Question (1). Given that $p_Y(y) > 0$, we have $p_{X|Y}(x|y) = p_{X,Y}(x,y)/p_Y(y)$. Summing over x, we have $\sum_x p_{X|Y}(x|y) = (1/p_Y(y))\sum_x p_{X,Y}(x,y)$. The sum on the right is precisely $p_Y(y)$ as mentioned in class, so the result follows.

Question (2). Recall that X_1 is the number of trials before a success is achieved and X_2 has the same distribution. Since they are independent, $X_1 + X_2$ is the number of trials before a second success. Given that a second success occurs at time n, we expect the first success to occur with probability $1/(n-1)$ at all of the previous times.

We verify this as follows:

$$\begin{aligned}
\mathbb{P}(X_1 = i|X_1 + X_2 = n) &= \frac{\mathbb{P}(X_1 = i, X_2 = n-i)}{\mathbb{P}(X_1 + X_2 = n)} \\
&= \frac{(1-p)^{i-1}p(1-p)^{n-i-1}p}{\sum_{j=1}^{n-1}\mathbb{P}(X_1 = j, X_2 = n-j)} \\
&= \frac{(1-p)^{n-2}p^2}{\sum_{j=1}^{n-1}(1-p)^{j-1}p(1-p)^{n-j-1}p} \\
&= \frac{(1-p)^{n-2}p^2}{(n-1)(1-p)^{n-2}p^2} = 1/(n-1).
\end{aligned}$$

Question (3). Given that $Y = 1$, X is 1 with probability 1/5; 2 with probability 3/5 and 3 with probability 1/5. $\mathbb{E}(X|Y = 1) = 2$.

Given that $Y = 2$, X is 1 with probability 2/3 and 3 with probability 1/3. $\mathbb{E}(X|Y = 2) = 5/3$.

Given that $Y = 3$, X is 2 with probability 3/5 and 3 with probability 2/5 so $\mathbb{E}(X|Y = 3) = 12/5$.

Question (4). Either: Using Q3, the expectation of X given the value of Y depends on the value of Y so X is not independent of Y.

Or: The probability that $X = 1$ is positive. The probability that $Y = 3$ is positive, but $\mathbb{P}(X = 1, Y = 3)$ is 0 so they are not independent.

Question (19). See book.

Question (27). Let X_1 denote the time of the first T; X_2 denote the time of the first TT; and X_3 denote the time of the first TTH.

We know $\mathbb{E}X_1 = 1/(1-p)$ as X_1 is a geometric distribution with parameter $1-p$.

We have $\mathbb{E}(X_2|X_1) = (1-p)(1+X_1) + p(1 + X_1 + \mathbb{E}X_2)$ as having achieved T, in the next go, we either get another T, in which case we have achieved TT (in $1+X_1$ steps) or we are back at the beginning, so having already used $1+X_1$ steps, the expected number of further steps is $\mathbb{E}X_2$. Taking expectations of everything, we have $\mathbb{E}X_2 = 1 + \mathbb{E}X_1 + p\mathbb{E}X_2$, so that $\mathbb{E}X_2 = 2 + 2\mathbb{E}X_1 = (2-p)/(1-p)^2$.

Once we have achieved TT, we may get T's for a while, but then we stop when we get our first H. It follows that X_3 is distributed as X_2 plus an independent geometric random variable with parameter p so $\mathbb{E}X_3 = 1/p + \mathbb{E}X_2 = 1/(p(1-p)^2)$.

Question (31). The trick is to condition on what comes first.

Then if a 1 comes first, the run length is one less than the position of the first 0. Given that a 1 came first, the position of the first 0 is one plus a geometric random variable with parameter $1-p$ so that given that a 1 comes first, the expected length of the first run is $1/(1-p)$. The second run is then a run of 0's and similarly its expected length is $1/p$.

On the other hand, if a 0 comes first, the first and second runs are of 0's and 1's respectively so the expected lengths are $1/p$ and $1/(1-p)$.

Using conditioning, the first run is therefore of expected length $p/(1-p)+(1-p)/p$. The second run is of expected length 2.

Question (32). There cannot be at least n successes and at least m failures before time $n+m$. Let T denote the total time to achieve at least n successes and at least m failures. We condition on the number X of successes up to time $n+m$.

We have $\mathbb{P}(X=k)=\binom{n+m}{k}p^k(1-p)^{n+m-k}$.

We see that if $X=n+r$, then there have been $n+r$ successes and $m-r$ failures, so it is necessary to wait until there have been r additional failures. The expected additional time is then $r/(1-p)$.

Similarly if $X=n-r$, there have been $n-r$ successes and $m+r$ failures, so it is necessary to wait for r additional successes, so the expected additional time is r/p.

We therefore have $\mathbb{E}(T|X=n-r)=n+m+r/p$, while $\mathbb{E}(T|X=n+r)=n+m+r/(1-p)$.

From this, we see that

$$\mathbb{E}T=n+m+\sum_{r=1}^{m}\binom{n+m}{n+r}p^{n+r}(1-p)^{m-r}r/(1-p)+\sum_{r=1}^{n}\binom{n+m}{n-r}p^{n-r}1-p^{m+r}r/p.$$