Random Variables

Def: is a real-valued function defined over each outcome for a given outcome space Ω.

Notation: X, Y, Z, etc. - random variables.

lowercase letter x, y, z. denote the specific values of the random variables.

E.g. - X: the # of heads in three coin toses, possible values of x are $\{0, 1, 2, 3\}$.

$x = 3$, x represents the actual outcome of three tosses of all heads.

- Y. the # obtained by rolling a dice

possible values of Y are $\{1, 2, 3, 4, 5, 6\}$

$y = 5$ if "5" is the observed value.

r.v. — a real valued function.

$X = X(w)$, where $w \in \Omega$

output → numerial values.

"input" outcome from Ω

"random" - uncertain about the outcome before an experiment.

"variable" - "numerial values"

Def. range of the random variable X: the set of all possible values of X

E.g. X be the # of heads obtained in three tosses of a coin,

Identify all outcomes that produce value x.

Sol: possible values x: 0, 1, 2, 3

? ? ? ?

Step 1: list all the outcomes corresponding value X.

HHH	3	THH	2
HHT	2	THT	1
HTH	2	TTH	1
HTT	1	TTT	0

(num of heads)

Step 2: list distinct values of x and collection of outcomes.

value x	collection of outcomes
0	{TTT}
1	{HTT, THT, TTH}
2	{HTH, HHT, THH}
3	{HHH}.

why list the collection of outcomes for each value x?

- sometimes helpful when assign pobabilities for each value x.

Events: defined by x

$x=0$	{TTT}
$x=1$	{HTT, THT, TTH}
$x=2$	{HTH, HHT, THH}
$x=3$	{HHH}.

The events corresponding to distinct value of x are distinct.

The union of event . . — . — . — . . — — — — is Ω.

i.e. the events form a partition of Ω.

$P(X=0 \text{ or } X=1) = P(X=0) + P(X=1)$.

Two types of random variables.

- Discrete: If a r.v. has finite number of values or infinitely many values that can be arranged in a sequence.

- Continuous: If a r.v. is capable of assuming all values in an interval.

  e.g. study time for a student per day. $x \in [0, 24]$ hours.

E.g 2. A student buys a single lottery ticket. Let x be the # of tickets the student purchases before he wins at least \$1000 on a ticket.

X. 1, 2, ········ → never. terminates. ⇒ Discrete.

→ countable

Def: The Probability Distribution of a Random Variable tells us.

- what are the possible values of x are
- how probabilities are assigned to those values.

For discrete R.V.

- a list of distinct values of x & their probabilities
- a formula can be used in place of a detailed list.

e.g $P(X=k) = \binom{n}{k} p^k q^{n-k}$

value x	Probability $p(x)$
x_1	$p(x_1)$
x_2	$p(x_2)$
$\vdots$	$\vdots$
x_k	$p(x_k)$
Total	1

Def: Probability mass function (pmf)

The function, $p(x) = p(X=x)$ which satisfies

① $0 \le p(x) \le 1$, for all value x

② $\sum_{\text{all } i} p(x) = 1$

is called the pmf

Caculate $p(x)$ $\quad p_x = p(X=x)$

$= p(\text{all } w \in \Omega : X(w) = x)$

e.g. X - # of heads in 3 tosses of a coin

$P(X=2) = P(\{HHT, HTH, THH\}) = \frac{3}{8}$

$P(X=0) = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$

Probability Distribution of X

value x	Probablities
1	$\frac{1}{8}$
2	$\frac{3}{8}$
3	$\frac{3}{8}$
4	$\frac{1}{8}$
Total	1

e.g. Let X be the sum of 2 points in 2 tosses of a dice.

(a) List the possible values of X

(b) obtain the probability distribution of X.

Sol: (a) Ω : $\{(1,1), (1,2), \text{———————}, (6,6)\}$.

$X = \{2, 3, 4, \cdots, 12\}$.

(b) $P(2) = P(1,1) = \frac{1}{36}$

$P(3) = P(\{(2,1), (1,2)\}) = \frac{2}{36} = \frac{1}{18}$

$P(4) = P(\{(2,2), (1,3), (3,1)\}) = \frac{1}{12}$

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X: sum of the points in two tosses of a dice

prob distribution

value x	2	3	4	5	6	7	8	9	10	11	12	
$P(x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$	1

E.g. 2. Examine if the specification represents a probability dist or not

↳ ① $0 \le P(X_i) \le 1$

② $\sum p(x_i) = 1$

a) $f(x) = \frac{1}{3}(x-3)$, for $x = 2, 3, 4, 5$

$f(2) = -\frac{1}{3}$ No

b) $f(x) = \frac{8}{15} \cdot \frac{1}{2^x}$, for $x = 0, 1, 2, 3$

x	0	1	2	3
$f(x)$	$\frac{8}{15}$	$\frac{4}{15}$	$\frac{2}{15}$	$\frac{1}{15}$

Sum = 1 ✓

Eg. 3. A probability distribution is partically given with additional information that the even values of x are equally likely, determine the missing values.

x	1	2	3	4	5	6
$p(x)$	0.1	P	0	P	0.3	P

$0.1 + 0.3 + 3 \times P = 1$ $\quad P = 0.2$.

E.g 4: A quiz contains three multiple-choice questions.

Q1 has 4 suggested answers

Q2 has 3 ......

Q3 has 2 ......

A completely unprepared student decides to choose the answers randomly.

Let x denote the number of questions the student answered correctly.

a) List the possible value of x

b) Find the probability distribution of x.

Sol: a) possible values: $x = 0, 1, 2, 3$.

b). $P(0) = P(x=0) = P(Q_1^c Q_2^c Q_3^c) = \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{4}$

$P(1) = \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$

$= \frac{11}{24}$

$$P(2) = \frac{1}{4}\times\frac{1}{3}\times\frac{1}{2} + \frac{1}{4}\times\frac{2}{3}\times\frac{1}{2} + \frac{3}{4}\times\frac{1}{3}\times\frac{1}{2}$$
$$= \frac{6}{24} = \frac{1}{4}$$
$$P(3) = \frac{1}{4}\times\frac{1}{3}\times\frac{1}{2} = \frac{1}{24}$$

Probability of an event defined by X

Let B be a generic subset of the range of random variables X, then

$$P(X \in B) = \sum_{x \in B} p(x)$$

E.g. Find the probability that at least 1 correct

$P(X \geq 1)$ $\qquad$ $X \geq 1$

$[P(X \in B)]$ $\qquad$ $B = \{1, 2, 3\}$.

$$p(X \geq 1) = 1 - P(0) = 1 - \frac{1}{4} = \frac{3}{4}$$

Def: Cumulative distribution function (cdf)

The cdf $F(x)$ of a discrete random variable X with probability $p(x)$ is $F(x) = P(X \leq x) = \sum_{y \leq x} p(y)$

"dummy variable"

$F(x)$ is the probability that the observed value of X will be at most x.

E.g. Let X: # of heads for 2 tosses of a coin

value x	0	1	2
$p(x)$	1/4	1/2	1/4

Find the cdf of random variable X.

$$F(0) = P(X \leq 0) = \frac{1}{4}$$

$F(1) = p(X \leq 1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$F(2) = p(X \leq 2) = 1$

$F(x)$ is an increasing function (In general)