OSSA'S THEOREM VIA THE KUNNETH FORMULA

ROBERT BRUNER, KHAIRIA MIRA, LAURA STANLEY AND VICTOR SNAITH

ABSTRACT. Let p be a prime. We calculate the connective unitary K-theory of the smash product of two copies of the classifying space for the cyclic group of order p, using a Künneth formula short exact sequence. As a corollary, using the Bott exact sequence and the mod 2 Hurewicz homomorphism we calculate the connective orthogonal K-theory of the smash product of two copies of the classifying space for the cyclic group of order two.

1. INTRODUCTION

This paper arose as a result of discussions during a graduate course at the University of Sheffield during 2008. In order to introduce Frank Adams' technique of constructing homology resolutions as realisations of iterated cofibrations of spectra a simpler example than the classical Adams spectral sequence was needed. We had the spectrum bu to hand but, in order to postpone the algebraic intricacies of spectral sequences, what was required was an example whose geometric resolution gave rise to a short exact sequence rather than a spectral sequence. As it happens the bu-resolution of $B\mathbb{Z}/2$ yields such an example, which was simple enough for the purposes of the course. At that point, John Greenlees mentioned the existence of [9], which prompted the writing of §2. In §2 we use the bu-resolution of §2.1 to calculate $bu_*(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)$ in terms of $bu_*(B\mathbb{Z}/p)$ and mod p Eilenberg-MacLane spaces (Theorem 2.12).

We shall merely compute connective K-groups. The papers [5], [7] and [9] derive equivalences of spectra involving $bu \wedge B\mathbb{Z}/p \wedge B\mathbb{Z}/p$ and $bo \wedge B\mathbb{Z}/p \wedge B\mathbb{Z}/p$ which imply our results upon taking homotopy groups. Importantly, unlike [5], [7] and [9], we do not resort to Adams spectral sequences to construct the essential algebraic homomorphism

$$\tilde{\mu}_* : bu_*(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \longrightarrow \vee_{i=1}^{p-1}\, bu_*(S^{2i} \wedge B\mathbb{Z}/p)$$

of §2.11.

Our method does not yield a homomorphism $\tilde{\mu}_*$ induced on homotopy from a map of spectra, it is merely an algebraic homomorphism, as explained in §2.13. However, since $\tilde{\mu}_*$ is closely related to the map induced by the multiplication on $B\mathbb{Z}/p$ it is virtually invariant under switching the $B\mathbb{Z}/p$-factors, which may prove useful in calculations of bu_* of other p-groups.

In [5] and [9] bo_*-analogues of the bu-result are offered when $p = 2$. Our bo_* calculations are consistent with the results proved in [5] and highlight the errors in

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the *bo*-analogue asserted in [9]. Consider the cofibration

$$\Sigma bo \xrightarrow{\eta} bo \xrightarrow{c} bu,$$

discovered by Raoul Bott during the proof of his famous Periodicity Theorem. Smashing this with X and taking homotopy groups yields the Bott sequence for X. In §3 we compute $bo_*(B\mathbb{Z}/2 \wedge B\mathbb{Z}/2)$ by comparing the Bott sequence for $B\mathbb{Z}/2 \wedge B\mathbb{Z}/2$ with that for $B\mathbb{Z}/2$ and with mod 2 homology. Our calculations are relevant to [6] and [11], for example.

2. The connective unitary case

2.1. Let bu_* denote connective unitary K-homology on the stable homotopy category of CW spectra [2] so that if X is a space without a basepoint its unreduced bu-homology is $bu_*(\Sigma^\infty X_+)$, the homology of the suspension spectrum of the disoint union of X with a base-point. In particular $bu_*(\Sigma^\infty S^0) = \mathbb{Z}[u]$ where $\deg(u) = 2$. Let p be a prime and consider the cofibration of pointed spaces

$$B\mathbb{Z}/p \xrightarrow{i} BS^1 \xrightarrow{\pi} W_p$$

where i is induced by the inclusion of the cyclic group of order p into the circle. This cofibration maps to the fibration

$$B\mathbb{Z}/p \xrightarrow{i} BS^1 \xrightarrow{Bp} BS^1$$

and the comparison for mod p and integral unreduced singular homology yields the following result:

Lemma 2.2.
For all j, $H_j(W_p;\mathbb{Z}) \cong H_j(BS^1;\mathbb{Z})$ being $\mathbb{Z}$ when $j \geq 0$ is even and zero otherwise.

From the Atiyah-Hirzebruch spectral sequence ([2] p.47) we obtain the following result, which also follows from the Thom isomorphism $bu_*(W_p) \cong bu_*(BS^1)$, since W_p is Thom complex of the p-th tensor power of the canonical complex line bundle, by §2.1.

Corollary 2.3.
Both $bu_*(\Sigma^\infty W_p)$ and $bu_*(\Sigma^\infty BS^1)$ are free modules over $bu_*(\Sigma^\infty S^0) = \mathbb{Z}[u]$.

2.4. The Atiyah-Hirzebruch spectral sequences for for bu_* and KU_* of $\Sigma^\infty B\mathbb{Z}/p$ both collapse for dimensional reasons and the map between them is injective so that $bu_*(\Sigma^\infty B\mathbb{Z}/p)$ injects into $KU_*(\Sigma^\infty B\mathbb{Z}/p)$ which, by the universal coefficient theorem for KU-theory [3] and the calculations of [4], is given by $KU_{2j+1}(\Sigma^\infty B\mathbb{Z}/p) \cong \oplus_{j=1}^{p-1} \mathbb{Z}/p^\infty$ ([9] §2; see also [8] Chapter I, §2) and is zero in even dimensions.

When p is odd it will be convenient to replace bu by $bu\mathbb{Z}_p$, connective unitary K-theory with p-adic integers coefficients and similarly for $KU\mathbb{Z}_p$. These p-adic spectra possess Adams decompositions [1] (see also [8])

$$bu\mathbb{Z}_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2} lu \text{ and } KU\mathbb{Z}_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2} LU$$

where $lu_*(\Sigma^\infty S^0) \cong \mathbb{Z}_p[v]$ where $\deg(v) = 2p-2$ corresponds to u^{p-1} and multiplication by u translates the summand $\Sigma^{2i-2} lu$ to $\Sigma^{2i} lu$ for $0 \leq i \leq p-2$ and $\Sigma^{2p-4} lu$ to lu. LU-theory is obtained from lu by localising to invert v. In addition there are canonical isomorphisms

$$bu_*(\Sigma^\infty B\mathbb{Z}/p) \cong (bu\mathbb{Z}_p)_*(\Sigma^\infty B\mathbb{Z}/p) \text{ and } KU_*(\Sigma^\infty B\mathbb{Z}/p) \cong (KU\mathbb{Z}_p)_*(\Sigma^\infty B\mathbb{Z}/p)$$

with $LU_{2j+1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^\infty$.

Corollary 2.5. Let p be a prime and let lu_* be as in §2.4 when p is odd or $lu = (bu\mathbb{Z}_2)_*$ when $p = 2$. Then, as a $\mathbb{Z}_p[v]$-module, where $\deg(v_{2i-1}) = 2i-1$,

$$lu_*(\Sigma^\infty B\mathbb{Z}/p) \cong \frac{\mathbb{Z}_p[u]\langle v_1, v_3, v_5, \ldots\rangle}{(pv_1, pv_3, \ldots, pv_{2p-3}, vv_{2i-1} - pv_{2(p-1)+2i-1})}.$$

Proof

The injection mentioned in §2.4 maps $lu_{2i-1}(\Sigma^\infty B\mathbb{Z}/p)$ into $LU_{2i-1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^\infty$. Therefore this group must be cyclic and an order-count in the collapsed Atiyah-Hirzebruch spectral sequence shows that the non-zero groups $lu_{2k(p-1)+2i-1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^{k+1}$ for $i = 1, \ldots, p-1$, generated by $v_{2k(p-1)+2i-1}$. In $KU_{2i+1}(\Sigma^\infty B\mathbb{Z}/p)$ the element $vv_{2k(p-1)+2i-1}$ has order p^{k+1}, by Bott periodicity, so we may choose $v_{2(k+1)(p-1)+2i-1}$ so that $vv_{2k(p-1)+2i-1} = pv_{2(k+1)(p-1)+2i-1}$. $\square$

Corollary 2.6.

The cofibration of §2.1 gives a free $\mathbb{Z}[u]$-module resolution

$$0 \longrightarrow bu_*(\Sigma^\infty BS^1) \xrightarrow{\pi_*} bu_*(\Sigma^\infty W_p) \longrightarrow bu_{*-1}(\Sigma^\infty B\mathbb{Z}/p) \longrightarrow 0$$

as well as similar resolutions for $bu\mathbb{Z}_p$ and lu.

2.7. If A is a $\mathbb{Z}$-graded group we write $A[n]$ for the graded group with $A[n]_j = A_{j+n}$ so that $bu_{*-1}(\Sigma^\infty B\mathbb{Z}/p)$ equals $bu_*(\Sigma^\infty B\mathbb{Z}/p)[-1]$. By a cell-by-cell induction, for all CW spectra of finite type X the external product gives isomorphisms

$$bu_*(\Sigma^\infty BS^1) \otimes_{\mathbb{Z}[u]} bu_*(X) \xrightarrow{\cong} bu_*(\Sigma^\infty BS^1 \wedge X),$$

$$bu_*(\Sigma^\infty W_p) \otimes_{\mathbb{Z}[u]} bu_*(X) \xrightarrow{\cong} bu_*(\Sigma^\infty W_p \wedge X).$$

Smashing the cofibration of §2.1 with X and applying the argument of [3] yields the following Künneth formula:

Theorem 2.8.

There is a natural short exact sequence

$$0 \longrightarrow bu_*(\Sigma^\infty B\mathbb{Z}/p) \otimes_{\mathbb{Z}[u]} bu_*(X) \longrightarrow bu_*(\Sigma^\infty B\mathbb{Z}/p) \wedge X)$$

$$\longrightarrow \mathrm{Tor}^1_{\mathbb{Z}[u]}(bu_*(\Sigma^\infty B\mathbb{Z}/p), bu_*(X))[1] \longrightarrow 0$$

as well as similar exact sequences for $bu\mathbb{Z}_p$ and lu.

Example 2.9. Let p be a prime. As in Corollary 2.5, let lu_* be as in §2.4 when p is odd or $lu = (bu\mathbb{Z}_2)_*$ when $p = 2$. In Theorem 2.8 set $X = \Sigma^\infty B\mathbb{Z}/p$. Then $lu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ comes entirely from the left-hand graded group which is generated by $\{v_{2i-1} \otimes v_{2j-1}\ i, j \geq 1\}$ but if $2i-1 > 2(p-1)$ then

$$\begin{aligned} pv_{2i-1} \otimes v_{2j-1} \quad &= vv_{2i-1-2(p-1)} \otimes v_{2j-1} \\ &= v_{2i-1-2(p-1)} \otimes vv_{2j-1} \\ &= pv_{2i-1-2(p-1)} \otimes v_{2(p-1)+2j-1} \end{aligned}$$

which is zero by induction and similarly if $2j-1 > 2(p-1)$. Therefore $lu_{2m}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ is the graded $\mathbb{F}_p$-vector space spanned by $v_1 \otimes v_{2m-1}, \ldots, v_{2m-1} \otimes v_1$

which are linearly independent, being detected by the canonical homomorphism to $H_{2m}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p);\mathbb{Z}/p)$. Therefore for each $m \geq 1$

$$\dim_{\mathbb{F}_p}(lu_{2m}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))) = \dim_{\mathbb{F}_p}(\pi_{2m}(\vee_{i,j>0}\ \Sigma^{2i+2j-2}H\mathbb{Z}/p)).$$

Similarly $lu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ comes entirely from the right-hand graded group

$$T_* = \mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_*(\Sigma^\infty B\mathbb{Z}/p), lu_*(\Sigma^\infty B\mathbb{Z}/p))[1].$$

For $1 \leq 2i-1 \leq 2p-3$ let Y_i denote the $\mathbb{Z}_p[v]$-submodule of $lu_*(\Sigma^\infty B\mathbb{Z}/p)$ generated by $\{v_{2k(p-1)+2i-1} \mid k \geq 0\}$ so that $lu_*(\Sigma^\infty B\mathbb{Z}/p) \cong \oplus_i Y_i$ with a corresponding decomposition $T_* \cong \oplus_i T_{i,*}$. This decomposition has a well-known geometric origin ([9] §2).

A free $\mathbb{Z}_p[v]$-module resolution is given by

$$0 \longrightarrow \oplus_{j=0}^\infty\ \mathbb{Z}_p[v]\langle a_j\rangle \xrightarrow{d} \oplus_{j=0}^\infty\ \mathbb{Z}_p[v]\langle b_j\rangle \xrightarrow{\epsilon} Y_i \longrightarrow 0$$

where a_j, b_j have internal degree $2j(p-1)+2i-1$, $\epsilon(b_j) = v_{2j(p-1)+2i-1}$, $d(a_0) = pb_0$ and $d(a_j) = pb_j - vb_{j-1}$ for $j \geq 1$. Therefore

$$T_{i,*} = \mathrm{Ker}(1 \otimes d : \oplus_{j=0}^\infty\ lu_*(\Sigma^\infty B\mathbb{Z}/p)\langle a_j\rangle \longrightarrow \oplus_{j=0}^\infty\ lu_*(\Sigma^\infty B\mathbb{Z}/p)\langle b_j\rangle).$$

For $2m \geq 2i-1$ write $2m-2i+1 = 2t(p-1)+2j-1$ with $1 \leq 2j-1 \leq 2p-3$. Then

$$T_{i,2m} = \mathbb{Z}/p^{t+1}\langle v_{2t(p-1)+2j-1}a_0 + v_{2(t-1)(p-1)+2j-1}a_1 + \ldots + v_{2j-1}a_t\rangle.$$

From this, for $1 \leq i \leq p-1$, one finds that

$$\mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_*(\Sigma^\infty B\mathbb{Z}/p), Y_i)[1] \cong lu_{2*+1}(\Sigma^{2i}B\mathbb{Z}/p)$$

and therefore

$$lu_{2*+1}(\Sigma^\infty B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \cong \oplus_{i=1}^{p-1}\ lu_{2*+1}(\Sigma^{2i}B\mathbb{Z}/p).$$

Adding together the suspensions of lu as in §2.4 yields a similar isomorphism for $bu\mathbb{Z}_p$ and therefore there is a $\mathbb{Z}[u]$-module isomorphism

$$bu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \cong \oplus_{i=1}^{p-1}\ bu_{2*+1}(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p))$$

and $\oplus_{i=1}^{p-1} bu_{2*}(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p)) = 0$.

From Example 2.9 we have an isomorphism

$$lu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \cong \pi_{2*}(\vee_{i,j>0}\ \Sigma^{2i+2j-2}H\mathbb{Z}/p)$$

and therefore

$$bu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \cong \pi_{2*}(\vee_{a=0}^{p-2} \vee_{i,j>0}\ \Sigma^{2a+2i+2j-2}H\mathbb{Z}/p).$$

Lemma 2.10.

The homomorphism induced by the multiplication μ in the group $\mathbb{Z}/p$ is injective

$$\mu_* : bu_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \longrightarrow bu_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p)).$$

Proof

For simplicity we prove this only for $p = 2$. The proof, which uses KU, may be modified for odd primes but requires a more careful analysis of the splittings of §2.4 and ([9] §2) in relation to the embedding

$$KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \subset \mathrm{Hom}(R(\mathbb{Z}/p \times \mathbb{Z}/p), \mathbb{Z}/p^\infty)$$

By Example 2.8, multiplication by u is injective in odd dimensions so that

$$bu_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \longrightarrow KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$$

is injective, because it is localisation by inverting u. We shall use this observation to show that

$$\mu_* : KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/2 \wedge B\mathbb{Z}/2)) \longrightarrow KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/2))$$

is injective, which suffices to prove the result when $p = 2$. Since $B\mathbb{Z}/2 = \mathbb{R}P^\infty$ a skeletal approximation to the multiplication gives

$$\mu : \Sigma^\infty(\mathbb{R}P^{2r} \wedge \mathbb{R}P^{2v}) \longrightarrow \Sigma^\infty\mathbb{R}P^{2r+2v}.$$

Consider the effect on reduced, periodic complex K-theory

$$\mu^* : \tilde{KU}^0(\mathbb{R}P^{2r+2v}) \cong \mathbb{Z}/2^{r+v} \longrightarrow \tilde{KU}^0(\mathbb{R}P^{2r} \wedge \mathbb{R}P^{2v}) \cong \mathbb{Z}/2^{\min(r,v)}.$$

If L is the Hopf line bundle then $\mu^*(L-1) = (L-1) \otimes (L-1)$ so that μ^* is onto and, by the universal coefficient formula for KU_*, KU^*,

$$\mu_* : \tilde{KU}_{2m+1}(\mathbb{R}P^{2r} \wedge \mathbb{R}P^{2v}) \cong \mathbb{Z}/2^{\min(r,v)} \longrightarrow \tilde{KU}_{2m+1}(\mathbb{R}P^{2r+2v}) \cong \mathbb{Z}/2^{r+v}$$

is injective. Letting r, s tend to infinity yields the result. $\square$

2.11. We have a cofibration of spectra $\Sigma^2 bu \longrightarrow bu \longrightarrow H\mathbb{Z}$. By Example 2.9 and Lemma 2.10 the composition

$$bu \wedge \Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \xrightarrow{1\wedge\mu} bu \wedge \Sigma^\infty(B\mathbb{Z}/p) \xrightarrow{k} H\mathbb{Z} \wedge \Sigma^\infty(B\mathbb{Z}/p)$$

is trivial on homotopy groups. Therefore, when $p = 2$, $(1 \wedge \mu)_*$ induces an isomorphism

$$\tilde{\mu}_* : bu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/2 \wedge B\mathbb{Z}/2)) \xrightarrow{\cong} bu_{2*+1}(\Sigma^\infty(S^2 \wedge B\mathbb{Z}/2)).$$

Similarly at odd primes, using the multiplication μ together with the stable homotopy splittings of $\Sigma^\infty B\mathbb{Z}/p$ [9], yields an isomorphism

$$\tilde{\mu}_* : bu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \xrightarrow{\cong} \oplus_{i=1}^{p-1}\ bu_{2*+1}(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p)).$$

By Example 2.9, the $\mathbb{F}_p$-vector space $bu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ is detected in mod p homology and there is a map of spectra

$$h : bu \wedge \Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \longrightarrow (\vee_{a=0}^{p-2}\ \vee_{i,j>0}\ \ \Sigma^{2a+2i+2j-2}H\mathbb{Z}/p)$$

which induces an isomorphism on even dimensional homotopy. Therefore we obtain the following result:

Theorem 2.12. *([9]; see also [5] and [7])*
There is an isomorphism

$$(\tilde{\mu}_*, h_*) : bu_*(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \xrightarrow{\cong}$$

$$\oplus_{i=1}^{p-1}\ bu_*(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p)) \oplus \oplus_{a=0}^{p-2} \oplus_{i,j>0}\ \pi_*(\Sigma^{2a+2i+2j-2}H\mathbb{Z}/p).$$

Remark 2.13. The composition of maps of spectra $k(1 \wedge \mu)$ used in §2.11 is not nullhomotopic, although is it zero on homotopy groups. It is for this reason that our method does not yield a homomorphism $\tilde{\mu}_*$ induced by a map of spectra.

3. The connective orthogonal case

3.1. In this section we shall concentrate on $p = 2$ and connective orthogonal K-theory bo. Consider the following commutative diagram of spectra of horizontal and vertical cofibrations in which c is complexification and η is multiplication by the generator of $\pi_1(bo)$. The notation for $bo\langle 1\rangle$ is taken from [5].

$$\begin{tikzcd}
\Sigma bo \arrow[r, "1"'] \arrow[d, "\tilde{\eta}"] & \Sigma bo \arrow[d, "\eta"'] & \\
bo\langle 1\rangle \arrow[r] \arrow[d, "\tilde{c}"] & bo \arrow[r] \arrow[d, "c"'] & H\mathbb{Z} \arrow[d, "1"'] \\
\Sigma^2 bu \arrow[r] & bu \arrow[r] & H\mathbb{Z}
\end{tikzcd}$$

We have the following table of (reduced) orthogonal connective K-theory groups:

$bo_{8n}(\mathbb{R}P^\infty)$	0
$bo_{8n+1}(\mathbb{R}P^\infty)$	$\mathbb{Z}/2$
$bo_{8n+2}(\mathbb{R}P^\infty)$	$\mathbb{Z}/2$
$bo_{8n+3}(\mathbb{R}P^\infty)$	$\mathbb{Z}/2^{4n+3}$
$bo_{8n+4}(\mathbb{R}P^\infty)$	0
$bo_{8n+5}(\mathbb{R}P^\infty)$	0
$bo_{8n+6}(\mathbb{R}P^\infty)$	0
$bo_{8n+7}(\mathbb{R}P^\infty)$	$\mathbb{Z}/2^{4n+4}$

The graded group $bo_*(\mathbb{R}P^\infty)$ is a module over

$$bo_*(S^0) = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

and multiplication by η is nontrivial from dimension $8n + 1$ to $8n + 2$ and from $8n + 2$ to $8n + 3$. Multiplication by α has kernel of order 4 from dimension $8n + 3$ to $8n + 7$ and is one-one from dimension $8n + 7$ to $8n + 11$. Multiplication by β is always one-one.

The central horizontal cofibration yields a long exact sequence of reduced homology theories

$$\ldots \longrightarrow bo\langle 1\rangle_i(\mathbb{R}P^\infty) \longrightarrow bo_i(\mathbb{R}P^\infty) \longrightarrow H_i(\mathbb{R}P^\infty; \mathbb{Z}) \longrightarrow \ldots$$

and there is a factorisation $bo \longrightarrow bu \longrightarrow H\mathbb{Z}$. Using the fact that $H_i(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}/2$ for odd $i > 0$ and is zero otherwise we may calculate $bo\langle 1\rangle_*(\mathbb{R}P^\infty)$. In addition we may double-check the results from the long exact homotopy sequence of the left-hand vertical fibration in the diagram of §3.1

$$\ldots \longrightarrow bo_{i-1}(\mathbb{R}P^{\infty}) \longrightarrow\longrightarrow bo\langle 1\rangle_i(\mathbb{R}P^{\infty}) \longrightarrow bu_{i-2}(\mathbb{R}P^{\infty}) \longrightarrow \ldots.$$

Diagram chasing yields the following table:

$bo < 1 >_{8n} (\mathbb{R}P^{\infty})\ n \geq 0$	$\mathbb{Z}/2 \cong H_{8n+1}(\mathbb{R}P^{\infty};\mathbb{Z}) \xrightarrow{\cong} bo < 1 >_{8n} (\mathbb{R}P^{\infty})$
$bo < 1 >_1 (\mathbb{R}P^{\infty})$	0
$bo < 1 >_{8n+1} (\mathbb{R}P^{\infty})\ n \geq 1$	$\mathbb{Z}/2 \cong bo < 1 >_{8n+1} (\mathbb{R}P^{\infty}) \xrightarrow{\cong} bo_{8n+1}(\mathbb{R}P^{\infty})$
$bo < 1 >_{8n+2} (\mathbb{R}P^{\infty})$	$\mathbb{Z}/2 \cong bo < 1 >_{8n+2} (\mathbb{R}P^{\infty}) \xrightarrow{\cong} bo_{8n+2}(\mathbb{R}P^{\infty})$
$bo < 1 >_{8n+3} (\mathbb{R}P^{\infty})$	$\mathbb{Z}/2^{4n+2} \cong bo < 1 >_{8n+3} (\mathbb{R}P^{\infty}) \xrightarrow{2(2s+1)} bu_{8n+3}(\mathbb{R}P^{\infty})$
$bo < 1 >_{8n+4} (\mathbb{R}P^{\infty})$	$\mathbb{Z}/2 \cong H_{8n+5}(\mathbb{R}P^{\infty};\mathbb{Z}) \xrightarrow{\cong} bo < 1 >_{8n+4} (\mathbb{R}P^{\infty})$
$bo < 1 >_{8n+5} (\mathbb{R}P^{\infty})$	0
$bo < 1 >_{8n+6} (\mathbb{R}P^{\infty})$	0
$bo < 1 >_{8n+7} (\mathbb{R}P^{\infty})$	$\mathbb{Z}/2^{4n+3} \cong bo < 1 >_{8n+7} (\mathbb{R}P^{\infty}) \xrightarrow{1-1} bo_{8n+7}(\mathbb{R}P^{\infty})$

We can now state the main result of this section (see also [5]), whose proof will be sketched in §3.6.

Theorem 3.2.
There are homomorphisms of graded groups

$$\tilde{\mu}_* : bo_*(\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}) \longrightarrow bo\langle 1\rangle_*(\mathbb{R}P^{\infty}),$$

characterised in §3.6, and

$$\tilde{h}_* : bo_m(\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}) \longrightarrow \pi_*(\vee_{i,j>0}\ \Sigma^{2i+4j-2} H\mathbb{Z}/2)$$

such that

$$(\tilde{\mu}_*, \tilde{h}_*) : bo_*(\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}) \longrightarrow bo\langle 1\rangle_*(\mathbb{R}P^{\infty}) \oplus \pi_*(\vee_{i,j>0}\ \Sigma^{2i+4j-2} H\mathbb{Z}/2)$$

is an isomorphism.

3.3. *The Bott sequence versus mod 2 homology*
The subalgebra $\mathcal{B}$ generated in the mod 2 Steenrod algebra by Sq^1 and Sq^2 has dimension eight and contained the exterior subalgebra $\mathcal{E} = E(Sq^1, Sq^{0,1}) = \{1, Sq^1, Sq^1Sq^2 + Sq^2Sq^1, Sq^2Sq^2\}$.
Consider the Bott sequence

$$\cdots \longrightarrow bo_i(X) \xrightarrow{c} bu_i(X) \longrightarrow bo_{i-2}(X) \xrightarrow{\eta_*} bo_{i-1}(X) \longrightarrow \cdots$$

which is isomorphic to the homotopy sequence of the cofibration

$$bo \wedge X \longrightarrow bo \wedge \Sigma^{-2}\mathbb{C}P^2 \wedge X \longrightarrow bo \wedge S^2 \wedge X,$$

where the middle spectrum is identified with $bu \wedge X$ via an equivalence due to Anderson-Wood [10].

The following commutative diagram is easy to establish.

$$\begin{tikzcd}
bo_i(X) \arrow[r] \arrow[d, "c"] & \mathrm{Hom}_{\mathcal{B}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2) \arrow[d, "\tilde{\phi}"'] \\
bu_i(X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{E}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2) \arrow[d, "\tilde{\lambda}"'] \\
bo_{i-2}(X) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(H^{i-2}(X;\mathbb{Z}/2),\mathbb{Z}/2)
\end{tikzcd}$$

The horizontal maps are induced by the canonical map $bo \longrightarrow H\mathbb{Z}/2$, $\tilde{\phi}(h) = h$ and $\tilde{\lambda}(g)(x) = g(Sq^2(x))$.

By Theorem 2.12 we know that the middle horizontal map is an isomorphism when i is even. The following result is straightforward.

Proposition 3.4.

(i) When $X = \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$ the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2) \stackrel{\tilde{\phi}}{\longrightarrow} \mathrm{Hom}_{\mathcal{E}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2)$$

$$\stackrel{\tilde{\lambda}}{\longrightarrow} \mathrm{Hom}_{\mathcal{B}}(H^{i-2}(X;\mathbb{Z}/2),\mathbb{Z}/2) \longrightarrow 0.$$

is exact.

(ii) For $i \geq 2$

$$\dim_{\mathbb{F}_2}(\mathrm{Hom}_{\mathcal{B}}(H^{2i}(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty;\mathbb{Z}/2),\mathbb{Z}/2)) = \begin{cases} (i-1)/2 \text{ if } i \text{ is odd,} \\ \\ 1 + (i/2) \text{ if } i \text{ is even} \end{cases}$$

(iii) For $i \geq 2$

$$\dim_{\mathbb{F}_2}(\mathrm{Hom}_{\mathcal{E}}(H^{2i}(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty;\mathbb{Z}/2),\mathbb{Z}/2)) = i.$$

We shall also need the following result.

Proposition 3.5.

Define

$$X_n = \#\{i, j \geq 1 \mid 2i - 1 + 4j - 1 = 2n\}.$$

Then

$$X_n = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ \\ (n-1)/2 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

3.6. *Sketch proof of Theorem 3.2*

The mod 2 Hurewicz homomorphism induces a homomorphism

$$h_* : bo_m(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(H^{2i}(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty; \mathbb{Z}/2), \mathbb{Z}/2).$$

The results in the central columns of the following table are proved by induction on dimension using Theorem 2.12, the map between Bott sequences induces by

$$\mu : \Sigma^\infty \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \longrightarrow \Sigma^\infty \mathbb{R}P^\infty$$

and the results of §3.3, Proposition 3.4 and Proposition 3.5 concerning mod 2 homology.

m	$bo_m(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty)$	h_*	$bo\langle 1\rangle_m(\mathbb{R}P^\infty)$
2	$\mathbb{Z}/2$	isom	$\mathbb{Z}/2$
$3 \leq 8n+3$	$\mathbb{Z}/2^{4n+3}$	$-$	$\mathbb{Z}/2^{4n+3}$
$4 \leq 8n+4$	$(\mathbb{Z}/2)^{2n+2}$	isom	$\mathbb{Z}/2$
$5 \leq 8n+5$	0	$-$	0
$6 \leq 8n+6$	$(\mathbb{Z}/2)^{2n+1}$	isom	0
$7 \leq 8n+7$	$\mathbb{Z}/2^{4n+4}$	$-$	$\mathbb{Z}/2^{4n+4}$
$8 \leq 8n$	$(\mathbb{Z}/2)^{2n+1}$	isom	$\mathbb{Z}/2$
$9 \leq 8n+1$	$\mathbb{Z}/2$	$-$	$\mathbb{Z}/2$
$10 \leq 8n+2$	$(\mathbb{Z}/2)^{2n+1}$	onto, not isom	$\mathbb{Z}/2$

The group $bo_m(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty) = 0$ for $m \leq 1$.

When $m = 8n+1, 8n+2, 8n+3, 8n+7$ it is important that the homomorphism $\tilde{\mu}_m$ is chosen so that the composition

$$bo_m(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty) \longrightarrow bo\langle 1\rangle_m(\mathbb{R}P^\infty) \subseteq bo_m(\mathbb{R}P^\infty)$$

is equal to μ_m.

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