ON THE POSTNIKOV TOWERS FOR REAL AND COMPLEX CONNECTIVE K-THEORY

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1. Introduction

The analysis of real connective K-theory is facilitated by the '$\eta c R$' cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko$$

relating real and complex K-theories [2]. Here we extend this relationship through the Postnikov towers, producing several useful ko-module maps in the process.

Theorem 1. *The $\eta c R$ sequence lifts to cofiber sequences relating the connective covers of ko and ku as follows:*

$$\begin{tikzcd}
\Sigma ko \arrow[r, "\eta"] & ko \arrow[r, "c"] & ku \arrow[r, "R"] & \Sigma^2 ko \\
\Sigma ko \arrow[u, equal] \arrow[r, "\eta_1"] & ko\langle 1\rangle \arrow[u] \arrow[r, "c_1"] & \Sigma^2 ku \arrow[u, "v"'] \arrow[r, "r"] & \Sigma^2 ko \arrow[u, equal] \\
\Sigma ko\langle 1\rangle \arrow[u] \arrow[r, "\eta_2"] & ko\langle 2\rangle \arrow[u] \arrow[r, "c_2"] & \Sigma^4 ku \arrow[u, "v"'] \arrow[r, "r_1"] & \Sigma^2 ko\langle 1\rangle \arrow[u] \\
\Sigma ko\langle 2\rangle \arrow[u] \arrow[r, "\eta_4"] & ko\langle 4\rangle \arrow[u] \arrow[r, "c_4"] & \Sigma^4 ku \arrow[u, equal] \arrow[r, "r_2"] & \Sigma^2 ko\langle 2\rangle \arrow[u] \\
\Sigma ko\langle 4\rangle \arrow[u] \arrow[r, "\eta_8"] & ko\langle 8\rangle \arrow[u] \arrow[r, "c_8"] & \Sigma^6 ku \arrow[u, "v"'] \arrow[r, "r_4"] & \Sigma^2 ko\langle 4\rangle \arrow[u] \\
\Sigma ko\langle 8\rangle \arrow[u] \arrow[r, "\Sigma^8\eta"] & ko\langle 8\rangle \arrow[u, equal] \arrow[r, "\Sigma^8 c"] & \Sigma^8 ku \arrow[u, "v"'] \arrow[r, "\Sigma^8 R"] & \Sigma^2 ko\langle 8\rangle \arrow[u]
\end{tikzcd}$$

In the sequence above, c is complexification, r is realification, and η is multiplication by $\eta \in ko_1$. The map R is an extension of realification r over the Bott map: $r = Rv$.

We will write $X\langle n\rangle \longrightarrow X$ for the n-connected cover of X. By this we mean that $\pi_i X\langle n\rangle = 0$ for $i < n$, while $\pi_i X\langle n\rangle \longrightarrow \pi_i X$ is an isomorphism for $i \geq n$. It will be useful to record the maps induced in cohomology. All the modules and maps we will deal with are in the image of induction from $\mathcal{A}(1)$-Mod,

$$\mathcal{A} \otimes_{\mathcal{A}(1)} - : \mathcal{A}(1)\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod},$$

so we will record the results in $\mathcal{A}(1)$-Mod, leaving it to the reader to tensor up.

The first lift, $\eta_1 c_1 r$, was brought to my attention by Vic Snaith ([3]). The remaining lifts appeared at one point to be useful in Geoffrey Powell's analysis of $ko^* BV_+$ ([4]), but in the end were unnecessary there.

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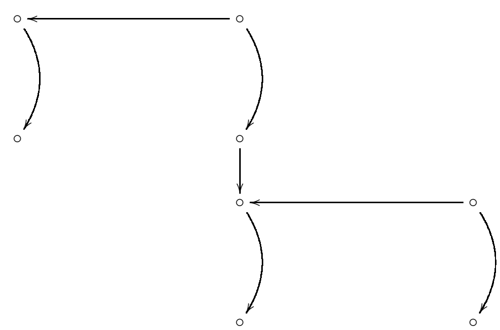
2. Complex Periodicity

In the complex case, periodicity and the Postnikov tower amount to the same thing. If we write $ku_* = \mathbf{Z}[v]$, with $|v| = 2$, then the Postnikov covers of ku are simply given by multiplication by powers of v.

$$\begin{tikzcd} \Sigma^2 ku \arrow[d, "\simeq"] \arrow[dr, "v"] & \\ ku\langle 2\rangle \arrow[r] & ku \end{tikzcd} \qquad \text{and more generally} \qquad \begin{tikzcd} \Sigma^{2i+2} ku \arrow[r, "v"] \arrow[d, "\simeq"] & \Sigma^{2i} ku \arrow[d, "\simeq"] \\ ku\langle 2i+2\rangle \arrow[r] & ku\langle 2i\rangle \end{tikzcd}$$

Proposition 2. *$ku \longrightarrow H\mathbf{Z} \longrightarrow \Sigma^3 ku$ induces the short exact sequence*

$$\mathcal{A}(1)/(Sq^1, Sq^3) \longleftarrow \mathcal{A}(1)/(Sq^1) \longleftarrow \Sigma^3\mathcal{A}(1)/(Sq^1, Sq^3)$$



3. Real Periodicity

In the real case, periodicity is broken into 4 steps. We write $ko_* = \mathbf{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$ with $|\eta| = 1$, $|\alpha| = 4$, and $|\beta| = 8$.

$$\begin{tikzcd} \Sigma^8 ko \arrow[r, "\simeq"] \arrow[ddddr, "\beta"'] & ko\langle 8\rangle \arrow[d] & \\ & ko\langle 4\rangle \arrow[r] \arrow[d] & \Sigma^4 H\mathbf{Z} \\ & ko\langle 2\rangle \arrow[r] \arrow[d] & \Sigma^2 H\mathbf{F}_2 \\ & ko\langle 1\rangle \arrow[r] \arrow[d] & \Sigma H\mathbf{F}_2 \\ & ko \arrow[r] & H\mathbf{Z} \end{tikzcd}$$

The following Proposition is well known. It is a simple way to show that a spectrum whose cohomology is $\mathcal{A}/\!\!/\mathcal{A}(1)$ must have 2-local homotopy additively isomorphic to $\pi_* ko$.

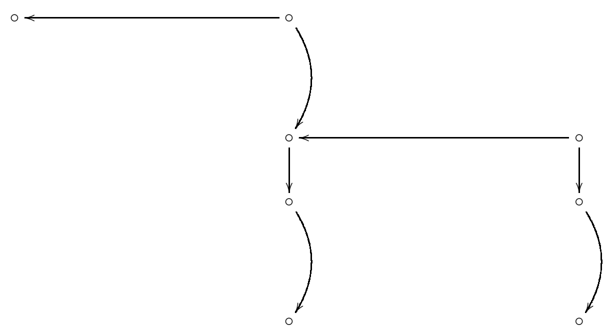
Proposition 3. *The maps induced in cohomology by the Postnikov tower for ko are as follows.*

(1)

$$ko \longrightarrow H\mathbf{Z} \longrightarrow \Sigma ko\langle 1\rangle$$

induces the short exact sequence

$$\mathbf{F}_2 \longleftarrow \mathcal{A}(1)/(Sq^1) \longleftarrow \Sigma^2\mathcal{A}(1)/(Sq^2)$$

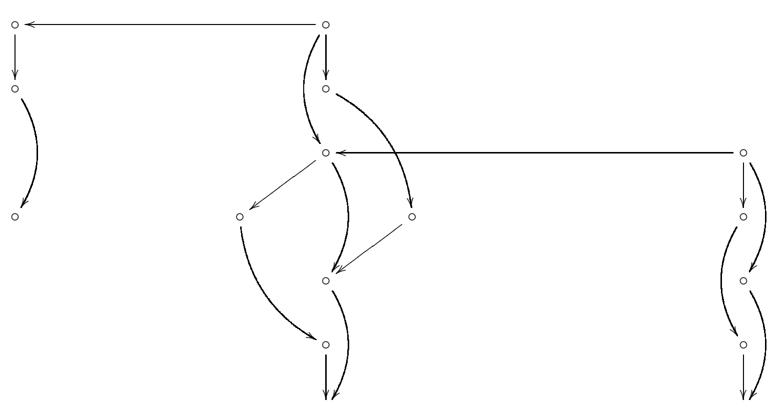


(2)

$$ko\langle 1\rangle \longrightarrow \Sigma H\mathbf{F}_2 \longrightarrow \Sigma ko\langle 2\rangle$$

induces the short exact sequence

$$\Sigma\mathcal{A}(1)/(Sq^2) \longleftarrow \Sigma\mathcal{A}(1) \longleftarrow \Sigma(Sq^2) \;\cong\; \Sigma^3\mathcal{A}(1)/(Sq^3)$$

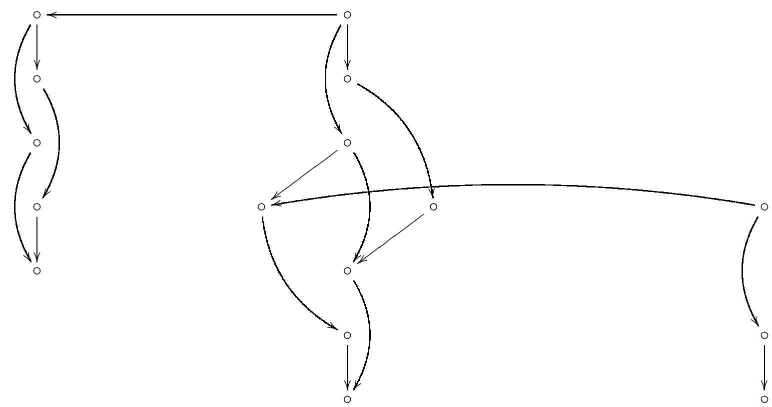


(3)

$$ko\langle 2\rangle \longrightarrow \Sigma^2 H\mathbf{F}_2 \longrightarrow \Sigma ko\langle 4\rangle$$

induces the short exact sequence

$$\Sigma^2\mathcal{A}(1)/(Sq^3) \longleftarrow \Sigma^2\mathcal{A}(1) \longleftarrow \Sigma^2(Sq^3) \;\cong\; \Sigma^5\mathcal{A}(1)/(Sq^1, Sq^2Sq^3)$$

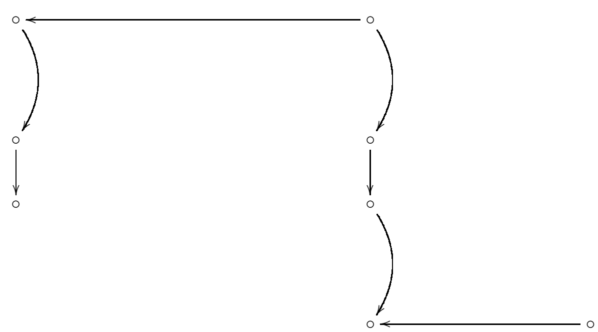


(4)

$$ko\langle 4\rangle \longrightarrow \Sigma^4 H\mathbf{Z} \longrightarrow \Sigma ko\langle 8\rangle$$

induces the short exact sequence

$$\Sigma^4\mathcal{A}(1)/(Sq^1, Sq^2Sq^3) \longleftarrow \mathcal{A}(1)/(Sq^1) \longleftarrow \Sigma^9\mathbf{F}_2$$

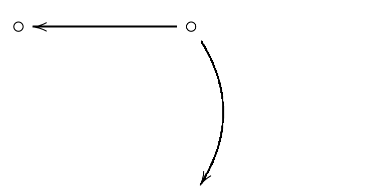


4. Maps of Postnikov towers

First, we record the maps induced in cohomology by our starting point, the ηcR sequence.

Proposition 4. $ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko$ *induces the short exact sequence*

$$\mathcal{A}(1)/(Sq^1, Sq^2) \longleftarrow \mathcal{A}(1)/(Sq^1, Sq^3) \longleftarrow \Sigma^2\mathcal{A}(1)/(Sq^1, Sq^2)$$



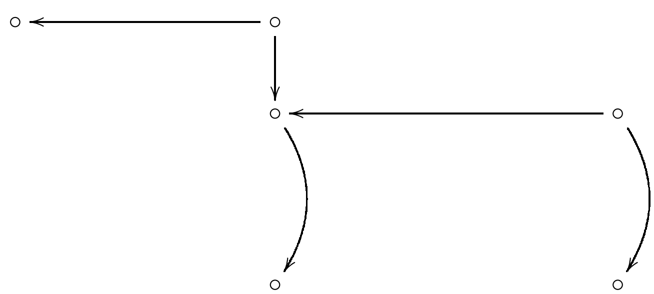
We will now prove Theorem 1 in a series of steps. We start with the braid of cofibrations induced by the composite $ko \xrightarrow{c} ku \longrightarrow H\mathbf{Z}$.

$$
\begin{tikzcd}
\Sigma ko \arrow[rr, bend left, "\eta"] \arrow[dr, "\eta_1"] & & ko \arrow[rr, bend left] \arrow[dr, "c"] & & H\mathbf{Z} \\
& ko\langle 1\rangle \arrow[ur] \arrow[dr, "c_1"] & & ku \arrow[ur] \arrow[dr, "R"] & \\
& & \Sigma^2 ku \arrow[ur, "v"] \arrow[rr, bend right, "r"] & & \Sigma^2 ko
\end{tikzcd}
$$

This gives the $\eta_1 c_1 r$ sequence. To continue to the next step, we will need to know the maps induced in cohomology by this one.

Proposition 5. $\Sigma ko \xrightarrow{\eta_1} ko\langle 1\rangle \xrightarrow{c_1} \Sigma^2 ku$ *induces the short exact sequence*

$$\Sigma \mathbf{F}_2 \longleftarrow \Sigma\mathcal{A}(1)/(Sq^2) \longleftarrow \Sigma^2\mathcal{A}(1)/(Sq^1, Sq^3)$$



Proof. These are the only maps which can make the long exact sequence exact. $\square$

From this we observe that we have a commutative square

$$
\begin{tikzcd}
\Sigma H\mathbf{F}_2 \arrow[r, "Sq^1"] & \Sigma^2 H\mathbf{Z} \\
ko\langle 1\rangle \arrow[u] \arrow[r, "c_1"] & \Sigma^2 ku \arrow[u]
\end{tikzcd}
$$

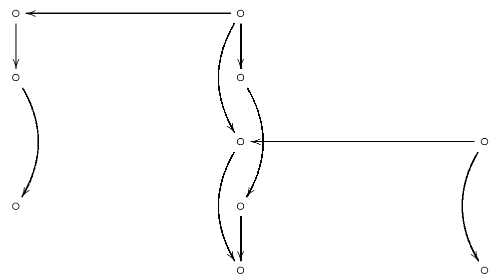
which induces the following map of cofiber sequences. The map induced in cohomology by η_1 implies that the left hand map $\Sigma ko \longrightarrow \Sigma H\mathbf{Z}$ is nontrivial. This implies that the fiber of c_2 is $\Sigma ko\langle 1\rangle$, giving the next Postnikov lift of the $\eta c R$ sequence.

$$
\begin{tikzcd}
\Sigma H\mathbf{Z} \arrow[r] & \Sigma H\mathbf{F}_2 \arrow[r, "Sq^1"] & \Sigma^2 H\mathbf{Z} \arrow[r] & \Sigma^2 H\mathbf{Z} \\
\Sigma ko \arrow[u] \arrow[r, "\eta_1"] & ko\langle 1\rangle \arrow[u] \arrow[r, "c_1"] & \Sigma^2 ku \arrow[u] \arrow[r, "r"] & \Sigma^2 ko \arrow[u] \\
\Sigma ko\langle 1\rangle \arrow[u] \arrow[r, "\eta_2"] & ko\langle 2\rangle \arrow[u] \arrow[r, "c_2"] & \Sigma^4 ku \arrow[u] \arrow[r, "r_1"] & \Sigma^2 ko\langle 1\rangle \arrow[u]
\end{tikzcd}
$$

Again we need to record the maps induced in cohomology for use in the next step.

Proposition 6. $\Sigma ko\langle 1\rangle \xrightarrow{\eta_2} ko\langle 2\rangle \xrightarrow{c_2} \Sigma^4 ku$ *induces the short exact sequence*

$$\Sigma^2\mathcal{A}(1)/(Sq^2) \xleftarrow[\eta_2^*]{} \Sigma^2\mathcal{A}(1)/(Sq^3) \xleftarrow[c_2^*]{} \Sigma^4\mathcal{A}(1)/(Sq^1, Sq^3)$$



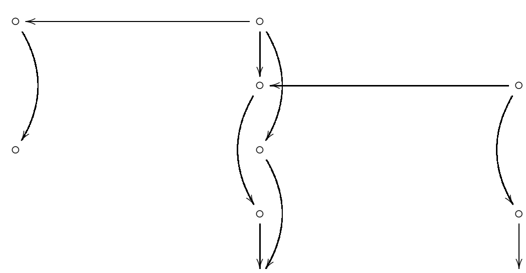
Now consider the braid of cofibrations induced by the composite $ko\langle 4\rangle \longrightarrow ko\langle 2\rangle \longrightarrow \Sigma^4 ku$.

$$\begin{tikzcd}
 & ko\langle 4\rangle \arrow[dr] \arrow[rr, bend left, "c_4"] & & \Sigma^4 ku \\
\Sigma ko\langle 2\rangle \arrow[ur, "\eta_4"] \arrow[dr] & & ko\langle 2\rangle \arrow[ur, "c_2"] \arrow[dr] & \\
 & \Sigma ko\langle 1\rangle \arrow[ur, "\eta_2"] \arrow[rr, bend right] & & \Sigma^2 H\mathbf{F}_2
\end{tikzcd}$$

Since η_2^* is nonzero in degree 2, the map $\Sigma ko\langle 1\rangle \longrightarrow \Sigma^2 H\mathbf{F}_2$ is nontrivial, and hence the the fiber of c_4 is $\Sigma ko\langle 2\rangle$. Again, we need to record the maps induced in cohomology, and again, they 'roll' one step to the left.

Proposition 7. $\Sigma^3 ku \xrightarrow{r_2} \Sigma ko\langle 2\rangle \xrightarrow{\eta_4} ko\langle 4\rangle$ *induces the short exact sequence*

$$\Sigma^3\mathcal{A}(1)/(Sq^1, Sq^3) \xleftarrow[r_2^*]{} \Sigma^3\mathcal{A}(1)/(Sq^3) \xleftarrow[\eta_4^*]{} \Sigma^4\mathcal{A}(1)/(Sq^1, Sq^2Sq^3)$$

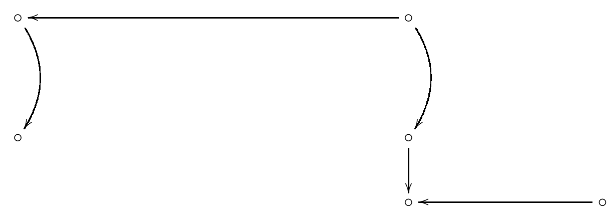


Since η_4^* sends the generator to Sq^1, we get a map of cofiber sequences whose fiber gives the next lift, $\eta_8 c_8 r_4$.

$$
\begin{tikzcd}
\Sigma^3 H\mathbf{Z} \arrow[r] & \Sigma^3 H\mathbf{F}_2 \arrow[r, "Sq^1"'] & \Sigma^4 H\mathbf{Z} \arrow[r] & \Sigma^4 H\mathbf{Z} \\
\Sigma^3 ku \arrow[u] \arrow[r, "r_2"] & \Sigma ko\langle 2\rangle \arrow[u] \arrow[r, "\eta_4"] & ko\langle 4\rangle \arrow[u] \arrow[r, "c_4"] & \Sigma^4 ku \arrow[u] \\
\Sigma^5 ku \arrow[u] \arrow[r, "r_4"] & \Sigma ko\langle 4\rangle \arrow[u] \arrow[r, "\eta_8"] & ko\langle 8\rangle \arrow[u] \arrow[r, "c_8"] & \Sigma^6 ku \arrow[u]
\end{tikzcd}
$$

Proposition 8. $\Sigma^5 ku \xrightarrow{r_4} \Sigma ko\langle 4\rangle \xrightarrow{\eta_8} ko\langle 8\rangle$ *induces the short exact sequence*

$$\Sigma^5\mathcal{A}(1)/(Sq^1, Sq^3) \xleftarrow[r_4^*]{} \Sigma^5\mathcal{A}(1)/(Sq^1, Sq^2Sq^3) \xleftarrow[\eta_8^*]{} \Sigma^8\mathbf{F}_2$$



Finally, consider the braid of cofibrations induced by the composite $\Sigma ko\langle 8\rangle \longrightarrow \Sigma ko\langle 4\rangle \xrightarrow{\eta_8} ko\langle 8\rangle$.

$$
\begin{tikzcd}
\Sigma ko\langle 8\rangle \arrow[rr, bend left, "\widetilde{\eta}"] \arrow[dr] & & ko\langle 8\rangle \arrow[rr, bend left, "c_8"] \arrow[dr, "\widetilde{c}"] & & \Sigma^6 ku \\
& \Sigma ko\langle 4\rangle \arrow[ur, "\eta_8"] \arrow[dr] & & \Sigma^8 ku \arrow[ur, "v"] \arrow[dr, "\widetilde{R}"] & \\
\Sigma^5 ku \arrow[ur, "r_4"] \arrow[rr, bend right] & & \Sigma^5 H\mathbf{Z} \arrow[ur] \arrow[rr, bend right] & & \Sigma^2 ko\langle 8\rangle
\end{tikzcd}
$$

Since r_4^* is an isomorphism on H^5, the map $\Sigma^5 ku \longrightarrow \Sigma^5 H\mathbf{Z}$ must be the bottom cohomology generator, justifying the appearance of $\Sigma^8 ku$ and v in this braid.

The result is a cofiber sequence $\Sigma^9 ko \longrightarrow \Sigma^8 ko \longrightarrow \Sigma^8 ku$. The maps are ko-module maps by construction, and agree with the 8-fold suspensions of η, c and R in homotopy, by the maps $X\langle 8\rangle \longrightarrow X$. The adjunction $F_{ko}(\Sigma^9 ko, \Sigma^8 ko) \simeq F(S^9, \Sigma^8 ko)$, shows that a ko-module map $\Sigma^9 ko \longrightarrow \Sigma^8 ko$ is determined by its effect on homotopy. Therefore, the first map, and hence the other two, are the 8-fold suspensions of η, c and R. $\square$

References

[1] R. R. Bruner, J. P. C. Greenlees, "The connective K-theory of finite groups", *Mem. Amer. Math. Soc.* **165** (2003), no. 785.

[2] R. R. Bruner, J. P. C. Greenlees, "The connective real K-theory of finite groups", *Math. Surveys and Monographs* **169**, 2010.

[3] Robert Bruner, Khaira Mira, Laura Stanley, Victor Snaith, “Ossa’s Theorem via the Kunneth formula”, arXiv:1008.0166.

[4] Geoffrey Powell, “On connective KO-theory of elementary abelian 2-groups”, arXiv:1207.6883.

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