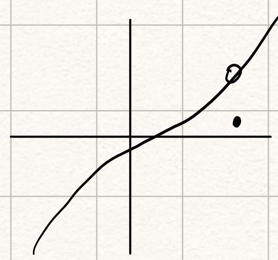
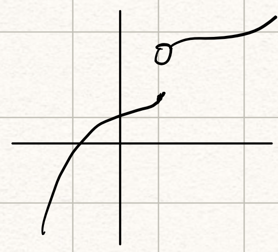
L5 01.31

## §5 Limits.

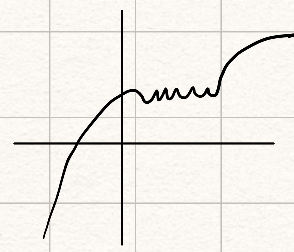
limit = value



limit ≠ value of f



limit = value one one side.



As x approaches a, f(x) approaches $l$.

'Approaches' should mean distance gets small.

Should have distance from f(x) to L.

$|f(x) - L|$ can be made as small as you like by taking $|x-a|$ small enough.

L6. 02.03

Want $\lim_{x \to a} f(x) = l$ if we can make $|f(x) - l|$ small by making the $|x-a|$ small enough.

Ex. let $f(x) = 2x$, $a=1$. Want $|f(x) - L|$ small. need $L=2$. Then $|f(x) - 2| = |2x-2| = 2|x-1|$

if we want $|f(x) - 1| < 1$, make $|x-a| < \frac{1}{2}$

i.e. if $|x-1| < \frac{1}{2}$, we have $|2x-2| = 2|x-1| < 2 \cdot \frac{1}{2} = 1$.

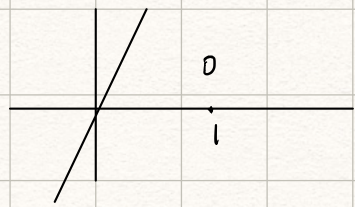
To get $|2x-2| < \varepsilon$, make $|x-1| < \frac{\varepsilon}{2}$.

**Def:** Let $f: (b,c) \to \mathbb{R}$, let $a \in (b,c)$

We say the limit of $f$ to $a$ is $L$ if

$\forall \varepsilon > 0,\ \exists \delta > 0$, s.t if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

Let $g(x) = \begin{cases} 2x & x \neq 1 \\ 0 & x = 1 \end{cases}$




E.x. $\lim\limits_{x \to 1} g(x) = 2$

Proof: Let $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. Suppose $x \in \mathbb{R}$ is s.t. $0 < |x-1| < \delta = \frac{\varepsilon}{2}$. Then, $x \neq 1$.

So. $|g(x) - 2| = |2x - 2| = 2|x-1| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$

$x \leq |x-a| + |a| < \delta + |a|$

E.x. Let $f(x) = x^2$, let $a \in \mathbb{R}$. Then $\lim\limits_{x \to a} f(x) = a^2$.

ROUGH WORK: Take $\varepsilon > 0$, need $\delta > 0$ s.t if $0 < |x-a| < \delta$, the $|x^2 - a^2| < \delta$.

Suppose $\delta < 1$: $|x+a||x-a| \leq \delta|x+a| \leq \delta(|x| + |a|)$

$\leq \delta(|a| + \delta + |a|)$

$\leq \delta(|a| + 1 + |a|) < \varepsilon$

if $\delta < \frac{\varepsilon}{2|a|+1}$, $\delta < 1$

Proof: Let $\varepsilon > 0$, let $\delta$ be s.t. $\delta < \min\left(\frac{\varepsilon}{2|a|+1}, 1\right)$

Let $x$ be s.t. $0 < |x-a| < \delta$

$|x| = |(x-a) + a| \leq \underbrace{|x-a|}_{<1} + |a| < |a| + 1$

As $|x-a| < 1$, we have $|x| < |a| + 1$.

tri ...

Then, $|x^2 - a^2| = |x+a||x-a| \leq (|x| + |a|)\delta$, by $\Delta$

$\leq \delta(2|a| + 1)$ as $|x| \leq |a| + 1$

$< \varepsilon$ as $\delta < \frac{1}{2|a|+1}$

E.x. Let $H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$ Then $\lim_{x\to 0} H(x)$ does not exist.

Need $\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$, s.t. $\forall \delta > 0$ $\exists x$ s.t. if $0 < |x-a| < \delta$ and $|f(x) - L| \ge \varepsilon$.

Two cases: $L > \frac{1}{2}$ and $L \le \frac{1}{2}$

Suppose $L > \frac{1}{2}$, then let $\delta > 0$. Let $x$ be s.t. $x \in (-\delta, 0)$.

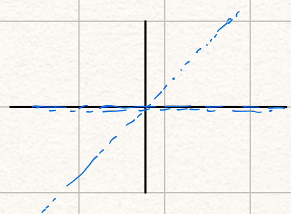
Then $H(x) = 0$, so $|H(x) - L| = |L| \ge \frac{1}{2}$

Suppose $L \le \frac{1}{2}$. then let $\delta > 0$. Let $x \in (0, \delta)$.

Then $H(x) = 1$, so $|H(x) - 1| \ge \frac{1}{2}$

Hence, $\forall L > 0$. $\exists \varepsilon = \frac{1}{2} > 0$ s.t. $\forall \delta > 0$, $\exists x$ s.t. $0 < |x - 0| < \frac{1}{2}$ and $|H(x) - L| \ge \frac{1}{2}$.

E.x. Let $g(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$



$\lim_{x\to 0} g(x) = 0$.

Proof: let $\varepsilon > 0$, let $\delta = \varepsilon$. Suppose $0 < |x| < \delta$

then $|g(x) - 0| = |g(x)| \le |x| < \delta = \varepsilon$

E.x. $\lim_{x\to a} g(x)$ does not exist if $a \ne 0$.

Proof: Suppose $L \ne 0$, Let $\varepsilon > 0$, then $\exists x \in (a-\delta, a+\delta)$ s.t. $g(x) = 0$

So, $|g(x) - L| = |L| > 0$

So, $\exists\ \varepsilon = |L|$ s.t. $\forall \delta > 0$ $\exists x$ s.t. $0 < |x-a| < \delta$ and $|g(x) - L| \ge \varepsilon$.

$L = 0$. E.x.

L7 02.05

Def: $\lim_{x\to a} f(x) = l$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-a| < \delta$, then $|f(x) - l| < \varepsilon$.

Thm 5.2: Suppose $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} f(x) = k$, then $l = k$. (Uniqueness of Limits)

Proof by contradiction.

without lose of generality.

Suppose by contradiction, $k \neq l$. WLOG, $k > l$

As $\lim_{x\to a} f(x) = l$. $\forall \varepsilon > 0, \exists \delta > 0$ s.t if $0 < |x-a| < \delta$, then $|f(a) - l| < \varepsilon$

Same for $k$.

Let $\varepsilon = \frac{k-l}{2}$. Note $\varepsilon > 0$.

So $\exists \delta_1 > 0$ if $0 < |x-a| < \delta_1$, then $|f(x) - l| < \varepsilon = \frac{k-l}{2}$.

As $\lim_{x\to a} f(x) = k$, $\exists \delta_2 > 0$ s.t. if $0 < |x-a| < \delta$, then $|f(x) - k| < \varepsilon = \frac{k-l}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$ Then $0 < |x-a| < \delta_1$ we have $|f(x) - l| < \varepsilon$ and $|f(x) - k| < \varepsilon$.

Let $x$ be s.t. $0 < |x-a| < \delta$. Then $0 < \underline{k-l} = k - f(x) + f(x) - l \leq |k - f(x) + f(x) - l|$.

Then, by $\Delta$ $\leq |f(x) - k| + |f(x) - l| < 2\varepsilon < \underline{k-l}$.

contradiction

Thm 5.3. Let $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = k$.

i) Sum rule. The $\lim_{x\to a} (f+g)(x) = l + k$.

ii) Product rule. The $\lim_{x\to a} (fg)(x) = lk$.

iii) Quotient rule. The $\lim_{x\to a} (\frac{f}{g})(x) = \frac{l}{k}$ if $k \neq 0$

Proof: i) Let $\varepsilon > 0$. Then as $\lim_{x\to a} f(x) = l$, $\exists \delta_1 > 0$ s.t. if $0 < |x-a| < \delta$, then $|f(x) - l| < \frac{\varepsilon}{2}$

As $\lim_{x\to a} g(x) = k$, $\exists \delta_2 > 0$ s.t. if $0 < |x-a| < \delta$, then $|g(x) - k| < \frac{\varepsilon}{2}$

Let $\delta = \min(\delta_1, \delta_2)$ Let $x$ be s.t. $0 < |x-a| < \delta$

Then $|f(x) - l| < \frac{\varepsilon}{2}$, $|g(x) - k| < \frac{\varepsilon}{2}$

So. $|(f+g)(x) - (l+k)| = |f(x) - l + g(x) - k|$

By $\Delta$, $\leq |f(x) - l| + |g(x) - k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

ii) Let $\varepsilon > 0$. Note $\frac{\varepsilon}{2(|k+l|)} > 0$, $\min(1, \frac{\varepsilon}{2(|l|+1)}) > 0$.

As $\lim_{x \to a} f(x) = l$, $\exists \delta_1 > 0$ s.t. if $0 < |x - a| < \delta_1$, then $|f(x) - l| < \frac{\varepsilon}{2(|k|+1)}$

As $\lim_{x \to a} g(x) = k$, $\exists \delta_2 > 0$ s.t. if $0 < |x-a| < \delta_2$, then $|g(x) - k| < \min(1, \frac{\varepsilon}{2(|l|+1)})$

Let $\delta = \min(\delta_1, \delta_2)$

Let $x$ be s.t. $0 < |x-a| < \delta$.

Then $|g(x) - k| < 1$, hence $|g(x)| < |k| + 1$

So, $|f(x)g(x) - kl| = |(f(x) - l)g(x) + l(g(x) - k)|$

$\leq |f(x) - l||g(x)| + |l||g(x) - k|$.

$\leq |f(x) - l|(|k|+1) + |g(x) - k|(|l|+1)$

$< \frac{\varepsilon}{2(|k|+1)} \cdot (|k|+1) + \frac{\varepsilon}{2(|l|+1)}(|l|+1) = \varepsilon$. ∎

iii) Claim: $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{k}$.

Assuming claim, $\frac{f}{g}(x) = (\frac{1}{g}) \cdot f(x) \xrightarrow{x \to a} \frac{l}{k}$ by product rule.

Let $\varepsilon > 0$. As $\lim_{x \to a} g(x) = k$ and $k \neq 0$, $\exists \delta_1 > 0$ s.t. if $0 < |x-a| < \delta_1$ then

$|g(x) - k| < \frac{|k|}{2}$. Hence, if $0 < |x-a| < \delta$, $|g(x)| > \frac{|k|}{2}$

As $\lim_{x \to a} g(x) = k$, $\exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2$

then $|g(x) - k| < \frac{\varepsilon |k|^2}{2}$ (note $\frac{\varepsilon |k|^2}{2} > 0$)

Let $\delta = \min(\delta_1, \delta_2)$. Let $x$ be s.t. $0 < |x-a| < \delta$.

So $|g(x)| > \frac{|k|}{2}$. Also, $|\frac{1}{g(x)} - \frac{1}{k}| = |\frac{g(x) - k}{k g(x)}|$

$\leq |g(x) - k| \frac{2}{|k|^2}$

$< \frac{\varepsilon |k|^2}{2} \cdot \frac{2}{|k|^2} = \varepsilon$. ∎

L8 02.07

Recall: Let $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = k$, then $\lim_{x\to a}(f+g)(x) = l+k$,

$\lim_{x\to a}(fg)(x) = lk$.

$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{l}{k}$. if $k \neq 0$

Proposition: Let $n \in \mathbb{N}$, then $\lim_{x\to a} x^n = a^n$.

Proof: by induction.

Let $n=1$. Then $x^n = x$, $\lim_{x\to a} x = a$ as required.

Let $n \in \mathbb{N}$, Suppose that $\lim_{x\to a} x^n = a^n$

Then $\lim_{x\to a} x^{n+1} = \lim_{x\to a} x^n \cdot x = a^n \cdot a = a^{n+1}$ by product rule & IH. ∎

Ex. let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ let $a \in \mathbb{R}$.

Then $\lim_{x\to a} p(x) = p(a)$.

Proof: by induction on $n$.

Base case. $n=0$. Then $p(x) = a_0$, so $\lim_{x\to a} p(a) = \lim_{x\to a} a_0 = a_0 = p(a)$

Suppose if $\deg(p) \le n$, then $\lim_{x\to a} p(x) = p(a)$

Suppose $\deg(p) = n+1$

Then $\lim_{x\to a} p(x) = \lim_{x\to a} (a_{n+1} x^{n+1} + \cdots + a_0)$

$= \lim_{x\to a} (a_{n+1} x^{n+1} + q(x))$ where $q(x) = a_n x^n + \cdots + a_0$.
is degree $\le n$.

Note $a_{n+1} \in \mathbb{R}$, so $\lim_{x\to a} a_{n+1} = a_{n+1}$,

thus, by product rule $\lim_{x\to a} a_{n+1} x^{n+1} = a_{n+1} a^{n+1}$ using previous propositions

So, by sum rule, $\lim_{x\to a} p(x) = \lim_{x\to a} (a_{n+1} a^{n+1} + q(a)) = p(a)$

by inductive hypothesis ∎

Ex, Let $r = \frac{p}{q}$ be rational function, $a \in \mathbb{R}$ s.t. $q(a) = 0$.

Then, $\lim_{x \to a} q(x) = q(a) \neq 0$, so by quotient rule, $\lim_{x \to a} r(x) = \frac{p(a)}{q(a)} = r(a)$

Case where $f, g$ not defined for $x \leq a$.

For $\lim_{x \to a} f(x) = l$, need $|f(x) - l| < \varepsilon$ $\forall x$ s.t. $0 < |x - a| < \delta$

i.e. $\forall x \in \{(a-\delta, a+\delta)\} \setminus \{a\}$.

As $f, g$ not defined for $x \leq a$, not possible.

**Def: (One-sided Limit)**

Let $a < b$, $f : (a, b) \to \mathbb{R}$. We say $\lim_{x \to a^+} f(x) = l$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t if $0 < x - a < \delta$, then $|f(x) - l| < \varepsilon$.

We say $\lim_{x \to b^-} f(x) = l$ if $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t. $0 < b - x < \delta$, then $|f(x) - l| < \varepsilon$.

**Thm:**

Let $f : (b, c) \to \mathbb{R}$, $a \in (b, c)$

Then, $\lim_{x \to a} f(x) = l$ iff $\lim_{x \to a^+} f(x) = l$ and $\lim_{x \to a^-} f(x) = l$.

Proof: Suppose first $\lim_{x \to a} f(x) = l$. Let $\varepsilon > 0$. then $\exists \delta > 0$ $0 < |x - a| < \delta$ s.t. $|f(x) - l| < \varepsilon$.

if $0 < x - a < \delta$, then $0 < |x - a| < \delta$, so $|f(x) - l| < \varepsilon$. Hence. $\lim_{x \to a^+} f(x) = l$.

If $0 < a - x < \delta$, then $0 < |x - a| < \delta$, so $|f(x) - l| < \varepsilon$. Hence $\lim_{x \to a^-} f(x) = l$.

So, $\lim_{x \to a} f(x) = l \Rightarrow \lim_{x \to a^+} f(x) = l$ and $\lim_{x \to a^-} f(x) = l$.

Now suppose $\lim_{x \to a^+} f(x) = l = \lim_{x \to a^-} f(x)$

Let $\varepsilon > 0$ As $\lim_{x \to a^+} f(x) = l$, $\exists \delta_1 > 0$ s.t. if $0 < x - a < \delta_1$, $|f(x) - l| < \varepsilon$.

As $\lim_{x \to a^-} f(x) = l$, $\exists \delta_2 > 0$ s.t. if $0 < a - x < \delta_2$, $|f(x) - l| < \varepsilon$.

Let $\delta = \min(\delta_1, \delta_2)$, Let $x$ be s.t. $0<|x-a|<\delta$

If $x>a$, then $0<x-a<\delta \leq \delta_2$, so $|f(x)-\ell|<\varepsilon$.

If $x<a$, then $0<a-x<\delta \leq \delta_1$, so $|f(x)-\ell|<\varepsilon$.

Consider $f(x)=\frac{1}{x}$ on $\mathbb{R}$, as $x$ gets big, $\frac{1}{x}$ gets small. In some sense, $\lim_{x\to\infty}\frac{1}{x}=0$.

Def: Let $f:(a,\infty)\to\mathbb{R}$, $L\in\mathbb{R}$, we say $\lim_{x\to\infty} f(x)=0$ if $\forall \varepsilon>0$, $\exists M>0$ if $x>M$, then $|f(x)-\ell|<\varepsilon$.

Def: Let $b<c$, $a\in(b,c)$ $f:(b,c)\to\mathbb{R}$. Then $\lim_{x\to a} f(x)=\infty$ if $\forall M>0$, $\exists \delta>0$ s.t. if $0<|x-a|<\delta$, then $f(x)>M$.