Blast! We might not be able to get away with the quasistatic approximation, because our frequency is so high. In that case, the magnetic field inside (and *outside*) a solenoid is *not* constant, but actually takes the form of radio waves. So we will have to calculate this again, using Maxwell's equations.

Regions

Here's the setup: Each of these regions represents an area covered by a different differential equation, which we'll have to stitch together with matching boundary conditions.

EM waves in free space: $0 \leq s < \frac{d}{2} - t$

EM waves in conductor: $\frac{d}{2} - t \leq s \leq \frac{d}{2}$

EM waves in free space: $\frac{d}{2} < s < \frac{D}{2}$

Sinusoidally varying current: $s = \frac{D}{2}$

EM waves in free space: $s > \frac{D}{2}$

Equations:

In free space

From Griffiths chapter 9,

$$\nabla^2 \mathrm{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathrm{E}}{\partial \mathrm{t}^2}, \quad \nabla^2 \mathrm{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathrm{B}}{\partial \mathrm{t}^2}$$

We're expecting the magnetic fields to be only pointing along z, and the electric fields should all be pointing along φ only. They should both vary sinusoidally with time. Also, they should be constant in z and φ. That should make our work considerably easier!

In cylindrical coordinates:

$$-\mu_0 \epsilon_0 \omega^2 E = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial E}{\partial s} \right)$$

$$-\mu_0 \epsilon_0 \mathrm{B} = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial B}{\partial s} \right)$$

Magnetic:

$$0 = \frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} + \mu_0 \epsilon_0 \omega^2 B$$

The speed of light is $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, so we can define $\omega\sqrt{\mu_0 \epsilon_0}$ as the wave number, $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$. Then:

$$0 = \frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} + \mathrm{k}^2 B$$

Electric:

$$0 = \frac{\partial^2 E}{\partial s^2} + \frac{1}{s}\frac{\partial E}{\partial s} + k^2 E$$

The solutions to these are, unfortunately, Bessel functions (aka the "cylindrical harmonics"). These functions always tend to appear in cylindrical coordinate problems. The solutions are:

$$B_z(s) = c_1 J_0(ks) + c_2 Y_0(ks)$$

$$E_\varphi(s) = c_3 J_0(ks) + c_4 Y_0(ks)$$

J_0 is a Bessel function "of the first kind" of order 0, and Y_0 is a Bessel function "of the second kind" of order 0. $C_{1,2,3,4}$ are just constants that we can use to stitch these equations together at the boundaries.

Also, B and E here represent the amplitudes of the electric fields, since the time-dependence cancelled out of the equation. Of course, they are time-varying.

The electric and magnetic fields are not independent though; they are related by

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$-\frac{\partial B}{\partial s} = -\frac{i\omega}{c^2} E$$

$$E = -\frac{ic^2}{\omega}\frac{\partial B}{\partial s}$$

$$E = -\frac{ic^2}{\omega}\frac{\partial}{\partial s}\big(c_1 J_0(ks) + c_2 Y_0(ks)\big)$$

$$E_\varphi(s) = \frac{ic^2}{\omega}\big(c_1 k J_1(ks) + c_2 k Y_1(ks)\big)$$

And since

$$k = \frac{\omega}{c}$$

$$\frac{ikc^2}{\omega} = ic$$

so

$$E_\varphi(s) = ic\big(c_1 J_1(ks) + c_2 Y_1(ks)\big)$$

In the conductor

From Griffiths, chapter 9, the electromagnetic field in a conductor is:

$$\nabla^2 E = \mu\epsilon \frac{d^2 E}{dt^2} + \mu\sigma \frac{\partial E}{\partial t}, \quad \nabla^2 B = \mu\epsilon \frac{\partial^2 B}{\partial t^2} + \mu\sigma \frac{\partial B}{\partial t}$$

In cylindrical coordinates:

$$\nabla^2 E = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial E}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 E}{\partial \varphi^2} + \frac{\partial^2 E}{\partial z^2} = \mu\epsilon \frac{\partial d^2 E}{dt^2} + \mu\sigma \frac{\partial E}{\partial t}$$

$$\nabla^2 B = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial B}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 B}{\partial \varphi^2} + \frac{\partial^2 B}{\partial z^2} = \mu\epsilon \frac{\partial^2 B}{dt^2} + \mu\sigma \frac{\partial B}{\partial t}$$

And then requiring them to be functions of s and t only,

$$\frac{\partial^2 E}{\partial s^2} + \frac{1}{s}\frac{\partial E}{\partial s} - \mu\epsilon \frac{\partial d^2 E}{dt^2} - \mu\sigma \frac{\partial E}{\partial t} = 0$$

$$\frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} - \mu\epsilon \frac{\partial d^2 B}{dt^2} - \mu\sigma \frac{\partial B}{\partial t} = 0$$

So now there's a bit of a complication. Unlike before, we have terms that are proportional to $-\frac{\partial E}{\partial t}$ and $-\frac{\partial B}{\partial t}$. You might recognize these as damping terms. When we plug in the time-dependence, it doesn't cancel out the way it did before. Using separation of variables, let

$$B = T_B(t) S_B(s)$$

$$T_B \frac{\partial^2 S_B}{\partial s^2} + T_B \frac{1}{s}\frac{\partial S_B}{\partial s} - \mu\epsilon S_B \frac{\partial d^2 T_B}{dt^2} - \mu\sigma S_B \frac{\partial T_B}{\partial t} = 0$$

$$\frac{1}{S_B}\frac{\partial^2 S_B}{\partial s^2} + \frac{1}{S_B}\frac{1}{s}\frac{\partial S_B}{\partial s} = \mu\epsilon \frac{1}{T_B}\frac{\partial d^2 T_B}{dt^2} + \mu\sigma \frac{1}{T_B}\frac{\partial T_B}{\partial t}$$

For the time half of this equation, we have the constant U that we have to solve for (with units of per-metres).

$$\mu\epsilon\frac{\partial d^2 T_B}{dt^2} + \mu\sigma\frac{\partial T_B}{\partial t} + T_B U^2 = 0$$

The characteristic polynomial of this differential equation is:

$$\mu\epsilon r^2 + \mu\sigma r - U^2 = 0$$

$$r = \frac{-\sigma\mu \pm \sqrt{(\sigma\mu)^2 - 4\epsilon\mu U^2}}{2\epsilon\mu} = \frac{-\sigma}{2\epsilon} \pm i\sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}$$

So the general solution for this is:

$$T_B(t) = c_5 \exp\left(\frac{-\sigma}{2\epsilon}t + i\sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right) + c_6 \exp\left(\frac{-\sigma}{2\epsilon}t - i\sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)$$

$$T_B(t) = e^{\frac{-\sigma}{2\epsilon}t}\left(c_5 \exp\left(i\sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right) + c_6 \exp\left(-i\sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)\right)$$

$$T_B(t) = e^{\frac{-\sigma}{2\epsilon}t}\left(c_5 \cos\left(\sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right) + c_6 \sin\left(\sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}\,t\right)\right)$$

$$T_B(t) = \left(c_5 \cos(\omega t) + c_6 \sin(\omega t)\right)e^{\frac{-\sigma}{2\epsilon}t}$$

Where I've set the frequency to be

$$\omega = \sqrt{\frac{U^2}{\mu\epsilon} - \left(\frac{\sigma}{2\epsilon}\right)^2}$$

$$\mu\epsilon\omega^2 + \mu\epsilon\left(\frac{\sigma}{2\epsilon}\right)^2 = U^2$$

Whew… so it's an exponentially decaying solution? That's not what we wanted at all! We need something that's just sinusoidal, but perhaps exponentially decaying in z. Clearly, I chose the wrong value of U. What value will cancel out the exponential decrease? I need

$$Re\left\{\frac{-\sigma}{2\epsilon}+i\sqrt{\frac{U^2}{\mu\epsilon}-\left(\frac{\sigma}{2\epsilon}\right)^2}\right\}=0$$

$$\frac{U^2}{\mu\epsilon}=\left(\frac{\sigma}{2\epsilon}\right)^2+\left(\frac{-i\sigma}{2\epsilon}-\omega\right)^2$$

$$\left(\frac{-i\sigma}{2\epsilon}-\omega\right)^2=\frac{-\sigma^2}{4\epsilon^2}+i\omega\frac{\sigma}{\epsilon}+\omega^2$$

$$U^2=i\omega\mu\sigma+\mu\epsilon\omega^2={k'}^2$$

Plugging this in,

$$T_B(t)=c_5\exp(-i\omega t)+c_6\exp\left(\frac{-\sigma}{\epsilon}t+i\omega t\right)$$

And now, since the solution is supposed to be just oscillating in time, we can say c_6=0.

$$T_B(t)=\exp(-i\omega t)$$

And it is worth noting also that

$$\mu\sigma-i\mu\epsilon\omega=\frac{{k'}^2}{i\omega}$$

Now, we just need to solve the spatial side:

$$\frac{\partial^2 S_B}{\partial s^2}+\frac{1}{s}\frac{\partial S_B}{\partial s}+{k'}^2S_B=0$$

This is just the same as in free space, except that the wave number is more complicated.

$$B_z(s)=c_5J_0(k's)+c_6Y_0(k's)$$

Now the magnetic field should be doing something like this:

$$B_z(s,t)=\left(c_5J_0(k's)+c_6Y_0(k's)\right)e^{-i\omega t}$$

The electric field is found from Maxwell's equations:

$$E_\varphi(s,t)=\left(c_7J_0(k's)+c_8Y_0(k's)\right)e^{-i\omega t}$$

$$\nabla\times B=\mu\epsilon\frac{\partial E}{\partial t}+\mu\sigma E=-i\mu\epsilon\omega E+\mu\sigma E=-\frac{\partial B}{\partial s}$$

$$-\frac{\partial B}{\partial s}=c_5k'J_1(k's)+c_6k'Y_1(k's)=E(\mu\sigma-i\mu\epsilon\omega)=E\frac{{k'}^2}{i\omega}$$

$$E_\varphi(s) = \frac{i\omega}{k'}\big(c_5 J_1(k's) + c_6 Y_1(k's)\big)$$

In region with current flow

$$I(t) = I_0 \sin(\omega t)\,, s = \frac{D}{2}$$

The electric field is continuous, but the magnetic field is not.

$$B|_{s=\frac{D}{2}-} - B|_{s=\frac{D}{2}+} = \mu_0 I_0 \frac{N}{l_p}$$

Matching up the boundary conditions

1. **EM waves in center area free space:** $0 \le s < \frac{d}{2} - t$

$$B_z(s) = \big(c_1 J_0(ks) + c_2 Y_0(ks)\big)e^{-i\omega t}$$

$$E_\varphi = ic\big(c_1 J_1(ks) + c_2 Y_1(ks)\big)e^{-i\omega t}$$

Condition: Must be finite as s->0

$$c_2 = 0$$

$$B_z(s) = c_1 J_0(ks)e^{i\omega t}$$

$$E_\varphi(s) = c_1 ic J_1(ks)e^{i\omega t}$$

2. **EM waves in conductor:** $\frac{d}{2} - t \le s \le \frac{d}{2}$

$$B_z(s,t) = \big(c_5 J_0(k's) + c_6 Y_0(k's)\big)e^{-i\omega t}$$

$$E_\varphi(s,t) = \frac{i\omega}{k'}\big(c_5 J_1(k's) + c_6 Y_1(k's)\big)e^{-i\omega t}$$

Conditions: Let's call $\frac{d}{2} - t = r_{in}$, and $\frac{d}{2} = r_{out}$ for notational simplicity.

Both E and B must be continuous, so

$$B: \quad c_5 J_0(k' r_{in}) + c_6 Y_0(k' r_{in}) = c_1 J_0(k r_{in})$$

$$E: \quad \frac{\omega}{k'}\big(c_5 J_1(k' r_{in}) + c_6 Y_1(k' r_{in})\big) = c_1 c J_1(k r_{in})$$

Let's try to solve c_5 and c_6 in terms of c_1.

$$c_5 = \frac{c_1 J_0(kr_{in}) - c_6 Y_0(k'r_{in})}{J_0(k'r_{in})}$$

$$\frac{\omega}{k'}\left(\frac{c_1 J_0(kr_{in}) - c_6 Y_0(k'r_{in})}{J_0(k'r_{in})} J_1(k'r_{in}) + c_6 Y_1(k'r_{in})\right) = c_1 c J_1(kr_{in})$$

$$\frac{c_1 J_0(kr_{in}) J_1(k'r_{in}) - c_6 Y_0(k'r_{in}) J_1(k'r_{in}) + c_6 Y_1(k'r_{in}) J_0(k'r_{in})}{J_0(k'r_{in})} = \frac{ck'}{\omega} c_1 J_1(kr_{in})$$

$$c_6\big(Y_1(k'r_{in})J_0(k'r_{in}) - Y_0(k'r_{in})J_1(k'r_{in})\big) = c_1\left(\frac{k'}{k} J_0(k'r_{in})J_1(kr_{in}) - J_0(kr_{in})J_1(k'r_{in})\right)$$

$$c_6 = c_1 \frac{\frac{k'}{k} J_0(k'r_{in})J_1(kr_{in}) - J_0(kr_{in})J_1(k'r_{in})}{Y_1(k'r_{in})J_0(k'r_{in}) - Y_0(k'r_{in})J_1(k'r_{in})}$$

Substitute back to solve for c_5.

$$c_5 = c_1 \frac{J_0(kr_{in}) - \dfrac{\frac{k'}{k} J_0(k'r_{in})J_1(kr_{in}) - J_0(kr_{in})J_1(k'r_{in})}{Y_1(k'r_{in})J_0(k'r_{in}) - Y_0(k'r_{in})J_1(k'r_{in})} Y_0(k'r_{in})}{J_0(k'r_{in})}$$

$$c_5 = c_1\left(\frac{J_0(kr_{in})}{J_0(k'r_{in})} - \frac{\frac{ck'}{\omega}\frac{J_1(kr_{in})}{J_1(k'r_{in})} - \frac{J_0(kr_{in})}{J_0(k'r_{in})}}{\frac{Y_1(k'r_{in})}{Y_0(k'r_{in})}\frac{J_0(k'r_{in})}{J_1(k'r_{in})} - 1}\right)$$

$$c_5 = c_1\left(\frac{J_0(kr_{in})}{J_0(k'r_{in})} - \frac{\frac{k'}{k} J_1(kr_{in})Y_0(k'r_{in}) - \frac{J_0(kr_{in})}{J_0(k'r_{in})} J_1(k'r_{in})Y_0(k'r_{in})}{Y_1(k'r_{in})J_0(k'r_{in}) - J_1(k'r_{in})Y_0(k'r_{in})}\right)$$

Simplify for the sake of sanity:

$$A_{15} \equiv \frac{J_0(kr_{in})}{J_0(k'r_{in})} - \frac{\frac{k'}{k} J_1(kr_{in})Y_0(k'r_{in}) - \frac{J_0(kr_{in})}{J_0(k'r_{in})} J_1(k'r_{in})Y_0(k'r_{in})}{Y_1(k'r_{in})J_0(k'r_{in}) - J_1(k'r_{in})Y_0(k'r_{in})}$$

$$A_{16} \equiv \frac{\frac{k'}{k} J_0(k'r_{in})J_1(kr_{in}) - J_0(kr_{in})J_1(k'r_{in})}{Y_1(k'r_{in})J_0(k'r_{in}) - Y_0(k'r_{in})J_1(k'r_{in})}$$

$$c_5 = c_1 A_{15}$$

$$c_6 = c_1 A_{16}$$

Continuing on...

3. EM waves between conductor and solenoid: $\frac{d}{2} < s < \frac{D}{2}$

$$B_z(s,t) = \big(c_7 J_0(ks) + c_8 Y_0(ks)\big)e^{-i\omega t}$$

$$E_\varphi(s,t) = \frac{ic^2 k}{\omega}\big(c_7 J_1(ks) + c_8 Y_1(ks)\big)e^{-i\omega t} = ic\big(c_7 J_1(ks) + c_8 Y_1(ks)\big)e^{-i\omega t}$$

Both E and B must be continuous, so

$$B: \quad c_7 J_0(kr_{out}) + c_8 Y_0(kr_{out}) = c_5 J_0(k' r_{out}) + c_6 Y_0(k' r_{out})$$

$$E: \quad c\big(c_7 J_1(kr_{out}) + c_8 Y_1(kr_{out})\big) = \frac{\omega}{k'}\big(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\big)$$

Let's try to solve c_7 and c_8 in terms of c_5 and c_6.

$$c_7 = \frac{c_5 J_0(k' r_{out}) + c_6 Y_0(k' r_{out}) - c_8 Y_0(kr_{out})}{J_0(kr_{out})}$$

$$c_5 J_0(k' r_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} + c_6 Y_0(k' r_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} - c_8\left(Y_0(kr_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} + Y_1(kr_{out})\right) = \frac{\omega}{ck'}\big(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\big)$$

$$c_8\left(Y_0(kr_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} + Y_1(kr_{out})\right) = c_5 J_0(k' r_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} + c_6 Y_0(k' r_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} - \frac{\omega}{ck'}\big(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\big)$$

$$c_8 = \frac{c_5 J_0(k' r_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} + c_6 Y_0(k' r_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} - \frac{\omega}{ck'}\big(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\big)}{Y_0(kr_{out})\frac{J_1(kr_{out})}{J_0(kr_{out})} + Y_1(kr_{out})}$$

$$c_8 = \frac{c_5 J_0(k' r_{out})J_1(kr_{out}) + c_6 Y_0(k' r_{out})J_1(kr_{out}) - \frac{\omega}{ck'}J_0(kr_{out})\big(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\big)}{Y_0(kr_{out})J_1(kr_{out}) + Y_1(kr_{out})J_0(kr_{out})}$$

$$c_8 = c_5\left(\frac{J_0(k' r_{out})J_1(kr_{out}) - \frac{\omega}{ck'}J_0(kr_{out})J_1(k' r_{out})}{Y_0(kr_{out})J_1(kr_{out}) + Y_1(kr_{out})J_0(kr_{out})}\right) + c_6\left(\frac{Y_0(k' r_{out})J_1(kr_{out}) - \frac{\omega}{ck'}J_0(kr_{out})Y_1(k' r_{out})}{Y_0(kr_{out})J_1(kr_{out}) + Y_1(kr_{out})J_0(kr_{out})}\right)$$

Substitute back to solve for c_7. It's a bit messy.

$$c_7 = \frac{c_5 J_0(k' r_{out}) + c_6 Y_0(k' r_{out}) - \frac{c_5 J_0(k' r_{out}) J_1(k r_{out}) + c_6 Y_0(k' r_{out}) J_1(k r_{out}) - \frac{\omega}{ck'} J_0(k r_{out})\left(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\right)}{Y_0(k r_{out}) J_1(k r_{out}) + Y_1(k r_{out}) J_0(k r_{out})} Y_0(k r_{out})}{J_0(k r_{out})}$$

$$c_7 = c_5 \frac{J_0(k' r_{out})}{J_0(k r_{out})} + c_6 \frac{Y_0(k' r_{out})}{J_0(k r_{out})} - \frac{c_5 \frac{J_0(k' r_{out})}{J_0(k r_{out})} J_1(k r_{out}) + c_6 \frac{Y_0(k' r_{out})}{J_0(k r_{out})} J_1(k r_{out}) - \frac{\omega}{ck'}\left(c_5 J_1(k' r_{out}) + c_6 Y_1(k' r_{out})\right)}{J_1(k r_{out}) + J_0(k r_{out}) \frac{Y_1(k r_{out})}{Y_0(k r_{out})}}$$

$$c_7 = c_5 \frac{J_0(k' r_{out})}{J_0(k r_{out})}\left(1 + \frac{\frac{\omega}{ck'} \frac{J_0(k r_{out})}{J_0(k' r_{out})} J_1(k' r_{out}) - J_1(k r_{out})}{J_1(k r_{out}) + J_0(k r_{out}) \frac{Y_1(k r_{out})}{Y_0(k r_{out})}}\right) + c_6 \frac{Y_0(k' r_{out})}{J_0(k r_{out})}\left(1 + \frac{\frac{\omega}{ck'} \frac{J_0(k r_{out})}{Y_0(k' r_{out})} Y_1(k' r_{out}) - J_1(k r_{out})}{J_1(k r_{out}) + J_0(k r_{out}) \frac{Y_1(k r_{out})}{Y_0(k r_{out})}}\right)$$

I'm afraid that it's still quite messy, but these expressions are just a bunch of constants piled together.

Simplify for the sake of sanity:

$$A_{57} \equiv \frac{J_0(k' r_{out})}{J_0(k r_{out})}\left(1 + \frac{\frac{\omega}{ck'} \frac{J_0(k r_{out})}{J_0(k' r_{out})} J_1(k' r_{out}) - J_1(k r_{out})}{J_1(k r_{out}) + J_0(k r_{out}) \frac{Y_1(k r_{out})}{Y_0(k r_{out})}}\right)$$

$$A_{67} \equiv \frac{Y_0(k' r_{out})}{J_0(k r_{out})}\left(1 + \frac{\frac{\omega}{ck'} \frac{J_0(k r_{out})}{Y_0(k' r_{out})} Y_1(k' r_{out}) - J_1(k r_{out})}{J_1(k r_{out}) + J_0(k r_{out}) \frac{Y_1(k r_{out})}{Y_0(k r_{out})}}\right)$$

$$A_{58} \equiv \frac{J_0(k' r_{out}) J_1(k r_{out}) - \frac{\omega}{ck'} J_0(k r_{out}) J_1(k' r_{out})}{Y_0(k r_{out}) J_1(k r_{out}) + Y_1(k r_{out}) J_0(k r_{out})}$$

$$A_{68} \equiv \frac{Y_0(k' r_{out}) J_1(k r_{out}) - \frac{\omega}{ck'} J_0(k r_{out}) Y_1(k' r_{out})}{Y_0(k r_{out}) J_1(k r_{out}) + Y_1(k r_{out}) J_0(k r_{out})}$$

$$c_7 = c_5 A_{57} + c_6 A_{67}$$

$$c_8 = c_5 A_{58} + c_6 A_{68}$$

4. EM waves outside of the solenoid: $s > \frac{D}{2}$

Sinusoidally varying current: $s = \frac{D}{2}$

Let's call $\frac{D}{2} = R$ for simplicity.

$$B_z(s) = c_9 J_0(ks) + c_{10} Y_0(ks)$$

$$E_\varphi = ic\left(c_9 J_1(ks) + c_{10} Y_1(ks)\right)$$

Also, the solution for s>R will be travelling waves, which happens to require a particular combination of J and Y so that, as $s \to \infty$ we get the solution for a plane wave. That is:

$$ic_9 = c_{10}$$

$$B_z(s) = c_9 J_0(ks) + ic_9 Y_0(ks)$$

$$E_\varphi = ic\left(c_9 J_1(ks) + ic_9 Y_1(ks)\right)$$

At R, E must be continuous, and B must be discontinuous (it gets boosted inside by the current), so

$$B(R_-) = \mu_0 I_0 \frac{N}{l_p} + B(R_+)$$

$$B: \quad c_7 J_0(kR_-) + c_8 Y_0(kR_-) = c_9 J_0(kR_+) + ic_9 Y_0(kR_+) + \mu_0 I_0 \frac{N}{l_p}$$

$$E: \quad c_7 J_1(kR_-) + c_8 Y_1(kR_-) = c_9 J_1(kR_+) + ic_9 Y_1(kR_+)$$

So two equations here, and the unknowns now are just c_1 and c_9.

$$c_9\left(J_0(kR_+) + iY_0(kR_+)\right) = c_7 J_0(kR_-) + c_8 Y_0(kR_-) - \mu_0 I_0 \frac{N}{l_p}$$

$$c_9 = \frac{c_7 J_0(kR_-) + c_8 Y_0(kR_-) - \mu_0 I_0 \frac{N}{l_p}}{J_0(kR_+) + iY_0(kR_+)}$$

$$c_9 = \frac{c_7 J_0(kR_-) + c_8 Y_0(kR_-) - \mu_0 I_0 \frac{N}{l_p}}{J_0(kR_+) + iY_0(kR_+)}$$

$$c_9 = c_7 \frac{J_0(kR)}{J_0(kR) + iY_0(kR)} + c_8 \frac{Y_0(kR)}{J_0(kR) + iY_0(kR)} - \mu_0 I_0 \frac{N}{l_p}\left(\frac{1}{J_0(kR) + iY_0(kR)}\right)$$

$$c_9 = c_7 \frac{1}{1 + i\frac{Y_0(kR)}{J_0(kR)}} + c_8 \frac{1}{\frac{J_0(kR)}{Y_0(kR)} + i} - I_0\mu_0 \frac{N}{l_p}\left(\frac{1}{J_0(kR) + iY_0(kR)}\right)$$

Simplifying:

$$A_{79} \equiv \frac{1}{1 + i\frac{Y_0(kR)}{J_0(kR)}}$$

$$A_{89} \equiv \frac{1}{\frac{J_0(kR)}{Y_0(kR)} + i}$$

$$B_{I09} \equiv -\mu_0 \frac{N}{l_p}\left(\frac{1}{J_0(kR) + iY_0(kR)}\right)$$

$$c_9 = c_7A_{79} + c_8A_{89} + I_0B_{I09}$$

Now it all comes together. It's basically a linear algebra problem from this point. We have to solve for c_1, using the electrical field:

$$c_7J_1(kR) + c_8Y_1(kR) = c_9J_1(kR) + ic_9Y_1(kR)$$

$$c_7J_1(kR) + c_8Y_1(kR) = (c_7A_{79} + c_8A_{89} + I_0B_{I09})\big(J_1(kR) + iY_1(kR)\big)$$

$$c_7\left(\frac{J_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{79}\right) + c_8\left(\frac{Y_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{89}\right) = I_0B_{I09}$$

$$(c_5A_{57} + c_6A_{67})\left(\frac{J_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{79}\right) + (c_5A_{58} + c_6A_{68})\left(\frac{Y_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{89}\right)$$
$$= I_0B_{I09}$$

$$c_5\left(A_{57}\left(\frac{J_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{79}\right) + A_{58}\left(\frac{Y_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{89}\right)\right)$$
$$+ c_6\left(A_{67}\left(\frac{J_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{79}\right) + A_{68}\left(\frac{Y_1(kR)}{\big(J_1(kR) + iY_1(kR)\big)} - A_{89}\right)\right)$$
$$= I_0B_{I09}$$

For simplicity:

$$A_{59} \equiv A_{57}\left(\frac{J_1(kR)}{\left(J_1(kR) + iY_1(kR)\right)} - A_{79}\right) + A_{58}\left(\frac{Y_1(kR)}{\left(J_1(kR) + iY_1(kR)\right)} - A_{89}\right)$$

$$A_{69} \equiv A_{67}\left(\frac{J_1(kR)}{\left(J_1(kR) + iY_1(kR)\right)} - A_{79}\right) + A_{68}\left(\frac{Y_1(kR)}{\left(J_1(kR) + iY_1(kR)\right)} - A_{89}\right)$$

$$c_5 A_{59} + c_6 A_{69} = I_0 B_{I09}$$

$$c_5 = c_1 A_{15}$$

$$c_6 = c_1 A_{16}$$

$$c_1 A_{15} A_{59} + c_1 A_{16} A_{69} = I_0 B_{I09}$$

$$c_1 (A_{15} A_{59} + A_{16} A_{69}) = I_0 B_{I09}$$

$$c_1 = I_0 \frac{B_{I09}}{A_{15} A_{59} + A_{16} A_{69}}$$

The problem is now solved, essentially.

Now we need to work backwards, now that the electric field is known everywhere.

$$E_\varphi(s,t) = \frac{i\omega}{k'}\left(c_5 J_1(k's) + c_6 Y_1(k's)\right)e^{-i\omega t}$$

All quantities here are known! Also, in conductive materials,

$$J = \sigma E$$

$$J_\varphi(s,t) = \frac{i\sigma\omega}{k'}\left(c_5 J_1(k's) + c_6 Y_1(k's)\right)e^{-i\omega t}$$

$$J_\varphi(s) = \frac{i\sigma\omega}{k'}\left(c_5 J_1(k's) + c_6 Y_1(k's)\right)$$

The total current is:

$$I_{ring} = 2\pi l_s \int_{r_{in}}^{r_{out}} J_\varphi(s) ds$$

$$I_{ring} = \frac{i2\pi l_s \sigma\omega}{k'} \int_{r_{in}}^{r_{out}} c_5 J_1(k's) + c_6 Y_1(k's) ds$$