Notes on Similar Matrices

As you have learned: for a matrix $\mathbf{A}$[1] to be diagonalizable means the following.

1. There is an invertible matrix $\mathbf{P} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix}$ of eigenvectors of $\mathbf{A}$ (so that $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i, 1 \leq i \leq n$).
2. It follows that
$$\mathbf{AP} = \begin{bmatrix} | & | & \cdots & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \\ | & | & \cdots & | \end{bmatrix} = \mathbf{PD}, \tag{1}$$
where $\mathbf{D}$ is the diagonal matrix with the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ down the diagonal.
3. Solving equation (1) for $\mathbf{A}$ or $\mathbf{D}$ respectively gives the equivalent equations
$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} \quad \text{and} \quad \mathbf{A} = \mathbf{PDP}^{-1}. \tag{2}$$

The relationship between $\mathbf{A}$ and $\mathbf{D}$ given in equation (2) is called *similarity*, and it is important for more than just diagonalization. The general definition[2] is this:

$$\boxed{\text{Matrix } \mathbf{A} \text{ is } \textit{similar} \text{ to matrix } \mathbf{B} \text{ if there exists an invertible matrix } \mathbf{P} \text{ such that } \mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}.}$$

The connection between similarity and diagonalizability is simply this: a matrix is diagonalizable if and only if it is similar to a diagonal matrix.

This handout discusses two basic properties of similarity; you will learn more about similarity in chapter 5. The most basic fact about similarity is that it is an equivalence relation.

Theorem 1 *For any $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$:*

[a]: *$\boldsymbol{A}$ is similar to itself.*

[b]: *If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $\boldsymbol{B}$ is similar to $\boldsymbol{A}$.*

[c]: *If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ and $\boldsymbol{B}$ is similar to $\boldsymbol{C}$, then $\boldsymbol{A}$ is similar to $\boldsymbol{C}$.*

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[1] All matrices in this handout will be understood to be $n \times n$.
[2] This definition is given in the text on p.194.

Proof of [a]: Let $\mathbf{P} = \mathbf{I}_n$: $\mathbf{A} = \mathbf{I}_n^{-1}\mathbf{A}\mathbf{I}_n$.

Proof of [b]: Suppose that $\mathbf{A}$ is similar to $\mathbf{B}$, so that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}. \tag{3}$$

Multiplying equation (3) on the left by $\mathbf{P}$ and on the right by $\mathbf{P}^{-1}$ gives

$$\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \left(\mathbf{P}^{-1}\right)^{-1}\mathbf{A}\mathbf{P}^{-1},$$

so that $\mathbf{B}$ is similar to $\mathbf{A}$ $\left(\textit{via}\text{ the invertible matrix }\mathbf{P}^{-1}\right)$.

Proof of [c]: Suppose that $\mathbf{A}$ is similar to $\mathbf{B}$ and that $\mathbf{B}$ is similar to $\mathbf{C}$; this means that there are invertible matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \quad \text{and} \quad \mathbf{B} = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}.$$

We can then calculate

$$\begin{array}{rrcl} & \mathbf{A} & = & \mathbf{P}^{-1}\mathbf{B}\mathbf{P} \\ & & = & \mathbf{P}^{-1}\left(\mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}\right)\mathbf{P} \\ & & = & \left(\mathbf{P}^{-1}\mathbf{Q}^{-1}\right)\mathbf{C}\left(\mathbf{Q}\mathbf{P}\right) \\ \text{Theorem 2.12 (2)} & \longrightarrow & = & \left(\mathbf{Q}\mathbf{P}\right)^{-1}\mathbf{C}\left(\mathbf{Q}\mathbf{P}\right), \end{array}$$

so that $\mathbf{A}$ is similar to $\mathbf{C}$ (*via* the invertible matrix $\mathbf{QP}$). ∎

Warning: This equivalence relation is completely different from the "row equivalence" equivalence relation.

- One difference is that row equivalence is defined on the set of $m \times n$ matrices for any $m \geq 1$ and $n \geq 1$, where as similarity is defined only for $n \times n$ matrices.

- Furthermore, even for square matrices, the two equivalence relations are completely different. For example: as we have seen, the $n \times n$ matrices that are row-equivalent to $\mathbf{I}_n$ are the invertible matrices, whereas the only $n \times n$ matrix that is similar to $\mathbf{I}_n$ is the matrix $\mathbf{I}_n$ itself.

Exercise 1 *Prove: if the $n \times n$ matrix $\boldsymbol{A}$ is similar to $\mathbf{I}_n$, then $\mathbf{A} = \mathbf{I}_n$.*

The second fact about similarity to be discussed here is that similar matrices always have the same characteristic polynomial (and therefore have the same eigenvalues).

Theorem 2 *If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $p_{\mathbf{A}}(x) = p_{\mathbf{B}}(x)$.*

Proof. Since $\mathbf{A}$ is similar to $\mathbf{B}$, there is an invertible matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$. We can then calculate:

$$
\begin{array}{rcl}
p_{\mathbf{A}}(x) & = & \det[x\mathbf{I}_n - \mathbf{A}] \\
 & = & \det\left[x\mathbf{I}_n - \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\right] \\
\mathbf{P}^{-1}(x\mathbf{I}_n)\,\mathbf{P} = \left(\mathbf{P}^{-1}\mathbf{P}\right)x\mathbf{I}_n = x\mathbf{I}_n \longrightarrow & = & \det\left[\mathbf{P}^{-1}(x\mathbf{I}_n)\,\mathbf{P} - \mathbf{P}^{-1}\mathbf{B}\mathbf{P}\right] \\
\text{factor out } \mathbf{P}^{-1} \longrightarrow & = & \det\left[\mathbf{P}^{-1}(x\mathbf{I}_n\mathbf{P} - \mathbf{B}\mathbf{P})\right] \\
\text{factor out } \mathbf{P} \longrightarrow & = & \det\left[\mathbf{P}^{-1}(x\mathbf{I}_n - \mathbf{B})\,\mathbf{P}\right] \\
\text{Theorem 3.7} \longrightarrow & = & \left(\det\left[\mathbf{P}^{-1}\right]\right)(\det[x\mathbf{I}_n - \mathbf{B}])\,(\det[\mathbf{P}]) \\
\text{Corollary 3.8} \longrightarrow & = & \left(\dfrac{1}{\det[\mathbf{P}]}\right)(\det[x\mathbf{I}_n - \mathbf{B}])\,(\det[\mathbf{P}]) \\
\text{cancel} \longrightarrow & = & \det[x\mathbf{I}_n - \mathbf{B}] \\
 & = & p_{\mathbf{B}}(x). \blacksquare
\end{array}
$$