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Linguistic characterization of time series ☆

Vilém Novák

University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, NSC IT4Innovations, 30. dubna 22, 701 03 Ostrava 1, Czech Republic

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Abstract

The goal of this paper is to provide an overview of applications of special soft computing theories — the fuzzy transform and fuzzy natural logic — to analysis, forecasting and mining information from time series. The focus is especially placed on the ability of methods of fuzzy natural logic to provide information in sentences of natural language. Our approach is based on the decomposition of time series into three components: the trend-cycle, seasonal component and noise. The trend-cycle is extracted using the F-transform, and its course is characterized by automatically generated linguistic description. The latter is then used to forecast the trend-cycle. The trend-cycle can be, furthermore, decomposed into trend (general tendency) and cycle. The former is computed again using the F-transform. Moreover, the F^1-transform makes it possible to estimate the direction of the trend, which can then be characterized by expressions of natural language (stagnating, slightly increasing, sharply decreasing, etc.). Finally, we focus on selected problems of mining information from time series. First, we suggest an algorithm for finding intervals of monotonous behavior and then show how the theory of intermediate quantifiers (a constituent of fuzzy natural logic) and generalized Aristotle's syllogisms can be applied to automatic summarization of information on time series.

Keywords: Time series decomposition; F-transform; Fuzzy natural logic; Mining information from time series; Forecasting; Intermediate quantifiers

1. Introduction

The goal of this paper is twofold: first, to put together results scattered over several papers ([34,35,37] and elsewhere) on applications of *fuzzy transform* (*F-transform*) and *fuzzy natural logic* (FNL) for analysis, forecasting and linguistic characterization of time series. Second, it brings new results in the area of mining information from time series with the aim of providing the obtained information in sentences of natural language.

Our approach is based on the assumption that the time series can be decomposed into three components: the trend-cycle, seasonal component and noise. A detailed analysis, moreover, reveals that the trend-cycle can be fur-

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E-mail address: Vilem.Novak@osu.cz.

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ther decomposed into trend and cycle. The F-transform is applied to extraction of the trend-cycle and can be also applied to extraction of the trend itself. It has, however, more potential. Namely, it is possible to estimate the direction of the trend in an arbitrary time interval (i.e., whether it is increasing, decreasing or stagnating) using the F^1-transform, which provides estimation of an average value of a first derivative of a given function over a specified domain (see [17]). This can be very useful because the direction of the trend of the time series may not be clear even when viewing the graph. It should be emphasized that the methods described in this paper are theoretically well justified (see [33,38,39]).

The fuzzy natural logic is applied in the following tasks:

(a) Generation of linguistic description of the trend-cycle and computation of its forecast. It was demonstrated in [35] that the obtained precision of our forecasting technique is fully comparable with precision of the top professional systems such as ForecastPro®. In addition, our system also provides linguistic comments explaining how forecast of the trend-cycle was obtained. For example, when examining the forecast of a time series summarizing the number of accidental deaths, we are able to generate conditional sentences such as the following:

    If the average number of accidental deaths in the recent 2 years is more or less low and the average decrease in the number of deaths in the same period is extremely big, the average increase in deaths is rather medium.

(b) Generation of linguistic characterization of the direction of the trend (obtained using the F^1-transform). Typical comments are the following:

    In the months 1–5, the trend is slightly decreasing.

(c) Segmentation of time series into subintervals with monotonous behavior. A typical result is as follows:

    In months 1–3, the trend is more or less decreasing, followed by 5 months of stagnation. In the final months 9–12, the trend is sharply increasing.

(d) Summarization of information on time series using formal theory of intermediate quantifiers (linguistic expressions such as *many, most, almost all, few*). We can summarize information either about one time series or about a set of them (typically several tens or hundreds).

    In most *periods is* slightly decreasing *trend followed by* clearly increasing *one*.

    Note that this is one of the typical tasks when mining information from time series (cf. [8]). We also propose to apply the theory of generalized Aristotle's syllogisms, which makes it possible to deduce new information based on information that has already been found before.

The paper is structured as follows. In the next section, we provide an overview of basic concepts of the F-transform and fuzzy natural logic that will be used later. Section 3 contains revamped presentation of our methods for forecasting and the trend evaluation of time series introduced earlier. Section 4 contains new results and is devoted to selected problems of mining information from time series. First, we suggest algorithm for finding intervals of monotonous behavior and in the second part, we show how the theory of intermediate quantifiers (constituent of fuzzy natural logic) and generalized Aristotle's syllogisms can be applied to automatic summarization of information on time series. Section 5 is the conclusion, in which we outline directions for further research.

2. Preliminaries: soft-computing techniques for time-series processing

In this section, we will briefly review two main soft-computing techniques that have been successfully applied to time series analysis and forecasting: the F-transform and special methods of fuzzy natural logic.

By a fuzzy set, we understand a function $A : U \longrightarrow [0, 1]$ where U is a universe and $[0, 1]$ is a support set of some standard algebra of truth values. The set of all fuzzy sets over U is denoted by $\mathscr{F}(U)$. If A is a fuzzy set in U, we will sometimes write $A \subsetapprox U$.

2.1. *Fuzzy transform*

2.1.1. *Basic F-transform*

The F-transform was invented by I. Perfilieva and published in detail in [38]. Its fundamental idea is to transform a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ to a finite vector of numbers (*direct F-transform*) and then to transform it back (*inverse F-transform*). The result is a function $\hat{f}$ approximating the original function f.

The first step is to form a *fuzzy partition* of the domain $[a, b]$. It consists of a finite set of fuzzy sets $\mathscr{P} = \{A_0, \ldots, A_n\}$, $n \geq 2$, defined over nodes $a = c_0, \ldots, c_n = b$. Properties of the fuzzy sets from $\mathscr{P}$ are specified by five axioms, namely: *normality, locality, continuity, unimodality, and orthogonality*. The membership functions $A_0, \ldots, A_n$ in a fuzzy partition $\mathscr{P}$ are called *basic functions*. A fuzzy partition $\mathscr{P}$ is called *h-uniform* if the nodes $c_0, \ldots, c_n$ are *h-equidistant*, i.e., $c_{k+1} = c_k + h$, where $h = (b - a)/n$ for all $k = 0, \ldots, n - 1$. A precise definition of fuzzy partition can be found in [13,38] and elsewhere.

Once the fuzzy partition $\mathscr{P}$ is selected, we define a *direct F-transform* of a continuous function f as a vector $\mathbf{F}[f] = (F_0[f], \ldots, F_n[f])$, where each k-th *component* $F_k[f]$ is equal to

$$F_k[f] = \frac{\int_a^b f(x) A_k(x)\, dx}{\int_a^b A_k(x)\, dx}, \qquad k = 0, \ldots, n.$$

The *inverse F-transform* of f with respect to $\mathbf{F}[f]$ is a continuous function[1] $\hat{f} : [a, b] \longrightarrow \mathbb{R}$ such that

$$\hat{f}(x) = \sum_{k=0}^{n} F_k[f] \cdot A_k(x), \qquad x \in [a, b]. \tag{1}$$

It is possible to set parameters to obtain a function $\hat{f}$ having desirable properties. It is also proved that the sequence $\{\hat{f}_n\}$ uniformly converges to f for $n \to \infty$. Details and full proofs can be found in [38,39].

2.1.2. *F^1-transform*

The F-transform introduced above is an F^0-transform (zero-degree F-transform) because its components are real numbers. If we replace them by polynomials of a degree $m \geq 1$, we arrive at the F^m transform. This generalization has been described in detail in [39]. In this paper, we need only the F^1-transform, whose brief description follows.

Definition 1. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and $\mathscr{P} = \{A_0, \ldots, A_n\}$, $n \geq 2$ be a fuzzy partition of $[a, b]$. The vector of linear functions

$$\mathbf{F}^1[f] = (\beta_1^0 + \beta_1^1(x - c_1), \ldots, \beta_{n-1}^0 + \beta_{n-1}^1(x - c_{n-1})) \tag{2}$$

is called the F^1-transform of f with respect to the fuzzy partition $\mathscr{P}$. If $\mathscr{P}$ is h-uniform given by the triangular-shaped basic functions then

$$\beta_k^0 = \frac{\int_{c_{k-1}}^{c_{k+1}} f(x) A_k(x) dx}{h}, \tag{3}$$

$$\beta_k^1 = \frac{12 \int_{c_{k-1}}^{c_{k+1}} f(x)(x - c_k) A_k(x) dx}{h^3}, \tag{4}$$

for every $k = 1, \ldots, n - 1$.

[1] By conventions of language, we indicate by inverse F-transform both the procedure as well as its result $\hat{f}$.

The fuzzy partition in the above definition does not need to be uniform. Then, formulas (3), (4) have to be modified (see [39] for the details).

The following theorem plays an important role in our application to time series trend evaluations.

Theorem 1. *(See [17].) Let $\mathscr{P} = \{A_1, \ldots, A_{n-1}\}$ be an h-uniform partition of $[a, b]$, let functions f and $A_k \in \mathscr{P}$, $k = 1, \ldots, n-1$, be four times continuously differentiable on $[a, b]$. Finally, let $\mathbf{F}^1[f]$ be the F^1-transform (2) of f. Then, for all $k = 1, \ldots, n-1$*

$$|\beta_k^1 - f'(c_k)| \leq Mh^2 \tag{5}$$

for some $M > 0$.

According to Theorem 1, each coefficient β_k^1 in (4) provides a reasonable estimation of an average value of the first derivative of f in the interval $[c_{k-1}, c_{k+1}]$. We will use this result in Subsection 3.3 where evaluation of the direction of trend of time series is described. The inverse F^1-transform of a function f is defined in the same way as in (1).

2.2. Fuzzy natural logic

Fuzzy natural logic (FNL)[2] is a group of mathematical theories that extend mathematical fuzzy logic in a narrow sense (cf. [28]). Its goal is to develop a mathematical model of special human reasoning schemes that employ natural language.[3] Therefore, FNL includes also a model of the semantics of some parts of natural language.

The main constituents of FNL are (until now) the following formal theories:

- Theory of evaluative linguistic expressions.
- Theory of fuzzy/linguistic IF–THEN rules and logical inference from them.
- Theory of fuzzy generalized quantifiers with emphasis on intermediate quantifiers, generalized Aristotle syllogisms and square of opposition.

2.2.1. Theory of evaluative linguistic expressions

Evaluative linguistic expressions are expressions such as *small, very big, rather medium, extremely strong, roughly important*, etc. The formal theory of their semantics formulated as a special theory of the higher-order fuzzy logic[4] was described in detail in [26]. We will only recall that this logic is developed over the standard Łukasiewicz MV-algebra $\mathscr{E}_Ł = \langle [0, 1], \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$ of truth values where $\otimes$ is the operation of Łukasiewicz conjunction and $\rightarrow$ that of Łukasiewicz implication.

We consider a set *EvExpr* of evaluative linguistic expressions. Because it is difficult to define it precisely,[5] we will classify a few kinds of expressions that definitely belong to *EvExpr*, but keep in mind that this classification is not exhaustive.

The most important subclass of *EvExpr* are *simple evaluative expressions*, which have the following general structure:

$$\langle \text{linguistic hedge} \rangle \langle \text{TE-adjective} \rangle \tag{6}$$

where ⟨TE-adjective⟩[6] is a special adjective having the following property: It is always accompanied by its *antonym*, and it is possible to form also a special expression in the position of, semantically, “a middle member”. We say that these adjectives form a *fundamental evaluative trichotomy*. Typical representatives of ⟨TE-adjective⟩s are *gradable adjectives* (big, cold, deep, fast, friendly, happy, high, hot, important, long, popular, rich, strong, tall, warm, weak,

[2] The concept of FNL continues the program of *fuzzy logic in a broader sense* (FLb-logic) introduced in [23]. It follows the concept of *natural logic*, which was initiated by G. Lakoff in [18].

[3] The model should be independent of a concrete language as much as possible. At the least, we can expect it to work for European languages.

[4] This logic is called *fuzzy type theory* (denoted as FTT) and was described in detail in [24].

[5] This suggests the idea that *EvExpr* is, in fact, a fuzzy set. This observation, however, is unimportant for the purposes of this paper.

[6] The short “TE” means trichotomous evaluative.

young), *evaluative adjectives* (good, bad, clever, stupid, ugly, etc.), and also adjectives such as *left, middle, medium*. Examples of an evaluative trichotomy are *low, medium, high*; *clever, average, stupid*; *good, normal, bad*, etc.

The ⟨linguistic hedge⟩ represents a class of adverbial modifications that includes a class of *intensifying adverbs* such as "very, roughly, approximately, significantly", etc. Two basic kinds of linguistic hedges can be distinguished in (6): *narrowing* hedges, for example, "extremely, significantly, very" and *widening* ones, for example, "more or less, roughly, quite roughly, very roughly". Note that narrowing hedges make the meaning of the whole expression more precise, while widening ones have the opposite effect. Thus, "very small" is more precise (more specific) than "small", which, on the other hand, is more precise than "roughly small". The situation in which the "⟨linguistic hedge⟩" is not present on the surface level is dealt with as a presence of *empty linguistic hedge*. In other words, all simple evaluative expressions have the form (6), including examples such as "small, long, deep".

Let us emphasize, however, that the situation in natural language is not as simple as (6) might suggest. Namely, we cannot combine an arbitrary TE-adjective with an arbitrary hedge. For example, "very medium, very roughly happy, approximately bad, roughly beautiful, etc." have no meaning. Therefore, we will mostly work with threesomes of adjectives *small, medium, big* and take them as *canonical* because they are mostly neutral with respect to objects falling in their extensions, and therefore, they are very flexible when combining them with linguistic hedges. Still narrowing hedges, for example, cannot be combined with "medium".

The evaluative expressions, in general, characterize parts of some linearly ordered scale. A special case of them are also *fuzzy numbers*, which vaguely characterize a special position on an ordered scale and do not form the evaluative trichotomy. Fuzzy numbers can also be combined with certain hedges, e.g., *approximately, roughly*.

In this paper, we will suppose that *EvExpr* always contains all meaningful combinations of *small (Sm), medium (Me), big (Bi), zero (Ze)* with the hedges *extremely (Ex), significantly (Si), very (Ve), rather (Ra), more or less (ML), roughly (Ro), very roughly (VR), quite roughly (QR), very very roughly (VV)*, where in the brackets is a simple short that is used to simplify specification of evaluative expressions in applications.

Moreover, there is a natural ordering $\lll$ of these expressions obtained as lexicographic ordering based on the natural ordering

$$Ze \text{ (zero)} \lll Sm \text{ (small)} \lll Me \text{ (medium)} \lll Bi \text{ (big)}$$

and the following ordering of hedges:

$$\begin{aligned} &Ex \text{ (extremely)} \lll Si \text{ (significantly)} \lll Ve \text{ (very)} \\ &\qquad \lll \langle\text{empty hedge}\rangle \lll ML \text{ more or less} \lll Ro \text{ (roughly)} \\ &\qquad \lll QR \text{ (quite roughly)} \\ &\qquad \lll VR \text{ (very roughly)} \lll VV \text{ (very very roughly)}. \end{aligned} \tag{7}$$

The linear lexicographic ordering $\lll$ of simple evaluative expressions (6) should be interpreted as "*to be more precise*" (or, "*to have a sharper meaning*").

We will distinguish abstract evaluative expressions, i.e., expressions such as *small, weak, very strong*, that alone do not address any specific objects and *evaluative linguistic predications* such as "temperature is high, expenses are extremely low, the building is quite ugly". A simplified form of the latter suitable for technical applications is

$$X \text{ is } \langle\text{simple evaluative expression}\rangle \tag{8}$$

where X is some variable taking values from $\mathbb{R}$. A deeper analysis reveals that the abstract evaluative expressions are, in fact, also evaluative predications where X is hidden because its values are dimensionless numbers from $[0, 1]$. Therefore, we can introduce the following definition.

Definition 2. The set of evaluative expressions *EvExpr* consists both of abstract evaluative expressions (6) as well as of evaluative predications (8). We will write $Ev \in EvExpr$ for an arbitrary element from *EvExpr*. The abstract evaluative expressions from *EvExpr* will be denoted by script letters $\mathscr{A}, \mathscr{B}, \ldots$.

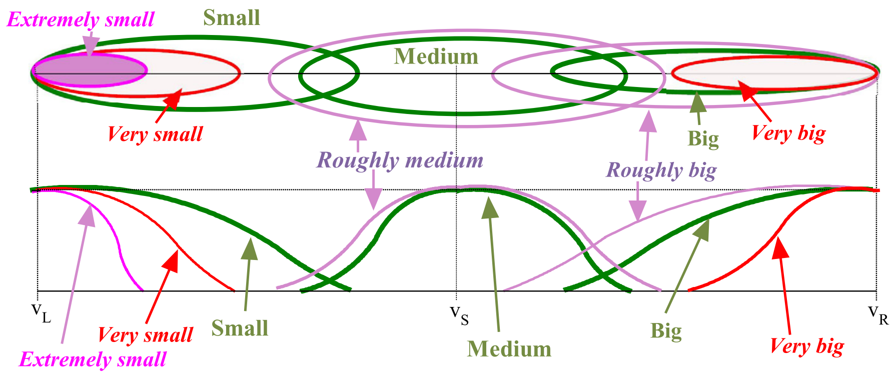

Fig. 1. Shapes of fuzzy sets representing extensions of a few canonical evaluative expressions in the context $\langle v_L, v_S, v_R\rangle$ are depicted in the lower part of the figure. The upper part depicts the intuitive meaning of these expressions. For example, "very small" are all values that lie somewhere on the left of the given context and more to the left than "small" values (note that each very small value is also small).

A mathematical model of the semantics of evaluative expressions is developed on the basis of logical analysis of their meaning. The essential concept is that of (linguistic) *context*, defined as follows.[7]

Definition 3. The linguistic context for evaluative expressions is determined by a triple of real numbers $v_L, v_S, v_R \in \mathbb{R}$, where $v_L < v_S < v_R$. These numbers represent the *smallest, typically medium* and the *largest* thinkable values, respectively. The context is a set

$$w = [v_L, v_S] \cup [v_S, v_R] \subset \mathbb{R} \tag{9}$$

together with the three distinguished points $DP(w) = \langle v_L, v_S, v_R\rangle$.[8] By W, we denote the set of all contexts (9).

The context w is the set (9), but the elements $DP(w)$ are equally important, and we cannot omit them from our reasoning. Therefore, we will usually write a context w as a triple $w = \langle v_L, v_S, v_R\rangle$, and $x \in w$ will mean that x belongs to the set (9). Note that v_S lies somewhere between v_L and v_R and not necessarily in the middle of the interval (v_R, v_L).

Furthermore, some evaluative expressions may also have a sign, namely, "positive" or "negative". The linguistic context for them is then a concatenation $w = w^- \sqcup w^+$ of a *positive* context w^+ and *negative* context w^-. The distinguished points in this case are $DP(w) = \langle v_R^-, v_S^-, v_L^+, v_S^+, v_R^+\rangle$, where $v_L^- = v_L^+$. We usually (but not necessarily) put $v_R^- = -v_R^+$ and $v_S^- = -v_S^+$.

Each evaluative expression $Ev \in EvExpr$ is assigned the meaning that is a function

$$\mathrm{Int}(Ev) : W \longrightarrow \mathscr{F}(\mathbb{R}).$$

We will call this function *intension* of the evaluative expression Ev. It assigns to each context $w \in W$ a fuzzy set $\mathrm{Ext}_w(Ev) \underset{\sim}{\subseteq} w$ called *extension* of the expression Ev in the context $w \in W$.

Shapes of extensions of few canonical evaluative expressions are depicted in Fig. 1.

2.2.2. Linguistic descriptions and logical inference

The *linguistic description* is defined as a finite set of fuzzy/linguistic IF–THEN rules with common X and Y:

$$\begin{array}{l} \text{IF } X \text{ is } \mathscr{A}_1 \text{ THEN } Y \text{ is } \mathscr{B}_1 \\ \dots\dots\dots\dots\dots\dots \\ \text{IF } X \text{ is } \mathscr{A}_m \text{ THEN } Y \text{ is } \mathscr{B}_m \end{array} \tag{10}$$

[7] This definition is a simplification because we consider here only the set of all real numbers as a universe. A more general definition takes the linguistic context as the function $w : L \longrightarrow M$, where L is the support of the algebra of truth values and M is an arbitrary set (see [26] for the details).

[8] The context w in (9) is written formally as a union of two adjacent intervals to stress the role of the middle point v_S.

We call the rules in (10) *fuzzy/linguistic* to emphasize that they are taken as *conditional clauses of natural language*.[9] The whole description can be taken as a *special piece of text* providing us with knowledge about some local situation, for example, a decision-making problem, local characterization of behavior of some system, etc. A complex knowledge can be provided by a *knowledge base* consisting of a system of interconnected linguistic descriptions.

Inference on the basis of a linguistic description is realized using a special procedure called *perception-based logical deduction* (PbLD) that was described in detail in [25,32]. The crucial role in it is played by the notion of a *local perception*. This can be understood as a linguistic characterization of a certain kind of "measurement" performed by people in a concrete situation.

Definition 4. The function of *local perception* is a partial function $LPerc : \mathbb{R} \times W \longrightarrow EvExpr$ defined as follows:

$$LPerc(u, w) = \begin{cases} X \text{ is } \mathscr{A} & \text{if } u \in w, \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{11}$$

If (11) is defined, then "X is $\mathscr{A}$" is the *sharpest* evaluative predication in the sense of the lexicographic ordering $\lll$ so that $u \in w \subset \mathbb{R}$ is the *most typical* element for the extension $\text{Ext}_w(X \text{ is } \mathscr{A})$.

Let us remark that this function makes it also possible to *learn* linguistic description from data. Learning linguistic description and inference on the basis of it is applied to time series forecasting (the details can be found in [29,36]).

2.2.3. Intermediate quantifiers

A well-developed theory belonging to FNL is that of intermediate quantifiers. These are expressions of natural language such as *most, many, almost all, a few, a large part of*. They refine classical quantification in the sense that their meaning lies between the limit cases *for all* (∀) and *exists* (∃). A deep semantic analysis of intermediate quantifiers is contained in the book [41]. A working formalization of them was first given in [27] and further elaborated in [19,22] within higher-order fuzzy logic. This theory differs from the other existing theories in its comprehensiveness because it is formulated syntactically, and thus, it captures general logical properties of intermediate quantifiers that hold in *all models*.

The intermediate quantifiers belong among generalized quantifiers whose theory has been studied already for many years (cf., for example, [16,40,42]). This theory was further generalized to the fuzzy approach (cf. [9,6,12,30]; a comprehensive overview of fuzzy generalized quantifiers can also be found in [5]). The intermediate quantifiers are special fuzzy generalized quantifiers of type $\langle 1, 1\rangle$ that are *isomorphism-invariant, conservative, and have extension properties*.

We will provide a very brief overview of some of the main concepts of our theory in Section 4.3. The presentation is in many places imprecise because precise formulations would require much more space.

The basic idea of the mathematical model of intermediate quantifiers consists in the assumption that intermediate quantifiers are just classical quantifiers ∀ or ∃, but the universe of quantification is modified by a suitable evaluative linguistic expression. This idea is formally expressed in the following definition.

Definition 5. An intermediate quantifier of type $\langle 1, 1\rangle$ interpreting the sentence

"$\langle$Quantifier$\rangle$ B are A"

is one of the following formulas:

$$(Q^{\forall}_{Ev}\, x)(Bx, Ax) := (\exists Z)((\mathbf{\Delta}(Z \subseteq B) \,\&\, (\forall x)(Zx \Rightarrow Ax)) \wedge Ev((\mu B)Z)), \tag{12}$$

$$(Q^{\exists}_{Ev}\, x)(Bx, Ax) := (\exists Z)((\mathbf{\Delta}(Z \subseteq B) \,\&\, (\exists x)(Zx \wedge Ax)) \wedge Ev((\mu B)Z)) \tag{13}$$

where x represents elements and Z, B, A are interpreted as fuzzy sets. The $(\mu B)Z$ represents a measure of the fuzzy set Z w.r.t. B and $Ev \in EvExpr$ is an evaluative expression.

[9] For simplicity of explanation, we consider in (10) one antecedent variable X only. Extension to more of them is straightforward.

Interpretation of these formulas is simple. Informally, without an explicit definition of interpretation in a model, we can explain them as follows: Take the largest fuzzy set $Z \subseteq B$ such that each element x having the property represented by Z also has the property A, and at the same time, the size of Z w.r.t. B (that is mathematically characterized by a measure $(\mu B)Z$) is evaluated by the evaluative expression Ev.

We can also introduce simple type $\langle 1 \rangle$ intermediate quantifiers:

$$(Q^{\forall}_{Ev}\, x)Ax := (\forall x)(x \in \mathrm{Supp}(A) \Rightarrow Ax) \wedge Ev((\mu V)A), \tag{14}$$

$$(Q^{\exists}_{Ev}\, x)Ax := (\exists x)(Ax \,\&\, x \in \mathrm{Supp}(A)) \wedge Ev((\mu V)A) \tag{15}$$

where V is a universe.

In the literature, 10 special intermediate quantifiers were formally introduced and studied in detail (see [19,21,22]):

A: All B are $A := (Q^{\forall}_{Bi\boldsymbol{\Delta}}x)(B, A)$ (*the biggest possible part of B has A*)

E: No B are $A := (Q^{\forall}_{Bi\boldsymbol{\Delta}}x)(B, \neg A)$ (*the biggest possible part of B has no A*)

P: Almost all B are $A := (Q^{\forall}_{BiEx}\, x)(B, A)$ (*extremely big part of B has A*)

B: Few B are $A := (Q^{\forall}_{BiEx}\, x)(B, \neg A)$ (*extremely big part of B has no A*)

T: Most B are $A := (Q^{\forall}_{BVe}\, x)(B, A)$ (*very big part of B has A*)

D: Most B are not $A := (Q^{\forall}_{BVe}\, x)(B, \neg A)$ (*very big part of B has no A*)

K: Many B are $A := (Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}\, x)(B, A)$ (*not small part of B has A*)

G: Many B are not $A := (Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}\, x)(B, \neg A)$ (*not small part of B has no A*)

I: Some B are $A := (Q^{\exists}_{Bi\boldsymbol{\Delta}}x)(B, A)$

O: Some B are not $A := (Q^{\exists}_{Bi\boldsymbol{\Delta}}x)(B, \neg A)$

All these quantifiers can be used in the summarization task discussed in Section 4.3.[10]

In a finite model, the measure $(\mu B)Z$[11] can be given, for example, by

$$(\mu B)Z = \begin{cases} 1 & \text{if } Z = B, \\ \frac{|Z|}{|B|} & \end{cases} \tag{16}$$

where[12] $|B| = \sum_{x \in V} B(x)$, $|Z| = \sum_{x \in V} Z(x)$.[13]

For example, the quantifier **P** says that we consider the greatest fuzzy set Z being a subset of B such that all its elements have the property A, and Z is "extremely big" (in the sense of the measure μ). The meaning of the other intermediate quantifiers is similar. Note that the quantifiers **A, E, I, O** coincide with the classical quantifiers defined in first-order classical logic (cf. [27]).

An important consequence of the formal theory of intermediate quantifiers is the formal theory of generalized Aristotle syllogisms. We proved the validity of over 120 generalized Aristotle's syllogisms with intermediate quantifiers. The proofs are syntactical, and therefore, the valid syllogisms are true in all situations (models) (for details and proofs, see [19,20,22]). This theory is applied to time series in Section 4.4.

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10 The $\boldsymbol{\Delta}$ is a special linguistic hedge that can be taken as "utmost". In a model, $\boldsymbol{\Delta}$ is interpreted by the Baaz delta operation that keeps the truth value 1 and sends all the other truth values to 0. The $\bar{\boldsymbol{\nu}}$ is the "empty hedge" that precedes simple expressions such as "small", but no surface expression corresponds to it.

11 There is also the possibility to characterize directly the absolute number of elements with respect to a given context w. This means that we formally put $(\mu B)Z$ equal to $w^{-1}x$. The details require precise formalization that is out of the scope of this paper.

12 For precise formulation of the interpretation of formulas (12) and (13), see the cited references. The evaluation is realized w.r.t. the standard context $\bar{w} = \langle 0, 0.5, 1 \rangle$.

13 The set V takes the role of the universe — a specific set in a chosen model.

3. Application of the F-transform and FNL to time series processing

3.1. Decomposition of time series

A time series is a stochastic process (see [1,11]) $X : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$ where Ω is a set of elementary random events and $\mathbb{T} = \{1, \dots, p\} \subset \mathbb{N}$ is a finite set whose elements are interpreted as time moments. Our basic assumption is that the time series can be decomposed into three components

$$X(t, \omega) = TC(t) + S(t) + R(t, \omega), \qquad t \in \mathbb{T}, \omega \in \Omega, \tag{17}$$

$TC(t)$ is a *trend-cycle* (ordinary real or complex function), $S(t)$ is a seasonal component that is a mixture of periodic functions and $R(t, \omega)$ is a random noise such that for each $t \in \mathbb{T}$, the $R(t)$ is a random variable with zero mean and finite variance. It was proved in [33] that the F-transform can be used for extraction of the trend-cycle TC with the error close to zero.

The seasonal component $S(t)$ in (17) is assumed to be a sum of periodic functions

$$S(t) = \sum_{j=1}^{r} P_j \sin(\lambda_j t + \varphi_j), \qquad t \in \mathbb{T}, \tag{18}$$

for some finite r where λ_j are frequencies, φ_j are phase shifts and P_j are amplitudes.[14]

If we fix $\omega \in \Omega$, then we obtain one realization of (17), and in this case, we can write $X(t)$ only. Moreover, if $\bar{\mathbb{T}} \subseteq \mathbb{T}$ is a subinterval of $\mathbb{T}$, then $X|\bar{\mathbb{T}}$ denotes the time series X in (17) restricted to $t \in \bar{\mathbb{T}}$. For consistency of this notation, we will sometimes also write $X|\mathbb{T}$, which is, in fact, the whole time series (17).

The following theorem assures us that we can find a fuzzy partition enabling us to estimate the trend cycle TC with high fidelity.

Theorem 2. *Let $X(t)$ be a realization of the stochastic process in (17) considered over the interval $[0, b]$. Let us construct an h-uniform fuzzy partition $\mathscr{P}$ over nodes $c_0, \dots, c_n$ with $h = d\,\bar{T}$, where $\bar{T} = \frac{2\pi}{\lambda}$ for a minimal $\lambda = \min\{\lambda_1, \dots, \lambda_r\}$ and a real number $d \geq 1$. If we compute a direct F-transform $F[X]$, then for the corresponding inverse F-transform $\hat{X}$ of X there is a certain small number D converging to 0 for $d \to \infty$ such that*

$$|\hat{X}(t) - TC(t)| \leq D, \qquad t \in [c_1, c_{n-1}]. \tag{19}$$

The precise expression for D and the proof of this theorem can be found in [33].

It follows from Theorem 2 that if we form the h-uniform fuzzy partition $\mathscr{P}$ with h corresponding to the largest periodicity of a periodic constituent occurring in the seasonal component of $S(t)$, then the latter is almost "wiped down", and also, the noise is significantly reduced. This means that the components $S(t)$ and $R(t, \omega)$ in (17) are almost completely removed. We thus obtain the following estimation of the trend cycle:

$$TC(t) \approx \hat{X}(t). \tag{20}$$

It can also be proved that D is minimal if $d \in \mathbb{N}$.

3.2. Forecasting trend-cycle of a time series using linguistic description

The linguistic description and PbLD inference method mentioned in Section 2.2.2 are applied in forecasting of the trend-cycle TC. The method was described in detail in [35], and so, we will only briefly review its main ideas.

Let $\mathbb{T}' \supset \mathbb{T}$ be a new time domain. Our task is to extrapolate values of X to $X|(\mathbb{T}' \setminus \mathbb{T})$ on the basis of the known values of $X|\mathbb{T}$. The method for finding the former is called *forecasting*. There are many forecasting methods mostly formulated using probability theory (cf. [2,11,15]). In this paper, we consider the methods described in [35] that are based on combination of the F-transform and fuzzy natural logic.

[14] Because $\cos x = \sin(x + \pi/2)$, it is sufficient to consider only sin.

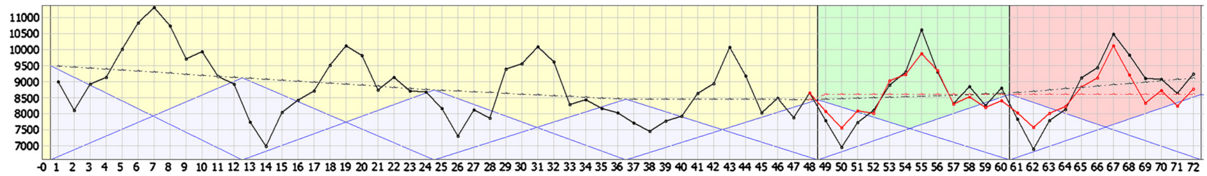

Fig. 2. In the figure, an example of an extraction of a trend-cycle of a time series (monthly accidental deaths in the USA in the years 1973–1978) using F-transform is presented. The time series is divided into learning, validation and testing (the last 12 months) parts. In the testing part, forecast of the values of the time series and its trend-cycle are depicted together with the real values and the trend-cycle computed from them.

Recall that the trend-cycle is obtained using the F-transform on the basis of an h-uniform fuzzy partition $\mathscr{P}$ where $h = d\,\bar{T}$ for some $d > 1$ and maximal periodicity $\bar{T}$ occurring in the seasonal component $S(t)$. The result of the direct F-transform is a vector of F-transform components

$$\mathbf{F}[X] = (F_1[X], \ldots, F_{n-1}[X]), \tag{21}$$

where each component $F_i[X]$ represents a weighted average of values of $X(t)$ in the area of width $2h$. The components (21) can be used as data for learning of a linguistic description. Then, using it and the PbLD method, we can forecast future F-transform components

$$F_n[X], \ldots, F_{n+l}[X] \tag{22}$$

and from them by the inverse F-transform (1) compute estimation of the future development of the trend cycle.

The learned linguistic description consists of fuzzy/linguistic rules of the form, for example,

$$\text{IF } \Delta^2 X_{i-1} \text{ is } \mathscr{A}_{\Delta^2 i-1} \text{ AND } \Delta X_{i-1} \text{ is } \mathscr{A}_{\Delta i-1} \text{ AND } X_i \text{ is } \mathscr{A}_i \text{ THEN } X_{i+1} \text{ is } \mathscr{B}_{i+1}, \tag{23}$$

where

$$\Delta F_i[X] = F_i[X] - F_{i-1}[X], \qquad i = 1, \ldots, n-1 \tag{24}$$

$$\Delta^2 F_i[X] = \Delta F_i[X] - \Delta F_{i-1}[X], \qquad i = 2, \ldots, n-1 \tag{25}$$

are first and second differences, respectively. Let us remark that in practice, all kinds of combinations of the F-transform components and their first and second differences can occur both in the antecedent as well as in the consequent.

The learned linguistic description provides us with information in linguistic form explaining how the forecast was obtained, i.e., what are the inner characteristics of the time series that led to the forecast. The differences (24) and (25) characterize dynamics of the time series as well as logical dependencies of the trend-cycle changes (hidden cycle influences).

Example 1. Let us consider the trend-cycle of the time series "Accidental deaths in the USA" depicted in Fig. 2 that is obtained using the F-transform with a fuzzy partition consisting of basic functions of the width 24 months. Its forecast was obtained using the following linguistic description that was learned from the data. In this special case, it consists of only 3 rules:

No.	$F_i[X] \& \Delta F_i[X] \Rightarrow \Delta F_{i+1}[X]$
1	*ExBi* & -*ExBi* ⇒ *VRBi*
2	*RaMe*& -*ExBi* ⇒ *RaMe*
3	*Ze* & -*RoBi* ⇒ *RaMe*

The used shorts were introduced in Subsection 2.2.1. For example, the full sentence represented by rule 1 in this table is:

If the average number of accidental deaths in the recent 2 years is extremely big and the average decrease in the number of deaths in the same period is also extremely big then the average increase in the number of deaths is very roughly big.

The details regarding how the forecast was obtained can be found in [35]. We only recall that the PbLD inference method was applied.

It is clear that we can apply this method also to knowledge about behavior of the time series in an arbitrary time period. In fact, such a description would provide information about autocorrelation inside the series expressed in a linguistic form.

Let us also remark that the learned linguistic description may consist of a large number of rules, and some of them may be redundant in the sense that omitting them has no influence on the derived conclusion. A sophisticated method for removing all the redundant rules is described in [7].

3.3. *Characterization of trend of time series using natural language*

The trend-cycle can be further decomposed into a *trend* $Tr(t)$ and a *cycle* $C(t)$, i.e., $TC(t) = Tr(t) + C(t)$. The trend can be estimated using the F-transform, again on the basis of Theorem 2 by setting $h = d\,T^{max}$ where T^{max} is the maximal periodicity of the time series X that can be found using, e.g., a periodogram (cf. [1]). In this section, we will not be interested in the trend itself but in its course. We may distinguish the trend of the whole time series as well as its trend in local time slots (for example, quarter of year, production period, etc.). Its course can be then estimated using the coefficients β^1 determined in (4) because they enable us to estimate the tangent of the transformed function in a given area. Let us remark that a similar problem has been solved also in [14]. The authors suggest approximating the general trend by a properly inclined line that is obtained by a heuristic and quite complicated algorithm.

The course of the trend is often in practice characterized using natural language. Below, we will show how such characterizations can be automatically generated using the function of local perception (11).

First, we must specify a context in which the (direction of) the trend is evaluated. This is clear because, for example, the increase in temperature in an Arctic area by 3 C° in, say, 2 hours can be taken as a "sharp increase", while the same in Sahara is a "very small increase". Therefore, we start with a specification of what "extreme increase (decrease)" means. The latter can be determined as the largest acceptable (or meaningful) difference in time series values with respect to a given (basic) time interval. The usual basic time interval is, e.g., 12 months, 31 days, and 1 hour, depending on the kind of time series. Thus, the context is determined by the three distinguished values v_L, v_S, v_R of the tangent. The largest tangent v_R is the mentioned extreme increase (decrease), while the smallest one is typically (but not always) $v_L = 0$. The typical medium value v_S is determined analogously as v_R. The result is the linguistic context for the (course of) trend that will be denoted by $w_{tg} = \langle v_L, v_S, v_R\rangle$. Moreover, because we distinguish between an increase and decrease in time series, we must also distinguish positive $w_{tg}^+ = \langle v_L, v_S, v_R\rangle$ from negative contexts $w_{tg}^- = \langle -v_R, -v_S, -v_L\rangle$ for its course. We usually put $-v_L = v_L = 0$.

In natural language, characterization of trend has one of two possible forms. First, we must realize that the basic characteristic of the trend is its *slope*. Its direction is characterized by a special word, namely, *increasing* (or *increase*) and *decreasing* (or *decrease*). This expression can be further completed by special words characterizing its degree. Moreover, the obtained expressions can be apparently ordered similarly as the "standard" evaluative expressions considered in Subsection 2.2.1. We conclude that the general syntactic form of expressions characterizing direction of trend is either direct, i.e., *(i) characterization of trend*, or indirect by characterizing the feature *(ii) sign of the trend*. We thus have the following syntactic form of both possibilities:

(i) *Characterization of trend*[15]:

$$\textit{Trend} \text{ is } \langle\text{direction}\rangle \tag{26}$$

[15] The pedantic characterization should be stated as "direction of trend", i.e., for example, "direction of trend is increasing". The real language, however, is not so strict, and so, we commonly say "the trend is increasing".

where

$$\langle\text{direction}\rangle := \textsf{stagnating}|\langle\text{hedge}\rangle\langle\text{sign}\rangle, \tag{27}$$

$$\langle\text{sign}\rangle := \textsf{increasing}|\textsf{decreasing} \tag{28}$$

and

$$\langle\text{hedge}\rangle := \emptyset|\textsf{negligibly}|\textsf{slightly}|\textsf{somewhat}|\textsf{clearly}|\textsf{roughly}|\textsf{sharply}|\textsf{significantly}.$$

The empty hedge $\emptyset$ is used if no more specific characterization of the direction of the trend is required.

(ii) *Characterization of increase (decrease) of trend*:

$$\langle\text{sign of trend}\rangle \text{ is } \langle\text{special hedge}\rangle \tag{29}$$

where $\langle\text{sign of trend}\rangle := \textsf{increase}|\textsf{decrease}$ and

$$\langle\text{special hedge}\rangle := \textsf{negligible}|\textsf{slight}|\textsf{small}|\textsf{clear}|\textsf{rough}|\textsf{large}|\textsf{fairly large}|\textsf{quite large}|\textsf{significant}|\textsf{huge}.$$

The expression (26) is more general because it also includes the possibility that the trend is stagnating. Otherwise, we can choose between (26) and (29) because the semantics of both expressions are the same.

This suggests the idea that both special evaluative predications (26) and (29) are semantically tantamount[16] to the standard form

$$\textit{Trend of } X|\bar{\mathbb{T}} \text{ is } \pm Ev[X|\bar{\mathbb{T}}] \tag{30}$$

where $Ev[X|\bar{\mathbb{T}}]$ an evaluative expression in which the TE-adjective is canonical (i.e., either of "small, medium, big" or "zero"). This leads to the following definition.

Definition 6. Let X be a time series (17) and $\bar{\mathbb{T}} \subset \mathbb{T}$ be a time interval. Let $\beta^1[X|\bar{\mathbb{T}}]$ be the coefficient (4) that provides estimation of the slope of trend of X over the period $\bar{\mathbb{T}}$. Finally, let $w_{tg}^- \sqcup w_{tg}^+$ be the corresponding context. Then, the evaluative expression $\pm Ev[X|\bar{\mathbb{T}}]$ in (30) is obtained as follows:

$$\pm Ev[X|\bar{\mathbb{T}}] := LPerc(\beta^1[X|\bar{\mathbb{T}}], w_{tg}^- \sqcup w_{tg}^+). \tag{31}$$

From (31) we can construct the intension of (30) and then its extension in the context of either w_{tg}^+ or w_{tg}^-, depending on the sign of the evaluative expression (31).

$$\text{Ext}_{w_{tg}}(\textit{Trend of } X|\bar{\mathbb{T}} \text{ is } \langle\text{direction}\rangle) \underset{\sim}{\subset} \mathbb{R}. \tag{32}$$

Recall that (32) is a fuzzy set of values of the slopes of $X|\bar{\mathbb{T}}$ estimated by (4). A few more details can be found in [31].

Generating linguistic evaluation of direction of trend

1. On the basis of Definition 6, generate the linguistic predication (30).
2. Check whether the trend is stagnating:

⟨direction⟩	short	$\pm Ev[X\|\bar{\mathbb{T}}]$ from (31)
stagnating	*Stag*	*zero*, $\pm$ *extremely small*

If yes, then finish. Otherwise, set ⟨sign⟩ of the direction of the trend according to the table below and then translate (30) either to (26) or to (29) according to 3. or 4.

[16] It is not clear whether the expressions (26) and (29) are indeed synonymous in the strict sense. We need further linguistic research to answer this question.

⟨sign⟩	short	sign of $Ev[X|\bar{\mathbb{T}}]$ from (31)
increasing	*Inc*	$+$
decreasing	*Dec*	$-$

3. Translation of (30) into (26): "*Trend of* $X|\bar{\mathbb{T}}$ is ⟨direction⟩" where

⟨direction⟩	short	$\pm Ev[X|\bar{\mathbb{T}}]$ from (31)
negligibly ⟨sign⟩	*Ne*	$\pm$ *significantly small*
slightly ⟨sign⟩	*Sl*	$\pm$ *very small*
somewhat ⟨sign⟩	*Sw*	$\pm$ *rather small*
clearly ⟨sign⟩	*Cl*	$\pm$ *medium or very roughly small*
roughly ⟨sign⟩	*Ro*	$\pm$ *very roughly big*
sharply ⟨sign⟩	*Sh*	$\pm$ *very big*
significantly ⟨sign⟩	*Si*	$\pm$ *significantly big*

This table is applied if more specific characterization of the degree of the trend slope is required.

4. Translation of (30) into (29): "⟨sign of trend⟩ is ⟨special hedge⟩" where the ⟨sign of trend⟩ is *increase* or *decrease* assigned analogously as in item 2. Furthermore, follow the table below:

⟨special hedge⟩	short	$Ev[X|\bar{\mathbb{T}}]$ from (31)
negligible	*Ne*	*significantly small*
slight	*Sl*	*very small*
small	*Sm*	*small*
clear	*Cl*	*very roughly small or medium*
rough	*Ro*	*very roughly big*
fairly large	*FL*	*roughly big*
quite large	*QL*	*rather big*
large	*La*	*big*
sharp	*Sh*	*very big*
significant	*Si*	*significantly big*
huge	*Hu*	*extremely big*

Remark 1. Let us remark that the translation tables above are considered for simplification. A pedantic approach would require explicit definition of the semantics of the proper expressions characterizing the trend (left columns of the tables). This can be done using the theory of evaluative expressions developed in [26], which is general enough to make it possible.

An example of the linguistic evaluation of the (direction of) trend generated using the method outlined above is in Figs. 3–5.[17] Fig. 3 demonstrates that the method indeed linguistically evaluates the direction of the trend in the given time slots as expected.

Fig. 4 demonstrates the evaluation of the trend in cases when the time series has a complicated course, and so, the direction of its trend may not be visually clear. We chose one such time series from the *M3-Competition* provided by the *International Institute of Forecasters* (the real content of the time series is not known). The linguistic context was set to $w_{tg} = \langle v_R = -3000/12, v_S = -1200/12, v_L = 0\rangle \sqcup \langle v_L = 0, v_S = 1200/12, v_R = 3000/12\rangle$ because the time series demonstrates clear periodicity of $T = 12$ (this was obtained using a periodogram). The slot 3 (time 116-127) is testing part of the time series that was never used for computation of the forecast but only for comparison of the forecasted values with the real ones. After applying our method for the generation of the evaluation of trend to the *predicted* values and also to the *real* values, we obtained in both cases the following evaluation:

[17] The results were obtained using the experimental software LFL Forecaster whose demo version is available at the web page http://irafm.osu.cz/.

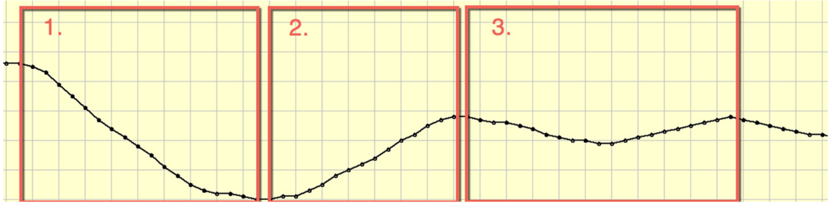

Fig. 3. Demonstration of evaluation of the direction of the trend in case that its slope is typical: Slot 1: *quite large decrease*, Slot 2: *rough increase*, Slot 3: *stagnating*.

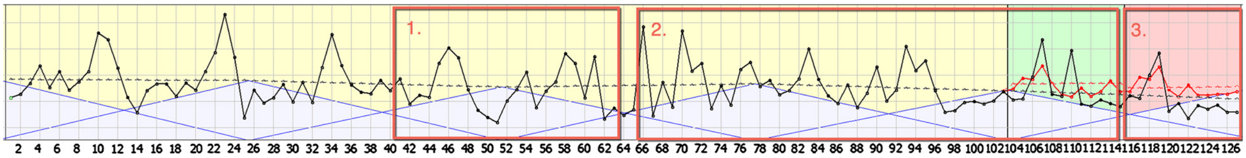

Fig. 4. Demonstration of evaluation of trend of various parts of a complicated time series. Trend of the whole series is *stagnating*. Slot 1 (time 41–63): *negligible increase*, Slot 2 (time 66–114): *slight decrease*, Slot 3 (time 116–127): both the real course and its forecast are evaluated as *clear decrease*.

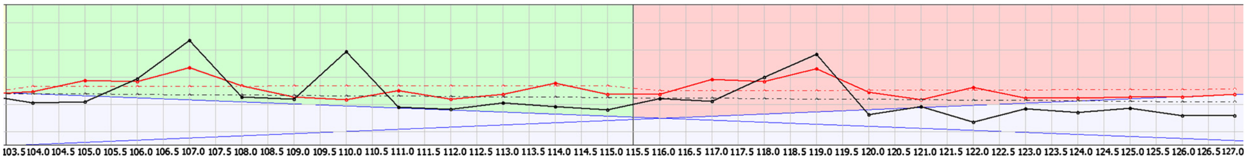

Fig. 5. Detail of the last part of the real time series and its forecast. The left side is the validation set (included in computation) and the right side is the testing set (not included in the computation). Both real and forecasted trends in both sets are evaluated as *clear decrease*.

Trend of $X|(\mathbb{T}' \setminus \mathbb{T})$ *is clearly decreasing.*

We see that both evaluations are in good agreement. We can take this result as further support for the quality of our forecasting method. A more detailed view of the real data and their forecast and also estimation of the real and forecasted trend-cycle, separately in the validation as well as testing set, are depicted in Fig. 5.

Let us remark that the trend-cycle was forecasted using the automatically generated linguistic description

Rule No.	$F_i[X]$	$F_{i-1}[X] \Rightarrow F_{i+1}[X]$
1	*ExBi*	*ExBi* $\Rightarrow$ *Ra Me*
2	*Ze*	*SiBi* $\Rightarrow$ *VRBi*
3	*Ra Me*	*Ze* $\Rightarrow$ *MLMe*

We argue that the linguistic evaluation of future course of the time series provides more information to a manager than concrete predicted numbers. In our case, the manager would be provided with the following information: "*clear decrease of values of time series is expected*". It must be emphasized that the forecast as well as its linguistic evaluation are obtained using well-founded mathematical methods.

4. Mining linguistic information from time series

In this section, we will describe how our methods can be applied to mining additional information from time series. The information is provided chiefly in sentences of natural language. The aim of this section is especially to outline further possibilities and to give inspiration for further research.

4.1. Initial processing

Let us consider a time series X from (17). We will divide $\mathbb{T}$ into m disjoint intervals so that $\bigcup\{\bar{\mathbb{T}}_1, \dots, \bar{\mathbb{T}}_m\} = \mathbb{T}$, where $\bar{\mathbb{T}}_j = \{t \mid t = (j-1)K, \dots, jK\}$ for some K (each $\bar{\mathbb{T}}_j$ has the length K). In case that p is not divisible by K, we must shorten $\mathbb{T}$ accordingly. It is natural to remove time moments from the beginning of $\mathbb{T}$ because the end of X is more important than its beginning. Therefore, we may assume that $\frac{p}{K} \in \mathbb{N}$.

The value of K (i.e., length of the intervals $\bar{\mathbb{T}}_j$) can be naturally set, for example, to 30 (one month), 7 (one week), etc. depending on the given time series. For the purpose of finding areas of monotonous trends, it is convenient to set K significantly smaller than p. Practical experience suggests setting $K \in \{4, 5\}$.

4.2. Behavior of time series

4.2.1. Finding intervals with monotonous behavior

In this subsection, we will suggest a special solution to the general problem of time series segmentation (cf. [8]). Our goal is to find time intervals in which the trend of the time series X exhibits monotonous behavior. This means that we must decompose the time domain $\mathbb{T}$ into a set of time intervals $\mathbb{T}_i \subseteq \mathbb{T}$, $i = 1, \dots, s$, with monotonous trend of X (increasing, decreasing, stagnating). Each interval $\mathbb{T}_i$ is a union of one or more adjacent intervals $\bar{\mathbb{T}}_j$. As a final result, direction of the trend of $X|\mathbb{T}_i$ is linguistically evaluated in the form (26) or (29).

Algorithm 1 enables us to find intervals $\mathbb{T}_i$ in which the trend of X is monotonous. Its idea is simple: Starting with the last interval $\bar{\mathbb{T}}_m$, we extend the candidate for the time slot $\mathbb{T}_i^{(k+1)}$ to the left and test whether sign of the slope of $X|\mathbb{T}_i^{(k+1)}$ either has not changed or is different from the sign of the slope of the following $X|\bar{\mathbb{T}}_{m-1}$. In the latter case, we finish with the previous $\mathbb{T}_i^{(k)}$, generate the linguistic evaluation of $X|\mathbb{T}_i^{(k)}$ and continue searching the next interval $\mathbb{T}_{i+1}$ to the left.

Let $\bar{T} \subseteq \mathbb{T}$ be a time interval. Then we put:

$$\mathrm{Sign}(\mathit{Tr}(X|\bar{\mathbb{T}})) = \begin{cases} 1, \text{if } \mathit{Trend\ of}\ X|\bar{\mathbb{T}} \text{ is } \mathit{Inc}, \\ 0, \text{if } \mathit{Trend\ of}\ X|\bar{\mathbb{T}} \text{ is } \mathit{Stag}, \\ -1, \text{if } \mathit{Trend\ of}\ X|\bar{\mathbb{T}} \text{ is } \mathit{Dec}. \end{cases} \tag{33}$$

The result of Algorithm 1 is the set of disjoint time intervals

$$\mathscr{T} = \{\mathbb{T}_i \mid i = 1, \dots, s\}, \qquad \bigcup \mathscr{T} = \mathbb{T} \tag{34}$$

with monotonous trend of X.

4.2.2. Linguistic characterization of time series

Combining the procedures described above, we can automatically generate the following linguistic characterization of the time series X:

(i) Linguistic evaluation of direction of the global trend of the whole time series X in the form (26) or (29). This is obtained from (30)

$$\mathit{Trend\ of}\ X|\mathbb{T} \text{ is } \mathit{Ev}[X|\mathbb{T}].$$

(ii) A list of linguistic evaluations of direction of the local trends of X in the time intervals $\mathbb{T}_i$ found using Algorithm 1:

$$\mathit{Trend\ of}\ X|\mathbb{T}_i \text{ is } \mathit{Ev}[X|\mathbb{T}_i], \qquad i = 1, \dots, s.$$

(iii) Statements such as "In the first (second, third, fourth) quarter of the year, the trend was slightly decreasing (increasing, stagnating)" are obtained from $\mathit{Trend\ of}\ X|\bar{\mathbb{T}}$ is $\mathit{Ev}[X|\bar{\mathbb{T}}]$, where $\bar{\mathbb{T}}$ is the considered period.

(iv) The longest period of *stagnating* trend lasted $\mathbb{T}_i$ time moments (days, weeks, months, etc.), where $\mathbb{T}_i$ is the longest of the intervals $\mathbb{T}_1, \dots, \mathbb{T}_s$, for which $\mathrm{Sign}(\mathit{Tr}(X|\mathbb{T}_i)) = 0$ (or $1, -1$).

Algorithm 1 Finding intervals, in which trend of the time series X is monotonous

$j := m; i := 1; k := 0; \mathbb{T}_i^{(k)} := \bar{\mathbb{T}}_m;$
repeat
  $\mathbb{T}_i^{(k+1)} := \bar{\mathbb{T}}_{j-1} \cup \mathbb{T}_i^{(k)};$
  if $\mathrm{Sign}(Tr(X|\mathbb{T}_i^{(k+1)})) \neq \mathrm{Sign}(Tr(X|\bar{\mathbb{T}}_{j-1}))$ **or** $\mathrm{Sign}(Tr(X|\mathbb{T}_i^{(k+1)})) \neq \mathrm{Sign}(X|\,Tr(\mathbb{T}_i^{(k)}))$ **then**
    $\mathbb{T}_i := \mathbb{T}_i^{(k)};$
    $Ev[X|\mathbb{T}_i] := LPerc(\beta^1[X|\mathbb{T}_i], w_{tg}^- \sqcup w_{tg}^+);$
    print *Trend of* $X|\mathbb{T}_i$ is $Ev[X|\mathbb{T}_i];$
    $k := 0;$
    $i := i + 1;$
    $j := j - 1;$
    $\mathbb{T}_i^{(k)} := \bar{\mathbb{T}}_j;$
    if $j = 1$ **then**
      $\mathbb{T}_i := \mathbb{T}_i^{(k)};$
      print *Trend of* $X|\mathbb{T}_i$ is $Ev[X|\mathbb{T}_i];$
    end if
  else
    $k := k + 1;$
    $j := j - 1;$
  end if
until $j = 1;$
$s := i$

(v) The longest period of slightly increasing, decreasing, etc. trend lasted $\mathbb{T}_i$ time moments (days, weeks, months, etc.) where $\mathbb{T}_i$ is the longest of the intervals $\mathbb{T}_1, \ldots, \mathbb{T}_s$, for which the sentence "*Trend of* $X|\mathbb{T}_i$ is *Sl Inc*", etc. is generated.

There are more possibilities for generating linguistic information about time series, which the reader will surely be able to suggest.

4.2.3. Example

The above algorithm was tested on a real and fairly volatile time series.[18] The width of the intervals $\bar{\mathbb{T}}_1, \ldots, \bar{\mathbb{T}}_m$ is $K = 5$. The context for the slope (tangent) is $w_{tg} = \langle -800/12, -320/12, 0\rangle \sqcup \langle 0, 320/12, 800/12\rangle$.

Evaluation of the trend of the whole original time series is: *somewhat decrease*. Furthermore, Algorithm 1 found 9 intervals with a monotonous trend:

interval	$Ev[X\|\mathbb{T}_i]$	interval	$Ev[X\|\mathbb{T}_i]$
$\mathbb{T}_1 = \{70, \ldots, 125\}$	*clear increase*	$\mathbb{T}_6 = \{30, \ldots, 35\}$	*huge decrease*
$\mathbb{T}_2 = \{65, \ldots, 70\}$	*somewhat increase*	$\mathbb{T}_7 = \{25, \ldots, 30\}$	*clear increase*
$\mathbb{T}_3 = \{55, \ldots, 65\}$	*huge increase*	$\mathbb{T}_8 = \{15, \ldots, 25\}$	*clear decrease*
$\mathbb{T}_4 = \{40, \ldots, 55\}$	*huge decrease*	$\mathbb{T}_9 = \{1, \ldots, 15\}$	*clear decrease*
$\mathbb{T}_5 = \{35, \ldots, 40\}$	*huge increase*		

The time series and intervals of monotonous trend are depicted in Fig. 6. Note that, because of the volatility of the time series, it may happen that though adjacent subintervals demonstrate each, e.g., decrease, their join can demonstrate a slight increase. This happened in the case of the intervals $\mathbb{T}_9$ and $\mathbb{T}_8$, where the direction of the trend of each of them is "clearly decreasing" but the direction of the trend in the interval $\{1, \ldots, 25\}$ (join of $\mathbb{T}_9$ and $\mathbb{T}_8$) is "somewhat increasing".

[18] The time series is denoted by NN105 and is taken from the INDUSTRY subset of time series on a monthly basis from the M3-Competition published on the Internet.

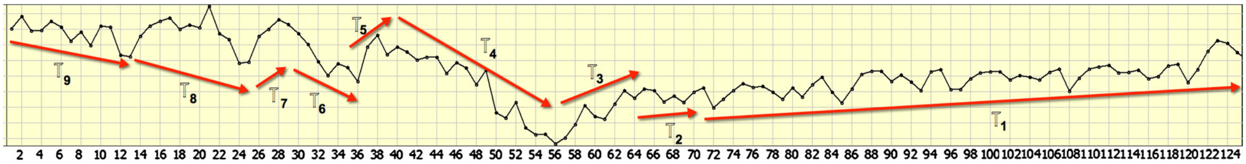

Fig. 6. Determination of intervals with monotonous behavior on the original values of the time series NN105.

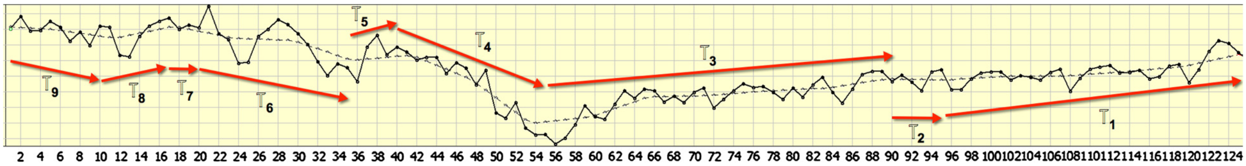

Fig. 7. Determination of intervals with monotonous behavior of the trend of the time series NN105.

To avoid the influence of outlying values, we can search monotonous intervals on the basis of the trend-cycle *TC*. The generated linguistic expression (31) is then replaced by $Ev[TC\,|\,\mathbb{T}_i]$. The following table contains the found intervals and the linguistic evaluation of the direction of trend on the basis of *TC*.

interval	$Ev[TC\,|\,\mathbb{T}_i]$	interval	$Ev[TC\,|\,\mathbb{T}_i]$
$\mathbb{T}_1 = \{95, \dots, 125\}$	*clear increase*	$\mathbb{T}_6 = \{20, \dots, 35\}$	*clear decrease*
$\mathbb{T}_2 = \{90, \dots, 95\}$	*negligible increase*	$\mathbb{T}_7 = \{15, \dots, 20\}$	*stagnating*
$\mathbb{T}_3 = \{55, \dots, 90\}$	*clear increase*	$\mathbb{T}_8 = \{10, \dots, 15\}$	*clear increase*
$\mathbb{T}_4 = \{40, \dots, 55\}$	*huge decrease*	$\mathbb{T}_9 = \{1, \dots, 10\}$	*clear decrease*
$\mathbb{T}_5 = \{35, \dots, 40\}$	*clear increase*		

The result is depicted in Fig. 7. Note that intervals $\mathbb{T}_1$ and $\mathbb{T}_3$ are interrupted by the short interval $\mathbb{T}_2$, demonstrating only a negligible increase. However, the evaluation of the direction of trend in the whole interval $\{55 \dots, 125\}$ (join of $\mathbb{T}_1$, $\mathbb{T}_2$ and $\mathbb{T}_3$) is "clear increase". Similarly, the evaluation of the trend in the whole interval $\{1 \dots, 55\}$ is "clear decrease".

We see that using the F^1-transform, we are able to detect the direction of the trend independently of how well it is visible in the graph. More detailed division into shorter subintervals depends on the length of the initial intervals $\bar{\mathbb{T}}_j$. Of course, the longer they are, the less the algorithm is sensitive to small deviations and volatility. Note also that evaluation of the direction of the trend of the whole original time series is *somewhat decrease*.

4.3. *Summarization*

Automatic summarization of knowledge about time series belongs among the interesting tasks that were addressed by several authors (see, e.g., [3,4,14]). The main concept in these papers is that of "linguistic label", which is the name for a fuzzy set being part of a partition of the universe.

In this paper, we suggest applying the sophisticated formal theory of *intermediate quantifiers* developed as a constituent of fuzzy natural logic (see Subsection 2.2.3). The summarized information may address either one time series or a set of time series. The theory includes a formalism on the basis of which we can develop a model of the meaning of linguistic statements containing quantified information, as is usual in natural language, but also human-like syllogistic reasoning that is based on the formal model of generalized Aristotle's syllogisms.

In the next sections, we will outline the main ideas and the way the quantified linguistic statements can be generated. A precise and detailed description would require introducing the formal system of higher-fuzzy logic. This will be the topic of some of the subsequent papers.

4.3.1. *One time series*

We assume that we have at disposal the set $\mathscr{T}$ from (34) obtained using Algorithm 1 and a set of evaluations $Ev[X|\mathbb{T}_i]$, $T_i \in \mathscr{T}$.

Let us now consider the following linguistic statement:

In *most* periods of monotonous behavior, the trend of time series X was *slightly increasing*. (*)

Using the theory of intermediate quantifiers, this proposition can be formalized by the formula

$$(Q^{\forall}_{BiVe}\mathbb{T}_i)(SlInc[X|\mathbb{T}_i]) \tag{35}$$

where $Q^{\forall}_{BiVe}$ is the quantifier **T** (formalization of "most") from Section 2.2.3 restricted to type $\langle 1\rangle$ (cf. definition (14)).

The proposition (*) has a truth value that can be computed on the basis of (35) as follows: The universe is the set $\mathscr{T}$. Furthermore, specify the context w_{tg}, and for all $\mathbb{T}_i \in \mathscr{T}$, find the truth value $\|SlInc[X|\mathbb{T}_i]\|$.[19] The next step is to determine the support $\mathrm{Supp}(SlInc[X|\mathbb{T}_i])$, compute the relative cardinality (16)

$$(\mu V)(SlInc[X|\bar{\mathbb{T}}_i]) = \frac{\sum_{i=1}^{s}\|SlInc[X|\mathbb{T}_i]\|}{s}$$

and determine the truth value $\|BiVe((\mu V)SlInc[X|\mathbb{T}_i])\|$ (i.e., the truth value of the statement "the relative cardinality $(\mu V)(SlInc[X|\mathbb{T}_i])$ is *very big*") in the standard context $w = \langle 0, 0.5, 1\rangle$. The truth value of (35) is finally obtained as

$$\min\{\|SlInc[X|\mathbb{T}_i]\|, \|BiVe((\mu V)(SlInc[X|\mathbb{T}_i]))\|\}. \tag{36}$$

Similarly, we should compute truth values of the other formulas, for example, $\|(Q^{\forall}_{Ev}\mathbb{T}_i)Stag(\mathbb{T}_i)\|$, $\|(Q^{\forall}_{Ev}\mathbb{T}_i)Inc(\mathbb{T}_i)\|$, $\|(Q^{\forall}_{Ev}\mathbb{T}_i)Dec(\mathbb{T}_i)\|$, where *Ev* is one of the evaluative expressions *very big, extremely big* or *not small*. The statement (*) is generated if the truth value (36) is the greatest among all the other ones.

This is the principle of how statements of the form (*) can be generated. There are many other possibilities of how summarizing linguistic statements can be generated about one time series. Below is a (non-exhaustive) list of a few of them:

- In *few* (many, almost all) periods of monotonous behavior, the increase of the trend is *small* (very small, medium, sharp, etc.). Formally:

  $$(Q^{\forall}_{BiEx}\mathbb{T}_i)(Inc(\mathbb{T}_i), \neg Sm(\mathbb{T}_i)),$$

  etc.
- In *almost all* (most, many, few) periods, *increasing* (decreasing) trend is followed by *sharply* (very slightly, clearly) *decreasing* (increasing) one. Formally:

  $$(Q^{\forall}_{BiEx}\mathbb{T}_i)(Inc(\mathbb{T}_i), ShDec(\mathbb{T}_{i+1})),$$

  etc.
- The trend was *often stagnating*, (increasing, very little decreasing), etc. in the past *n* years. The meaning of *often* can be formally modeled by the quantifier $(Q^{\forall}_{Bi}\mathbb{T}_i)$.
- *Short* periods of *sharp increase* of trend were *often* followed by *very long periods* of *stagnation*.

4.3.2. A set of time series

Using the methods outlined above and the theory of intermediate quantifiers, we can generate summarizing linguistic statements about sets of time series. Without going into formal details, we only give here the following (non-exhaustive) list of possibilities:

- *Most* (many, few) analyzed time series stagnated recently but their future trend is *slightly increasing*.
- There is evidence of a *huge (slight, clear)* decrease in trend of *almost all* time series in the recent quarter of the year.
- *Most* (many, few) analyzed time series *stagnated* recently but their expected trend is *slightly increasing*.

[19] This truth value is obtained from the extension (32). More details require precise formalization in higher-order fuzzy logic, which is out of the scope of this paper.

- Trend of *most* of the inspected time series in the past 3 months was *sharply increasing*.
- *Many* time series are expected to be *stagnating* in the following 6 months.
- During the last year, *few* time series *sharply increased* more than once.

Then, we generate comments for interesting time intervals, or we can also determine intervals giving answer to questions, for example, “in which period was the time series *sharply increasing*”, “how long was the time series *stagnating or decreasing before sharp increase*”, etc.

4.4. Syllogistic reasoning with time series

Summarization using intermediate quantifiers suggests the idea of applying the theory of syllogisms mentioned in Subsection 2.2.3. In this section, we will only give a few examples of valid syllogisms in four different figures (I–IV). Recall that the first line is the major premise (P_1), the second one is the minor premise (P_2) and the third line is the conclusion (C).

$$\textbf{ATT-I}\quad \begin{array}{l} \textit{All carefully inspected time series are financial} \\ \textit{Most available time series are carefully inspected} \\ \hline \textit{Most available time series are financial} \end{array}$$

$$\textbf{ETO-II}\quad \begin{array}{l} \textit{In no period was there an increasing time series from the car industry} \\ \textit{In most periods there was an inspected time series from the car industry} \\ \hline \textit{In some periods, the inspected time series was not increasing} \end{array}$$

$$\textbf{BKO-III}\quad \begin{array}{l} \textit{In few cases, the increase of time series is small} \\ \textit{In many cases, the increase of time series is clear} \\ \hline \textit{In some cases, the clear increase of time series is not small} \end{array}$$

$$\textbf{PPI-III}\quad \begin{array}{l} \textit{Almost all periods when TS is decreasing occurred during the past 3 years} \\ \textit{Almost all periods when TS is decreasing are short} \\ \hline \textit{Some short periods when TS is decreasing occurred during the past 3 years} \end{array}$$

$$\textbf{TAI-IV}\quad \begin{array}{l} \textit{Most financial time series were sharply decreasing last month} \\ \textit{All sharply decreasing time series last month are now stagnating} \\ \hline \textit{Some time series that are now stagnating are financial} \end{array}$$

Translation of the above syllogisms into concrete formulas of the formal theory of intermediate quantifiers is a more or less straightforward task. Still, there are various subtle problems that should be solved. We leave the details and careful formal elaboration of this task to the next paper.

5. Conclusion

In this paper, we discussed applications of special soft computing methods, namely, the F-transform and fuzzy natural logic, on the basis of which it is possible to automatically generate information about time series in sentences of natural language. The F-transform is used mainly for extraction of the trend-cycle and trend and for estimation of the slope of trend in arbitrary time intervals. The methods based on the results of the fuzzy natural logic are applied to the generation of linguistic description characterizing the direction of the trend of time series, to forecasting of its future development, and to generation of linguistic comments characterizing its trend in arbitrary time intervals. A combination of these methods and the mentioned results can be used for the mining of interesting information from the time series and also for the mining of information from sets of them. All the mined information is provided in sentences of natural language.

There are various problems left to be solved in the future. First, application of the theory of intermediate quantifiers and their syllogisms to time series needs to be elaborated in detail. The work on this topic also reveals the necessity of analyzing in more detail the semantics of special subclasses of evaluative expressions to be able to develop a more realistic mathematical model of it. Another problem is the segmentation of time series discussed in [8]. Our algorithm for segmentation of time series introduced in Subsection 4.2 focused only on finding intervals with monotonous behavior. In time series, however, we can find more complicated patterns (see, e.g., [10]) that can be identified and used for more precise description of the time series behavior and also for its forecast. Last but not least, we must elaborate in detail the formalization of methods for summarization of information from time series on the basis of the theory of intermediate quantifiers and their syllogisms. All these problems will be discussed in future papers.

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