Homework: Quantum Field Theory #5

Yingsheng Huang

December 28, 2016

1. Complete table at P&S P71.

For the first three colomn, they're already done in class so I simply give the results. (We use the shorthand in P&S: $(-1)^\mu = 1$ for $\mu = 0$ and $(-1)^\mu = -1$ for $\mu = 1, 2, 3$.[1])

$$P\bar{\psi}\psi P^{-1} = +\bar{\psi}\psi(t, -\mathbf{x})$$

$$T\bar{\psi}\psi T^{-1} = +\bar{\psi}\psi(-t, \mathbf{x})$$
$$C\bar{\psi}\psi C^{-1} = +\bar{\psi}\psi(t, \mathbf{x})$$
$$CPT\bar{\psi}\psi CPT^{-1} = +\bar{\psi}\psi(-t, -\mathbf{x})$$
$$P\bar{\psi}\gamma^5\psi P^{-1} = -\bar{\psi}\gamma^5\psi(t, -\mathbf{x})$$
$$T\bar{\psi}\gamma^5\psi T^{-1} = -\bar{\psi}\gamma^5\psi(-t, \mathbf{x})$$
$$C\bar{\psi}\gamma^5\psi C^{-1} = +\bar{\psi}\gamma^5\psi(t, \mathbf{x})$$
$$CPT\bar{\psi}\gamma^5\psi CPT^{-1} = +\bar{\psi}\gamma^5\psi(-t, -\mathbf{x})$$
$$P\bar{\psi}\gamma^\mu\psi P^{-1} = (-1)^\mu\bar{\psi}\gamma^\mu\psi(t, -\mathbf{x})$$
$$T\bar{\psi}\gamma^\mu\psi T^{-1} = (-1)^\mu\bar{\psi}\gamma^\mu\psi(-t, \mathbf{x})$$
$$C\bar{\psi}\gamma^\mu\psi C^{-1} = -\bar{\psi}\gamma^\mu\psi(t, \mathbf{x})$$
$$CPT\bar{\psi}\gamma^\mu\psi CPT^{-1} = -\bar{\psi}\gamma^\mu\psi(-t, -\mathbf{x})$$

Now we calculate the rest.

Given

$$P\psi P^{-1} = \eta\gamma^0\psi(t, -\mathbf{x})$$
$$P\bar{\psi}P^{-1} = \eta^*\bar{\psi}(t, -\mathbf{x})\gamma^0$$

we have

$$\begin{aligned} P\bar{\psi}\gamma^\mu\gamma^5\psi P^{-1} &= |\eta|^2\bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi \\ &= -(-1)^\mu\bar{\psi}\gamma^\mu\gamma^5\psi \end{aligned}$$

and

$$\begin{aligned} P\bar{\psi}\sigma^{\mu\nu}\psi P^{-1} &= \frac{i}{2}\bar{\psi}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0\psi \\ &= \frac{i}{2}(-1)^\mu(-1)^\nu\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi \\ &= (-1)^\mu(-1)^\nu\bar{\psi}\sigma^{\mu\nu}\psi \end{aligned}$$

[1] And I write $(CPT)^{-1}$ as CPT^{-1} for short.

and similarly

$$P\bar{\psi}\partial_\mu\psi P^{-1} = (-1)^\mu\bar{\psi}\partial_\mu\psi$$

Define

$$\mathscr{T} \equiv i\gamma^1\gamma^3$$

and

$$\begin{gathered}
T\psi T^{-1} = \mathscr{T}\psi \\
T\bar{\psi}T^{-1} = \bar{\psi}\mathscr{T}^{-1} \\
\mathscr{T}(\gamma^\mu)^*\mathscr{T}^{-1} = \gamma_\mu = (-1)^\mu\gamma^\mu \\
\mathscr{T}(\gamma^5)^*\mathscr{T}^{-1} = \gamma^5 \\
\mathscr{T} = \mathscr{T}^{-1} = \mathscr{T}^\dagger
\end{gathered}$$

we have

$$\begin{aligned}
T\bar{\psi}\gamma^\mu\gamma^5\psi T^{-1} &= \bar{\psi}\mathscr{T}^{-1}(\gamma^\mu\gamma^5)^*\mathscr{T}\psi \\
&= \bar{\psi}\mathscr{T}^{-1}\gamma^{\mu *}\mathscr{T}^{-1}\mathscr{T}\gamma^{5*}\mathscr{T}\psi \\
&= \bar{\psi}\gamma_\mu\gamma^5\psi \\
&= (-1)^\mu\bar{\psi}\gamma^\mu\gamma^5\psi
\end{aligned}$$

and

$$\begin{aligned}
T\bar{\psi}\sigma^{\mu\nu}\psi T^{-1} &= -\frac{i}{2}T\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi T^{-1} \\
&= -\frac{i}{2}\bar{\psi}\mathscr{T}[\gamma^\mu,\gamma^\nu]^*\mathscr{T}^{-1}\psi \\
&= -(-1)^\mu(-1)^\nu\bar{\psi}\sigma^{\mu\nu}\psi
\end{aligned}$$

and

$$T\bar{\psi}\partial_\mu\psi T^{-1} = -(-1)^\mu\bar{\psi}\partial_\mu\psi$$

Define

$$\mathscr{C} \equiv i\gamma^2\gamma^0$$

and

$$\begin{gathered}
C\psi C^{-1} = \mathscr{C}\bar{\psi}^T \\
C\bar{\psi}C^{-1} = \psi^T\mathscr{C} \\
\mathscr{C}(\gamma^\mu)^T\mathscr{C}^{-1} = -\gamma^\mu \\
\mathscr{C}(\gamma^5)^T\mathscr{C}^{-1} = \gamma^5 \\
\mathscr{C}^\dagger = \mathscr{C}^{-1} = -\mathscr{C} = \mathscr{C}^T
\end{gathered}$$

thus

$$\begin{aligned}
&(\mathscr{C}(\gamma^\mu)^T\mathscr{C}^{-1})^\dagger = -(\gamma^\mu)^\dagger \\
=&(\mathscr{C}^{-1})^\dagger(\gamma^\mu)^*\mathscr{C}^\dagger \\
=&-\mathscr{C}^\dagger(\gamma^\mu)^*\mathscr{C}^{-1} \\
=&\mathscr{C}(\gamma^\mu)^*\mathscr{C}^{-1} = -(\gamma^\mu)^\dagger
\end{aligned}$$

and

$$\mathscr{C}\gamma^5\mathscr{C}^{-1} = \gamma^5$$

Then we have

$$\begin{aligned} C\bar{\psi}\gamma^\mu\gamma^5\psi C^{-1} &= \psi^T\mathscr{C}\gamma^\mu\gamma^5\mathscr{C}\bar{\psi}^T \\ &= \psi^T\gamma^{\mu T}\gamma^{5T}\bar{\psi}^T \\ &= -(\bar{\psi}\gamma^5\gamma^\mu\psi)^T \\ &= \bar{\psi}\gamma^\mu\gamma^5\psi \end{aligned}$$

and

$$\begin{aligned} C\bar{\psi}\sigma^{\mu\nu}\psi C^{-1} &= \frac{i}{2}\psi^T\mathscr{C}[\gamma^\mu,\gamma^\nu]\mathscr{C}\bar{\psi}^T \\ &= -\frac{i}{2}\psi^T[\gamma^{\mu T},\gamma^{\nu T}]\bar{\psi}^T \\ &= \frac{i}{2}(\bar{\psi}[\gamma^\nu,\gamma^\mu]\psi)^T \\ &= -\bar{\psi}\sigma^{\mu\nu}\psi \end{aligned}$$

and

$$C\bar{\psi}\partial_\mu\psi C^{-1} = \bar{\psi}\partial_\mu\psi$$

CPT is to multiply all those coefficients and too trival to list here.

2. Calculate the Dirac propagator.

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s u_a^s(p)\bar{u}_b^s(p)e^{-ip\cdot(x-y)} = (i\partial\!\!\!/_x + m)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}$$

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s v_a^s(p)\bar{v}_b^s(p)e^{-ip\cdot(x-y)} = -(i\partial\!\!\!/_x + m)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{ip\cdot(x-y)}$$

The definition of Dirac propagator

$$S_F(x-y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$$

the two poles (if $\epsilon = 0$) are located in $p^0 = \omega_p \equiv \sqrt{\mathbf{p}^2+m^2}$ and $p^0 = -\omega_p$.

For $x^0 - y^0 > 0$, we have

$$\begin{aligned} \int\frac{\mathrm{d}p^0}{2\pi}\frac{i(p\!\!\!/ + m)}{p^2-m^2+i\epsilon}e^{-ip^0(x^0-y^0)} &= \int\frac{\mathrm{d}p^0}{2\pi}\frac{i(p\!\!\!/ + m)}{p_0^2-\omega_p^2+i\epsilon}e^{-ip^0(x^0-y^0)} \\ &= \int\frac{\mathrm{d}p^0}{2\pi}\frac{i(p\!\!\!/ + m)}{(p_0-\omega_p+i\epsilon)(p_0+\omega_p-i\epsilon)}e^{-ip^0(x^0-y^0)} \\ &= \frac{1}{2(\omega_p-i\epsilon)}\int\frac{\mathrm{d}p^0}{2\pi}i(p\!\!\!/ + m)(\frac{1}{p_0-\omega_p+i\epsilon}-\frac{1}{p_0+\omega_p-i\epsilon})e^{-ip^0(x^0-y^0)} \\ &= \frac{i\partial\!\!\!/_x + m}{2(\omega_p-i\epsilon)}\int\frac{\mathrm{d}p^0}{2\pi}i(\frac{1}{p_0-\omega_p+i\epsilon}-\frac{1}{p_0+\omega_p-i\epsilon})e^{-ip^0(x^0-y^0)} \end{aligned}$$

use residue theorem, and the contour is closed below (only one singularity on the right)

$$
\begin{aligned}
&= \frac{i\not{\partial}_x + m}{2(\omega_p - i\epsilon)} e^{-ip^0(x^0-y^0)} \\
&= \frac{i\not{\partial}_x + m}{2\omega_p} e^{-ip^0(x^0-y^0)}
\end{aligned}
$$

wihch means

$$S_F(x-y) = \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle$$

Similarly, when $x^0 - y^0 < 0$, the contour is closed above and

$$S_F(x-y) = -\frac{i\not{\partial}_x + m}{2\omega_p} e^{ip^0(x^0-y^0)} = -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$$

Appendix

	γ^0	γ^1	γ^2	γ^3	γ^5
T	1	-1	1	-1	1
-1	1	-1	-1	-1	1
*	1	1	-1	1	1
†	1	-1	-1	-1	1