L22 03.11

§11. Mean Value Theorem

Def 11.1: Let $A \neq \emptyset$, $f: A \to \mathbb{R}$. We say $x \in A$ is a maximum point for f if $f(x) \geq f(y)$ $\forall y \in A$

We say $f(x)$ is the maximum value of f

Note: max value is unique, max point may not be.

e.g. $f(x) = x^2$ on $[-1, 1]$

max value: 1, max point: $-1, 1$

Thm 11.2: Let $f: (a,b) \to \mathbb{R}$ be diff'ble at $x \in (a,b)$ and suppose f has a max at $x \in (a,b)$, then $f'(x) = 0$

Proof: Let $h > 0$. Then as f has a max at x, we obtain (for $h < b - x$)

$$\frac{f(x+h)-f(x)}{h} \leq 0$$

Hence, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \leq 0$

Let $h < 0$, then as f has a max at x, we see $\frac{f(x+h)-f(x)}{h} \geq 0$

So, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h} \geq 0$

Hence, $f'(x) = 0$ ∎

Def 11.3: Let $A \subseteq \mathbb{R}$ be non-empty. $A \subseteq \text{dom} f$. $\text{Range}(f) \subseteq \mathbb{R}$. Then we say f has a local maximum at $x \in A$. If $\exists \delta > 0$, s.t. x is a max for f on $A \cap (x - \delta, x + \delta)$

Thm 11.4: Let $f: (a,b) \to \mathbb{R}$ have a local max at $x \in (a,b)$ and suppose f is diff'ble at x.

Then $f'(x) = 0$.

Proof: Exercise $|h| < \delta$.

Clearly, Thm 11.4 is still true if we replace max w/ min

Rmk 11.5: Converse is not true, $f'(x)=0$ does not imply f has max or min at x.

e.g. $f(x)=x^3$ has $f'(0)=0$, 0 is not max or min.

Def 11.6: We say f has a critical pt. at x if $f'(x)=0$ and call $f(x)$ a critical value.
(Finding local max or min)

Extreme Value Theorem

Suppose $f:[a,b]\to\mathbb{R}$ cts. Then max & min of f exists by EVT

These occur at pts: ① $f'(x)=0$ ② $x=a$ or b ③ x s.t. f is not diff'ble at x.

Note: if x is a max or min in ① or ②

Then $x\in(a,b)$ and by Thm 11.4 and x not diff'ble at x (or we would have $f'(x)=0$)

So, x is in ③

Ex (i) Let $f(x)=x^2+2x-1$ on $[-2,2]$

So f is diff'ble on $(-2,2)$, so ③ is empty.

Check $f'(x)=2x+2$. So $f'(x)=0$ iff $x=-1$

So, ① $=\{-1\}$ ② $=\{-2,2\}$ ③ $=\emptyset$

So, max or min occur in $\{-1,2,-2\}$

$f(-2)=-1$ $f(-1)=-2$ $f(2)=7$

So, $f(x)$ has a min at -1, max at 2.

(ii) $g(x)=\frac{1}{1-x^2}$ on $(-1,1)$, g is not bdd! g is diff'ble on $(-1,1)$ and $g'(x)=\frac{2x}{(1-x^2)^2}$

So, if $g(x)$ has a max or min in $(-1,1)$ at x, $g'(x)=0$. So $x=0$.

As $x\to\pm1$, $g(x)\to\infty$

So, $\exists a,b$ s.t. $-1<a<0<b<1$ s.t. if $-1<y<a$ or $b<y<-1$, then $g(y)>g(0)$

So, on $[a,b]$ min or max can only exist at 0, and as $g(a), g(b) > g(0)$.

0 is a min pt. of g is $g(0)=1$.

MVT: $f:[a,b]\to\mathbb{R}$ is on $[a,b]$, diff'ble on (a,b)

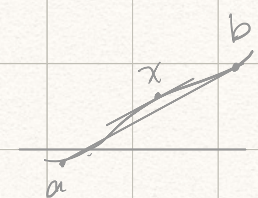
$f'(x)$ should be "rate of change" of f at x

Overall, f change by amount $f(b)-f(a)$ over distance $b-a$

So, "mean" rate of change is $\frac{f(b)-f(a)}{b-a}$

$L(x)=f(a)+(x-a)\frac{f(b)-f(a)}{b-a}$. Then $f'(x)=\frac{f(b)-f(a)}{b-a}$

MVT: $\exists x\in(a,b)$ s.t. $f'(x)=\frac{f(b)-f(a)}{b-a}$.



Thm (MVT): Let $f:[a,b]\to\mathbb{R}$ be cts on $[a,b]$, diff'ble on (a,b).
Then $\exists x\in(a,b)$ s.t. $f'(x)=\frac{f(b)-f(a)}{b-a}$

Cor 11.9: Let $f:(a,b)\to\mathbb{R}$ be diff'ble and suppose $f'(x)=0$ $\forall x\in(a,b)$. Then f is a constant

Proof: Let $x,y\in(a,b)$ s.t. $x<y$. As f is diff'ble on (a,b),
it's cts on $[x,y]$ and diff'ble on (x,y).
So by MVT, $\exists z\in(x,y)$ s.t. $\frac{f(y)-f(x)}{y-x}=f'(z)=0$
As $y\neq x$, this gives $f(y)-f(x)=0$ i.e. $f(y)=f(x)$ ∎

In general, $f'=0$ on dom(f) $\not\Rightarrow$ f constant

e.g. $f(x)=\begin{cases}1 & x\in(-1,0)\\ 0 & x\in(0,1)\end{cases}$ Then $f'=0$ $\forall x\in$ dom f, f not constant

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Thm 11.7 (Rolle's Theorem) Let $f: [a,b] \to \mathbb{R}$ is cts on $[a,b]$ and diff'ble on (a,b)

Suppose $f(a) = f(b)$. Then $\exists x \in (a,b)$ s.t. $f'(x) = 0$

Recall: Thm 11.4. Let f have a local max or min at x & suppose f is diff'ble at a

then $f'(x) = 0$

Proof. As f is cts on $[a,b]$.

By EVT, $\exists x_+, x_- \in [a,b]$ s.t. $f(x_-) \le f(x) \le f(x_+)$ $\forall x \in [a,b]$

i.e. x_- is a min pt, x_+ is a max pt.

If $x_- \in (a,b)$, then by Thm 11.4 $f'(x_-) = 0$ & we are done

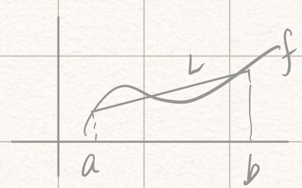
If $x_+ \in (a,b)$, then $f'(x_+) = 0$ & we are done.

If $x_-, x_+ \in \{a,b\}$, so $\max f = f(x_+) = f(a) = f(b) = f(x_-) = \min f$

So, f is a constant f, Hence $f'(x) = 0$ $\forall x \in (a,b)$ ■

Thm 11.8 (MVT) Let $f: [a,b] \to \mathbb{R}$ be cts on $[a,b]$ & diff'ble on (a,b)

Then $\exists x \in (a,b)$ s.t. $f'(x) = \dfrac{f(b) - f(a)}{b-a}$



$l(x) = f(a) + (x-a)\dfrac{f(b)-f(a)}{b-a}$

? $\exists x$ s.t. $f'(x) = l'(x)$ $\to$? $\exists x$ s.t. $(f-l)'(x) = 0$

Proof: Let f be as above & define $l(x) = f(a) + (x-a)\dfrac{f(b)-f(a)}{b-a}$

Consider $g(x) = f(x) - l(x)$

As f, l are cts on $[a,b]$ & diff'ble on (a,b), so is g

$g(a) = f(a) - (f(a) + \overset{0}{(a-a)}\cdots) = f(a) - f(a) = 0$

$g(b) = f(b) - \left(f(a) + (b-a)\dfrac{f(b)-f(a)}{b-a}\right) = 0$

So, $g(a) = g(b)$. Hence by Rolle's thm. $\exists x \in (a,b)$ s.t. $g'(x) = 0$

Hence, $0 = f'(x) - l'(x)$, i.e. $f'(x) = \dfrac{f(b) - f(a)}{b-a}$ ▨

Cor 11.10: If $f, g: (a,b) \to \mathbb{R}$, $f' = g'$ on (a,b), then $g(x) = f(x) + C$ on (a,b).

Proof. Exercise. Consider $f - g$. ▨

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Def 11.11: f is increasing on (a,b) if whenever $x, y \in (a,b)$ and $x \le y$, then $f(x) \le f(y)$

If inequality is strict, the f is strictly increasing.

Similar def for decreasing.

Constant function is increasing and decreasing.

Cor 11.12 Suppose $f'(x) > 0$ $\forall x \in (a,b)$. Then f is strictly increasing on (a,b).

Proof Let $x, y \in (a,b)$ s.t. $x < y$.

By MVT, $\exists z \in (x,y)$ s.t. $f'(z) = \dfrac{f(y) - f(x)}{y-x}$

So, $f(y) - f(x) = f'(z)(y-x) > 0$, i.e. $f(y) > f(x)$ ▨

Note: f is strictly increasing does not imply $f' > 0$

e.g. $f(x) = x^3$ imply $f' \geq 0$ $f'(x) = 3x^2$ $f'(0) = 0$

Graph Sketching: with f', can find regions where f is increasing/decreasing & critical pts.

Determing type of critical pt x:

① if $f' < 0$ to left of x & $f' > 0$ to the right of x, x is a min pt.

② If $f' > 0$ to the left of x & $f' < 0$ to the right of x, x is a max pt.

③ If f' has the same sign on both sides of x,

x is neither a max nor a min.

Pic:

2nd derivatives properties.

Thm 11.13: Suppose $f'(a) = 0$ and f is twice diff'ble at a.

If $f''(a) > 0$, f has a local min at a

If $f''(a) < 0$, f has a local max at a.

Proof: $f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h} = \lim_{h \to 0} \frac{f'(a+h)}{h}$

Suppose $f''(a) < 0$. Then, as $\lim_{h \to 0} \frac{f'(a+h)}{h} < 0$,

$\exists\, \delta > 0$ s.t $\forall |h| < \delta$, $h \neq 0$ $\quad \frac{f'(a+h)}{h} < 0$

So, if $|h| < \delta$, $h > 0$, then $f'(a+h) < 0$. Hence $f' < 0$ to the right of a.

if $|h| < \delta$, $h < 0$, then $f'(a+h) > 0$. Hence $f' > 0$ to the left of a.

Thus, f is increasing to the left of a, decreasing to the right.

So, a is a max pt.

Case $f''(a) > 0 \Rightarrow a$ min is similar ■

$=0 \quad x^3?$

Thm 11.14: Suppose $f''(a)$ exist. If f has a local min at a, $f''(a) \geq 0$.

If f has a local max at a, $f''(a) \leq 0$.

Proof: by contradiction.

Suppose f has a local min at a, but $f''(a) < 0$.

As f has a local min at a & $f''(a)$ exists, $f'(a) = 0$

Hence, by previous thm, $f(a)$ is a local max.

So, a is local max and local min.

Hence, f is locally constant. So, $f''(a)=0$.

※ to $f''(a)<0$. ■

Thm 11.15. Suppose f cts at a and f' exists an interval containing a $(a-\delta, a+\delta)$, but possibly not at a.
Suppose $\lim_{x\to a} f'(x)$ exists, then f is diff'ble at a and $f'(a) = \lim_{x\to a} f'(x)$

derivatives cannot have removable discontinuities

Proof: Let $L = \lim_{x\to a} f'(x)$ need to show $\lim_{h\to 0} \frac{f(a+h)-f(h)}{h} = L$.

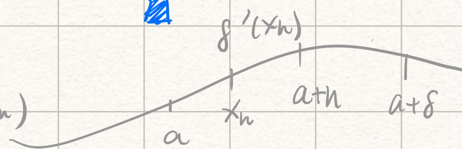
Let $\varepsilon>0$, $\exists \delta>0$ s.t. if $|x-a|<\delta$, $x\neq a$, then $|f'(x) - \ell| < \varepsilon$

Let $|h|<\delta$, $h\neq 0$. Then as f is cts on $[a, a+h]$ & diff'ble on $(a, a+h)$
(or, on $(a+h, a)$ if $h<0$), then by MVT, $\exists x_h \in (a, a+h)$
s.t. $f'(x_h) = \frac{f(a+h)-f(a)}{h = a+h-a}$

So, if $|h|<\delta$, $h\neq 0$, $\left|\frac{f(a+h)-f(a)}{h} - \ell\right| = |f'(x_h) - \ell| < \varepsilon$ as $|x_h - a| < |h| < \delta$

Hence, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \ell$ ■

$\frac{f(a+h)-f(a)}{h} = f'(x_h)$



As $h\to 0$, $x_h \to a$, so $f'(x_h) \to \ell$ i.e. $f'(a) = \ell$.

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Recall: A point a is a removable discty of f if f is discts at a, but $\lim_{x\to a} f(x)$ exists.

Ex. Let $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x\neq 0 \\ 0 & x=0 \end{cases}$ Then, $f'(x)$ exists & $f'(x) = \begin{cases} 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x\neq 0 \\ 0 & x=0 \end{cases}$

which is discts at 0. Discty is not removable, which agrees with Thm 11.15.

Thm (Darboux's Thm) Suppose f is diff'ble on (a,b) & f has one-sided derivaties at a, b.

Suppose $f'(a) < c < f'(b)$, then $f'(x) = c$ for some $x \in (a,b)$

Remark: this is a form of IVT for derivities.

Note it does not assume that f' is cts.

Thm: Suppose f is diff'ble on an interval containing a but f' is discts at a.

Then at least one of the one-sided limits $\lim_{x\to a^-} f'(x)$ & $\lim_{x\to a^+} f'(x)$ does not exist.

Exercise 11-60, 61 for proof.

Recall: MVT: Let f $[a,b] \to \mathbb{R}$ be cts on $[a,b]$, diff'ble on $[a,b]$

Then $\exists x \in (a,b)$ s.t. $f'(x) = \dfrac{f(b)-f(a)}{b-a}$

or $f'(x)(b-a) = f(b) - f(a)$

Suppose f, g satisfies MVT assumptions.

Applying MVT to f, $\exists x \in (a,b)$ s.t. $f'(x)(b-a) = f(b) - f(a)$

By MVT for g, $\exists y \in (a,b)$ s.t. $g'(y)(b-a) = g(b) - g(a)$

So. $\dfrac{f'(x)}{g'(y)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Q: can we take $x = y$?

Thm 11.16 (Cauchy MVT) Let f, g $[a,b] \to \mathbb{R}$ be cts on $[a,b]$, diff'ble on (a,b)

Then $\exists x \in (a,b)$ s.t. $(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x)$

Note: if $g'(x) \neq 0$ & $g(b) - g(a) \neq 0$. Then $\dfrac{f'(x)}{g'(x)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

if $g(x) = x$, this is the usual MVT.

Proof: Let $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$

Then $h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a))$

$h(a) = f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a)$

$\quad = f(a)g(b) - g(a)f(b)$

$h(b) = f(b)g(b) - f(b)g(a) - g(b)f(b) + g(b)f(a)$

$\quad = g(b)f(a) - g(a)f(b)$

$h(a) = h(b).$

By MVT / Rolle's Thm. we get $\exists\, x \in (a,b)$ s.t. $h'(x) = 0$

i.e. $f'(x)(g(b) - g(a)) = g'(x)(f(b) - f(a))$.

Thm (L'Hôpital) Suppose $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \alpha$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \alpha$ also.

Proof: WLOG, assume $f(a) = g(a) = 0$ (so f, g are cts at a)

As $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, $\exists\, \delta > 0$ s.t. f', g' exists on $(a-\delta, a+\delta) \setminus \{a\}$

also $g'(x) \neq 0 \quad \forall x \in (a-\delta, a+\delta)\setminus\{a\}$.

Claim: For $x \in (a-\delta, a+\delta)\setminus\{a\}$ $g(x) \neq 0$

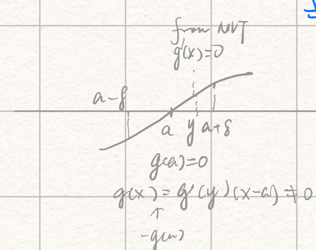
Proof: Let $x \in (a-\delta, a+\delta)\setminus\{a\}$

WLOG, $x > a$. As g is diff'ble on (a,x) and cts on $[a,x]$,

we can apply MVT to obtain $y \in (a,x)$ s.t. $g(x) = g(x) - g(a) = g'(y)(x-a)$
$\neq 0$

where we have used $g(a) = 0$, $g'(y) \neq 0$

So $g(x) \neq 0$ for $x \in (a-\delta, a+\delta)\setminus\{a\}$



Now let $x \in (a-\delta, a+\delta)\setminus\{a\}$

We apply Cauchy MVT to f, g to obtain $\frac{g(x) - g(a)}{f(x) - f(a)} = \frac{f'(x)}{g'(x)} = \frac{f'(\alpha_x)}{f'(\alpha_x)}$

for some α_x between x and a

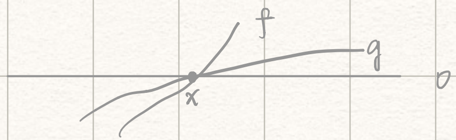
As $\alpha_x \in (x, a)$ (or (a,x) if $x > a$), then, as $x \to a$, $\alpha_x \to a$ also.

(ε-δ proof last lecture)

So, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \lim_{\alpha_x \to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \alpha$ as required ■

Ex. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof: $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \lim_{x \to 0} \cos x = 1$ ■



$f(x) \simeq f(a) + f'(a)(x-a)$

$g(x) \simeq g(a) + g'(a)(x-a)$

if $f(a) = g(a) = 0$, then

$f(x) \simeq f'(a)(x-a)$

$g(x) \simeq g'(a)(x-a)$