ES.1803 Topic 28 Notes

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28 Linearization of nonlinear systems

28.1 Nonlinear Systems

A general first order autonomous (2×2) system has the following form

$$x' = f(x, y) \quad (1)$$
$$y' = g(x, y) \quad (2)$$

Vector Field: This defines a vector field $(f(x, y), g(x, y))$ that attaches the velocity vector to each point (x, y) in the *phase plane.*

By definition a critical point is one where $x' = 0$ and $y' = 0$. That is, it is a point (x_0, y_0) where

$$f(x_0, y_0) = 0, \text{ and } g(x_0, y_0) = 0.$$

Equivalently, it is an *equilibrium solution* $x(t) = x_0$, $y(t) = y_0$. This is a solution whose trajectory is a single point.

28.2 Linearization around a critical point

We'll start by presenting the method of linearization to sketch the phase portrait. First we'll use it in an example. After that we'll justify the method.

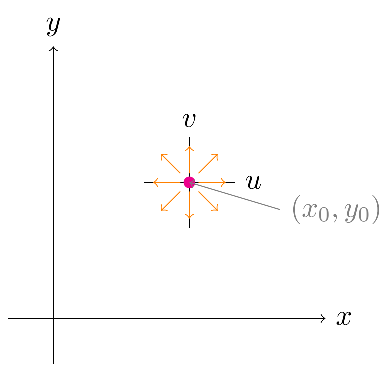
Jacobian. At a critical point (x_0, y_0) of the system 1 we define the Jacobian by

$$J(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}.$$

This gives the linearization around the critical point (x_0, y_0)

$$\begin{bmatrix} u \\ v \end{bmatrix}' = J(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}$$

In general, the nonlinear system behaves like the linearized one. (We will learn the exceptions later.) That is, if we center our uv-axes on (x_0, y_0) then the linear vector field near the uv origin approximates the nonlinear field near (x_0, y_0)



Near a critical point the nonlinear system, is approximately linear.

Example 28.1. Find the critical points for the following system.

$$x' = 14x - \frac{1}{2}x^2 - xy$$
$$y' = 16y - \frac{1}{2}y^2 - xy$$

answer: We solve the equations $x' = 0$, $y' = 0$.

$$x' = x\left(14 - \frac{1}{2}x - y\right) = 0 \Rightarrow x = 0 \text{ or } 14 - \frac{1}{2}x - y = 0$$
$$y' = y\left(16 - \frac{1}{2}y - x\right) = 0 \Rightarrow y = 0 \text{ or } 16 - \frac{1}{2}y - x = 0.$$

Looking at the product for x' we see $x' = 0$ when $x = 0$ or $14 - x/2 - y = 0$. Likewise, $y' = 0$ when $y = 0$ or $16 - y/2 - x = 0$. This leads to four sets of equations for critical points.

$$\begin{Bmatrix} x = 0 \\ y = 0 \end{Bmatrix} \qquad \begin{Bmatrix} x = 0 \\ 16 - y/2 - x = 0 \end{Bmatrix} \qquad \begin{Bmatrix} 14 - x/2 - y = 0 \\ y = 0 \end{Bmatrix} \qquad \begin{Bmatrix} 14 - x/2 - y = 0 \\ 16 - y/2 - x = 0 \end{Bmatrix}$$

The first three sets are easy to solve by inspection. The fourth requires a small computation. We get the following four critical points:

$$(0, 0),\ (0, 32),\ (28, 0),\ (12, 8).$$

Example 28.2. (Continued from previous example.) Linearize the system at each of the critical points and determined the type of the linearized critical point.

answer: The linearized system at (x_0, y_0) is $\begin{bmatrix} u' \\ v' \end{bmatrix} = J(x_0, y_0)\begin{bmatrix} u \\ v \end{bmatrix}$.

First we compute the Jacobian:

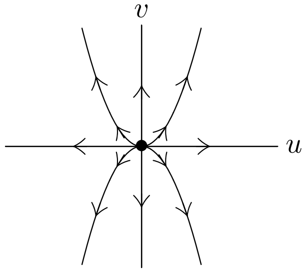
$$J(x, y) = \begin{bmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{bmatrix}$$

Next we look at each of the critical points in turn.

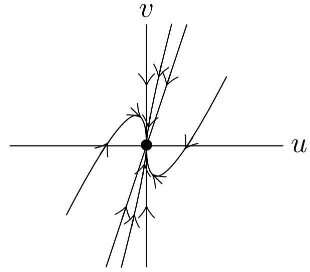
Critical point $(0, 0)$:

$$J(0, 0) = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix}; \text{ eigenvalues } 14,\ 16; \text{ eigenvectors } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

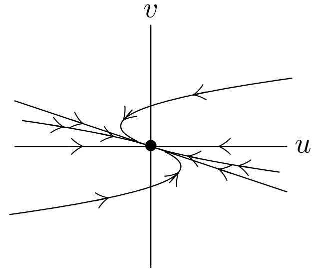
This is a source node, Its sketch on uv-axes is shown in the left-most figure below.



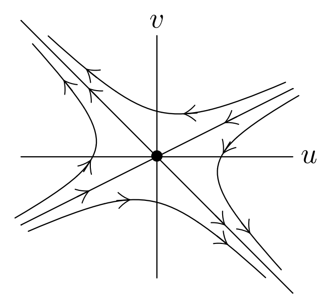
Source node



Sink node 1



Sink node 2



Saddle

Critical point $(0, 32)$:

$$J(0,32) = \begin{bmatrix} -18 & 0 \\ -32 & -16 \end{bmatrix}; \text{ eigenvalues } -18, -16; \text{ corresponding eigenvectors } \begin{bmatrix} 1 \\ 16 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This is a sink node. Its sketch is shown in the 'Sink node 1' figure above.

Critical point $(28, 0)$:

$$J(28,0) = \begin{bmatrix} -14 & -28 \\ 0 & -12 \end{bmatrix}, \text{ eigenvalues } -14, -12; \text{ corresponding eigenvectors } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -14 \\ 1 \end{bmatrix}$$

This is a sink node. Its sketch is shown in the 'Sink node 2' figure above.

Critical point $(12, 8)$:

$$J(12,8) = \begin{bmatrix} -6 & -12 \\ -8 & -4 \end{bmatrix}; \text{ eigenvalues } -5 \pm \sqrt{97} \approx -15, 5.$$

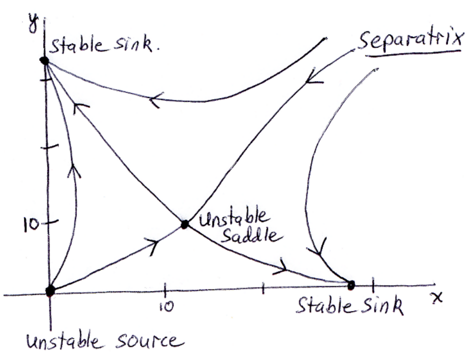
$$\text{Eigenvectors:For } \lambda = -5 - \sqrt{97} : \begin{bmatrix} 1 + \sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$\text{For } \lambda = -5 + \sqrt{97} : \begin{bmatrix} 1 - \sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

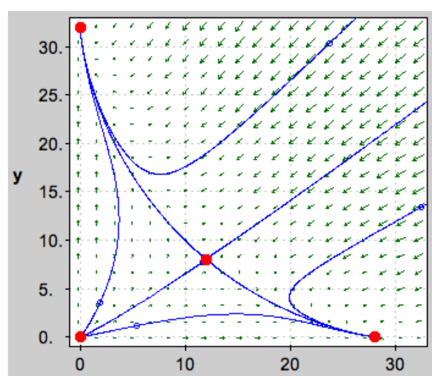
This is a saddle. Its sketch is shown in the 'Saddle' figure above.

To make a rough sketch of the nonlinear system's phase portrait we: 1. Sketch the phase portrait near each critical point, using the linearization.
2. Connect these sketches together in a consistent manner.

We do this below and compare it with the sketch made by a Matlab program called PPlane.



Hand sketch of the phase plane.



PPlane plot of the phase plane.

28.2.1 Justification for using linearization

We'll go through this in detail. One key fact is that the change of variables $u = x - x_0$, $v = y - y_0$ puts the uv origin at (x_0, y_0).

We will use the tangent plane, i.e. linear approximations of f and g. You might recall this from 18.02. If not, notice that is just a multivariable version of the single variable approximation

$$f(x) \approx f(x_0) + f'(x_0)\Delta x,$$

where $\Delta x = x - x_0$.

For small changes $(x - x_0) = \Delta x$ and $(y - y_0) = \Delta y$ the tangent plane approximations for f and g near (x_0, y_0) are

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0) + f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y \\ g(x,y) &\approx g(x_0,y_0) + g_x(x_0,y_0)\,\Delta x + g_y(x_0,y_0)\,\Delta y \end{aligned}$$

Now let $u = x - x_0 = \Delta x$ and $v = y - y_0 = \Delta y$). Note two things.
1. This puts the origin of the uv-plane at (x_0, y_0).
2. As functions of t: $u' = x'$, $v' = y'$ (since x_0 and y_0 are constants).

Using u and v

$$\begin{aligned} f(x_0+u, y_0+v) &\approx f(x_0,y_0) + f_x(x_0,y_0)\,u + f_y(x_0,y_0)\,v \\ g(x_0+u, y_0+v) &\approx g(x_0,y_0) + g_x(x_0,y_0)\,u + g_y(x_0,y_0)\,v \end{aligned}$$

Writing these in matrix form we see the Jacobian appear:

$$\begin{aligned} \begin{bmatrix} f(x_0+u,y_0+v) \\ g(x_0+u,y_0+v) \end{bmatrix} &\approx \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix} + \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix} + J(x_0,y_0) \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

If (x_0, y_0) is a critical point the first term on the right is 0, i.e

$$\begin{bmatrix} f(x_0+u,y_0+v) \\ g(x_0+u,y_0+v) \end{bmatrix} \approx J(x_0,y_0) \begin{bmatrix} u \\ v \end{bmatrix}.$$

Now, $u = x - x_0$ can be rewritten $x = x_0 + u$. Remembering that $u' = x'$, $v' = y'$ we put everything together as

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x_0+u,y_0+v) \\ g(x_0+u,y_0+v) \end{bmatrix} \approx J(x_0,y_0) \begin{bmatrix} u \\ v \end{bmatrix}$$

Using just the first and last terms from the above gives the linearization formula

$$\begin{bmatrix} u' \\ v' \end{bmatrix} \approx J(x_0,y_0) \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is a linear system with coefficient matrix $J(x_0, y_0)$. We call it the linearization of the system around the critical point.