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§7. Theorems on continuity.

Thm 7.1 (Intermediate value theorem)

s.t. $f(a)<0<f(b)$. Then $\exists c\in(a,b)$ s.t. $g(c)=0$.

Let $f:[a,b]\to\mathbb{R}$ be cts at $[a,b]$ and s.t. $f(a)<f(b)$

Then $\forall v\in(f(a),f(b))$, $\exists c\in(a,b)$ s.t. $f(c)=v$.

Ex. $f(x)=\begin{cases} x\sin(\frac{1}{x}) & x\neq 0 \\ 0 & x=0 \end{cases}$.

IVT says f attains every value $v\in\mathbb{R}$.

Thm 7.2 (Extreme Value Thm)

Let $f:[a,b]\in\mathbb{R}$ be cts on $[a,b]$,

then $\exists M>0$ s.t. $\forall x\in[a,b]$, $|f(x)|\leq M$ (i.e. f is bounded)

Also, $\exists x_-, x_+\in[a,b]$. s.t. $\forall x\in[a,b]$, $f(x_-)\leq f(x)\leq f(x_+)$

Note if $f(a)=f(b)$, $\nexists v\in(f(a),f(b))$.

So still true (vacuously) that $\forall v\in(f(a),f(b))$, $\exists c\in(a,b)$ s.t. $g(c)=v$.

Suppose we relax cty at a single point in $[a,b]$

Ex. Heaviside function $H(x)=\begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases}$

on $[-1,1]$, $H(x)$ cts at $x\neq 0$.

$H(-1)=0$, $H(1)=1$.

Note $\frac{1}{2}\in(H(-1),H(1))$, but $\nexists x\in(-1,1)$ s.t. $H(x)=\frac{1}{2}$.

For IVT, need closed $[a,b]$ or cannot define $(f(a), f(b))$.

For EVT, we made 2 assumption: closed $[a,b]$, cty on $[a,b]$.

Ex. on open (a,b), f cts, not bounded.

e.g. $f(x) = \frac{1}{x}$ on $(0,1)$ is cts and not bounded.

Ex. $f(x)$ cts, bdd, not attain extreme values on (a,b).

e.g $f(x) = x$ on $(0,1)$

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Cor 7.3: Let $g: [a,b] \to \mathbb{R}$ be cts on $[a,b]$ s.t. $f(a) < f(b)$

Then $\forall v \in (f(a), f(b))$, $\exists c \in (a,b)$ s.t. $f(c) = v$.

Proof: Let v be s.t. $f(a) < v < f(b)$

Define $g(x) = f(x) - v \quad \forall x \in [a,b]$

By Thm 7.1, $\exists c \in (a,b)$ s.t. $g(c) = 0$ as g is cts on $[a,b]$

and $g(a) < 0 < g(b)$ Hence $f(c) = g(c) + v = v$ ∎

Cor 7.4: Let $f: [a,b] \to \mathbb{R}$ be cts on $[a,b]$ s.t. $f(a) > f(b)$ Then $\forall v \in (f(a), f(b))$,

s.t. $f(c) = v$.

Proof: Let $g(x) = -f(x)$. Then g is cts on $[a,b]$.

$g(a) = -f(a) < -f(b) = g(b)$

Let $v \in (f(b), f(a))$ Then $-v \in (g(a), g(b))$

By IVT. $\exists c \in (a,b)$ s.t. $g(c) = -v$. Hence $f(c) = v$ ∎

Thm 7.5. (Existence of roots).

i) Let $x>0$, $\exists y>0$ s.t. $y^2=x$

ii) Let $x>0$, $n\in\mathbb{N}$. Then $\exists y>0$ s.t. $y^n=x$

if $n\in\mathbb{N}$ - n odd, $x<0$, $\exists y<0$ s.t. $y^n=x$

We define $\sqrt{x}$ for $x>0$ as $y\geq 0$ s.t. $y^2=x$

Note: if $0<y_1<y_2$ then $y_2^2>y_1^2$ Hence y s.t. $y^2=x$ is unique,

This square root is well-defined

Proof i) Let $f(y)=y^2$ Then f is cts.

Note $f(0)=0<x$ and $f(1+x)=(1+x)^2$.

If $x<1$, then $(1+x)^2\geq 1>x$.

If $x\geq 1$, then $x^2\geq x$ So $(1+x)^2=1+2x+x^2>1+x^2\geq 1+x>x$.

So $f(x+1)>x$

As f is cts on $[0,1+x]$, by IVT, $\exists y\in(0,1+x)$ s.t. $f(y)=x$, $y^2=x$.

ii) Let $g(y)=y^n$

Note $g(0)=0<x$

If $x<1$, then $(1+x)^n\geq 1>x$.

If $x\geq 1$, then $(1+x)^n\geq 1+x^n\geq 1+x>x$

As g is cts on $[0,1+x]$, $\exists y\in(0,1+x)$ s.t. $g(y)=x$ i.e. $y^n=x$.

If n odd, $x<0$, $-x>0$, So $\exists y>0$ s.t. $y^n=-x$.

Thus $(-y)^n=(-1)^n y^n=-y^n=x$.

Thm 7.6 Let $n \in \mathbb{N}$ be odd, let $p(x) = x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$.

Then the equation $p(x) = 0$ has at least one solution.

Proof: Note that p is cts on $\mathbb{R}$

We want $x_1, x_2 \in \mathbb{R}$ s.t. $x_1 < x_2$ $\quad P(x_1) < 0 < p(x_2)$

Then, applying IVT on $[x_1, x_2]$, done.

Recall : $\lim_{x \to \infty} x^n = \infty$,

$\lim_{x \to -\infty} x^n = -\infty$ for n odd (Ex)

Now $p(x) = x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)$ for $x \neq 0$

Claim: $\exists M > 0$ s.t. if $|x| > M$, then $\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) \geq \frac{1}{2}$.

Let $M > \max\left(1, \frac{|a_{n-1}| + \cdots + |a_0|}{2}\right)$

Then, if $|x| > M$, we have

$$\left| \frac{a_{n-1}}{|x|} + \cdots + \frac{|a_0|}{|x^n|} \right| \leq \left|\frac{a_{n-1}}{x}\right| + \cdots + \left|\frac{a_0}{x^n}\right|$$

$$\leq \frac{1}{|x|}(|a_{n-1}| + \cdots + |a_0|)$$

$$\leq \frac{1}{2}.$$

Hence, for $x > M$, $p(x) = x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)$

$$\geq x^n \left(1 - \left|\frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right|\right)$$

$$\geq \frac{x^n}{2} > 0$$

Similarly, if $x < -M$, $p(x) \leq \frac{x^n}{2} < 0$

As p is cts on $[-M-1, M+1]$ by IVT, $\exists x \in (-M-1, M+1)$ s.t. $p(x) = 0$ ∎

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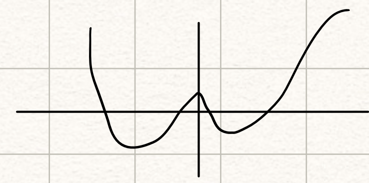
Thm 7.7. Let $n \in \mathbb{N}$ be even, $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 n + a_0$

Then $\exists y \in \mathbb{R}$ s.t. $f(y) \leq f(x)$ $\forall x \in \mathbb{R}$.



Proof: by proof of thm 7.6. $\exists \tilde{M} > 0$ s.t. if $|x| > \tilde{M}$, then $(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}) \geq \frac{1}{2}$

So if $|x| > \tilde{M}$, then $p(x) = x^n(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}) \geq \frac{x^n}{2}$

So, Let $M > 0$ s.t. $M > \tilde{M}$, $M^n > 2p(0)$

Then if $|x| > M$, $p(x) \geq \frac{x^n}{2} \geq \frac{M^n}{2} > p(0)$

As p is cts on $[-M, M]$, $\exists y \in [-M, M]$ s.t. $\forall x \in [-M, M]$ $p(x) \geq p(y)$

Also, if $|x| > M$, we have $p(x) > p(0) \geq p(y)$ as we have $0 \in [-M, M]$

So, $\forall x \in \mathbb{R}$, $p(x) \geq p(y)$ ∎

Thm 7.8 Let $n \in \mathbb{N}$ be even, $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$

$\exists M \in \mathbb{R}$ s.t. if $c \geq M$, then $\exists x$ s.t. $p(x) = c$.

and if $c < m$, then $\nexists x$ s.t. $p(x) = c$.

Proof: By Thm 7.7, $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$, $p(x) \geq p(y)$

Let $m = p(y)$ So, if $c < m$, $\nexists x$ s.t. $p(x) = c$.

If $c = m$, then $p(y) = c$

If $c > m$, then as $\lim_{x \to \infty} p(x) = \infty$ (by previous proof)

$\exists b > y$ s.t. $p(b) > c$

As p is cts on $[y, b]$ and $p(y) = m < c < p(b)$

$\exists x \in (y, b)$ s.t. $p(x) = c$ by IVT ∎