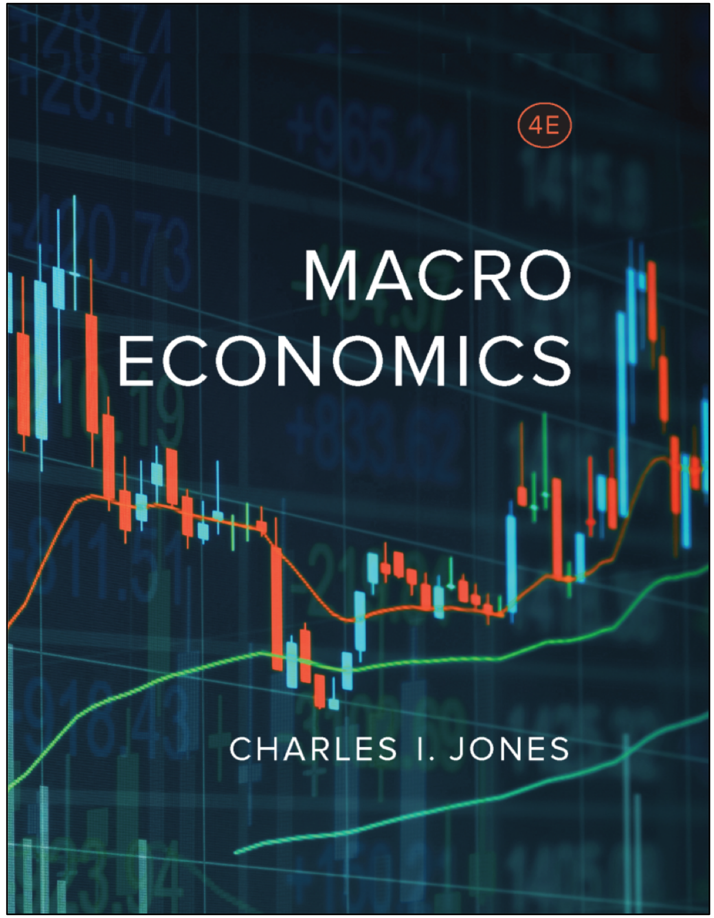
Chapter 4

A Model of Production



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4.1 Introduction

GDP per capita

$\frac{Y}{L}$ relationship $\frac{K}{L}$ = Capital - labor ratio (capfital per person)

capital

people

- In this chapter, we learn:
  - How to set up and solve a macroeconomic model
  - The purpose of a production function and its use for understanding differences in GDP per capita across countries.
  - The role of capital per person and technology in explaining differences in economic growth.
  - The relevance of "returns to scale" and "diminishing marginal products"
  - How to look at economic data through the lens of a macroeconomic model

Introduction

- A model:
  - a mathematical representation of a hypothetical world that we use to study economic phenomena
  - Consists of equations and unknowns with real-world interpretations
- Macroeconomists:
  - Document facts
  - Build a model to understand the facts
    × realistic → useful.
  - Examine the model to see how effective it is

4.2 A Model of Production

- Vast oversimplifications of the real world in a model can still allow it to provide important insights.
- Consider the following model:
  - single, close economy
  - assume consumption good
- Inputs in the production process:
  - Labor ($\bar{L}$) workers
  - Capital ($\bar{K}$) equipments — infrastructures
- Production function:
  - Shows how much output (*Y*) can be produced given any number of inputs

Production Function

easy to keep track

Labor share = 2/3

$\bar{Y} = \bar{A} K^{\alpha} L^{1-\alpha}$

$$Y = F(K, L) = \bar{A} K^{1/3} L^{2/3}$$

how efficient → input → output

(assume all countries has same $\bar{A}$)

└ assume $\bar{A} = 1$ (2/5)

$Y =$	$F(K,L)$	$= \bar{A}$	$K^{1/3} L^{2/3}$
Output		productivity Parameter	Inputs

Model

- Output growth corresponds to changes in Y.
- There are three ways that Y can change:

1. Capital stock (K) changes
2. Labor force (L) changes
3. Ability to produce goods with given resources (K, L) changes

   - Technological advances occur (changes in A)
   - TFP is assumed to be exogenous in the Solow model

Cobb-Douglas Production Function

- The Cobb-Douglas production function is the particular production function that takes the form of:
  - $Y = K^{\alpha} L^{1-\alpha}$
  - α is assumed to be $1/3$
- And $F(K,L)$ is increasing in both K and L
  - More inputs yield more output.
  - $\frac{\partial F}{\partial K} > 0$ and $\frac{\partial F}{\partial L} > 0$
- A production function exhibits constant returns to scale if doubling each input exactly doubles output.

Constant Returns to Scale (CRS)

difference with changing $\bar{A}$?

- If increase K and L by x%
  - ➔ Y also increases by x%

$Y = K^{\alpha} L^{1-\alpha}$.

$(\lambda K)^{\alpha} (\lambda L)^{1-\alpha}$

$= \lambda K^{\alpha} L^{1-\alpha}$

$= \lambda Y$

- Mathematically,
  - $F(\alpha K, \alpha L) = \alpha F(K, L)$

Homogeneous function (of degree 1)

- Standard replication argument

Output per Person (Intensive Form)

$y = \frac{Y}{L} = \frac{K^\alpha L^{1-\alpha}}{L} = \frac{K^\alpha}{L^\alpha} = \left(\frac{K}{L}\right)^\alpha = k^\alpha$

- Divide output by the number of workers

detensive

$$\frac{Y}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = F\left(\frac{K}{L}, 1\right)$$

  - per capita = per person = per worker
  - Lowercase letters denote per capita variables
- We can rewrite output per person as rescale

$y = f(k) = k^\alpha$

where $y = \frac{Y}{L}$ and $k = \frac{K}{L}$

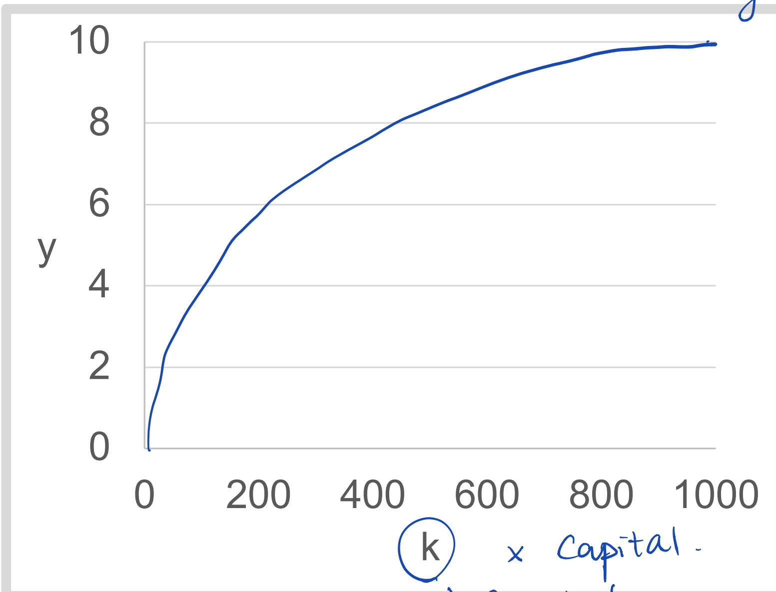
Typical Production Function

$k \uparrow$

① diminishing marginal return $\uparrow$

② $y \uparrow$

- Graph of $y = f(k) = k^{1/3}$
- Note: if $k=0$, $y=f(k)=0$



× capital.

capital per worker

Returns to Scale Comparison

If exponents associated with **inputs** in production function

- sum to 1 ⇒ constant returns to scales
- sum to > 1 ⇒ increasing returns to scales
- sum to < 1 ⇒ decreasing returns to scales

firms

Allocating Resources

- $$\max_{K,L} \Pi = F(K,L) - rK - wL$$
  - π : profits
  - r : rental rate of capital
  - w : wage rate

unit : ＄ in wages.

$w=1$.

$F(K,L) - rK - L.$

in wages.

- The rental rate and wage rate are taken as given under perfect competition
  - Hire capital until the MPK = r
  - Hire labor until MPL = w

numeraire = output

- For simplicity, the price of the output is normalized to one

Marginal Products

$$\frac{\partial MPL}{\partial L} = \underbrace{(1-\alpha)}_{+}\underbrace{(-\alpha)}_{-}\underbrace{K^{\alpha}}_{+}\underbrace{L^{-1-\alpha}}_{+} < 0$$

- The marginal product of labor (MPL):

how many wage needed to pay for ~~one~~ more labor

$$= \frac{\partial Y}{\partial L} = \frac{\partial F(K,L)}{\partial L} = (1-\alpha)K^{\alpha}L^{-\alpha} = \boxed{(1-\alpha)k^{\alpha} = w}$$

- The marginal product of capital (MPK)

$$= \frac{\partial Y}{\partial K} = \alpha K^{\alpha-1}L^{1-\alpha} = \boxed{\alpha k^{\alpha-1} = r}$$

proportional

Average product of labor

$$APL = \frac{Y}{L} = \frac{K^{\alpha}L^{1-\alpha}}{L} = \frac{K^{\alpha}}{L^{\alpha}} = y = k^{\alpha}$$

$$= \frac{MPL}{(1-\alpha)}$$

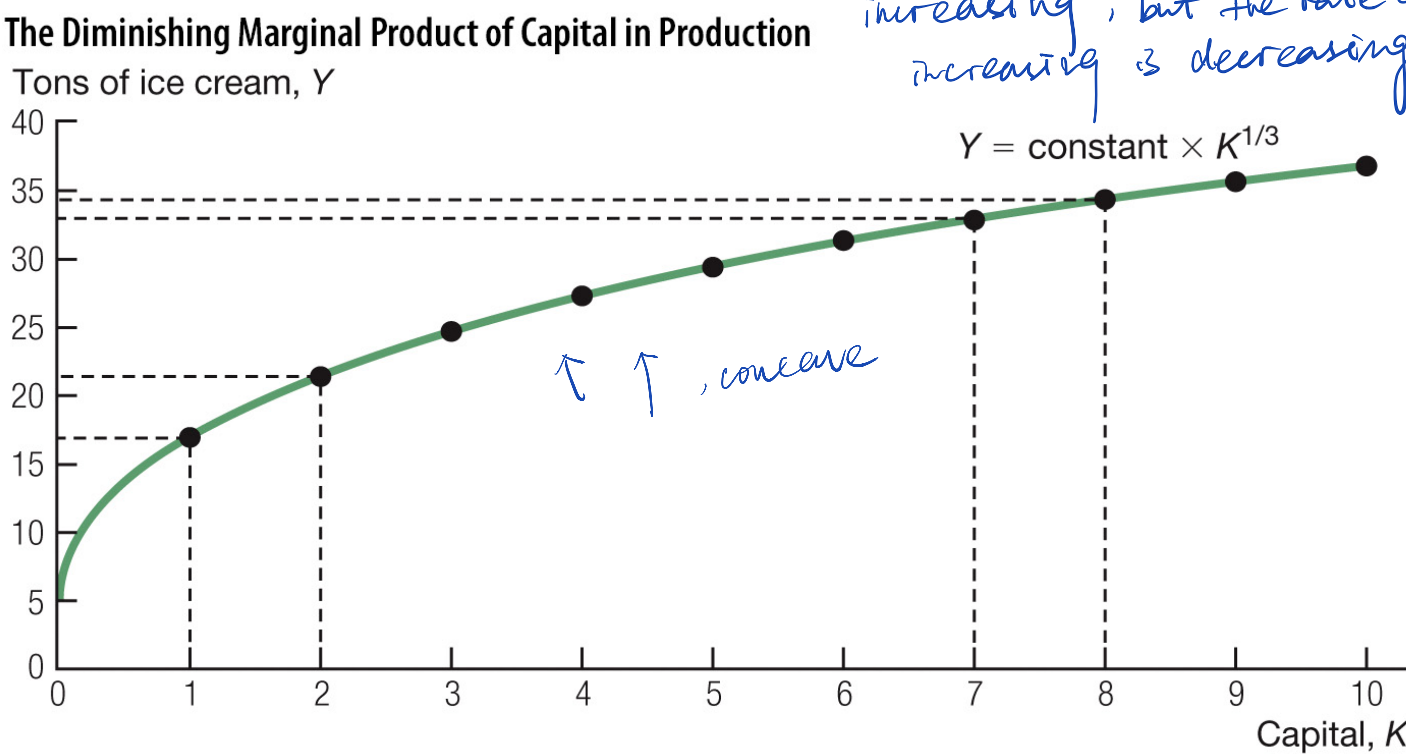
—— of capital

$$APK = \frac{Y}{K} = \frac{K^{\alpha}L^{1-\alpha}}{K} = k^{\alpha-1}$$

$$= \frac{MPK}{\alpha}$$

Diminishing Marginal Product of Capital in Production

increasing, but the rate of increasing is decreasing



Diminishing Returns

- Formally, a function exhibits diminishing returns if:

$$\frac{\partial^2 F}{\partial K^2} > 0 \quad , \quad \frac{\partial^2 F}{\partial L^2} < 0 \quad \left(\frac{\partial MPL}{\partial L}\right).$$

- Suppose we have one unit of *K* and one unit of *L*.
- Assume this results in one unit of *Y*.
  - Add a second unit of *L*.
  - Previously, each unit of *L* had one unit of *K* to work with.
  - Now, each unit of *L* has $\frac{1}{2}$ unit of *K* to work with.
  - We are assuming that this will make each unit of *L* less productive, and output will increase, but not double.

Solving the Model: General Equilibrium

- Five Endogenous Variables
  - Output (Y)
  - The amount of capital (K)
  - The amount of labor (L)
  - The wage (w)
  - The rental price of capital (r)
- Five Equations
  - The production function
  - The rule for hiring capital
  - The rule for hiring labor
  - Supply equals the demand for labor
  - Supply equals the demand for capital

Solving the Model: General Equilibrium (cont'd)

firm want:

by choosing K, L

$$\max_{K,L} \Pi = F(K,L) - rK - wL$$

eg 一直提K 直到收支打平

$$\frac{\partial \Pi(K,L)}{\partial K} = 0 = \alpha K^{\alpha-1} L^{1-\alpha} - r$$

$$r^* = \alpha K^{\alpha-1} L^{1-\alpha}$$

$$\frac{\partial \Pi(K,L)}{\partial L} = (1-\alpha) K^{\alpha} L^{-\alpha} - w = 0$$

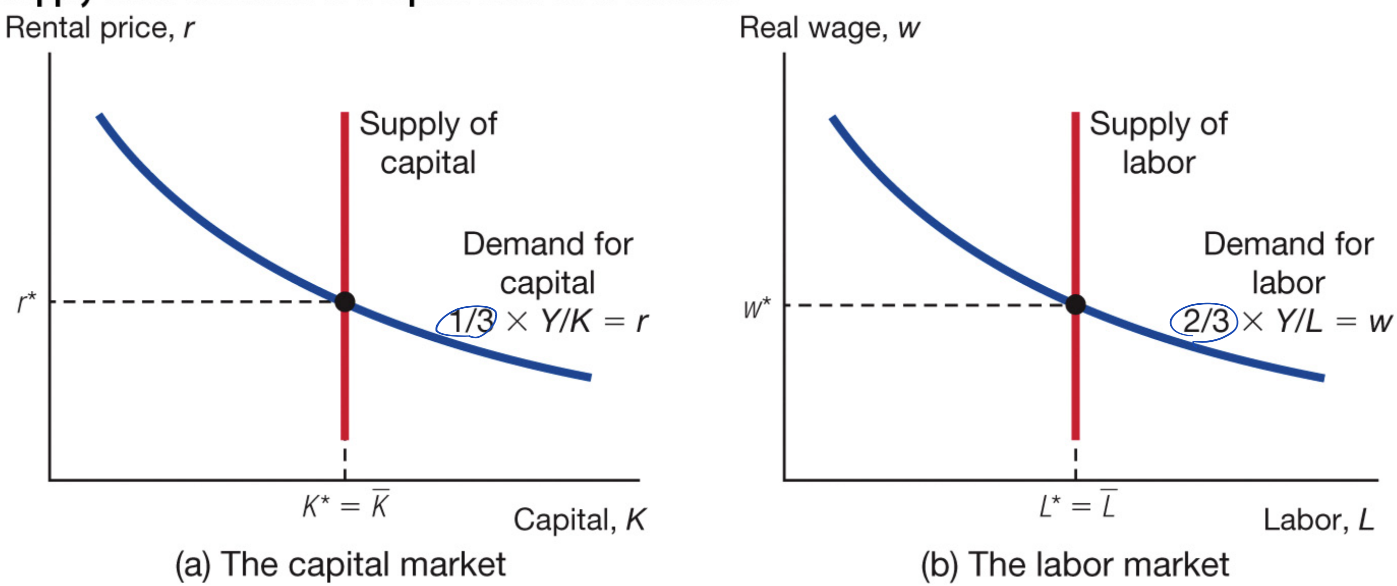
$$w^* = (1-\alpha) K^{\alpha} L^{-\alpha}$$

$$\frac{\alpha L^{1-\alpha} K^{\alpha-1}}{(1-\alpha) K^{\alpha} L^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{L}{K} = \frac{r}{w}$$

→ ratio of capital / Labor want to use depends on the price of r and

Supply and Demand in the Capital and Labor Markets

Supply and Demand in the Capital and Labor Markets



In This Model…

- The solution implies:
  - firms employ all the supplied capital and labor in the economy
  - the production function is evaluated with the given supply of inputs

$$Y^* = F(\bar{K}, \bar{L}) = \bar{A}\bar{K}^{1/3}\bar{L}^{2/3}$$

  - $w = MPL$ evaluated at the equilibrium values of *Y*, *K*, and *L*
  - $r = MPK$ evaluated at the equilibrium values of *Y*, *K*, and *L*

Interpreting the Solution

wage, L offset effect.

$L^* = \bar{L}$

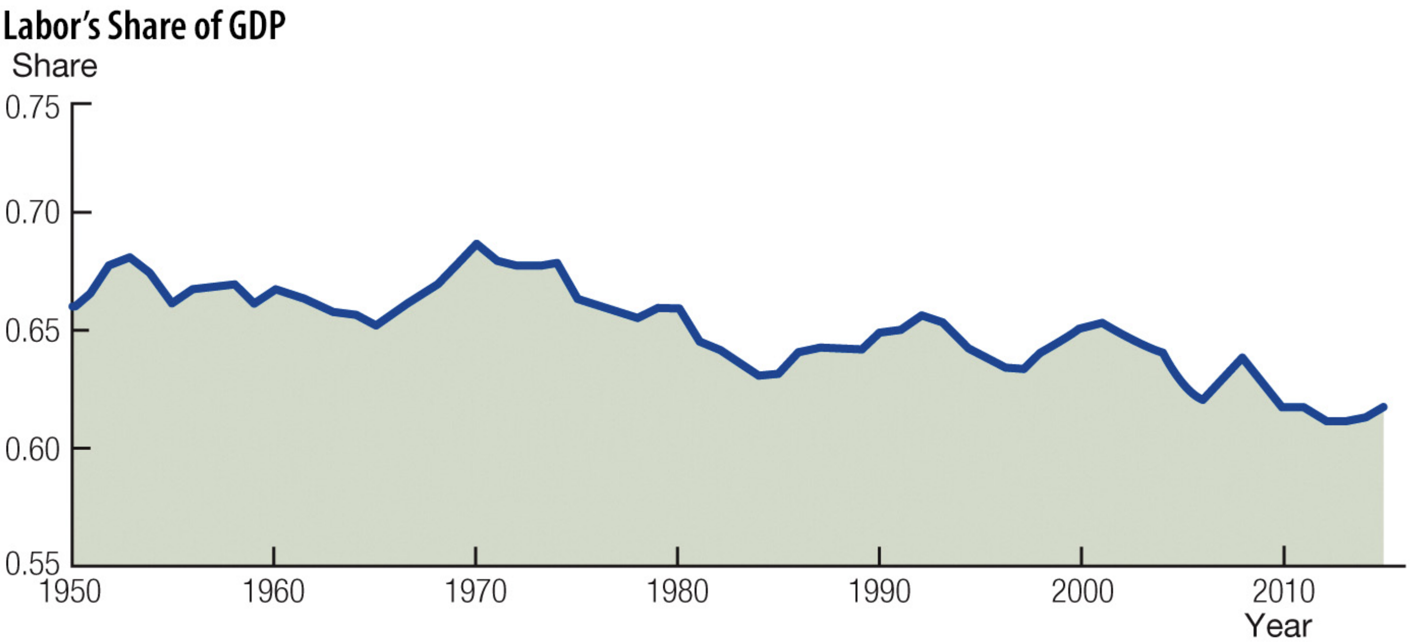
$K^* = \bar{K}$

$w^* = (1-\alpha)\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}$

$r^* = \alpha\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha-1}$

- The equilibrium wage is proportional to out put per worker
  - Output per worker $= \frac{Y}{L}$
- The equilibrium rental rate is proportional to output per capital
  -
- In the United States, empirical evidence shows:
  - 2/3 production is paid to labor $\frac{w^* L^*}{Y^*} = \frac{2}{3}$ $\qquad \frac{w^* L^*}{Y^*} = \frac{(1-\alpha)\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}\bar{L}}{\bar{K}^{\alpha}\bar{L}^{1-\alpha}} = 1-\alpha$
  - 1/3 capital $\frac{r^* K^*}{Y^*} = \frac{1}{3}$.
  - The factor shares of the payments are equal the exponents on the inputs in production function

Labor's Share of Income



Equilibrium

- All income is paid to capital or labor.
  - Results in zero profit in the economy
  - This verifies the assumption of perfect competition
  - Also verifies that production equals spending equals income.

$$w^*L^* + r^*K^* = Y^*$$

Income = Production

4.3 Analyzing the Production Model

- Development accounting:
  - The use of a model to explain differences in incomes across countries

$$y^* = \bar{A}\bar{k}^{1/3}$$

  - Setting the productivity parameter = 1

$$y^* = \bar{k}^{1/3}$$

The Empirical Fit of the Production Function

- If the productivity parameter is 1, the model overpredicts GDP per capita. marginal return to capital poor > rich
  ↑ in low capital countries.
- Diminishing returns to capital implies that:
  - Countries with low *K* will have a high MPK
  - Countries with a lot of *K* will have a low MPK, and cannot raise GDP per capita by much through more capital accumulation

Tabel 4.3 in ~~tex~~textbook

The Model's Prediction for Per Capita GDP (United States = 1)

TABLE 4.3

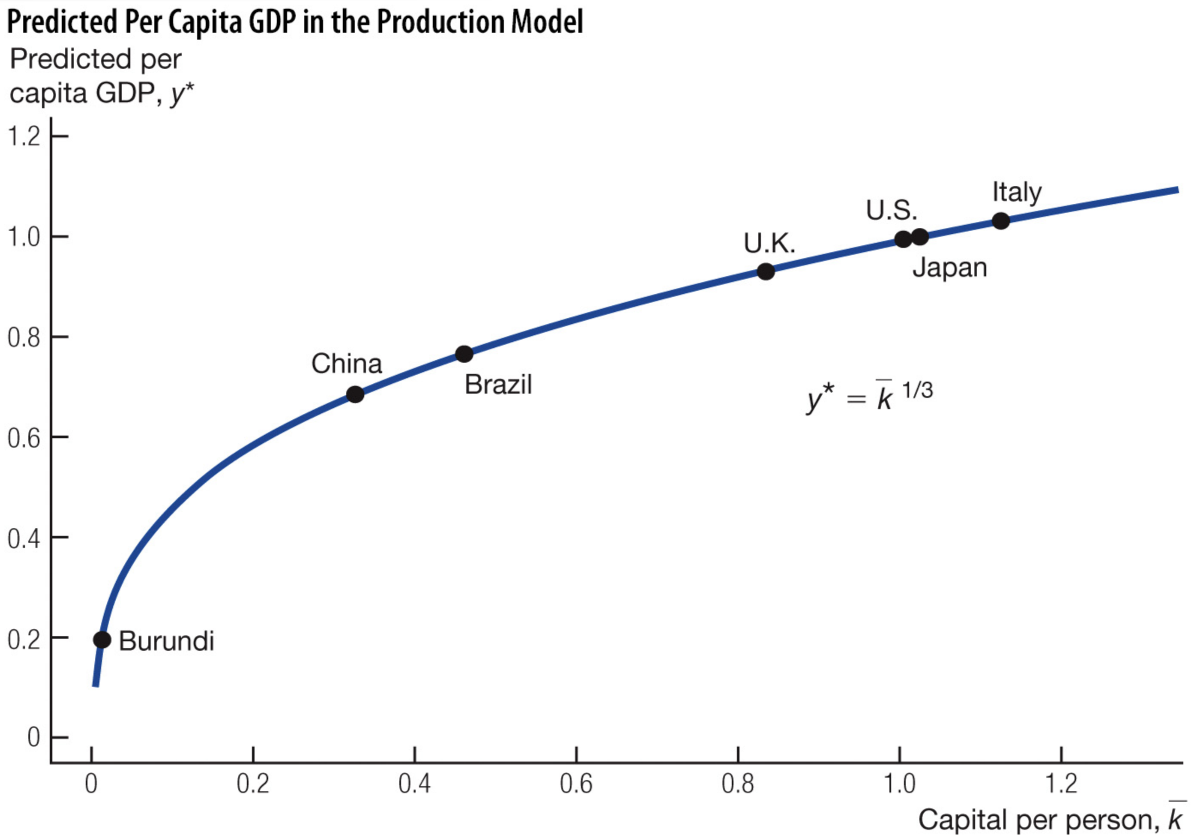
The Model's Prediction for Per Capita GDP (U.S. = 1)

Country	Observed capital per person, $\bar{k}$	Predicted per capita GDP $y = \bar{k}^{1/3}$	Observed per capita GDP
United States	1.000	1.000	1.000
Switzerland	1.416	1.123	1.147
Japan	1.021	1.007	0.685
Italy	1.124	1.040	0.671
Spain	1.128	1.041	0.615
United Kingdom	0.832	0.941	0.733
Brazil	0.458	0.771	0.336
China	0.323	0.686	0.241
South Africa	0.218	0.602	0.232
India	0.084	0.437	0.105
Burundi	0.007	0.192	0.016

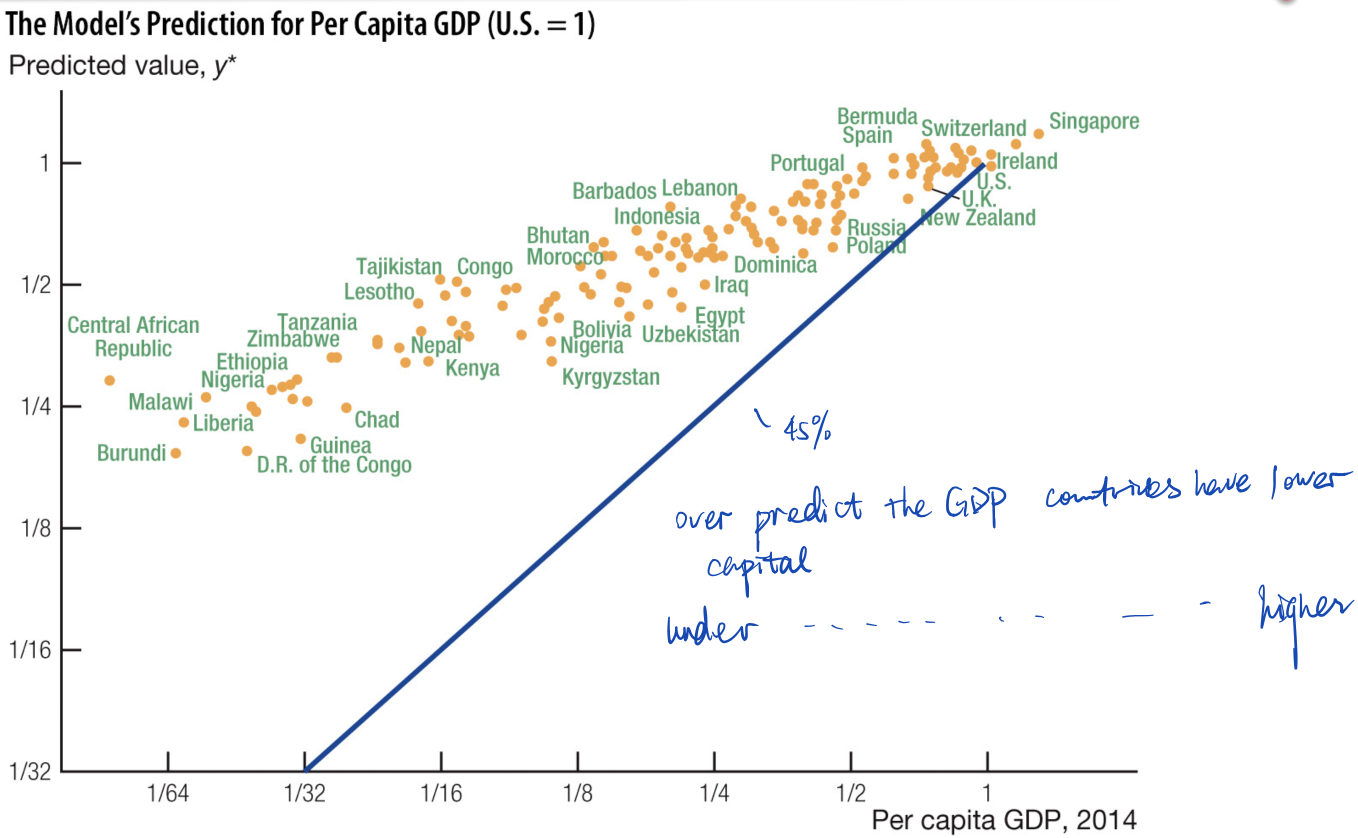
Predicted per capita GDP is computed as $\bar{k}^{1/3}$, that is, assuming no differences in productivity across countries. Data correspond to the year 2014 and are divided by the values for the United States.

Source: Penn World Tables, Version 9.0.

Predicted Per Capita GDP in the Production Model



The Model's Prediction for Per Capita GDP



Case Study: Why Doesn't Capital Flow from Rich to Poor Countries?

- If MPK is higher in poor countries with low *K*, why doesn't capital flow to those countries?
  - Short Answer: Simple production model $MPK^R > MPK^P$
    with no difference in productivity across countries
    is misguided.
  - also consider $\bar{A}$.

Productivity Differences: Improving the Fit of the Model

- The productivity parameter measures how efficiently countries are using their factor inputs.
- Total factor productivity (TFP) ($\bar{A}$)
- If TFP is different from 1 ➔ need better model

$\bar{A}$ same amount capital labor

↳ whether more return

eg. US

Burine use much less efficiently

Total Factor Productivity

- Data on TFP is not collected. (not measured in data).
  - It can be calculated because we have data on output and capital per person. $Y_t^i = A^i K_t^{i\,\alpha} L_t^{i\,1-\alpha}$
  - TFP is referred to as the "residual." $A = \frac{Y_t^i}{K_t^{i\,\alpha} L_t^{i\,1-\alpha}}$
- A lower level of TFP
  - implies that workers produce less output for any given level of capital per person.

Measuring TFP So the Model Fits Exactly—1

$y_t^i = A_t^i (k_t^i)^\alpha$

TABLE 4.4

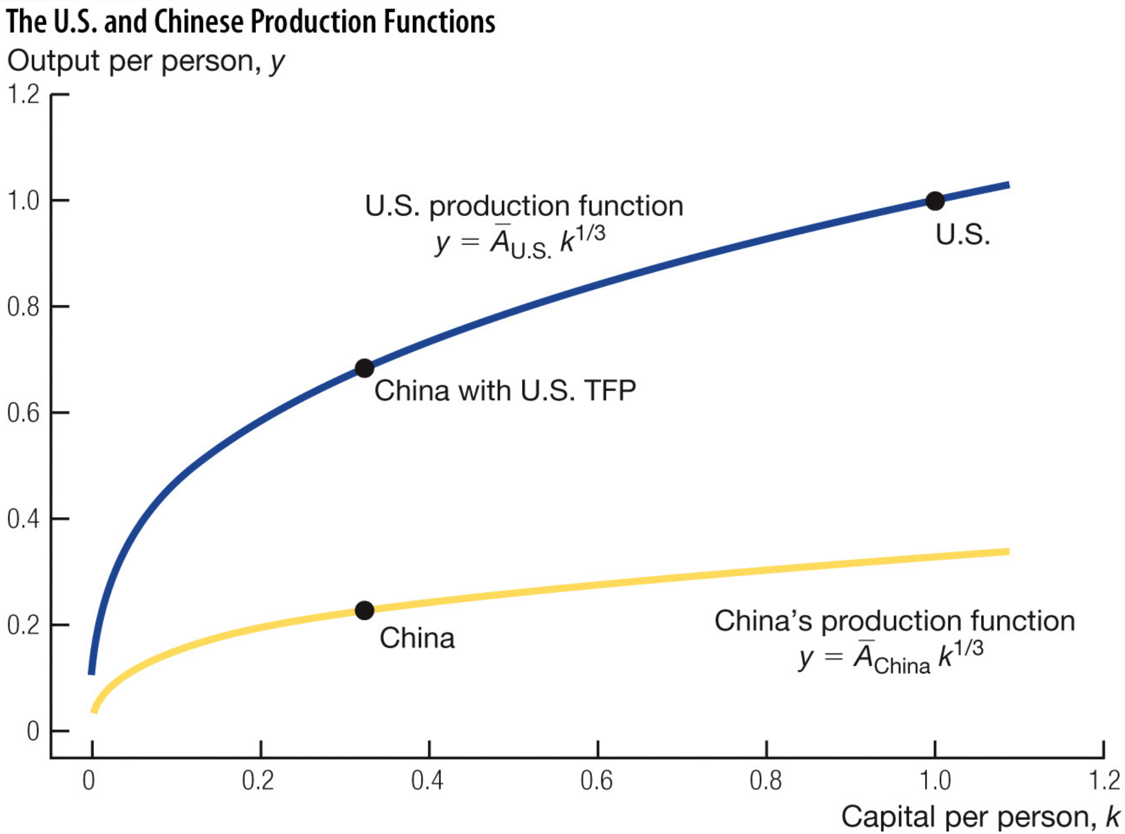
Measuring TFP So the Model Fits Exactly

Country	Per capita GDP (y)	$\bar{k}^{1/3}$	Implied TFP ($\bar{A}$)
United States	1.000	1.000	1.000
Switzerland	1.147	1.123	1.022
United Kingdom	0.733	0.941	0.779
Japan	0.685	1.007	0.680
Italy	0.671	1.040	0.646
Spain	0.615	1.041	0.590
Brazil	0.336	0.771	0.436
South Africa	0.232	0.602	0.386
China	0.241	0.686	0.351
India	0.105	0.437	0.240
Burundi	0.016	0.192	0.085

less efficient

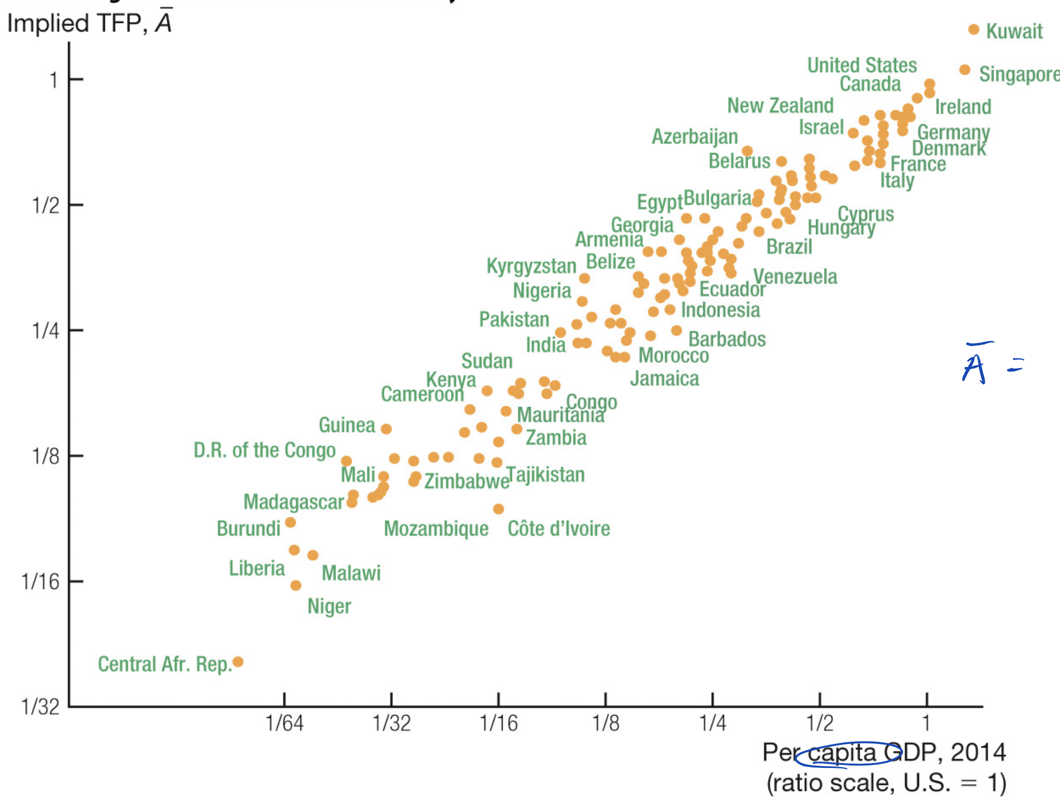
Calculations are based on the equation $y = \bar{A}\bar{k}^{1/3}$. Implied productivity $\bar{A}$ is calculated from data on y and $\bar{k}$ for the year 2010, so that this equation holds exactly as $\bar{A} = y/\bar{k}^{1/3}$.

United States and Chinese Production Functions



Measuring TFP So the Model Fits Exactly—2

Measuring TFP So the Model Fits Exactly



$$\bar{A} = \frac{y}{k^{\frac{1}{3}}}$$

4.4 Understanding TFP Differences

- Output differences between the richest and poorest countries? (top-5 vs. bottom-5)
  - differences in capital per person explain about one quarter of the differences
  - TFP explains the remaining three quarters
- Thus, rich countries are rich because:
  - they have more capital per person.
  - they use labor / capital more efficient.
  - **Why are some countries more efficient at using capital and labor?**

Understanding TFP Differences

- Human Capital
- Technology (blue print of economy)
- Misallocation
- Institutions,

Let y^*_{rich} be GDP per capita in rich countries

Let y^*_{poor} be per capita in poor countries.

$$\frac{y^*_{rich}}{y^*_{poor}} = \frac{A_{rich}}{A_{poor}} \times \left(\frac{k^*_{rich}}{k^*_{poor}}\right)^{\frac{1}{3}}$$

Human Capital

- Human capital
  - Stock of skills that individuals accumulate to make them more productive
  - Education and training

poor countries less school years.
less quality of labor.

- Returns to education
  - Value of the increase in wages from additional schooling
- Accounting for human capital reduces the residual from a factor of 14 to a factor of 7 (rough approximation).

Technology

- Richer countries may use more modern and efficient technologies than poor countries.
- Why?
  -
  -

" Economist" YouTube channel interview with

Institutions

- Even if human capital and technologies are better in rich countries, why do they have these advantages?
- Institutions are in place to foster human capital and technological growth.
    -
    -
    -
    -

Misallocation

- Misallocation
    -
- Examples
    -
    -

4.5 Evaluating the Production Model

- Per capita GDP is higher if capital per person is higher and if factors are used more efficiently.
- Constant returns to scale imply that output per person can be written as a function of capital per person.
- Capital per person is subject to strong diminishing returns because the exponent is much less than one.

Weaknesses of the Model

- In the absence of country-specific TFP, the production model incorrectly predicts differences in income.
- The model does not provide an answer as to **why** countries have different TFP levels.