GAME THEORY

- Players
- Act^n → Ac^n Profile.
- Payoffs | Preferences — well defined — $u(A) = u(B)$ or $u(A) > u(B)$ or $u(A) < u(B)$ — rational players
  — complete: $u(A) > u(B)$, $u(B) > u(C)$ $\Rightarrow u(A) > u(C)$
  + Temporarily consistent
- Outcome.

- Ordinal preference | payoffs
  $>, =, <$ — do not consider level of difference
  $5<6$, $4<6$, $3<6$ — wrt 6 | 5, 4, 3 are same

- Players → {A, B}
  Actions → {(C, D), (C, D)}
  Action profile → {CC, CD, DC, DD}

		B: C	B: D
A	C	(1,1)	(0,10)
	D	(10,0)	(5,5)

-

		B: C	B: D
A	C	1,1	10,0
	D	0,10	5,5

$u_A(CC) = 1$
$u_A(CD) = 10$
$u_A(DC) = 0$
$u_A(DD) = 5$
$u_B(CC) = 1$
$u_B(CD) = 0$
$u_B(DC) = 10$
$u_B(DD) = 5$

$u_A(DC) > u_A(CC) > u_A(DD) > u_A(CD)$

Q/

		G: F	G: M
B	F	(2,1)	(0,0)
	M	(0,0)	(1,2)

Players – {B, G}
Actions – {(F, M), (F, M)}
~~Payoffs~~
Action Profile – {FF, FM, MF, MM}
Payoffs – {(1,2), (0,0), (0,0), (2,1)}

$u_B(FF) = 2 = u_g(MM)$
$0 = u_B(FM) = u_B(MF) = u_g(MF) = u_g(FM)$

Q.

	E	N
E	2,2	0,3
N	3,0	1,1

(Rows: A; Columns: B)

Good qua. — Payoff = 4
Mod. " — " = 3
Poor " — " = 2

→ One shot
→ complete info.
→ Simulta.

Player – {A, B}

Actions – {(E,N) ; (E,N)}.

Action - profile → {EE, EN, NE, NN}

Payoff – {(2,2), (0,3), (3,0), (1,1)}

$u_B(NE) = u_A(EN) = 3$

$u_A(EE) = u_B(EE) = 2.$

$u_A(NN) = 1 = u_B(N,N)$

$u_A(EN) = 0 = u_B(NE)$

Q.

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

(Rows: A; Columns: B)

∵ Sum of all (each) boxes is 0 ⇒ Zero sum game.
↓
one player gaining on loss of someone.

Matching Pennies

Player – {A, B}

Actions – {(H,T) ; (T,H)}

Acti^n profile → {HH, HT, TH, TT}

Payoff – {(1,-1), (-1,1), (-1,1), (1,-1)}

$u_A(HH) = u_B(TH) = u_B(HT) = u_A(TT) = 1.$

$u_A(HT) = u_A(TH) = u_B(HH) = u_B(TT) = -1$

* Nash Equilibrium

1.

	C	D
C	1,1	10,0
D	0,10	5,5.

* Unilateral move.

· More than 1 Nash equilibrium possible in any game

2.

	F	C
F	2,1	0,0
C	0,0	1,2

Q

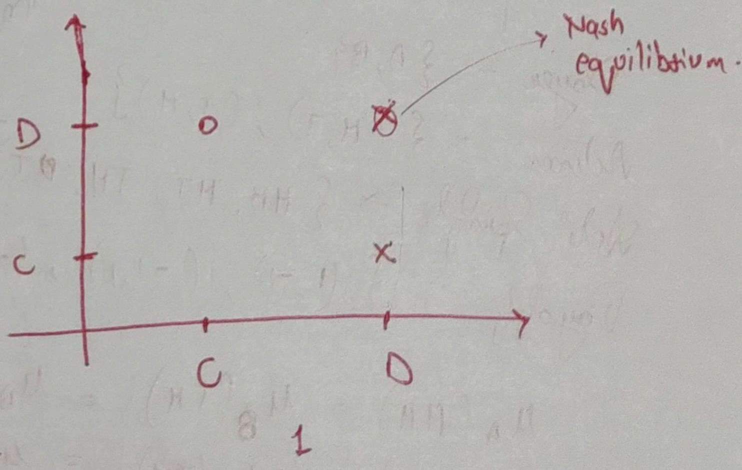
B

A	H	T
H	1,-1	-1,1
T	1,-1	1,-1

No Nash pure strategy nash equilibrium

Q

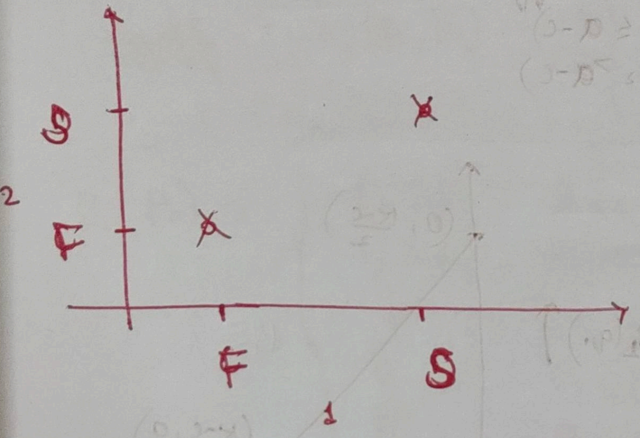
1	C	D
C	-2,2	0,3
D	3,0	1,1



D is dominating C as Payoff is always more [redacted] [redacted] [redacted]

Q

		2: F	S
1	F	1,2	0,0
	S	0,0	2, 1



Dominating Action
Dominated action — strictly → NE ✗
— weakly → NE ✓

Q Players → F_1, F_2

Actions → q_1, q_2

Payoff → $p(q)$ = $\begin{cases} K-q & K>q \\ 0 & \text{else} \end{cases}$

$p(q)$ → price function

want to maximize the profit.

$u_1(q_1, q_2)$
$= (K - (q_1+q_2))q_1 - cq_1$ $K \geq (q_1+q_2)$
$-q_1^c$ (else)

$u_2(q_1, q_2) = (K - (q_1+q_2))q_2 - cq_2$ $K \geq (q_1+q_2)$
$-q_2^c$ (else)

		F_2: q_1	q_2
F_1	q_1		
	q_2		

✗ (Not possible as q_1 has infinite numbers).
same for q_2...

$\pi_1(q_1, q_2) = [a - (q_1+q_2)]q_1 - cq_1$ — (i)

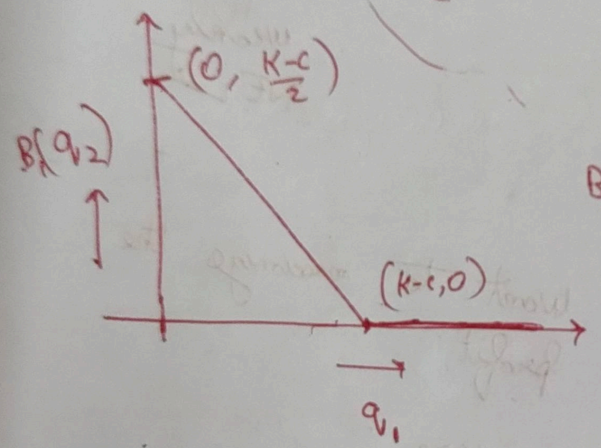
$\pi_2(q_1, q_2) = [a - (q_1+q_2)]q_2 - cq_2$ — (ii)

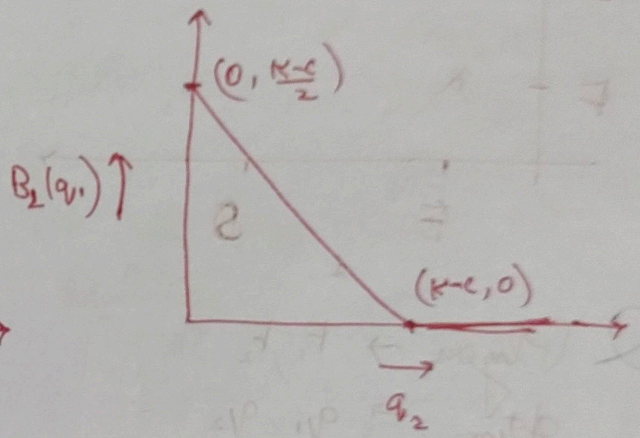
$B_1(q_2)$ = Best response of player 1 wrt q_2 (when player (2) has opted q_2.)

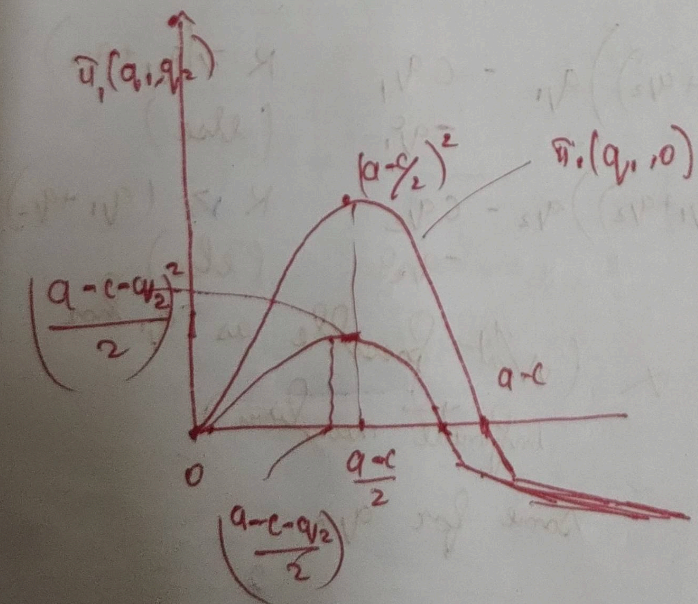
$B_1(q_2) = \frac{a-c-q_2}{2}$ $(q_2 \le a-c)$ (Differentiate (i) wrt q_1)

0 $(q_2 > a-c)$

$B_2(q_1) = \frac{a-c-q_1}{2}$







$\pi_1(q_1, 0) = (a-q_1)q_1 - cq_1$

Q. Price and Demand

$$D(p) = \begin{cases} k - p & i + k \geq p \\ 0 & \text{else} \end{cases}$$

Players - 1, 2

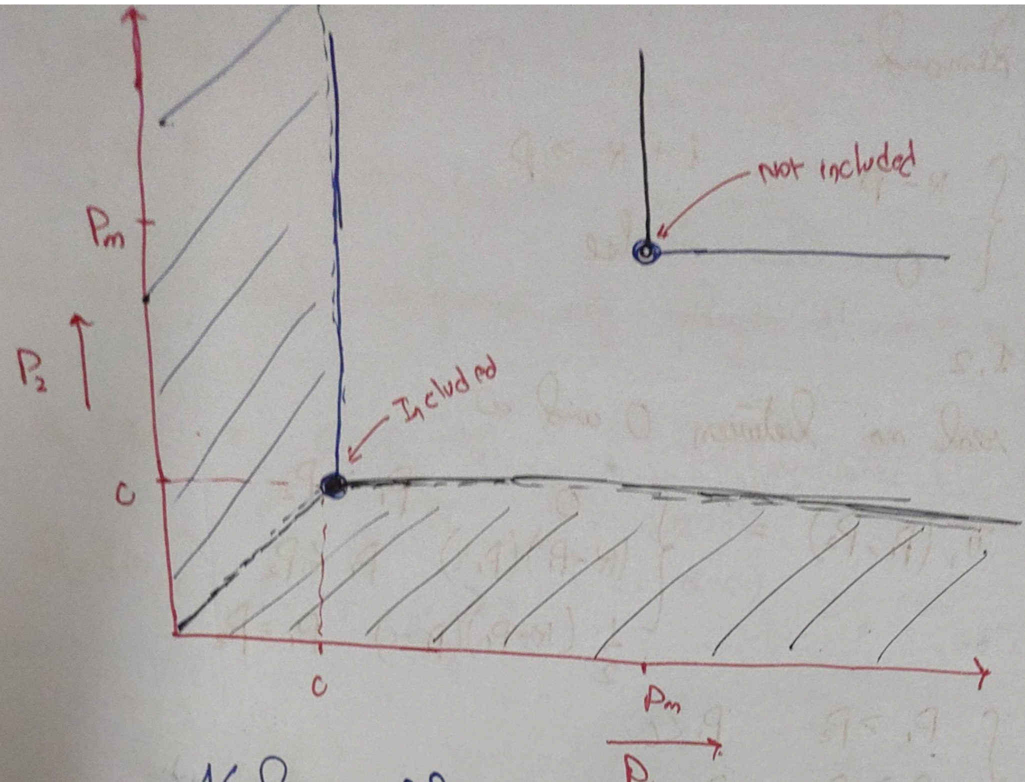
Actions - real no. between 0 and ∞

Payoffs - $$\pi_1(P_1, P_2) = \begin{cases} 0 & P_1 > P_2 \\ (k - P_1)(P_1 - c) & P_1 < P_2 \\ \frac{1}{2}(k - P_1)(P_1 - c) & P_1 = P_2 \end{cases}$$

$$B_1(P_2) = \begin{cases} P_1 > P_2 & P_2 < c \\ P_1 \geq P_2 & P_2 = c \\ \phi & << P_2 < P^m \\ \phi & P_2 = P^m \\ P^m & P_2 > P^m \end{cases}$$

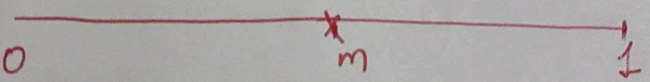
$+\frac{h}{4}$

$$B_2(P_1) = \begin{cases} P_2 > P_1 & P_1 < c \\ P_2 \geq P_2 & P_1 = c \\ \phi & << P_2 < P^m \\ \phi & P_2 = P^m \\ P^m & P_1 > P^m \end{cases}$$



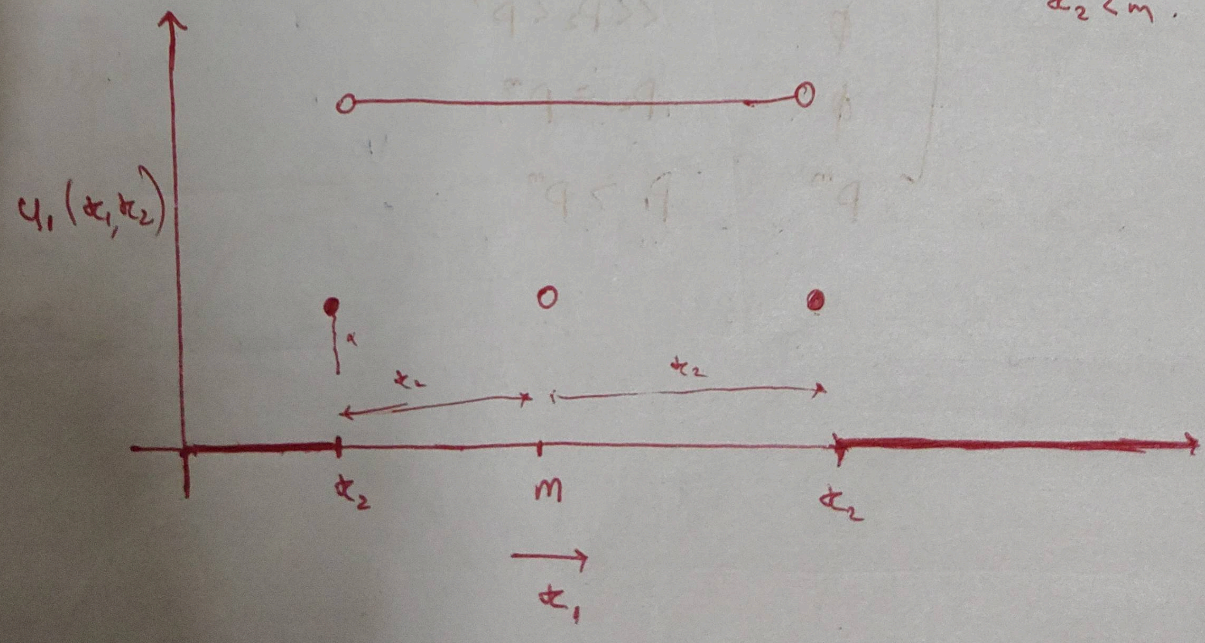
Nash equilibrium $= (c, c)$

Q



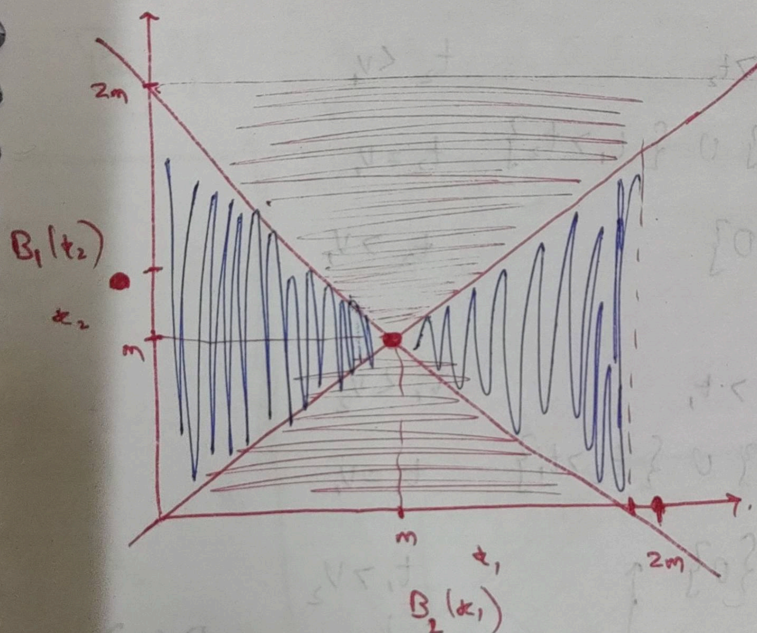
Players → C_1, C_2
Actions → Finite number

$x_2 < m.$



$$B_1(x_2) = \begin{cases} (x_2,\ 2m - x_2) & x_2 < m \\ m & x_2 = m \\ (2m - x_2,\ x_2) & x_2 > m \end{cases}$$

$$B_2(x_1) = \begin{cases} (x_1,\ 2m - x_1) & x_1 < m \\ m & x_1 = m \\ (2m - x_1,\ x_1) & x_1 > m \end{cases}$$

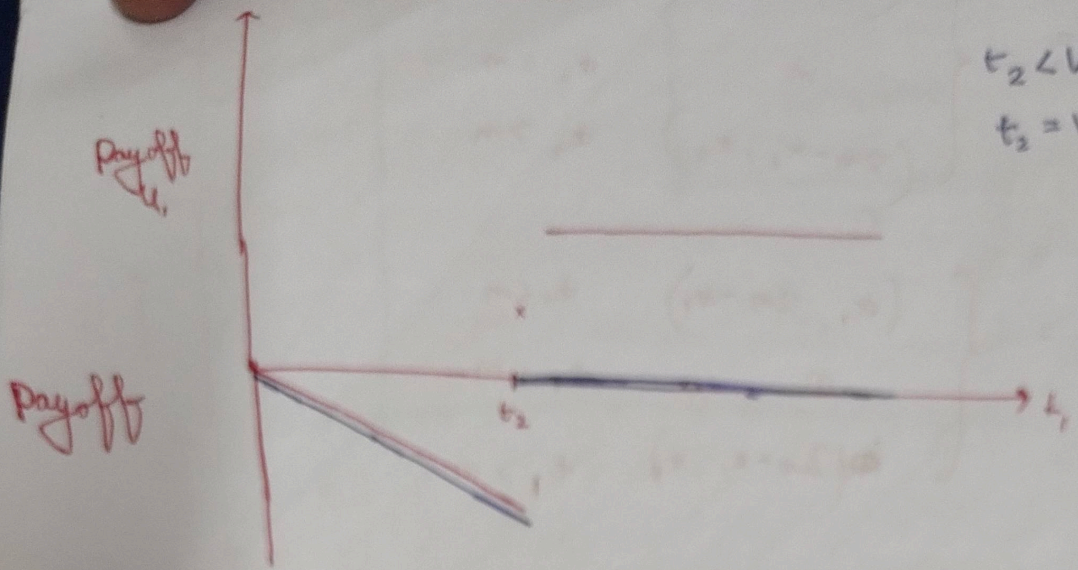


Q War of Attribution

Players = 1, 2.

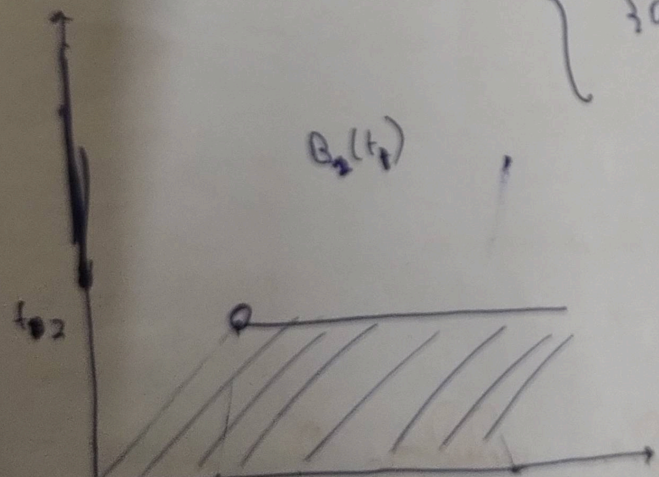
Actions = positive real number including 0.

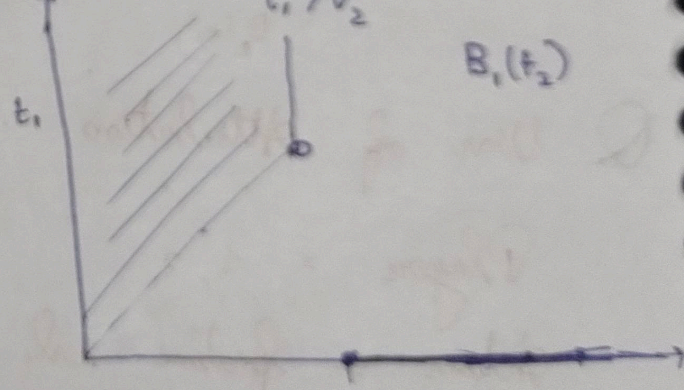
$t_2 < V_1$ - Red
$t_2 = V_1$ - Blue.



$$B_1(t_2) = \begin{cases} t_1 > t_2 & t_2 < v_1 \\ \{0\} \cup \{t_1 > t_2\} & t_2 = v_1 \\ \{0\} & t_2 > v_1 \end{cases}$$

$$B_2(t_1) = \begin{cases} t_2 > t_1 & t_1 < v_2 \\ \{0\} \cup \{t_2 > t_1\} & t_1 = v_1 \\ \{0\} & t_1 > V_2 \end{cases}$$





$V_1 > V_2$

* Auctions

SPSBA - Second prize seal based bid auction

FPSBA - First " " " "

$v_1, v_2 \ldots\ldots v_n$

$b_1, b_2 \ldots\ldots b_n$

b_K - max SPSBA to $b^{K^{th}}$ person at b'_K prize

b'_K - 2nd max FPSBA to b_K^{th} person at b_K prize

Players → 1, ... n

(X) valuat^n $v_1, v_2 \ldots\ldots v_n$

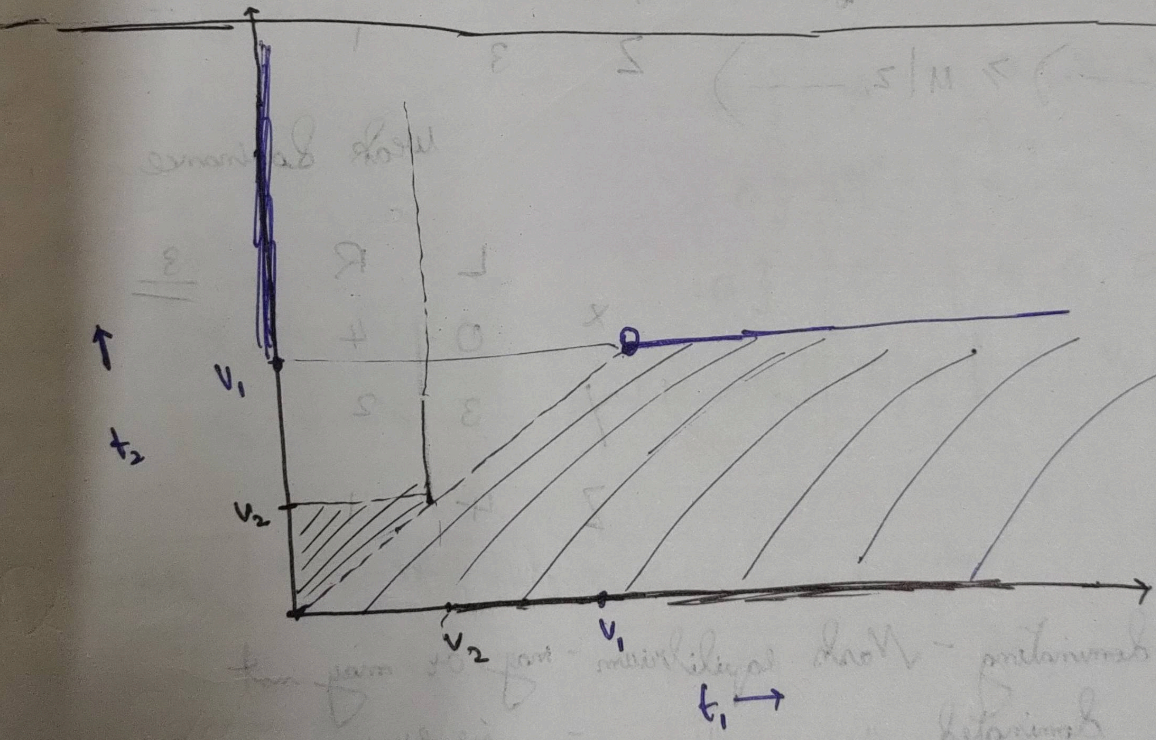
$b_1, b_2 \ldots\ldots b_n$ bids

Actions → All +ve number including 0

Payoff - $v_1 - b_2, 0, 0, 0 \ldots\ldots$

b_1 = max

b_2 = 2nd max



$$u_1(y,L) = 3 \qquad u_1(z,L) = 2$$

$$u_1(y,R) = 2 \qquad u_1(z,R) = 1$$

$$u(y, \text{---}) > u(z, \text{---})$$ Strict dominance

1

1 \ 2	L	R
X	0	4
Y	3	2
Z	2	1

2

	L	R
X	0	4
Y	3	2
Z	3	1

$$u_1(y,L) = 3 \qquad u_1(z,L) = 3$$

$$u_1(y,R) = 2 \qquad u_1(z,L) = 1$$

$$u(y, \text{---}) \geq u(z, \text{---})$$

weak dominance

3

	L	R
X	0	4
Y	3	2
Z	4	1

- Strictly dominating - Nash equilibrium - may or may not
- Strictly dominated " " - never.
- weakly dominating, dominated - Nash equi. - can be.

SPSBA

$\{\overset{b_1}{v_1''}, \overset{b_2}{v_2''} \ldots \ldots \overset{b_n}{v_n''}\}$ — (iv)

$\{v_1, 0, 0 \ldots \ldots 0\}$ — (iii)

$\{v_1, v_n, v_n \ldots \ldots v_n\}$

$\{v_1, b_2, b_3 \ldots \ldots b_n\}$ s.t. $(b_2 \ldots b_n) < v_1 / b_1$

$\{v_2, v_1, v_3 \ldots \ldots v_n\}$ — (i)

p=0
$v_1+\delta$

v_1
p=0

(Not getting object) $\{v_1, v_2, 0 \ldots \ldots 0\}$ — (i)

A $\{v_1, v_2, v_3 \ldots \ldots v_n\}$
B $\{v_1, 0, 0 \ldots \ldots 0\}$
C $\{v_1, v_2, 0 \ldots \ldots 0\}$
D $\{v_2, v_1, v_3 \ldots \ldots v_n\}$

$A' = \{v_1, v_2, v_3\}$
$B' = \{v_2, v_1, 0\}$
$C' = \{v_1, 0, 0\}$
$D' = \{v_1, v_2, 0\}$

FPSBA

$v_1 > v_2 > v_3 \ldots \ldots > v_n$

$b_1, b_2, \ldots \ldots b_n$

Q

Mixed Strategy

Players → A, B

Actions → $\begin{Bmatrix} C, D ; P \\ C, P ; \bar{P} \end{Bmatrix}$ $P, P' \in [0,1]$

Action profile $\{(P), (P')\}$

$\boxed{q}$

B

A		q	$1-q$
	P	$\boxed{2,1}$ pq	$\boxed{0,0}$ $p(1-q)$
	$1-P$	$\boxed{0,0}$ $(1-p)q$	$\boxed{1,2}$ $(1-p)(1-q)$

Payoff of A

$= \cancel{pq \cdot 2} + p(1-p) \cdot 0$
$+ q(1-p) \cdot 0 + (1-p)(1-q) \cdot 1$

Q

B

A		H	T
	H	1, -1	-1, 1
	T	-1, 1	1, -1

Payoff of A

Q

B

A		L	R
	p T	pq	$p(1-q)$
	$1-p$ B	$(1-p)q$	$(1-p)(1-q)$

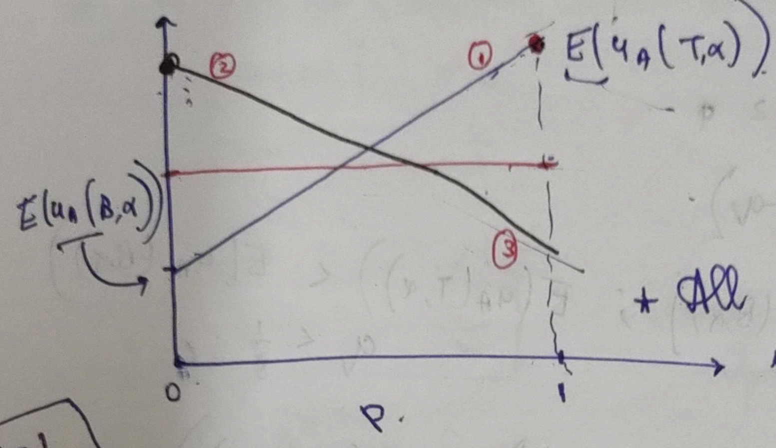
$u_A(p,q) = pq\, u_A(TL)$
$+ p(1-q)\, u_A(TR)$
$+ (1-p)q\, u_A(BL)$
$+ (1-p)(1-q)\, u_A(BR)$

$u_B(p,q) = pq\, u_B(TL) + p(1-q)\, u_B(TR)$
$(1-q)(1-p)\, u_B(BR) + u_B(BL)(1-p) \cdot q$

$$u_A(p,q) = p\left(q\,u_A(TL) + (1-q)\,u_A(TR)\right) + (1-p)\left(q\,u_A(BL) + (1-q)\,u_A(BR)\right)$$

Expectat^n

$$= p\left(E(u_A(T,\alpha))\right) + (1-p)\left(E(u_A(B,\alpha))\right)$$



① $E(u_A(T,\alpha)) > E(u_A(B,\alpha))$

② $E(u_A(T,\alpha)) < E(u_A(B,\alpha))$

③ $E(u(T,\alpha)) = E(u_A(B,\alpha))$

* All three may or may not intersect.

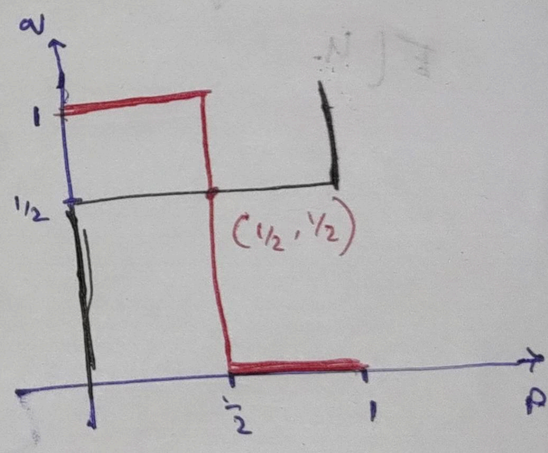
$\boxed{p=1}$

$$E(u_A(H,\alpha)) = q\cdot 1 + (-1)(1-q) = 2q-1.$$

$$E(u_A(T,\alpha)) = (-1)\cdot q + (1-q)\cdot 1 = 1-2q$$

Best response

$$B_A(q) = \begin{cases} p=1 & 2q-1 > 1-2q, \quad q > 1/2. \\ p=0 & 2q-1 < 1-2q, \quad q < 1/2. \\ p\in[0,1] & q = 1/2 \end{cases}$$



$$B_B(p) = \begin{cases} 0 & p > 1/2 \\ 1 & p < 1/2 \\ q \in [0,1] & p = 1/2 \end{cases}$$

Q.

		q L	$1-q$ R
p	T	2,1	0,0
$1-p$	B	0,0	1,2

BOS

$E[u_A(T,q)] = q \cdot 2$

$E[u_A(B,q)] = (1-q)$

$E[u_A(T,\alpha)] > E[u_A(B,\alpha)]$; $E(u_A(T,\alpha)) < E[u_A(B,\alpha)]$

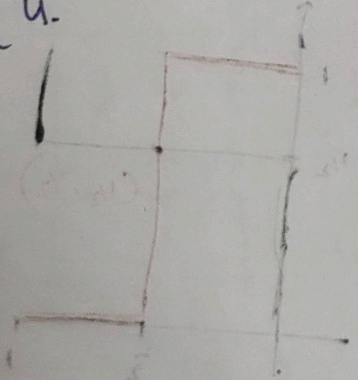
$q > \frac{1}{3}$ $\qquad$ $q < \frac{1}{3}$

$E(u_A(T,\alpha)) = E(u_A(B,\alpha))$

$q = 1/3$

$E(u_B(\alpha, k)) = 1 \cdot p$

$E[u.$



* Equal payoff — Non zero probability
* one of them less than equal payoff — assign 0 probability

* **Reporting of a crime**

$w \rightarrow$ ~~val~~ to all when someone reports

$w - c \rightarrow$ to the one who reports.

$$w - c = \omega(1 - P(\text{no one reports})) + 0\,P(\text{No one else report})$$

$$w - w(1-P) = c$$

$$\frac{w-c}{w} = 1 - P$$

$$P = 1 - \frac{w-c}{w}$$

$$\boxed{P = \frac{c}{w}}$$

(no one reports) $P = (1-p)^{n-1}$

$$\boxed{\frac{c}{w} < 1}$$

$$(1-p)^{n-1} = \frac{c}{w}$$

$$\boxed{1 - \left(\frac{c}{w}\right)^{\frac{1}{n-1}} = p}$$

$n \uparrow \quad p \downarrow$

Car - repair

Q

Car $\rightarrow$ Big problem - (θ)

Car $\rightarrow$ small " - ($1-\theta$)

$\downarrow$

Mechanic — Honest

Mechanic — Dishonest.

Players $\rightarrow$ O, m

Action $\rightarrow$ $O \rightarrow \{A, R\}$

$M \rightarrow \{H, D\}$

$C_1 > C^{\alpha}_1 > C_1'$

$> C'$

profit - Honest

~~Bada profit~~

		A^{p}	O	R^{1-p}
M	α H		$\pi_1 - (\alpha C + (1-\alpha)C')$	$(1-\alpha)\pi_1 - (\alpha C_1 + (1-\alpha)C_1')$
	$1-\alpha$ D		$\pi'(1-\alpha) + \alpha\pi_1 - C$	$0, -(\alpha C_1 + (1-\alpha)C_1')$

1st mech. chor

Bada problem (dusra mechanism)

profit (chote profit ↓ Bada bataka)

2nd mech. change

small problem dusra mechanic

B → B → π

S → S → π

S → B → π'

$\boxed{B \to S\ \alpha}$

$\alpha = (0.5, 0.5, 0)$

→ If some dominates α (strictly)

⇒ α cannot be part of mixed equilibrium.

⇒ ~~Strictly~~

If some dominates α (weakly)

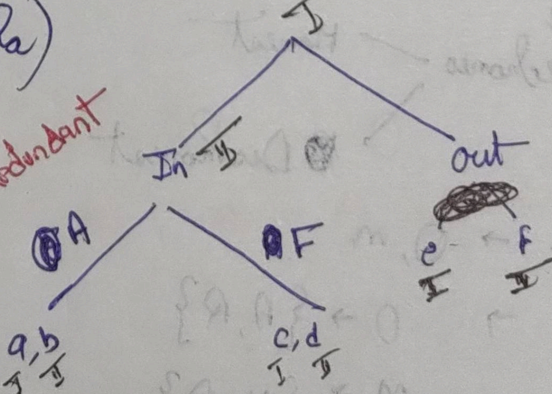
⇒ α may or maynot be part of mixed equilibrium.

* Sequence

Commercial war (Pepsi vs Cola)

- Players - $\{C^{I}, P^{II}\}$
- Actions - C → {In, out}
  P → {F, A}

Redundant



Terminal histories

TH = {In, F}, {In, A}, {out}

Player funct^n → $P(\phi) = C$ & $P(In) = P$

Payoff → $u_c\{In, A\} = a$ $u_p\{In, A\} = b$.

$u_c\{In, F\} = c$ $u_p\{In, F\} = d$

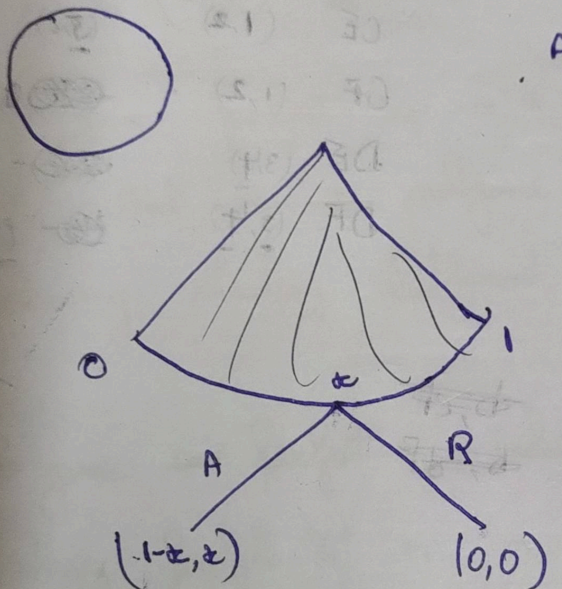
$u_c\{out\} = e$ $u_p\{out\} = f$

Nash equilibrium
(In, A);
(out, F)

$\{In, A\}$ – solut^n
(Do from reverse)

$a = 2$
$b = 1$
$c = 0$
$d = 0$
$e = 1$
$f = 2$.

Q



$P(\phi) = 1$ → player 1
$P(x) = 2$ → player 2.

TH – $\{x, A\}, \{x, R\}$.

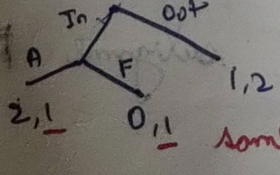
$x \in [0,1]$

$u_1(x\,A) = 1 - x$ $u_2(x, A) = x$

$u_1(x\,R) = 0$ $u_2(x, R) = 0$

$\{0, Accept\}$ – solut^n

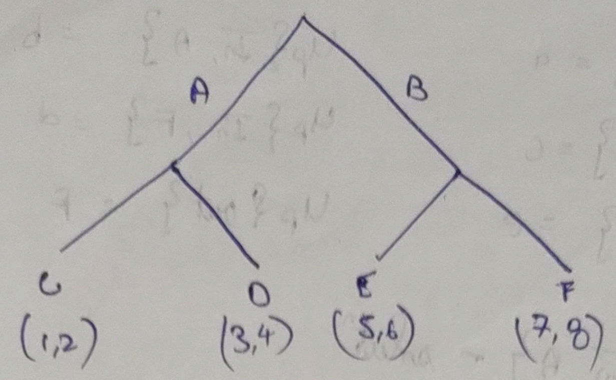
Drawbacks :-

1)


same = ?

2) Do not work for infinite games.

Q



NE (not SGPE)

b, cf

b, df.

NE (SGPE)

CE

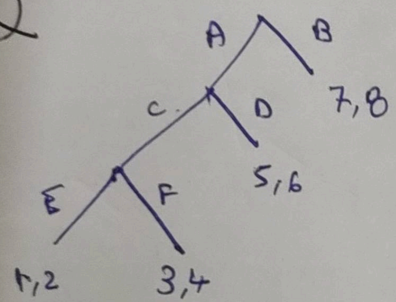
CF

DE

DF

	A	B
CE	(1,2)	(5,6)
CF	(1,2)	(7,8)
DE	(3,4)	(5,6)
DF	(3,4)	(7,8)

Q

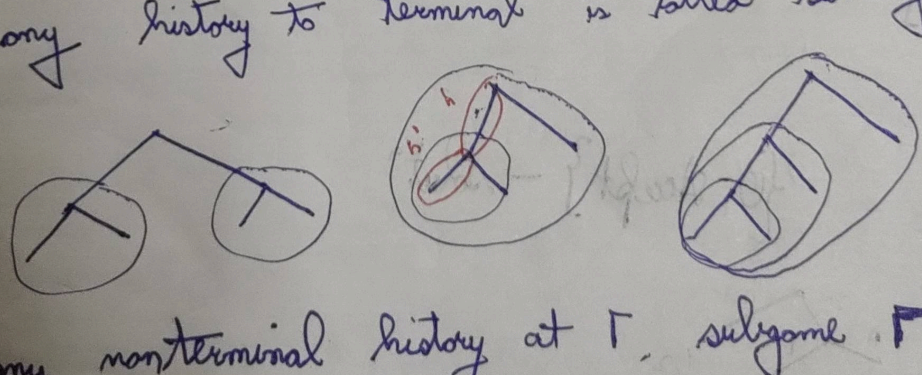


~~b, cf~~

~~b, df~~

* Sub game perfect equilibrium

From any history to terminal is called sub-game



For any nonterminal history at Γ, subgame $\Gamma(h)$ following the history h is defined as

Players → The players in Γ

Terminal histories → set of sequences h' s.t. (h, h') is TH of Γ

Player F^n → the players $P(h, h')$ is assigned to each proper sub history h'' of a TH.

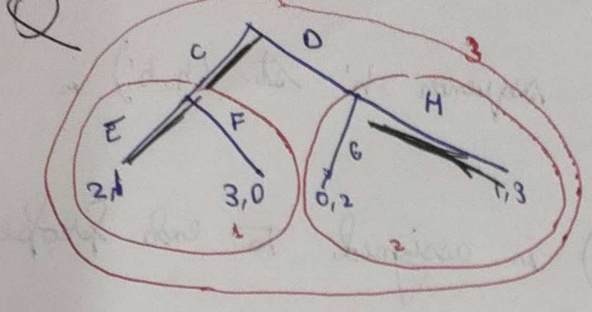
Preferences → each player prefers h' to h'' iff the preference (h, h') to (h, h'') in Γ

A sub game perfect eq. is a strategy profile s^* with the property that in no subgame any player can do better by choosing strategy different from s_i^* given everybody else.

- All sub game ~~sou~~ perfect eq. induces nashes eq. in all sub game
- The strategy profile s^* in an extensive game with perfect info. is a subgome perfect eq. if for every player i, every history h after which its players i turn to move $(P(h) = i)$ and every strategy s_i of player i the TH $O_H(s^*)$ generated by s^* after the history h is at least as good according to i, s preference to TH $O(s_i, s_{-i}^*)$ in which i choses s_i and every body else s_{-i}^*

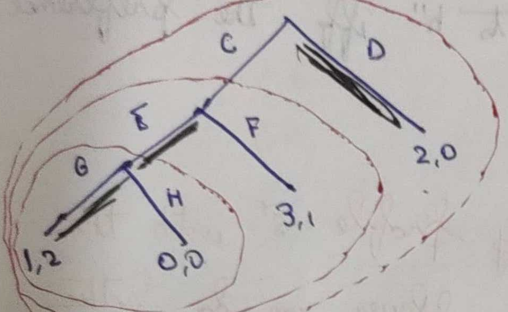
$$u_i(O_h(s^*)) \geq u_i(O_h(s_i, s_{-i}^*))$$

Q



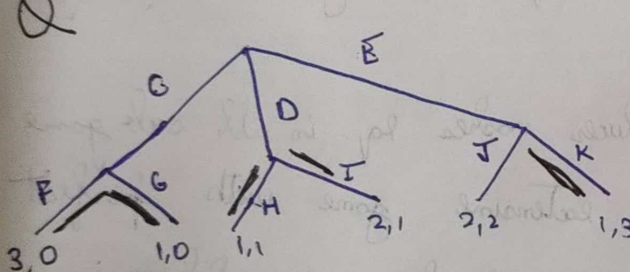
$\{C, EH\}$ — sub game perfect equilibrium

Q



$\{DG, E\}$

Q



$\{FHK\}$ - C

$\{GHK\}$ - C, D, E

$\{FIK\}$ - C

$\{GIK\}$ - D

$\therefore$ (C, FHK), (C, FIK), (C, GHK), (D, GHK), (E, GHK), (D, GIK)

Buying Votes

Q K members — vote for x → x → v_x; y → y → v_y

Interest group (x bribes 1st)

K - odd

1 vote each + everyone has to vote

Players → x, y

TH → (x, y)

PF → P(x) = y

P(φ) = x

P(φ) = x

P(x) = y

Payoff → $\begin{cases} v_x - (x_1 + x_2 \dots) \\ -(x_1 + x_2 + x_3 \dots) \end{cases}$

$\boxed{K = 3}$

$v_x = 300$

$v_y = 100$

~~$V_x = 300$~~

~~$V_y = 100$~~

X 51, 51, 0

Y 0 52, 1

X 51 51, 51

Y 0 0 0

sub game perfect equilibrium

X - 0 0 0

Y - 1 1 0

X cannot ~~will~~ win because its move come 1st. — X —

∴ He bribes all — Y —

but (Y bribes $\frac{K+1}{2}$)

$V_x = 300$ I

$V_y = 100$

$V_x = 100$ II

$V_y = 300$

$V_x = 300$ III

$V_y = 300$