L3 01.28

Nash equilibrium.

The action profile a^* in a strategic game with ordinal preference is a nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a_{-i}^*), in which player i chooses a_i while every other player i chooses a_i while every other player j chooses a_j^*.

Equivalently, for every player i, $u_i(a^*) \geq u_i(a_i, a_{-i}^*)$ for every action a_i, of player i.

Best Response Function

— can be no BR.

$b_i(a_{-i}) = \{a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \ \forall a_i' \in A_i\}$.

given other's action
output own response

any other action

a_i is a best Response (BR) to a_{-i}

a^* is NE if a_i^* is BR to a_{-i}^* for all players i

E.g. in L2. Best Response

NE Assumptions

- No communication
- one-time interaction (payoff only base on current action, no future..)
- Players from large population
- Beliefs from experience
- simultaneous choice of strategy.
- Common knowledge of Rationality and of the game.

different levels.

wiki common knowledge, puzzle (blue/green eyes, k days for k people to leave the island)

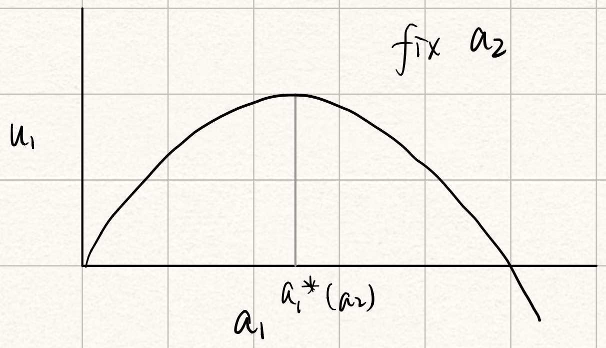
Sometime payoff matrix might not work!

Synergy Game. (2 player game)

乘法，非2式.

① each player payoffs: $u_i(a_1, a_2) = a_i(c - a_i + a_j)$ $\quad c > 0$.

strategies: $a_1 + a_2 \geq 0$

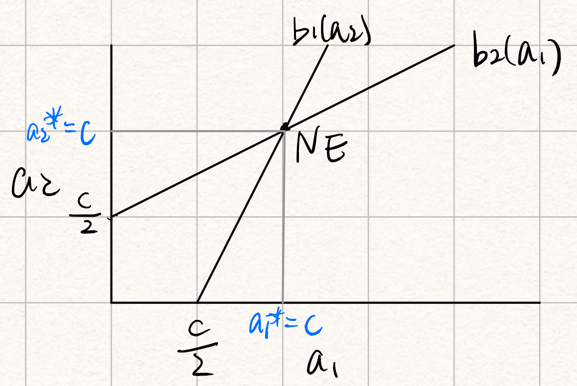


$$u_1 = a_1 c - a_1^2 + a_1 a_2.$$

$$\frac{\partial u_1}{\partial a_1} = c - 2a_1 + a_2 = 0.$$

$$b_1(a_2) = \frac{c + a_2}{2}$$

Similarly: $b_2(a_1) = \dfrac{c + a_1}{2}$



Solve: $a_1 = \dfrac{c + \frac{c + a_1}{2}}{2}$

$$a_1 = \frac{c}{2} + \frac{c}{4} + \frac{a_1}{4}$$

$$a_1 = c$$

Similarly, $a_2 = c$

② Maximize total surplus.

$$ca_1 - a_1^2 + a_1a_2 + ca_2 - a_2^2 + a_1a_2$$

$$= c(a_1 + a_2) - (a_1 - a_2)^2.$$

as big as you want w/ ↗ choose $a_1 = a_2$.

no max utility.

L4 01.30

Focal Point. ← what we think might happen by various equilibrium.

1. Heads or Tails : heads.
2. A, B, C in some order : ABC
3. split money game . \$0 and \$100 .. \$50

Strict Best Responses.

a_i is a strict best response to a_{-i} if

$u(a_i, a_{-i}) > u(a_i', a_{-i}) \quad \forall a_i' \neq a_i$

a^* is a strict nash equilibrium if a_i^* is a Best Response for all player.

e.g. Prisoner's Dilemma. ✓

split & steal ✗.

Strictly Dominated action

can not by a combination

a_i is strictly dominated by a_i' if $u(a_i', a_{-i}) > u(a_i, a_{-i})$ for all a_{-i}.

SD action is never a BR, never used in NE.

e.g.

	~~L~~	M	~~R~~
U	5,2	1,6	3,4
S	6,1	1,6	2,5
~~D~~	1,6	0,7	0,7

strictly dominated.

NE (U, M)
(S, M)

iterated deletion of SD strategies.

Weakly Dominated Strategy.

a_i is weakly dominated by a_i' if $u(a_i', a_{-i}) \geq u(a_i, a_{-i}) \quad \forall a_{-i}$

and $u(a_i', a_{-i}) > u(a_i, a_{-i})$ for some a_{-i}

e.g. i) split, steal : split is weakly dominated by steal.

weakly dominated strategy can still be in NE.

ii)

	L	C	R
T	2,2	1,3	0,1
M	3,1	0,0	0,0
B	1,0	0,0	0,0

B is WD by T, M.

(T, C)

(M, L)

(B, R)

iii) Voter game.

n voters, vote for A or B,

each person has a preferred candidate

NE: every voter vote A / B,

any outcome except votes for A − votes for B ≤ 2, vice versa.

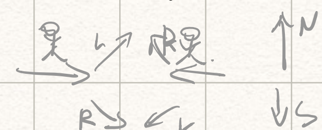
lots of NE involve WD strategies

Games.

Symmetric Game.

- set of actions is the same for all players.
- payoffs are symmetric $u_1(a', a'') = u_2(a'', a')$
- SG has a symmetric equilibrium if all the players choose same action.

-e.g. on the street, facing other person. how to choose



	R	L
R	1,1	-1,-1
L	-1,-1	1,1

(R, R), (L, L).

symmetric equilibrium

	N	S
N	-1,-1	1,1
S	1,1	-1,-1

(N, S), (S, N)

not symmetric equilibrium.

⇒ game that is not symmetric can get symmetric Equilibrium.

Zero-Sum (constant-sum) Game.   conflict win/lose.

payoff for any outcome add up to same value.

To solve: minimax solution (of zero-sum game).

choose action that maximize the minimum payoff.

e.g.

		1	6	4
		L	M	R
1	U	5,2	1,6	3,4
1	S	6,1	1,6	2,5
0	D	1,6	0,7	0,7

(U, M) or (S, M) is minimax solution.

The NE of Zero-Sum Game is also a minimax solution

		-3	$\boxed{0}$	-4
		A	B	C
-2	A	3,-3	-2,2	2,-2
$\boxed{-1}$	B	-1,1	0,0	4,-4
-4	C	-4,4	-3,3	1,-1

minimax: (B, B).

No NE.

L5 02.04

Public Good Game

2 plays, have wealth, w_1, w_2. contribute C_1, C_2. $0 \le C_i \le W_i$

$u_i(C_1, C_2) = \underbrace{V_i(C_1+C_2)}_{\text{private}} + \underbrace{(W_i - C_i)}_{\text{public.}}$

$\Leftarrow$ $V_1(C_1, C_2) = 3(C_1+C_2) - (C_1+C_2)^2$

$V_2(C_1, C_2) = 5(C_1+C_2) - (C_1+C_2)^2$

↑ need boundary. so that not enter the extreme case.

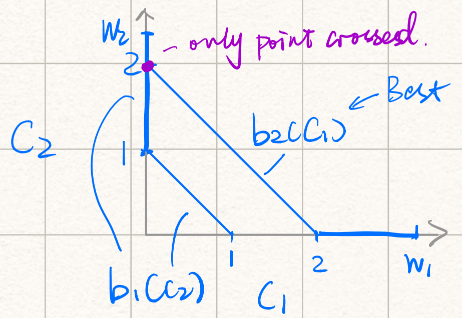
$u_1 = 3(C_1+C_2) - (C_1+C_2)^2 + (W_1 - C_1)$

$= 2C_1 - (C_1+C_2)^2 + W_1 - 3C_2.$

$\frac{\partial u_1}{\partial C_1} = 2 - 2(C_1+C_2) \overset{\text{set}}{=} 0$ (to max u_1) $\Rightarrow$ $C_1 = 1 - C_2$. ① Best Response function.

$u_2 = 5(C_1+C_2) - (C_1+C_2)^2 + (W_2 - C_2)$

$\frac{\partial u_2}{\partial C_2} = 4 - 2(C_1+C_2) \overset{\text{set}}{=} 0$ $\Rightarrow$ $C_2 = 2 - C_1$ ②.



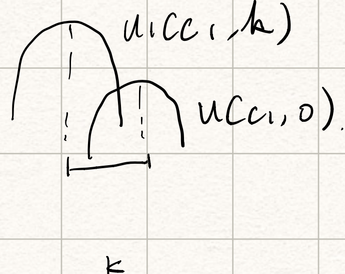
(value the contribution more)

$\Rightarrow$ one pay for all public good as others free-riding.

NE: $C_2 = 2$, $C_1 = 0$

$u_1(C_1, k) = u_1(C_1 + k, 0) + k$

(how C_2 influence C_1)

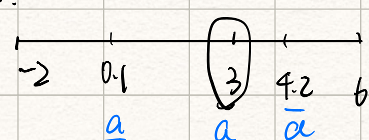


Public Median Investment Games

odd number of voters

each voters vote for particular policy a.

median policy is the one chosen. e.g.



each player has preferred point x_i^*.

$u_i = -|x_i^* - x|$
↖ chosen policy.

never worse-off to choose preferred point.

Various NE including weekly dominated strategy

Any strategy but choosing preferred point here is weekly dominated.

Xavier and Yvonne

(i) choose price

$Q_X = 44 - 2P_X + P_Y$ Q for quantity, P for price.

$Q_Y = 44 - 2P_Y + P_X$. Substitute for each other.

Assume has constant marginal cost.

$U_X = (P_X - 8)(\underbrace{44 - 2P_X + P_Y}_{Q_X})$ $U_Y = (P_Y - 8)(44 - 2P_Y + P_X)$

$= 44P_X - 2P_X^2 + P_XP_Y - 352 + 16P_X - 8P_Y$

$= -2P_X^2 + 60P_X + P_XP_Y - 352 - 8P_Y$.

$\frac{\partial U_X}{\partial P_X} = -4P_X + 60 + P_Y \overset{set}{=} 0$

Best Response for X: $P_X = 15 + \frac{1}{4}P_Y$.

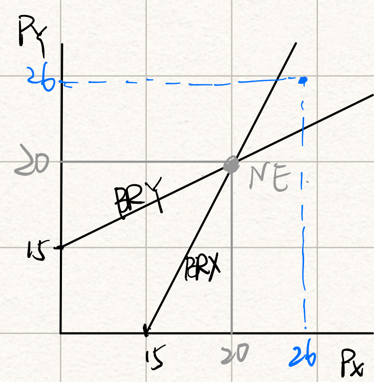
Similarly, ... Y: $P_Y = 15 + \frac{1}{4}P_X$.

$P_X = 15 + \frac{1}{4}(15 + \frac{1}{4}P_X)$

$\frac{15}{16}P_X = 15 \cdot \frac{5}{4}$

$P_X = 20$

$P_Y = 20$



(ii) Cartel Outcome.

2 firms cooperate, max $U_X + U_Y = U_T$

$U_T = -2P_X^2 - 2P_Y^2 + 52P_X + 52P_Y + 2P_XP_Y - 704$

$\frac{\partial U_T}{\partial P_X} = 52 + 2P_Y - 4P_X \overset{set}{=} 0$

$\frac{\partial U_T}{\partial P_Y} = 52 + 2P_X - 4P_Y \overset{set}{=} 0$.

$P_X = P_Y = 26$.

Cournot Competition. (next time)

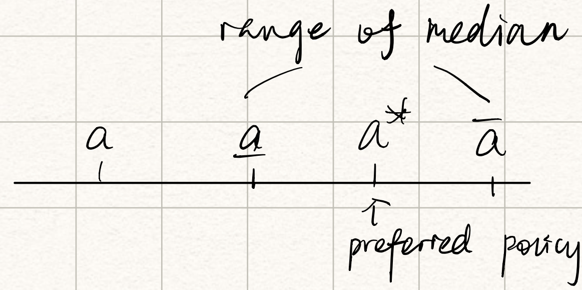
n firms, choose quantities, $q_1, \dots, q_n$.

common demand curve. $P = P(Q)$, $Q = \sum_i q_i$ (more produce, lower price).

cost function $C_i(q_i)$

$\max_{q_i} u_i = q_i P(Q) - C_i(q_i)$

Lb 02.06



Last Lecture: Median Investment

when $a \notin [\underline{a}, \bar{a}]$.

a^* better than a $\quad \underline{a} \le a^* \le \bar{a}$

same $\quad a^* < \underline{a}$ or $a^* > \bar{a}$ $\Rightarrow$ weakly dominated

when $a \in [\underline{a}, \bar{a}]$. $\quad a^*$ better than a.

Stag & Hunt

extend to a group of players.

i) - need n players to catch stag,

- value of stag is R, of hare is 1
- stag hunters divide R, $\frac{R}{n}$ is preferred to 1 $\left(\frac{R}{n} > 1\right)$
- S_i stag hunters.

$S = 0$ NE

$0 < S < n-1$: not NE some S players will switch to H.

$S = n-1$ $\quad S \to H$, $H \to S$

$S = n$ $\quad$ NE.

ii) need fewer hunters to catch the stag, $2 \le m < n$, $\frac{R}{n} > 1$

$S=0$ NE

iii) $R=k$ $\frac{R}{k+1} < 1$, $\frac{R}{k} > 1$, $2 \le m \le R < n$. (Discussion).

Dividing money Game

- 2 players, choose $m_1, m_2 \in (0,10)$
- payoff if $m_1 + m_2 \le 10$, $u_1 = m_1$, $u_2 = m_2$

  if $m_1 + m_2 > 10$, $m_1 > m_2$: $u_1 = 10 - m_2$ $u_2 = m_2$

  $m_1 < m_2$: $u_1 = m_1$. $u_2 = 10 - m_1$

  $m_1 = m_2$: $u_1 = u_2 = 5$.

BR_1 $m_2 \le 5$: $10 - m_2$

$m_2 = 6$ $BR = 5$ or 6

$m_2 \geq 7$ $BR = m_2 - 1$

NE $(5,5)$ $(6,6)$ $(5,6)$ $(6,5)$

Competition Game

Cournot Game

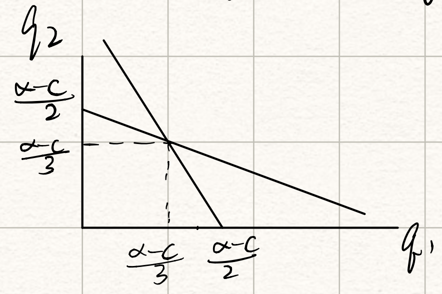
- choose quantities, n firms
- $P = P(Q)$, $Q = \sum_i q_i$
- cost : $C_i(q_i)$
- $\max_{q_i} u_i = q_i p(Q) - C_i(q_i)$

i) 2 players $P(Q) = \alpha - Q$, $C_i(q_i) = cq_i$, $u_i = q_i(\alpha - q_1 - q_2) - cq_i$

constant marginal cost

To max $\frac{\partial u_1}{\partial q_1} = \alpha - 2q_1 - q_2 - c \overset{set}{=} 0$ $\Rightarrow$ BR: $q_1 = \frac{\alpha - c - q_2}{2}$

$\frac{\partial u_2}{\partial q_2} = \alpha - 2q_2 - q_1 - c \overset{set}{=} 0$ $\Rightarrow$ BR $q_2 = \frac{\alpha - c - q_1}{2}$



Solve: $q_1 = \frac{\alpha - c}{2} - \frac{\alpha - c - q_1}{4}$ (Nash Equilibrium)

$q_1 = q_2 = \frac{\alpha - c}{3}$

$u_1 = u_2 = \frac{(\alpha - c)^2}{9}$

ii) Cartel: $\max_Q (Q(\alpha - Q) - cQ)$

$\frac{\partial u_T}{\partial Q} = \alpha - 2Q - c \overset{set}{=} 0$

$Q = \frac{\alpha - c}{2}$

Profit: $\frac{(\alpha - c)^2}{4}$

producing less $\Rightarrow$ get higher output.

iii) n players

$\frac{\partial u_i}{\partial q_i} = \alpha - c - 2q_i - q_j - Q_{-i,-j} \overset{set}{=} 0$

$\frac{\partial u_j}{\partial q_j} = \alpha - c - 2q_j - q_i - Q_{-i,-j} \overset{set}{=} 0$

$\Rightarrow$ For any 2 player of n players, NE is equal (symmetric $C_i(q_i)$, $P(Q)$)

BR: $q_i^* = q_j^* = q^* = \frac{\alpha - c}{n+1}$

Total Q: $\frac{n}{n+1}(\alpha - c)$ Price $= c$

L7 02.11

Rationaliability

- eliminate strictly dominated strategy.
- eliminate strategies that are never BR - only BR is chose in NE.
- continue process until nothing else can be eliminated.

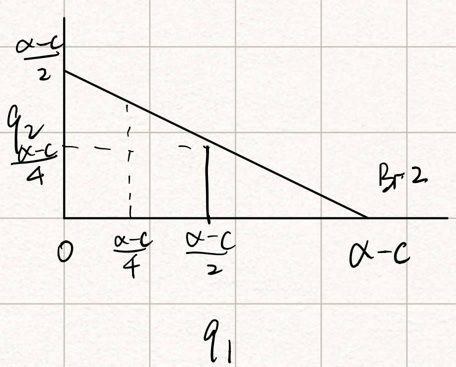
上次 cornow. perfect competition (n player)

Book P151

First C4 not best response for any row. (no strictly dominated stratagies).

Then R4 is not BR to anything
↑
only BR to C4.

Cannot: $q_2 = \frac{\alpha - c - q_1}{2}$



any option $> \frac{\alpha-c}{2}$ not rationalizable. (for q_2)

know q_1 with same BR curve, not choose option

$> \frac{\alpha-c}{2} \Rightarrow q_2$: option $> \frac{\alpha-2}{4}$ not rationalizable

...... → reach one point.

Bertrand Competition

- n firms, choose $p_1, \dots, p_n$.
- consumers buy from lowest price firm(s)
- profits: m firms at lowest price: earning $p_i \frac{D(p_i)}{m} - c_i\left(\frac{D(p_i)}{m}\right)$

  others: 0.

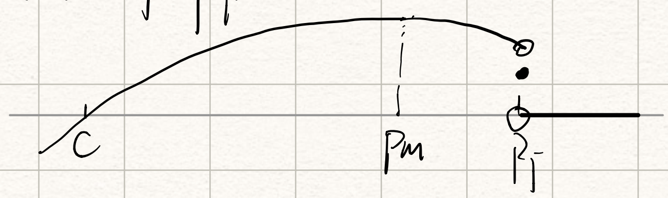
i) 2 Firms: assume constant MC $C_i = cq_i$; Demand: $D = \alpha - p$.

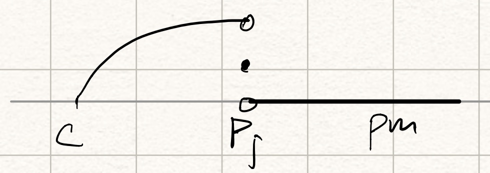
profit function: If $p_i < p_j$: $(p_i - c)(\alpha - p_i)$

$p_i = p_j$: $(p_i - c)\left(\frac{\alpha - p_i}{2}\right)$

$p_i > p_j$: 0

pm: monopoly price





BR_i: $p_j > p_m$: $p_i = p^m$

$p_m > p_j > c$: if continuous, no BR (a little bit less but not defined)

if discrete: BR: $p_i = p_j - 1$

$p_j = c$: $p_i \geq c$ ($=c$, $p_i - c = 0$, profit $= 0$, $> c$: no market, 0, $< c$, $p_i - c < 0$, lose money).

$p_j < c$: $p_i > p_j$

NE: Continuous: Both charge at c (c, c)

(no BR when $p_i > c$,
for $p_j > p_m$, if i choose p_m, p_j choose $< p_i$ and under cut each other in the case $p_m > p_j > c$ until c.
$< c$: choose c to avoid negative profit).

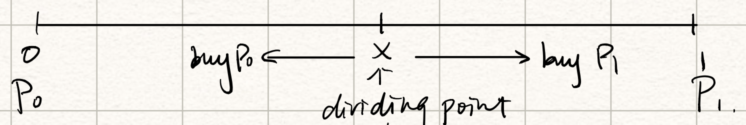
Discrete: (c, c) $(c+1, c+1)$

↑
c: no profit $p_i - c = 0$ $< c$: negative
$> c$: no market.

n players: same to 2 players.

Spatial Dimension

Hotelling (Exercise 57 p.168)



Consumer at x be indifferent for P_0, P_1

Consumer on continuous $[0,1]$.
Icecream (product) at 0, 1.

- $MC = 0.25$
- Continuous of Consumers each buy 1
- Transport costs $0.5\,d^2$
- $P_0 + 0.5x^2 = P_1 + 0.5(1-x)^2$

$\Rightarrow x = P_1 - P_0 + 0.5$ $\quad P_0 \uparrow \Rightarrow x \Rightarrow P_0$; $P_1 \uparrow \Rightarrow x \Rightarrow P_1$

Firm 0 : $\max\,(P_0 - 0.25)(P_1 - P_0 + 0.5)$

$$\frac{\partial u_0}{\partial p_0} = P_1 - 2P_0 + 0.5 + 0.25 \overset{\text{set}}{=} 0 \qquad P_0 = \frac{P_1}{2} + \frac{3}{8}.$$

Firm 1 : $P_1 = \frac{P_0}{2} + \frac{3}{8}$. (symmetric).

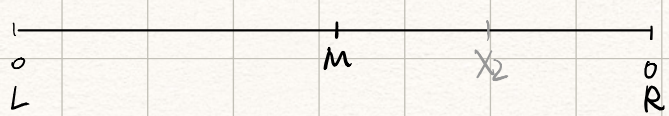
$\Rightarrow P_0 = P_1 = \frac{3}{4}$. ← equally divide market.

Cartel: can increase their charge if they change the same price.
$\rightarrow \infty$

Location → NE → next to each other ?

L8 02.13

Hoteling (Location) Model – Electoral competition.



- Players are candidates.
- strategies: position x_1, x_2.
- voters choose the closer candidate. ← uniform distributed or not.
- Candidates wins by getting most votes.

PMF.
choose M based on

BR: $x_2 > m$: $1-x_2 < x_1 < x_2$.

$x_2 < m$: $x_2 < x_1 < 1-x_2$.

$x_2 = m$: $x_1 = m$.

NE: (m, m) can always improve when at least one not at m.

Not at m: if lose, go to m to ensure at least a tie.
if tie, go to m to win.
if win, go to m will not improve

at $x_i \neq m$ not better off.

Auctions

n bidders

bids: $b_1, \ldots, b_n \geq 0$.

valuations: $v_1, \ldots, v_n \geq 0$

Second Price Auction, seal-bid

- Everyone submit a bit simultaneous.
- the highest bidder, pay the 2^{nd} highest price b_j
- represent ascending auctions.

payoff: winner : $v_i - b_j$

losers : 0

Assume $v_1 \geq v_2 \geq \cdots \geq v_n \geq 0$

NE: for example: $(v_1, 0, 0, \ldots, 0)$

$(v_2, v_1, 0, \ldots, 0)$

$(v_3, v_3, v_1, \ldots, 0)$

Bidding $b_i = v_i$ weakly dominate any other bid.

fix b', Let $\bar{b}$ be the highest bid of others.

fix $b' < b$, $b'' > v$.

	$\bar{b} < b'$	$b' < \bar{b} < v$	$v < \bar{b} < b''$	$\bar{b} > b''$
b'	$v - \bar{b}$	$\underset{\sim}{0}$ smaller	0	0
v	$v - \bar{b}$	$v - \bar{b}$	0	0
b''	$v - \bar{b}$	$v - \bar{b}$	$v - \bar{b} < 0$	0

First Price Auction

The highest bidder wins and pay their bid. sealed-bid.

$b_i'' > v_i$ is weakly dominated by $b_i = v_i$ is weakly dominated by $b_i' < v_i$

$\hookrightarrow b_i'' \le v_i \le b_i'$

Player 1 always wins (player with highest value)

winning bid between b_1, b_2 $\quad V_2 \le b_1 \le V_1$.

break tie: player with higher value win.

L9 02.20

highest bid of others.

	$\bar{b} < b'$	$b' < \bar{b} < v$	$v < \bar{b} < b''$	$b'' < \bar{b}$
$v > b'$	$v - b'$	0	0	0
v	$v - v = 0 < v - b'$	$v - v = 0$	0	0
$v < b''$	$v - b'' < 0$	$v - b'' < 0$	$v - b'' < 0$	0

cases for "=" belong to lower or higher cases based on the tie breaker.

NE:

1. top 2 bid the same.
2. player 1 always win.
3. winning bid is between V_1 and V_2 : $V_1 \geq b_1 \geq V_2$

All pay auction

everyone pay their own bid. no NE for first-paid in all pay auction

War of attrition, drop out.

pay for the second highest if win, pay his own cost if lose.

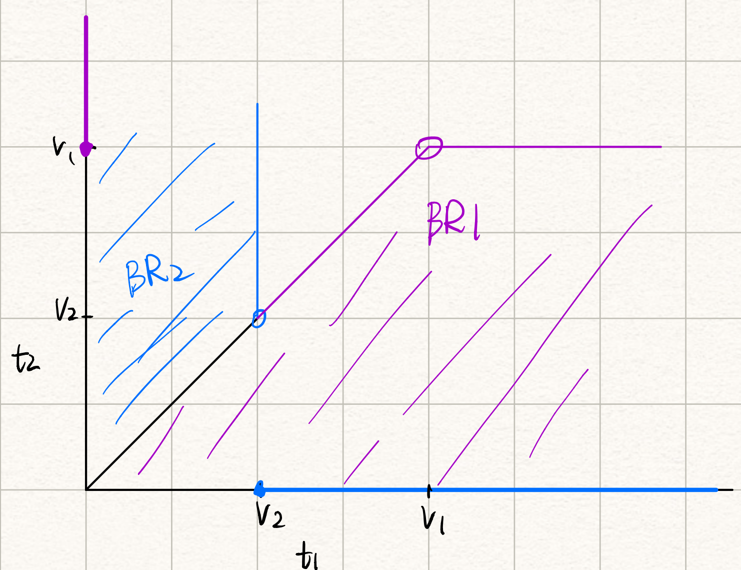
- 2 players, strategies $t_1, t_2 \geq 0$,

Payoff for 1:

$t_1 > t_2$	$V_1 - t_2$
$t_1 = t_2$	$0.5V_1 - t_1$
$t_1 < t_2$	$-t_1$

BR for 1:

$t_2 < V_1$	any $t_1 > t_2$ to get payoff $= V_1 - t_2$.
$t_2 = V_1$	$t_1 > t_2$ or $t_1 = 0$.
$t_2 > V_1$	$t_1 = 0$.



NE: $t_1 = 0$, $t_2 \geq V_1$

$t_1 \geq V_2$, $t_2 = 0$

do not depend on value.

Real world: value change,

sequential.

swoopo.