Game Theory

Game

- Players
- Payoffs / Preferences
- Actions
- Outcomes

Rationality Temporally consident & transient

$< < <$

$U(A)$

$\hookrightarrow$ payoff when A is chosen

Ordinal Preferences

$> \quad = \quad <$

Assignment Problem

Pay off matrix

A \ B	E	N
E	2,2	3,0
N	0,3	1,1

payoff for hardwork 2

Action matrix $\{(E,E)\ (N,N)\ (E,N)\ (N,E)\}$

$$u_A(E,E) = 2$$

$$u_A(E,N) = 3$$

$$u_A(N,N) = 1$$

$$u_A(N,E) = 0$$

Zero Sum games

- One shot
- Symml
- Complete information

A \ B	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Action matrix $\{(H,H), (H,T)\, (T,H)\, (T,T)\}$

$$u_A(H,H) = 1$$

$$u_A(H,T) = -1$$

$u_A(T,H) = -1$ $\quad u_A(T,T) = 1$

Nash - Profile - Equilibrium

- Unilateral - One player can change other keep constant

	C	D
C	~~1,1~~	~~10,0~~
D	~~0,10~~	5,5

Nash Equilibrium

- as both D

A is D then only if he can go from D → C but B will stay at D

- more than one are possible

$a = \{a_1, a_2, a_3, \ldots\ldots a_n\}$

i

$M(a_1, a_2, \ldots\ldots\ldots a_n)$

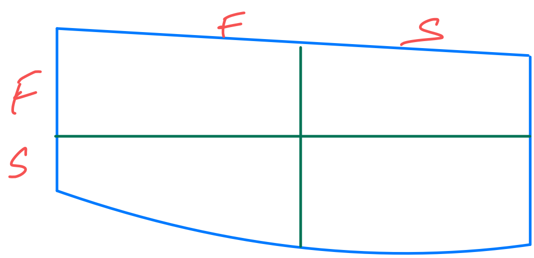
$M(d)$

a_{-i} - all players except i

$M_i(a_i^*, a_{-i}^*)$ → nash equilibrium profile

$$M_i(a_i^*, a_{-i}^*) \geq M_i(a_i, a_{-1}^*)$$

↳ this player has it's choice open



$M_1(F, S) \geq M_1(S, S)$

	C_1	C_2	C_3	C_4
R_1				
R_2				
R_3				
R_4				
R_5				

$$^{4}C_1 \times {}^{5}C_1 = \underline{\underline{20}}$$

$$u_1(R_3, C_3) \geq u_1(R_1, C_3)$$

$$u_1(R_3, C_3) \geq u_1(R_2, C_3)$$

$$u_1(R_3, C_3) \geq u_1(R_4, C_3)$$

$$u_1(R_3, C_3) \geq u_1(R_5, C_3)$$

$$u_1(R_3, C_3) \geq u_1(R_3, C_3)$$

	C_1	C_2
R_1	1, 1	0, 0
R_2	0, 0	0, 0

$\geq$ non strict nash / nash equilibrium

$>$ strict nash equilibrium

Best Response

1, 1	2, 0	3, 4
2, 1	0, 0	5, 6
2, 1	3, 5	0, 1

Profit for player 1 $\Pi_1(q_1, q_2) = [a - (q_1+q_2)]\, q_1 - cq_1$ (1)

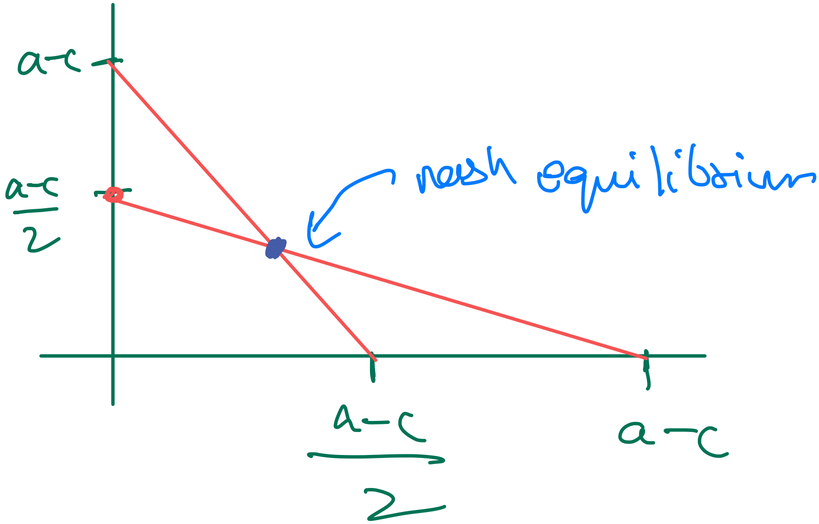
$\Pi_2(q_1, q_2) = [a - (q_1+q_2)]\, q_2 - cq_2$ (2)

$B_1(q_2)$: Best response of Player 1 when player 2 is opting q_2

$B_2(q_1)$: Best response of Player 2 when player 1 is opting q_1

(1) diff. wrt q_1 to get $B_1(\ _2)$

$$B_1(q_2) = \frac{a - c - q_2}{2}$$

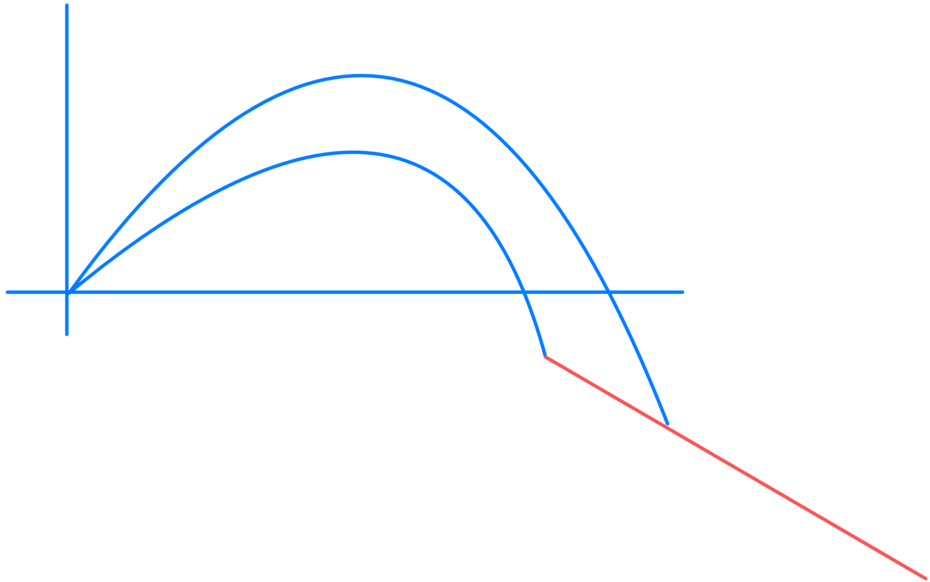


$$[a-(q_1+q_2)]q_1 - cq_1 \qquad a > c,\ q_1$$

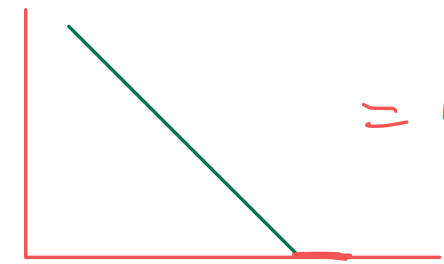
$$- \; cq_1 \qquad \text{else}$$

$$B_1(q_2) = \begin{cases} \dfrac{a-c-q_2}{2} & q_2 \leq a-c \\ 0 & q_2 > a-c \end{cases}$$

$$P(Q) = \begin{cases} a-Q & \text{if } a > Q \quad (Q = q_1+q_2) \end{cases}$$



Price and demand



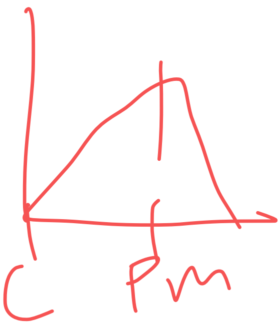
$= k - p$

$$\begin{cases} k - p & \text{if } k \geq p \\ 0 & \text{else} \end{cases}$$

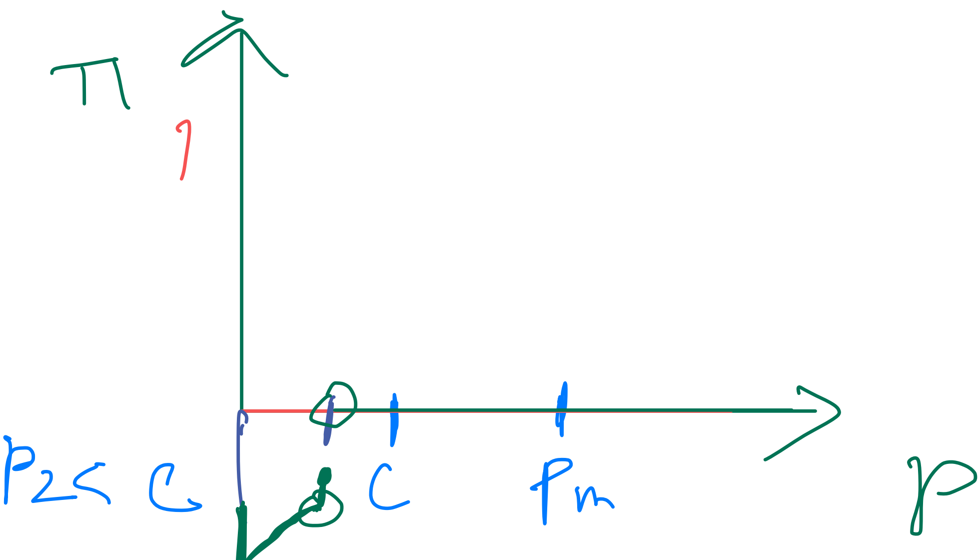
Players $\longrightarrow$ A^i & B^j

Actions $\longrightarrow$ Real no. between $0 - \infty$

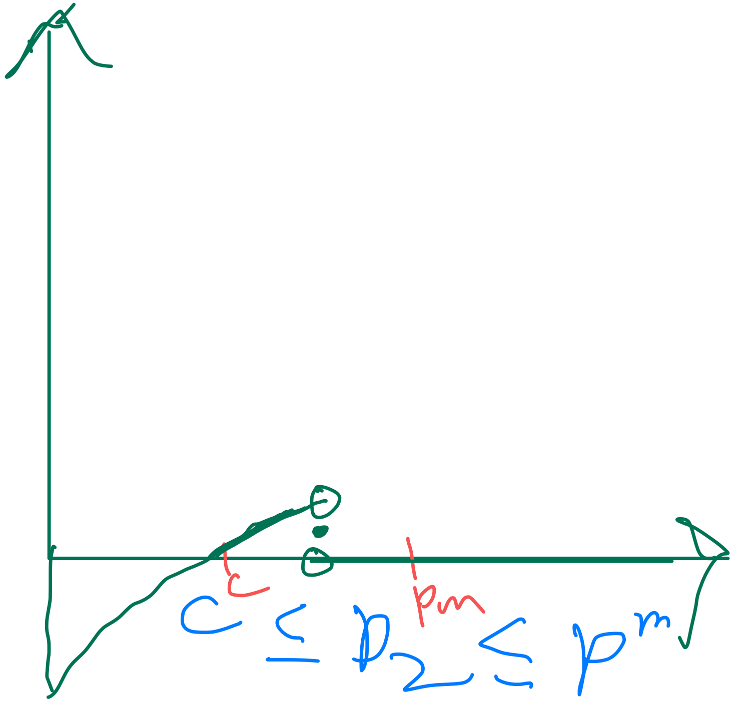
Payoff $\longrightarrow$ $\Pi(p_1, p_2)$



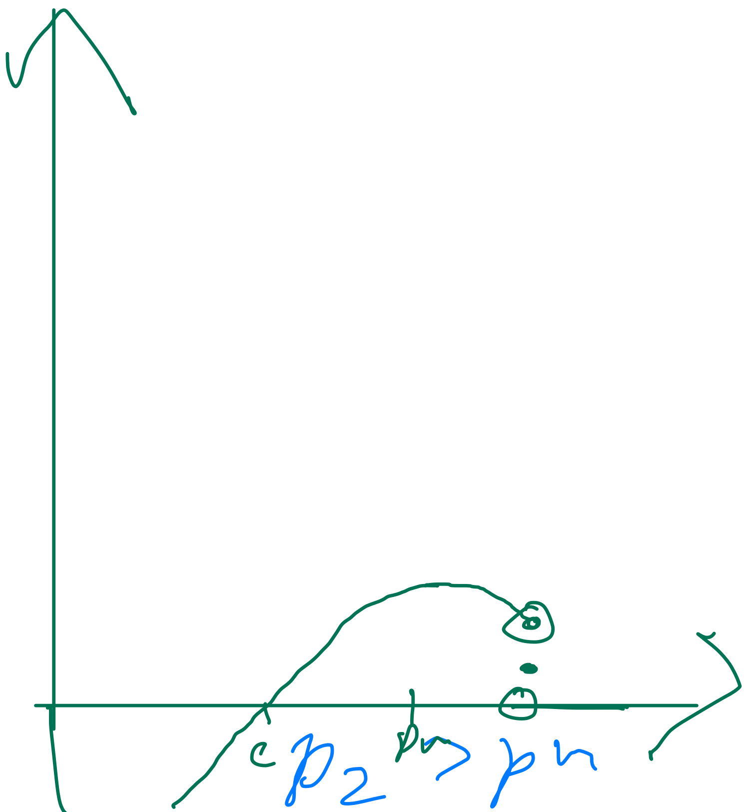
$$\Pi = \begin{cases} 0 & p_1 > p_2 \\ (k - p_1)(p_1 - c) & p_1 < p_2 \\ \frac{1}{2}(k - p_1)(p_1 - c) & p_1 = p_2 \end{cases}$$



π_2

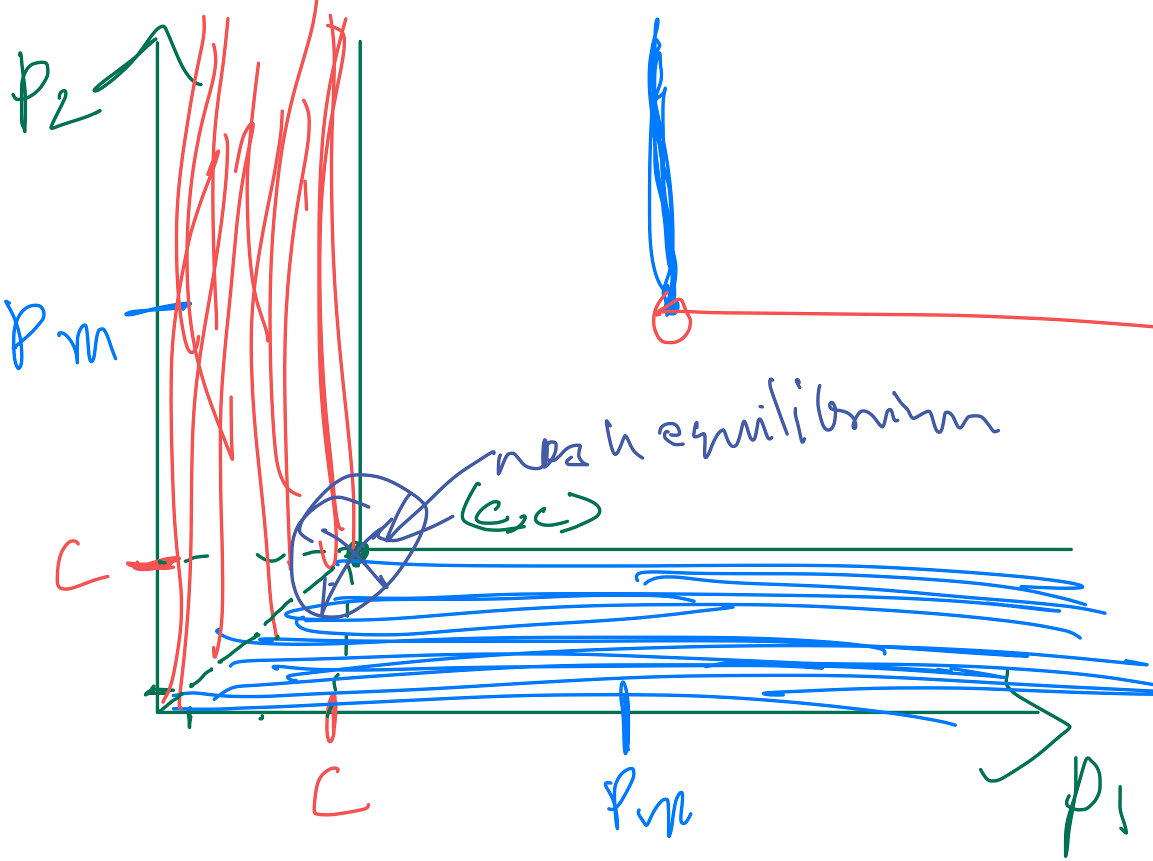


π_3



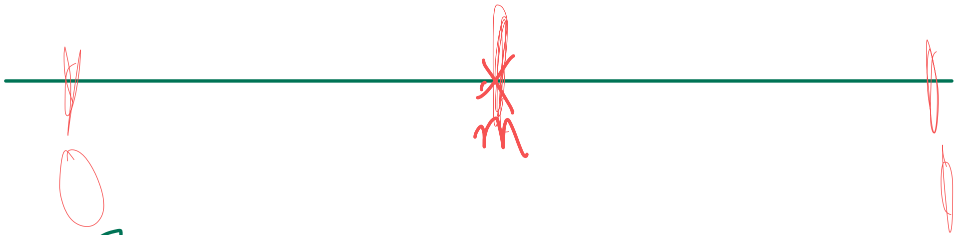
$$B_1(p_2) = \begin{cases} p_1 > p_2 & p_2 < c \\ p_1 \geq p_2 & p_2 = c \\ \emptyset & c < p_2 < p^m \\ \emptyset & p_2 = p^m \\ p^m & p_2 > p^m \end{cases}$$

unilateral



multiple Best response can exist

Voting



Players $\longrightarrow$ c_1, c_2

Actions $\longrightarrow$ finite number

$u_1(x_1, x_2) =$

1

0

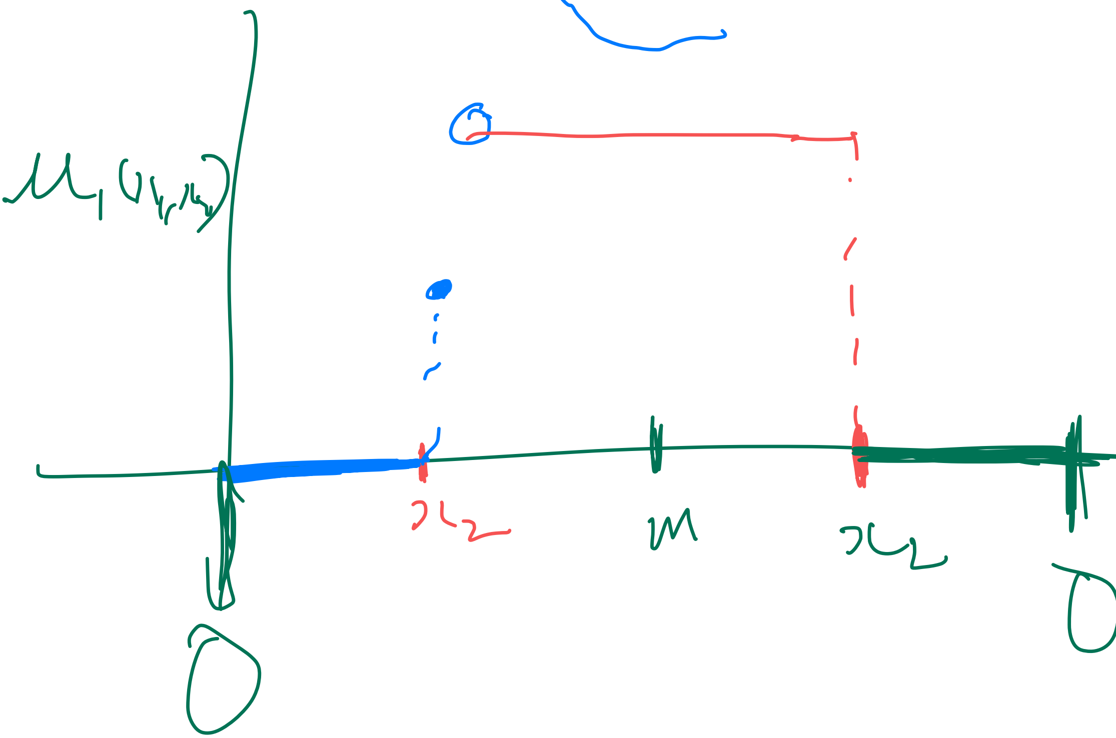
$1/2$

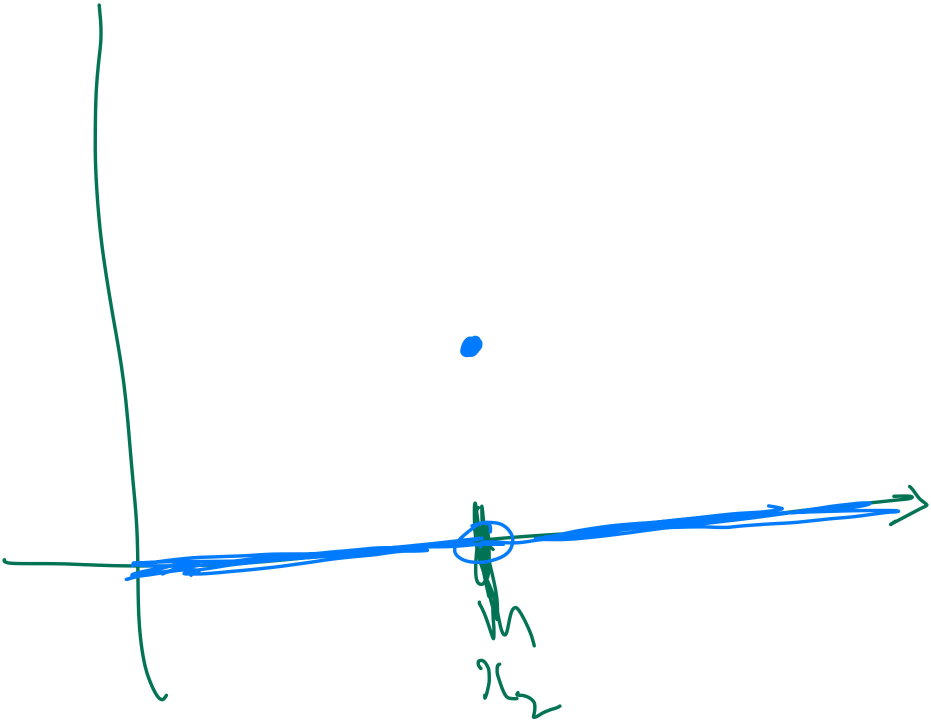
$$B_1(x_2)$$

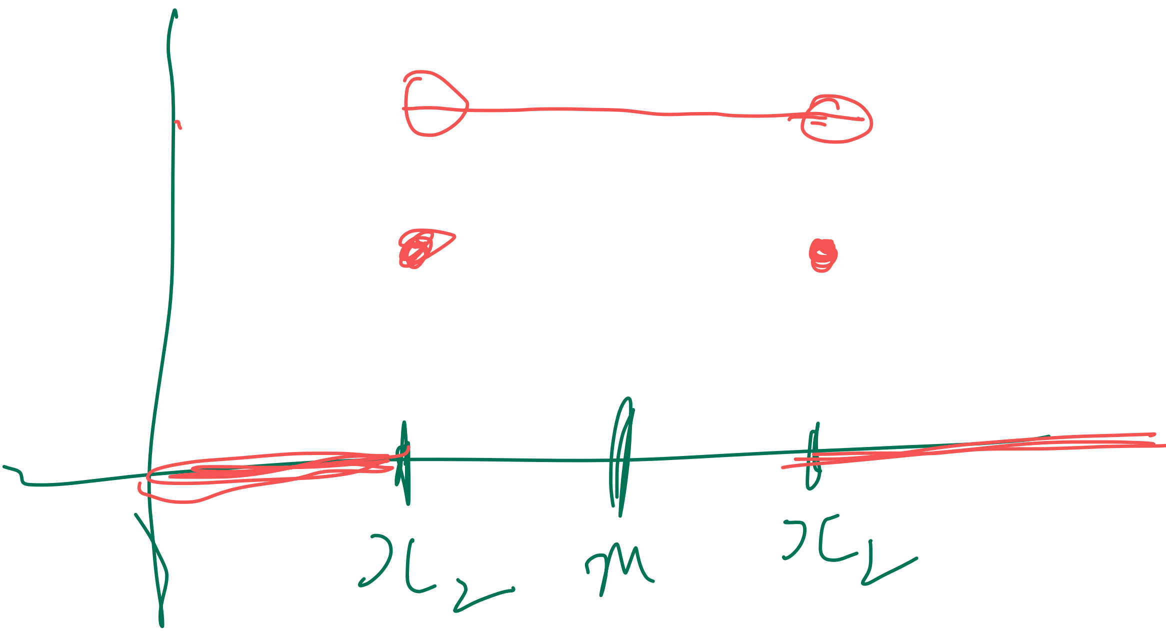
$$B_1(x_1)$$

$$B_1(x_2) = \begin{cases} (x_2, 2m - x_2) & x_2 < m \\ m & x_2 = m \\ (2m - x_2, x_2) & x_2 > m \end{cases}$$

$u_1(x_1, x_2) =$




m
x_2


x_2
m
x_1

0-100

$$\frac{\text{avg}(\_\_\_\_)}{3}$$

Hotling

$$\frac{99}{3} = 33$$

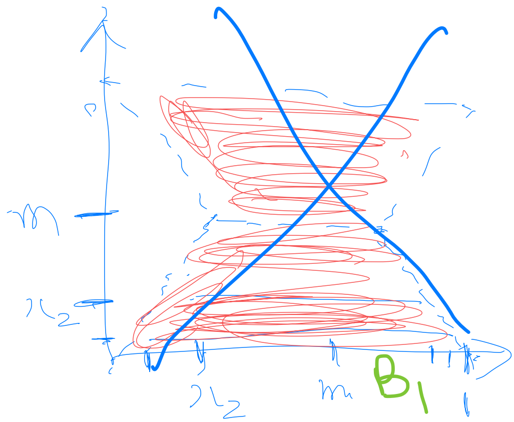
1

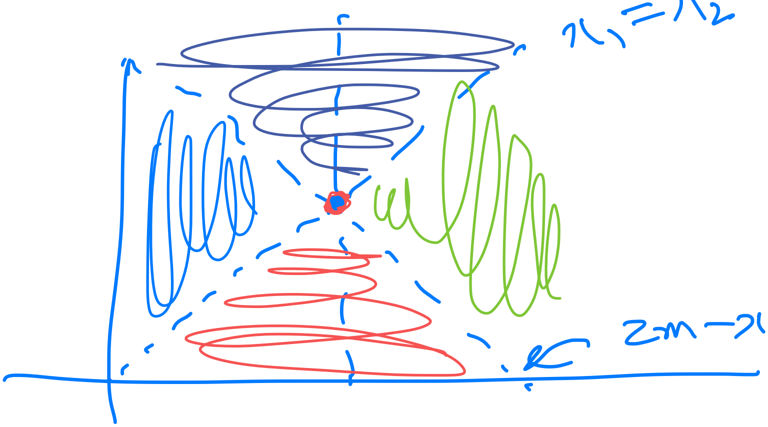
nash eq 0

99 → 33 → 11 → . . . → 0 ≠ 10

unilaterally deviate

$$B_1(x_2) = \begin{cases} x_1 \in (x_2, 2m - x_2) & x_2 < m \\ m & x_2 = m \\ x_1 \in (2m - x_2, x_1) & x_2 > m \end{cases}$$

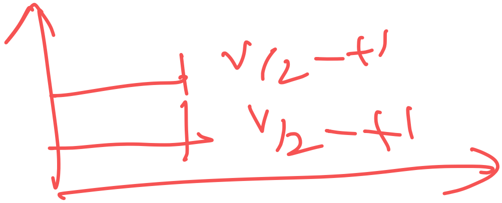




War of attrition

θ_1 $\boxed{0}$ θ_2

1 2



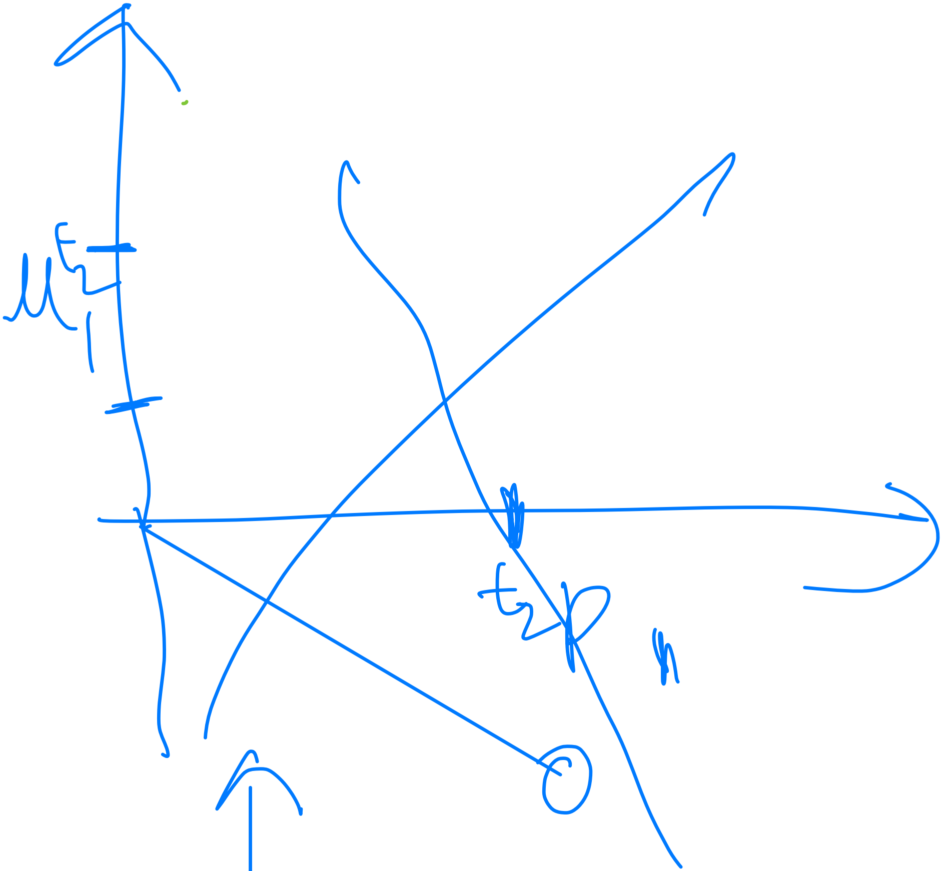
$t_1 > t_2$

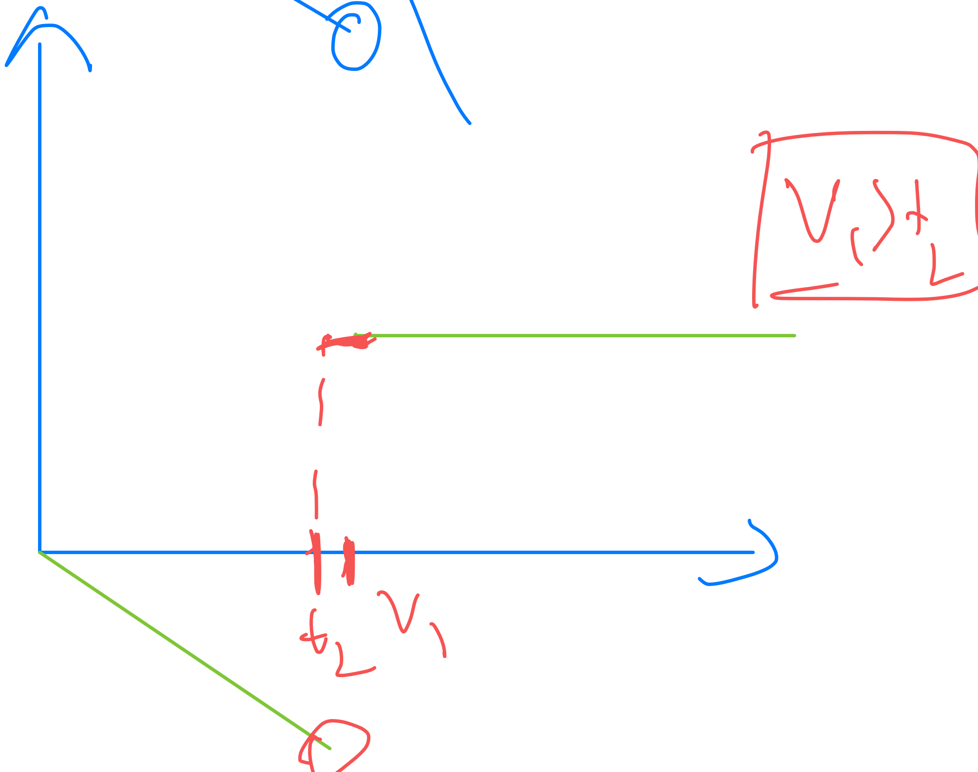
$v_1 - t_2$ $-t_2$

Players = (p_1, p_2)

$$p_1 = \begin{cases} -t_1 & t_1 < t_2 \\ v/2 - t_2 & t_1 = t_2 \\ v/2 - t_2 & t_2 < t_1 \end{cases}$$

Actions : $[0, \infty)$ all positive real no.s


$u_1^{t_2}$
t_2 p

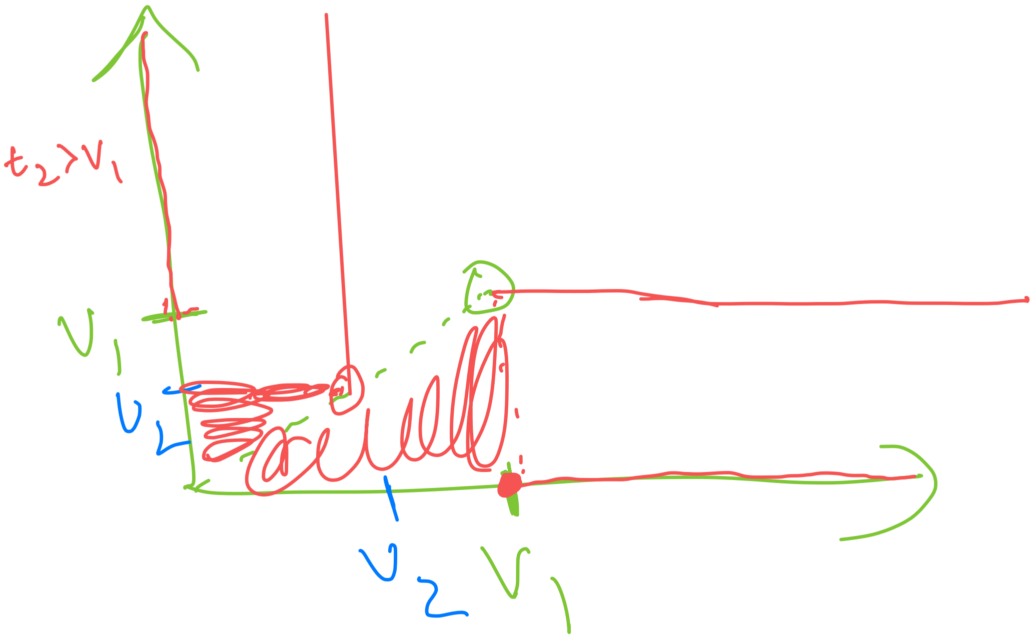

$V_1 t_2$
t_2 V_1

Profits $v_2 - t_1$ $- t_1$

Total $v_{2f} = v_2 - t_1$

Valuation — for an object how much are you ready to pay

$$B_1(t_2) = \begin{cases} \{0\} & t_1 > t_2 \quad t_2 < v_1 \\ \{0\} \cup \{t_1 > t_2\} & t_2 = v_1 \\ \{0\} & t_2 > v_1 \end{cases}$$



t2>V1
V1
V2
V2 V1

Auction

SPSBA $\longrightarrow$ $b_1 \; b_2 \; \dots\dots\dots \; b_n$

FPSBA $\longrightarrow$

first pitseal bit option

b_k

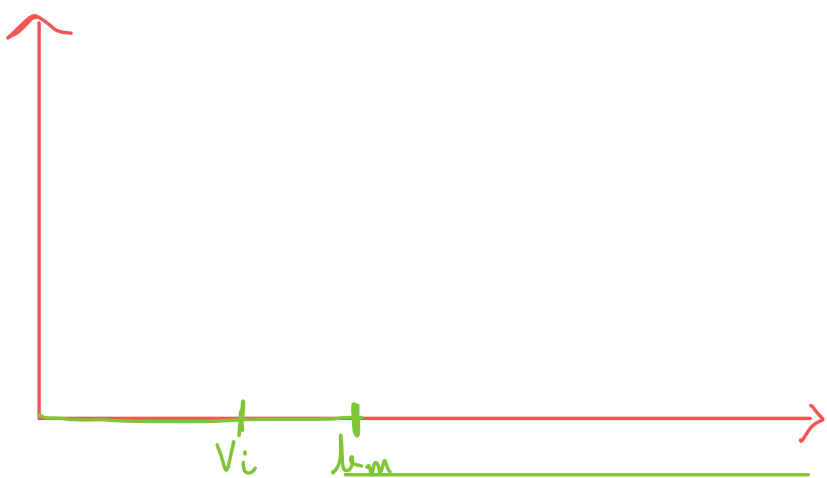
$b_{k'}$

Players $\longrightarrow$ $1, \dots\dots n$

Action $\longrightarrow$ All possible numbers including 0

Payoff $\longrightarrow$ $v_1 - b_2, 0, 0, 0, \dots$

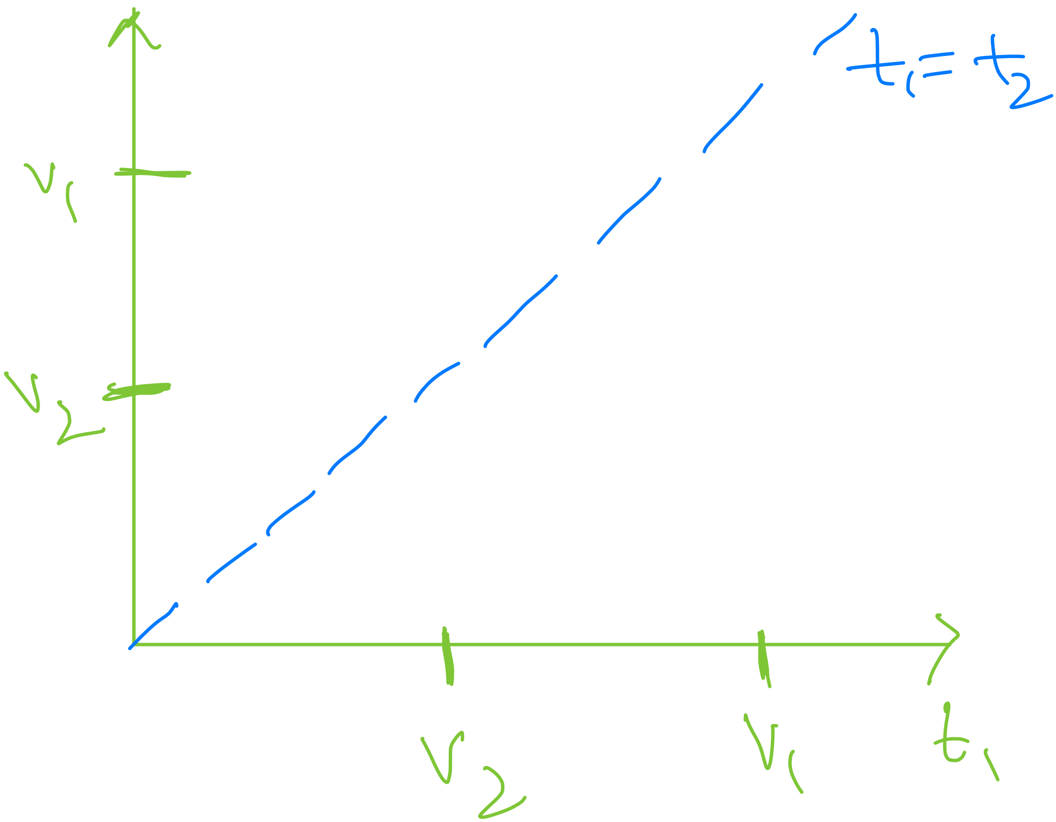
Profit ki tarah



$b_m < v_i$

$v_i - b_m$


un = di
un


t1 = t2
v1
v2
v2
v1
t1

Concept

Pay off — Player 1

strategy 2

Strategy P_1

	L	R
x	0	4
y	3	2
z	2	1

$u_1(y, \dots) > u_1(z, \dots)$

$u_1(y, A) = 3$

$u_1(y, B) = 2$

$u_1(z, L) = 2$

$u_1(z, R) = 1$

Strictly dominating

$u_1(y, \dots) > u_1(z, \dots)$

for all possible conditions

* strict domination cannot be part of nash equilibri-

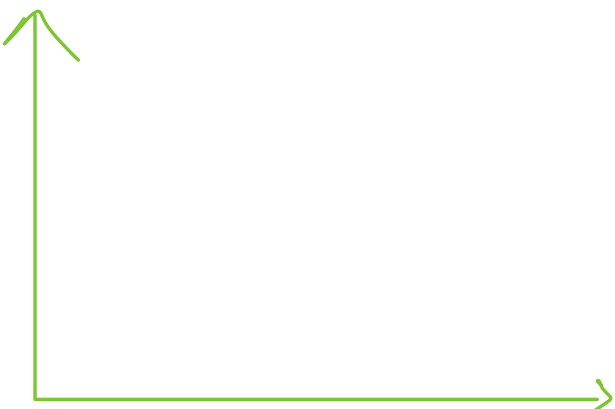
* Weakly dominating can be part of nash equilibrium ?

SPSBA

$$\{v_1, v_2, \ldots\ldots, v_n\}$$

$$\begin{array}{cccc} \| & \| & \| & \| \\ \underbrace{u_1}_{p_1}, & \underbrace{u_2}_{p_2}, & \underbrace{u_3}_{p_3}, & \ldots \\ \| & \| & \| & \| \end{array}$$

$$\{v_1, 0, 0, \ldots\ldots 0\}$$



$v_1 > v_2 > v_3 > v_4$ - - -

known

$\{ v_2, v_1, v_3, \ldots v_n \}$

$v_2 = h_1$, $v_1 = h_2$ - - -

→ actions you are choosing

$\{v_2, v_1, v_3, \ldots\ldots v_n\}$

$\hookrightarrow$ Payoff $= 0$

P1 $\quad v_2 \longrightarrow v_1 \longrightarrow v_1 + \delta$

Payoff $\quad 0 \longrightarrow 0 \longrightarrow 0$

P 2 $\quad v_1 \longrightarrow (v_2, v_1)$

Price $\quad v_2 \longrightarrow v_2$

Payoff $\quad 0 \longrightarrow 0$

P2 $\quad v_1 \longrightarrow v_2$

Price $\quad v_2 \longrightarrow$ doesn't get

Payoff $\quad 0 \longrightarrow 0$

$\Longrightarrow$ nash equilibrium

$A' \{v_1, v_2, \ldots v_n\}$

$B' \{v_1, 0, 0, 0, 0 \ldots\}$

$C' \{v_1, v_2, \ldots\}$

$D' \{v_2, v_1, v_3, \cdots v_n\}$

In A', all players are having weakly dominating strategy over any other N.E action profile

v_1 is weakly dominating v_2

Player 1

Payoff$(v_1) \geq$ payoff (any other)

First-Price - Seal - Bid-auction

$v_1 > v_2 > v_3 \ldots \ldots \ldots > v_n$

Players $\{1, 2, \ldots \ldots \ldots n\}$

Actions Profiles $[\,(v_n, v_1)\,]$

p C	q D	Players → A,B Actions (C, D; p)

Action Profil

$\{(p), (p')\}$; $p, p' \in [0, 1]$

probability distribution of C over D.

$$\mathbb{E}(X) = \sum p \,(\text{Payoff of } C)$$

Ex

B

		q	$1-q$
A	p	2, 1 pq	0, 0 $p(1-q)$
	$1-p$	$(1-p)q$ 0, 0	$(1-p)(1-q)$ 1, 2

$$2pq + p(1-p)\,0$$

$$+ q(1-p) \times 0 + 1 \times (1-p)(1-q)$$

$u_1(p,q)$

VNM preference

Best response

$$\begin{array}{c|cc} & H & T \\ \hline H & 1,-1 & -1,1 \\ T & -1,1 & 1,-1 \end{array}$$

Matching pennies

Nonzero

zero sum

~~p1 + pc~~

$$pq' + p(1-q')(-1) + (1-p)(1-q)$$

$$> q(1-p)q(-1)$$

$H > T$

$$pp' + \cancel{p} > \cancel{p} + pq'$$

$$> -q + qp$$

$$+1 + pq - \cancel{p} - q$$

$$pp' + pq' > 2pq + 1 - 2q$$

$$u_A(p, q) = pq - p(1-q)$$

Payoff Player 1

$u_A = (p, q)$

$$= pq - p(1-q)$$
$$- q(1-p) + (1-p)(1-q)$$

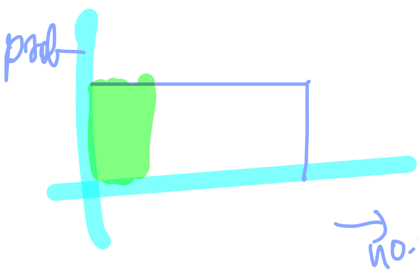
$\frac{\partial}{\partial q}$ why

Random number generator

$x = \text{Rand}$

generates between a interval

those no.s come from equal probability



0, 1, 2, ... 9

generated 100 100 100 . . . 100

			B: q	$1-q$
A	p	T	pq	$p(1-q)$
	$1-p$	B	$(1-p)q$	$(1-p)(1-q)$

B → Action

Imp X is a r.v X is defined as ?

$\mathbb{E}(X) =$ Ry a prob you payoff + ...

so expected value

→ what will be the max ~~prob~~ payoff if q varies

my p be what should

$$u_A(p,q) = pq\,(u_A)(TL) + p(1-q)\,u_A(TR) + (1-p)q\,u_A(BL) + (1-p)(1-q)\,u_A(BR)$$

$$p\left[q\,u_A(TL) + (1-q)\,u_A(TR)\right] + (1-p)\left[q\,u_A(BL) + (1-q)\,u_A(BR)\right]$$

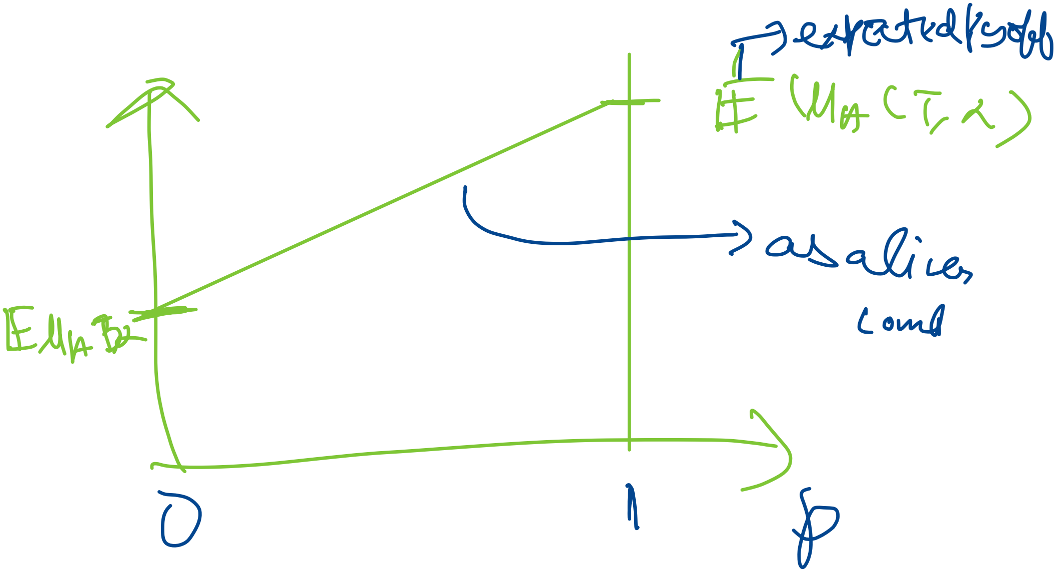
$$\mathbb{E}[u_A(T,\alpha)]$$

$$\alpha = (q, 1-q)$$

$$p\,\mathbb{E}(u_A(T,\alpha)) + (1-p)\,\mathbb{E}(u_A(B,\alpha))$$

if

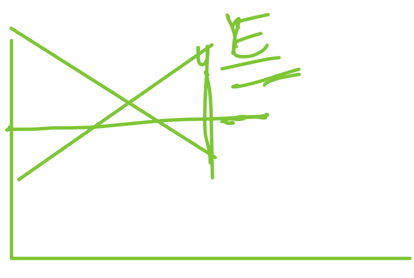
$$\mathbb{E}[u_A(T,\alpha)] > \mathbb{E}(u_A(\beta,\alpha))$$



~~assuming~~

why straight line ??

$y = mx + c$



Strategy of 1

when 2 is randomising

All strategies

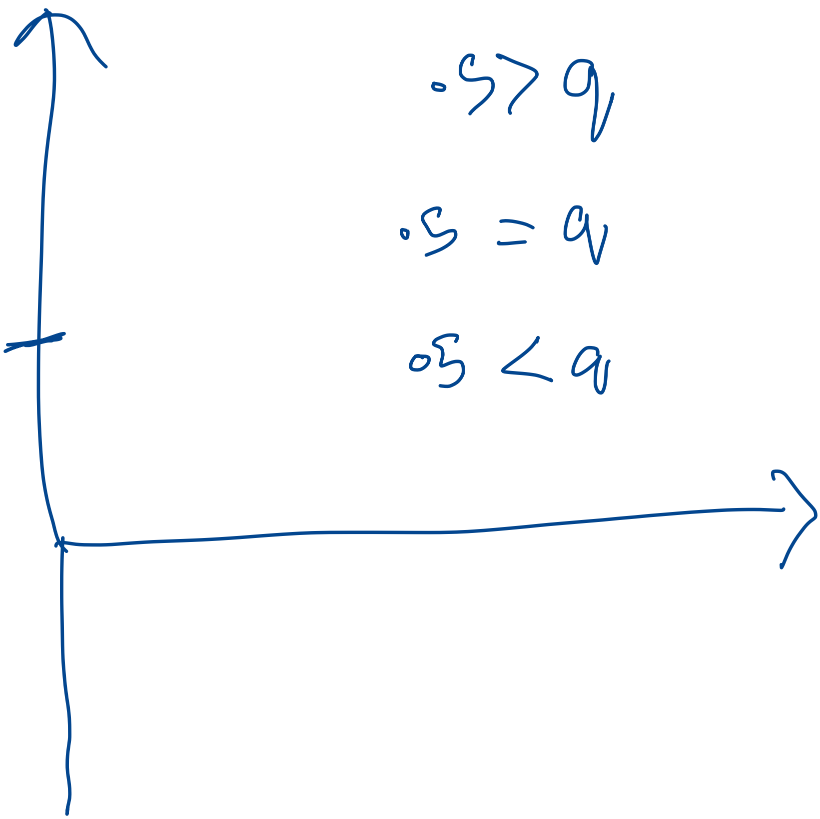
either = > ≤ only 3 things to do

$\mathbb{E}(u_A(T,\alpha))$

$= q(1) + (1-q)\times -1$

$= 2q - 1$ $\underline{\underline{<0}}$

$\mathbb{E}(u_A(B,\alpha)) = 1 - 2q$ $\underline{\underline{>0}}$



A choose ~~head~~ $\longrightarrow$ $p \; \mathbb{E}(\; , \;)$

p $\longrightarrow$ prob of head

$\mathbb{E}(\; , \;)$ $\longrightarrow$ payoff of A

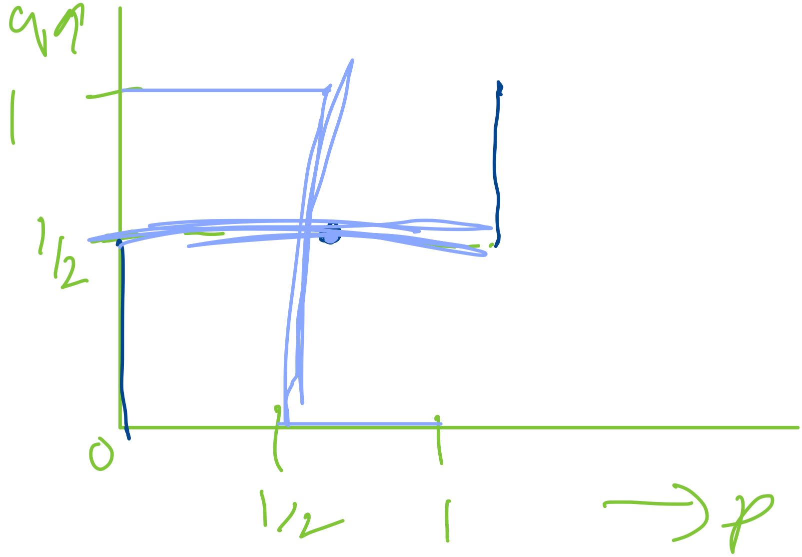
Cant we directly say

$\mathbb{E}$ payoff jisika zada ?

$B_A(q, \mathbb{E})$

$\longrightarrow$ Best response kis kis strategy mai likhta hai

$$B_A(q) = \begin{cases} p = 1 & q > \frac{1}{2} \\ p = 0 & q < \frac{1}{2} \\ p \in [0,1] & q = \frac{1}{2} \end{cases}$$


q ↑
1
1/2
0
1/2
1
→ p

		q	$1-q$
		L	R
p	T	2, 1	0, 0
$1-p$	H	0, 0	1, 2

$$u_A(p,q) = pq \times 2 + 0 \times p(1-q) + 0 + (1-p)(1-q) \times 1$$

$$= 2pq + 1 + pq - p - q$$

$$= 3pq - p - q + 1$$

$$= p\,u_A(T,q) + (1-p)\,u_A(H,q)$$

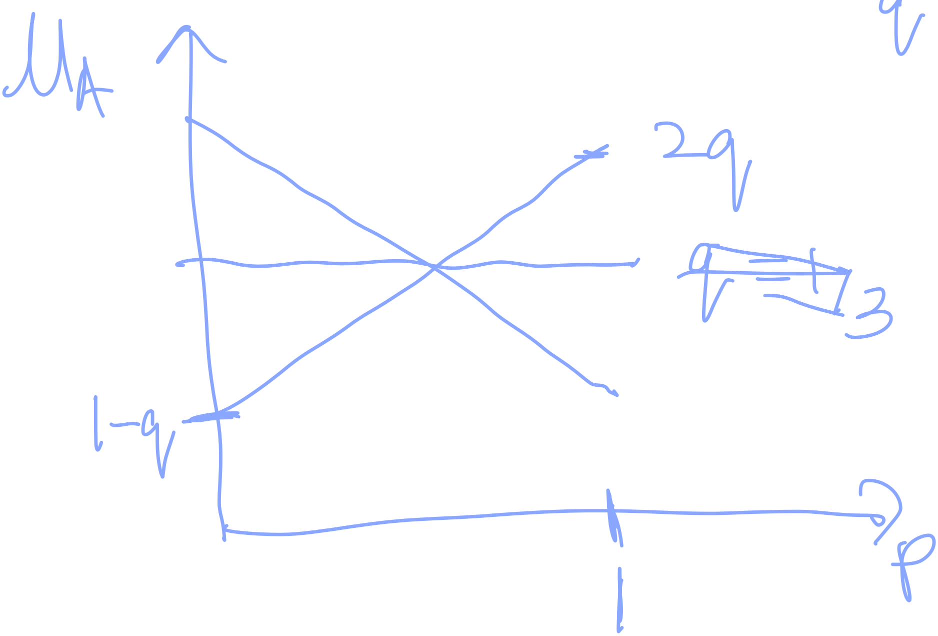
$$u_A(T,q) > u_A(H,q)$$

$$2q > 1-q$$

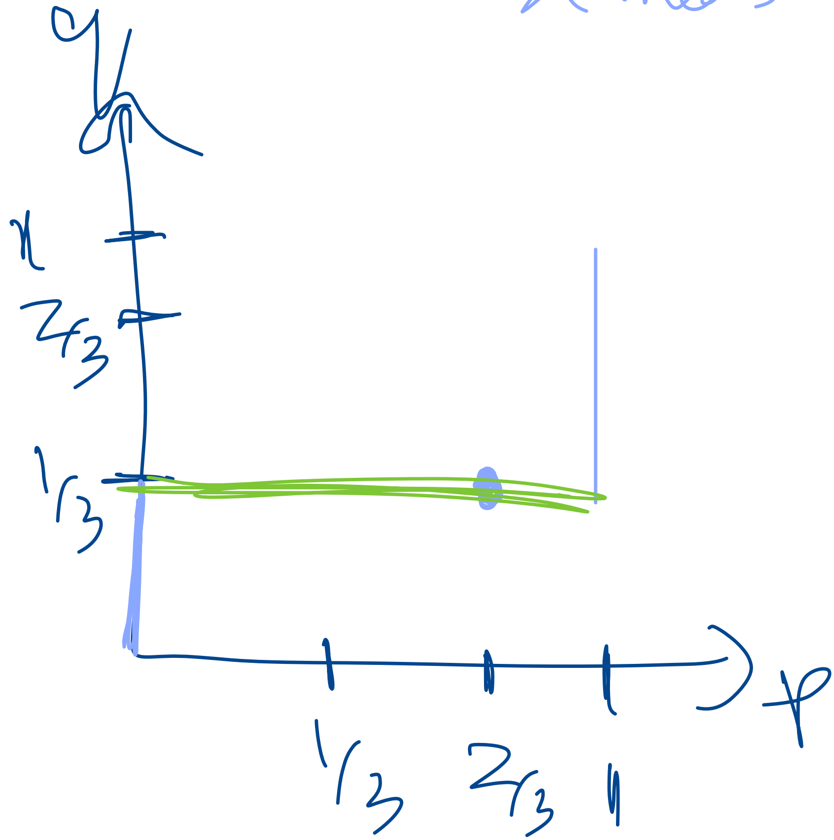
$$q > 1/3$$

$$B_A(q) = \begin{cases} \cancel{p = 2/3} \quad p \in [0,1] & q = \frac{1}{3} \\ p = 1 & q > \frac{1}{3} \\ p = 0 & q < \frac{1}{3} \end{cases}$$

$q = 1/3$



α means the q



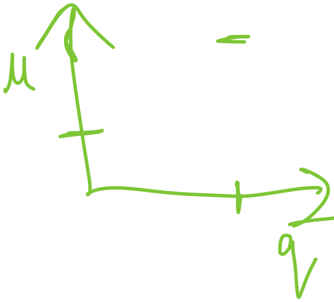
Case I

$$\mathbb{E}_B(L, \alpha)^A > \mathbb{E}_B(R, \alpha)^A$$

$$\cancel{2}\,1 \times p > (1-p) \times 2$$

$$p > 2 - 2p$$

$3p > 2$



$p > 2/3$

$p \in [0, 1]$

$p = 2/3$

$p > 2/3$

$p < 2/3$

$\mathbb{E}_A(\quad\cdot\quad) = \mathbb{E}_A(\cdot)$

I can take as prob

1) choosing $\underline{\underline{T}}$ & B is randomising

Ame

2) If $\mu_A(T) = \mu_A(H)$

Assign equal probabilities

3) If $\mu_A(T) \geq \mu_A(H)$

$\Rightarrow$ prob for H is zero

by me

Reporting a crime

n people

$w \to$ payoff

$= w - c$

$c \to$ cost

report hogya $= w$ to all − who rep

reported person $= w - c$

else $= 0$

$w > c$ $\quad$ $w = c$

$w < c$

$w < c$

R	NR
$\otimes$	0

$(p_1, p_2, \ldots \ldots p_n)$

arrives in order

$\frac{\alpha}{p_1}(w-c) \quad w \quad w \quad w \quad w \quad (w)\frac{\alpha}{p_n}$

$p_1 \rightarrow p_2 \qquad \frac{w-c}{p_1} \qquad \frac{\overset{w}{\cancel{w-c}}}{p_2}$

$p_2 < p_1 \Rightarrow w$

$\frac{w-c}{p_1}$ it will drop

$\frac{w-c}{p_1} \longrightarrow \frac{w}{p_1}$

payoff increases

$w = (1-p_r(1-p_2)(1-p_3))$

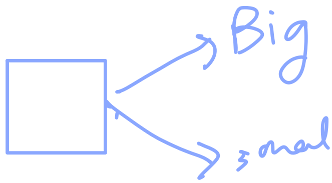
p is the prob of reporting for any

$\frac{c}{w} = (1-p)^{n-1}$

$$\boxed{1-\sqrt[n-1]{\frac{c}{w}}} = p$$

Car rep.

H



D

Players $\longrightarrow$ owner, mechanic

Actions $\longrightarrow$ $O \longrightarrow \{A, R\}$

$M \longrightarrow \{H, D\}$

$c_1 > c > c_1' > c$

prob sad

prob small yet goes to next

	A	R
H	$\pi, -[\gamma\, c + (1-\gamma)\, c']$	$(1-\gamma)\pi, [(1-\gamma)\, c']_{k-1} + [\gamma\, c]_{-1} \rightarrow c_1$
D	$\pi'(1-\gamma)+\pi\gamma, -c$	$0, -[\gamma\, c_1 + (1-\gamma)\, c_1']$

c, c' first mechanic cost

c_1, c_1' is of second mechanic

		q A	$1-q$ R
p	H	$\pi, -\gamma c - (1-\gamma)c'$	$\pi(1-\gamma), -\gamma c + -(1-\gamma)c'$
$1-p$	D	$\gamma\pi + (1-\gamma)\pi', -c$	$0, -\gamma c_1 - (1-\gamma)c_1'$

$$u_1(H, q) = \pi p + (1-\gamma)\pi p$$

$$u_2(T, q) = \gamma\pi(1-p) + (1-\gamma)\pi'(1-p) + 0$$

$$E_1(A) > E_1(R)$$

$$\pi q > \pi(1-\gamma)(1-q)$$

$$0 > -\pi\gamma$$

$$\boxed{\pi\gamma > 0} \implies \cancel{\pi > 0}$$

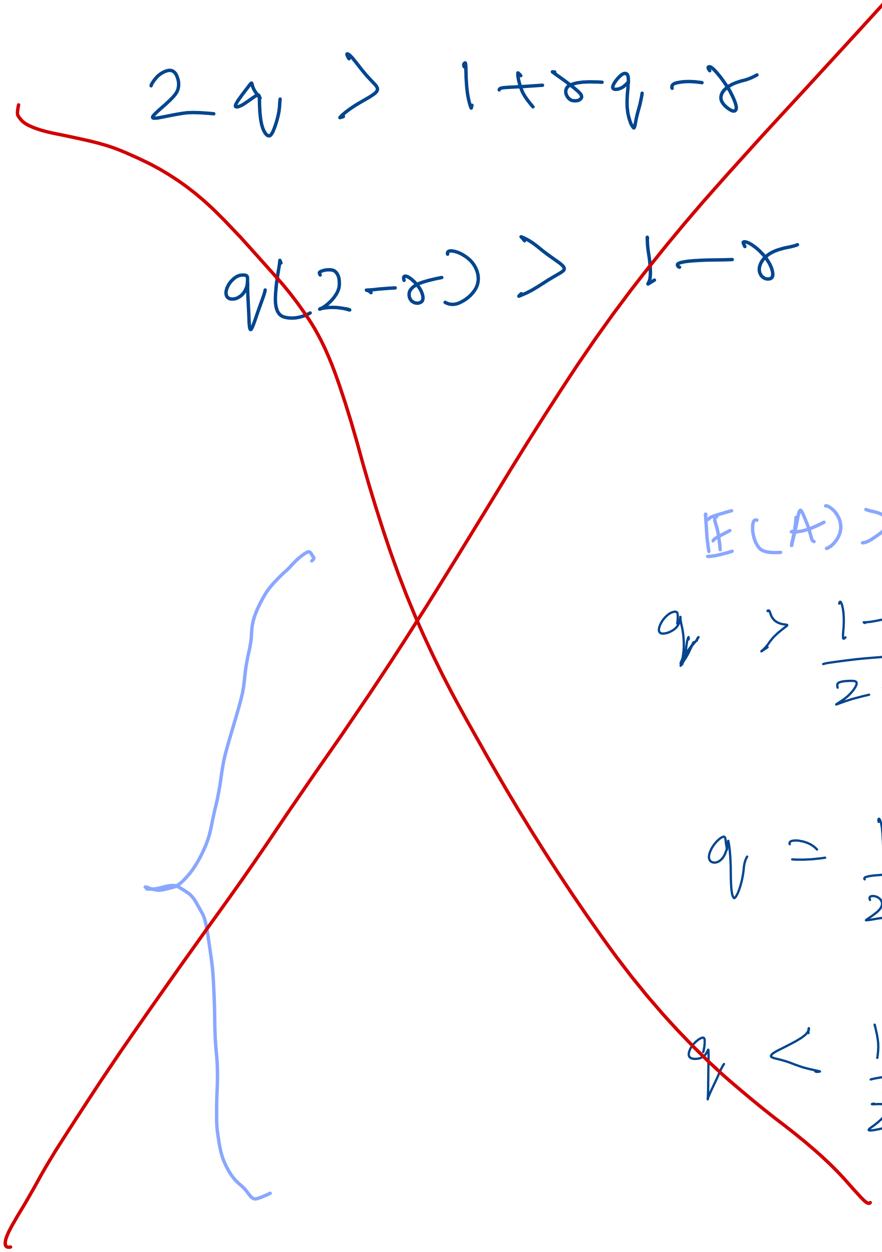
as γ cannot be neg

$$\pi q > \pi(1 + \gamma q - \gamma - q)$$

$$2\pi q > \pi c$$

$$2q > 1 + \gamma q - \gamma$$

$$q(2-\gamma) > 1-\gamma$$



$$\mathbb{E}(A) > \mathbb{E}(R)$$

$$q > \frac{1-\gamma}{2-\gamma}$$

$$q = \frac{1-\gamma}{2-\gamma}$$

$$q < \frac{1-\gamma}{2-\gamma}$$

		A (q)	R ($1-q$)
p	H	$\Pi, -\gamma c - (1-\gamma)c'$	$\Pi(1-\gamma), -\gamma c - (1-\gamma)c'$
$1-p$	D	$\gamma\Pi + (1-\gamma)\Pi', -c$	$0, -\gamma c_1 - (1-\gamma)c_1'$

$$\mathbb{E}_1(H, q) \qquad\qquad \mathbb{E}_1(D, q)$$

$$\Pi q + \Pi(1-\gamma)(1-q) \qquad\qquad q(\gamma\Pi + (1-\gamma)\Pi') + 0 \times (1-q)$$

$$q(1-\gamma)\Pi - q\Pi'(1-\gamma) + \Pi(1-\gamma)(1-q) \qquad\qquad 0$$

$$q(1-\gamma)(\Pi - \Pi') + (1-\gamma)(1-q)(\Pi)$$

$$(1-\gamma)\left(q\cancel{\pi} - \pi' q - \cancel{\pi'} q + \pi\right)$$

$$(1-\gamma)(\pi - \pi' q) \qquad 0$$

+ve $\qquad \gamma$ not 1 or 0

$\cancel{\pi - \pi' q}$

$$\pi - \pi' q \qquad 0$$

$\cancel{\pi}$

$$\frac{\pi}{\pi'} > q$$

$$\frac{\pi}{\pi'} < q$$

$$q = \frac{\pi}{\pi'}$$

condition → Best response → graph

↓

intersection

nash equilibrium

End nalli ? ? Z

weakly dominating dominant

? ? ?

? ? ? ? ?

Dominating dominated

	L	R
T	2	1
M	5	0
B	0	4

variable

$$\mathbb{E}\left(u_0(\alpha, A)\right)$$

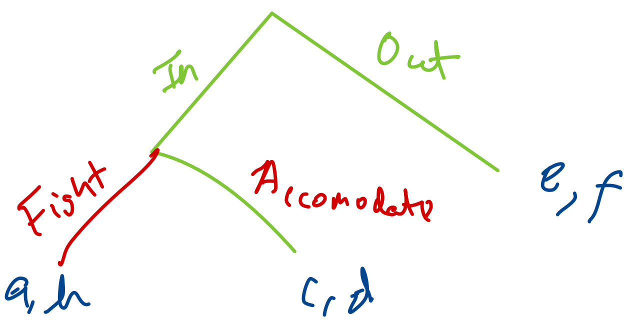
Actions sequencially

Read Coca cola   Pepsi (Pale)

P   is in market

C → Out

C → In



Terminal history   In fight, in accomodate

T.H = {In, F} {In, A} Out