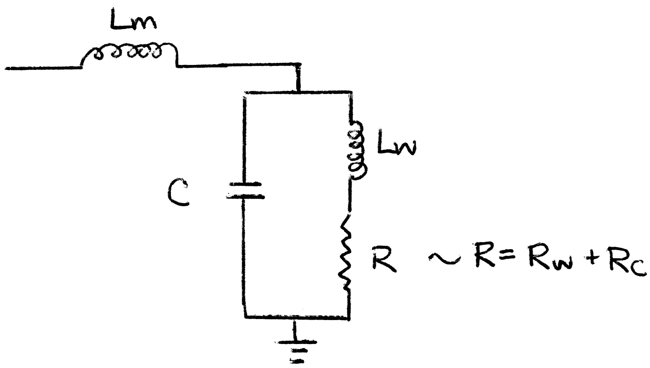
Resonant Tank Circuit



$$Z_{total} = j\omega L_m + \frac{1}{j\omega C} // (j\omega L_w + R)$$

$$= j\omega L_m + \frac{\frac{1}{j\omega C}(R + j\omega L_w)}{\frac{1}{j\omega C} + R + j\omega L_w} \times \frac{j\omega C}{j\omega C}$$

$$= j\omega L_m + \frac{R + j\omega L_w}{1 + R(j\omega C) - \omega^2 C L_w} \times \frac{(1-\omega^2 C L_w) - j\omega C R}{(1-\omega^2 C L_w) - j\omega C R}$$

$$= j\omega L_m + \frac{R(1-\omega^2 C L_w) - j\omega C R^2 + j\omega L_w(1-\omega^2 C L_w) + \omega^2 C R L_w}{(1-\omega^2 C L_w)^2 + (\omega C R)^2}$$

$$= j\omega L_m + \frac{R - \omega^2 C R L_w - j\omega C R^2 + j\omega L_w - j\omega^3 C L_w^2 + \omega^2 C R L_w}{(1-\omega^2 C L_w)^2 + (\omega C R)^2}$$

$$= \frac{j\omega L_m[(1-\omega^2 C L_w)^2 + (\omega C R)^2] + (R - \omega^2 C \cancel{R} L_w + \omega^2 C \cancel{R} L_w) + j(\omega L_w - \omega^3 C L_w^2 - \omega C R^2)}{(1-\omega^2 C L_w)^2 + (\omega C R)^2}$$

$$= \frac{j\omega L_m[(1 - 2\omega^2 C L_w + \omega^4 C^2 L_w^2) + \omega^2 C^2 R^2] + R + j(\omega L_w - \omega^3 C L_w^2 - \omega C R^2)}{(1-\omega^2 C L_w)^2 + (\omega C R)^2}$$

$$= \frac{(R + \omega^2 C^2 R^2) + j[\omega L_m - 2\omega^3 C L_m L_w + \omega^5 C^2 L_m L_w^2 + \omega^3 C^2 R^2 L_m + \omega L_w - \omega^3 C L_w^2 - \omega C R^2]}{(1-\omega^2 C L_w)^2 + (\omega C R)^2}$$

set the imaginary part to zero:

$$\omega L_m - 2\omega^3 C L_m L_w + \omega^5 C^2 L_m L_w^2 + \omega^3 C^2 R^2 L_m + \omega L_w - \omega^3 C L_w^2 - \omega C R^2 = 0$$

$$\omega^5 C^2 L_m L_w^2 + \omega^3 (C^2 R^2 L_m - C L_w^2 - 2 C L_m L_w) + \omega (L_m + L_w - C R^2) = 0$$

$$\omega\left[\omega^4 C^2 L_m L_\omega^2 + \omega^2(L_m C^2 R^2 - C L_\omega^2 - 2C L_m L_\omega) + (L_m + L_\omega - CR^2)\right] = 0$$

$\omega = 0$ OR

$$\omega^2 = \frac{-(L_m C^2 R^2 - C L_\omega^2 - 2C L_m L_\omega) \pm \sqrt{\underbrace{(L_m C^2 R^2 - C L_\omega^2 - 2C L_m L_\omega)^2}_{①} - \underbrace{4C^2 L_m L_\omega^2 (L_m + L_\omega - CR^2)}_{②}}}{2 L_m C^2 L_\omega^2}$$

Note:

① $(L_m C^2 R^2 - C L_\omega^2 - 2C L_m L_\omega)^2$

$$= L_m^2 C^4 R^4 - C^3 R^2 L_m L_\omega^2 - 2C^3 R^2 L_m^2 L_\omega - C^3 R^2 L_m L_\omega^2 + C^2 L_\omega^4 + 2C^2 L_m L_\omega^3 - 2C^3 R^2 L_m^2 L_\omega + 2C^2 L_m L_\omega^3 + 4C^2 L_m^2 L_\omega^2$$

$$= C^2 L_\omega^4 + 4C^2 L_m L_\omega^3 + 4C^2 L_m^2 L_\omega^2 - C^3 R^2 L_m L_\omega^2 - 4C^3 R^2 L_m^2 L_\omega - C^3 R^2 L_m L_\omega^2 + C^4 R^4 L_m^2$$

$$= C^2 L_\omega^4 + 4C^2 L_m L_\omega^3 + 4C^2 L_m^2 L_\omega^2 - 2C^3 R^2 L_m L_\omega^2 - 4C^3 R^2 L_m^2 L_\omega + C^4 R^4 L_m^2$$

$$① + ② = C^2 L_\omega^4 + 4C^2 L_m L_\omega^3 + 4C^2 L_m^2 L_\omega^2 - 2C^3 R^2 L_m L_\omega^2 - 4C^3 R^2 L_m^2 L_\omega + C^4 R^4 L_m^2 - 4C^2 L_m^2 L_\omega^2 - 4C^2 L_m L_\omega^3 + 4C^3 R^2 L_m L_\omega^2$$

$$= C^2 L_\omega^4 + 2C^3 R^2 L_m L_\omega^2 - 4C^3 R^2 L_m^2 L_\omega + C^4 R^4 L_m^2$$

$$\omega^2 = -\frac{L_m C^2 R^2}{2C^2 L_m L_\omega^2} + \frac{C L_\omega^2}{2C^2 L_m L_\omega^2} + \frac{2C L_m L_\omega}{2C^2 L_m L_\omega^2} \pm \sqrt{\frac{C^2 L_\omega^4}{4C^4 L_m^2 L_\omega^4} + \frac{2C^3 R^2 L_m L_\omega^2}{4C^4 L_m^2 L_\omega^4} + \frac{-4C^3 R^2 L_m^2 L_\omega + C^4 R^4 L_m^2}{4C^4 L_m^2 L_\omega^4}}$$

$$= -\frac{R^2}{2L_\omega^2} + \frac{1}{2CL_m} + \frac{1}{CL_\omega} \pm \sqrt{\frac{1}{4C^2 L_m^2} + \frac{R^2}{2C L_m L_\omega^2} - \frac{R^2}{C L_\omega^3} + \frac{R^4}{4L_\omega^4}}$$

let $L_\omega = L_m = L$

$$\omega_0^2 = \frac{1}{2CL} + \frac{1}{CL} - \frac{R^2}{2L^2} \pm \sqrt{\frac{1}{4C^2L^2} + \frac{R^2}{2CL^3} - \frac{R^2}{CL^3} + \frac{R^4}{4L^4}}$$

$$= \frac{3L - CR^2}{2CL^2} \pm \sqrt{\frac{L^2 + 2R^2CL - 4R^2CL + C^2R^4}{4C^2L^4}}$$

$$= \frac{3L - CR^2 \pm \sqrt{(L - CR^2)^2}}{2CL^2}$$

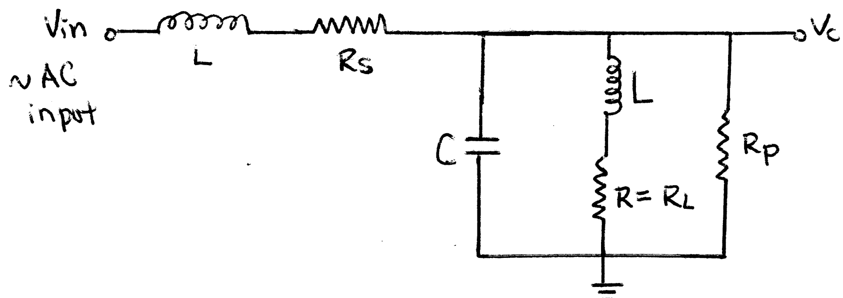
$$= \frac{3L - CR^2 \pm (L - CR^2)}{2CL^2}$$

$$\omega_0 = \sqrt{\frac{3L - CR^2 + L - CR^2}{2CL^2}} = \sqrt{\frac{4L - 2CR^2}{2CL^2}}$$

$$\omega_0 = \sqrt{\frac{3L - CR^2 - L + CR^2}{2CL^2}} = \sqrt{\frac{2L}{2CL^2}} = \sqrt{\frac{1}{LC}} \quad (\checkmark)$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \longrightarrow f_c = \frac{1}{2\pi\sqrt{LC}}$$

Taking component losses and loaded Q into account (@ RF):



R_S = component loss of the matching inductor → treat it as a source resistance

R_L = load resistance $= R_W + R_C$

R_P = the true load that the resonant circuit sees → equivalent parallel resistance of R_S and R_L

Suppose

1) f_c = centre frequency = 500kHz
$f_2 - f_1 = 60\text{kHz}$

$$\text{loaded } Q = \frac{f_c}{f_2-f_1} = \frac{500\text{kHz}}{60\text{kHz}} = 8.333$$

2) $R_S = 1\text{k}\Omega$ where $R_{\text{work coil}}$ and R_S should be the same ∵ $L_m = L_W$

$R_L = R_W + R_C$
$= 1k + 500$
$= 1.5\text{k}\Omega$

3) Quality Factor of the inductor: $Q = 85$ (the higher the Q factor is, the closer it approaches the behavior of an ideal inductor is)

Soln: Find values of L and C

1) Q factor:

$$Q_P X_P = R_P \longrightarrow 85 X_P = R_P \quad \text{—(1)}$$

2) loaded Q:

$$\text{loaded } Q = \frac{R_{total}}{X_P} = \frac{R_S // R_L // R_P}{X_P}$$

where $R_S // R_L // R_P = \frac{1000(1500)}{1000+1500} // R_P$
$= 600 // R_P$

$$= \frac{\frac{600(R_p)}{600+R_p}}{X_p} \quad \text{------} (2)$$

$$(X_p)8.33 = \frac{600(85X_p)}{600+85X_p}$$

$$5100X_p = X_p(708.05X_p + 4998)$$

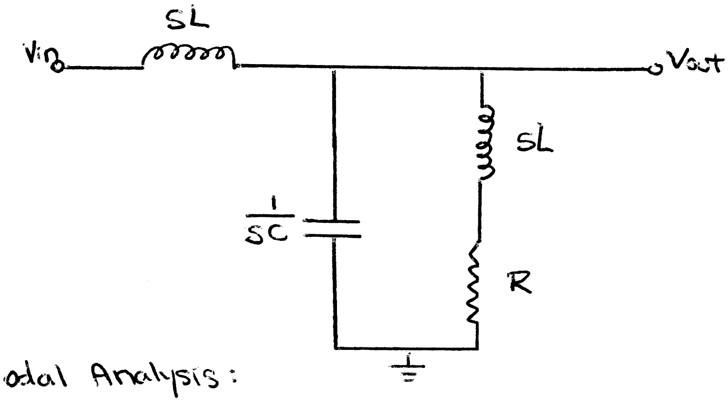
$$X_p(708.05X_p - 4602) = 0$$

$$X_p = 64.97\,\Omega \quad \text{and} \quad R_p = 5522.45\,\Omega$$

$$L = \frac{X_p}{\omega} = \frac{64.97}{2\pi(500\times10^3)} = \underline{\underline{20.68\,\mu H}}$$

$$C = \frac{1}{\omega X_p} = \frac{1}{2\pi(500\times10^3)(64.97)} = \underline{\underline{4.899\,nF}}$$

Transfer Function and Bode Plot



Nodal Analysis:

$$\frac{V_o - V_{in}}{SL} + \frac{V_o}{\frac{1}{SC}} + \frac{V_o}{SL+R} = 0$$

$$V_o\left(\frac{1}{SL} + sC + \frac{1}{SL+R}\right) = \frac{V_{in}}{SL}$$

$$V_o\left(\frac{SL+R+SL+SC\cdot SL(SL+R)}{SL(SL+R)}\right) = \frac{1}{SL}V_{in}$$

$$V_o\left(\frac{2SL+R+SC\cdot(S^2L^2+SRL)}{SL(SL+R)}\right) = \frac{1}{SL}V_{in}$$

$$V_o\left(\frac{S^3CL^2+S^2CRL+2SL+R}{SL(SL+R)}\right) = \frac{1}{SL}V_{in}$$

$$H(S) = \frac{V_o(S)}{V_{in}(S)} = \frac{SL+R}{S^3CL^2+S^2CRL+S(2L)+R}$$

$$= \frac{\cancel{R}(1+S\frac{L}{R})}{\cancel{R}\left[S^3\left(\frac{CL^2}{R}\right)+S^2(CL)+S\left(\frac{2L}{R}\right)+1\right]}$$

zero: $S = -R/L$

poles: $S^3\left(\frac{CL^2}{R}\right)+S^2(CL)+S\left(\frac{2L}{R}\right)+1 = 0$

$L = 20.68\,\mu H$

$C = 4.899\,nF$ $\xrightarrow{\text{MATLAB}}$

$R = 1.5k\Omega$

$S_1 = -7.2398\times10^7 = -\underbrace{7.2398\times10^7}_{\omega_1}$

$S_2 = -0.0068+j0.3144 = -\underbrace{(0.0068-j0.3144)}_{\omega_2}$

$S_3 = -0.0068-j0.3144 = -\underbrace{(0.0068+j0.3144)}_{\omega_3}$

$$H(s) = \frac{1+S\frac{L}{R}}{(S+\omega_1)(S+\omega_2)(S+\omega_3)} = \frac{1+S\frac{L}{R}}{\omega_1\left(1+\frac{S}{\omega_1}\right)\omega_2\left(1+\frac{S}{\omega_2}\right)\omega_3\left(1+\frac{S}{\omega_3}\right)}$$

$S = j\omega$

$$H(j\omega) = \frac{1}{\omega_1\omega_2\omega_3}\,\frac{1+j\left(\frac{L}{R}\omega\right)}{\left(1+j\frac{\omega}{\omega_1}\right)\left(1+j\frac{\omega}{\omega_2}\right)\left(1+j\frac{\omega}{\omega_3}\right)}$$

- Phase:

$$\angle H(j\omega) = \angle\left(\frac{1}{\omega_1\omega_2\omega_3}\right) + \angle\left(1+j\frac{L}{R}\omega\right) - \angle\left(1+j\frac{\omega}{\omega_1}\right) - \angle\left(1+j\frac{\omega}{\omega_2}\right) - \angle\left(1+j\frac{\omega}{\omega_3}\right)$$

- Magnitude:

$$20\log_{10}(|H(j\omega)|) = 20\log_{10}\left(\left|\frac{1}{\omega_1\omega_2\omega_3}\right|\right) + 20\log_{10}\left(\left|1+j\frac{L}{R}\omega\right|\right) - 20\log_{10}\left(\left|1+j\frac{\omega}{\omega_1}\right|\right)$$

$$-20\log_{10}\left(\left|1+j\frac{\omega}{\omega_2}\right|\right) - 20\log_{10}\left(\left|1+j\frac{\omega}{\omega_3}\right|\right)$$

At this point, please refer to the Bode Plot attached in the email:
$\omega_r = 3.14\times10^6$ rad/s (the peak frequency)

$$f_r = \frac{3.14\times10^6 \text{ rad/s}}{2\pi} \approx 500\text{kHz}$$

Role of the PLL:

the reference signal

synchronizes frequency of V_{inv} (V_{in}) with that of V_c (V_{out}) so they operate at the same frequency → PLL will lock onto the resonant frequency ($f_r = 500$kHz) and maintain a phase difference of 90° between V_c and V_{inv}.

However, looking at the Phase Plot, at f_r, the output lags the input signal by 180°...