Prisoner's Dilemma: stage game

	C	D
Cooperate	2,2	0,3
Defect	3,0	1,1

Repeated game: players,

(extensive form) terminal histories $(a_1, a_2, \cdots a_T)$ finitly repeated (action in round 2)

$(a_1, a_2, \cdots)$ infinitly repeated

Actions, same in each round

Payoffs discounted (δ) value payoffs in later lower than that of current.

Sum: $S = 1 + \delta + \delta^2 + \cdots\cdots$, $0 < \delta < 1$

$$S = \frac{1-\delta^T}{1-\delta}$$

Discounted average payoff: $(1-\delta) \sum_{t=1}^{T} \delta^{t-1} u(a_t)$

Grim trigger. (strategy)

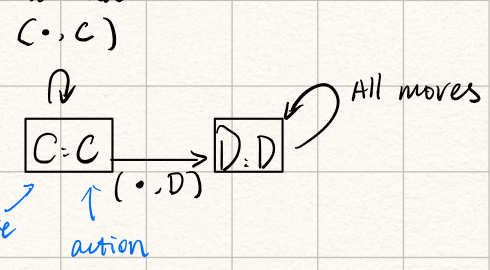
In repeated prisoner's dilemma: pay C until other player plays D

Extension: "the evolution of Cooperation"

Strategies: Automaton

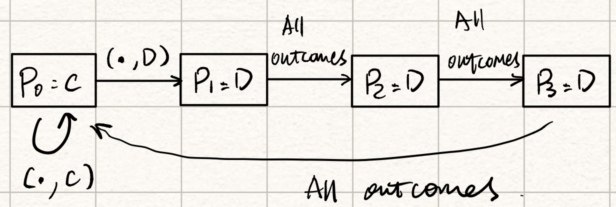
E.g. Prionser's Dilemma

Grim Trigger:



(punish defect forever)

Three-period punishment:



Tit - for - Tat:
(copy)

$$\begin{tikzcd} \boxed{C:C} \arrow[loop below, "(\cdot,C)"] \arrow[r, bend left, "(\cdot,D)"] & \boxed{D:D} \arrow[loop below, "(\cdot,D)"] \arrow[l, bend left, "(\cdot,C)"] \end{tikzcd}$$

Whether some strategies will let people cooperate?

Finitely repeated game: NE: Always (D,D) on eq path

SPE: Always (D,D)

Infinitely repeated game: (no end for backward indution)

⇒ Both Grim Trigger strategy: CC → CC → ...... forever

payoff: 2,2 → 2,2 ...... DAP = 2.

if defeat, defeat at first round.

1 D, 2 Grim Trigger: DC → DD → ...... $DAP = (1-\delta)(3+\delta+\delta^2+\dots)$

3,0 1,1 ...... $= (1-\delta)3 + \delta$.

↓

$(1-\delta)3+\delta < 2$ for Grim Trigger to be NE ⇒ $\delta \geq \frac{1}{2}$.

(no deviate)

⇒ If play k-period punishment: CC, CC, ... DAP=2 (if infinitely)

2,2, 2,2 ......

If one defeat, will do in first period. $DC, \overbrace{DD, DD \dots DD}^{k}, DC \dots$

$2(1-\delta^{k+1}) \geq (3+\delta+\delta^2+\dots\delta^k)(1-\delta) = 3(1-\delta)+\delta(1-\delta^k)$ not to deviate

$2\delta \geq 1+\delta^{k+1}$ for k-period publishment to be NE.

$k=1 \Rightarrow 1-2\delta+\delta^2 \leq 0 \quad \delta=1$

$k=2 \Rightarrow \delta \geq 0.62$

$k=3 \Rightarrow \delta \geq 0.55$

$k \to \infty \Rightarrow \delta \geq 0.5$

⇒ If Tit-for-tat   CC, CC, CC, ----   DAP = 2

Deviate: ① DC, DD, DD....

Not deviate: $\delta \geq 1/2$ (see grim trigger)

② DC, CD, DC $DAP = (1-\delta)(3+3\delta^2+3\delta^4+\cdots) = \frac{3(1-\delta)}{1-\delta^2} = \frac{3}{1+\delta}$

Not deviate: $2 \geq \frac{3}{1+\delta}$ $2(1+\delta) \geq 3$ $\delta \geq 1/2$

NE if $\delta \geq 1/2$

Subgame Perfect Equilibrium ⇐ in infinity, no last round

↳ One deviation property (no one will change/deviate in any point in the game).

For grim trigger, cooperate: CC, CC, ----

deviate: DC, DD

↑ modified ↘

$$C \xrightarrow{(D,\cdot) \text{ or } (\cdot,D)} D$$

Following a history with D: CD, DD, DD ---- .. ⇐ SPE if $\delta \geq 1/2$.

↓ deviate DD, DD ---

K-period punishment: will deviate

Tit-for-tat SPE: ① (CC), CC -- if cooperate   if $\delta \geq 1/2$ if NE

deviate: DC, CD -- . ↑ payoff < not deviate

② (CD), DC, CD, ----

deviate: CC, CC, ----   not deviate: $\delta \leq 1/2$

③ (DC), CD, DC, ---- $\frac{3\delta}{1+\delta} \geq 1 \Rightarrow \delta \geq 1/2$.

deviate DD, DD, DD-----

④ (DD), DD, - - - -

$\delta \le 1/2$.

deviate $\to$ CD, DC, - - - -

$\Rightarrow \delta = 1/2$ in SPE.

Evolutionarily stable strategy

Strategy comes from genes,

probability of "mutation" ε

Higher payoff $\to$ survival reproduction.

Monomorphic equilibrium

a is evolutionary stable strategy (ESS) if

$$u(a^*, a^*)\cdot(1-\varepsilon) + u(a^*, b)\cdot\varepsilon > (1-\varepsilon)u(b, a^*) + \varepsilon u(b,b) \quad \forall b \ne a^*$$

E.g.

	X	Y ← original
X	2,2	0,0
Y	0,0	1,1

$1-\varepsilon$ play X and ε play Y

payoff of X: $2(1-\varepsilon) + 0\cdot\varepsilon = 2(1-\varepsilon) > u_Y$

Y: $0(1-\varepsilon) + 1\cdot\varepsilon = \varepsilon$

X plays $1-\varepsilon$

X is ESS if $2(1-\varepsilon) + 0\varepsilon > 0(1-\varepsilon) + \varepsilon$ $(u_X > u_Y)$ $\quad \varepsilon < 2/3$

Y is ESS if $1(1-\varepsilon) + 0\cdot\varepsilon > 0\cdot(1-\varepsilon) + 2\varepsilon$ $\quad \varepsilon < 1/3$

Y plays $1-\varepsilon$

E.g.

	X	Y
X	2,0	0,0
Y	0,0	0,0

$2(1-\varepsilon) > 0$, so X is ESS

$0 < 2\varepsilon$, so Y is not ESS

Condition for ESS: $(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, b) > (1-\varepsilon)u(b, a^*) + \varepsilon u(b, b)$

a^* is ESS if $u(a^*, a^*) > u(b, a^*)$ $\forall b \neq a^*$

$\Leftrightarrow$ (a^*, a^*) being a strict NE

a^* is ESS if $u(a^*, a^*) = u(b, a^*)$ and $u(a^*, b) > u(b, b)$ $\forall b \neq a^*$ that are BR to a^*.

E.g. Prisoner's Dilemma

	C	D
C	2,2	0,3
D	3,0	1,1

Consider: $2(1-\varepsilon) + 0\cdot\varepsilon < 3(1-\varepsilon) + 1\cdot\varepsilon$

$\Rightarrow$ C not ESS

Consider: $1\cdot(1-\varepsilon) + 3\varepsilon > 0(1-\varepsilon) + 2\varepsilon$

$\Rightarrow$ D is ESS

Hawk - Dove

	A	P
A	$\frac{v-c}{2}, \frac{v-c}{2}$	$v, 0$
P	$0, v$	$\frac{v}{2}, \frac{v}{2}$

$v > c$ (A, A) is the only strict NE.

A is the only ESS

$v = c$ (A, A) is the NE, but not strict

$u(A, P) > u(P, P)$?

$v > \frac{v}{2}$ $\Rightarrow$ A is the only ESS

$v < c$ (A, P) and (P, A) are the only pure strategy NE.

no ESS in monomorphic eq.

Polymophic Equilibrium

α^* is ESS when

1. (α^*, α^*) is a NE
2. $u(\beta, \beta) < u(\alpha^*, \beta)$ $\forall \beta \neq \alpha^*$ and is BR to α^*

E.g. (i)

	X	Y	Z
X	2,2	1,2	1,2
Y	2,1	0,0	3,3
Z	2,1	3,3	0,0

Pure strategies

$u(X, X) = u(Y, X)$

$u(X, Y) > u(Y, Y)$

} X is ESS (in pure strategies)

Mixed: $\alpha^* = X$ $\quad \beta = \frac{1}{2}Y + \frac{1}{2}Z$

$u(X, X) = u(\beta, X)$

$u(X, \beta) = 1 \qquad u(\beta, \beta) = \frac{1}{4}\cdot 0 + \frac{1}{4} 0 + \frac{1}{4}\cdot 3 + \frac{1}{4}\cdot 3 = \frac{3}{2}$

X is not ESS in mixed strategies.

(ii) BoS

		p	1-p
		M	L
2/3	M	0,0	2,1
1/3	L	1,2	0,0

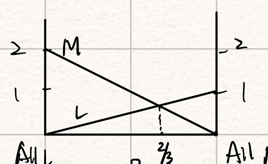
✓ NE in mixed strategy.

$P_M = \frac{2}{3}$, $P_L = \frac{1}{3}$: α^*

$P_M = p$, $P_L = 1-p$: β

$u(\alpha^*, \beta) = \frac{2}{3}p \cdot 0 + \frac{2}{3}(1-p)\cdot 2 + \frac{1}{3}p \cdot 1 + \frac{1}{3}(1-p)\cdot 0$

$= \frac{4}{3} - p$



M 越多 → payoff ↓

L 越多, payoff ↓

→ evolutionary stable.

$u(\beta, \beta) = p^2 \cdot 0 + p(1-p)\cdot 2 + (1-p)p \cdot 1 + (1-p)(1-p)\cdot 0$

$= 3p(1-p)$

α^* will be ESS if $\frac{4}{3} - p > 3p(1-p)$ $\quad \forall p \neq \frac{2}{3}$

$\frac{4}{3} - p > 3p - 3p^2$, $3p^2 - 4p + \frac{4}{3} >$ $(p - \frac{2}{3})^2 > 0$ $\forall p \neq \frac{2}{3}$.

α^* is ESS.

(iii)

		X	Y
p	X	2, 2	0, 0
(1-p)	Y	0, 0	1, 1

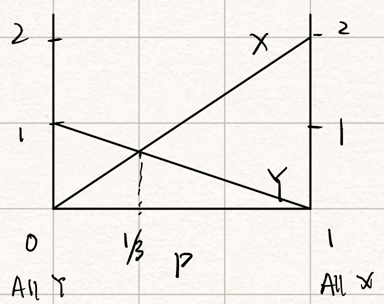
Mixed strategy $2(p) = 1\cdot(1-p) \Rightarrow p = \frac{1}{3}$. (symmetric)

$\uparrow \alpha^*$

mutant always play x

$P(\alpha^*, X) = 2\cdot\frac{1}{3} + 0 = \frac{2}{3}$

$P(X, X) = 2 > \frac{2}{3}$

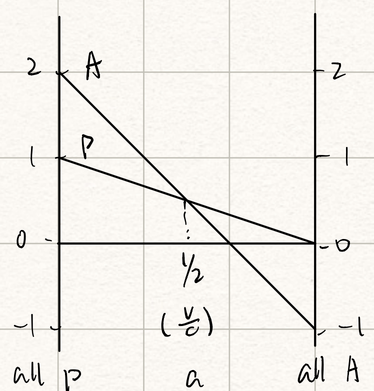


Y越多 → payoff ↑ for Y   X越多 → payoff ↑ for X

not evolutionarily stable.
(除 all X or all Y)

(iv) Hawk - Dove

		A	P
a	A	-1, -1	2, 0
1-a	P	0, 2	1, 1



P↑, UP↓ ← all P   all A → A↑, UA↓

$v=2,\ c=4$

stable

General Hawk - Dove

		A	P
a	A	$\frac{v-c}{2}, \frac{v-c}{2}$	$v, 0$
1-a	P	$0, v$	$\frac{v}{2}, \frac{v}{2}$

play A   play P   (使另一方 indifferent between A/P)

Mixed strategy: $\frac{v-c}{2}\cdot a + v(1-a) = 0\cdot a + \frac{v}{2}(1-a)$

$\frac{v}{2}(1-a) = \frac{c-v}{2}a$

$\frac{v}{2} - \frac{va}{2} = \frac{ca}{2} - \frac{va}{2}$

$a = \frac{v}{c}$ ⇒ ESS

(v) Tit-tat-cat (T), always defect (A) in Prisoner's Dilemma.

two times

	T	A
T	4,4	1,4
A	4,1	2,2

TA: CD, DD (0,3), (1,1)

A is ESS

A is a strict best response to itself.

$u(T,A) < u(A,A)$, T is not a ESS

three times

	T	A
T	6,6	2,5
A	5,2	3,3

TA = CD, DD, DD (0,3) (1,1) (1,1)

T, A are both strict best Response to themself.

T and A are best Responses.

(vi) Never defeat (always C) (N), T, A

Thrice:

		N	T	A
$1-\pi$	N	6,6	6,6	0,9
π	T	6,6	6,6	2,5
	A	9,0	5,2	3,3

A is a strict BR to itself.

T is no longer evolutionary stable

Mixed strategy: mix of T and N = α $u(\alpha,\alpha) > u(A,\alpha)$

$6 > 5\pi + 9(1-\pi)$

$4\pi > 3 \rightarrow \pi > 3/4$

(vii)

	R	P	S
R	γ,γ	$-1,1$	$1,-1$
P	$1,-1$	γ,γ	$-1,1$
S	$-1,1$	$1,-1$	γ,γ

gamma $\gamma>0$

NE: $(1/3, 1/3, 1/3)$

compare to pure strategy, this is not ESS.

play R, always get

$U(R, R) > U(R, NE)$.

(viii)

	X	Y
X	1,1	1,1
Y	1,1	1,1

Do not have any ESS.

(not able to drive out any action)