We might not be able to get away with the quasistatic approximation, because our frequency is so high. In that case, the magnetic field inside (and *outside*) a solenoid is *not* constant, but actually takes the form of radio waves. So we will have to calculate this again, using Maxwell's equations.

Regions

Here's the setup: Each of these regions represents an area covered by a different differential equation, which we'll have to stitch together with matching boundary conditions.

EM waves in free space: $0 \leq s < \frac{d}{2} - t$

EM waves in conductor: $\frac{d}{2} - t \leq s \leq \frac{d}{2}$

EM waves in free space: $\frac{d}{2} < s < \frac{D}{2}$

Sinusoidally varying current: $s = \frac{D}{2}$

EM waves in free space: $\frac{d}{2} < s < \frac{d_s}{2}$

EM waves in shield: $s = \frac{d_s}{2}$

EM waves in free space: $s > d_s$

Equations:

In free space

From Griffiths chapter 9,

$$\nabla^2 \mathrm{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2}, \quad \nabla^2 \mathrm{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathrm{B}}{\partial t^2}$$

We're expecting the magnetic fields to be only pointing along z, and the electric fields should all be pointing along φ only. They should both vary sinusoidally with time. Also, they should be constant in z and φ. That should make our work considerably easier!

In cylindrical coordinates:

$$-\mu_0\epsilon_0\omega^2 E = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial E}{\partial s}\right)$$

$$-\mu_0\epsilon_0 \mathrm{B} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial B}{\partial s}\right)$$

Magnetic:

$$0 = \frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} + \mu_0\epsilon_0\omega^2 B$$

The speed of light is $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$, so we can define $\omega\sqrt{\mu_0\epsilon_0}$ as the wave number, $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$. Then:

$$0 = \frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} + \mathrm{k}^2 B$$

Electric:

$$0 = \frac{\partial^2 E}{\partial s^2} + \frac{1}{s}\frac{\partial E}{\partial s} + k^2 E$$

The solutions to these are, unfortunately, Bessel functions (aka the "cylindrical harmonics"). These functions always tend to appear in cylindrical coordinate problems. The solutions are:

$$B_z(s) = c_1 J_0(ks) + c_2 Y_0(ks)$$

$$E_\varphi(s) = c_3 J_0(ks) + c_4 Y_0(ks)$$

J_0 is a Bessel function "of the first kind" of order 0, and Y_0 is a Bessel function "of the second kind" of order 0. $C_{1,2,3,4}$ are just constants that we can use to stitch these equations together at the boundaries.

Also, B and E here represent the amplitudes of the electric fields, since the time-dependence cancelled out of the equation. Of course, they are time-varying.

The electric and magnetic fields are not independent though; they are related by

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$-\frac{\partial B}{\partial s} = -\frac{i\omega}{c^2}E$$

$$E = -\frac{ic^2}{\omega}\frac{\partial B}{\partial s}$$

$$E = -\frac{ic^2}{\omega}\frac{\partial}{\partial s}\big(c_1 J_0(ks) + c_2 Y_0(ks)\big)$$

$$E_\varphi(s) = \frac{ic^2}{\omega}\big(c_1 k J_1(ks) + c_2 k Y_1(ks)\big)$$

And since

$$k = \frac{\omega}{c}$$

$$\frac{ikc^2}{\omega} = ic$$

so

$$E_\varphi(s) = ic\big(c_1 J_1(ks) + c_2 Y_1(ks)\big)$$

In the conductor

From Griffiths, chapter 9, the electromagnetic field in a conductor is:

$$\nabla^2 E = \mu\epsilon \frac{d^2 E}{dt^2} + \mu\sigma \frac{\partial E}{\partial t}, \quad \nabla^2 B = \mu\epsilon \frac{\partial^2 B}{\partial t^2} + \mu\sigma \frac{\partial B}{\partial t}$$

In cylindrical coordinates:

$$\nabla^2 E = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial E}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 E}{\partial \varphi^2} + \frac{\partial^2 E}{\partial z^2} = \mu\epsilon \frac{\partial d^2 E}{dt^2} + \mu\sigma \frac{\partial E}{\partial t}$$

$$\nabla^2 B = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial B}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 B}{\partial \varphi^2} + \frac{\partial^2 B}{\partial z^2} = \mu\epsilon \frac{\partial^2 B}{dt^2} + \mu\sigma \frac{\partial B}{\partial t}$$

And then requiring them to be functions of s and t only,

$$\frac{\partial^2 E}{\partial s^2} + \frac{1}{s}\frac{\partial E}{\partial s} - \mu\epsilon \frac{\partial d^2 E}{dt^2} - \mu\sigma \frac{\partial E}{\partial t} = 0$$

$$\frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} - \mu\epsilon \frac{\partial d^2 B}{dt^2} - \mu\sigma \frac{\partial B}{\partial t} = 0$$

So now there's a bit of a complication. Unlike before, we have terms that are proportional to $-\frac{\partial E}{\partial t}$ and $-\frac{\partial B}{\partial t}$. You might recognize these as damping terms. When we plug in the time-dependence, it doesn't cancel out the way it did before. Using separation of variables, let

$$B = T_B(t) S_B(s)$$

$$T_B \frac{\partial^2 S_B}{\partial s^2} + T_B \frac{1}{s}\frac{\partial S_B}{\partial s} - \mu\epsilon S_B \frac{\partial d^2 T_B}{dt^2} - \mu\sigma S_B \frac{\partial T_B}{\partial t} = 0$$

$$\frac{1}{S_B}\frac{\partial^2 S_B}{\partial s^2}+\frac{1}{S_B}\frac{1}{s}\frac{\partial S_B}{\partial s}=\mu\epsilon\frac{1}{T_B}\frac{\partial d^2 T_B}{dt^2}+\mu\sigma\frac{1}{T_B}\frac{\partial T_B}{\partial t}$$

For the time half of this equation, we have the constant U that we have to solve for (with units of per-metres).

$$\mu\epsilon\frac{\partial d^2 T_B}{dt^2}+\mu\sigma\frac{\partial T_B}{\partial t}+T_B U^2=0$$

The characteristic polynomial of this differential equation is:

$$\mu\epsilon r^2+\mu\sigma r-U^2=0$$

$$r=\frac{-\sigma\mu\pm\sqrt{(\sigma\mu)^2-4\epsilon\mu U^2}}{2\epsilon\mu}=\frac{-\sigma}{2\epsilon}\pm i\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}$$

So the general solution for this is:

$$T_B(t)=c_5\exp\left(\frac{-\sigma}{2\epsilon}t+i\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)+c_6\exp\left(\frac{-\sigma}{2\epsilon}t-i\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)$$

$$T_B(t)=e^{\frac{-\sigma}{2\epsilon}t}\left(c_5\exp\left(i\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)+c_6\exp\left(-i\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)\right)$$

$$T_B(t)=e^{\frac{-\sigma}{2\epsilon}t}\left(c_5\cos\left(\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)+c_6\sin\left(\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)\right)$$

$$T_B(t)=\left(c_5\cos(\omega t)+c_6\sin(\omega t)\right)e^{\frac{-\sigma}{2\epsilon}t}$$

Where I've set the frequency to be

$$\omega=\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}$$

$$\mu\epsilon\omega^2+\mu\epsilon\left(\frac{\sigma}{2\epsilon}\right)^2=U^2$$

Whew… so it's an exponentially decaying solution? That's not what we wanted at all! We need something that's just sinusoidal, but perhaps exponentially decaying in z. Clearly, I chose the wrong value of U. What value will cancel out the exponential decrease? I need

$$Re\left\{\frac{-\sigma}{2\epsilon}+i\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\right\}=0$$

$$\frac{U^2}{\mu\epsilon}=\left(\frac{\sigma}{2\epsilon}\right)^2+\left(\frac{-i\sigma}{2\epsilon}-\omega\right)^2$$

$$\left(\frac{-i\sigma}{2\epsilon}-\omega\right)^2=\frac{-\sigma^2}{4\epsilon^2}+i\omega\frac{\sigma}{\epsilon}+\omega^2$$

$$U^2=i\omega\mu\sigma+\mu\epsilon\omega^2={k'}^2$$

Plugging this in,

$$T_B(t)=c_5\exp(-i\omega t)+c_6\exp\left(\frac{-\sigma}{\epsilon}t+i\omega t\right)$$

And now, since the solution is supposed to be just oscillating in time, we can say c_6=0.

$$T_B(t)=\exp(-i\omega t)$$

And it is worth noting also that

$$\mu\sigma-i\mu\epsilon\omega=\frac{{k'}^2}{i\omega}$$

Now, we just need to solve the spatial side:

$$\frac{\partial^2 S_B}{\partial s^2}+\frac{1}{s}\frac{\partial S_B}{\partial s}+{k'}^2 S_B=0$$

This is just the same as in free space, except that the wave number is more complicated.

$$B_z(s)=c_5J_0(k's)+c_6Y_0(k's)$$

Now the magnetic field should be doing something like this:

$$B_z(s,t)=\big(c_5J_0(k's)+c_6Y_0(k's)\big)e^{-i\omega t}$$

The electric field is found from Maxwell's equations:

$$E_\varphi(s,t)=\big(c_7J_0(k's)+c_8Y_0(k's)\big)e^{-i\omega t}$$

$$\nabla\times B=\mu\epsilon\frac{\partial E}{\partial t}+\mu\sigma E=-i\mu\epsilon\omega E+\mu\sigma E=-\frac{\partial B}{\partial s}$$

$$-\frac{\partial B}{\partial s}=c_5k'J_1(k's)+c_6k'Y_1(k's)=E(\mu\sigma-i\mu\epsilon\omega)=E\frac{{k'}^2}{i\omega}$$

$$E_\varphi(s)=\frac{i\omega}{k'}\big(c_5J_1(k's)+c_6Y_1(k's)\big)$$

In region with current flow

$$I(t) = I_0 \sin(\omega t)\,, s = \frac{D}{2}$$

The electric field is continuous, but the magnetic field is not.

$$B|_{s=\frac{D}{2}-} - B|_{s=\frac{D}{2}+} = \mu_0 I_0 \frac{N}{l_p}$$

In the shield

Because the skin depth is shallow at high frequencies, the shield is modelled as a copper or aluminum foil with negligible thickness, and a finite resistance Rsh. The electric field is continuous, but the magnetic field is not. The shield has the same length as the primary.

$$I_{sh} = \frac{\pi d_{sh} E}{R_{sh}}$$

$$B|_{s=\frac{d_{sh}}{2}-} - B|_{s=\frac{d_{sh}}{2}+} = \frac{\mu_0 I_{sh}}{l_p} = \frac{\mu_0 \pi d_{sh} E(r_{sh})}{R_{sh} l_p}$$

$$E_\varphi(s) = ic\big(c_{11} J_1(ks) + ic_{11} Y_1(ks)\big)$$

Matching up the boundary conditions

EM waves in center area free space: $0 \leq s < \frac{d}{2} - t$

$$B_z(s) = \big(c_1 J_0(ks) + c_2 Y_0(ks)\big) e^{-i\omega t}$$

$$E_\varphi = ic\big(c_1 J_1(ks) + c_2 Y_1(ks)\big) e^{-i\omega t}$$

Condition: Must be finite as s->0

$$c_2 = 0$$

$$B_z(s) = c_1 J_0(ks) e^{i\omega t}$$

$$E_\varphi(s) = c_1 ic J_1(ks) e^{i\omega t}$$

EM waves in conductor: $\frac{d}{2} - t \leq s \leq \frac{d}{2}$

$$B_z(s,t) = \big(c_5 J_0(k's) + c_6 Y_0(k's)\big) e^{-i\omega t}$$

$$E_\varphi(s,t) = \frac{i\omega}{k'}\big(c_5 J_1(k's) + c_6 Y_1(k's)\big) e^{-i\omega t}$$

Conditions: Let's call $\frac{d}{2} - t = r_{in}$, and $\frac{d}{2} = r_{out}$ for notational simplicity.

Both E and B must be continuous, so

$$B: \quad c_5 J_0(k' r_{in}) + c_6 Y_0(k' r_{in}) = c_1 J_0(k r_{in})$$

$$E: \quad \frac{k}{k'}\big(c_5 J_1(k' r_{in}) + c_6 Y_1(k' r_{in})\big) = c_1 J_1(k r_{in})$$

Continuing on...

EM waves between conductor and solenoid: $\frac{d}{2} < s < \frac{D}{2}$

$$B_z(s,t) = \big(c_7 J_0(ks) + c_8 Y_0(ks)\big)e^{-i\omega t}$$

$$E_\varphi(s,t) = \frac{ic^2 k}{\omega}\big(c_7 J_1(ks) + c_8 Y_1(ks)\big)e^{-i\omega t} = ic\big(c_7 J_1(ks) + c_8 Y_1(ks)\big)e^{-i\omega t}$$

Both E and B must be continuous, so

$$B: \quad c_7 J_0(k r_{out}) + c_8 Y_0(k r_{out}) = c_5 J_0(k' r_{out}) + c_6 Y_0(k' r_{out})$$

$$E: \quad c\big(c_7 J_1(k r_{out}) + c_8 Y_1(k r_{out})\big) = \frac{\omega}{k'}\big(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\big)$$

$$E: \quad c_7 J_1(k r_{out}) + c_8 Y_1(k r_{out}) = \frac{k}{k'}\big(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\big)$$

EM waves between solenoid and shield: $\frac{D}{2} < s < \frac{d_{sh}}{2}$

Sinusoidally varying current: $s = \frac{D}{2}$

Let's call $\frac{D}{2} = R$ for simplicity.

$$B_z(s) = c_9 J_0(ks) + c_{10} Y_0(ks)$$

$$E_\varphi = ic\big(c_9 J_1(ks) + c_{10} Y_1(ks)\big)$$

At R, E must be continuous, and B must be discontinuous (it gets boosted inside by the current), so

$$B(R_-) = \mu_0 I_0 \frac{N}{l_p} + B(R_+)$$

$$B: \quad c_7 J_0(kR_-) + c_8 Y_0(kR_-) = c_9 J_0(kR_+) + c_{10} Y_0(kR_+) + \mu_0 I_0 \frac{N}{l_p}$$

$$E: \quad c_7 J_1(kR) + c_8 Y_1(kR) = c_9 J_1(kR) + c_{10} Y_1(kR)$$

EM waves outside shield: $s > \frac{d_{sh}}{2} = r_{sh}$

$$B_z(s) = c_{11}J_0(ks) + c_{12}Y_0(ks)$$

$$E_\varphi(s) = ic\big(c_{11}J_0(ks) + c_{12}Y_0(ks)\big)$$

Also, the solution for s>dsh/2 will be travelling waves, which happens to require a particular combination of J and Y so that, as $s \to \infty$ we get the solution for a plane wave. That is:

$$ic_{11} = c_{12}$$

$$B_z(s) = c_{11}J_0(ks) + ic_{11}Y_0(ks)$$

$$E_\varphi = ic\big(c_{11}J_1(ks) + ic_{11}Y_1(ks)\big)$$

And using the boundary condition,

$$B\big|_{s=\frac{d_{sh}}{2}-} - B\big|_{s=\frac{d_{sh}}{2}+} = \frac{\mu_0 I_{sh}}{l_p} = \frac{\mu_0 \pi d_{sh} E(r_{sh})}{R_{sh} l_p}$$

$$B: \quad c_9 J_0(kr_{sh_-}) + c_{10}Y_0(kr_{sh_-}) = c_{11}J_0(kr_{sh_+}) + ic_{11}Y_0(kr_{sh_+}) + \frac{\mu_0 \pi d_{sh} E(r_{sh})}{R_{sh} l_p}$$

$$E(r_{sh}) = ic\big(c_{11}J_1(kr_{sh}) + ic_{11}Y_1(kr_{sh})\big)$$

$$0 = c_{11}\left(J_0(kr_{sh_+}) + \frac{\mu_0 \pi d_{sh}}{R_{sh} l_p} icJ_1(kr_{sh}) + iY_0(kr_{sh}) - \frac{\mu_0 \pi d_{sh}}{R_{sh} l_p} cY_1(kr_{sh})\right) - c_9 J_0(kr_{sh_-}) - c_{10}Y_0(kr_{sh_-})$$

$$E: \; c_9 J_1(kr_{sh}) + c_{10}Y_1(kr_{sh}) = c_{11}J_1(kr_{sh}) + ic_{11}Y_1(kr_{sh})$$

Matrix form

Rather than using straight substitution, maybe a better plan would be to put these constants and equations into matrix form.

List of (mostly) unknowns

$$0 = c_1 J_0(kr_{in}) - c_5 J_0(k' r_{in}) - c_6 Y_0(k' r_{in})$$

$$0 = c_1 J_1(kr_{in}) - \frac{k}{k'}\left(c_5 J_1(k' r_{in}) + c_6 Y_1(k' r_{in})\right)$$

$$0 = c_5 J_0(k' r_{out}) + c_6 Y_0(k' r_{out}) - \ c_7 J_0(kr_{out}) - c_8 Y_0(kr_{out})$$

$$0 = \frac{k}{k'}\left(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\right) - c_7 J_1(kr_{out}) - c_8 Y_1(kr_{out})$$

$$\mu_0 I_0 \frac{N}{l_p} = c_7 J_0(kR_-) + c_8 Y_0(kR_-) - c_9 J_0(kR_+) - c_{10} Y_0(kR_+)$$

$$0 = c_7 J_1(kR) + c_8 Y_1(kR) - \ c_9 J_1(kR) - c_{10} Y_1(kR)$$

$$0 = c_9 J_1(kr_{sh}) + c_{10} Y_1(kr_{sh}) - c_{11} J_1(kr_{sh}) - ic_{11} Y_1(kr_{sh})$$

$$0 = c_9 J_0\left(kr_{sh_-}\right) + c_{10} Y_0\left(kr_{sh_-}\right) + c_{11}\left(\frac{\mu_0 \pi d_{sh}}{R_{sh} l_p} c Y_1(kr_{sh}) - J_0\left(kr_{sh_+}\right) - \frac{\mu_0 \pi d_{sh}}{R_{sh} l_p} ic J_1(kr_{sh}) - iY_0(kr_{sh})\right)$$

Electromagnetic field equations in Matrix Form

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mu_0 I_0 \dfrac{N}{l_p} \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix}
J_0(kr_{in}) & -J_0(k'r_{in}) & -Y_0(k'r_{in}) & 0 & 0 & 0 & 0 & 0 \\
J_1(kr_{in}) & -\dfrac{k}{k'}J_1(k'r_{in}) & -\dfrac{k}{k'}Y_1(k'r_{in}) & 0 & 0 & 0 & 0 & 0 \\
0 & J_0(k'r_{out}) & Y_0(k'r_{out}) & -J_0(kr_{out}) & -Y_0(kr_{out}) & 0 & 0 & 0 \\
0 & \dfrac{k}{k'}J_1(k'r_{out}) & \dfrac{k}{k'}Y_1(k'r_{out}) & -J_1(kr_{out}) & -Y_1(kr_{out}) & 0 & 0 & 0 \\
0 & 0 & 0 & J_0(kR) & Y_0(kR) & -J_0(kR) & -Y_0(kR) & 0 \\
0 & 0 & 0 & J_1(kR) & Y_1(kR) & -J_1(kR) & -Y_1(kR) & 0 \\
0 & 0 & 0 & 0 & 0 & J_1(kr_{sh}) & Y_1(kr_{sh}) & -J_1(kr_{sh}) - iY_1(kr_{sh}) \\
0 & 0 & 0 & 0 & 0 & J_0(kr_{sh}) & Y_0(kr_{sh}) & \dfrac{\mu_0 \pi c d_{sh}}{R_{sh} l_p}\left(Y_1(kr_{sh}) - iJ_1(kr_{sh})\right) - J_0(kr_{sh}) - iY_0(kr_{sh})
\end{bmatrix}
\begin{bmatrix} c1 \\ c5 \\ c6 \\ c7 \\ c8 \\ c9 \\ c10 \\ c11 \end{bmatrix}
$$

However, matrix is singular to numerical precision. This shouldn't be the case. Either there's a problem with the math, or the computer program is not properly implemented, or we need to use a tool other than Octave to get higher-precision simulation results.

When the problem is solved without the shield, the solution is numerically stable below ~1 MHz for reasonable values of r, R, etc., and then breaks down. But with the shield it appears to be singular for all frequencies.