

Maximum-width empty square and rectangular annulus [☆]Sang Won Bae ^a, Arpita Baral ^b, Priya Ranjan Sinha Mahapatra ^{b,*}^a Division of Computer Science and Engineering, Kyonggi University, Suwon, Republic of Korea^b Department of Computer Science and Engineering, University of Kalyani, India

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ABSTRACT

An annulus is, informally, a ring-shaped region, often described by two concentric circles. The maximum-width empty annulus problem asks to find an annulus of a certain shape with the maximum possible width that avoids a given set of n points in the plane. This problem can also be interpreted as the problem of finding an optimal location of a ring-shaped obnoxious facility among the input points. In this paper, we study square and rectangular variants of the maximum-width empty annulus problem, and present first nontrivial algorithms. Specifically, our algorithms run in $O(n^3)$ and $O(n^2 \log n)$ time for computing a maximum-width empty axis-parallel square and rectangular annulus, respectively. Both algorithms use only $O(n)$ space.

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1. Introduction

The problem of computing a minimum-size geometric object that encloses an input point set P is one of the central research problems in computational geometry. This type of problem has been extensively studied with direct applications to location of desirable facilities to customers P , for a variety of different geometric shapes including circles [19], rectangles [22], and annuli [1–3,5,13,18].

On the other hand, in some applications, the facility to be built among P is considered *obnoxious*, that is, every member in P wants to be as far away from it as possible. The problem of locating an obnoxious facility is often interpreted as the problem of finding a maximum-size empty geometric object among P . For examples, the center of a largest circle or square that is empty of P corresponds to an optimal location of a point obnoxious facility that maximizes the Euclidean or L_∞ distance, respectively, from its closest point in P . A largest empty circle or square can be found in optimal $O(n \log n)$ time using the Voronoi diagram [21], and the best known algorithm that computes an empty axis-parallel rectangle of maximum area runs in $O(n \log^2 n)$ time by Aggarwal and Suri [4]. The widest empty corridor problem, in which one wants to find a widest empty strip among P of arbitrary orientation, is another interesting problem in this concept. After Houle and Maciel [15] presented an $O(n^2)$ -time algorithm for this problem, a lot of variants and extensions have been addressed, including the widest L-shaped corridor problem [8], and the widest 1-corner corridor problem [10]. Note that these problems are equivalent to those of finding an optimal location of an obnoxious facility whose shape is of a line, a line segment, or a polygonal chain.

In this paper, along this line of research, we study the *maximum-width empty annulus problem*. Informally, an annulus is a ring-shaped region, often described by two concentric circles. Thus, the maximum-width empty annulus problem is to find

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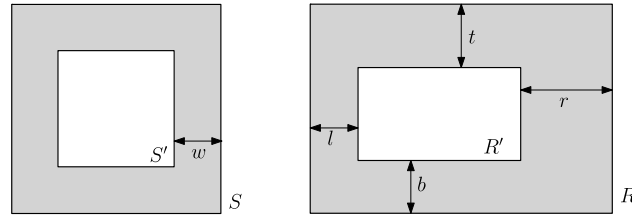


Fig. 1. A square annulus of width w with outer and inner squares S and S' (left) and a rectangular annulus with outer and inner rectangles R and R' whose top-, bottom-, left-, right-widths are t , b , l , and r , respectively (right).

an optimal location of a ring-shaped obnoxious facility among the input points P . Specifically, we discuss its square and rectangular variants, and present first nontrivial algorithms. Our algorithms run in $O(n^3)$ and $O(n^2 \log n)$ time for computing a maximum-width axis-parallel square and rectangular annulus, respectively, that is empty of a given set P of n points in the plane. Both algorithms use only $O(n)$ space.

There has been a little work on the maximum-width empty annulus problem. Díaz-Báñez et al. [11] first studied the problem for circular annulus, and proposed an $O(n^3 \log n)$ -time and $O(n)$ -space algorithm to solve it. To our best knowledge, there was no known correct algorithm in the literature for the maximum-width empty square or rectangular annulus problem. Mahapatra [17] considered the maximum-width empty rectangular annulus problem and claimed an incorrect $O(n^2)$ -time algorithm. There is a missing argument in Observation 2 of [17], which incorrectly claimed that the total number of potential outer rectangles forming an empty rectangular annulus is $n - 1$.

Unlike the maximum-width empty annulus problem, the problem of finding a minimum-width annulus that encloses P has recently attained intensive interests from researchers. As a classical one, circular annuli have been studied earlier with applications to the roundness problem [12,20,23], and the currently best known algorithm runs in $O(n^{3/2+\epsilon})$ time [2,3]. Computing a minimum-width axis-parallel square or rectangular annulus that encloses n points P can be done in $O(n \log n)$ or $O(n)$ time, respectively [1,13]. Mukherjee et al. [18] considered the problem of identifying a rectangular annulus of minimum width that encloses P in arbitrary orientation, and presented an $O(n^2 \log n)$ -time algorithm. Bae [5] studied a minimum-width square annulus in arbitrary orientation and showed that it can be solved in $O(n^3 \log n)$ time and further improved it to $O(n^3)$ time [6].

The rest of the paper is organized as follows: In Section 2, we introduce some definitions and notations, and precisely define our problems. Our algorithms are described in the following sections: Section 3 for computing a maximum-width empty axis-parallel square annulus and Section 4 for computing a maximum-width empty axis-parallel rectangular annulus. Finally, we finish the paper with concluding remarks in Section 5.

2. Problem definition and terminologies

Throughout the paper, we consider a Cartesian coordinate system of the plane \mathbb{R}^2 with the x - and y -axes. For any point p in the plane \mathbb{R}^2 , we denote by $x(p)$ and $y(p)$ its x - and y -coordinates. For an axis-parallel rectangle or square, its four sides are naturally identified by *top*, *bottom*, *left*, and *right* sides, respectively.

The intersection point of the two diagonals of an axis-parallel square, is called its *center*, and its *radius* is half its side length. For an axis-parallel square S , the (*inward*) *offset* of S by a real number $\delta \geq 0$ is a smaller axis-parallel square obtained by sliding each side of S inwards by δ . An *axis-parallel square annulus* is the region between an axis-parallel square S and its offset S' by some $\delta \geq 0$. We call S and S' the *outer* and *inner* squares, respectively, of the annulus, and δ the *width* of the annulus. See Fig. 1(left) for an illustration. In this paper, we allow the outer and inner squares of an axis-parallel square annulus to have one or more sides *at infinity*. A side of an axis-parallel square (resp. an axis-parallel rectangle) is said to be at infinity if its associated coordinate value is infinite. For examples, an axis-parallel strip (between two parallel lines) is also considered a special form of an axis-parallel square annulus.

For the rectangular annulus, we adopt the definition of Mukherjee et al. [18]. An *axis-parallel rectangular annulus* is the region obtained by subtracting the interior of an axis-parallel rectangle R' from another axis-parallel rectangle R such that $R' \subseteq R$. We call R and R' the *outer rectangle* and *inner rectangle* of the annulus, respectively. Consider a rectangular annulus A defined by its outer and inner rectangles, R and R' . By our definition, note that R and R' defining annulus A do not have to be concentric, so that A may not be a symmetric shape. The *top-width* of A is the vertical distance between the top sides of R and R' , and the *bottom-width* of A is the vertical distance between their bottom sides. Analogously, the *left-width* and *right-width* of A are defined to be the horizontal distance between the left sides of R and R' and the right sides of R and R' , respectively. Then, the *width* of A is defined to be the minimum of the four values: the top-width, bottom-width, left-width, and right-width of A . See Fig. 1(right) for an illustration. We also allow the outer and inner rectangles of an axis-parallel rectangular annulus to have one or more sides at infinity. If two corresponding sides of an axis-parallel rectangular annulus are at infinity then the corresponding width is not defined. Then the width of the annulus is the minimum among those defined widths.

In this paper, we only discuss squares, rectangles, square annuli, and rectangular annuli that are axis-parallel. Hence, we shall drop the term “axis-parallel”, and any square, rectangle, or annulus we discuss is assumed to be axis-parallel.

Let P be a set of n points in \mathbb{R}^2 . A square or rectangular annulus A is said to be *empty* of P , or just *empty* when there is no confusion, if the interior of A does not contain any point in P . Consider any empty square or rectangular annulus A . Then, A induces a partition of P into two subsets P_{out} and P_{in} such that P_{in} is the set of points in P lying in the interior or on the boundary of the inner square or rectangle of A , and $P_{\text{out}} = P \setminus P_{\text{in}}$. If both P_{out} and P_{in} are nonempty, then we say that A is *valid*. In this paper, we address the following problems:

MAXWIDTHEMPTY SQUARE ANNULUS (MaxESA)

Input: A set of points P in \mathbb{R}^2

Output: A valid empty square annulus A of maximum width

MAXWIDTHEMPTY RECTANGULAR ANNULUS (MaxERA)

Input: A set of points P in \mathbb{R}^2

Output: A valid empty rectangular annulus A of maximum width

The constraint that the resulting empty annulus should be valid is essential to make the problem nontrivial; the same constraint has often been considered in the problem of computing empty objects of maximum size [8,10,11,16]. Throughout the paper, we are interested only in valid empty annuli, so we shall drop the term “valid” unless stated otherwise.

3. Maximum-width empty square annulus

In this section, we present an algorithm that computes a maximum-width valid empty square annulus for a given set P of n points.

Consider any empty square annulus A . Keeping the same partition of P by A , one can enlarge the outer square and shrink the inner square so that some points of P lie on the boundary of the outer and inner squares. This process implies the following observation.

Observation 1. *There exists a maximum-width empty square annulus such that one side of its inner square contains a point of P and one of the following holds: (i) There are a pair of opposite sides of its outer square, each of which contains a point of P , (ii) there are two adjacent sides of its outer square, each of which contains a point of P , and the other two sides are at infinity, or (iii) One side of its outer square contains a point of P and the other three sides are at infinity.*

Proof. Consider any maximum-width empty square annulus A with outer square S , inner square S' , and center c . If there is no point on the boundary of the inner square S' , then we can shrink it with the same center, since A is valid, so that we can strictly increase its width, a contradiction. So, there must be at least one point lying on the boundary of S' . Similarly, if S contains no point of P on its boundary, then we can enlarge it with the same center, keeping S' the same, so that we can strictly increase its width, a contradiction. So, there must be at least one point $p \in P$ lying on the boundary of S .

Without loss of generality, we assume that p lies on the top side of S . If the other three sides of S , contain no point of P , then we move the center c in the direction away from p , keeping p on the top side of S and the width of A unchanged, until one of the other three sides of S hits a point of P . If this enlarging process does not stop, then we end with the three sides of S being at infinity, resulting in case (iii). Otherwise, we now have another side of S containing a point $p' \in P$. There are two cases: either it is the bottom side of S , which is now in case (i), or is the left or right side of S .

In the latter case, we have two adjacent sides of S each of which contains a point of P . Without loss of generality, we assume the second side is the left side. Now, we again move the center in the direction away from the top-left corner of S , keeping p on the top side of S , p' on the left side of S , and the width of A unchanged, until one of the bottom and right sides hits a point of P . If this enlarging process does not stop, then we end with the two sides of S being at infinity, resulting in case (ii). Otherwise, we now have the third side of S containing a point of P , and this falls in case (i). \square

By Observation 1, we now have three different configurations of empty square annuli to search for. If this is case (iii), then observe that it corresponds to a maximum-width empty horizontal or vertical strip, which also can be reduced to the problem of finding the maximum gap in $\{x(p) \mid p \in P\}$ or in $\{y(p) \mid p \in P\}$. Hence, case (iii) can be handled in $O(n)$ time after sorting P .

On the other hand, if this is case (ii), then the resulting square annulus corresponds to a maximum-width empty “axis-parallel” L-shaped corridor. It is known that a maximum-width empty L-shaped corridor over all orientations can be computed in $O(n^3)$ time with $O(n^3)$ space by Cheng [8], while we are seeking only for axis-parallel ones. Here, we give a simple $O(n^2 \log n)$ time algorithm for this problem.

Theorem 1. *Given n points in the plane, one can compute a widest empty axis-parallel corridor in $O(n^2 \log n)$ time using $O(n)$ space.*

Proof. Consider the induced grid by points in P obtained by drawing the horizontal and vertical lines through every point in P . For each grid point o , consider the four quadrants at o obtained by the vertical and horizontal lines through o . Consider

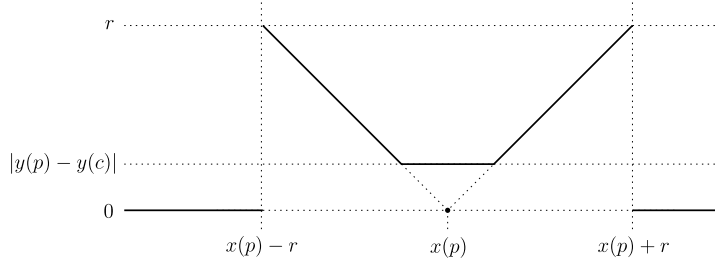


Fig. 2. Illustration of the graph of function f_p for $p \in P$.

the first quadrant, or equivalently the top-right quadrant, Q . By finding out the point with the smallest x -coordinate and the point with the smallest y -coordinate among $P \cap Q$, we can identify a **candidate L-shape empty corridor** with corner o . This query can be handled in $O(\log n)$ time by the segment dragging data structure by Chazelle [7] which can be built in $O(n \log n)$ time using $O(n)$ space.

Since there are $O(n^2)$ grid points, we can compute a maximum-width empty axis-parallel corridor in $O(n^2 \log n)$ time. \square

Now, we suppose that the solution falls in case (i) of Observation 1, so that both two opposite sides of its outer square contains a point of P . Without loss of generality, we assume that each of the top and bottom sides of the outer square of our target annulus contain a point of P . The other case can be handled in a symmetric way.

First, as preprocessing, we sort the given point set P in decreasing order of their y -coordinates, so $P = \{p_1, p_2, \dots, p_n\}$, where $y(p_1) \geq \dots \geq y(p_n)$. We also maintain the list of points in P sorted in their x -coordinates. Our algorithm runs repeatedly for all pairs of indices (i, j) with $1 \leq i < j - 1 < n$, and finds a maximum-width empty square annulus such that the top and bottom sides of its outer square contain p_i and p_j , respectively.

From now on, we assume i and j are fixed. Let $P_{ij} := \{p_{i+1}, \dots, p_{j-1}\}$, $r := (y(p_i) - y(p_j))/2$, and ℓ be the horizontal line with y -coordinate $y(\ell) = (y(p_i) + y(p_j))/2$. Provided that p_i lies on the top side and p_j lies on the bottom side of the outer square, the possible locations of its center is constrained to be on ℓ . For a possible center $c \in \ell$, let $S(c)$ be the square centered at c with radius r . Then, the corresponding inner square $S'(c)$ is determined by center c and the farthest point among those points in P_{ij} lying in the interior of $S(c)$. Here, the distance is measured by the L_∞ metric. More precisely, the radius of $S'(c)$ is exactly $\max_{p \in P_{ij} \cap S(c)} \|p - c\|_\infty$, where $\|\cdot\|_\infty$ denotes the L_∞ norm, and we want to minimize this over the relevant segment $C \subset \ell$ such that $S(c)$ for $c \in C$ contains p_i and p_j on its top and bottom sides. Note that the length of segment $C \subset \ell$ is exactly $2r - |x(p_i) - x(p_j)|$.

For the purpose, we define $f_p(c)$ for each $p \in P_{ij}$ and all $c \in \ell$ to be

$$f_p(c) = \begin{cases} \|p - c\|_\infty & \text{if } \|p - c\|_\infty < r \\ 0 & \text{otherwise} \end{cases},$$

and let $F(c) := \max_{p \in P_{ij}} f_p(c)$ be their upper envelope. Note that our goal is to minimize the upper envelope F over $C \subset \ell$.

Note that $x(p) - r < x(c) < x(p) + r$, or equivalently, $p \in S(c)$ if and only if $\|p - c\|_\infty < r$ for any $p \in P_{ij}$. As also observed in Bae [5], the function f_p is piecewise linear with at most three pieces over $c \in \ell$ such that $p \in S(c)$, and the three pieces have slopes -1 , 0 , and 1 in this order. Moreover, the height of the part of slope 0 is exactly $|y(p) - y(c)| = |y(p) - y(\ell)|$ and the extensions of the two pieces of slope -1 and 1 always cross at the point of x -coordinate $x(p)$ and height 0 . See Fig. 2. These properties of f_p can be easily verified from the behavior of the L_∞ norm. By the above observations, one can explicitly compute F by computing the upper envelope of $O(n)$ line segments in $O(n \log n)$ time [14]. Applying this to all possible pairs (i, j) yields an $O(n^3 \log n)$ time algorithm.

In the following, we show how to improve this to $O(n^3)$ time by decomposing the function f_p into two functions g_p and h_p . For each $p \in P_{ij}$ and $c \in \ell$, define

$$g_p(c) = |x(p) - x(c)| \quad \text{and} \quad h_p(c) = |y(p) - y(c)| \quad \text{if } \|p - c\|_\infty < r,$$

and $g_p(c) = h_p(c) = 0$, otherwise. Also, let

$$G(c) := \max_{p \in P_{ij}} g_p(c) \quad \text{and} \quad H(c) := \max_{p \in P_{ij}} h_p(c).$$

As $\|p - c\|_\infty = \max\{|x(p) - x(c)|, |y(p) - y(c)|\}$, it is obvious that $f_p(c) = \max\{g_p(c), h_p(c)\}$, and hence $F(c) = \max\{G(c), H(c)\}$. We now show that the functions G and H can be explicitly computed in $O(n)$ time.

Lemma 1. *The functions G and H can be explicitly computed in $O(n)$ time.*

Proof. First recall that we know the list of points in P sorted in their x -coordinates in the preprocessing. From this list, we can find out the list of points in P_{ij} sorted in the x -coordinates in linear time.

By above discussions, the graph of function g_p consists of exactly two line segments, except of the two parts of zero, of slopes -1 and 1 , and the breakpoint lies at $x(p)$. That is, all the functions g_p are translates of this V-shaped segments. We first compute the upper envelope of all segments of slope -1 of g_p . This can be done in $O(n)$ time since we know their sorted order. Analogously, we compute the upper envelope of all segments of slope 1 of g_p in $O(n)$ time. Merging these two envelope into the upper envelope G can be done in $O(n)$ time. Thus, we can compute G in $O(n)$ time.

To compute H , observe that the graph of function h_p consists of two zero parts and a horizontal line segment of length $2r$. Since every non-zero segment in functions h_p for $p \in P_{ij}$ has the same length and we know the sorted order, we can compute their upper envelope H in $O(n)$ time. \square

Since the function F is the upper envelope of G and H , we can compute F in $O(n)$ time using the explicit description of functions G and H . Note that the three functions F , G , and H are piecewise linear with $O(n)$ breakpoints. Consequently, we can compute F and find a lowest point of F over $C \subset \ell$ in $O(n)$ time, and hence a maximum-width empty square annulus of case (i) can be found in $O(n^3)$ time. Finally, we conclude the following theorem.

Theorem 2. *Given n points in the plane, a maximum-width empty square annulus can be computed in $O(n^3)$ time using $O(n)$ space.*

Proof. Recall Observation 1 stating that there are three cases of a possible maximum-width empty square annulus. In the overall algorithm, we check all the three possibilities as follows: For case (i), we run the above algorithm in $O(n^3)$ time. For case (ii), we compute a maximum-width empty axis-parallel L-shaped corridor in $O(n^2 \log n)$ time by Theorem 1. For case (iii), we compute a maximum-width empty axis-parallel strip in $O(n)$ time. Thus, the correctness directly follows from Observation 1. All these routines use $O(n)$ space. \square

4. Maximum-width empty rectangular annulus

In this section, we present an algorithm computing a maximum-width empty rectangular annulus. First we give several basic observations on maximum-width empty rectangular annuli.

4.1. Configurations of empty rectangular annuli

Consider any empty rectangular annulus A and the partition of P induced by A . As done for square annuli before, one can enlarge the outer rectangle of A and shrink its inner rectangle, while keeping the partition of P and not decreasing the width of A . This results in the following observation.

Observation 2. *There exists a maximum-width empty rectangular annulus such that each side of its outer rectangle either contains a point of P or lies at infinity, and every side of its inner rectangle contains a point of P .*

Proof. Let A be any empty rectangular annulus with outer rectangle R_{out} and inner rectangle R_{in} . We first enlarge R_{out} to obtain a new outer rectangle R'_{out} as follows: If the top side of R_{out} contains no point of P , then we slide it upwards until it hits a point of P . If this process stops at some point, then the top side contains a point of P ; otherwise, it is now at infinity. Afterwards, we perform the same process for the other three sides, one by one. Hence, each side of the resulting rectangle R'_{out} either contains a point of P or lies at infinity.

Next, we shrink R_{in} by sliding each side of R_{in} inwards until it hits a point of P , and let R'_{in} be the resulting rectangle. Since A is assumed to be valid, there is at least one point of P in the interior or on the boundary of R_{in} , which implies that every side of R'_{in} contains a point of P .

Now, consider a new rectangular annulus A' defined by its outer rectangle R'_{out} and inner rectangle R'_{in} . By the above processes, it is obvious that A' is also empty and the width of A' is not smaller than that of A . Moreover, A' satisfies the condition described in the statement. By our construction, we show that there always exists such an annulus A' for any empty rectangular annulus, so the observation follows. \square

In Observation 2, note that each side of a rectangle is considered to include its endpoints. Thus, a point $p \in P$ can be contained in two adjacent sides of a rectangle if p is located at a corner. See Fig. 3(left).

A rectangular annulus A is said to be *width-uniform*, or simply *uniform*, if its top-width, bottom-width, left-width, and right-width are all equal to its width. In the following observation, we show that we can focus only on uniform rectangular annuli to solve our problem.¹

¹ Recently Bae [6] have studied the minimum-area rectangular and square annulus problems where two different definitions of rectangular annuli: uniform and non-uniform are considered. The uniform and non-uniform versions are different in “min-area” problems.

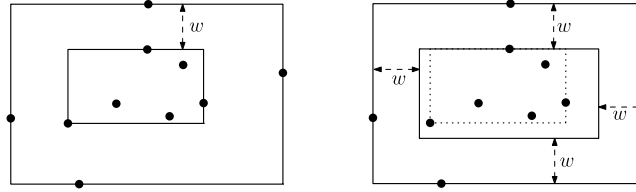


Fig. 3. Empty rectangular annuli of maximum width w : (left) Each side of the inner and outer rectangles contains a point. (right) A maximum-width empty rectangular annulus that is uniform and also top-anchored.

Observation 3. *There exists a maximum-width empty rectangular annulus A that is uniform such that the following property holds: each side of its outer rectangle either contains a point of P or lies at infinity, and at least one side of its inner rectangle contains a point of P .*

Proof. Let A be a maximum-width empty rectangular annulus with the property described in Observation 2. Let R_{out} and R_{in} be the outer and inner rectangles of A , and w be the width of A .

Without loss of generality, we assume that w is equal to the top-width of A , that is, the top-width is the smallest among the other three. We then enlarge the inner rectangle R_{in} to have a new inner rectangle R'_{in} by sliding the left side to the left, the right side to the right, and the bottom side downwards so that the resulting annulus A' formed by R_{out} and R'_{in} is uniform and the width of A' is equal to w , the width of A . See Fig. 3 for an illustration. Since $R_{in} \subseteq R'_{in}$ and thus $A' \subseteq A$, we conclude that A' is also empty of P . The proof is done by also observing that every side of R_{out} contains at least one point of P and the top side of R'_{in} contains at least one point of P by Observation 2. \square

See Fig. 3(right) for an illustration of Observation 3. This observation suggests a specific configuration of annuli for us to solve the problem. First of all, we do not have to consider non-uniform annuli. Moreover, candidate outer rectangles are defined by at most four points in P . If we fix an outer rectangle R , then the inner rectangle that maximizes the width is also determined by searching points in $P \cap R$. This already yields an $O(n^5)$ -time algorithm for our problem.

Let A be an empty rectangular annulus satisfying the condition described in Observation 2. We call A *top-anchored* (or, *bottom-anchored*, *left-anchored*, *right-anchored*) if both the top sides (or, bottom sides, left sides, right sides, resp.) of the outer and inner rectangles of A contain a point of P . For example, Fig. 3(right) shows an empty top-anchored rectangular annulus.

Observation 4. *There exists a maximum-width empty rectangular annulus A that satisfies the condition described in Observation 3 and is either top-anchored, bottom-anchored, left-anchored, or right-anchored.*

Proof. Let A be a maximum-width empty rectangular annulus that satisfies the condition described in Observation 3. Note that A is guaranteed to be uniform. Now, suppose that A is neither top-anchored, bottom-anchored, left-anchored, nor right-anchored. By Observation 3, at least one side of the inner rectangle of A contains a point of P . Without loss of generality, assume that the top side of the inner rectangle contains a point of P . Since A is not top-anchored, the top side of the outer rectangle contains no point of P and is thus at infinity. This implies that its top-width is unbounded. Since A is uniform, this should be the case for the other three bottom-width, left-width, and right-width. This means that all sides of the outer rectangle of A lie at infinity, and that no point of P lies on the boundary or outside of the outer rectangle. This contradicts to the assumption that A is valid. \square

Our algorithm will find an empty anchored and uniform rectangular annulus of maximum width, which is the correct answer to our problem by Observation 4. In the following, we assume without loss of generality that there exists a maximum-width empty annulus that is uniform and top-anchored, and describe our algorithm for this case. The other three cases can be handled analogously.

Let $P = \{p_1, p_2, \dots, p_n\}$ be the given set of points, sorted in the descending order of their y -coordinates, that is, $y(p_1) \geq y(p_2) \geq \dots \geq y(p_n)$. Consider any empty top-anchored rectangular annulus A that satisfies the condition of Observation 3. Let $p_i \in P$ be the point lying on the top side of the outer rectangle of A . By Observation 3, either the bottom side of the outer rectangle is at infinity or there is another point $p_j \in P$ for $i < j \leq n$ on it. If the bottom side is at infinity, then we say that a point p_∞ at infinity in the $(-y)$ -direction lies on the bottom side. Thus, in either case, there is p_j on the bottom side of the outer rectangle for $i < j \leq n$ or $j = \infty$.

Since A is top-anchored, there is a third point $p_k \in P$ on the top side of the inner rectangle of A . Observe that the width of A is determined by the y -difference of p_i and p_k , that is, $y(p_i) - y(p_k)$. Thus, the maximum width for top-anchored empty rectangular annuli is one among $O(n^2)$ values $\{y(p_i) - y(p_k) \mid 1 \leq i \leq k \leq n\}$.

The problem becomes even simpler if we fix p_i on the top side of the outer rectangle, since the number of possible widths is reduced to at most n . An outlook of our algorithm that computes a maximum-width empty top-anchored rectangular annulus is as follows: (1) For each $p_i \in P$, find an empty annulus A_i^* with p_i lying on the top side of its outer

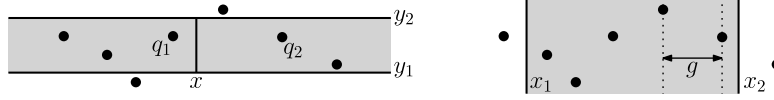


Fig. 4. (left) the y -range x -neighbor query for (x, y_1, y_2) and its answer q_1 and q_2 , (right) the range maximum-gap query for (x_1, x_2) and the maximum gap is g .

rectangle whose width is the maximum among the set $\{y(p_i) - y(p_k) \mid i < k \leq n\}$ and then (2) output the one with maximum width among A_i^* for all $i \in \{1, \dots, n\}$. In order to compute A_i^* , we try all possible points p_j that bound the bottom side of the outer rectangle.

In the following subsections, we first study the case where two points p_i and p_j on the top and bottom sides are fixed, and then move on to the case where only a point p_i on the top side is fixed. More precisely, we discuss a decision algorithm when two points on the top and bottom sides are fixed, and exploit it as a sub-procedure to solve the other case.

4.2. Decision when two points on top and bottom are fixed

Suppose that we are given p_i and p_j with $2 \leq i+1 < j \leq n$ or $j = \infty$, and we consider only empty rectangular annuli whose outer rectangle contains p_i and p_j on its top and bottom sides, respectively.

Here, we consider the following decision problem.

Given: A positive real $w > 0$
 Task: Does there exist an empty rectangular annulus of width at least w whose outer rectangle contains p_i and p_j on its top and bottom sides, respectively?

Let $D_{ij}(w)$ denote the outcome of the above decision problem.

Observation 5. If $D_{ij}(w)$ is TRUE, then $D_{ij}(w')$ is TRUE for any $w' \leq w$. On the other hand, if $D_{ij}(w)$ is FALSE, then $D_{ij}(w')$ is FALSE for any $w' \geq w$.

Let $P_{ij} := \{p_{i+1}, \dots, p_{j-1}\}$ for $i < j \leq n$, and $P_{i\infty} := \{p_{i+1}, \dots, p_n\}$. In the following, we show that the decision problem for a given width $w > 0$ can be solved by a combination of certain operations on points P_{ij} , namely, the y -range x -neighbor query and the range maximum-gap query. Each of the two operations is described as follows:

- (i) The y -range x -neighbor query: Given three real numbers (x, y_1, y_2) , this operation is to find two points q_1 and q_2 in P_{ij} such that q_1 is the rightmost one among points $P_{ij} \cap [-\infty, x] \times [y_1, y_2]$ and q_2 is the leftmost one among points $P_{ij} \cap [x, \infty] \times [y_1, y_2]$ (Fig. 4). Either q_1 and q_2 may be undefined if there is no point of $P_{i,j}$ in the corresponding range. If q_1 is undefined, then we return q_1 as a point at infinity such that $x(q_1) = -\infty$ and $y(q_1) = y_1$; if q_2 is undefined, then we return q_2 such that $x(q_2) = \infty$ and $y(q_2) = y_1$.
- (ii) The range maximum-gap query (in x -coordinates): Given two real numbers (x_1, x_2) , find the maximum gap in the set of real numbers $\{x(p) \mid x_1 \leq x(p) \leq x_2, p \in P_{ij}\} \cup \{x_1, x_2\}$, where $x(p)$ denotes the x -coordinate of point p . The maximum gap in a set X of real numbers is the maximum difference between two consecutive elements when X is sorted. Notice that x_1 and x_2 are also included in the above set. Here, the output of the range maximum-gap query is to be the pair of values that define the maximum gap.

We describe our algorithm for the decision problem as Algorithm 1.

Our decision algorithm, Algorithm 1, evaluates $D_{ij}(w)$ for a given w . As described in Algorithm 1, the decision is made by four calls of the y -range x -neighbor queries and the range maximum-gap queries. Thus, its running time depends on how efficiently we can handle these queries. If the algorithm decides that $D_{ij}(w)$ is TRUE, then it also returns a corresponding rectangular annulus, that is, an empty uniform annulus of width w with p_i and p_j on the top and bottom sides of the outer rectangle. This can be done by constructing its outer rectangle since its width w is fixed.

In the following, we show the correctness of our decision algorithm.

Lemma 2. Algorithm 1 correctly computes $D_{ij}(w)$ for any given $w > 0$ in time $O(T)$, where T is an upper bound on time needed to perform a y -range x -neighbor query or a range maximum-gap query. Moreover, if $D_{ij}(w)$ is TRUE, then an empty rectangular annulus of width w such that p_i and p_j lie on the top and bottom side of its outer rectangle can be found in the same time bound.

Proof. It is straightforward to see that the time complexity is $O(T)$, since it runs the y -range x -neighbor queries and the range maximum-gap queries at most four times. Hence, we focus on the correctness proof. Without loss of generality, we assume that $x(p_i) \leq x(p_j)$.

Algorithm 1: Decision algorithm.

Input: a width $w > 0$
Output: $D_{ij}(w)$, and an empty rectangular annulus of width w with p_i and p_j lying on the top and bottom sides of its outer rectangle, respectively, if $D_{ij}(w)$ is TRUE

```

1 if  $y(p_i) - y(p_j) < 2w$  then
2   | Return FALSE.
3 Perform a  $y$ -range  $x$ -neighbor query for  $(x(p_i), y(p_i) - w, y(p_i))$ , and let  $q_l$  and  $q_r$  be the output with  $x(q_l) \leq x(p_i) \leq x(q_r)$ .
4 Perform a  $y$ -range  $x$ -neighbor query for  $(x(p_j), y(p_j), y(p_j) + w)$ , and let  $q'_l$  and  $q'_r$  be the output with  $x(q'_l) \leq x(p_j) \leq x(q'_r)$ .
5 Let  $p_l$  be the rightmost one in  $\{q_l, q'_l\}$  and  $p_r$  be the leftmost one in  $\{q_r, q'_r\}$ .
6 if  $\min\{x(p_i), x(p_j)\} < x(p_l) < \max\{x(p_i), x(p_j)\}$  or  $\min\{x(p_i), x(p_j)\} < x(p_r) < \max\{x(p_i), x(p_j)\}$  then
7   | Return FALSE.
8 Perform a range maximum-gap query for  $(x(p_l), \min\{x(p_i), x(p_j)\} + w)$ , and let  $(l, l')$  be the output and  $g_l := l' - l$  be the corresponding maximum gap.
9 Perform a range maximum-gap query for  $(\max\{x(p_i), x(p_j)\} - w, x(p_r))$ , and let  $(r', r)$  be the output and  $g_r := r - r'$  be the corresponding maximum gap.
10 if  $g_l \geq w$  and  $g_r \geq w$  then
11   | Return TRUE, and the rectangular annulus of width  $w$  whose outer rectangle is defined by the top and bottom sides through  $p_i$  and  $p_j$ , respectively, and the left and right sides at  $x=l$  and  $x=r$ , respectively.
12 else
13   | Return FALSE.

```

First, we show that if Algorithm 1 for input w returns FALSE, then $D_{ij}(w)$ is FALSE. There are two cases where Algorithm 1 returns FALSE: Algorithm 1 returns FALSE either (1) at line 2, (2) at line 7 or (3) at line 13. If $y(p_i) - y(p_j) < 2w$, then we cannot build a rectangular annulus of width w with p_i and p_j on the top and bottom sides of its outer rectangle. Thus, if Algorithm 1 returns FALSE either at line 2, then this is clearly correct.

Suppose case (2), so Algorithm 1 returns FALSE at line 7. From lines 3 and 4, we have the following:

- q_l is the rightmost point in $P_{ij} \cap ([-\infty, x(p_i)] \times [y(p_i) - w, y(p_i)])$.
- q_r is the leftmost point in $P_{ij} \cap ([x(p_i), \infty] \times [y(p_i) - w, y(p_i)])$.
- q'_l is the rightmost point in $P_{ij} \cap ([-\infty, x(p_j)] \times [y(p_j), y(p_j) + w])$.
- q'_r is the leftmost point in $P_{ij} \cap ([x(p_j), \infty] \times [y(p_j), y(p_j) + w])$.

Also, by line 5, p_l is the rightmost one among $\{q_l, q'_l\}$, and p_r is the leftmost one among $\{q_r, q'_r\}$. Then, in this case, we have

$$x(p_i) < x(p_l) < x(p_j) \quad \text{or} \quad x(p_i) < x(p_r) < x(p_j),$$

by the condition in line 6. Assume the first condition holds, so $x(p_i) < x(p_l) < x(p_j)$. This implies that $p_l = q'_l$. By our construction, we have $y(p_l) = y(q'_l) \in [y(p_j), y(p_j) + w]$. Consider now any rectangular annulus A of width w whose outer rectangle R_{out} contains p_i and p_j on its top and bottom sides, respectively, and observe that A must contain the point p_l . Thus, such an annulus A cannot be empty, so $D_{ij}(w)$ is FALSE. The other case where it holds that $x(p_i) < x(p_r) < x(p_j)$ can be handled analogously.

Now, suppose case (3), so Algorithm 1 returns FALSE at line 13. Then, by lines 8 and 9, so that we have

- (l, l') defines the maximum gap $g_l = l' - l$ in range $[x(p_l), x(p_i) + w]$.
- (r', r) defines the maximum gap $g_r = r - r'$ in range $[x(p_j) - w, x(p_r)]$.

In this case, we have

$$g_l < w \quad \text{or} \quad g_r < w,$$

by the condition in line 10. Assume that the first condition holds, so $g_l < w$. This means that there is no vertical strip of width at least w among points in P_{ij} in the x -range $[x(p_l), x(p_i) + w]$. Consider now any rectangular annulus A of width w whose outer rectangle R_{out} contains p_i and p_j on its top and bottom sides, respectively. If the left side of R_{out} is to the left of p_l , then A contains p_l by the construction; otherwise, any vertical strip of width w between $[x(p_l), x(p_i) + w]$ contains a point in P_{ij} . Since the left side of R_{out} cannot be to the right of p_i , this implies that no such annulus A is empty in this case, so $D_{ij}(w)$ is indeed FALSE. The other case where it holds that $g_r < w$ can be handled analogously.

Next, we show that if Algorithm 1 for input w returns TRUE, then $D_{ij}(w)$ is correctly TRUE. If Algorithm 1 for input w returns TRUE, then we have the following:

$$x(p_l) \leq x(p_i), \quad x(p_r) \geq x(p_j), \quad g_l \geq w, \quad g_r \geq w,$$

by the conditions in lines 6 and 10. Then, we can construct a rectangle R_{out} such that its top and bottom sides go through p_i and p_j , respectively, and the x -coordinates of its left and right sides are l and r , respectively. Let A be the rectangular annulus of width w whose outer rectangle is R_{out} . Then, by our construction, A is empty with respect to P_{ij} . Since A is completely contained in the horizontal strip defined by the horizontal lines through p_i and p_j , A is also empty with respect

Algorithm 2: Computing A_i^* .

Input: A set $P = \{p_1, \dots, p_n\}$ of points sorted by y -coordinates, and a point $p_i \in P$
Output: A_i^* with p_i and its width w_i^*

- 1 Set k to be $i+1$, w to be 0, and A to be any annulus of width 0.
- 2 Build the data structure \mathcal{D} for P and initialize \mathcal{X} to be $\mathcal{X}_{i,i}$. // Recalling data structures for \mathcal{D} and \mathcal{X} from section 4.2
- 3 **for** each $j = i+2, \dots, n$ and $j = \infty$ **do**
- 4 Insert p_{j-1} if $j \leq n$, or p_n if $j = \infty$, into \mathcal{X} , so that now $\mathcal{X} = \mathcal{X}_{ij}$.
- 5 **while** $D_{ij}(y(p_i) - y(p_k))$ is TRUE and $k \leq j$ **do**
- 6 Set A to be the corresponding annulus of width $y(p_i) - y(p_k)$.
- 7 Set w to be $y(p_i) - y(p_k)$.
- 8 Increase k by 1.
- 9 Return the current A as A_i^* and the current w as w_i^* .

to the whole set P . This shows the existence of an empty annulus A of width w such that the top and bottom sides of its outer rectangle contains p_i and p_j , respectively. Therefore, we conclude that $D_{ij}(w)$ is TRUE. In this case, Algorithm 1 indeed returns the annulus A . This completes the proof. \square

Note that the two operations can be easily done in linear time. We now show how to perform them in logarithmic time with an aid of the following data structures.

- Let \mathcal{D} be the data structure on P described in Chazelle [7] that supports a segment dragging query for vertical line segments dragged by two horizontal rays. A segment dragging query is given by a segment and a direction along two rays and it returns the first point in P that is hit by the dragged segment. This structure can be constructed in $O(n \log n)$ time using $O(n)$ storage
- Let \mathcal{X}_{ij} be a 1D range tree for the x -coordinates of points in P_{ij} with an additional field $\text{maxgap}(v)$ at each node v , where $\text{maxgap}(v)$ denotes the maximum gap in the canonical subset of v [9]. Note that $\text{maxgap}(v) = 0$ if the canonical subset of v consists of only one element. The structure \mathcal{X}_{ij} can be constructed using storage $O(|P_{ij}|)$ [9].

Now, suppose that we have already built these two structures \mathcal{D} and \mathcal{X}_{ij} . Then, the two operations can be handled in $O(\log n)$ time as follows:

- (i) For a y -range x -neighbor query for (x, y_1, y_2) , we perform two segment dragging queries on \mathcal{D} for a vertical line segment with endpoints (x, y_1) and (x, y_2) to both the left and the right directions. These two queries result in the rightmost point q_1 in the range $[-\infty, x] \times [y_1, y_2]$ and the leftmost point q_2 in the range $[x, \infty] \times [y_1, y_2]$.
- (ii) For a range maximum-gap query for (x_1, x_2) , we perform a 1D range search for the x -range $[x_1, x_2]$ on \mathcal{X}_{ij} again to obtain a collection C of $O(\log n)$ nodes. The maximum gap in the x -coordinates of the points in P_{ij} in the range $[x_1, x_2]$ can be found by comparing $O(\log n)$ values: $\text{maxgap}(v)$ for all $v \in C$ and every gap between two consecutive canonical subset.

Therefore, we conclude the following:

Lemma 3. Suppose that we already have two tree structures \mathcal{D} and \mathcal{X}_{ij} . Then, Algorithm 1 correctly computes $D_{ij}(w)$ for any given $w > 0$ in time $O(\log n)$.

4.3. Optimization when only a point on top is fixed

Next, we describe how to find a maximum-width empty top-anchored rectangular annulus such that p_i lies on the top side of the outer rectangle.

Let w_i^* be the width of A_i^* . Observe that w_i^* lies in the set $W_i := \{y(p_i) - y(p_k) \mid i \leq k \leq n\}$. Instead of solving the optimization problem for each pair (p_i, p_j) , we can rather solve the optimization problem when only a point p_i on top is fixed. Our algorithm that computes A_i^* and its width w_i^* is presented as in Algorithm 2.

Lemma 4. Algorithm 2 can be implemented in $O(n \log n)$ time and $O(n)$ space for a fixed $p_i \in P$. Also, it correctly computes w_i^* and A_i^* .

Proof. In Algorithm 2, we maintain several variables:

- A is the currently best empty top-anchored annulus with p_i on the top side of its outer rectangle.
- w is the width of A .
- k is the index such that the next larger width $y(p_i) - y(p_k)$ is currently being tried. It always holds that $w = y(p_i) - y(p_{k-1})$.

- \mathcal{X} is the 1D range tree with additional field $\text{maxgap}(v)$ at each node defined as above on the x -coordinates of the points in P_{ij} , that is, $\mathcal{X} = \mathcal{X}_{ij}$ in the main loop.

After initializing these variables properly in lines 1-2, the main loop runs for $j = i + 2, \dots, n$. Let w_{ij} be defined as above. For each j , we keep the best annulus we have seen so far as A , not computing the exact value of w_{ij} . That is, we keep

$$w = \max\{0, w_{i(i+2)}, w_{i(i+3)}, \dots, w_{ij}\}$$

as the loop invariant for each j . In this way, after the main loop has finished at $j = \infty$, we have

$$w = \max_{j=i+2, \dots, n, \infty} w_{ij} = w_i^*$$

and the corresponding annulus of width $w = w_i^*$ is stored in variable A .

Hence, the correctness of Algorithm 2 is guaranteed. The time complexity is bounded by $O(n \log n)$ as follows.

Lines 1-2 in Algorithm 2 takes $O(n \log n)$ time since \mathcal{D} can be built in $O(n \log n)$ time [7] and \mathcal{X}_{ii} is initialized in $O(1)$ time as $P_{i,i} = \emptyset$. In line 4, a point p_{j-1} or p_n is inserted into two tree data structures \mathcal{T} and \mathcal{X} . The structures \mathcal{X} is known to support an insertion in logarithmic time [9]. The additional information $\text{maxgap}(v)$ at each node v of \mathcal{X} can be correctly updated in the bottom-up fashion through the path to the root from the newly inserted node. Hence, line 4 can be implemented in $O(\log n)$ time. In line 5, we call the decision algorithm, Algorithm 1, and one execution of line 5 takes $O(\log n)$ time by Lemma 3, since the necessary data structures \mathcal{D} and $\mathcal{X} = \mathcal{X}_{ij}$ are provided. Lines 6-8 takes only $O(1)$ time.

To conclude the total running time, observe that line 4 is executed at most $n - 1$ times. We assert that the “while” loop in line 5 is executed at most $2n - 3$ times. To see why, observe that whenever the loop condition fails, j is incremented and therefore false case is bounded by $n - 2$ whereas when the loop condition holds, k is incremented and thus true case is bounded by $n - 1$.

Consequently, the time complexity of Algorithm 2 is bounded by $O(n \log n)$. Since the segment dragging query structure \mathcal{D} and the 1D range tree \mathcal{X} use $O(n)$ space [7,9], the space usage of Algorithm 2 is bounded by $O(n)$. \square

4.4. Putting it all together

We are now ready to describe the overall algorithm to solve the MaxERA problem. Under the assumption that there exists a maximum-width empty rectangular annulus A^* that satisfies the condition of Observation 3 and is top-anchored, we execute Algorithm 2 for each $i = 1, \dots, n - 1$ and choose the one with the maximum width as A^* . Its correctness is guaranteed by Lemma 4. The other three cases where there is a maximum-width empty rectangular annulus that satisfies the condition of Observation 3 and is either bottom-anchored, left-anchored, or right-anchored, can be handled in a symmetric way. Thus, the overall algorithm runs for the four cases and outputs one with the maximum width.

Theorem 3. *Given a set P of n points in the plane, a maximum-width rectangular annulus that is empty with respect to P can be computed in $O(n^2 \log n)$ time and $O(n)$ space.*

Proof. The correctness follows from the above discussion and Observation 4.

For the time complexity, we mainly call Algorithm 2 $O(n)$ times. Thus, it takes $O(n^2 \log n)$ time by Lemma 4. The space usage can be bounded by $O(n)$ again by Lemma 4. \square

5. Concluding remarks

In this paper, we addressed the problem of computing a maximum-width empty square or rectangular annulus that avoids a given set P of n points in the plane, and presented two efficient algorithms. Our algorithms run in $O(n^3)$ and $O(n^2 \log n)$ time, respectively, for a square and rectangular annulus. Note that our algorithms are the first nontrivial algorithms that solve these problems, and considered to be efficient compared to the currently best algorithm for the circular counterpart, which runs in $O(n^3 \log n)$ time [11].

There are two obvious open questions. One is whether there is an asymptotically faster algorithm for either problem, and the other is to obtain a lower bound on the computational complexity of the empty annulus problem. The circular, square, and rectangular versions of the problem are now known to be solved in $O(n^3 \log n)$, $O(n^3)$, and $O(n^2 \log n)$ time, respectively. At this moment, it seems difficult to improve each of these upper bounds. On the other hand, no nontrivial lower bound, other than $\Omega(n)$, is known for these problems. This simply means that nobody fully understand the intrinsic complexity of this type of problems for now.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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