The algorithms to derive the frequency and weight of roles within interconnection motifs with intra-guild interactions for connector species

Notation

1. For binary networks

The two subnetworks are represented by P and Q and three intra-guild association matrices A, B and C:

$$P_{ab} = \begin{cases} 1 & \text{if } a \text{ interact to } b \\ 0 & \text{if } a \text{ not interact to } b \end{cases}$$

$$Q_{bc} = \begin{cases} 1 & \text{if } b \text{ interact to } c \\ 0 & \text{if } b \text{ not interact to } c \end{cases}$$

$$A_{aa'} = \begin{cases} 1 & \text{if } a \text{ interact to } a' \\ 0 & \text{if } a \text{ not interact to } a' \end{cases}$$

$$B_{bb'} = \begin{cases} 1 & \text{if } b \text{ interact to } b' \\ 0 & \text{if } b \text{ not interact to } b' \end{cases}$$

$$C_{cc'} = \begin{cases} 1 & \text{if } c \text{ interact to } c' \\ 0 & \text{if } c \text{ not interact to } c' \end{cases}$$

We define A represents the number of rows of P, and B represents both the number of columns of P and the number of rows of Q; C represents the number of columns of Q.

$\mathcal{A}_{aa'} = \begin{cases} 1 & if\ A_{aa'} = 0 \\ 0 & if\ A_{aa'} > 0 \end{cases}$ is the complement of A, and $\mathcal{A}_{aa'} = 0$ while $a = a'$.

$\mathcal{B}_{bb'} = \begin{cases} 1 & if\ B_{bb'} = 0 \\ 0 & if\ B_{bb'} > 0 \end{cases}$ is the complement of B, and $\mathcal{B}_{bb'} = 0$ while $b = b'$.

$\mathcal{C}_{cc'} = \begin{cases} 1 & if\ C_{cc'} = 0 \\ 0 & if\ C_{cc'} > 0 \end{cases}$ is the complement of C, and $\mathcal{C}_{cc'} = 0$ while $c = c'$.

$O_b = \sum_{a=1}^{A} P_{ab}$ defines a vector of length B, in which each element is the sum of each column of P and represents the weighted degree of the nodes of group b in the P subnetwork.

$R_b = \sum_{c=1}^{C} Q_{bc}$ defines a vector of length B, in which each element is the sum of each row of Q and represents the weighted degree of the nodes of group b in the Q subnetwork.

$G_b = \frac{1}{2}\sum_{a=1}^{A} (P^T \cdot A)_{ba} {*} P_{ab}{}^T$ defines a vector of length B, in which each element represents the sum of pairwise connected intra-guild degrees of the nodes of group b in the P subnetwork.

$\mathcal{G}_b = \frac{1}{2}\sum_{a=1}^{A} (P^T \cdot \mathcal{A})_{ba} P_{ab}{}^T$ defines a vector of length B, in which each element represents the sum of pairwise disconnected intra-guild degrees of the nodes of group b in the P subnetwork.

$W_b = \frac{1}{2}\sum_{c=1}^{C} (Q \cdot C)_{bc} {*} Q_{bc}$ defines a vector of length B, in which each element represents the sum of pairwise connected intra-guild degrees of the nodes of group b in the Q subnetwork.

$\mathcal{W}_b = \frac{1}{2}\sum_{c=1}^{C} (Q \cdot \mathcal{C})_{bc} {*} Q_{bc}$ defines a vector of length B, in which each element represents the sum of pairwise disconnected intra-guild degrees of the nodes of group b in the Q subnetwork.

Explanation: contributing to matrix P.

$$\mathcal{P}_{ab} = \begin{cases} 1 & if\ P_{ab} = 0 \\ 0 & if\ P_{ab} > 0 \end{cases}$$ is the complement of P

$$U_{bb'} = \sum_{a}^{A} P_{ab} P_{ab'} = (P^T P)_{bb'}$$

U: this is a matrix of dimension $B \times B$. For two columns b, b' in P, i.e. two nodes b, b' in the b-node group, entry bb' gives the following: it counts the sum of products of pairwise weighted degrees of nodes in the a-node group, which are adjacent to both b and b'.

$$D = [\mathrm{vec}(P_b \cdot P_b^T \circ A)^T]_{b=1}^{B} \in \mathbb{R}^{b \times a^2}$$

$$E = [\mathrm{vec}(P_b \cdot \mathcal{P}_b^T \circ A)^T]_{b=1}^{B} \in \mathbb{R}^{b \times a^2}$$

$$F = [\mathrm{vec}(\mathcal{P}_b \cdot P_b^T \circ A)^T]_{b=1}^{B} \in \mathbb{R}^{b \times a^2}$$

$$\mathcal{D} = [\mathrm{vec}(P_b \cdot P_b^T \circ \mathcal{A})^T]_{b=1}^{B} \in \mathbb{R}^{b \times a^2}$$

$$\mathcal{E} = [\mathrm{vec}(P_b \cdot \mathcal{P}_b^T \circ \mathcal{A})^T]_{b=1}^{B} \in \mathbb{R}^{b \times a^2}$$

$$\mathcal{F} = [\mathrm{vec}(\mathcal{P}_b \cdot P_b^T \circ \mathcal{A})^T]_{b=1}^{B} \in \mathbb{R}^{b \times a^2}$$

$$H_{bb'} = (EF^T)_{bb'}$$

$$\mathcal{H}_{bb'} = (\mathcal{E}\mathcal{F}^T)_{bb'}$$

$$I_{bb'} = (DF^T)_{bb'}$$

$$\mathcal{I}_{bb'} = (\mathcal{D}\mathcal{F}^T)_{bb'}$$

$$K_{bb'} = \frac{1}{2}(DD^T)_{bb'}$$

$$\mathcal{K}_{bb'} = \frac{1}{2}(\mathcal{D}\mathcal{D}^T)_{bb'}$$

Explanation: contributing to matrix Q.

$$\mathcal{Q}_{bc} = \begin{cases} 1 & if\ Q_{bc} = 0 \\ 0 & if\ Q_{bc} > 0 \end{cases}$$ is the complement of Q

$$V_{bb'} = \sum_{c}^{C} Q_{bc} Q_{b'c} = (QQ^T)_{bb'}$$

V: this is a matrix of dimension $B \times B$. For two rows in Q, b, b', i.e. two nodes b, b' in the b-node group, entry bb' gives the number of columns, or nodes in the c-node group, which are adjacent to both b and b'.

$$L = [\text{vec}(Q_b Q_b{}^T \circ C)^T]_{b=1}^{B} \in \mathbb{R}^{b \times c^2}$$

$$M = [\text{vec}(Q_b \mathcal{Q}_b{}^T \circ C)^T]_{b=1}^{B} \in \mathbb{R}^{b \times c^2}$$

$$N = [\text{vec}(\mathcal{Q}_b Q_b{}^T \circ C)^T]_{b=1}^{B} \in \mathbb{R}^{b \times c^2}$$

$$\mathcal{L} = [\text{vec}(Q_b Q_b{}^T \circ \mathcal{C})^T]_{b=1}^{B} \in \mathbb{R}^{b \times c^2}$$

$$\mathcal{M} = [\text{vec}(Q_b \mathcal{Q}_b{}^T \circ \mathcal{C})^T]_{b=1}^{B} \in \mathbb{R}^{b \times c^2}$$

$$\mathcal{N} = [\text{vec}(\mathcal{Q}_b Q_b{}^T \circ \mathcal{C})^T]_{b=1}^{B} \in \mathbb{R}^{b \times c^2}$$

$$S_{bb'} = (MN^T)_{bb'}$$

$$\mathcal{S}_{bb'} = (\mathcal{M}\mathcal{N}^T)_{bb'}$$

$$T_{bb'} = (LN^T)_{bb'}$$

$$\mathcal{T}_{bb'} = (\mathcal{L}\mathcal{N}^T)_{bb'}$$

$$Z_{bb'} = \frac{1}{2}(LL^T)_{bb'}$$

$$\mathcal{Z}_{bb'} = \frac{1}{2}(\mathcal{L}\mathcal{L}^T)_{bb'}$$

All the formulae for counting motif roles

Role1:

$$O_b R_b$$

Role2:

$$O_b \mathcal{W}_b$$

Role3:

$$O_b W_b$$

Role4:

$$\mathcal{G}_b R_b$$

Role5:

$$G_b R_b$$

Role6:

$$\mathcal{G}_b \mathcal{W}_b$$

Role7:

$$G_b \mathcal{W}_b$$

Role8:

$$\mathcal{G}_b W_b$$

Role9:

$$G_b W_b$$

Role10:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * V_{bb'} * \mathcal{B}_{bb'}$$

Role11:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * V_{bb'} * B_{bb'}$$

Role12:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * \mathcal{S}_{bb'}$$

Role13:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * B_{bb'} * \mathcal{S}_{bb'}$$

Role14:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * S_{bb'}$$

Role15

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * B_{bb'} * S_{bb'}$$

Role16:

$$\sum_{b'=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * \mathcal{T}_{bb'}$$

Role17:

$$\sum_{b=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * \mathcal{T}_{bb'}$$

Role18:

$$\sum_{b'=1}^{B} U_{bb'} * B_{bb'} * \mathcal{T}_{bb'}$$

Role19:

$$\sum_{b=1}^{B} U_{bb'} * B_{bb'} * \mathcal{T}_{bb'}$$

Role20:

$$\sum_{b'=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * T_{bb'}$$

Role21:

$$\sum_{b=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * T_{bb'}$$

Role22:

$$\sum_{b'=1}^{B} U_{bb'} * B_{bb'} * T_{bb'}$$

Role23:

$$\sum_{b=1}^{B} U_{bb'} * B_{bb'} * T_{bb'}$$

Role24:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * \mathcal{Z}_{bb'}$$

Role25:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * B_{bb'} * \mathcal{Z}_{bb'}$$

Role26:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * \mathcal{B}_{bb'} * Z_{bb'}$$

Role27:

$$\frac{1}{2}\sum_{b'=1}^{B} U_{bb'} * B_{bb'} * Z_{bb'}$$

Role28:

$$\frac{1}{2}\sum_{b'=1}^{B} \mathcal{H}_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role29:

$$\frac{1}{2}\sum_{b'=1}^{B} H_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role30:

$$\frac{1}{2}\sum_{b'=1}^{B} \mathcal{H}_{bb'} * B_{bb'} * V_{bb'}$$

Role31:

$$\frac{1}{2}\sum_{b'=1}^{B} H_{bb'} * B_{bb'} * V_{bb'}$$

Role32:

$$\sum_{b'=1}^{B} \mathcal{I}_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role33:

$$\sum_{b=1}^{B} \mathcal{I}_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role34:

$$\sum_{b'=1}^{B} I_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role35:

$$\sum_{b=1}^{B} I_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role36:

$$\sum_{b'=1}^{B} \mathcal{I}_{bb'} * B_{bb'} * V_{bb'}$$

Role37:

$$\sum_{b=1}^{B} \mathcal{I}_{bb'} * B_{bb'} * V_{bb'}$$

Role38:

$$\sum_{b'=1}^{B} H_{bb'} * B_{bb'} * V_{bb'}$$

Role39:

$$\sum_{b=1}^{B} H_{bb'} * B_{bb'} * V_{bb'}$$

Role40:

$$\frac{1}{2} \sum_{b'=1}^{B} \mathcal{K}_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role41:

$$\frac{1}{2} \sum_{b'=1}^{B} K_{bb'} * \mathcal{B}_{bb'} * V_{bb'}$$

Role42:

$$\frac{1}{2} \sum_{b'=1}^{B} \mathcal{K}_{bb'} * B_{bb'} * V_{bb'}$$

Role43:

$$\frac{1}{2} \sum_{b'=1}^{B} K_{bb'} * B_{bb'} * V_{bb'}$$

For weighted networks

We also define A represents the number of rows of PW, and B represents both the number of columns of PW and the number of rows of QW; C represents the number of columns of QW.

$$PW_{ab} = \begin{cases} \mathbb{R}^+ & \text{if } a \text{ interact to } b \\ 0 & \text{if } a \text{ not interact to } b \end{cases}$$

$$QW_{bc} = \begin{cases} \mathbb{R}^+ & \text{if } b \text{ interact to } c \\ 0 & \text{if } b \text{ not interact to } c \end{cases}$$

$$AW_{aa'} = \begin{cases} \mathbb{R}^+ & \text{if } a \text{ interact to } a' \\ 0 & \text{if } a \text{ not interact to } a' \end{cases}$$

$$BW_{bb'} = \begin{cases} \mathbb{R}^+ & \text{if } b \text{ interact to } b' \\ 0 & \text{if } b \text{ not interact to } b' \end{cases}$$

$$CW_{cc'} = \begin{cases} \mathbb{R}^+ & \text{if } c \text{ interact to } c' \\ 0 & \text{if } c \text{ not interact to } c' \end{cases}$$

$OW_b = \sum_{a=1}^{A} PW_{ab}$ defines a vector of length B, in which each element is the sum of each column of P_{ab} and represents the weighted degree of the nodes of group b in the P subnetwork.

$RW_b = \sum_{c=1}^{C} QW_{bc}$ defines a vector of length B, in which each element is the sum of each row of Q_{bc} and represents the weighted degree of the nodes of group b in the Q subnetwork.

$$GW_b = \frac{1}{2}\sum_{a=1}^{A}\sum_{a'=1}^{A}(PW_{ab} + PW_{a'b} + AW_{aa'})^*P_{ab}{}^*P_{a'b}{}^*A_{aa'}$$

$$\mathcal{G}W_b = \frac{1}{2}\sum_{a=1}^{A}\sum_{a'=1}^{A}(PW_{ab} + PW_{a'b})^*P_{ab}{}^*P_{a'b}{}^*\mathcal{A}_{aa'}$$

$$WW_b = \frac{1}{2}\sum_{c=1}^{C}\sum_{c'=1}^{C}(QW_{bc} + QW_{bc'} + CW_{cc'})^*Q_{bc}{}^*Q_{bc'}{}^*C_{cc'}$$

$$\mathcal{W}W_b = \frac{1}{2}\sum_{c=1}^{C}\sum_{c'=1}^{C}(QW_{bc} + QW_{bc'})*\mathcal{Q}_{bc}*\mathcal{Q}_{bc'}*\mathcal{C}_{cc'}$$

Explanation: contributing to matrix PW.

$$\mathcal{P}_{ab} = \begin{cases} 1 & if\ PW_{ab} = 0 \\ 0 & if\ PW_{ab} > 0 \end{cases}$$ is the complement of PW

$$UW_{bb'} = \sum_{a}^{A}(PW_{ab} + PW_{ab'})*P_{ab}*P_{ab'}$$

$$DW = \left[\mathrm{vec}\left(P_b \cdot PW_b^T*PW_b\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times a^2}$$

$$EW = \left[\mathrm{vec}\left(PW_b \cdot \mathcal{P}_b{}^T\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times a^2}$$

$$FW = \left[\mathrm{vec}\left(\mathcal{P}_b \cdot PW_b{}^T\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times a^2}$$

$$\mathcal{D}W = \left[\mathrm{vec}\left(\left(P_b \cdot PW_b{}^T*PW_b\right) \circ \mathcal{A}\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times a^2}$$

$$\mathcal{E}W = \left[\mathrm{vec}\left(PW_b \cdot \mathcal{P}_b{}^T \circ \mathcal{A}\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times a^2}$$

$$\mathcal{F}W = \left[\mathrm{vec}\left(\mathcal{P}_b \cdot PW_b{}^T \circ \mathcal{A}\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times a^2}$$

$$HW_{bb'} = \sum_{a=1}^{A^2}(EW_{ba} + FW_{b'a} + \mathrm{vec}(\mathrm{AW})_a)*E_{ba}*F_{b'a}*\mathrm{vec}(\mathrm{A})_a$$

$$\mathcal{H}W_{bb'} = \sum_{a=1}^{A^2}(\mathcal{E}W_{ba} + \mathcal{F}W_{b'a})*\mathcal{E}_{ba}*\mathcal{F}_{b'a}$$

$$IW_{bb'} = \sum_{a=1}^{A^2}(DW_{ba} + FW_{b'a} + \mathrm{vec}(\mathrm{AW})_a)*D_{ba}*F_{b'a}*\mathrm{vec}(\mathrm{A})_a$$

$$\mathcal{I}W_{bb'} = \sum_{a=1}^{A^2}(\mathcal{D}W_{ba} + \mathcal{F}W_{b'a})*\mathcal{D}_{ba}*\mathcal{F}_{b'a}$$

$$KW_{bb'} = \frac{1}{2}\sum_{a=1}^{A^2}(DW_{ba} + DW_{b'a} + \mathrm{vec}(\mathrm{AW})_a)*D_{ba}*D_{b'a}*\mathrm{vec}(\mathrm{A})_a$$

$$\mathcal{K}W_{bb'} = \frac{1}{2}\sum_{a=1}^{A^2}(\mathcal{D}W_{ba} + \mathcal{D}W_{b'a})*\mathcal{D}_{ba}*\mathcal{D}_{b'a}$$

Explanation: contributing to matrix QW.

$$\mathcal{Q}_{bc} = \begin{cases} 1 & if\ QW_{bc} = 0 \\ 0 & if\ QW_{bc} > 0 \end{cases}$$ is the complement of QW

$$VW_{bb'} = \sum_{c}^{C} (QW_{bc} + QW_{b'c})*Q_{bc}*Q_{b'c}$$

$$LW = \left[\operatorname{vec}\left(Q_b \cdot QW_b{}^T*QW_b\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times c^2}$$

$$MW = \left[\operatorname{vec}\left(QW_b \cdot \mathcal{Q}_b{}^T\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times c^2}$$

$$NW = \left[\operatorname{vec}\left(\mathcal{Q}_b \cdot QW_b{}^T\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times c^2}$$

$$\mathcal{L}W = \left[\operatorname{vec}\left(\left(Q_b \cdot QW_b{}^T*QW_b\right) \circ \mathcal{C}\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times c^2}$$

$$\mathcal{M}W = \left[\operatorname{vec}\left(QW_b \cdot \mathcal{Q}_b{}^T \circ \mathcal{C}\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times c^2}$$

$$\mathcal{N}W = \left[\operatorname{vec}\left(\mathcal{Q}_b \cdot QW_b{}^T \circ \mathcal{C}\right)^T\right]_{b=1}^{B} \in \mathbb{R}^{b\times c^2}$$

$$SW_{bb'} = \sum_{c=1}^{C^2} (MW_{bc} + NW_{b'c} + \operatorname{vec}(C\mathrm{W})_c)*M_{bc}*N_{b'c}*\operatorname{vec}(C)_c$$

$$\mathcal{S}W_{bb'} = \sum_{c=1}^{C^2} (\mathcal{M}W_{bc} + \mathcal{N}W_{b'c})*\mathcal{M}_{bc}*\mathcal{N}_{b'c}$$

$$TW_{bb'} = \sum_{c=1}^{C^2} (LW_{bc} + MW_{b'c} + \operatorname{vec}(C\mathrm{W})_c)*L_{bc}*M_{b'c}*\operatorname{vec}(C)_c$$

$$\mathcal{T}W_{bb'} = \sum_{c=1}^{C^2} (\mathcal{L}W_{bc} + \mathcal{M}W_{b'c})*\mathcal{L}_{bc}*\mathcal{M}_{b'c}$$

$$ZW_{bb'} = \frac{1}{2}\sum_{c=1}^{C^2} (LW_{bc} + LW_{b'c} + \operatorname{vec}(C\mathrm{W})_c)*L_{bc}*L_{b'c}*\operatorname{vec}(C)_c$$

$$\mathcal{Z}W_{bb'} = \frac{1}{2}\sum_{c=1}^{C^2} (\mathcal{L}W_{bc} + \mathcal{L}W_{b'c})*\mathcal{L}_{bc}*\mathcal{L}_{b'c}$$

All the formulae for the weight vector of the nodes with a role in a particular motif

Role1:

$$\frac{1}{2}(OW_b R_b + RW_b O_b)$$

Role2:

$$\frac{1}{3}(OW_b \mathcal{W}_b + \mathcal{W}W_b O_b)$$

Role3:

$$\frac{1}{4}(OW_b W_b + WW_b O_b)$$

Role4:

$$\frac{1}{3}(\mathcal{G}W_b R_b + RW_b \mathcal{G}_b)$$

Role5:

$$\frac{1}{4}(GW_b R_b + RW_b G_b)$$

Role6:

$$\frac{1}{4}(\mathcal{G}W_b \mathcal{W}_b + \mathcal{W}W_b \mathcal{G}_b)$$

Role7:

$$\frac{1}{5}(GW_b \mathcal{W}_b + \mathcal{W}W_b G_b)$$

Role8:

$$\frac{1}{5}(\mathcal{G}W_b W_b + WW_b \mathcal{G}_b)$$

Role9:

$$\frac{1}{6}(GW_b W_b + WW_b G_b)$$

Role10:

$$\frac{1}{8}\sum_{b'=1}^{B}(UW_{bb'}*V_{bb'} + VW_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role11:

$$\frac{1}{10}\sum_{b'=1}^{B}(UW_{bb'}*V_{bb'} + VW_{bb'}*U_{bb'})*B_{bb'} + BW_{bb'}*U_{bb'}*V_{bb'}$$

Role12:

$$\frac{1}{8}\sum_{b'=1}^{B}(UW_{bb'}*\mathcal{S}_{bb'} + \mathcal{S}W_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role13:

$$\frac{1}{10}\sum_{b'=1}^{B}(UW_{bb'}*\mathcal{S}_{bb'}+\mathcal{S}W_{bb'}*U_{bb'})*B_{bb'}+U_{bb'}*\mathcal{S}_{bb'}*BW_{bb'}$$

Role14:

$$\frac{1}{10}\sum_{b'=1}^{B}(UW_{bb'}*S_{bb'}+SW_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role15:

$$\frac{1}{12}\sum_{b'=1}^{B}(UW_{bb'}*S_{bb'}+SW_{bb'}*U_{bb'})*b_{bb'}+U_{bb'}*S_{bb'}*BW_{bb'}$$

Role16:

$$\frac{1}{5}\sum_{b'=1}^{B}(UW_{bb'}*\mathcal{T}_{bb'}+\mathcal{T}W_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role17:

$$\frac{1}{5}\sum_{b=1}^{B}(UW_{bb'}*\mathcal{T}_{bb'}+\mathcal{T}W_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role18:

$$\frac{1}{6}\sum_{b'=1}^{B}(UW_{bb'}*\mathcal{T}_{bb'}+\mathcal{T}W_{bb'}*U_{bb'})*B_{bb'}+U_{bb'}*\mathcal{T}_{bb'}*BW_{bb'}$$

Role19:

$$\frac{1}{6}\sum_{b=1}^{B}(UW_{bb'}*\mathcal{T}_{bb'}+\mathcal{T}W_{bb'}*U_{bb'})*B_{bb'}+U_{bb'}*\mathcal{T}_{bb'}*BW_{bb'}$$

Role20:

$$\frac{1}{6}\sum_{b'=1}^{B}(UW_{bb'}*T_{bb'}+TW_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role21:

$$\frac{1}{6}\sum_{b=1}^{B}(UW_{bb'}*T_{bb'}+TW_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role22:

$$\frac{1}{7}\sum_{b'=1}^{B}(UW_{bb'}*T_{bb'}+TW_{bb'}*U_{bb'})*b_{bb'}+U_{bb'}*T_{bb'}*BW_{bb'}$$

Role23:

$$\frac{1}{7}\sum_{b=1}^{B}(UW_{bb'}*T_{bb'}+TW_{bb'}*U_{bb'})*b_{bb'}+U_{bb'}*T_{bb'}*BW_{bb'}$$

Role24:

$$\frac{1}{12}\sum_{b'=1}^{B}(UW_{bb'}*\mathcal{Z}_{bb'}+\mathcal{Z}W_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role25:

$$\frac{1}{14}\sum_{b'=1}^{B}(UW_{bb'}*\mathcal{Z}_{bb'}+\mathcal{Z}W_{bb'}*U_{bb'})*B_{bb'}+U_{bb'}*\mathcal{Z}_{bb'}*BW_{bb'}$$

Role26:

$$\frac{1}{14}\sum_{b'=1}^{B}(UW_{bb'}*Z_{bb'}+ZW_{bb'}*U_{bb'})*\mathcal{B}_{bb'}$$

Role27:

$$\frac{1}{16}\sum_{b'=1}^{B}(UW_{bb'}*Z_{bb'}+ZW_{bb'}*U_{bb'})*b_{bb'}+U_{bb'}*Z_{bb'}*BW_{bb'}$$

Role28:

$$\frac{1}{8}\sum_{b'=1}^{B}(\mathcal{H}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{H}_{bb'})*\mathcal{B}_{bb'}$$

Role29:

$$\frac{1}{10}\sum_{b'=1}^{B}(HW_{bb'}*V_{bb'}+VW_{bb'}*H_{bb'})*\mathcal{B}_{bb'}$$

Role30:

$$\frac{1}{10}\sum_{b'=1}^{B}(\mathcal{H}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{H}_{bb'})*B_{bb'}+\mathcal{H}_{bb'}*V_{bb'}*BW_{bb'}$$

Role31:

$$\frac{1}{12}\sum_{b'=1}^{B}(HW_{bb'}*V_{bb'}+VW_{bb'}*H_{bb'})*B_{bb'}+H_{bb'}*V_{bb'}*BW_{bb'}$$

Role32:

$$\frac{1}{5}\sum_{b=1}^{B}(\mathcal{I}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{I}_{bb'})*\mathcal{B}_{bb'}$$

Role33:

$$\frac{1}{5}\sum_{b'=1}^{B}(\mathcal{I}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{I}_{bb'})*\mathcal{B}_{bb'}$$

Role34:

$$\frac{1}{6}\sum_{b=1}^{B}(IW_{bb'}*V_{bb'}+VW_{bb'}*I_{bb'})*\mathcal{B}_{bb'}$$

Role35:

$$\frac{1}{6}\sum_{b'=1}^{B}(IW_{bb'}*V_{bb'}+VW_{bb'}*I_{bb'})*\mathcal{B}_{bb'}$$

Role36:

$$\frac{1}{6}\sum_{b=1}^{B}(\mathcal{I}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{I}_{bb'})*B_{bb'}+\mathcal{I}_{bb'}*V_{bb'}*BW_{bb'}$$

Role37:

$$\frac{1}{6}\sum_{b'=1}^{B}(\mathcal{I}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{I}_{bb'})*B_{bb'}+\mathcal{I}_{bb'}*V_{bb'}*BW_{bb'}$$

Role38:

$$\frac{1}{7}\sum_{b=1}^{B}(IW_{bb'}*V_{bb'}+VW_{bb'}*I_{bb'})*B_{bb'}+I_{bb'}*V_{bb'}*BW_{bb'}$$

Role39:

$$\frac{1}{7}\sum_{b'=1}^{B}(IW_{bb'}*V_{bb'}+VW_{bb'}*I_{bb'})*B_{bb'}+I_{bb'}*V_{bb'}*BW_{bb'}$$

Role40:

$$\frac{1}{12}\sum_{b'=1}^{B}(\mathcal{K}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{K}_{bb'})*\mathcal{B}_{bb'}$$

Role41:

$$\frac{1}{14}\sum_{b'=1}^{B}(KW_{bb'}*V_{bb'}+VW_{bb'}*K_{bb'})*\mathcal{B}_{bb'}$$

Role42:

$$\frac{1}{14}\sum_{b'=1}^{B}(\mathcal{K}W_{bb'}*V_{bb'}+VW_{bb'}*\mathcal{K}_{bb'})*B_{bb'}+\mathcal{K}_{bb'}*V_{bb'}*BW_{bb'}$$

Role43:

$$\frac{1}{16}\sum_{b'=1}^{B}(KW_{bb'}*V_{bb'}+VW_{bb'}*K_{bb'})*B_{bb'}+K_{bb'}*V_{bb'}*BW_{bb'}$$