L1 01.22.2020

Thm. There does not exist a rational p such that $p^2 = 2$

Proof:

Suppos for a contradiction that there exists $p \in \mathbb{Q}$ such that $p^2 = 2$

As $p \in \mathbb{Q}$, $\exists m, n \in \mathbb{Z}$ s.t. $n \neq 0$, m, n have no common factors and $p = \frac{m}{n}$

As $p^2 = 2$, $\frac{m^2}{n^2} = 2$, i.e. $m^2 = 2n^2$

As squars of odd numbers are odd and m^2 is even, m is even.

So, $\exists k \in \mathbb{Z}$ s.t. $m = 2k$.

Hence, as $m^2 = 2n^2$, $4k^2 = 2n^2$, i.e. $2k^2 = n^2$.

So n is also even.

Hence, 2 is a common factor of m, n.

Contradiction as m and n has no common factors

So $\nexists p \in \mathbb{Q}$ s.t. $p^2 = 2$ $\blacksquare$

Writing Proofs.

- Complete sentences.
- State clear assumptions.
- Define notation.
- Be neat.
- Make sense when read from beginning to end
- Be concise

§1 Number Axioms.

We start with 9 axioms of real numbers.

Let $a, b, c \in \mathbb{R}$

A 1) Associativity $a+(b+c) = (a+b)+c$. need to justify all cases.

2) Commutativity $a+b = b+a$.

3) Identity: $\exists\, 0 \in \mathbb{R}$ s.t. $a+0 = a$.

4) Inverses: $\exists\, (-a) \in \mathbb{R}$ s.t. $a+(-a) = 0$.

M 1) Associativity $a\cdot(bc) = (ab)\cdot c$

2) Commutativity $ab = ba$

3) Identity: $\exists\, 1 \in \mathbb{R}$ s.t. $1\cdot a = a$, $1 \neq 0$

4) Inverses: $\exists\, a^{-1} \in \mathbb{R}$ s.t. $a\cdot a^{-1} = 1$ & $a \neq 0$

Distributivity: $a(b+c) = ab+ac$

Prop 1.1

i) Let $x, a \in \mathbb{R}$, $a+x = a$, Then $x = 0$.

ii) Suppose $a \neq 0$, $b, c \in \mathbb{R}$, $ab = ac$. Then $b = c$.

iii) Suppose $ab = 0$. Then either $a=0$ or $b=0$. ($\forall$ $a=0, b=0$, $a=0$ & $b=0$)

iv) Suppose $a \in \mathbb{R}$. Then $a\cdot 0 = 0$

Proof

i) Let $a+x = a$

Adding $(-a)$ to this. $(-a)+a+x = (-a)+a$.

by A4, $0+x = 0$. So $x = 0$ by A3.

ii) Let $a \neq 0$, $ab = ac$.

As $a \neq 0$, $\exists a^{-1} \in \mathbb{R}$.

Multiplying by a^{-1} $(a^{-1}a)b = (a^{-1}a)c$

So by M4, $1 \cdot b = 1 \cdot c$

So by M3, $b = c$.

iii) Suppose $ab = 0$

Suppose $a \neq 0$, we will show $b = 0$.

As $a \neq 0$, by M4, $\exists a^{-1} \in \mathbb{R}$.

Multiplying by a^{-1}, we get $(a^{-1}a)b = a^{-1} \cdot 0 = 0$ by (iv)

So by M4, $1 \cdot b = 0$, so by M3 $b = 0$.

Similarly, if $b \neq 0$, $a = 0$.

Finally, if $a = 0 = b$, then $ab = 0$.

iv) Let $a \in \mathbb{R}$

Then $a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$ by D

$= a \cdot 0$ by A3.

So by i) $a \cdot 0 = 0$ ■

L2 01.24.2020

Axioms for positivity

P 1) $\exists$ a set P of numbers such that if $a \in \mathbb{R}$, exactly one of the following holds.

i) $a = 0$ ii) $a \in P$ iii) $-a \in P$. $Q = \{a > 0, -1\} \setminus \{1\}$.

2) If $a, b \in P$, then $a + b \in P$ 对P1 不一定全正数.

3) If $a, b \in P$, then $ab \in P$

Inequalities

We define $<, >, \leq, \geq$ by

$a < b$ iff $b - a \in P$

$a > b$ iff $a - b \in P$

$a \leq b$ iff $b - a \in P$ or $b = a$

$a \geq b$ iff $a - b \in P$ or $b = a$

If $a > 0$, we say a is positive

$a < 0$ negative

$a \geq 0$ non-negative

$a \leq 0$ non-positive

Absolute Value

For $a \in \mathbb{R}$, we define $|a|$ as $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$

Proposition 1.2 (traingle inequality)

Let $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$

Proof by case analysis.

Consider 4 cases. i) $a \geq 0$, $b \geq 0$

ii) $a \geq 0$, $b < 0$

iii) $a < 0$, $b \geq 0$

iv) $a < 0$, $b < 0$.

i) Suppose $a, b \geq 0$, then by P2, $a+b \geq 0$. So $|a+b| = a+b = |a| + |b|$

iv) Suppose $a, b < 0$, then by P2, $a+b < 0$. So $|a+b| = -(a+b)$

$= (-a) + (-b)$

$= |a| + |b|$.

ii) Suppose $a \geq 0, b < 0$, two further cases: $a+b \geq 0$, $a+b < 0$.

If $a+b \geq 0$, we have $|a+b| = a+b < a+b+(-b)$ as $b<0$, $-b>0$.

$= a < a+(-b) = |a| + |b|$

If $a+b<0$, we have $|a+b| = -(a+b) = -a+(-b) = a+(-a)+(-b)$ as $a \geq 0$

$= -b \leq a+(-b) = |a| + |b|$.

iii) is similar to ii

Lemma 1.3

Let $a, b \in \mathbb{R}$ s.t. $a-b = b-a$, then $a = b$

Proof: Suppose $a-b = b-a$.

then $a+a-b = b-a+a = b$,

So $a+a = b+b$.

hence $a(1+1) = b(1+1)$, So $a = b$.

used $1+1 \neq 0$, does not follow A1) – A4), M1) – M4), D.

Fact 1.4. $1+1>0$.

Proof: Recall $1>0$, thus $1+1>0$ by P2).

Note: if $a \neq 0$, $a \cdot a > 0$

Case 1: $a>0$, then by P3), $a \cdot a > 0$

Case 2: $a<0$, then $-a>0$

So $(-a)\cdot(-a)>0$ i.e. $a \cdot a > 0$.

As $1 \cdot 1 = 1$, we have $1>0$