Solow Growth Model

Long-Term Growth

$$Y_t^i = \bar{A}_t^i N_t^i h_t.$$

$$Y_t^i = (\bar{A}_t^i N_t^i h_t)^{1-\alpha} K_t^{\alpha}$$

$\leftarrow$ capital

$$Y_t^i = (A_t^i N_t h_t)^{1-\alpha} K_t^{\alpha}$$

3 step

① add K_t

② human capital

③ romon model.

Ch2-3.

Solow Growth Model

identify the observation x fundamental.

- Simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert Solow and Trevor Swan.
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar model emphasized potential dysfunctional aspects of growth: e.g. how growth could go hand-in-hand with increasing unemployment.
- Solow model demonstrated why the Harrod-Domar model was not an attractive place to start.
- At the center of the Solow growth model is the *neoclassical aggregate production function*.

Solow Growth Model

Basic Elements

- Aggregate production function with two inputs:
  1. capital input
  2. labor
- Two key assumptions about production function:
  1. constant return to scale CRS
  2. diminishing return to input. DMR.
- Basic resource constraint or "social accounting"
- Simple behavioral assumption household have saving rate.
- Households are not optimizing

SOLOW V.S. NEOCLASSICAL GROWTH MODEL.

Dynamic programming.

Assume all labor works,
everyone is working full time.

Labor

Denoted by $L(t)$

- $L(t)$ is given exogenously
- No disutility from labor effort
- No endogenous fertility

Physical Capital

Denoted by $K(t)$

Idea

Today's output can be consumed or invested (added to capital) to produce more output (and maybe even more investment and hence capital) tomorrow.

price consumption 1

capital 1

normalized.

Example

① eat.

1. Agricultural Output, wheat. ② keep some for next period.
2. Industrial , steel. final consumption
   investment, rebuilt the building.

Aggregate Production Function

make everything ↑ productive. (total factor).

$$Y = \underbrace{A}_{TFP} \times F\left(\underbrace{K}_{capital}, \underbrace{L}_{labor}, \text{education, land}, \ldots\right)$$

Describes the amount of goods produced with a given amount of inputs (capital, labor, education, land, . . .) and a given technology (denoted by A).

An Important Example: Cobb-Douglas Production Function

$$
\begin{aligned}
Y &= AF(K, L) \\
&= AK^{\alpha}L^{1-\alpha}. \\
&= AK^{\alpha}(\underbrace{Lh}_{H})^{1-\alpha}.
\end{aligned}
$$

physical, Human, capital

Some Properties of the Cobb Douglas Production Function

1. constant returns to scale
2. decreasing marginal returns
3. constant factor shares

Constant Factor Shares

Consider a (representative) competitive firm producing with the technology $F(K, L)$. As usual, firms choose inputs to maximize their profits:

$$\max_{K \geq 0, L \geq 0} Y - (r + \delta)K - wL, \text{ or}$$

$$\max_{K \geq 0, L \geq 0} F(K, L) - (r + \delta)K - wL$$

where $r + \delta$ is the rental rate of capital and w is the wage rate.

Constant Factor Shares (cont'd)

How much capital and labor would this firm use? It would use capital and labor up to the point in which the increase in output equals the increase in costs. Formally,

$$\text{MPK.} \qquad \frac{\partial F(K, L)}{\partial K} = r + \delta.$$

$$\text{MPL.} \qquad \frac{\partial F(K, L)}{\partial L} = w$$

Constant Factor Shares (cont'd)

When production technology is Cobb-Douglas

$$\frac{(r+\delta)K}{Y} = \alpha.$$

How much of total output Y is used to pay for labor? What is the labor income share $\frac{wL}{Y}$?

Using $\frac{\partial F(K,L)}{\partial L} = w$ and $Y = F(K, L)$:

$$\frac{wL}{Y} = \frac{\frac{\partial F(K,L)}{\partial L} L}{Y}$$

$$= \frac{(1-\alpha)K^{\alpha}L^{-\alpha}L}{Y} = \frac{(1-\alpha)K^{\alpha}L^{1-\alpha}}{Y}$$

$$= 1-\alpha.$$

Constant Factor Shares (cont'd)

When production technology is Cobb-Douglas

- The income share of labor is constant!
- By analogy, capital's share is constant and equals α.
- Is this a desirable property?

Constant Factor Shares (cont'd)

Evidence for US and UK (1935-85) from Gollin (2002)

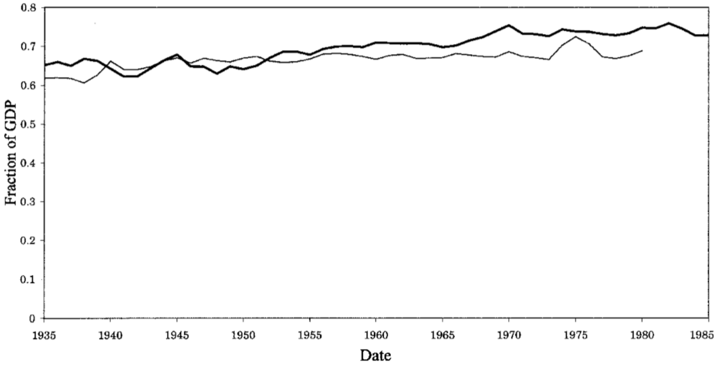
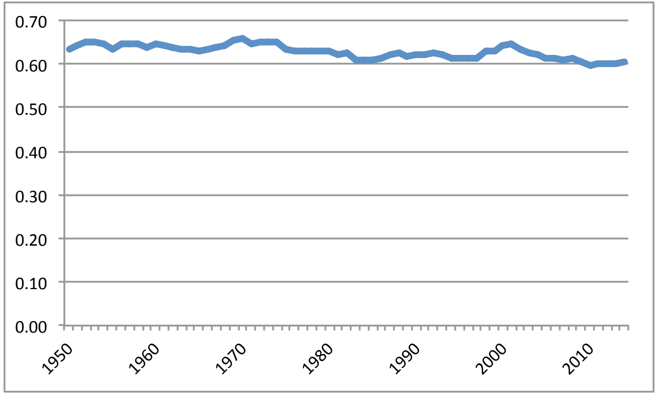


FIG. 1.—Employee compensation share of GNP, United States and Great Britain, 1935–85. Source: U.S. Department of Commerce (1986, 1990). British data: Mitchell (1988), pp. 823–25.

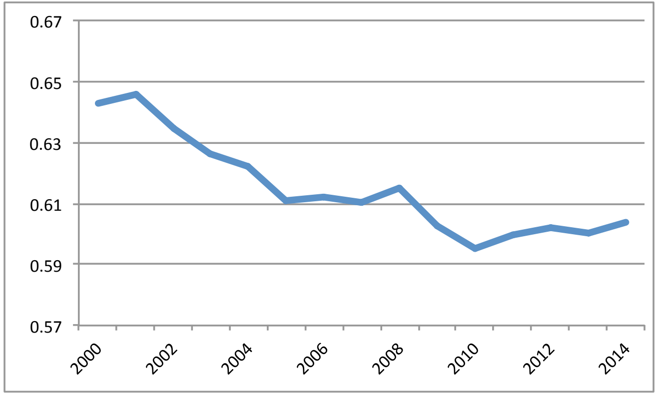
Constant Factor Shares (cont'd)

The US updated using Penn World Tables (1950 - 2014)



Constant Factor Shares (cont'd)

More recently . . .



① labor in general is losing out w.r.t capital.

not sure.

② capital share ↓.

Back to the Solow Growth Model

- Aggregate Production Function: $Y(t) = F(K(t), L(t))$
- Basic resource constraints or "social accounting"
- Simple behavioral assumption (constant saving rate)

Basic Social Accounting

In a *closed* economy the resource constraint is

$$\underbrace{Y(t)}_{\text{output}} = \underbrace{C(t)}_{\text{consumption}} + \underbrace{I(t)}_{\text{investment}}$$

and the stock of physical capital (machines, non-residential structures, transport equipment) evolves according to the law of motion:

$$K(t+1) - K(t) = I(t) - \delta K(t).$$

Basic Social Accounting

For small increments of time Δ, the law of motion is

A Func of time.

$$\dot{K}(t) = I(t) - \delta K(t).$$

where $\dot{K}(t) = \frac{dK(t)}{dt} = \frac{K(t+\Delta)-K(t)}{\Delta}$ $= \frac{\partial \dot{K}(t)}{\partial t}$..

Back to the Solow Growth Model

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Simple Behavioral Assumption

Constant saving rate

$$C(t) = (1 - s)\, Y(t)$$

or, using that in a closed economy saving equals investment, i.e., $Y(t) - C(t) = I(t)$,

$$I(t) = sY(t).$$

Basic Equation of the Solow Growth Model

Putting all the elements together

$$\dot{K}(t) = sY(t) - \delta K(t).$$

$$= s \cdot F(K(t), L(t)) - \delta K(t).$$

Assuming (for now) that population (and hence labor input) is constant, i.e. $L(t) = L$, and exploiting the CRS property of the aggregate production:

$$(k(t))$$

$$\dot{k}(t) = sF\left(\frac{K(t)}{L}, \frac{L}{L}\right) - \frac{\delta K(t)}{L}$$

$$f(k(t)) = F\left(\frac{K(t)}{L(t)}, 1\right) \qquad = sF(k(t), 1) - \delta k(t).$$

$$= sf(k(t)) - \delta k(t).$$

Solow Growth Model with Population Growth

$\frac{\dot{K}(t)}{L(t)} \neq \dot{k}(t).$

How does the analysis change if population, and the number of workers, is growing? For example, if L is growing at a constant rate n, i.e., $L(t) = L(0)\, e^{nt}$, then:

invest perperson

$$\dot{k}(t) = \dot{\left(\frac{K(t)}{L(t)}\right)} = sf(k(t)) - (\delta + n)\, k(t).$$

change of the ratio not invidual

investment.

Intuitively, the stock of machines that each worker can use depreciates for two reasons:

1. change in $\dot{K}(t)$
2. change in $\dot{L}(t)$

Some *log* "magic"

$$= \frac{\dot{K}(t)}{K(t)} \frac{K(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)} \frac{K(t)}{L(t)}.$$

↳ g of K

g of Labor.

The logarithm of a product is the sum of the logarithms:

$$\log(k(t)) = \log\left(\frac{K(t)}{L(t)}\right) = \log(K(t)) - \log(L(t))$$

$$\frac{\partial \frac{K(t)}{L(t)}}{\partial t} = \frac{\dot{K}(t)}{L(t)} - \frac{k(t)}{L(t)} \frac{\dot{L}(t)}{L(t)}$$

Differentiate $\log(k(t))$ with respect to time:

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}$$

$$\frac{\partial K(t) L(t)^{-1}}{\partial t}.$$

$$= \dot{K}(t) L(t)^{-1} - K(t) L(t)^{-2} \cdot \dot{L}(t)$$

① $\dot{K}(t) = sY(t) - \delta K(t)$

$Y(t) = A(t) K(t)^{\alpha} L(t)^{1-\alpha}$

$y(t) = \frac{Y(t)}{L(t)} = A(t) k(t)^{\alpha}$

$\dot{k}(t) \neq \frac{\dot{K}(t)}{L(t)}.$

$\dot{k}(t) = \dot{\left(\frac{K(t)}{L(t)}\right)} = \frac{\partial\left(\frac{K(t)}{L(t)}\right)}{\partial t.} = \frac{\partial K(t)}{\partial t} L(t)^{-1} - K(t) L(t)^{-2} \frac{\partial L(t)}{\partial t.}$

$\dot{k}(t) = \dot{K}(t)\, L(t)^{-1} - K(t)\, L(t)^{-2}\, \dot{L}(t)$

$\dot{k}(t) = \frac{\dot{K}(t)}{K(t)} k(t) - k(t) \frac{\dot{L}(t)}{L(t)}$

change rate of $K(t)$, $L(t)$.

$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}$ growth rate of $k(t)$ = growth rate of K - growth rate of $L(t)$

$k(t) = \frac{K(t)}{L(t)}$

② $\ln(k(t)) = \ln(K(t)) - \ln(L(t)).$

$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}$

Solow Growth Model with Population Growth (cont'd)

Evolution equation for the stock of capital:

$$\dot{K}(t) = sY(t) - \delta K(t)$$

Divide by $K(t)$

$$\frac{\dot{K}(t)}{K(t)} = s\frac{Y(t)}{K(t)} - \delta$$

and use $\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}$ (from previous slide)

$$\frac{\dot{k}(t)}{k(t)} + \frac{\dot{L}(t)}{L(t)} = s\frac{Y(t)}{K(t)} - \delta$$

Solow Growth Model with Population Growth (cont'd)

Replace $\frac{Y(t)}{K(t)}$ with $\frac{Y(t)/L(t)}{K(t)/L(t)} = \frac{y(t)}{k(t)}$:

$$\frac{\dot{k}(t)}{k(t)} = s\frac{y(t)}{k(t)} - \delta - \frac{\dot{L}(t)}{L(t)} = n$$

Rearrange and use $\frac{\dot{L}(t)}{L(t)} = n$ (constant population growth):

$$\dot{k}(t) = sy(t) - (\delta + n)\,k(t)$$

more people dilute the capital.

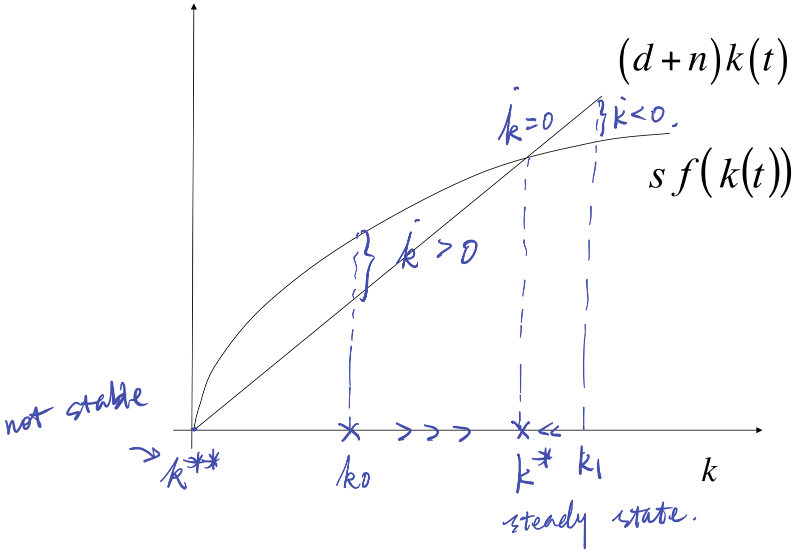
Finally, replace $y(t)$ with $f(k(t))$

$$\dot{k}(t) = sf(k(t)) - (\delta + n)\,k(t)\,.$$

Fundamental Equation of the Solow Growth Model with Population Growth

$$\dot{k}(t) = sf\big(k(t)\big) - (\delta + n)\,k(t)$$

Dynamics of the Solow Growth Model



Fundamental Equation of the Solow Growth Model with Population Growth in Growth Rates

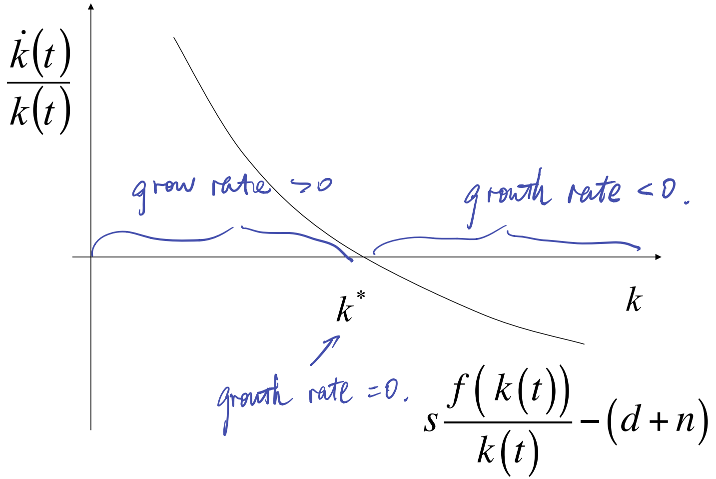
Recall: $f(k(t)) = k(t)^{\alpha}$

$\frac{f(k(t))}{k(t)} = k(t)^{\alpha - 1}$ negative. $(\alpha \in (0,1))$

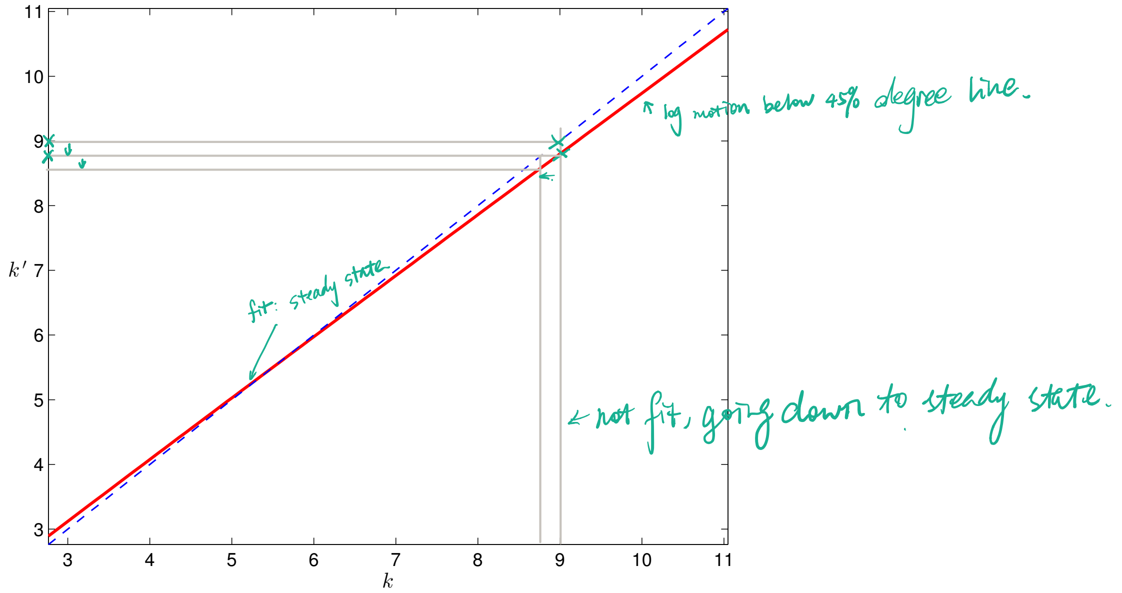
$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (\delta + n)$$

$\left.\frac{\dot{k}(t)}{k(t)}\right|_{k(t)=k^*} = 0. \rightarrow \uparrow k(t) \Rightarrow \downarrow \frac{\dot{k}(t)}{k(t)}.$ vice versa.

Dynamics of Growth Rates in the Solow Growth Model



Law of Motion for Capital (per capita)



Analyzing the Law of Motion

- General approach: Plot the law of motion together with the 45-degree line.
- If the law of motion is above the 45-degree line, we have $k_{t+1} > k_t$: Capital is increasing over time.
- If the law of motion is below the 45-degree line, we have $k_{t+1} < k_t$: Capital is decreasing over time.
- If the law of motion crosses the 45-degree line, we have $k_{t+1} = k_t$: Capital stays constant.
- When capital stays constant over time we say that capital has reached a *steady state*.

Analyzing the Law of Motion

- Steady states can be *stable* or *unstable*.
- A steady state is stable if capital approaches the steady state from any point close to the steady state.
- A steady state is unstable if capital moves away even when starting very close to the steady state.
- We can check the stability of a steady state by determining in which direction the law of motion crosses the 45-degree line:
  - Law of motion crosses from above the 45-degree line:
    - Capital increase below / decreases above steady state.
    - Capital therefore moves toward steady state.
    - Steady state is stable.
  - Law of motion crosses from below the 45-degree line:
    - Capital decreases below and increases above the steady state.
    - Capital therefore moves away from the steady state.
    - Steady state is unstable.

Properties of the Law of Motion in the Solow Model

- For $k_t = 0$, we have $k_{t+1} = 0$.
- For k_t small, we have $k_{t+1} > k_t$:
  - We can check that by looking at the first derivative of the law of motion with respect to k_t:

$$\frac{\partial k_{t+1}}{\partial k_t} = 1 - (n + \delta) + s\alpha k_t^{\alpha-1}$$

  - Derivative approaches infinity as k_t approaches zero from above.
- For k_t large, we have $k_{t+1} < k_t$:
  we can check that by noting that the first derivative approaches $1 - (n + \delta)$ as k_t approaches infinity, which is smaller than the slope of the 45-degree line (which is one).

The Steady State in the Solow Model

- Law of motion crosses 45-degree line exactly once from above: We have a unique positive steady state.
- Steady state is stable.
- Economy approaches steady state from any positive initial level of capital.

Solving for the Steady State

- We can compute the steady state by plugging the steady-state level of capital k^* into both sides of the law of motion. $\dot{k}(t) = sf(k(t)) - (\delta+n)k(t) = 0$
- This works because if $k_t = k^*$, we have $k_{t+1} = k^*$ as well.
- Plugging in k^*: $sf(k^*) = (\delta + n)k^* = sA(k^*)^\alpha$

  Simplifying: $\frac{sA}{\delta+n} = (k^*)^{1-\alpha}$
- Interpretation: At the steady state, investment is just equal to depreciation, so that capital stays constant.
- Solving for k^*: $k^* = \left(\frac{sA}{\delta+n}\right)^{\frac{1}{1-\alpha}}$

Properties of the Steady State

Steady-state level of capital k^* is increasing in the savings rate s and decreasing in the depreciation rate δ and population growth rate n.

and productivity.

Growth Rate of Output Per-Capita?

It is easy to derive a simple expression for the case of a Cobb-Douglas production function

$$y(t) = k(t)^{\alpha}.$$

$\Rightarrow \ln(y(t)) = \alpha \ln(k(t))$

Using *log* magic we obtain

$$\frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{k}(t)}{k(t)}.$$

U.S. might in steady state.

MP constant for several years.

growth rate is constant.