(p)$ .

EXERCISE: Find the operators that implement the changes of basis (A) when replacing  $\overline{p}$  by  $\overline{p}'$ , and (B) when replacing  $\Lambda(p)$  by  $\Lambda'(p)$ .

EXERCISE: Suppose that, for a particle, there existed a state  $|\overline{p}_{\perp}\rangle$  different from all the  $p = \Lambda \overline{p}$ . Prove then that  $\langle \overline{p}_{\perp} | \Lambda \overline{p} \rangle = 0$  for all  $\Lambda$ , and that the representation turns out to be reducible.

Finally, the analysis of this section may appear excessively abstract to the reader. This could be overcome by returning to it after having gone over the next two sections.

#### 7.3. Relativistic states of massive particles

The idea behind Wigner's method is actually very simple, at least for particles with mass. In this case, one chooses a reference system with  $\overline{p}_0 = m$ ,  $\overline{p}_i = 0$ , that is to say, the reference system in which the particle is at rest. Here, nonrelativistic quantum mechanics is manifestly valid, which suggests to us that we take the quantum numbers  $\zeta$  to be the values of the third component of spin. In this case, we will use the label  $\lambda$  instead of  $\zeta$ . We thus start by considering the states at rest,

$$|\overline{p},\lambda\rangle$$
.

The little group of  $\overline{p}$  consists of ordinary three-dimensional rotations, which we denote by R rather than  $\Gamma$ . The matrices D(R) are just the standard  $D^{(s)}(R(\boldsymbol{\theta}))$ , for a particle with total spin s. They are

$$D^{(s)}(R(\boldsymbol{\theta})) = \exp \frac{-\mathrm{i}}{\hbar} \boldsymbol{\theta} \mathbf{S},$$

where **S** are the familiar spin operators. For s = 1/2,

$$D^{(1/2)}(R(\boldsymbol{\theta})) = e^{-i\boldsymbol{\sigma}\boldsymbol{\theta}/2}.$$

For arbitrary s, the values of the matrix elements  $D_{\lambda\lambda'}^{(s)}(R)$  of  $D^{(s)}$  can be found in Wigner (1959). We then have

$$U(R)|\overline{p},\lambda\rangle = \sum_{\lambda'} D_{\lambda'\lambda}^{(s)}(R)|\overline{p},\lambda'\rangle.$$

For states in an arbitrary reference system, with momentum p, we may boost by a L(p) such that  $L(p)\overline{p} = p$ .

Then the states  $|L(p), \lambda\rangle$  are defined as

$$|L(p), \lambda\rangle \equiv U(L(p))|\overline{p}, \lambda\rangle,$$

and we normalize them to

$$\langle L(p), \lambda | L(p'), \lambda' \rangle = 2p_0 \delta(\mathbf{p} - \mathbf{p}') \delta_{\lambda \lambda'}.$$

To find the transformation properties of the  $|L(p), \lambda\rangle$  under an arbitrary Lorentz transformation  $\Lambda$ , we proceed as follows:  $\Lambda$  will carry p over  $\Lambda p$ . Therefore we (a) go to the reference system where the

particle is at rest decelerating by  $L^{-1}(p)$ , (b) see how the state transforms there and (c) boost now by  $L(\Lambda p)$ . In formulas,

$$U(\Lambda)|L(p),\lambda\rangle = U(\Lambda)U(L(p))|\overline{p},\lambda\rangle$$
$$= U(L(\Lambda p))U(L(\Lambda p)^{-1})U(\Lambda)U(L(p))|\overline{p},\lambda\rangle$$
$$= U(L(\Lambda p))U(R(p,\lambda))|\overline{p},\lambda\rangle,$$

where

$$R(p, \Lambda) = L(\Lambda p)^{-1} \Lambda L(p)$$

is called a Wigner rotation; it is a rotation since  $R(p,\Lambda)\overline{p}=\overline{p}$ . We obtain the result

$$\begin{split} U(\Lambda)|L(p),\lambda\rangle &= U(L(\Lambda p))U(R(p,\Lambda))|\overline{p},\lambda\rangle \\ &= U(L(\Lambda p))\sum_{\lambda'}D_{\lambda'\lambda}^{(s)}(R(p,\Lambda))|\overline{p},\lambda'\rangle \\ &= \sum_{\lambda'}D_{\lambda'\lambda}^{(s)}(R(p,\Lambda))|L(\Lambda p),\lambda'\rangle, \end{split}$$

so that

$$U(\Lambda)|\Lambda(p),\lambda\rangle = \sum_{\lambda'} D_{\lambda'\lambda}^{(s)}(R(p,\Lambda))|L(\Lambda p),\lambda'\rangle,$$
$$R(p,\Lambda) = L(\Lambda p)^{-1}\Lambda L(p).$$

Of course, we have already seen this in the previous section. The basis  $|L(p), \lambda\rangle$  is sometimes called the *covariant spin* basis. Another useful basis is the *helicity* basis. To build it, we choose, instead of pure boosts L(p), the transformations H(p) defined as follows: first, take a pure boost  $L(p^z)$  that carries  $\overline{p}$  over  $p^z$  with  $p_0^z = p_0$ ,  $p_1^z = p_2^z = 0$ ,  $p_3^z = p_3$ . Then, let  $R(\mathbf{z} \to \mathbf{p})$  be a rotation around the axis  $\mathbf{z} \times \mathbf{p}$  that carries the OZ axis over  $\mathbf{p}$ . We define

$$H(p) \equiv R(\mathbf{z} \to \mathbf{p})L(p^z), |H(p), \eta = \zeta\rangle = U(H(p))|\overline{p}, \zeta\rangle.$$

The corresponding states  $|H(p), \eta = \zeta\rangle$  are the helicity states, since  $\eta$  is the projection of the spin on the vector  $\mathbf{p}$ .

The analysis is fairly straightforward for massive particles. The reason why we gave the general discussion of the previous section is its usefulness in studying the case of massless particles.

The nonrelativistic limit is obtained when  $|\mathbf{p}| \ll m$ , so that  $p_0 \simeq m$ . The normalization becomes (taking the covariant spin case for definiteness)

$$\langle L(p), \lambda | L(p'), \lambda' \rangle \simeq_{NP} 2m \delta_{\lambda \lambda'} \delta(\mathbf{p} - \mathbf{p}'),$$

so that

$$|L(p),\lambda\rangle = \sqrt{2p_0}|\mathbf{p},\lambda\rangle_{NR} \underset{NR}{\sim} \sqrt{2m}|\mathbf{p},\lambda\rangle_{NR}, \quad _{NR}\langle\mathbf{p},\lambda|\mathbf{p}',\lambda'\rangle_{NR} = \delta_{\lambda\lambda'}\delta(\mathbf{p}-\mathbf{p}').$$

Because of this some authors define

$$|L(p), \lambda\rangle_{\rm I} = \frac{1}{\sqrt{2m}} |L(p), \lambda\rangle,$$

or

$$|L(p), \lambda\rangle_{\mathrm{II}} = \frac{1}{\sqrt{2p_0}} |L(p), \lambda\rangle.$$

Here we will stick to our conventions. Choice I presents the problem of collapsing for massless particles; choice II is not relativistically invariant. Our choice is valid for massless as well as massive particles, and is relativistically invariant; the price to pay is a factor  $\sqrt{2p_0}$  between relativistic and NR normalization, a price that is quite justified.

Next we turn to the discrete symmetries  $\mathcal{C}$ ,  $\mathcal{P}$ ,  $\mathcal{T}$ .  $\mathcal{C}$  is defined trivially by setting

$$\mathcal{C}|p,\lambda\rangle \equiv \eta_C|\overline{p,\lambda}\rangle,$$

where  $|\overline{p,\lambda}\rangle$  denotes the state of an antiparticle with the same momentum p and spin  $\lambda$  as the particle  $|p,\lambda\rangle$ .  $\mathcal{P}$  and  $\mathcal{T}$  are not given by the previous analysis; but we can use the same method, with slight modifications. Beginning with parity, we define the operator  $\mathcal{P}$  by considering that it is the representative of space reversal,  $I_s$ ,  $(I_s x)_{\mu} = g_{\mu\mu} x_{\mu}$ :  $\mathcal{P} = U(I_s)$ . We then write

$$\begin{split} \mathcal{P}|L(p),\lambda\rangle &= U(I_s)U(L(p))|\overline{p},\lambda\rangle \\ &= U(L(I_sp))U(L(I_sp)^{-1}I_sL(p))|\overline{p},\lambda\rangle. \end{split}$$

Now,  $L(I_s p)^{-1} I_s L(p)$  leaves  $\overline{p}$  invariant. It is not a rotation, because its determinant is (-1); but then

$$R(p, I_s) \equiv L(I_s p)^{-1} I_s L(p) I_s$$

is a rotation. In the nonrelativistic case,

$$\mathcal{P}|\overline{p},\lambda\rangle = \eta_P|\overline{p},\lambda\rangle,$$

so that, finally,

$$\mathcal{P}|L(p),\lambda\rangle = \eta_P \sum_{\lambda'} D_{\lambda'\lambda}^{(s)}(R(p,I_s))|L(I_sp),\lambda'\rangle.$$

For time reversal we can repeat the analysis with the modifications due to the antiunitary character of  $\mathcal{T}$ . Using that

$$\mathcal{T}P_{\mu}\mathcal{T}^{-1} = (I_s P)_{\mu},$$

we find that

$$\mathcal{T}|L(p),\lambda\rangle = \eta_T \sum_{\lambda'} D_{\lambda',-\lambda}^{(s)}(R(p,I_s))(-\mathrm{i})^{2\lambda}|L(I_s p),\lambda'\rangle.$$

EXERCISE: Evaluate  $\mathcal{P}|H(p), \zeta\rangle, \mathcal{T}|H(p), \zeta\rangle$ .

## 7.4. Massless particles

This case is essentially different from the previous one, not merely the limit as  $m \to 0$ , something that could already have been imagined from what one finds for massless particles with the wave function formalism. To begin with, since a particle without mass cannot be at rest, the choice of  $\overline{p}$  is less helpful than before. What we do is merely define our spatial axes so that  $\overline{\mathbf{p}}$  points in a convenient direction, say, along OZ: we thus take

$$\overline{p}_1 = \overline{p}_2 = 0, \ \overline{p}_3 = \overline{p}_0.$$

The particular value of  $\overline{p}_0$  is (for systems with a single particle) irrelevant; we may get  $\overline{p}_0 = 1$  by a boost, or by just taking  $\overline{p}_0$  as the unit of energy.

Let us now consider the little group of this  $\overline{p}$ ,  $\mathcal{W}(\overline{p})$ . If  $\Gamma$  is in  $\mathcal{W}(\overline{p})$ , we can represent it as before. We then decompose  $\Gamma$  as

$$\Gamma = \Lambda_t R_z(\theta),$$

where  $R_z(\theta)$  is a rotation around OZ by an angle  $\theta$ , so that the corresponding matrix  $(\Gamma)$  is

$$(\Gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi & \eta & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma_{31} = \xi \cos \theta - \eta \sin \theta, \quad \Gamma_{32} = \xi \sin \theta + \eta \cos \theta.$$

The first term in the expression for  $(\Gamma)$ , viz.,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi & \eta & 1 \end{pmatrix},$$

corresponds to  $\Lambda_t$ ; the second one to  $R_z(\theta)$ . Because the product of two transformations  $\Gamma_1$ ,  $\Gamma_2$  in  $\mathcal{W}(\overline{p})$  lies in  $\mathcal{W}(\overline{p})$ , it follows that we can write

$$\Gamma_i = \Lambda_{it} R_z(\theta_i), \ i = 1, 2,$$

and

$$\Gamma_1 \Gamma_2 = \Lambda_{12t} R_z(\theta_{12}),$$

where the angle  $\theta_{12}$  will depend on  $\Gamma_1$ ,  $\Gamma_2$ :

$$\theta_{12} = \theta_{12}(\Gamma_1, \Gamma_2).$$

EXERCISE: Prove that, with self-explanatory notation,

$$\theta_{12}(\Gamma_1, \Gamma_2) = \theta_1 + \theta_2, \xi_{12}(\Gamma_1, \Gamma_2) = \xi_1 + (\cos \theta_1)\xi_2 - (\sin \theta_1)\eta_2, \quad \eta_{12}(\Gamma_1, \Gamma_2) = \eta_1 + (\cos \theta_1)\eta_2 + (\sin \theta_1)\xi_2.$$

To get a representation of the Poincaré group we require a representation of this little group,  $\mathcal{W}(\overline{p})$ . This little group is actually isomorphic to the Euclidean group in two dimensions, and its representations can be studied by the same methods we are using to find the representations of the Poincaré group. The details may be found in Wigner (1939)<sup>19</sup>; we will take from there, and without proof, the following result. If we want to have particles with discrete spin values, then the representation must be of the form

$$D(\Gamma) = D(R_z(\theta)),\tag{1}$$

i.e., we must have

$$D(\Lambda_t) \equiv 1. \tag{2}$$

Moreover, the representation  $D(R_z(\theta))$  can be at most double-valued, so that

$$D(R_z(2\pi)) = \pm 1.$$

This is because the covering group of the Lorentz group, SL(2,C), is simply connected and covers twice  $\mathcal{L}$ .

There is no physical reason for excluding particles with continuous spins (which have been studied by Wigner, 1963); but it is a fact that all particles found in nature have discrete spin values. We will therefore require (2).

With the help of this the analysis is easily completed. The *irreducible* representations of the  $R_z(\theta)$ , rotations around a fixed (OZ) axis, are trivial. Since the group is Abelian, Schur's lemma implies that these representations must be one-dimensional. From this it follows that the index  $\lambda$  in the classification of the states,

$$|\overline{p},\lambda\rangle$$
,

can only take one value. The matrices  $D_{\lambda'\lambda}(\Gamma)$  are therefore just numbers, equal to  $\delta_{\lambda\lambda'}d_{\lambda}(\theta)$ . Because the representation has to be unitary, these numbers are of modulus unity and we can write

$$d_{\lambda}(\theta) = e^{-\mathrm{i}\lambda\theta}.$$

<sup>&</sup>lt;sup>19</sup>Or in Wightman (1960), Bogoliubov, Logunov and Todorov (1975).

The fact that the representation is at most two-valued, implies that the number  $\lambda$  is integer or half integer. Its interpretation is readily accomplished by comparing the expression for  $d(\theta)$  with that for a rotation around the OZ axis in terms of the  $S_z$  component of the spin operator,

$$U(R_z(\theta)) = e^{-i\theta S_z/\hbar}$$
:

 $\lambda$  is the spin component along OZ (or along  $\overline{\mathbf{p}}$ , since it coincides with the OZ axis). This is the helicity. Because there is only one possible value of  $\lambda$ , it follows that, for massless particles, the helicity is relativistically invariant, something that can be seen in specific cases with the wave function formalism.

Once the transformation properties of the states  $|\overline{p}, \lambda\rangle$  under the little group  $\mathcal{W}(\overline{p})$ ,

$$U(\Gamma)|\overline{p},\lambda\rangle = e^{-i\lambda\theta(\Gamma)}|\overline{p},\lambda\rangle,$$

are known, we have to specify the family of transformations  $\Lambda(p)$  with  $\Lambda(p)\overline{p}=p$  to extend the analysis to arbitrary transformations. Choose  $\overline{p}_0=1$ ; for an arbitrary p we set

$$\Lambda(p) = H(p),$$
  
 $H(p) = R(\mathbf{z} \to \mathbf{p})L(p^z).$ 

 $L(p^z)$  is the pure boost along OZ such that

$$L(p^{z})\overline{p} = p^{z},$$
 
$$p_{0}^{z} = p_{0}, \ p_{1}^{z} = p_{2}^{z} = 0, \ p_{3}^{z} = p_{0};$$

 $R(\mathbf{z} \to \mathbf{p})$  is the rotation around the axis  $\mathbf{z} \times \mathbf{p}$  that carries OZ over  $\mathbf{p}$ . We then define

$$|p,\lambda\rangle \equiv U(H(p))|\overline{p},\lambda\rangle,$$

and we find that

$$U(\Lambda)|p,\lambda\rangle = e^{-\mathrm{i}\lambda\theta(p,\Lambda)}|\Lambda p,\lambda\rangle;$$

the angle  $\theta(p, \Lambda)$  is the angle of the OZ rotation contained in

$$\Gamma(p, \Lambda) = H(\Lambda p)^{-1} \Lambda H(p),$$

when we decompose it as

$$\Gamma(p,\Lambda) = \Lambda_t R_z(\theta(p,\Lambda)).$$

The normalization is

$$\langle p, \lambda | p, \lambda \rangle = 2p_0 \delta(\mathbf{p}, \mathbf{p}').$$

Next we consider the discrete symmetries  $\mathcal{P}$ ,  $\mathcal{T}$ . Starting with parity, the corresponding operator should satisfy

$$\mathcal{P}P_0\mathcal{P}^{-1} = P_0, \quad \mathcal{P}P\mathcal{P}^{-1} = -P,$$
  
 $\mathcal{P}L\mathcal{P}^{-1} = L, \quad \mathcal{P}S\mathcal{P}^{-1} = S;$ 

from this, and for the helicity operator

$$S_{\bf p} = (1/|{\bf p}|) \, {\bf PS},$$

we obtain

$$\mathcal{P}S_{\mathbf{p}}\mathcal{P}^{-1} = -S_{\mathbf{p}}.$$

Therefore we would have to postulate that

$$\mathcal{P}|p,\lambda\rangle = \eta_P|I_s p, -\lambda\rangle.$$

In general this will be impossible: because the value of  $\lambda$  is now *invariant*, this requires that there exist two independent states, a state with helicity  $\lambda$  and another with  $-\lambda$ . In nature we find two kinds of particle. In one class we have particles like the photon, gluons or, presumably, the graviton, which can exist in the two helicity states:  $\pm 1$  for the first two,  $\pm 2$  for the last. In the second class we have

particles,<sup>20</sup> like the neutrinos, which exist only with helicity -1/2; or the antineutrinos which always carry helicity +1/2. For these particles parity is not defined and indeed the interactions that involve them violate parity.

For neutrinos and antineutrinos we can define a combined operation,  $\mathcal{CP}$ , the product of parity and particle–antiparticle conjugation that carries neutrinos (with helicity -1/2) into antineutrinos (with helicity +1/2), and vice versa<sup>21</sup>. There is a third class, that of particles with helicity  $\lambda$  for which neither particles or antiparticles with helicity  $-\lambda$  existed, which is mathematically possible but of which no representative has been found in nature.

For time reversal,

$$T\mathbf{S}T^{-1} = -\mathbf{S}, \ T\mathbf{P}T^{-1} = -\mathbf{P},$$

so that

$$\mathcal{T} S_{\mathbf{p}} \mathcal{T}^{-1} = S_{\mathbf{p}},$$

and we can define the antiunitary operator  $\mathcal{T}$  with

$$\mathcal{T}|p,\lambda\rangle = \eta_T(-\mathrm{i})^{2\lambda}|I_s p,\lambda\rangle;$$

the phase  $(-i)^{2\lambda}$  is introduced for aesthetic reasons, to make the massless case similar to the massive one.

Let us return to parity. If the state  $|I_s p, -\lambda\rangle$  exists, we will have to double our Hilbert space of states to make room for it. We define total spin as  $s = \max |\lambda|$ , and chirality  $\delta$  as  $\delta = \lambda/s = \pm 1$ . We may label the states as

$$|p,s,\delta\rangle$$
,

and the transformation properties can then be written as

$$U(\Lambda)|p, s, \delta\rangle = e^{-i\delta s\theta(p, \Lambda)}|\Lambda p, s, \delta\rangle,$$
  
$$\mathcal{P}|p, s, \delta\rangle = \eta_P|I_s p, s, -\delta\rangle.$$

The representation is reducible as a representation of the Poincaré group because the subspaces with  $\delta = 1$  and  $\delta = -1$  are separately invariant; it is irreducible as a representation of the orthochronous (but not proper) group obtained adjoining space reversal,  $I_s$ , with  $U(I_s) \equiv \mathcal{P}$ , to the orthochronous, proper Poincaré group.

## 7.5. Connection with the wave function formalism

The construction of relativistic states with well-defined position,  $|\mathbf{r}, t, a\rangle$  (t is the time, and a represents possible extra labels) does not make much physical sense. Therefore, the connection between the abstract ket formalism and the wave function formalism is now less straightforward than in the nonrelativistic case, where we simply have  $\Psi_a(\mathbf{r},t) = \langle \mathbf{r}, t, a | \Psi \rangle$ . Now, we will connect with the momentum space wave functions; these can be then linked, via the appropriate Fourier transformations, to x-space ones.

We then want to establish the correspondence between ket states and (multicomponent) wave functions  $\psi_a^{(\mathbf{k},\lambda)}(\mathbf{p})$ , corresponding to momentum  $\mathbf{k}$  and spin component  $\lambda$  (note that here  $\mathbf{p}$  is the variable). We will work in the Heisenberg representation, so the  $\psi$  are time independent. Time dependence can be introduced, if so wished, by writing

$$\Psi_a^{(\mathbf{k},\lambda)}(\mathbf{p},t) = e^{-ik_0t}\psi_a^{(\mathbf{k},\lambda)}(\mathbf{p}), \ k_0 = \sqrt{m^2 + \mathbf{k}^2}.$$

Here we work in natural units,  $\hbar = c = 1$ .

<sup>&</sup>lt;sup>20</sup>We are here neglecting neutrino masses.

<sup>&</sup>lt;sup>21</sup>One can prove quite generally that the product  $\mathcal{CPT}$  is always a symmetry for any relativistic theory of local fields. For the proof see, for example, the text of Bogoliubov, Logunov and Todorov (1975).

The case of spinless particles is simple. We just have

$$\varphi^{(\mathbf{k})}(\mathbf{p}) = \langle p|k\rangle = 2k_0\delta(\mathbf{p} - \mathbf{k}),$$

but spin poses nontrivial problems. We will only consider the spin 1/2 case; the generalization to higher spins is straightforward, for  $m \neq 0$ , and can be found in Moussa and Stora (1968), Weinberg (1964) and Zwanziger (1964a,b). (The latter also treat the massless case).

The wave function of a particle of spin 1/2, with third component of covariant spin  $s_3$  and momentum  $\mathbf{k}$  can be written (extracting the time dependence) as

$$\psi^{(k,s_3)}(\mathbf{p}) = D(L(k))u(0,s_3)2k_0\delta(\mathbf{k} - \mathbf{p}).$$

Taking into account that

$$u(0, 1/2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u(0, -1/2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

it becomes convenient for our calculations to change the labels  $s_3 = \pm 1/2$  to  $\tau = 1, 2$ , so that  $1/2 \to 1$ ,  $-1/2 \to 2$ . Then we may write  $u_a(0,\tau) = \delta_{a\tau}$ , and (6.6.3) adopts the simple form

$$\psi_a^{(\mathbf{k},\tau)}(\mathbf{p}) = D_{a\tau}(L(k))2k_0\delta(\mathbf{k} - \mathbf{p}),$$

and we then have the explicit expression

$$u_a(k,\tau) = D_{a\tau}(L(k)).$$

 $D_{ab}(L(k))$  is the ab matrix element of the matrix D(L(k)); we will here use the Weyl representation of the  $\gamma$  matrices, so that

$$\gamma_{\mu}^{W} = \begin{pmatrix} 0 & \tilde{\sigma}_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix}, \quad \tilde{\sigma}_{i} = -\sigma_{i}, \ \tilde{\sigma}_{0} = \sigma_{0} = 1.$$

We have

$$D(L(k)) \equiv D(L(\mathbf{k})) = \frac{1}{\sqrt{m}} (k_0 + \mathbf{k}\boldsymbol{\alpha})^{1/2} = \frac{1}{\sqrt{m}} (k \cdot \gamma \gamma_0)^{1/2},$$

a formula valid in any representation. In Weyl's, this becomes

$$D^{W}(L(k)) = \frac{1}{\sqrt{m}} \begin{pmatrix} (k \cdot \tilde{\sigma})^{1/2} & 0\\ 0 & (k \cdot \sigma)^{1/2} \end{pmatrix}. \tag{1}$$

This is of course the reason why the Weyl representation is useful for us: the matrix  $D^{W}$  is "box-diagonal". Taking into account that the matrix that leads from the Pauli to the Weyl representation is

$$\frac{1}{\sqrt{2}}(\gamma_0^{\rm P} + \gamma_5^{\rm P}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and the known expression for the spinors in the Pauli relization (see, e.g., Ynduráin, 1996) we find for the spinors  $u(0,\tau)$ , in the Weyl realization,

$$u^{W}(0,1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}, \quad u^{W}(0,2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix}.$$
 (2)

In what follows we suppress the label "W".

We may rewrite the wave function as

$$\psi = \begin{pmatrix} \varphi \\ \tilde{\varphi} \end{pmatrix} \quad \psi_a^{(\mathbf{k},\tau)}(\mathbf{p}) = \varphi_\alpha^{(k,\tau)}(\mathbf{p}), \ a = \alpha = 1, 2; \quad \psi_b^{(\mathbf{k},\tau)}(\mathbf{p}) = \tilde{\varphi}_{\dot{\beta}}^{(k,\tau)}(\mathbf{p}), \ b = \dot{\beta} + 2 = 3, 4,$$

with

$$\varphi_{\alpha}^{(k,\tau)}(\mathbf{p}) = \frac{1}{\sqrt{2m}} ((k \cdot \tilde{\sigma})^{1/2})_{\alpha\tau} 2k_0 \delta(\mathbf{p} - \mathbf{k}),$$

$$\tilde{\varphi}_{\dot{\beta}}^{(k,\tau)}(\mathbf{p}) = \frac{1}{\sqrt{2m}} ((k \cdot \tilde{\sigma})^{1/2})_{\dot{\beta}\tau} 2k_0 \delta(\mathbf{p} - \mathbf{k})$$
(3)

(the notation with dotted indices, such as  $\dot{\beta}$ , for the components  $\tilde{\varphi}_{\dot{\beta}}$  is the traditional one).

Because  $\psi$  satisfies the Dirac equation, it follows that we can get  $\tilde{\varphi}$  in terms of  $\varphi$  (or vice versa). Indeed, we have

$$\tilde{\varphi}_{\dot{\beta}}^{(k,\tau)}(\mathbf{p}) = \sum_{a} \left(\frac{k \cdot \sigma}{m}\right)_{\dot{\beta}\alpha} \varphi_{\alpha}^{(k,\tau)}(\mathbf{p}). \tag{4}$$

EXERCISES: i) Prove (4) by verifying that the identity  $(k \cdot \sigma)(k \cdot \tilde{\sigma}) = k \cdot k$  implies that (3) is equivalent to the Dirac equation  $(k \cdot \gamma - m)\psi^{(\mathbf{k},\tau)}(\mathbf{p}) = 0$ . ii) Check that

$$(k \cdot \tilde{\sigma})^{1/2} = [2(k_0 + m)]^{-1/2}(m + k_0 + \mathbf{k}\boldsymbol{\sigma}).$$

Owing to this relation (4), it is sufficient to establish the connection between the states  $|k, \tau\rangle$  and the wave functions  $\varphi_{\alpha}^{(k,\tau)}(\mathbf{p})$ . This is achieved by introducing the so-called *spinorial states*,  $|p,\alpha\rangle$ , defined to be such that

$$\varphi_{\alpha}^{(k,\tau)}(\mathbf{p}) \equiv \langle p, \alpha | k, \tau \rangle.$$

Taking into account the explicit form of the  $\varphi$ , we obtain the formula that links the spinorial states to the familiar states with given covariant spin  $|k, \tau\rangle$ : it is

$$|p,\alpha\rangle = \sum_{\tau} \int \frac{d^3k}{2k_0} \left( \left( \frac{k \cdot \tilde{\sigma}}{2m} \right)^{1/2} \right)_{\tau\alpha} 2k_0 \delta(\mathbf{p} - \mathbf{k}) |k,\tau\rangle,$$

and we have used the Hermiticity of the matrix  $(k \cdot \tilde{\sigma})^{1/2}$ .

The matrix  $(k \cdot \tilde{\sigma}/m)^{1/2}$  is not unitary. The basis  $|p,\alpha\rangle$  is therefore not orthogonal; rather one has

$$\langle p', \alpha' | p, \alpha \rangle = \frac{(p \cdot \tilde{\sigma})_{\alpha'\alpha}}{2m} 2p_0 \delta(\mathbf{p} - \mathbf{p}')$$

The index  $\alpha$  does not correspond to any quantum number.

EXERCISE: Prove that  $d^3p/2p_0$ ,  $2p_0\delta(\mathbf{p}-\mathbf{p}')$  are invariant by writing, for  $p_0>0$ ,

$$\delta_4(p-p') = \delta(p^2 - p'^2) 2p_0 \delta(\mathbf{p} - \mathbf{p}').$$

EXERCISE: Find  $R(p, \Lambda)$  in the NR limit, including corrections  $O(v^2/c^2)$ .

EXERCISE: Find  $R(p, I_s)$  for  $\Lambda(p) = L(p)$ . Find  $|H(p), \lambda\rangle$  in terms of  $|L(p), \eta\rangle$ , and viceversa.

EXERCISE: Let  $W^{\mu} = \epsilon^{\mu\nu\rho\sigma} P_{\nu} M_{\rho\sigma}$  (Pauli-Lubanski vector). Prove that  $W^2 = \text{invariant} = -m^2 s(s+1)$ , s the spin.

EXERCISE: Verify that, for any  $\Lambda$ ,

$$U(\Lambda): \varphi_{\alpha}^{(k,\tau)}(p) \to \sum_{\alpha'} D_{\alpha\alpha'}^{(1/2)}(\Lambda) \varphi_{\alpha'}^{(k,\tau)}(\Lambda^{-1}p),$$

$$U(\Lambda): \varphi_{\alpha}^{(k,\tau)}(p) \to \sum_{\alpha'} D_{\tau'\tau}^{(1/2)}(R(k,\Lambda)) \varphi_{\alpha}^{(\Lambda k,\tau')}(p).$$

Here,  $D(\Lambda) = D(L)D(R)$ , for  $\Lambda = LR$ , with

$$D_{\alpha\beta}^{(1/2)}(L(p)) = m^{-1}(p \cdot \tilde{\sigma})_{\alpha\beta}^{1/2}, \ D_{\alpha\beta}^{(1/2)}(R(\boldsymbol{\theta})) = \left(e^{-i\boldsymbol{\theta}\boldsymbol{\sigma}/2}\right)_{\alpha\beta}, \text{ etc.}$$

# 7.6. Two-Particle States. Separation of the Center of Mass Motion. States with Well-Defined Angular Momentum

Although the subject of this subsection has little to do with groups, we include it here for completeness.

Let us consider two free particles (which for simplicity we take to be distinguishable), A, B, with masses  $m_A$ ,  $m_B$ . A state of these two particles can be specified by giving the momenta  $\mathbf{p}_A$ ,  $\mathbf{p}_B$  and spin quantum numbers (for example, the helicities) to be denoted by  $\alpha$ ,  $\beta$ : we thus write it as

$$|p_A, \alpha; p_B, \beta\rangle$$
,  $p_{A0} \equiv \sqrt{m_A^2 + \mathbf{p}_A^2}$ ,  $p_{B0} \equiv \sqrt{m_B^2 + \mathbf{p}_B^2}$ 

with normalization

$$\langle p_A', \alpha'; p_B', \beta' | p_A, \alpha; p_B, \beta \rangle = \delta_{\alpha\alpha'} 2p_{A0} \delta(\mathbf{p}_A - \mathbf{p}_A') \times \delta_{\beta\beta'} 2p_{B0} \delta(\mathbf{p}_B - \mathbf{p}_B').$$

The same state can be specified by giving the total four-momentum,  $p = p_A + p_B$ , the direction of the relative three-momentum,  $\mathbf{k} = (\mathbf{p}_A - \mathbf{p}_B)/2$ , and the spin labels  $\alpha$ ,  $\beta$ :

$$|p_A, \alpha; p_B, \beta\rangle = |p; \mathbf{k}; \alpha, \beta\rangle;$$

we write  $\mathbf{k}$ , which is redundant (just as  $p_{A0}$ ,  $p_{B0}$  were redundant before) instead of  $\Omega_{\mathbf{k}}$  (the angular variables of  $\mathbf{k}$ ) for simplicity of notation.

EXERCISE: Show that, given p,  $\Omega_{\mathbf{k}}$  we can reconstruct  $p_A$ ,  $p_B$ .

The tensor product notation is at times convenient, and we will thus write

$$|p_A, \alpha\rangle \otimes |p_B, \beta\rangle = |p_A, \alpha; p_B, \beta\rangle = |p; \mathbf{k}; \alpha, \beta\rangle = |p\rangle \otimes |\mathbf{k}; \alpha, \beta\rangle.$$

The scalar product can be easily expressed in terms of the new variables: first,

$$\delta(\mathbf{p}_A - \mathbf{p}_A')\delta(\mathbf{p}_B - \mathbf{p}_B') = \delta(\mathbf{p} - \mathbf{p}')\delta(\mathbf{k} - \mathbf{k}');$$

then, we can use the relation

$$\delta(\mathbf{k} - \mathbf{k}') = \frac{1}{\mathbf{k}^2} \delta(|\mathbf{k}| - |\mathbf{k}'|) \delta(\Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'}) = \frac{1}{\mathbf{k}^2} J^{-1} \delta(p_0 - p_0') \delta(\Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'}),$$

where J is the Jacobian  $J = \partial |\mathbf{k}|/\partial p_0$ , to get

$$\delta(\mathbf{p}_A - \mathbf{p}_A')\delta(\mathbf{p}_B - \mathbf{p}_B') = (1/J\mathbf{k}^2)\delta(p_0 - p_0')\delta(\Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'}).$$

We will only need the relative motion (described by  $\mathbf{k}$ ) in the center of mass (c.m.) system,  $\mathbf{p}=0$ . Here,  $p_0=p_{A0}+p_{B0}=(m_A^2+\mathbf{k}^2)^{1/2}+(m_B^2+\mathbf{k}^2)^{1/2}$  so that

$$J = \partial |\mathbf{k}|/\partial p_0 = p_{A0}p_{B0}/p_0|\mathbf{k}|,$$

and finally we obtain

$$\langle p'_A, \alpha'; p'_B, \beta' | p_A, \alpha; p_B, \beta \rangle = \langle p'; \mathbf{k}'; \alpha', \beta' | p; \mathbf{k}; \alpha, \beta \rangle = \frac{4p_0}{|\mathbf{k}|} \delta_4(p - p') \delta(\Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'}) \delta_{\alpha\alpha'} \delta_{\beta\beta'},$$
$$\delta(\Omega - \Omega') \equiv \delta(\cos \theta - \cos \theta') \delta(\phi - \phi'),$$

with  $\theta$ ,  $\phi$  the polar angles corresponding to the solid angle  $\Omega$ . We write this also as

$$\langle p'|p\rangle = \delta_4(p'-p), \ \langle \mathbf{k}';\alpha',\beta'|\mathbf{k};\alpha,\beta\rangle = \frac{4p_0}{|\mathbf{k}|}\delta(\Omega_{\mathbf{k}'}-\Omega_{\mathbf{k}})\delta_{\alpha\alpha'}\delta_{\beta\beta'}.$$

This will allow us to introduce a completeness relation once we ascertain the range of the variables  $p_0$ ,  $\mathbf{p}$ . Clearly,  $\mathbf{p}$  varies over all space; but  $p_0$  is limited by

$$p_0 = p_{A0} + p_{B0} = \sqrt{m_A^2 + \mathbf{p}_A^2} + \sqrt{m_B^2 + \mathbf{p}_B^2} = \sqrt{p^2 + \mathbf{p}^2},$$
  
 $p^2 \ge (m_A + m_B)^2.$ 

We can thus write the four-dimensional delta as

$$\delta_4(p-p') = 2p_0\delta(\mathbf{p} - \mathbf{p}')\delta(p^2 - p'^2),$$

so that the completeness relation can be expressed separating the c.m. piece, which behaves as a composite particle with (variable) squared mass  $p^2$  and momentum  $\mathbf{p}$ , and the relative motion, described by  $\mathbf{k}$ , as follows:

$$1 = \sum_{\alpha\beta} \int \frac{\mathrm{d}^{3}p_{A}}{2p_{A0}} \int \frac{\mathrm{d}^{3}p_{B}}{2p_{B0}} |p_{A}, \alpha; p_{B}, \beta\rangle \langle p_{A}, \alpha; p_{B}, \beta|$$

$$= \sum_{\alpha\beta} \int \mathrm{d}^{4}p \int d\Omega_{\mathbf{k}} \frac{|\mathbf{k}|}{4p_{0}} |p; \mathbf{k}; \alpha, \beta\rangle \langle p; \mathbf{k}; \alpha, \beta|$$

$$= \int_{(m_{A}+m_{B})^{2}}^{\infty} \mathrm{d}(p^{2}) \int \frac{\mathrm{d}^{3}p}{2p_{0}} |p\rangle \langle p| \otimes \sum_{\alpha\beta} \int d\Omega_{\mathbf{k}} \frac{|\mathbf{k}|}{4p_{0}} |\mathbf{k}; \alpha, \beta\rangle \langle \mathbf{k}; \alpha, \beta|$$

$$= 1_{\text{CM}} \otimes 1_{\text{rel}}.$$

In the c.m. system one can construct states with well-defined *orbital* angular momentum l, and third component M as in the nonrelativistic case: we have

$$|l, M; \alpha, \beta\rangle = \int d\Omega_{\mathbf{k}} Y_M^l(\Omega_{\mathbf{k}}) |\mathbf{k}; \alpha, \beta\rangle.$$

The completeness relation can again be expressed in terms of the states  $|l, M; \alpha, \beta\rangle$ : separating c.m. and relative motion, we get

$$\begin{split} 1 &= 1_{\text{c.m.}} \otimes 1_{\text{rel}}; \\ 1_{\text{c.m.}} &= \int \mathrm{d}^4 p |p\rangle \langle p|, \\ 1_{\text{rel}} &= \sum_{\alpha\beta} \int \mathrm{d}\Omega_{\mathbf{k}} \frac{|\mathbf{k}|}{4p_0} |\mathbf{k}; \alpha, \beta\rangle \langle \mathbf{k}; \alpha, \beta| \\ &= \frac{|\mathbf{k}|}{4p_0} \sum_{\alpha\beta} \sum_{lM} |l, M; \alpha, \beta\rangle \langle l, M; \alpha, \beta|. \end{split}$$

One can, if so wished, compose the angular momentum and spins; we leave the subject here (see e.g. Ynduráin, 1996).

#### -ELEMENTS OF GROUP THEORY-

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