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Induction heater + temperature sensor mathematical model

Assume a current-controlled, high-frequency sinusoidal power source in the primary coil.

Primary coil has N turns of wire, over a coil length l_p. Current in primary:

$$I_p = I_0 \sin(\omega t)$$

The coil is wound around a glass tube of outer diameter D = 6 mm. To calculate the magnetic flux through the tube produced by the primary, we'll start with the quasistatic approximation:

$$B = \frac{\mu_0 N I_p}{l_p}$$

$$\Phi_{p,p} = BA_p = \frac{\mu_0 N I_p}{l_p} \cdot \frac{\pi D^2}{4} = \frac{\mu_0 \pi D^2 N I_p}{4 l_p}$$

Inside is a narrow metal ring of outer diameter d = 3mm, with a wall thickness *t*. This is the secondary "coil". If the ring is at the center of the primary, the flux contained inside (due to the primary) is

$$\Phi_{s,p} = \Phi_p \frac{A_s}{A_p} = \frac{\mu_0 \pi D^2 N I_p}{4 l_p} \cdot \frac{d^2}{D^2}$$

The mutual inductance of two coils, M, is defined as:

$$M = \frac{\Phi_{s,p}}{I_p} = \frac{\mu_0 \pi D^2 N}{4 l_p} \cdot \frac{d^2}{D^2} = \frac{\mu_0 \pi d^2 N}{4 l_p}$$

By Faraday's law,

$$V_s = -\frac{d\Phi_s}{dt} = -\frac{d(I_s L_s + I_p M)}{dt}$$

$$V_s = -L_s \frac{dI_s}{dt} - M \frac{dI_p}{dt}$$

The secondary coil has a resistance R_s, which is a function of temperature and also frequency. By Ohm's law,

$$V_s = I_s R_s$$

$$I_s R_s + L_s \frac{dI_s}{dt} = -M\frac{dI_p}{dt}$$

What's the inductance of the secondary coil? Can we determine it? I think so.

If a current spontaneously appears in the secondary, it will create a flux:

$$B_{s,s} = \frac{\mu_0 I_s}{L_s}$$

$$\Phi_{s,s} = BA_s = \frac{\mu_0 I_s}{l_s} \cdot \frac{\pi D^2}{4} = \frac{\mu_0 \pi d^2 I_s}{4 l_s}$$

That flux will induce a voltage in the loop that opposes it. That induced voltage will be

$$V_{s,s} = -\frac{d\Phi_{s,s}}{dt} = -\frac{\mu_0 \pi d^2}{4 l_s}\frac{dI_s}{dt}$$

Treating it as an inductor, the voltage that would appear is

$$V_{s,s} = L_s \frac{dI_s}{dt}$$

So the inductance of the secondary ought to be

$$L_s = \frac{\mu_0 \pi d^2}{4 l_s}$$

Dimensional analysis sanity check: It does have units of henries...

$$I_s R_s + \frac{\mu_0 \pi d^2}{4 l_s}\frac{dI_s}{dt} = -\frac{\mu_0 \pi d^2 N}{4 l_p}\frac{dI_p}{dt}$$

The current in the secondary has to be at the same frequency as the driving coil, so it's of the form:

$$I_s(t) = I_s \sin(\omega t + \emptyset)$$

So

$$I_s \sin(\omega t + \emptyset)\, R_s + \frac{\omega \mu_0 \pi d^2}{4 l_s} I_s \cos\,(\omega t + \emptyset) = -\frac{\mu_0 \pi d^2 N}{4 l_p} I_0 \omega \cos(\omega t)$$

This equation has two unknowns: I_s and $\emptyset$. But, we also know that it must be true for all times. Can we use two values of t to obtain two equations?

Plug in $t = \frac{\pi}{2\omega}$…

$$I_s \sin\left(\frac{\pi}{2} + \emptyset\right) R_s + \frac{\omega\mu_0\pi d^2}{4l_s} I_s \cos\left(\frac{\pi}{2} + \emptyset\right) = 0$$

$$4R_s l_s I_s \sin\left(\frac{\pi}{2} + \emptyset\right) + \omega\mu_0\pi d^2 I_s \cos\left(\frac{\pi}{2} + \emptyset\right) = 0$$

And of course,

$$\sin\left(\frac{\pi}{2} + \emptyset\right) = \cos(\emptyset)$$

$$\cos\left(\frac{\pi}{2} + \emptyset\right) = -\sin(\emptyset)$$

$$4R_s l_s I_s \cos(\emptyset) = \omega\mu_0\pi d^2 I_s \sin(\emptyset)$$

This looks promising!

$$\tan(\emptyset) = \frac{4R_s l_s I_s}{\omega\mu_0\pi d^2 I_s} = \frac{4R_s l_s}{\omega\mu_0\pi d^2}$$

$$\emptyset = \tan^{-1}\left(\frac{4R_s l_s}{\omega\mu_0\pi d^2}\right)$$

Plug in t = 0…

$$I_s \sin(\emptyset) R_s + \frac{\omega\mu_0\pi d^2}{4l_s} I_s \cos(\emptyset) = -\frac{\mu_0\pi d^2 N}{4l_p} I_0\omega$$

$$4R_s l_s \sin(\emptyset) + \omega\mu_0\pi d^2 \cos(\emptyset) + \mu_0\omega\pi d^2 N \frac{l_s}{l_p}\frac{I_0}{I_s} = 0$$

$$4R_s l_s \sin\left(\tan^{-1}\left(\frac{4R_s l_s}{\omega\mu_0\pi d^2}\right)\right) + \omega\mu_0\pi d^2 \cos\left(\tan^{-1}\left(\frac{4R_s l_s}{\omega\mu_0\pi d^2}\right)\right) + \mu_0\omega\pi d^2 N \frac{l_s}{l_p}\frac{I_0}{I_s} = 0$$

Simplifying, using

$$\sin\left(\tan^{-1}(x)\right) = \frac{x}{\sqrt{x^2+1}}$$
$$\cos\left(\tan^{-1}(x)\right) = \frac{1}{\sqrt{x^2+1}}$$

$$4R_s l_s \frac{\frac{4R_s l_s}{\omega\mu_0\pi d^2}}{\sqrt{\frac{4R_s l_s}{\omega\mu_0\pi d^2}^2+1}} + \omega\mu_0\pi d^2 \frac{1}{\sqrt{\frac{4R_s l_s}{\omega\mu_0\pi d^2}^2+1}} + \mu_0\omega\pi d^2 N \frac{l_s}{l_p}\frac{I_0}{I_s} = 0$$

Simplified:

$$(4R_s l_s)^2 + (\omega\mu_0\pi d^2)^2 + \omega\mu_0\pi d^2 N \frac{l_s}{l_p}\frac{I_0}{I_s}\sqrt{(4R_s l_s)^2 + (\omega\mu_0\pi d^2)^2} = 0$$

Defining these constants:

$$A = 4R_s l_s$$

$$B = \omega\mu_0\pi d^2$$

$$A^2 + B^2 + BN\frac{l_s}{l_p}\frac{I_0}{I_s}\sqrt{A^2+B^2} = 0$$

Therefore

$$I_s = -I_0 N \frac{l_s}{l_p}\frac{B\sqrt{A^2+B^2}}{A^2+B^2}$$

$$I_s = -I_0 N \frac{l_s}{l_p}\frac{B}{\sqrt{A^2+B^2}}$$

The current in the secondary is thus:

$$I_s(t) = -I_0 N \frac{l_s}{l_p}\frac{B}{\sqrt{A^2+B^2}}\sin(\omega t + \emptyset)$$

$$\emptyset = \tan^{-1}\left(\frac{A}{B}\right)$$

or

$$I_s(t) = -I_0 N \frac{l_s}{l_p}\frac{\omega\mu_0\pi d^2}{\sqrt{(4R_s l_s)^2 + (\omega\mu_0\pi d^2)^2}}\sin(\omega t + \emptyset)$$

The power transformed into heat in the secondary will be

$$P_s = {I_s}^2 R_s$$

$$P_s(t) = R_s \left[I_0 N \frac{l_s}{l_p} \frac{B}{\sqrt{A^2 + B^2}} \sin(\omega t + \emptyset) \right]^2$$

$$P_s(t) = (I_0 N)^2 \left(\frac{l_s}{l_p} \right)^2 R_s \frac{B^2}{A^2 + B^2} \sin^2(\omega t + \emptyset)$$

This is the instantaneous power at a given time, t, but we'd rather know the power over an entire cycle. Integrating this expression over one period, and dividing by the period, gives:

$$P_s = \frac{R_s}{2} \left[I_0 N \frac{l_s}{l_p} \frac{B\sqrt{A^2 + B^2}}{A^2 + B^2} \right]^2$$

$$P_s = \frac{(I_0 N)^2}{2} \left(\frac{l_s}{l_p} \right)^2 R_s \frac{B^2}{A^2 + B^2}$$

To un-simplify this expression a little, so that it doesn't falsely appear to be linear with R_s,

$$P_s = \frac{(I_0 N)^2}{2} \left(\frac{l_s}{l_p} \right)^2 R_s \frac{(\omega \mu_0 \pi d^2)^2}{(4 R_s l_s)^2 + (\omega \mu_0 \pi d^2)^2}$$

Next step: How does this load appear to the primary coil?

The primary coil is assumed to be current-controlled, so the voltage that appears across it should be pure Faraday's law: the sum of the terms from the primary's self-inductance, and from the flux generated by the current in the secondary. And also from the coil resistance, to be precise.

$$V_p = I_p R_p + L_p \frac{dI_p}{dt} + M \frac{dI_s}{dt}$$

And from before:

$$M = \Phi_p \frac{A_s}{A_p} = \frac{\mu_0 \pi d^2 N I_p}{4 l_p}$$

So

$$\Phi_{p,s} = \frac{\mu_0 \pi d^2 N}{4 l_p} I_s$$

The voltage on the primary should therefore be:

$$V_p = I_p R_p + L_p \frac{dI_p}{dt} + \frac{d \frac{\mu_0 \pi d^2 N}{4l_p} I_s}{dt}$$

$$V_p = I_p R_p + I_0 L_p \omega \cos(\omega t) + \frac{\mu_0 \pi d^2 N}{4l_p} \frac{dI_s}{dt}$$

The inductance of the primary is

$$L_p = \frac{\Phi_{p,p}}{I_p} = \frac{\mu_0 \pi D^2 N}{4l_p}$$

So

$$V_p = I_0 \sin(\omega t) R_p + I_0 \omega \frac{\mu_0 \pi D^2 N}{4l_p} \cos(\omega t) + \frac{\mu_0 \pi d^2 N}{4l_p} \frac{dI_s}{dt}$$

And, since

$$I_s(t) = -I_0 N \frac{l_s}{l_p} \frac{\omega \mu_0 \pi d^2}{\sqrt{(4R_s l_s)^2 + (\omega \mu_0 \pi d^2)^2}} \sin(\omega t + \emptyset)$$

$$\frac{dI_s}{dt} = -I_0 N \frac{l_s}{l_p} \frac{\omega^2 \mu_0 \pi d^2}{\sqrt{(4R_s l_s)^2 + (\omega \mu_0 \pi d^2)^2}} \cos(\omega t + \emptyset)$$

We can solve for the primary voltage:

$$V_p = I_0 \sin(\omega t) R_p + I_0 \omega \frac{\mu_0 \pi D^2 N}{4l_p} \cos(\omega t) - \frac{\mu_0 \pi d^2 N}{4l_p} I_0 N \frac{l_s}{l_p} \frac{\omega^2 \mu_0 \pi d^2}{\sqrt{(4R_s l_s)^2 + (\omega \mu_0 \pi d^2)^2}} \cos(\omega t + \emptyset)$$

$$V_p = I_0 \sin(\omega t) R_p + I_0 \omega \frac{\mu_0 \pi N}{4l_p} \left[D^2 \cos(\omega t) - d^2 N \frac{l_s}{l_p} \frac{\omega \mu_0 \pi d^2}{\sqrt{(4R_s l_s)^2 + (\omega \mu_0 \pi d^2)^2}} \cos(\omega t + \emptyset) \right]$$

Now the problem becomes that R_p and R_s are both functions of temperature. How do we know which temperature we're measuring? Well, we can change our input current by adding a DC offset:

$$I_p = I_0 \sin(\omega t) + I_1$$

This won't affect anything in the AC end of the calculation, since it doesn't affect dI_p/dt. But it does let us measure the resistance of the primary coil!

$$V_p = R_p(I_0 \sin(\omega t) + I_1)$$

$$+I_0\omega\frac{\mu_0\pi N}{4l_p}\left[D^2\cos(\omega t) - d^2N\frac{l_s}{l_p}\frac{\omega\mu_0\pi d^2}{\sqrt{(4R_sl_s)^2 + (\omega\mu_0\pi d^2)^2}}\cos(\omega t + \emptyset)\right]$$

Then splitting the measured coil voltage into AC and DC frequencies:

$$V_{pDC} = I_1R_p$$

$$V_{pAC} = I_0\sin(\omega t)R_p + I_0\omega\frac{\mu_0\pi N}{4l_p}\left[D^2\cos(\omega t) - d^2N\frac{l_s}{l_p}\frac{\omega\mu_0\pi d^2}{\sqrt{(4R_sl_s)^2 + \left(\omega\mu_0\pi d^2\right)^2}}\cos(\omega t + \emptyset)\right]$$

Sensed AC and DC Voltage across primary coil

The area below this line is a work in progress.

Next up: We have to include a model of R_s as a function of temperature, frequency, geometry, and material properties.

For a thin-walled cylinder secondary, the resistance at DC is expected to just be:

$$R_{sDC} = \frac{\pi d\rho}{tl}$$

But actually this is only true if the current flow is uniform through the material. In fact, that's not the case because the current on the outside layer of the material cancels out some of the flux inside that layer. So there's less current deeper in the material compared with on the surface. The resistance seen by the current should be the average of the resistance over the area in which the current is flowing.

From Griffiths, chapter 9, the electromagnetic field in a conductor is:

$$\nabla^2 E = \mu\epsilon \frac{d^2 E}{dt^2} + \mu\sigma \frac{\partial E}{\partial t}, \quad \nabla^2 B = \mu\epsilon \frac{\partial^2 B}{\partial t^2} + \mu\sigma \frac{\partial B}{\partial t}$$

Since we're talking about a tube, we'll have to solve this in cylindrical coordinates. Oh my. The distance from the axis is s, the angle is φ, and the distance along the axis is z.

$$\nabla^2 E = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial E}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 E}{\partial \varphi^2} + \frac{\partial^2 E}{\partial z^2} = \mu\epsilon \frac{\partial d^2 E}{dt^2} + \mu\sigma \frac{\partial E}{\partial t}$$

$$\nabla^2 B = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial B}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 B}{\partial \varphi^2} + \frac{\partial^2 B}{\partial z^2} = \mu\epsilon \frac{\partial^2 B}{dt^2} + \mu\sigma \frac{\partial B}{\partial t}$$

Before we solve this, let's first figure out what kind of solutions we're interested in. We're expecting the magnetic fields to be only pointing along z, and the electric fields should all be pointing along φ only. They should both vary sinusoidally with time. Also, they should be constant in z and φ. That should make our work considerably easier!

Magnetic:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial B}{\partial s}\right)\sin(\omega t) = -\mu\epsilon\omega^2 B\sin(\omega t) + \mu\sigma\omega B\cos\left(\omega t + \emptyset_B(s)\right)$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial B}{\partial s}\right) = -\mu\epsilon\omega^2 B \ + \mu\sigma\omega B\cot\left(\omega t + \emptyset_B(s)\right)$$

Electric:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial E}{\partial s}\right)\sin(\omega t + \emptyset) = -\mu\epsilon\omega^2 E\sin(\omega t + \emptyset) + \mu\sigma\omega E\cos\left(\omega t + \emptyset_E(s)\right)$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial E}{\partial s}\right) = -\mu\epsilon\omega^2 E + \mu\sigma\omega E\cot\left(\omega t + \emptyset_E(s)\right)$$

Now we need some boundary conditions.

First of all, "s" is the radial coordinate, and this solution only applies inside a conductor. So it can only be valid for where "s" is a conductor, which is to say, it only applies inside the ring. That's from:

$\frac{d}{2} - t \leq s \leq \frac{d}{2}$

Outside of this region, the magnetic field should be a (time-varying) constant, supplied by the primary coil.

$$B|_{s=d/2} = B_0\sin(\omega t)$$

Inside the ring, the magnetic field is still a time-varying constant, but it's partially cancelled by current in the ring.

$$B|_{s=d/2} = B_0 \sin(\omega t) - \frac{\mu_0 I_s}{l_s}$$

Expressing I_s as a current density, rather than a current:

$$Is(t) = l_s \int_{\frac{d}{2}-t}^{\frac{d}{2}} J(s,t)\, ds$$

$$B|_{s=d/2} = B_0 \sin(\omega t) - \mu_0 \int_{\frac{d}{2}-t}^{\frac{d}{2}} J(s,t)\, ds$$

$$B|_{s=d/2} = B_0 \sin(\omega t) - \sigma\mu_0 \int_{\frac{d}{2}-t}^{\frac{d}{2}} E(s,t)\, ds$$

Uh oh – as expected, the magnetic field past the ring depends on the current flowing in the ring. So it depends on the solution of the equation that it's a boundary condition for… Does this make it unsuitable to use as a boundary condition? Perhaps we'd be better off using a boundary condition like this one:

$$\frac{\partial}{\partial s} B\bigg|_{s=d/2} = 0$$

I'm not sure what to use for boundary conditions on the electric field, because it's less obvious what they are. We'll have to check Maxwell's equations I guess…. Do that later?

Now for inside the tube itself:

$$\frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} = -\mu\epsilon\omega^2 B + \mu\sigma\omega B \cot(\omega t + \emptyset_B(s))$$

$$\frac{\partial^2 E}{\partial s^2} + \frac{1}{s}\frac{\partial E}{\partial s} = -\mu\epsilon\omega^2 E + \mu\sigma\omega E \cot(\omega t + \emptyset_E(s))$$

I'm afraid that the solutions to these are Bessel functions, aka the "cylindrical harmonics".

$$B(s) = c_3 J_0\left(s\sqrt{\mu\epsilon\omega^2 - \mu\sigma\omega\cot(\omega t + \emptyset_B(s))}\right) + c_4 Y_0\left(s\sqrt{\mu\epsilon\omega^2 - \mu\sigma\omega\cot(\omega t + \emptyset_B(s))}\right)$$

$$E(s) = c_1 J_0\left(s\sqrt{\mu\epsilon\omega^2 - \mu\sigma\omega\cot(\omega t + \emptyset_E(s))}\right) + c_2 Y_0\left(s\sqrt{\mu\epsilon\omega^2 - \mu\sigma\omega\cot(\omega t + \emptyset_E(s))}\right)$$

Now match to the boundary conditions. What's the derivative of a Bessel function?

$$\frac{\partial}{\partial z}(J_n(z)) = \frac{J_{n-1}(z) - J_{n+1}(z)}{2}$$

$$-\mu_0\epsilon_0\omega^2 E = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial E}{\partial s}\right)$$

$$-\mu_0\epsilon_0 \mathrm{B} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial B}{\partial s}\right)$$

But, fortunately, the magnetic field should be constant outside this region.

So what we'll have to do is solve both of these sets of equations, then match up the solutions on the boundary. The outside equations are actually easier to solve, so let's do that first.

$$0 = \frac{\partial^2 E}{\partial s^2} + \frac{1}{s}\frac{\partial E}{\partial s} + \mu_0\epsilon_0\omega^2 E$$

$$0 = \frac{\partial^2 B}{\partial s^2} + \frac{1}{s}\frac{\partial B}{\partial s} + \mu_0\epsilon_0 \mathrm{B}$$

I'm afraid that the solutions to these are Bessel functions, aka the "cylindrical harmonics".

$$E(s) = c_1 J_0\left(\sqrt{\mu_0\epsilon_0}s\right) + c_2 Y_0\left(\sqrt{\mu_0\epsilon_0}s\right)$$

$$B(s) = c_3 J_0\left(\sqrt{\mu_0\epsilon_0}s\right) + c_4 Y_0\left(\sqrt{\mu_0\epsilon_0}s\right)$$

J_0 is a Bessel function of the first kind, Y_0 is a Bessel function of the second kind. Fortunately, Y_0 explodes at the origin, which means we can take $c_2 = 0$ and $c_4 = 0$. Also, J_0 is a special case, where

$$J_0(x) = \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}$$

(It looks really ugly, but we wouldn't have gotten away so easily with the other Bessel functions…)

$$R_s \approx \frac{\rho}{\delta}\left(\frac{L}{\pi(d-\delta)}\right) \approx \frac{\rho}{\delta}\left(\frac{L}{\pi d}\right)$$

Where the skin depth is

$$\delta = \sqrt{\frac{2\rho}{\omega\mu}}$$

And for all materials we're interested in, it's likely that

$$\mu = \mu_0$$

So substituting:

$$R_s \approx \rho\left(\frac{L}{\pi d}\right)\sqrt{\frac{\omega\mu}{2\rho}} = \left(\frac{L}{\pi d}\right)\sqrt{\frac{\rho\omega\mu}{2}}$$

This assumes that the skin depth $\ll d$, and also $\delta \ll t$. This second one is the assumption least likely to be valid, so we ought to come back and check this later, and replace our expression with a more exact one for a thin-walled cylinder. (This is where the question of "magnetic transparency" comes in).

The resistivity of metal varies as a function of temperature, which can be fit to a polynomial:

$$\rho = \rho_0(1 + \alpha_1(T - T_0) + \alpha_2(T - T_0)^2 + \cdots)$$

Usually T_0 = 0°C or 20°C, and the nonlinear terms are small enough to be neglected.

Substance	$\boldsymbol{\rho_0}$ **[Ω·m] at 20°C**	$\boldsymbol{\alpha_1}$**[°C^{-1}] at 20°C**
Aluminum (Al)	2.82×10^{-8}	0.00429
Copper (Cu)	1.68×10^{-8}	0.0043
Gold (Au)	2.44×10^{-8}	0.004
Iron (Fe)	1.0×10^{-7}	0.00651

Lead (Pb)	2.2x10^{-7}	0.0042
Nickel (Ni)	6.99x10^{-8}	0.0067
Nichrome	1.10x10^{-6}	0.00017
Platinum (Pt)	1.06x10^{-7}	0.003927
Tungsten (W)	5.60x10^{-8}	0.0048

Substituting into the resistivity relationship:

$$R_s \approx \left(\frac{L}{\pi d}\right)\sqrt{\frac{\rho_0(1+\alpha_1(T-T_0))\omega\mu_0}{2}} = \left(\frac{L}{\pi d}\right)\sqrt{\frac{\rho_0\omega\mu_0}{2}}\sqrt{1+\alpha_1(T-T_0)}$$

For copper, aluminum, gold, or lead at a temperature of 260°C,

$1+\alpha_1(T-T_0) = 1+\ 0.0043\times 240 = 2.032$

So we ought to expect resistivity to double, which means the resistance should increase by about 43%, taking into account the skin effect.

We also should determine the sensitivity of the signal about small changes in the temperature. A further increase of 1°C will lead to

$$1+\alpha_1(T-T_0) = 1+\ 0.0043\times 241 = 2.0363$$

The resistance increase of the primary for one degree C about 260°C is therefore

$$R_{261} = R_{260}\sqrt{\frac{2.0363}{2.032}}$$

$$\frac{R_{261}}{R_{260}} = 1.00106$$

In other words, we should aim for our electronics to be sensitive enough to detect a 0.1% change in the resistance of the secondary. We can plug this into the voltage expression found earlier to determine what change in voltage that will result in across the primary, but that might be best done numerically (the equation is getting big enough as it is).

Thin-walled tube:

Magnetic field inside the tube:

$$B_{in} = \frac{B_0}{\sqrt{1+\left(\frac{dt}{2\delta^2}\right)^2}}$$