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LOCALLY COMPACT GROUPS WITHOUT DISTINCT ISOMORPHIC CLOSED SUBGROUPS[1]

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ABSTRACT. In this note the structure of those locally compact topological groups which do not contain distinct isomorphic closed subgroups is determined.

In [8], Szele showed that the subgroups of the rational circle (denoted by Q/Z) are characterized within the class of all groups by the property that they do not contain distinct isomorphic subgroups. It is the purpose of this note to investigate the corresponding property within the class of locally compact topological groups (which we always assume to be Hausdorff).

We call a topological group an *S-group* if and only if it contains no two distinct closed subgroups which are topologically isomorphic. Thus Szele's result says that the discrete S-groups are precisely the subgroups of Q/Z. In what follows we will first consider the locally compact abelian (LCA) S-groups, after which we will pass to the case of a general locally compact group.

1. **The locally compact abelian case.** The discrete abelian groups which we mention frequently are the integers Z, the rationals Q, the cyclic groups $Z(n)$ of order n, and the quasicyclic groups $Z(p^\infty)$ where p is a prime. The full direct product (with the product topology) of the LCA groups G_i, where i runs over an index set I, is denoted by $\mathrm{P}_{i\in I} G_i$. The local direct product of the LCA groups G_i relative to the compact open subgroups O_i of G_i will be indicated by $\mathrm{LP}_{i\in I}(G_i:O_i)$ (see [5, 6.16] for the definition of this group and its topology). Our index set I will always be the set $\mathscr{P}$ of primes. Throughout, topological isomorphism will be denoted by "$\cong$".

We first make the preliminary observation that, since every closed subgroup of an S-group is again an S-group and since Z is not an S-group, every element of a locally compact S-group must be compact (see [5, 9.1

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[1] We are thankful to the referee for pointing out to us the feasibility of including the nonabelian groups in our discussion.

and 9.9]). Therefore the character group $\hat{G}$ of an LCA S-group G is totally disconnected [**5**, 24.17]. This fact will be useful in the sequel.

Let us now describe a class of LCA S-groups which will figure in later developments.

PROPOSITION 1. *Let G be the local direct product over the set $\mathcal{P}$ of primes of the discrete groups $Z(p^{n_p})$ (where n_p is a nonnegative integer or ∞) with respect to the compact open subgroups $Z(p^{m_p})$ (where m_p is a nonnegative integer). Then G is an LCA S-group.*

PROOF. It follows from the definition of local direct products that G is locally compact. Now let H_1 and H_2 be closed subgroups of G with $H_1 \simeq H_2$. We must show that $H_1 = H_2$. Now G is a topological torsion group. (That is, $\lim_{n\to\infty} (n!)x = 0$ for each x in G; see [**6**, 3.1]. This is equivalent to the condition that both G and $\hat{G}$ be totally disconnected.) Hence H_1 and H_2 are also topological torsion groups, so by [**6**, 3.21] or by [**2**, III, 1, Théorème 1], both H_1 and H_2 may be written as the local direct product of their p-summands: $H_i \simeq \mathrm{LP}_{p\in\mathcal{P}} (H_p^i : O_p^i)$ where O_p^i is a compact open subgroup of H_p^i, which is in turn a subgroup of $Z(p^{n_p})$, for each prime p and $i = 1, 2$. By [**2**, III, 1, Proposition 4] there exists, for each p, a topological isomorphism f_p from H_p^1 onto H_p^2 such that $f_p(O_p^1) = O_p^2$ for all but a finite number of primes p. Since the groups O_p^i and H_p^i are all isomorphic to subgroups of $Z(p^\infty)$, we conclude that $H_p^1 = H_p^2$ for all p and that $O_p^1 = O_p^2$ for all but a finite number of primes p. Hence, by the definition of local direct product, $H_1 = H_2$, proving that G is an S-group.

With the help of this proposition we can already give many examples of nondiscrete LCA S-groups, both compact and noncompact. We now seek to describe all the compact abelian S-groups. In point of fact, these are already completely known, via duality. To be explicit, let us call an LCA group G an *S^*-group* if and only if from $G/H_1 \simeq G/H_2$, where H_1 and H_2 are closed subgroups of G, it follows that $H_1 = H_2$. It is a simple matter to verify that an LCA group G is an S^*-group if and only if its character group $\hat{G}$ is an S-group. In particular it follows immediately that every quotient of an S^*-group by a closed subgroup is also an S^*-group. Now the discrete S^*-groups are completely known:

LEMMA 1. *Let G be a discrete abelian group. Then G is an S^*-group if and only if either* (1) *G is isomorphic to a reduced subgroup of Q/Z, or* (2) *G is isomorphic to a subgroup of Q all of whose elements have finite p-height for each prime p.*

PROOF. The statement of the lemma occurs as Exercise 25 on p. 202 of [**3**]. We outline a proof as follows. First observe that G can have no quotients of the form $Z(p^\infty)$ or $Z(p) \times Z(p)$. If G is a p-group, it must be indecomposable, so $G \simeq Z(p^n)$ for some n (see [**3**, Corollary 24.4]). Hence, if G is a torsion group, it must be a direct sum of groups $Z(p^n)$ and therefore a reduced subgroup of Q/Z. Using our first observation above it is not hard to see that if G is not torsion it must be torsion-free and have the form (2) of the lemma. Conversely, it is easy to see that groups of the form (1) or (2) are S^*-groups.

PROPOSITION 2. *Let G be a compact abelian group. Then G is an S-group if and only if either* (1) $G \simeq \mathrm{P}_{p\in\mathscr{P}} Z(p^{n_p})$, *where n_p is a nonnegative integer for each $p \in \mathscr{P}$, or* (2) *G is a compact connected group of dimension* 1 *such that for each $p \in \mathscr{P}$ the set of elements in G whose order is a power of p* (*i.e. the p-component of G*) *is dense in G.*

PROOF. This follows from Lemma 1 by dualization. The first case is practically obvious. As for (2) the fact that G must be a connected group of dimension 1 follows from [**5**, 24.25 and 24.28]. As to the rest of the description, a simple duality argument shows that if for a given p the intersection of the subgroups of $\hat{G}$ of the form $p^n\hat{G}$ is $\{0\}$, then the subgroup of G of elements of order a power of p must be dense in G. The converse is proved similarly by duality.

REMARK 1. It is not hard to show that a group G of type (2) above must contain a dense subgroup of the form $Z(p^\infty)$ for each prime p (see [**1**, Theorem 3]). The authors of [**1**] call an LCA group *p-thetic* if it contains a dense subgroup of the form $Z(p^\infty)$. Thus a group G of type (2) may be referred to as a compact group of dimension 1 which is p-thetic for each prime p. (It is easy to see that a compact p-thetic group is automatically connected.)

We next state a lemma which shows that the groups described in Proposition 1 are precisely the S-groups which are topological torsion groups, and which leads us to a complete description of the class of LCA S-groups.

LEMMA 2. *Let the LCA group G be a topological torsion group. Then G is an S-group if and only if it is of the form given in Proposition* 1.

PROOF. If the S-group G is a topological torsion group, we invoke [**6**, 3.21] to write $G \simeq \mathrm{LP}_{p\in\mathscr{P}} (G_p : H_p)$, where H_p is a compact open subgroup of G_p and G_p is a topological p-group (i.e. $\lim_{n\to\infty} p^n x = 0$ for each x in G; see [**6**, 3.1]). Now each H_p is a compact S-group, so we conclude from Proposition 2 that H_p must have the form $Z(p^n)$ for some nonnegative integer n. But then G_p must be discrete, since H_p is open in G_p. Szele's result then forces G_p to have the form $Z(p^{n_p})$, where n_p is either ∞ or a nonnegative integer. Hence G must have the form given in Proposition 1. The converse has already been given in Proposition 1.

We can now proceed to the main result of this section, which we state for S-groups. The dual statement for S^*-groups is easily formulated and is omitted.

THEOREM 1. *An LCA group G is an S-group if and only if either* (1) *G is of the form given in Proposition* 1, *or* (2) *G is a compact group of dimension* 1 *which is p-thetic for each prime p.*

PROOF. Let G be an LCA S-group. As observed at the beginning of this section, $\hat{G}$ must be totally disconnected. If G is totally disconnected as well, then G is a topological torsion group [**6**, 3.15], and so Lemma 2 leads us to case (1) of the theorem. Suppose, then, that G is not totally disconnected, and let C be the identity component of G. Now since every element of G is compact, it follows from [**5**, 9.14] that C is compact, so C has the form (2) of Proposition 2. We wish to show that $G=C$. Now by [**5**, 5.14] there is an open compactly generated subgroup U of G which contains C; since U is also an S-group it follows from [**5**, 9.8] that U is compact. Since U contains C, and is hence not totally disconnected, we conclude from Proposition 2 that U is already connected, and so $U=C$. Thus C is a compact open subgroup of G, and so it is a topological direct factor of G [**5**, 24.45]. Write $G=C\times G_0$, where G_0 is a discrete S-group. Since, by Szele's result, G_0 is a subgroup of Q/Z, it must contain a subgroup of the form $Z(p)$ for some prime p if it is not trivial. But according to Proposition 2, C must contain a copy of $Z(p)$ for every prime p. Thus if G_0 is not trivial, G cannot be an S-group. Therefore $G=C$ and, taking Remark 1 into account, we have case (2) of the theorem.

Finally, let us observe that the converse of the theorem has already been proved in Propositions 1 and 2.

REMARK 2. We may deduce from this theorem that the class of LCA S-groups is closed under the formation of quotients by closed subgroups and that, dually, a closed subgroup of an LCA S^*-group is an S^*-group.

2. **The general locally compact case.** The aim of this section is to show that a locally compact S-group must be abelian, so that Theorem 1 gives a complete description of the locally compact S-groups. We first observe that if G is an S-group, then every closed subgroup of G is normal, since conjugation is a topological automorphism. A group G in which every closed subgroup is normal is called a topological hamiltonian group in [**7**]. Now the author of [**7**] gives a complete description of the nonabelian locally compact topological hamiltonian groups. We will not need the complete description, since Lemma 7 of [**7**], which states that a nonabelian locally compact topological hamitonian group must contain a copy H of the 8-element quaternion group (see p. 23 of [**4**] for the definition), is all that is really necessary. Now it is simple to check that H is not an S-group. Therefore no locally compact S-group can contain a copy of H.

Hence every locally compact S-group must already be abelian. We sum up our results as follows.

THEOREM 1′. *Let G be a locally compact group. Then G is an S-group if and only if G is of type* (1) *or* (2) *in Theorem* 1.

REMARK 3. Our proof that a locally compact S-group must already be abelian, when applied in the discrete case, yields an argument quite different from that used by Szele to show that a discrete S-group must be abelian.

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