Streaming Algorithm: Filtering & Counting Distinct Elements

CompSci 590.02
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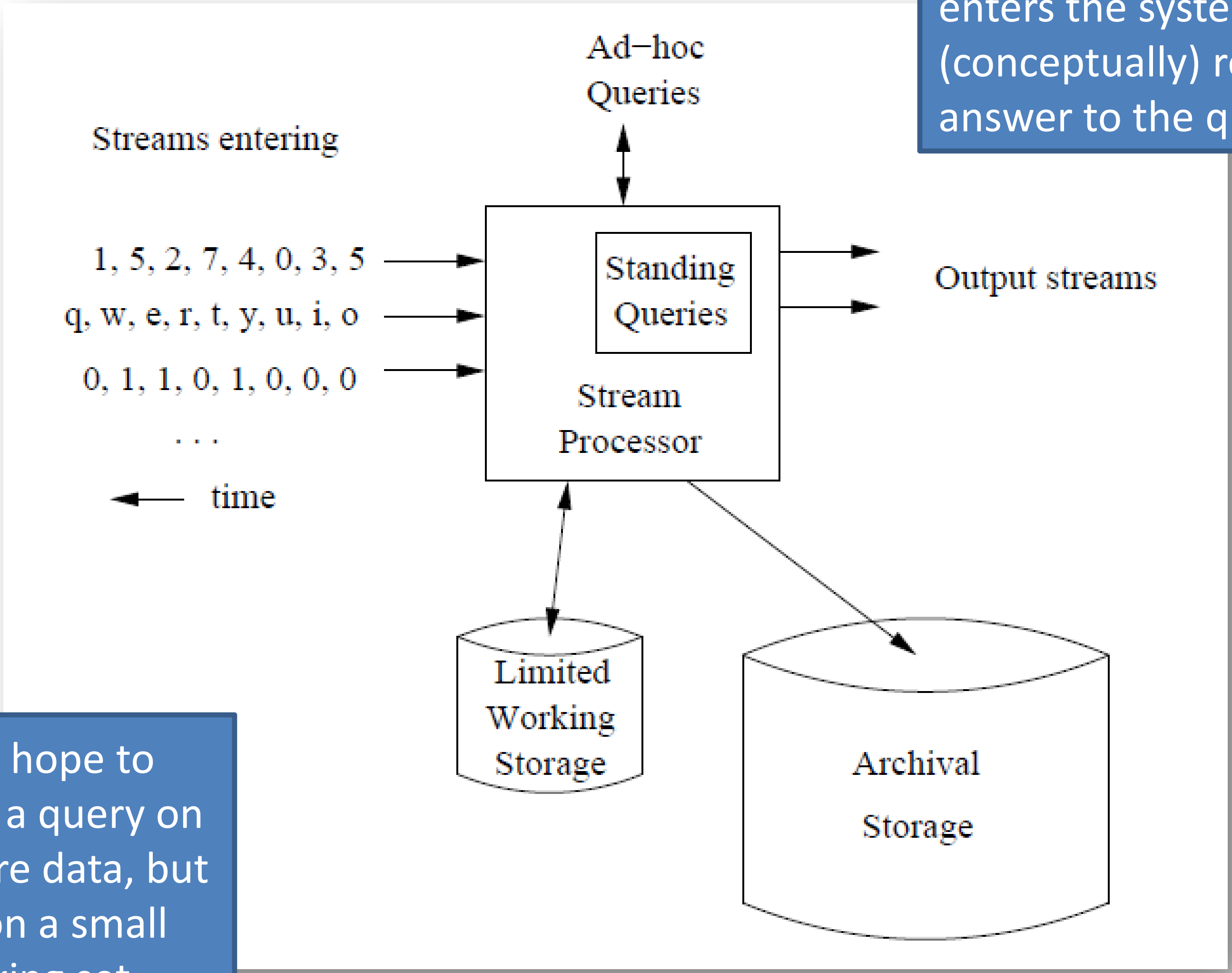
Streaming Databa

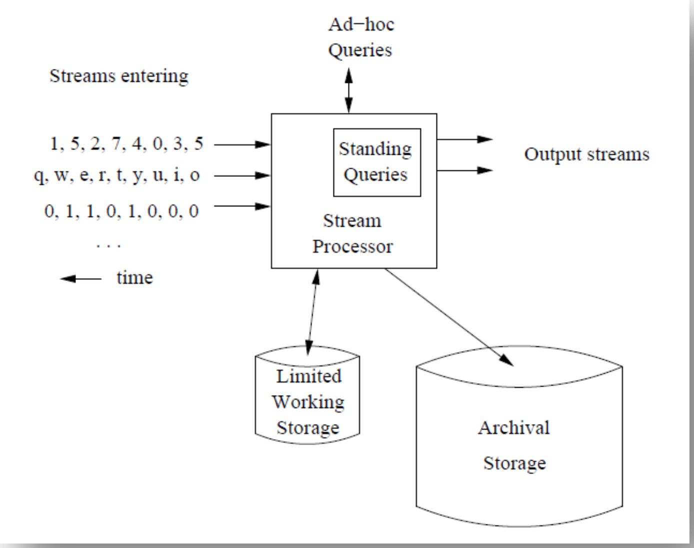
Continuous/Standing Queries: Every time a new data item enters the system, (conceptually) re-evalutate the answer to the query



Can't hope to process a query on the entire data, but only on a small working set.

Examples of Streaming Data

- Internet & Web traffic
  - Search/browsing history of users: Want to predict which ads/content to show the user based on their history.
    Can't look at the entire history at runtime
- Continuous Monitoring
  - 6 million surveillance cameras in London
  - Video feeds from these cameras must be processed in real time
- Weather monitoring
- …



Processing Streams

- Summarization
  - Maintain a small size *sketch* (or summary) of the stream
  - Answering queries using the sketch
  - E.g., random sample
  - *later in the course* – AMS, count min sketch, etc
  - Types of queries: # distinct elements, most frequent elements in the stream, aggregates like sum, min, max, etc.

- Window Queries
  - Queries over a recent k size window of the stream
  - Types of queries: alert if there is a burst of traffic in the last 1 minute, denial of service identification, alert if stock price > 100, etc.

Streaming Algorithms

- Sampling
  - We have already seen this.
- Filtering
  - "… does the incoming email address appear in a set of white listed addresses … "
- Counting Distinct Elements
  - "… how many unique users visit cnn.com …"
- Heavy Hitters
  - "… news articles contributing to >1% of all traffic …"
- Online Aggregation
  - "… Based on seeing 50% of the data the answer is in [25,35] …"

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This Class

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FILTERING

Problem

- A set S containing *m* values
  - A whitelist of a billion non-spam email addresses

- Memory with *n* bits.
  - Say 1 GB memory

- Goal: Construct a data structure that can efficient check whether a new element is in S
  - Returns TRUE with probability 1, when element is in S
  - Returns FALSE with high probability (1-ε), when element is not in S

Bloom Filter

- Consider a set of hash functions $\{h_1, h_2, .., h_k\}$, h_i: S → [1, n]

Initialization:

- Set all *n* bits in the memory to 0.

Insert a new element 'a':

- Compute $h_1(a)$, $h_2(a)$, …, $h_k(a)$. Set the corresponding bits to 1.

Check whether an element 'a' is in S:

- Compute $h_1(a)$, $h_2(a)$, …, $h_k(a)$.
  If all the bits are 1, return TRUE.
  Else, return FALSE

Analysis

If a is in S:

- If $h_1(a)$, $h_2(a)$, …, $h_k(a)$ are all set to 1.
- Therefore, Bloom filter returns TRUE with probability 1.

If a not in S:

- Bloom filter returns TRUE if each hi(a) is 1 due to some other element

Pr[bit j is 1 after m insertions] = 1 – Pr[bit j is 0 after m insertions]

= 1 – Pr[bit j was not set by k x m hash functions]

$= 1 - (1 - 1/n)^{km}$

Pr[Bloom filter returns TRUE] $= \{1 - (1 - 1/n)^{km}\}^k\} \approx (1 - e^{-km/n})^k$

Example

- Suppose there are $m = 10^9$ emails in the white list.
- Suppose memory size of 1 GB (8×10^9 bits)

k = 1

- Pr[Bloom filter returns TRUE | a not in S] $= 1 - e^{-m/n}$

$$= 1 - e^{-1/8} = 0.1175$$

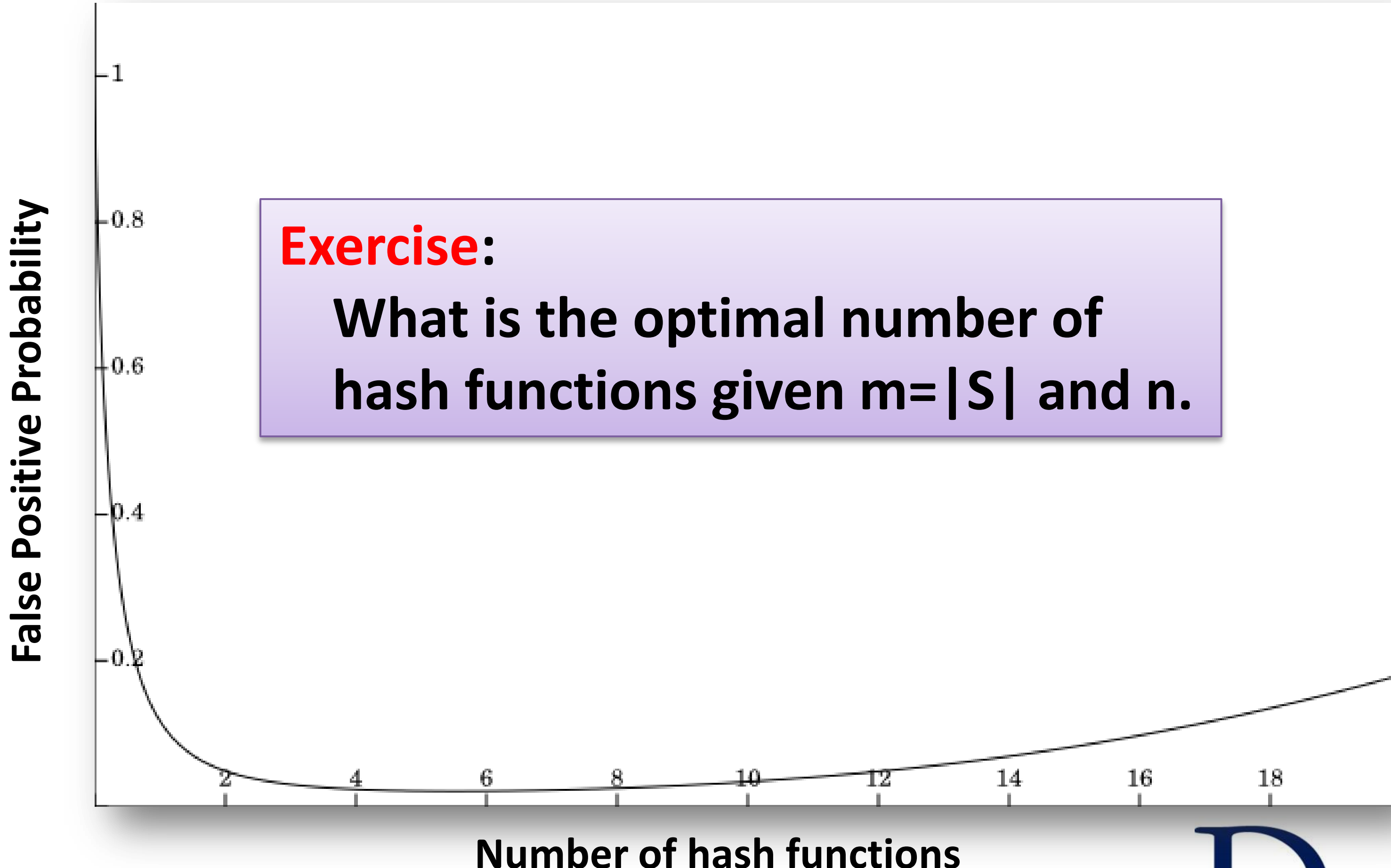
k = 2

- Pr[Bloom filter returns TRUE | a not in S] $= (1 - e^{-2m/n})^2$

$$= (1 - e^{-1/4})^2 \approx 0.0493$$

Example

- Suppose there are $m = 10^9$ emails in the white list.
- Suppose memory size of 1 GB (8×10^9 bits)



Exercise:
What is the optimal number of hash functions given m=|S| and n.

Summary of Bloom Filters

- Given a large set of elements S, efficiently check whether a new element is in the set.

- Bloom filters use hash functions to check membership
  - If a is in S, return TRUE with probability 1
  - If a is not in S, return FALSE with high probability
  - False positive error depends on |S|, number of bits in the memory and number of hash functions

COUNTING DISTINCT ELEMENTS

Distinct Elements

INPUT:

- A stream S of elements from a domain D
  - A stream of logins to a website
  - A stream of URLs browsed by a user
- Memory with n bits

OUTPUT

- An estimate of the number of distinct elements in the stream
  - Number of distinct users logging in to the website
  - Number of distinct URLs browsed by the user

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FM-sketch

- Consider a hash function $h: D \rightarrow \{0,1\}^L$ which uniformly hashes elements in the stream to L bit values

- IDEA: The more distinct elements in S, the more distinct hash values are observed.

- Define: $Tail_0(h(x))$ = number of trailing consecutive 0's
  - $Tail_0(101001) = 0$
  - $Tail_0(101010) = 1$
  - $Tail_0(001100) = 2$
  - $Tail_0(101000) = 3$
  - $Tail_0(000000) = 6$ (=L)

FM-sketch

Algorithm

- For all $x \in S$,
  - Compute $k(x) = Tail_0(h(x))$
- Let $K = \max_{x \in S} k(x)$
- Return $F' = 2^K$

Analysis

Lemma: Pr[$Tail_0(h(x)) \geq j$] = 2^{-j}

Proof:

- $Tail_0(h(x)) \geq j$ implies at least the last j bits are 0
- Since elements are hashed to L-bit string uniformly at random, the probability is $(½)^j = 2^{-j}$

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Analysis

- Let F be the true count of distinct elements, and let c>2 be some integer.

- Let k_1 be the largest k such that $2^k < cF$
- Let k_2 be the smallest k such that $2^k > F/c$

- If K (returned by FM-sketch) is between k_2 and k_1, then

$$F/c \leq F' \leq cF$$

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Analysis

- Let $z_x(k) = 1$ if $\text{Tail}_0(h(x)) \geq k$
  $= 0$ otherwise
- $E[z_x(k)] = 2^{-k}$ $\quad Var(z_x(k)) = 2^{-k}(1 - 2^{-k})$
- Let $X(k) = \Sigma_{x \varepsilon S}\, z_x(k)$
- We are done if we show with high probability that
  $X(k1) = 0$ and $X(k2) \neq 0$

Analysis

Lemma: $\Pr[X(k_1) \geq 1] \leq 1/c$

Proof: $\Pr[X(k_1) \geq 1] \leq E(X(k_1))$ *Markov Inequality*

$= F\,2^{-k1} \leq 1/c$

Lemma: $\Pr[X(k2) = 0] \leq 1/c$

Proof: $\Pr[X(k2) = 0] = \Pr[X(k2) - E(X(k2)) = E(X(k2))]$

$\leq \Pr[|X(k2) - E(X(k2))| \geq E(X(k2))]$

$\leq \mathrm{Var}(X(k2)) / E(X(k2))^2$ *Chebyshev Ineq.*

$\leq 2^{k2}/F \leq 1/c$

Theorem: If FM-sketch returns F', then for all c > 2,

$F/c \leq F' \leq cF$ with probability 1-2/c

Boosting the success probability

- Construct s independent FM-sketches (F'_1, F'_2, …, F'_s)
- Return the median F'_{med}

Q: For any δ, what is the value of s s.t. $P[F/c \le F'_{med} \le cF] > 1 - \delta$?

Analysis

- Let $c > 4$, and $x_i = 0$ if $F/c \leq F'_i \leq cF$, and 1 otherwise
- $\rho = E[x_i]$
  $= 1 - \Pr[F/c \leq F'_i \leq cF] \leq 2/c < ½$

- Let $X = \Sigma_i x_i$ $E(X) = s\rho$

Lemma: If $X < s/2$, then $F/c \leq F'_{med} \leq cF$ (Exercise)

We are done if we show that $\Pr[X \geq s/2]$ is small.

Analysis

$$
\begin{aligned}
\Pr[\, X \geq s/2 \,] &= \Pr[\, X - E(X) = s/2 - E(X) \,] \\
&\leq \Pr[\, |X - E(X)| \geq s/2 - s\rho \,] \\
&= \Pr[\, |X - E(X)| \geq (1/2\rho - 1)\, s\rho \,] \\
&\leq 2\exp(-(1/2\rho - 1)^2\, s\rho/3\,) \qquad \textit{Chernoff bounds}
\end{aligned}
$$

Thus, to bound this probability by δ, we need s to be:

$$s \geq \frac{3\rho}{\left({}^{1}\!/_{2} - \rho\right)^2} \ln\left(\frac{2}{\delta}\right)$$

Boosting the success probability

In practice,

- Construct sk independent FM sketches
- Divide the sketches into s groups of k each
- Compute the mean estimate in each group
- Return the median of the means.

Summary

- Counting the number of distinct elements exactly takes O(N) space and Ω(N) time, where N is the number of distinct elements
- FM-sketch estimates the number of distinct elements in O(log N) space and Θ(N) time
- FM-sketch: maximum number of trailing 0s in any hash value
- Can get good estimates with high probability by computing the median of many independent FM-sketches.