Hydrogen

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1 Matching

QED Lagrangian is

$$\mathcal{L}_{QED} = \bar{l}(i\not{D} - m)l + \bar{N}(iD^0)N - \mathcal{L}_\gamma \tag{1}$$

Set the NRQED Lagrangian as (take large M limit where M is the mass of the proton/hydrogen nucleus)

$$\mathcal{L}_{NRQED} = \psi^\dagger(iD_0 + \frac{\mathbf{D}^2}{2m})\psi + \bar{N}(iD_0)N + \mathcal{L}_{4-fer} + \mathcal{L}_\gamma \tag{2}$$
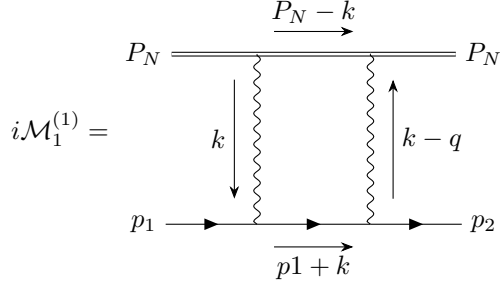
In tree level[1]

$$i\mathcal{M}^{(0)}_{QED} = \qquad\qquad = -e^2\bar{u}_N(P_N)v^0 u_N(P_N)\frac{i}{\mathbf{q}^2}\bar{u}_e(p_2)\gamma_0 u_e(p1)$$

$$i\mathcal{M}^{(0)}_{NRQED} = \qquad\qquad = -e^2\bar{u}_N(P_N)v^0 u_N(P_N)\frac{i}{\mathbf{q}^2}\psi^\dagger(p_2)\psi(p1)$$

The box diagram for NRQED process is

$$i\mathcal{M}^{(1)}_{NRQED} =$$

$$= e^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int[\mathrm{d}k]\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)$$

$$= -ie^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m})}\psi(p_1)$$

$$= -ie^4\bar{u}_N(P_N)\frac{1+\gamma^0}{2}u_N(P_N)\psi^\dagger(p_2)\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2(E_1-\frac{\mathbf{k}^2}{2m})}\psi(p_1)$$
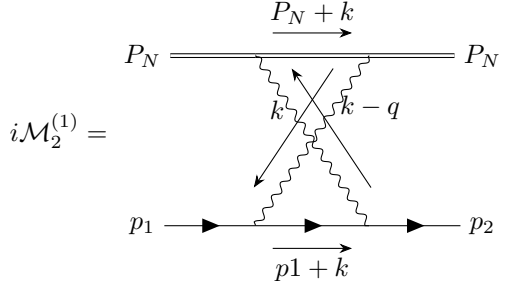
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[1]Note that there's no Gamma matrice in the heavy particle side, they can only appear in the QED side.

1

The box and crossed box diagram for QED process is

$$i\mathcal{M}_1^{(1)} =$$

$$= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{(\slashed{p}_1 + \slashed{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1)$$

$$= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{2p_1^0 + \slashed{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](-k^0 + i\epsilon)} u_e(p_1)$$

$$= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 + \sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2 + m^2 - p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}]} u_e(p_1)$$

$$= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 + \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 - p_1^0\sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)$$

$i\mathcal{M}_1^{(1)}$ has infrared log divergence and no ultraviolet divergence.

$$i\mathcal{M}_2^{(1)} =$$

$$= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{(\slashed{p}_1 + \slashed{k} + m)\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1)$$

$$= e^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int [\mathrm{d}k] \frac{2p_1^0 + \slashed{k}\gamma^0}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(p_1+k)^2 - m^2 + i\epsilon](k^0 + i\epsilon)} u_e(p_1)$$

$$= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{p_1^0 + k_i \gamma^i \gamma^0 - \sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}}{2\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2[(\mathbf{k}+\mathbf{p_1})^2 + m^2 + p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2 + m^2}]} u_e(p_1)$$

$$= -ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{p_1^0 + (k_i - p_{1i})\gamma^i \gamma^0 - \sqrt{\mathbf{k}^2 + m^2}}{2(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 + p_1^0\sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)$$

$i\mathcal{M}_2^{(1)}$ has no infrared or ultraviolet divergence.

$$i\mathcal{M}_1^{(1)} + i\mathcal{M}_2^{(1)} = ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{p_1^{0\,2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 + m^2 - p_1^{0\,2}]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1)$$

$$= ie^4 \bar{u}_N(P_N) \frac{1+\gamma^0}{2} u_N(P_N) u_e^\dagger(p_2) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{p_1^{0\,2} + k^2 + m^2 + (k_i - p_{1i})p_1^0 \gamma^i \gamma^0}{(\mathbf{k}-\mathbf{p_1})^2(\mathbf{k}-\mathbf{p_2})^2[\mathbf{k}^2 - \mathbf{p_1}^2]\sqrt{\mathbf{k}^2 + m^2}} u_e(p_1)$$
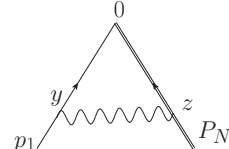
Note that after the expansion over external momentum, k^i can be converted into p^i so it's actually at p^1 order.

Now consider operator product expansion.

Tree level matching: p_1 x 0 P_N & p_1 0 P_N

$$\langle 0|T\psi(x)N(0)|pP_N\rangle = \quad \text{} \quad = u_e(p)u_N(P_N)e^{-ip\cdot x}$$

$$\langle 0|T\psi_e(x)N(0)|pP_N\rangle = \quad \text{} \quad = \psi_e(p)u_N(P_N)$$
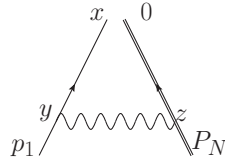
At leading order $u_e(p) = \begin{pmatrix} \psi_e(p) \\ 0 \end{pmatrix}$. (If we're only interested in the hard region contribution, which is independent of states, the leading order is independent of any on-shell momentums.)

One loop scenario for NRQED case:

$$\langle 0|\psi_e(0)N(0)e\int d^4y\bar{\psi}_e\psi_e A^0 e\int d^4z\bar{N}NA^0|eN\rangle = e^2 u_N(v_N)\int[dk]\frac{1}{\mathbf{k}^2(-k^0+i\epsilon)(p_1^0+k^0-m-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)$$

$$= -ie^2 u_N(v_N)\int\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{(\mathbf{p_1}+\mathbf{k})^2}{2m}+i\epsilon)}\psi(p_1)$$

$$= -ie^2 u_N(v_N)\int\frac{d^3k}{(2\pi)^3}\frac{1}{(\mathbf{k}-\mathbf{p_1})^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1)$$

drop p_1

$$= -ie^2 u_N(v_N)\int\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2(E_1-\frac{\mathbf{k}^2}{2m}+i\epsilon)}\psi(p_1) = \pi ie^2\sqrt{\frac{2m}{E_1}}u_N(v_N)\psi(p_1)$$

For QED case:

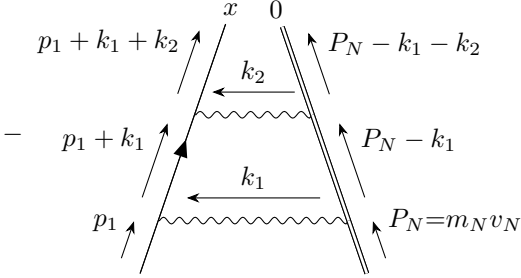$$\langle 0|\psi(x)N(0)e\int d^4y\bar{\psi}\gamma^0\psi A^0 e\int d^4z\bar{N}NA^0|eN\rangle\,^2 = e^2 u_N(v_N)\int[dk]e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{(\not{p}_1+\not{k}+m)\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)$$

$$= e^2 u_N(v_N)\int[dk]e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{2p_1^0+\not{k}\gamma^0}{\mathbf{k}^2[(p_1+k)^2-m^2+i\epsilon](-k^0+i\epsilon)}u_e(p_1)$$

$$= ie^2 u_N(v_N)\int\frac{d^3k}{(2\pi)^3}e^{-i(\mathbf{k}+\mathbf{p_1})\cdot\mathbf{x}}\frac{p_1^0+k_i\gamma^i\gamma^0+\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}}{2\mathbf{k}^2[(\mathbf{k}+\mathbf{p_1})^2+m^2-p_1^0\sqrt{(\mathbf{k}+\mathbf{p_1})^2+m^2}]}u_e(p_1)$$

$$= ie^2 u_N(v_N)\int\frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{p_1^0+(k_i-p_{1i})\gamma^i\gamma^0+\sqrt{\mathbf{k}^2+m^2}}{2(\mathbf{k}-\mathbf{p_1})^2[\mathbf{k}^2+m^2-p_1^0\sqrt{\mathbf{k}^2+m^2}]}u_e(p_1)$$

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drop $\mathbf{p_1}$

$$= ie^2 u_N(v_N) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{m + \sqrt{\mathbf{k}^2 + m^2}}{2\mathbf{k}^2[\mathbf{k}^2 + m^2 - m\sqrt{\mathbf{k}^2 + m^2}]} u_e(p_1)$$

$$= ie^2 u_N(v_N) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{m + \sqrt{\mathbf{k}^2 + m^2}}{2\mathbf{k}^2\sqrt{\mathbf{k}^2 + m^2}[\sqrt{\mathbf{k}^2 + m^2} - m]} u_e(p_1)$$

we can see there's no UV divergence here. IR divergence is presented but it's exactly the same with the NRQED one so there's nothing that needs to concern about.

Two loop scenario for QED case $\langle 0|T\psi(x)N(0)e\int \mathrm{d}^4 y_1 \bar\psi\gamma^0\psi A^0 e\int \mathrm{d}^4 z_1 \bar N N A^0 e\int \mathrm{d}^4 y_2 \bar\psi\gamma^0\psi A^0 e\int \mathrm{d}^4 z_2 \bar N N A^0|eN\rangle$:

$$=e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_1^0 + i\epsilon} \frac{\slashed{p}_1 + \slashed{k}_1 + \slashed{k}_2 + m}{(p_1+k_1+k_2)^2 - m^2 + i\epsilon} \gamma^0 \frac{\slashed{p}_1 + \slashed{k}_1 + m}{(p_1+k_1)^2 - m^2 + i\epsilon} \gamma^0 u_N(v_N) u_e(p_1)$$

$$=e^4 \int [dk_1][dk_2] e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} \frac{4p_1^{0\,2} + 2p_1^0 k_1^0 + 2\mathbf{p_1}\cdot\mathbf{k_1} + 2(p_1^0 + k_1^0)(\slashed{k}_1 + \slashed{k}_2)\gamma^0 - \slashed{k}_2\slashed{k}_1}{[(p_1+k_1+k_2)^2 - m^2 + i\epsilon][(p_1+k_1)^2 - m^2 + i\epsilon][-k_1^0 - k_2^0 + i\epsilon][-k_1^0 + i\epsilon]} u_N(v_N) u_e(p_1)^3$$

$$=ie^4 \int [dk_1] \frac{\mathrm{d}^3 \mathbf{k_2}}{(2\pi)^3} \frac{2(k_1^0 + p_1^0)[(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2} - k_1^0 - p_1^0) + (k_2^i\gamma_i + \slashed{k}_1)\gamma^0] - [\gamma^0(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2} - k_1^0 - p_1^0) - k_2^i\gamma_i]\slashed{k}_1}{2\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2}(\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2} - p_1^0 + \frac{2((\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2) + 2\sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2} - p_1^0}{2((\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2)} i\epsilon)}$$

$$\frac{1}{-k_1^0 + i\epsilon} \frac{1}{(p_1+k_1)^2 - m^2 + i\epsilon} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

define $a = (\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2$ and $b = \sqrt{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2 + m^2} - k_1^0 - p_1^0 + k_2^i\gamma_i\gamma^0 = \sqrt{a} - k_1^0 - p_1^0 + k_2^i\gamma_i\gamma^0$ (pole location is $\sqrt{a} - k_1^0 - p_1^0 - \frac{i\epsilon}{2\sqrt{a}}$), and note that the long coefficient of the first ϵ above is positive

$$=ie^4 \int [dk_1] \frac{\mathrm{d}^3 \mathbf{k_2}}{(2\pi)^3} \frac{4p_1^{0\,2} + 2p_1^0 k_1^0 + 2\mathbf{p_1}\cdot\mathbf{k_1} + 2(k_1^0 + p_1^0)[b + \slashed{k}_1\gamma^0] - \gamma^0 b\slashed{k}_1}{2\sqrt{a}(\sqrt{a} - p_1^0)[-k_1^0 + i\epsilon][(p_1+k_1)^2 - m^2 + i\epsilon]} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

also define b' so that $b = b' - k_1^0$ ($b' = \sqrt{a} - p_1^0 + k_2^i\gamma_i\gamma^0$) and $a' = (\mathbf{p_1}+\mathbf{k_1})^2 + m^2$

$$=ie^4 \int [dk_1] \frac{\mathrm{d}^3 \mathbf{k_2}}{(2\pi)^3} \frac{4p_1^{0\,2} + 2p_1^0 k_1^0 + 2\mathbf{p_1}\cdot\mathbf{k_1} + 2(k_1^0 + p_1^0)[b' + k_1^i\gamma_i\gamma^0] - \gamma^0(b' - k_1^0)(k_1^0\gamma^0 + k_1^i\gamma_i)}{2\sqrt{a}(\sqrt{a} - p_1^0)[-k_1^0 + i\epsilon][(p_1+k_1)^2 - m^2 + i\epsilon]|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

the pole location is $\sqrt{a'} - p_1^0 - \frac{i\epsilon}{2\sqrt{a}}$

$$= -e^4 \int \frac{\mathrm{d}^3 \mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3 \mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} - p_1^0 + (k_1+k_2)^i\gamma_i\gamma^0) - (\sqrt{a'} - \sqrt{a} + k_2^i\gamma_i\gamma^0)(\sqrt{a'} - p_1^0 - k_1^i\gamma_i\gamma^0) + 2p_1^{0\,2} + 2\mathbf{k_1}\cdot\mathbf{p_1} + 2\sqrt{a'}p_1^0}{4\sqrt{a}\sqrt{a'}(\sqrt{a'} - p_1^0)(\sqrt{a} - p_1^0)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}$$
$$u_N(v_N) u_e(p_1) e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

shift both loop momentum[4] so that $a = |\mathbf{k_2}|^2 + m^2$ and $a' = |\mathbf{k_1}|^2 + m^2$, now $b = \sqrt{a} - k_1^0 + (k_2 - k_1)^i\gamma_i\gamma^0$ and $b' = \sqrt{a} + (k_2 - k_1)^i\gamma_i\gamma^0$

$$= -e^4 \int \frac{\mathrm{d}^3 \mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3 \mathbf{k_2}}{(2\pi)^3} u_N(v_N) u_e(p_1) e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$
$$\frac{2\sqrt{a'}(\sqrt{a} - p_1^0 + (k_2 - p_1)^i\gamma_i\gamma^0) - (\sqrt{a'} - \sqrt{a} + (k_2 - k_1)^i\gamma_i\gamma^0)(\sqrt{a'} - p_1^0 - (k_1 - p_1)^i\gamma_i\gamma^0) + 2m^2 + 2\mathbf{k_1}\cdot\mathbf{p_1} + 2\sqrt{a'}p_1^0}{4\sqrt{a}\sqrt{a'}(\sqrt{a'} - p_1^0)(\sqrt{a} - p_1^0)|\mathbf{k_1} - \mathbf{p_1}|^2|\mathbf{k_2} - \mathbf{k_1}|^2}$$

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[2] $\langle 0|\psi(x)N(0)e\int \mathrm{d}^4 y \bar\psi\gamma^0\psi A^0 e\int \mathrm{d}^4 z \bar N N A^0|eN\rangle = e^2 \int \mathrm{d}^4 y \int \mathrm{d}^4 z \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{\mathbf{k}^2} e^{-ik\cdot(z-y)} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \tilde{S}_e(k_1) e^{-ik_1\cdot(y-x)} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \tilde{S}_N(k_2) u_N(v_N) u_e(p) e^{-ip_1\cdot y}.$

drop $\mathbf{p_1}$

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} - m + k_2^i\gamma_i\gamma^0) - (\sqrt{a'} - \sqrt{a} + (k_2 - k_1)^i\gamma_i\gamma^0)(\sqrt{a'} - m - k_1^i\gamma_i\gamma^0) + 2m^2 + 2\sqrt{a'}m}{4\sqrt{a}\sqrt{a'}(\sqrt{a'} - m)(\sqrt{a} - m)|\mathbf{k_1}|^2|\mathbf{k_2} - \mathbf{k_1}|^2}$$

$$u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

rewrite it with $a_1 = a'$ and $a_2 = a$, now $a_1 = \sqrt{|\mathbf{k_1}|^2 + m^2}$ and $a_2 = \sqrt{|\mathbf{k_2}|^2 + m^2}$
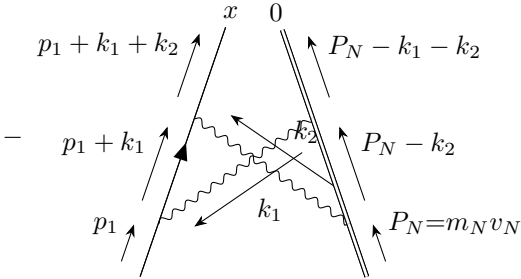
$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a_1}(\sqrt{a_2} - m + k_2^i\gamma_i\gamma^0) - (\sqrt{a_1} - \sqrt{a_2} + (k_2 - k_1)^i\gamma_i\gamma^0)(\sqrt{a_1} - m - k_1^i\gamma_i\gamma^0) + 2m^2 + 2\sqrt{a_1}m}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1} - m)(\sqrt{a_2} - m)|\mathbf{k_1}|^2|\mathbf{k_2} - \mathbf{k_1}|^2}$$

$$u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

to investigate the divergent behaviour of the integral, rewrite the integral before the shift ($a_1 = (\mathbf{p_1} + \mathbf{k_1})^2 + m^2$ and $a_2 = (\mathbf{p_1} + \mathbf{k_1} + \mathbf{k_2})^2 + m^2$)

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a_1}(\sqrt{a_2} - p_1^0 + (k_1 + k_2)^i\gamma_i\gamma^0) - (\sqrt{a_1} - \sqrt{a_2} + k_2^i\gamma_i\gamma^0)(\sqrt{a_1} - p_1^0 - k_1^i\gamma_i\gamma^0) + 2p_1^{0^2} + 2\mathbf{k_1}\cdot\mathbf{p_1} + 2\sqrt{a_1}p_1^0}{4\sqrt{a_2}\sqrt{a_1}(\sqrt{a_1} - p_1^0)(\sqrt{a_2} - p_1^0)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}$$

$$u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

we can see it's UV logarithm divergent and IR logarithm divergent (only for the $p_1^{0^2} \approx m^2$ term). Now we must regularize it to dimention $(d-1)$

$$= -e^4 \int \frac{\mathrm{d}^{d-1}\mathbf{k_1}}{(2\pi)^{d-1}} \frac{\mathrm{d}^{d-1}\mathbf{k_2}}{(2\pi)^{d-1}} \frac{2\sqrt{a_1}(\sqrt{a_2} - m + k_2^i\gamma_i\gamma^0) - (\sqrt{a_1} - \sqrt{a_2} + (k_2 - k_1)^i\gamma_i\gamma^0)(\sqrt{a_1} - m - k_1^i\gamma_i\gamma^0) + 2m^2 + 2\sqrt{a_1}m}{4\sqrt{a_1}\sqrt{a_2}(\sqrt{a_1} - m)(\sqrt{a_2} - m)|\mathbf{k_1}|^2|\mathbf{k_2} - \mathbf{k_1}|^2}$$

$$u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

$$=e^4 \int [dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0 - k_2^0 + i\epsilon} \frac{1}{-k_2^0 + i\epsilon} \frac{\not{p}_1 + \not{k}_1 + \not{k}_2 + m}{(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon}\gamma^0 \frac{\not{p}_1 + \not{k}_1 + m}{(p_1 + k_1)^2 - m^2 + i\epsilon}\gamma^0 u_N(v_N)u_e(p_1)$$

$$=e^4 \int [dk_1][dk_2]e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} \frac{4p_1^{0^2} + 2p_1^0k_1^0 + 2\mathbf{p_1}\cdot\mathbf{k_1} + 2(p_1^0 + k_1^0)(\not{k}_1 + \not{k}_2)\gamma^0 - \not{k}_2\not{k}_1}{[(p_1 + k_1 + k_2)^2 - m^2 + i\epsilon][(p_1 + k_1)^2 - m^2 + i\epsilon][-k_1^0 - k_2^0 + i\epsilon][-k_2^0 + i\epsilon]} u_N(v_N)u_e(p_1)$$

$$=ie^4 \int [dk_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{4p_1^{0^2} + 2p_1^0k_1^0 + 2\mathbf{p_1}\cdot\mathbf{k_1} + 2(k_1^0 + p_1^0)[b + \not{k}_1\gamma^0] - \gamma^0b\not{k}_1}{2\sqrt{a_2}(\sqrt{a_2} - k_1^0 + i\epsilon)\left[k_1^0 + p_1^0 - \sqrt{a_2} + \frac{i\epsilon}{2\sqrt{a_2}} + i\epsilon\right][(p_1 + k_1)^2 - m^2 + i\epsilon]} \frac{1}{|\mathbf{k_1}|^2|\mathbf{k_2}|^2} u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a} - p_1^0 + (k_1 + k_2)^i\gamma_i\gamma^0) + (\sqrt{a'} + \sqrt{a} + k_2^i\gamma_i\gamma^0)(\sqrt{a'} + p_1^0 + k_1^i\gamma_i\gamma^0) - 2p_1^{0^2} - 2\mathbf{k_1}\cdot\mathbf{p_1} + 2\sqrt{a'}p_1^0}{4\sqrt{a_1}\sqrt{a_2}\left(\sqrt{a_1} + \sqrt{a_2}\right)\left(\sqrt{a_2} - p_1^0\right)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}$$

$$u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

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[3]With a $u_e(p_1)$ on the right hand side, $(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0(\not{p}_1 + \not{k}_1 + m)\gamma^0 = (\not{p}_1 + \not{k}_1 + \not{k}_2 + m)[2(p_1^0 + k_1^0)\gamma^0 - (\not{p}_1 + \not{k}_1 - m)] = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (\not{p}_1 + \not{k}_1 + \not{k}_2 + m)(\not{p}_1 + \not{k}_1 - m) = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\not{k}_1 = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (2p_1 \cdot k_1 - \not{k}_1\not{p}_1 + \not{k}_2\not{k}_1 + \not{k}_1 m) = 2(p_1^0 + k_1^0)(\not{p}_1 + \not{k}_1 + \not{k}_2 + m)\gamma^0 - (2p_1 \cdot k_1 + \not{k}_2\not{k}_1) = 2(p_1^0 + k_1^0)[2(p_1^0 + k_1^0) - \gamma^0(\not{p}_1 + \not{k}_1 + \not{k}_2 - m)] - (2p_1 \cdot k_1 + \not{k}_2\not{k}_1) = 2(p_1^0 + k_1^0)[2p_1^0 + (\not{k}_1 + \not{k}_2)\gamma^0] - (2p_1 \cdot k_1 + \not{k}_2\not{k}_1) = 4p_1^{0^2} + 2p_1^0k_1^0 + 2\mathbf{p_1}\cdot\mathbf{k_1} + 2(p_1^0 + k_1^0)(\not{k}_1 + \not{k}_2)\gamma^0 - \not{k}_2\not{k}_1$.

[4]$k_1 \to k_1' = k_1 + p_1$ and $k_2 \to k_2' = k_1 + k_2 + p_1 = k_1' + k_2$.

5

do the shift

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a}-p_1^0+(k_1+k_2)^i\gamma_i\gamma^0)+(\sqrt{a'}+\sqrt{a}+k_2^i\gamma_i\gamma^0)(\sqrt{a'}+p_1^0+k_1^i\gamma_i\gamma^0)-2p_1^{0^2}-2\mathbf{k_1}\cdot\mathbf{p_1}+2\sqrt{a'}p_1^0}{4\sqrt{a_1}\sqrt{a_2}\left(\sqrt{a_1}+\sqrt{a_2}\right)\left(\sqrt{a_2}-p_1^0\right)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}$$

$$u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

drop $\mathbf{p_1}$

$$= -e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{2\sqrt{a'}(\sqrt{a}-p_1^0+(k_1+k_2)^i\gamma_i\gamma^0)+(\sqrt{a'}+\sqrt{a}+k_2^i\gamma_i\gamma^0)(\sqrt{a'}+p_1^0+k_1^i\gamma_i\gamma^0)-2p_1^{0^2}-2\mathbf{k_1}\cdot\mathbf{p_1}+2\sqrt{a'}p_1^0}{4\sqrt{a_1}\sqrt{a_2}\left(\sqrt{a_1}+\sqrt{a_2}\right)\left(\sqrt{a_2}-p_1^0\right)|\mathbf{k_1}|^2|\mathbf{k_2}|^2}$$

$$u_N(v_N)u_e(p_1)e^{-i\mathbf{k_2}\cdot\mathbf{x}}$$

The sum of QED diagram at NNLO is

$$=e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} u_N(v_N)u_e(p_1)e^{-i(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})\cdot\mathbf{x}}$$

For NRQED case ($\langle 0|\psi_e(0)N(0)e\int \mathrm{d}^4y_1\bar{\psi}_e\psi_e A^0 e\int \mathrm{d}^4z_1\bar{N}NA^0 e\int \mathrm{d}^4y_2\bar{\psi}_e\psi_e A^0 e\int \mathrm{d}^4z_2\bar{N}NA^0|eN\rangle$)

$$=e^4 \int [dk_1][dk_2] \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0-k_2^0+i\epsilon} \frac{1}{-k_1^0+i\epsilon} \frac{1}{p_1^0+k_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1})^2}{2m}+i\epsilon} \frac{1}{p_1^0+k_1^0+k_2^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+i\epsilon} \psi_e(p_1)u_N(v_N)^{[5]}$$

$$=ie^4 \int [dk_1]\frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{-k_1^0+i\epsilon} \frac{1}{p_1^0+k_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1})^2}{2m}+i\epsilon} \frac{1}{p_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+2i\epsilon} \psi_e(p_1)u_N(v_N)$$

$$=-e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}|^2} \frac{1}{p_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1})^2}{2m}+2i\epsilon} \frac{1}{p_1^0-m-\frac{(\mathbf{p_1}+\mathbf{k_1}+\mathbf{k_2})^2}{2m}+2i\epsilon} \psi_e(p_1)u_N(v_N)$$

do the shift as above

$$=-e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}-\mathbf{p_1}|^2} \frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2} \frac{1}{p_1^0-m-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon} \frac{1}{p_1^0-m-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon} \psi_e(p_1)u_N(v_N)$$

drop $\mathbf{p_1}$

$$=-e^4 \int \frac{\mathrm{d}^3\mathbf{k_1}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k_2}}{(2\pi)^3} \frac{1}{|\mathbf{k_1}|^2} \frac{1}{|\mathbf{k_2}-\mathbf{k_1}|^2} \frac{1}{-\frac{|\mathbf{k_1}|^2}{2m}+2i\epsilon} \frac{1}{-\frac{|\mathbf{k_2}|^2}{2m}+2i\epsilon} \psi_e(p_1)u_N(v_N)$$

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[5]Clearly in this line, if this NRQCD diagram is crossed, the second pole would become $-k_2^0+i\epsilon$ and the whole formula is zero (since both poles of k_1^0 would be in the same side).

2 HSET

2.1 Lagrangian

For scalar QED
$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi$$

where
$$D_\mu = \partial_\mu + ieA_\mu$$

In Schwartz's QFT (Chap. 35) he mentioned a choice of χ_v and $\tilde{\chi}_v$:

$$\phi(x) = e^{-imv \cdot x} \frac{1}{\sqrt{2m}} (\chi_v(x) + \tilde{\chi}_v(x)) \tag{3}$$

$$\chi_v(x) = e^{imv \cdot x} \frac{1}{\sqrt{2m}} (iv \cdot D + m)\phi(x), \ \tilde{\chi}_v(x) = e^{imv \cdot x} \frac{1}{\sqrt{2m}}(-iv \cdot D + m)\phi(x) \tag{4}$$

Put (3) into (4), a simple relation is derived:

$$(-iv \cdot D)\chi_v(x) = (2m + iv \cdot D)\tilde{\chi}_v(x)$$

It can also be writen as
$$2m\tilde{\chi}_v = (-iv \cdot D)(\chi_v + \tilde{\chi}_v)$$

Use this result

$$\mathcal{L} = \frac{1}{2m}\Big\{ \{[D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger + imv^\mu(\chi_v + \tilde{\chi}_v)^\dagger\}\{[D_\mu(\chi_v + \tilde{\chi}_v)] - imv_\mu(\chi_v + \tilde{\chi}_v)\} - m^2(\chi_v + \tilde{\chi}_v)^\dagger(\chi_v + \tilde{\chi}_v)\Big\}$$

$$= (\chi_v + \tilde{\chi}_v)^\dagger(iv \cdot D)(\chi_v + \tilde{\chi}_v) + \frac{1}{2m}[D^\mu(\chi_v + \tilde{\chi}_v)]^\dagger D_\mu(\chi_v + \tilde{\chi}_v) \tag{5}$$

$$= (\chi_v(x) + \tilde{\chi}_v(x))^\dagger(iv \cdot D)(\chi_v(x) + \tilde{\chi}_v(x)) + \mathcal{O}(\frac{1}{m}) \tag{6}$$

(note that $D_\mu\phi = e^{-imv \cdot x}[D_\mu(\chi_v + \tilde{\chi}_v) - imv_\mu(\chi_v + \tilde{\chi}_v)]$ and $-imv^\mu[D_\mu(\chi_v + \tilde{\chi}_v)]^\dagger(\chi_v + \tilde{\chi}_v) = imv^\mu(\chi_v + \tilde{\chi}_v)^\dagger D_\mu(\chi_v + \tilde{\chi}_v) -$ *total derivative term*)

Use the leading order of (5)

$$\begin{aligned}
\mathcal{L}^{(0)} &= (\chi_v + \tilde{\chi}_v)^\dagger(iv \cdot D)(\chi_v + \tilde{\chi}_v) \\
&= \chi_v^\dagger iv \cdot D\chi_v + \tilde{\chi}_v^\dagger iv \cdot D(\chi_v + \tilde{\chi}_v) + \chi_v^\dagger iv \cdot D\tilde{\chi}_v \\
&= \chi_v^\dagger iv \cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + (iv \cdot D\chi_v)^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv \cdot D\chi_v - 2m\tilde{\chi}_v^\dagger\tilde{\chi}_v + [(-2m - iv \cdot D)\tilde{\chi}_v]^\dagger\tilde{\chi}_v \\
&= \chi_v^\dagger iv \cdot D\chi_v - \tilde{\chi}_v^\dagger(iv \cdot D + 4m)\tilde{\chi}_v
\end{aligned}$$

We can have the final form[6]
$$\mathcal{L} = \chi_v^\dagger iv \cdot D\chi_v - \tilde{\chi}_v^\dagger(iv \cdot D + 4m)\tilde{\chi}_v + \mathcal{O}(\frac{1}{m})$$

2.2 Quantization

2.2.1 HQET as an example

The leading term of HQET Lagrangian is

$$\mathcal{L} = \bar{Q}_v(iv \cdot D)Q_v$$

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[6]With one problem: if we can tolerate coupled particle-anti particle pair, we can trade $iv \cdot D$ for mass term, so the leading part is the same but the anti-particle part could be different with the mixing?

$$Q_v(x) = e^{imv \cdot x} \frac{1 + \slashed{v}}{2} \psi(x)$$

In free Dirac fermion theory we know that

$$\left\{ \psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y}) \right\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_{ab}$$

$$\{a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

also the plane wave expansion of ψ is

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}} u(p) e^{-ip \cdot x}$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2mv^0}} a_v \sqrt{m} u(v) e^{-imv \cdot x - ik \cdot x}$$

using normalization of states $u(k) = \sqrt{m} u(v)$[7], $\langle p'|p \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\mathbf{p}' - \mathbf{p})$ and $\langle v', k'|v, k \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta^{(3)}(\mathbf{k}' - \mathbf{k})$ we have $|p\rangle = \sqrt{m} |v\rangle$ ($|p\rangle = \sqrt{2E_p} a_{\mathbf{p}}^\dagger |0\rangle$ while $|v, k\rangle = \sqrt{2v^0} a_{v,\mathbf{k}}^\dagger |0\rangle$)

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v u(v) e^{-imv \cdot x - ik \cdot x}$$

Using the definition of $Q_v(x)$

$$Q_v(x) = e^{imv \cdot x} \frac{1 + \slashed{v}}{2} \psi(x)$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v \frac{1 + \slashed{v}}{2} u(v) e^{-ik \cdot x}$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2v^0}} a_v u(v) e^{-ik \cdot x}$$

The commutation relation should be

$$\{Q_{va}(\mathbf{x}), Q_{v'b}(\mathbf{x}')\} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{\sqrt{4v^0 v'^0}} \{a_v, a_{v'}^\dagger\} u_a(v) u_b^\dagger(v') e^{-ik \cdot x + ik' \cdot x'}$$

using $\sum_s u_a(v) u_b^\dagger(v) = \frac{1}{m} \sum_s u_a(p) u_b^\dagger(p) = [(\slashed{v} + 1)\gamma^0]_{ab}$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{\sqrt{4v^0 v'^0}} \{a_v, a_{v'}^\dagger\} [(\slashed{v} + 1)\gamma^0]_{ab} e^{-ik \cdot x + ik' \cdot x'}$$

assuming $\{a_v, a_{v'}\} = (2\pi)^3 \delta_{vv'} \delta^{(3)}(\mathbf{k} - \mathbf{k}')$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2v^0} [(\slashed{v} + 1)\gamma^0]_{ab} e^{-ik \cdot (x - x')} \delta_{vv'}$$

$$= [\frac{(\slashed{v} + 1)\gamma^0}{2v^0}]_{ab} \delta_{vv'} \delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

2.2.2 HSET

The anti-particle field is decoupled so we don't have to consider that for now. The equation-of-motion is

$$\begin{cases} v \cdot D \chi_v^\dagger = 0 \\ v \cdot D \chi_v = 0 \end{cases}$$

By definition

$$\chi_v(x) = \frac{e^{imv \cdot x}}{\sqrt{2m}} (iv \cdot D + m) \phi(x)$$

$$= \frac{1}{\sqrt{2m}} (iv \cdot D + 2m) e^{imv \cdot x} \phi(x)$$

Obviously the plane wave expansion should be irrelevant of the heavy particle mass, which means the exponential part is $e^{-ik \cdot x}$ where k marks the offshellness.

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[7]The relation $\bar{u}^s(p) \gamma^\mu u^s(p) = 2p^\mu$ can be derived using Gordon identity, same for $\bar{u}^s(v) \gamma^\mu u^s(v) = 2v^\mu$, but it's actually $\bar{u}u$.