

Statistics 100A

Homework 6 Solutions

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Chapter 5

32. The time (in hours) required to repair a machine is an exponential distributed random variable with paramter $\lambda = \frac{1}{2}$. What is

Let X denote the time (in hours) required to repair a machine. $X \sim \text{Exp}(\frac{1}{2})$. The PDF for the exponential distribution is

$$f(x) = \lambda e^{-\lambda x}$$

- (a) the probability that a repair time exceeds 2 hours;

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - \int_0^2 \frac{1}{2} e^{-\frac{1}{2}x} dx \\ &= 1 + \left[e^{-\frac{1}{2}x} \right]_0^2 \\ &= \boxed{e^{-1}} \end{aligned}$$

- (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

We want to find $P(X \geq 10 | X > 9)$.

$$\begin{aligned} P(X \geq 10 | X > 9) &= \frac{P(X \geq 10)}{P(X > 9)} \\ &= \frac{\int_{10}^{\infty} \frac{1}{2} e^{-\frac{1}{2}x} dx}{\int_9^{\infty} \frac{1}{2} e^{-\frac{1}{2}x} dx} \\ &= \frac{2e^{-\frac{1}{2}} \Big|_{10}^{\infty}}{2e^{-\frac{1}{2}} \Big|_9^{\infty}} \\ &= \frac{-\lim_{x \rightarrow \infty} \left(e^{-\frac{1}{2}x} \right) + \frac{1}{2}e^{-5}}{-\lim_{x \rightarrow \infty} \left(e^{-\frac{1}{2}x} \right) + \frac{1}{2}e^{-\frac{9}{2}}} \\ &= \frac{\frac{1}{2}e^{-5}}{\frac{1}{2}e^{-\frac{9}{2}}} \\ &= \boxed{e^{-\frac{1}{2}}} \end{aligned}$$

33. The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?

The phrase *additional* may seem confusing because we do not know how long the radio has already been “alive.” Let t be the number of years that the radio has been alive. We want to find $P(X > 8 | X > t)$.

But t does not matter! This is the **memoryless** property of the exponential distribution. That is, for some t and some shift s ,

$$P(X > s + t | X > t) = P(X > s)$$

In the case of our problem, t is unknown and $s = 8$.

$$\begin{aligned} P(X > 8) &= \int_8^{\infty} \frac{1}{8} e^{-\frac{1}{8}x} dx \\ &= \frac{1}{8} \int_8^{\infty} e^{-\frac{1}{8}x} dx \\ &= -\frac{1}{8} \left[\lim_{x \rightarrow \infty} \left(-8e^{-\frac{1}{8}x} \right) - 8e^{-1} \right] \\ &= -\frac{1}{8} [-8e^{-1}] \\ &= \boxed{e^{-1}} \end{aligned}$$

34. Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter $\lambda = \frac{1}{20}$. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed but rather is (in thousands of miles) uniformly distributed over $(0, 40)$.

Let X be the number of miles the car can be driven before being junked.

- (a) First, let $X \sim \text{Exp}\left(\frac{1}{20}\right)$.

We know that the car has been driven 10,000 miles and we want to find the probability that it will last an *additional* 20,000 miles. We use the memoryless property of exponential distribution, where $t = 10000$ and $s = 20000$.

Then, $P(X \geq 10000 + 20000 | X > 10000) = P(X \geq 20000)$.

$$\begin{aligned} P(X \geq 30000 | X > 10000) &= P(X \geq 20000) \\ &= \int_{20}^{\infty} e^{-\frac{1}{20}x} dx \\ &= \boxed{e^{-1}} \end{aligned}$$

(b) Let $X \sim U(0, 40)$. We want to find $P(X > 30 | X > 10)$. By Bayes' Rule,

$$\begin{aligned}
 P(X > 30 | X > 10) &= \frac{P(X > 30)}{P(X > 10)} \\
 &= \frac{\int_{30}^{40} \frac{1}{40} dx}{\int_{10}^{40} \frac{1}{40} dx} \\
 &= \frac{x]_{30}^{40}}{x]_{10}^{40}} \\
 &= \frac{40 - 30}{40 - 10} \\
 &= \boxed{\frac{1}{3}}
 \end{aligned}$$

40. If X is uniformly distributed over $(0, 1)$, find the density function of $Y = e^X$.

First, we start with the cumulative density function $F(y)$, since we want to find the distribution function for Y .

$$\begin{aligned}
 F_Y(y) &= P(Y \leq y) \\
 &= P(e^X \leq y) \\
 &\quad \text{since } Y = e^X. \text{ Next, isolate } X \\
 &= P(X \leq \ln y) \\
 &= F_X(\ln y)
 \end{aligned}$$

which is the cumulative density function with respect to X , but we are not done yet. We want the PDF, so take the derivative.

$$f_X(x) = F'_X(\ln y) = f_X(\ln y) \cdot \left| \frac{d}{dy} \ln y \right| = f_X(\ln y) \left| \frac{1}{y} \right|$$

But $f_X(\ln y) = 1$ because $X \sim U(0, 1)$. Also, note that the domain of y is $1 < y < e$ (which is positive) because the domain of x is $0 < x < 1$ and $Y = e^X$. So,

$$f_Y(y) = \begin{cases} \frac{1}{y}, & 1 < y < e \\ 0, & \text{elsewhere} \end{cases}$$

Chapter 5 Theoretical Exercises

- 12c. The median of a continuous random variable having distribution function F is that value m such that $F(m) = \frac{1}{2}$. The is, a random variable is just as likely to be larger than its median as it is to be smaller. Find the median of X if X is exponential with rate λ .

Recall that $f(x) = \lambda e^{-\lambda x}$. To find percentiles, we must find the cumulative density function $F(x)$ and set it equal to $\frac{1}{2}$.

$$\begin{aligned}\frac{1}{2} &= \int_0^x \frac{1}{2} e^{-\frac{1}{2}x} dx \\ &= -\left[e^{-\frac{1}{2}x}\right]_0^x \\ &= -\left[e^{-\frac{1}{2}x} - 1\right] \\ &= 1 - e^{-\frac{1}{2}x} \\ \frac{1}{2} &= e^{-\frac{1}{2}x} \\ -\ln 2 &= -\frac{1}{2}x \\ 2\ln 2 &= x\end{aligned}$$

Thus, $\boxed{x = 2\ln 2}$ is the median.

14. If X is an exponential random variable with parameter λ , and $c > 0$, show that cX is exponential with parameter $\frac{\lambda}{c}$.

As with problem 40 earlier, we start with the CDF and introduce our new random variable cX .

$$\begin{aligned}P(X \leq x) &= 1 - e^{-\lambda x} \\ P(cX \leq x) &= P\left(X \leq \frac{x}{c}\right) \\ &= 1 - e^{-\lambda(\frac{x}{c})}\end{aligned}$$

Then take the derivative to get the PDF of our new random variable.

$$\begin{aligned}f\left(\frac{x}{c}\right) &= \left[1 - e^{-\lambda(\frac{x}{c})}\right]' \\ &= \frac{c}{\lambda} e^{-\lambda(\frac{x}{c})} \\ &= \boxed{\text{Exp}\left(\frac{c}{\lambda}\right)}\end{aligned}$$

20. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

This is one of those problems that can be solved using what I call the “meet in the middle” method. We start on the left hand side $\Gamma\left(\frac{1}{2}\right)$, do some massaging, and then move to the right hand side (or in this case, use an intermediary step), do some work and then note that both computations yield the same result.

Proof. By the hint given in the text,

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx \\ &\quad \text{Let } y = \sqrt{x}. \text{ Then } dy = \frac{1}{\sqrt{2x}}. \\ &= \sqrt{2} \int_0^\infty e^{-\frac{y^2}{2}} dy\end{aligned}$$

That was the left hand side. Now we consider the fact that $\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1$.

$$\begin{aligned}1 &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ \sqrt{2\pi} &= 2 \int_{-\infty}^\infty e^{-\frac{y^2}{2}} dy\end{aligned}$$

by symmetry.

$$\sqrt{\pi} = \sqrt{2} \int_0^\infty e^{-\frac{y^2}{2}} dy$$

by dividing both sides by $\sqrt{2}$ and rationalizing.

Note that the two bolded expressions above are equal, thus we have “met in the middle.” We have proven that $\Gamma\left(\frac{1}{2}\right) = \sqrt{2} \int_0^\infty e^{-\frac{y^2}{2}} dy = \sqrt{\pi}$.

□

Chapter 6

2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the i th ball selected is white and let it equal 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2 ;

We must enumerate every combination of 0/1 values for X_1 and X_2 and determine the combination of draws that result in those values.

First consider $p(0, 0) : X_1 = 0, X_2 = 0$. This means that the first ball drawn is not white and the second ball drawn is not white. This means both balls drawn are red. Initially, there are 8 red balls in the urn and 13 balls today. On the second draw, there are 7 red balls and 12 balls total. So, $p(0, 0) = \frac{8 \cdot 7}{13 \cdot 12} = \boxed{\frac{14}{39}}$.

Note that the denominator will always be the same since on each draw we draw one ball, so I will exclude this from my derivation.

$p(0, 1) : X_1 = 0, X_2 = 1$ means that the first ball is not white and the second ball is white. Initially there are 8 red balls. On the second draw, we draw a white ball of which there are 5 since none of them had been drawn yet. So $p(0, 1) = \frac{8 \cdot 5}{13 \cdot 12} = \boxed{\frac{10}{39}}$.

$p(1, 1) : X_1 = 1, X_2 = 1$ means that the first ball is white and the second ball is white. Initially there are 5 white balls in the urn. On the second draw, after one of the white balls is chosen, there are 4 remaining. So $p(1, 1) = \frac{5 \cdot 4}{13 \cdot 12} = \boxed{\frac{5}{39}}$.

$P(1, 0) : X_1 = 1, X_2 = 0$ means that the first ball is white and the second ball is red. Initially there are 5 white balls. On the second draw there are 8 red balls. So $p(1, 0) = \frac{5 \cdot 8}{13 \cdot 12} = \boxed{\frac{10}{39}}$.

(b) X_1, X_2, X_3

Now we are working with the joint distribution of 3 discrete random variables, but the premise is the same as with part (a). We consider all possible combinations of 0/1 values for X_i and compute the corresponding probabilities.

$p(0, 0, 0)$	$=$	$\frac{8 \cdot 7 \cdot 6}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{28}{143}}$	$p(1, 0, 1)$	$=$	$\frac{5 \cdot 8 \cdot 4}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{40}{429}}$
$p(0, 0, 1)$	$=$	$\frac{8 \cdot 7 \cdot 5}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{70}{429}}$	$p(1, 1, 0)$	$=$	$\frac{5 \cdot 4 \cdot 8}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{40}{429}}$
$p(0, 1, 0)$	$=$	$\frac{8 \cdot 5 \cdot 7}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{70}{429}}$	$p(1, 1, 1)$	$=$	$\frac{5 \cdot 4 \cdot 3}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{5}{143}}$
$p(1, 0, 0)$	$=$	$\frac{5 \cdot 8 \cdot 7}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{70}{429}}$	$p(0, 1, 1)$	$=$	$\frac{8 \cdot 5 \cdot 4}{13 \cdot 12 \cdot 11}$	$=$	$\boxed{\frac{40}{429}}$

8. The joint probability density function of X and Y is given by

$$f(x, y) = c(y^2 - x^2)e^{-y} \quad -y \leq x \leq y, 0 \leq y \leq \infty$$

(a) Find c .

To find c , just integrate the double integral using the appropriate limits of integration, set the result equal to 0 and solve for c . Suppose we integrate with respect to x first. Note that $-y \leq x \leq y$ so those should be our limits for integration. Additionally, note that $0 < y < \infty$, so 0 and ∞ are our limits of integration when we integrate with respect to y .

$$\begin{aligned} \int_0^\infty \int_{-y}^y c(y^2 - x^2)e^{-y} dx dy &= c \int_0^\infty \int_{-y}^y y^2 e^{-y} - x^2 e^{-y} dx dy \\ &= c \int_0^\infty xy^2 e^{-y} + \frac{x^3}{3} e^{-y} \Big|_{-y}^y dy \\ &= c \int_0^\infty y^3 e^{-y} - \frac{y^3}{3} e^{-y} + y^3 e^{-y} - \frac{y^3}{3} e^{-y} dy \\ &= 2c \int_0^\infty y^3 e^{-y} - \frac{y^3}{3} e^{-y} dy \\ &= \frac{4}{3} c \int_0^\infty y^3 e^{-y} dy \\ &\quad \text{Let } u = y^3, dv = e^{-y} \text{ so } du = 3y^2, v = -e^{-y} \\ &= \frac{4}{3} c \left[-y^3 e^{-y} + 3 \int y^2 e^{-y} dy \right]_0^\infty \\ &\quad \text{Let } u = y^2, dv = e^{-y} \text{ so } du = 2y, v = -e^{-y} \\ &= \frac{4}{3} c \left[-y^3 e^{-y} + 3 \left(-y^2 e^{-y} + 2 \int y e^{-y} dy \right) \right]_0^\infty \\ &\quad \text{Let } u = y, dv = -e^{-y} \text{ so } du = dy, v = -e^{-y}. \\ &= \frac{4}{3} c \left[-y^3 e^{-y} + 3 \left(-y^2 e^{-y} + 2 \{ -y e^{-y} - e^{-y} \} \right) \right]_0^\infty \\ &= \frac{4}{3} c \left[-y^3 e^{-y} - 3y^2 e^{-y} - 6y e^{-y} - 6e^{-y} \right]_0^\infty \\ &= \frac{4}{3} c \left[-e^{-y} \{ y^3 + 3y^2 + 6y + 6 \} \right]_0^\infty \\ &= -4c \left[-e^{-y} \left\{ \frac{1}{3} y^3 + y^2 + 2y + 2 \right\} \right]_0^\infty \\ &= 0 - 4c \left[-e^{-y} \left\{ \frac{1}{3} y^3 + y^2 + 2y + 2 \right\} \right]_{x=0} \\ 8c &= 1 \\ c &= \frac{1}{8} \end{aligned}$$

Thus, $\boxed{c = \frac{1}{8}}$.

(b) Find the marginal densities of X and Y .

To find the marginal distribution with respect to some variable u_i integrate $f(u_1, u_2, \dots, u_d)$ with respect to the variables that we are not interested in $u_{i \neq j}$. Then, choose the limits of integration such that the resulting distribution is in terms of the variable of interest u_i .

To find the marginal with respect to x , we integrate out y . The domain $-y \leq x \leq y$ can be rewritten as $x \leq |y| \Rightarrow |x| \leq |y| \Rightarrow |y| \geq |x| \Rightarrow y \geq |x|$ since y is non-negative. So we integrate from $|x|$ to ∞ .

$$\begin{aligned}
f_X(x) &= \int_{|x|}^{\infty} \frac{1}{8} (y^2 - x^2) e^{-y} dy \\
&= \frac{1}{8} y^2 e^{-y} - \frac{1}{8} x^2 e^{-y} dy \\
&= \frac{1}{8} \int_{|x|}^{\infty} y^2 e^{-y} dy - \frac{1}{8} x^2 \int_{|x|}^{\infty} e^{-y} dy \\
&\quad \text{Let } u = y^2, dv = e^{-y}, \text{ so } du = 2y dy, v = -e^{-y}. \\
&= \frac{1}{8} \left[-y^2 e^{-y} - \int -2y e^{-y} dy \right]_{|x|}^{\infty} - \frac{1}{8} x^2 [-e^{-y}]_{|x|}^{\infty} \\
&= \frac{1}{8} \left[-y^2 e^{-y} + 2 \int y e^{-y} dy \right]_{|x|}^{\infty} + \frac{1}{8} x^2 [e^{-y}]_{|x|}^{\infty} \\
&= \frac{1}{8} \left[-y^2 e^{-y} + 2 \left\{ -y e^{-y} - \int -e^{-y} dy \right\} \right]_{|x|}^{\infty} - \frac{1}{8} x^2 e^{-|x|} \\
&= \frac{1}{8} [-y^2 e^{-y} - 2y e^{-y} - 2e^{-y}]_{|x|}^{\infty} - \frac{1}{8} x^2 e^{-|x|} \\
&= -\frac{1}{8} [-|x|^2 e^{-|x|} - 2|x| e^{-|x|} - 2e^{-|x|}] - \frac{1}{8} x^2 e^{-|x|} \\
&\quad \text{Note that } x^2 = |x|^2. \\
&= \frac{1}{4} |x| e^{-|x|} + \frac{1}{4} e^{-|x|} \\
&= \boxed{\frac{1}{4} e^{-|x|} (|x| + 1)}
\end{aligned}$$

To find the marginal density of Y , integrate out X and choose limits of integration that result in a function in terms of y . We have the domain is $-y \leq x \leq y$ so integrate from $-y$ to y .

$$\begin{aligned}
f_Y(y) &= \frac{1}{8} \int_{-y}^y (y^2 - x^2) e^{-y} dx \\
&= \frac{1}{8} \int_{-y}^y y^2 e^{-y} - x^2 e^{-y} dx \\
&= \frac{1}{8} \left[xy^2 e^{-y} - \frac{x^3}{3} e^{-y} \right]_{-y}^y \\
&= \frac{1}{8} \left[y^3 e^{-y} - \frac{y^3}{3} e^{-y} - \left(-y^3 e^{-y} + \frac{y^3}{3} e^{-y} \right) \right] \\
&= \frac{1}{8} \left[2y^3 e^{-y} - \frac{2}{3} y^3 e^{-y} \right] \\
&= \boxed{\frac{1}{6} e^{-y} y^3}
\end{aligned}$$

(c) Find $E(X)$.

To find the expected value of X we use the *marginal distribution of X* and integrate over the domain $(0, \infty)$.

$$E(X) = \int_0^{\infty} x \left[\frac{1}{4} e^{-|x|} |x| + \frac{1}{4} e^{-|x|} \right] dx$$

The absolute value function poses a small challenge. By definition,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

However, note that $0 \leq x < \infty$ so the marginal of X is non-negative, so we do *not* have to split up the integral.

$$\begin{aligned} E(X) &= \int_0^{\infty} \frac{1}{4} x e^{-|x|} (|x| + 1) \\ &= \int_0^{\infty} \frac{1}{4} x e^{-x} (x + 1) dx \\ &= \int_0^{\infty} \frac{1}{4} x^2 e^{-x} + \frac{1}{4} x e^{-x} dx \\ &= \frac{1}{4} \int_0^{\infty} x^2 e^{-x} dx + \frac{1}{4} \int_0^{\infty} x e^{-x} dx \end{aligned}$$

For the first integral, let $u = x^2$, $dv = e^{-x}$ so $du = 2x dx$, $v = -e^{-x}$

For the second integral, let $u = x$, $dv = e^{-x}$ so $du = dx$, $v = -e^{-x}$.

$$\begin{aligned} &= \dots \\ &= \frac{1}{4} [-x^2 e^{-x} - 3e^{-x}(x + 1)]_0^{\infty} \\ &= -\frac{1}{4} \lim_{x \rightarrow \infty} \frac{x^2 + 3(x + 1)}{e^x} \end{aligned}$$

which is indeterminate so apply l'Hôpital's Rule once, and then once more.

$$\frac{\underline{\underline{L'H^2}}}{0}$$

So $\boxed{E(X) = 0}$.

9. The joint probability density function of X and Y is given by

$$f(x, y) = \frac{6}{7} \left(x^2 + \frac{xy}{2} \right) \quad 0 < x < 1, 0 < y < 2$$

(a) Verify that this is indeed a joint density function.

Proof. To show $f(x, y)$ is a joint PDF, we show that the double integral is equal to 0.

$$\begin{aligned} \int_0^2 \int_0^1 \frac{6}{7} \left(x^2 + \frac{xy}{2} \right) dx dy &= \int_0^2 \frac{6}{7} \left(\frac{x^3}{3} + \frac{x^2 y}{4} \right) \Big|_0^1 dy \\ &= \int_0^2 \frac{6}{7} \left(\frac{1}{3} + \frac{y}{4} \right) dy \\ &= \frac{6}{7} \left(\frac{1}{3} y + \frac{y^2}{8} \right) \Big|_0^2 \\ &= \boxed{1} \end{aligned}$$

□

(b) Compute the density function of X .

The density function of X is the same thing as the marginal density function for X . We integrate out y . We can simply use the limits of the domain of x as the limits of integration because we will still get a function in terms of x .

$$\begin{aligned} f_X(x) &= \int_0^2 \frac{6}{7} \left(x^2 + \frac{xy}{2} \right) dy \\ &= \frac{6}{7} \left(x^2 y + \frac{xy^2}{4} \right) \Big|_0^2 \\ &= \boxed{\frac{6}{7} (2x^2 + x)} \end{aligned}$$

(c) Find $P(X > Y)$.

To find the probability that $X > Y$ we integrate the joint distribution. We have to be careful about the limits of integration. First, we know that $0 < x < 1$ and that $0 < y < 2$. Since $X > Y$, the value of y can only vary from its lower bound of 0 up to its upper bound which is x . X has no restriction so we just integrate over the domain for X : from 0 to 1.

$$\begin{aligned} P(X > Y) &= \int_0^1 \int_0^x \frac{6}{7} \left(x^2 + \frac{xy}{2} \right) dy dx \\ &= \int_0^1 \frac{6}{7} \left(x^2 y + \frac{xy^2}{4} \right) \Big|_0^x dx \\ &= \int_0^1 \frac{6}{7} \left(x^3 + \frac{x^3}{4} \right) dx \\ &= \frac{6}{7} \left(\frac{5}{16} x^4 \right) \Big|_0^1 \\ &= \boxed{\frac{15}{56}} \end{aligned}$$

(d) Find $P\left(Y > \frac{1}{2} \mid X < \frac{1}{2}\right)$.

Using Bayes' Rule how we would usually use it would produce the following expression.

$$P\left(Y > \frac{1}{2} \mid X < \frac{1}{2}\right) = \frac{P\left(Y > \frac{1}{2} \cap X < \frac{1}{2}\right)}{P\left(X < \frac{1}{2}\right)}$$

In terms of the joint distribution, the intersection in the numerator is just the integral of the joint over the restricted domain. The denominator is also the integral of the joint, where the limit for integration over x is restricted by $X < \frac{1}{2}$ and the limit for integrating over y is free to vary over its entire domain. That is,

$$\begin{aligned} P\left(Y > \frac{1}{2} \mid X < \frac{1}{2}\right) &= \frac{P\left(Y > \frac{1}{2} \cap X < \frac{1}{2}\right)}{P\left(X < \frac{1}{2}\right)} \\ &= \frac{\int_{\frac{1}{2}}^2 \int_0^{\frac{1}{2}} \frac{6}{7} \left(x^2 + \frac{xy}{2}\right) dx dy}{\int_0^{\frac{1}{2}} \int_0^2 \frac{6}{7} \left(x^2 + \frac{xy}{2}\right) dy dx} \\ &= \frac{\int_{\frac{1}{2}}^2 \left(\frac{x^3}{3} + \frac{x^2 y}{4}\right) \Big|_0^{\frac{1}{2}}}{\int_0^{\frac{1}{2}} \left(x^2 y + \frac{xy^2}{4}\right) \Big|_0^2 dx} \\ &= \frac{\int_{\frac{1}{2}}^2 \frac{1}{24} + \frac{y}{16} dy}{\int_0^{\frac{1}{2}} 2x^2 + x dx} \\ &= \frac{\frac{1}{24}y + \frac{1}{32}y^2 \Big|_{\frac{1}{2}}^2}{\frac{2}{3}x^3 + \frac{x^2}{2} \Big|_0^{\frac{1}{2}}} \\ &= \boxed{\frac{69}{80}} \end{aligned}$$

(e) Find $E(X)$.

Recall that

$$E(X) = \int_a^b x f_X(x) dx$$

$$\begin{aligned} E(X) &= \int_0^1 x \frac{6}{7} (2x^2 + x) dx \\ &= \frac{6}{7} \int_0^1 2x^3 + x^2 dx \\ &= \frac{6}{7} \left[\frac{x^4}{2} + \frac{x^3}{3} \right]_0^1 \\ &= \boxed{\frac{5}{7}} \end{aligned}$$

(f) Find $E(Y)$.

Recall that

$$E(Y) = \int_a^b x f_Y(y) dy$$

But we do not know $f_Y(y)$ so we must find it. Yay! We integrate out x from the joint density and integrate from 0 to 1, the domain of x which will give us a function in terms of y .

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{6}{7} \left(x^2 + \frac{xy}{2} \right) dx \\ &= \left[\frac{6}{7} \left(\frac{x^3}{3} + \frac{x^2 y}{4} \right) \right]_0^1 \\ &= \frac{6}{7} \left[\frac{1}{3} + \frac{y}{4} \right]_0^1 = \frac{6}{7} \left[\frac{4 + 3y}{12} \right] \end{aligned}$$

Now we can find $E(Y)$,

$$\begin{aligned} E(Y) &= \int_0^2 \frac{6}{7} \left[\frac{4y + 3y^2}{12} \right] dy \\ &= \frac{1}{14} \int_0^2 4y + 3y^2 dy \\ &= \frac{1}{14} [2y^2 + y^3]_0^2 \\ &= \boxed{\frac{8}{7}} \end{aligned}$$

10. The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)} \quad 0 \leq x \leq \infty, 0 \leq y < \infty$$

Find

(a) $P(X < Y)$

This is the opposite of problem 9c. Since $X < Y$, we integrate x from 0 to y , and y is free to vary on its domain. Then,

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_0^y e^{-(x+y)} dx dy \\ &= \int_0^\infty -e^{-(x+y)} \Big|_0^y dy \\ &= \int_0^\infty -e^{-2y} + e^{-y} dy \\ &= \int_0^\infty e^{-y} - e^{-2y} dy \\ &= \left[-e^{-y} + \frac{1}{2}e^{-2y} \right]_0^\infty \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

(b) $P(X < a)$

To find $P(X < a)$ integrate the joint density with respect to y over its domain and with respect to x from its domain's lower bound, 0, to a , some constant.

$$\begin{aligned} P(X < a) &= \int_0^a \int_0^\infty e^{-(x+y)} dy dx \\ &= \int_0^a \left[-e^{-(x+y)} \right]_0^\infty dx \\ &= \int_0^a e^{-x} dx \\ &= -e^{-x} \Big|_0^a \\ &= \boxed{1 - e^{-a}} \end{aligned}$$