Math 23a
Phase Portraits
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1 Introduction

There is already a fantastic supplement made by Jake Carr on exponentiating 2×2 matrices, so in this supplement, I'd like to explore what the solutions to these equations mean in the form of phase portraits. Jake already explores this idea, but I hope this supplement is able to provide more detail. Consider a system of linear differential equations, **evolving in time**, that can be written in the following form:

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

The solution to such an equation is simply:

$$\vec{x} = \vec{x_0}e^{At}$$

as the exponential solution in the single variable case does nicely generalize to the multivariable case. Let's consider the case of when the two components of $\vec{x}$ are the two Cartesian coordinates x, y, so that these solutions have geometric interpretation. Notice that $\frac{d\vec{x}}{dt}$ can be thought of mathematically as a **vector field**. We can formally define a vector field as follows:

Definition A **vector field** $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, assigns to every point in space, $\mathbf{u} \in \mathbb{R}^n$, an associated vector, $\vec{v} = \vec{F}(\mathbf{u})$.

Basically, $A\vec{x}$ can be thought of as a vector field. It is a function that assigns to every point in space $\mathbf{x}$, and associated vector $A\vec{x}$, which happens to be the derivative of $\vec{x}$ as a function of time, under the system of differential equations described by A! A plot of this vector field is useful, because then we'll be able to see where the particle will go at any point in space. These plots are called **phase portraits**.

2 Phase Portraits with Real Eigenvalues

A **phase portrait** is a helpful way to visualize the solutions to a system of differential equations. Let's look at a sample phase portrait of several systems of differential equations. First, let's consider one with positive, real eigenvalues.

2.1 Positive, Real Eigenvalues

Consider the system of differential equations:

$$\begin{aligned}\dot{x} &= x + y \\ \dot{y} &= -2x + 4y\end{aligned}$$

This system can be written as:

$$\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 1 \\ -2 & 4\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}$$
$$\dot{\vec{x}} = A\vec{x}$$

The eigenvectors/eigenvalues of this matrix A are:

$$\vec{v_1} = \begin{bmatrix}1 \\ 1\end{bmatrix} \text{ with eigenvalue } \lambda_1 = 2$$
$$\vec{v_2} = \begin{bmatrix}1 \\ 2\end{bmatrix} \text{ with eigenvalue } \lambda_2 = 3$$

Notice that both of these eigenvalues are positive, but not equal! Let's look at the phase portrait of this system.

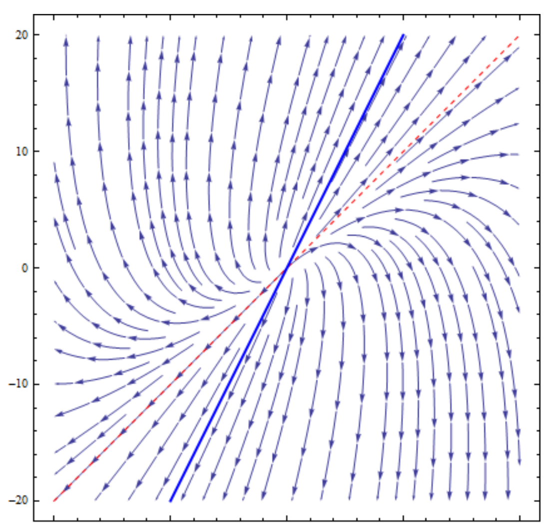

Figure 1: The red dotted line is along the eigenvector with eigenvalue 2, and the thick blue line is along the eigenvector with eigenvalue 3

There are several interesting features about this phase portrait, that we should note:

- **Positions along eigenvectors have trajectories that stay along the eigenvector.** To make sense of this, we simply need to use the definition of eigenvector. Consider starting at an initial position $\vec{x}(0) = \vec{x_0}$ that is equal to one of my eigenvectors, $\vec{v_1}$. In some small time step, Δt, I will move an amount proportional to the time derivative of my vector. Namely:

$$
\begin{aligned}
\vec{x}(\Delta t) &\approx \vec{x}(0) + \underbrace{\frac{d\vec{x}(0)}{dt}}_{A\vec{x}(0)} \Delta t \\
&\approx \vec{v_1} + A\vec{v_1}\Delta t \\
&\approx \vec{v_1} + \lambda_1 \vec{v_1}\Delta t \\
&\approx (1 + \lambda_1 \Delta t)\vec{v_1}
\end{aligned}
$$

  Namely, my position is still in the $\vec{v_1}$ direction!

- **Any deviation from the eigenvector all tend toward the eigenvector with highest eigenvalue**. Notice, that eventually, when you get far enough away, despite some vectors being very close in direction to the red dashed eigenvector, they all start curving to the blue thick eigenvector. This is because this blue thick eigenvector has a greater eigenvalue!

- **The origin is an unstable fixed point in all directions**. Notice, that if you start at the origin, then $A\vec{x} = \frac{d\vec{x}}{dt} = 0$, and you will be stuck there forever. This is known as a **fixed point**. However, this fixed point is **unstable**, because if you move away from the origin (even by a tiny amount), your particle will no longer be confined, and will be free to run off in the direction of the eigenvector with the highest eigenvalue. This is because **neither eigenvalue is negative**.

Now, let's move to another case, where there is a possibility of a fixed point.

2.2 Positive and Negative Real Eigenvalues

Consider the set of differential equations:

$$
\begin{aligned}
\dot{x} &= -5x + 3y \\
\dot{y} &= -6x + 4y
\end{aligned}
$$

This can be written as:

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

where

$$A = \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}$$

We can diagonalize this matrix, and obtain the following eigenvectors and eigenvalues:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ with eigenvalue } \lambda_1 = -2$$
$$\vec{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ with eigenvalue } \lambda_2 = 1$$

Here is what the phase portrait looks like:

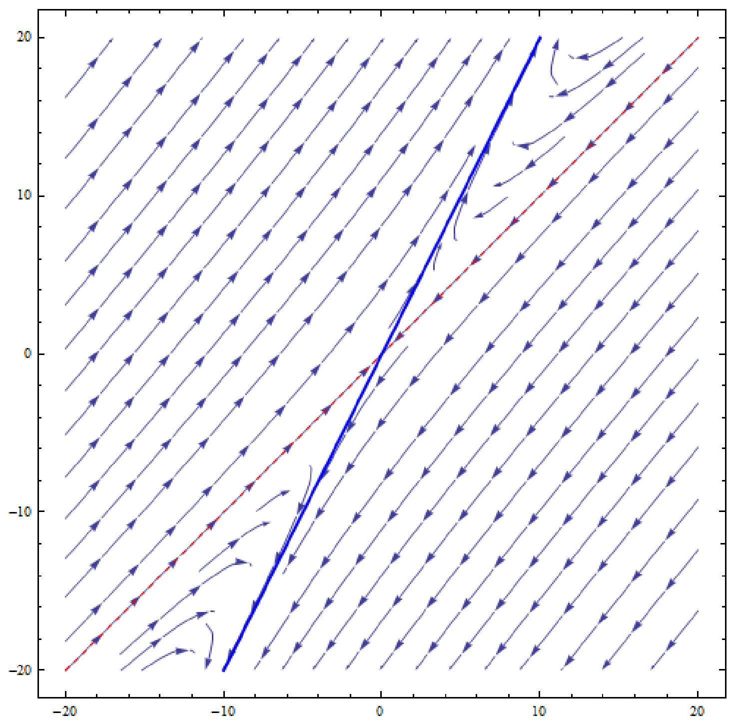

Figure 2: The red dotted line is along the eigenvector with eigenvalue -2, and the thick blue line is along the eigenvector with eigenvalue 1

Once again, let's note some key features about this plot.

- **Along the eigenvector with negative eigenvalue, the trajectory is toward the origin. This leads to a fixed point that is stable to movement along that direction, but not any other direction**. It is not hard to see why that is. This results, because

$$\frac{d\vec{v_1}}{dt} = A\vec{v_1} = -2\vec{v_1}$$

Therefore, starting along this first eigenvector will cause movement toward the origin.

- **On long time scales, the direction of motion is along the eigenvector with largest positive eigenvalue**. This is still true in this case, as it was in the last case! Namely, that over time, unless you are directly on the red dashed line, your trajectory will tend toward the thick blue line.

Now, we can infer what will happen in the case of both negative, real eigenvalues. Wherever you are, the trajectory should be toward the origin, and the origin is a **stable fixed point**. A set of differential equations that produce this result is the following:

$$\dot{x} = -3x + y$$
$$\dot{y} = -2x$$

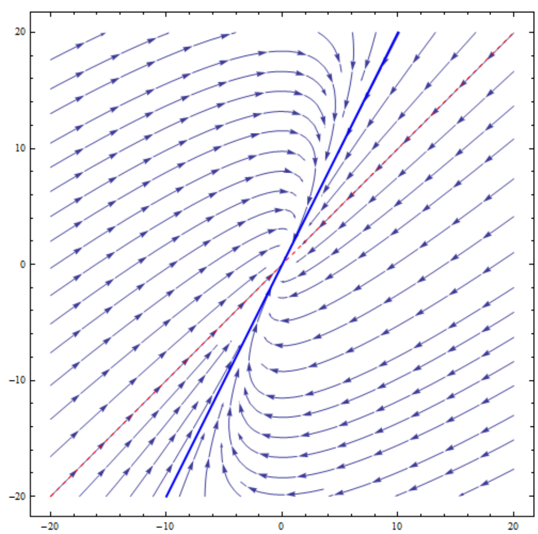

Figure 3: Both eigenvectors have negative eigenvalues, so the origin is a stable fixed point

Lastly, we will deal with the case with one zero eigenvalue.

2.3 One Eigenvalue of Zero

Consider the system of differential equations:

$$\dot{x} = -8x + 4y$$
$$\dot{y} = -4x + 2y$$

We can diagonalize the associated matrix, and obtain the following eigenvectors and eigenvalues:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ with eigenvalue } \lambda_1 = 0$$
$$\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ with eigenvalue } \lambda_2 = -6$$

The phase portrait looks like the following:

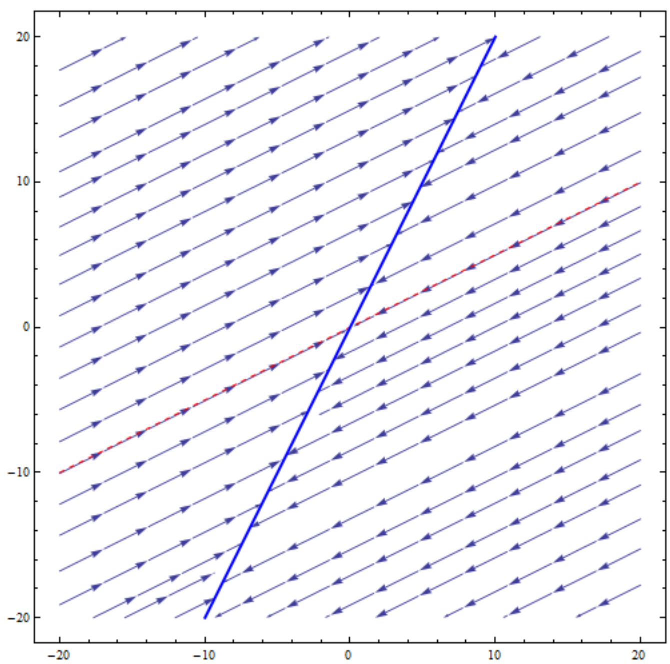

Figure 4: The eigenvalue along the red dashed eigenvector is -6, and the eigenvalue along the blue thick eigenvector is 0

In this case, notice several interesting features:

- **There exists a fixed "line". The entire line is composed of fixed points**. This is simply because the entire line is along the direction of the eigenvector with eigenvalue 0. This entire line will result in no movement, because $A\vec{v_1} = \vec{0}$.

- **All trajectories follow exactly the direction as given by the eigenvector with nonzero eigenvalue.** To see this, consider any general point $\mathbf{w} = a\vec{v_1} + b\vec{v_2}$. To find the time derivative of our vector at this point, we apply A to $\mathbf{w}$. We obtain

$$A\mathbf{w} = A(a\vec{v_1} + b\vec{v_2}) = -6b\vec{v_2}$$

which is along $\vec{v_2}$. Any general point will lead to a trajectory toward the fixed "line" along the direction by $\vec{v_2}$.

Now, we are ready to explore cases where there is no real eigenbasis. Namely, the cases of a matrix with a single eigenvector, and with complex eigenvectors and eigenvalues.

3 Lack of Eigenbasis and Complex Eigenvectors

First, we'll consider the case where there is no eigenbasis.

3.1 No Eigenbasis

Consider the system of differential equations:

$$\begin{aligned} \dot{x} &= 3x - y \\ \dot{y} &= x + y \end{aligned}$$

This can be written as a matrix:

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

This matrix has just a single eigenvector:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ with eigenvalue } \lambda_1 = 2$$

The phase portrait looks like the following:

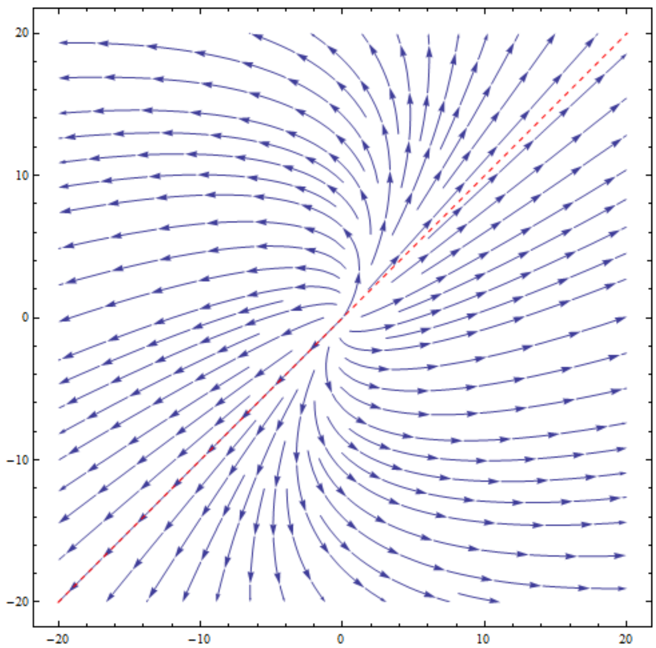

Figure 5: There is only one preferred direction in space, namely the lone eigenvector

Notice that in this case there is only one preferred direction in space, and because the eigenvalue is positive, the direction of travel along this line is **away from the origin**. Next, we will explore the case of complex eigenvalues.

3.2 Complex Eigenvalues/Eigenvectors

Consider the following system of differential equations:

$$\dot{x} = y$$
$$\dot{y} = -4x$$

This can be written as a matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

This matrix has complex eigenvalues $\pm 2i$. Let's look at its phase portrait:

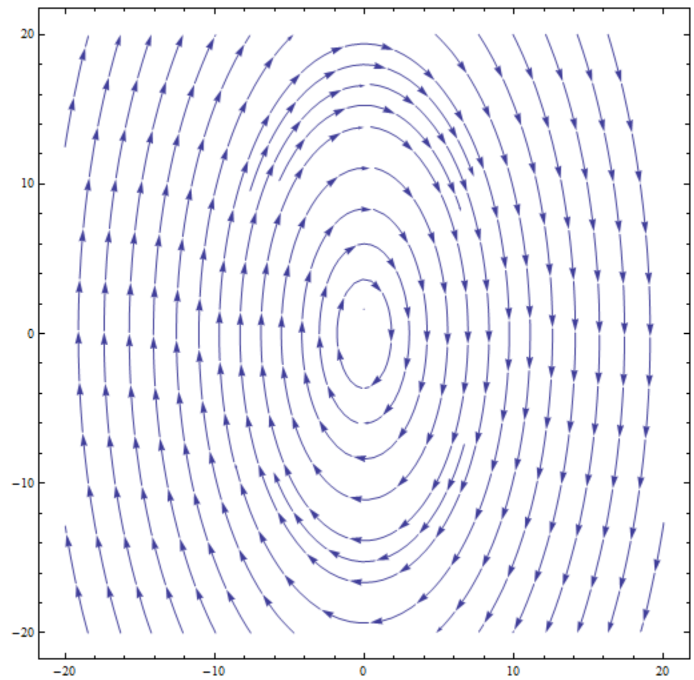

Figure 6: Spirals and circular motion are characteristic of the case with complex eigenvalues. There is clearly no preferred direction in space (no real eigenvector).

There are some key features to note about this plot:

- **There are no preferred directions in space.** This phase portrait has no preferred direction in space. If there was, then it would have a real eigenvector.

- **We observe spiraling motion.** Notice that after looking at this phase portrait, the case of no eigenbasis was an intermediate between this and standard phase portraits of systems with a basis of real eigenvectors. To understand why we obtain spiraling motion, we need to look at the most general solution to a problem with complex eigenvalues. Every matrix A that has complex eigenvalues, $a \pm bi$ and eigenvectors has an associated conformal matrix:

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

In the proper basis, $\vec{v_1} = \vec{e_1}$, and $\vec{v_2} = (\frac{A-aI}{b})\vec{e_1}$, the transformation can be represented by this conformal matrix. In general, if P is the change of basis matrix from this new basis to the standard basis, then:

$$e^{At} = Pe^{Ct}P^{-1}$$

To understand the phase portrait, let's seek to understand it in the new basis. You can show that:

$$e^{Ct} = e^{at} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$$

This can be interpreted as a rotation in time with a frequency associated with the imaginary part of the eigenvalue, and a scaling associated with the real part of the eigenvalue! A combination of rotation and scaling results in a spiral pattern in time, so the phase portrait does indeed reflect these trajectories! In this particular system above, the solutions are closed loops, because the real part of the eigenvalue is 0, so there is no scaling, only rotation. These loops are not perfect circles, because a pure, clean rotation in the new basis, is not necessarily a clean rotation in the standard basis.

4 Summary

A linear system of differential equations can be represented visually as a **phase portrait**, which simply plots the **time derivative vector field** at points in space. These phase portraits describe the trajectory of the particle in time (given that time is the parameter which the derivative is taken with respect to). In a system with a real eigenbasis, the trajectory will tend toward the eigenvector with largest positive eigenvalue. Along directions where the eigenvalue is negative, the system tends toward the origin. In a system without an eigenbasis, but with a single real eigenvector, the system tends toward the single eigenvector, but spirals otherwise. In a system with only complex eigenvectors/eigenvalues, the system simply undergoes "circular" motion of sorts, governed by the imaginary part of the eigenvalue, while it gets scaled by the real part of the eigenvalue.