Probability distribution

⇒ A probability distribution over Ω is a function of subsets of Ω satisfying the following three rules:

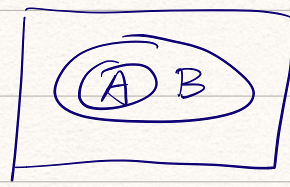
1. Non-negative: For every event A, $0 \le P(A) \le 1$
2. Addition: If $B_1 \cdots B_n$ is a partition of B,
   then $P(B) = P(B_1) + \cdots P(B_n)$
3. Total one: $P(\Omega) = 1$

⇒ Properties.

1. complement rule: $P(\text{not } A) = P(\bar{A}) = 1 - P(A)$.

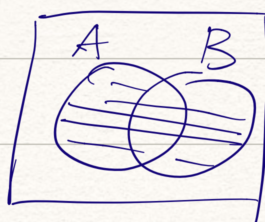
   Proof: A, A^c is a partition of Ω
   By addition rule, $P(\Omega) = P(A) + P(A^c)$
   By total one rule, $1 = P(A) + P(A^c)$
   $P(A^c) = 1 - P(A)$.

2. Difference Rule:
   If occurance event A implies occurance of event B,
   then $P(A) \le P(B)$

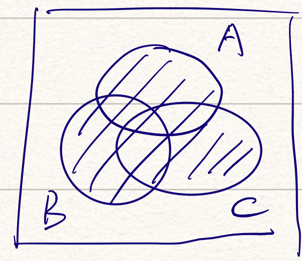
   and the difference between $P(A)$, $P(B)$ is
   $P(B \text{ and not } A) = P(BA^c) = P(B) - P(A)$

3. Inclusion - Exclusion rule

   $P(A \cup B) = P(A) + P(B) - P(AB)$

Three events.



$P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) -$

$P(AB) - P(AC) - P(BC) + P(ABC)$

$A_1, A_2, \cdots, A_n$

$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i_1 < i_2} P(A_{i_1} A_{i_2})$

$+ \cdots + (-1)^{n+1} P(A_{i_1} A_{i_2} \cdots A_{i_n})$

e.g. $P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$= P(A_1) + P(A_2) + P(A_3) + P(A_4)$

$- P(A_1A_2) - P(A_1A_3) - P(A_1A_4) - P(A_2A_3) - P(A_2A_4) - P(A_3A_4)$

$+ P(A_1A_2A_3) + P(A_1A_2A_4) + P(A_1A_3A_4) + P(A_2A_3A_4)$

$- P(A_1A_2A_3A_4)$

e.g. Among all students on campus majoring stat or Math

$A = \{$ taking stat / math 309 $\}$

$B = \{$ taking stat 461 $\}$

$C = \{$ taking stat 333 $\}$

$P(A) = 0.3 \quad P(B) = 0.3 \quad P(C) = 0.2$

$P(AB) = 0.1 \quad P(AC) = 0.1 \quad P(BC) = 0.1$

$P(A \cup B \cup C) = 0.6$.

What is $P(A \cap B \cap C)$?

$P(ABC) = P(A \cup B \cup C) + P(AB) + P(AC) + P(BC) - P(A) - P(B) - P(C)$.

$= 0.6 + 0.1 + 0.1 + 0.1 - 0.3 - 0.3 - 0.2$

$= 0.1$

Counting Techniques.

⇒ Addition rule

If $B_1, B_2 \cdots B_n$ is a partition of B, then

$$\#(B) = \#(B_1) + \#(B_2) + \cdots \#(B_n)$$

⇒ Multiplication Rule
(Product)

A set of ordered pairs, n_1 choices for the 1st element, for each choice of the 1st element, there are n_2 choices for 2nd element, and for each choice of 1st and 2nd element, ...
n_k possible choices of the kth element.

Then, the total num of choices for the K-tuple is

$$n_1 n_2 n_3 \cdots n_k$$

e.g.1. Menu in a Chinese Restaurant.

Ingredients	chicken, beef, shrimp.
Made	Kong-Pao, Ma-Po, Orange, Sesame

If one dish is going to be ordered,
how many choices are there?

$3 \times 4 = 12$

2. Birthday Problem.

In a classroom, there are $N = 60$ students.
What is the probability that at least 2 students have the same birthday?

A^c: every student have different birthday.

$$P(A^c) = \frac{\#(A^c)}{\# \Omega} = \frac{366 \times \cdots 307}{366^{60}} = 0.0059.$$

$$P(A) = 1 - P(A^c) = 1 - 0.0059 = 0.9941.$$

⇒ Permutation: an ordered sequence of k distinct objects taken from a set of n distinct objects is called a permutation of k

The total number of permutations of size k from n objects is devoted by. $P_{k,n}$ or $(n)_k$ or $_nP_k$

$$= \underbrace{n(n-1) \cdots (n-k+1)}_{k.}$$

if $k = n$, $P_{n,n} = (n)_n = {}_nP_n = n(n-1) \cdots 1 = n!$

$$P_{k,n} = n(n-1) \cdots (n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

⇒ Combination

An unordered subset of size k from a set of n distinct objects is called a combination.

The number of combinations of size k from n distinct objects is denoted by $C_n^k = \binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$

e.g. Roll a fair dice 20 times. What is the probability that the sum of all 20 numbers is 22?

$1 \times 19 + 3$ or $1 \times 18 + 2 \times 2$;

$\downarrow$

20 $\qquad\qquad$ $\binom{20}{2} = 190.$

$\#(A) = 20 + 190 = 210.$

$P(A) = \frac{\#(A)}{\#(\Omega)} = \frac{210}{6^{20}}$