Endogenous Growth Theory

"Variety" Model

Based on Romer (1990)

Model with two sectors:

1. Production Sector
2. Research Sector

Production Sector

- Goods produced with the following technology:

$$Y(t) = K(t)^{\alpha}\left(A(t)\, L_Y(t)\right)^{1-\alpha}$$

- L_Y is the labor employed to produce goods.
- As before, the stock of capital evolves according to:

$$\dot{K}(t) = s_K Y(t) - dK(t)\,.$$

$$= \left(\frac{\dot{k}(t)}{k(t)} + n\right) K(t)$$

$$\frac{\dot{k}(t)}{k(t)} = s_K \frac{Y(t)}{K(t)} - d - n.$$

$$\dot{k}(t) = s_K \frac{Y(t)}{K(t)} \cdot \frac{K(t)}{L(t)} - (d+n)k(t)$$

$$= s_K \frac{Y(t)}{L(t)} - (d+n)\, k(t).$$

$$= s_K\, y(t) - (d+n)\, k(t)$$

Production Sector

- In per-capita terms,

$$y(t) = k(t)^{\alpha}\left(A(t)\frac{L_Y(t)}{L(t)}\right)^{1-\alpha}$$

and

$$\dot{k}(t) = \dot{\left(\frac{K(t)}{L(t)}\right)} = \frac{\partial \frac{K(t)}{L(t)}}{\partial t}$$

$$= \frac{\dot{K}(t)}{L(t)} - \frac{K(t)}{L(t)^2}\cdot \dot{L}(t)$$

$$= \frac{\dot{K}(t)}{K(t)}\cdot\frac{K(t)}{L(t)} - \frac{K(t)}{L(t)}\cdot\frac{\dot{L}(t)}{L(t)}$$

$$= \frac{\dot{K}(t)}{K(t)}\cdot k(t) - n\cdot k(t)$$

- As before, we assume that population grows at the constant rate n:

$$\frac{\dot{L}(t)}{L(t)} = n$$

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - n.$$

$$\dot{K}(t) = \left(\frac{\dot{k}(t)}{k(t)} + n\right)K(t).$$

Research Sector

- Individuals can spend time searching for ideas. Let L_A be the number of people employed in R&D.
- New ideas increase the effective units of labor for *all* workers:

$$\dot{A}(t)$$

- Each Researcher generates one new idea at some rate $\bar{\theta}(\ldots)$

some function.

Research Sector (cont'd)

- The stock of ideas evolves according to:

$$\dot{A}(t) = \bar{\theta}(.)\, L_A(t)$$

- ?: what kinds of property should my function $\bar{\theta}(..)$ have?

Arrival Rate of Ideas

Properties of $\bar{\theta}$

$$\bar{\theta}(A(t), L_A(t), \dots)$$

1). existing ideas are inputs in the development of new ideas.

2) Researchers are at risk of duplication.

(duplication
replication of ideas
has no economics
value.

Old Ideas as Inputs

$$\bar{\theta}(A, \ldots) = \theta A^{\varphi} A^{-\gamma} \leftarrow \text{fishing out.}$$

$\uparrow$ standing on the shoulder of giants. $\varphi > 0$

Two forces:

1. Newton's *"standing on the shoulders of giants"* force.
2. *"Fishing out"* effect $\rightarrow$ harder to develop new ideas.

θ is the flow of productivity of researchers.

Old Ideas are Inputs to Produce New Ideas

Net effect is:

$$\bar{\theta}(A, \ldots) = \theta A^{\phi}$$

$\phi = \varphi - \gamma$

measure the importance of "standing on the shoulders" effect relative to the "fishing out" effect.

Duplication of Ideas

$$\bar{\theta}(A, L_A, \ldots) = \theta A^{\phi} \underbrace{L_A^{\lambda-1}}_{\text{negative externality}}$$

$0 < \lambda < 1$ implies $\lambda - 1 < 0$.

Intuition for negative externality:

- only original ideas are economically valuable
- if research is done by multiple individuals, than the probability of duplication is greater.

Aggregate Evolution of Ideas

Adding up these three effects:

$$\begin{aligned}
\dot{A}(t) &= \bar{\theta}\big(A(t), L_A(t)\big) \times L_A(t) \\
&= \theta A(t)^{\phi} L_A(t)^{\lambda-1} L_A(t) \\
&= \theta A(t)^{\phi} L_A(t)^{\lambda}
\end{aligned}$$

Externalities

- Let i index individual workers on the interval $[0, L(t)]$.
- Let $\mathbf{1}_A(i,t)$ denote worker i's choice to become a researcher.
- Then, $L_A(t) = \int_0^{L(t)} \mathbf{1}_A(i,t) di$. labor are indifferent between being worker / researcher.
- Individual's decision (worker vs. researcher):

  $w = \bar{\theta}(A(t), L_A(t)) \cdot P$ — probability of finding a new idea.

  wage

- Individual i takes occupational choice of others as given and ignores effect of *own* choice on the others' productivity:
  - direct effect on today's arrival of ideas (per researcher):

    $$\frac{\partial \bar{\theta}}{\partial L_A(t)} = \theta(\lambda-1) A_t^{\phi} L_A(t)^{\lambda-2} < 0$$

    negative $\lambda \in (0,1)$

    not help you today but be import into new idea tomorrow.

  - indirect effect on the productivity of *all* researchers tomorrow: one more enter $L_A(t)$, $A(t)\uparrow$ tomorrow.

    $$\frac{\partial(A(t+\Delta))}{\partial L_A(t)} = \frac{\partial[A(t) + \Delta A(t)]}{\partial L_A(t)} = \Delta\theta\lambda A^{\phi} L_A(t)^{\lambda-1} > 0.$$

    old ideas no changes.

Externalities

11/11

Society's choice:

$$
\begin{aligned}
\dot{A}(t) &= \bar{\theta}(A(t), L_A(t))\, L_A(t) \\
&= \theta A(t)^{\phi}\, L_A(t)^{\lambda-1}\, L_A(t) \\
&= \underbrace{\theta A(t)^{\phi}\, L_A(t)^{\lambda}}_{\text{internalizes direct / indirect externalities}}
\end{aligned}
$$

trade off: produce today or foster the development of new idea
↑ output tomorrow

Simple Case with Constant Research Effort

Constant fraction s_R of the population works as researchers:

$$L_A = s_R L$$

What is the Rate of Technological Progress?

$$
\begin{aligned}
\dot{A}(t) &= \theta A(t)^{\phi} L_A(t)^{\lambda} \\
&= \theta A(t)^{\phi} \left(s_R L(t)\right)^{\lambda}
\end{aligned}
$$

Dividing by $A(t)$:

$$
\begin{aligned}
\frac{\dot{A}(t)}{A(t)} &= \theta s_R^{\lambda} A(t)^{\phi-1} L(t)^{\lambda} \\
&= \theta s_R^{\lambda} \frac{L(t)^{\lambda}}{A(t)^{1-\phi}}
\end{aligned}
$$

more people generate new idea.

Balanced Growth Path

Along a balanced growth path (i.e., when endogenous variables grow at a constant rate), the presumed growth rate of technology growth is:

$$g_A = \frac{\dot{A}(t)}{A(t)} = \theta \, s_R^{\lambda} \frac{L(t)^{\lambda}}{A(t)^{1-\phi}}$$

Balanced Growth Path

Taking logs: $\ln(g_A) = \log(\theta) + \chi \log(S_R) + \lambda \log(L(t)) - (1-\phi)\log(A(t))$

Differentiating with respect to time t:

$$0 = 0 + 0 + \lambda \underbrace{\frac{\dot{L}(t)}{L(t)}}_{\equiv n.} - (1-\phi)\frac{\dot{A}(t)}{A(t)}.$$

And rearranging:

$$g_A = \frac{\dot{A}(t)}{A(t)} = \frac{\lambda n}{(1-\phi)} \rightarrow \text{balanced growth exists.}$$

Intuition

1. $\frac{\dot{A}(t)}{A(t)}$ is increasing in λ: lambda
   larger λ implies less duplication of ideas (lower *negative* externalities)

2. $\frac{\dot{A}(t)}{A(t)}$ is increasing in ϕ: the
   larger ϕ implies stronger 'standing in the shoulder of giants' effect (higher *positive* externalities of past research) and/or weaker 'fishing-out effect'

3. $\frac{\dot{A}(t)}{A(t)}$ is increasing in n:
   faster population growth implies more growth in the number of researchers

Growth Rate of Income Along Balanced Growth Path

$$\ln(y(t)) = \alpha \ln(k(t)) + (1-\alpha)[\ln(A(t)) + \ln(1-s_R)]$$

$$y(t) = k(t)^{\alpha}\left(A(t)(1-s_R)\right)^{(1-\alpha)}$$

$$\frac{\dot{y}(t)}{y(t)} = \alpha\frac{\dot{k}(t)}{k(t)} + (1-\alpha)\frac{\dot{A}(t)}{A(t)}$$

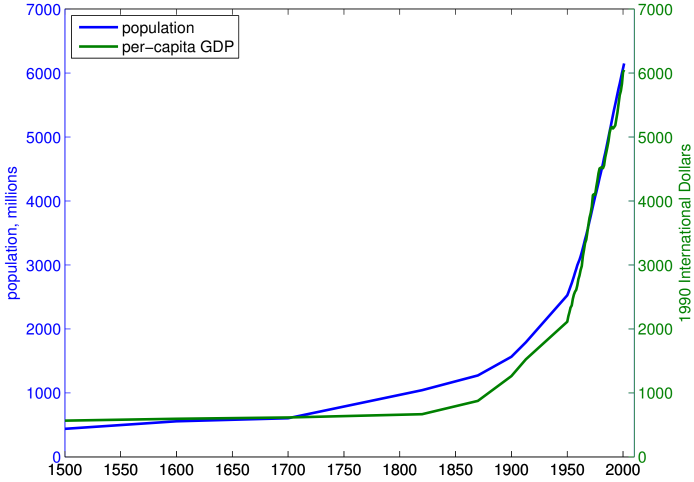
$$= \alpha\frac{\dot{A}(t)}{A(t)} + (1-\alpha)\frac{\dot{A}(t)}{A(t)} = \frac{\dot{A}(t)}{A(t)} = \frac{\lambda n}{(1-\phi)}$$

$$\dot{k}(t) = s_K y(t) - (\delta+n)k(t)).$$

$$\frac{\dot{k}(t)}{k(t)} = s_K\frac{y(t)}{k(t)} - (\delta+n). = s_K\frac{k(t)^{\alpha}(A(t)(1-s_R))^{1-\alpha}.}{k(t)} - (\delta+n). = s_K(1-s_R)^{1-\alpha}\left(\frac{A(t)}{k(t)}\right)^{1-\alpha} - (\delta+n)$$

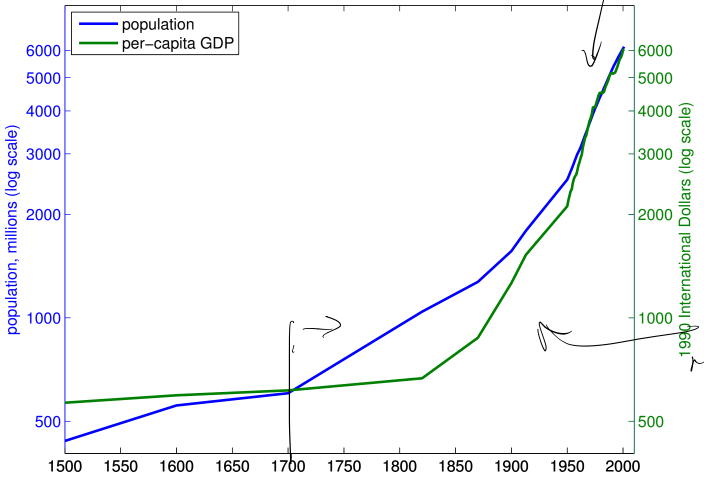
[to be constant, $A(t)$, $k(t)$ should grow in the same rate.

Population and Per-Capita Income Growth: Historical Perspective



Population and Per-Capita Income Growth: Historical Perspective

long run: move proportioned to each other



gdp per capita grow right after the population growth.

Basic Equation of Technological Progress

Off the balanced growth path the rate of technological progress is given by:

$$\frac{\dot{A}(t)}{A(t)} = \theta s_R^{\lambda} \frac{L(t)^{\lambda}}{A(t)^{1-\phi}}$$

balanced growth path: reach some kind of relationship s.t $\frac{L(t)^{\lambda}}{A(t)^{1-\phi}} = g$ (constant).

In the *special case* $1 - \phi = \lambda$ this simplifies to:

$$\frac{\dot{A}(t)}{A(t)} = \theta s_R^{1-\phi} \left(\frac{L(t)}{A(t)} \right)^{1-\phi}$$

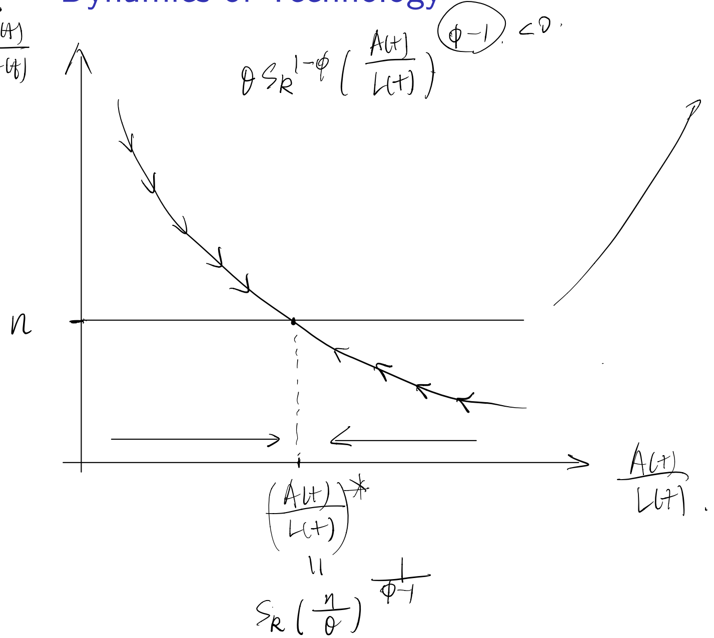
$g_A = \theta f\left(s_R, \frac{L(t)}{A(t)}\right)$

e.g $\uparrow(L(t)), \downarrow(A(t)) \to \uparrow \frac{\dot{A}(t)}{A(t)} \to g.A(t)\uparrow$.

$\to \left(\frac{L(t)}{A(t)}\right)\downarrow$.

Dynamics of Technology

$g_A = \frac{\dot{A}(t)}{A(t)}$



$g_A^* = \frac{\lambda n}{1-\phi}$

$\lambda = 1-\phi$

$g_A^* = n.$

$n = \theta S_R^{1-\phi}\left(\frac{A(t)}{L(t)}\right)^{\phi-1}.$

$\left(\frac{A(t)}{L(t)}\right) = \left(\frac{n}{\theta S_R^{(1-\phi)}}\right)^{\frac{1}{\phi-1}} = \left(\frac{n}{\theta}\right)^{\frac{1}{\phi-1}} S_R.$

only depend by params

time invariant.

Law of Motion for Physical Capital per Person

Physical capital accumulation:

$$\frac{\dot{k}(t)}{k(t)} = s_K \left(\frac{k(t)}{A(t)}\right)^{\alpha-1} (1 - s_R)^{1-\alpha} - (\delta + n)$$

Dynamics of Physical Capital per Person

Dynamics of the Model with Endogenous Technology $(\lambda = 1 - \phi)$

Technological progress:

$$\frac{\dot{A}(t)}{A(t)} = \theta s_R^{1-\phi} \left(\frac{A(t)}{L(t)} \right)^{\phi-1}$$

Physical capital accumulation:

$$\frac{\dot{k}(t)}{k(t)} =$$

Balanced Growth with Endogenous Technology $(\lambda = 1 - \phi)$

Technological progress:

$$= \theta s_R^{1-\phi} \left(\frac{A(t)}{L(t)} \right)^{\phi - 1}$$

Physical capital accumulation:

Balanced Growth with Endogenous Technology ($\lambda = 1 - \phi$)

Technological progress:

Physical capital accumulation:

Balanced Growth with Endogenous Technology ($\lambda = 1 - \phi$)

$$y(t) = k(t)^{\alpha}(A(t)(1 - s_R))^{1-\alpha}$$

$$=$$

$$=$$

$$=$$

Micro-Foundations of the Model with Endogenous Technological Progress

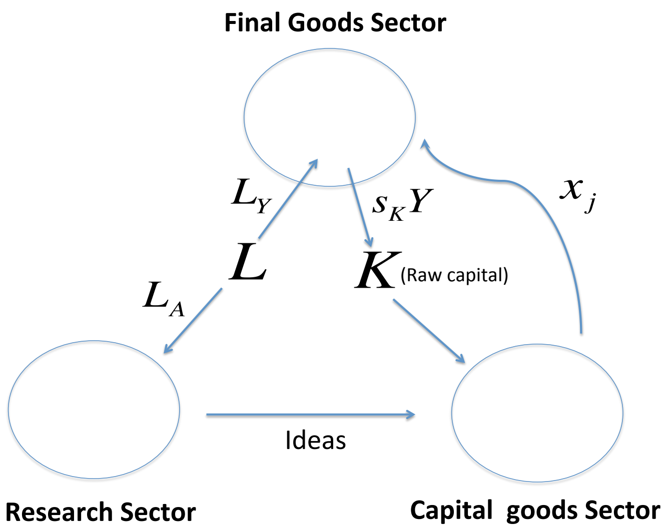
- So far, model "predicts" growth rate of GDP per capita for a given allocation of workers to research (s_R) and production $(1 - s_R)$.
- How does the market allocate workers?

Micro-Foundations of the Model with Endogenous Technological Progress

Three Productive Sectors:

1. Final Good Producers (Competitive)
2. Intermediate Good Producers (Monopolistic Competition)
3. Research Sector (Free entry$\rightarrow$Competitive)

Flow of Factors of Production Across Sectors



Final Good Producer(s)

Aggregate production function

decreasing return to L_Y and each x_j.

A — very large ⇒ x_j be relatively small section.

$$Y = L_Y^{1-\alpha} \sum_{j=0}^{A} x_j^{\alpha}$$

A is the number of different capital goods used in the production of the final-good, x_j is the quantity of the capital good j used in the production of the final good.

Final Good Producer(s)

decreasing marginal product.

use all of them

$$\frac{\partial Y}{\partial L_Y} = (1-\alpha) L_Y^{-\alpha} \int_0^A x^\alpha dj.$$

<0

$$\frac{\partial Y}{\partial x_j} = L_Y^{1-\alpha} \alpha x_j^{\alpha-1} \int_{0,\, \neq j}^{A} x_j^{\alpha} dj$$

<0

perfectly competitive.

Aggregate production function

$$Y = L_Y^{1-\alpha} \int_0^A x_j^\alpha dj$$

more variety.

where A is the *measure* of different capital goods used in the production of the final-good, x_j is the quantity of the capital good j used in the production of the final good.

Final Good Producer(s): Input Demands

Competitive firms choose the labor input L_Y and the quantity of each capital good to maximize profit:

$$\max_{L_Y, x_j} L_Y^{1-\alpha} \int_0^A x_j^\alpha dj - \overset{\text{wage}}{wL_Y} - \overset{\text{cost of purchasing } x_j.}{\int_0^A p_j x_j dj}$$

Final Good Producer(s): Input Demands

Competitive firms choose labor and capital goods to maximize profits, i.e.,

$$\begin{aligned} MPL &= w \\ (1-\alpha)\frac{Y}{L_Y} &= w \end{aligned}$$

and

$$\begin{aligned} MPx_j &= p_j \\ \alpha L_Y^{1-\alpha} x_j^{\alpha-1} &= p_j \end{aligned}$$

$\rightarrow$ if $\uparrow p_j \Rightarrow \downarrow x_j$

i.e., we obtain a demand for labor and capital goods x_j, $j \in [0,1]$.

Capital Good Producers

- Capital goods are produced with "raw" capital only (no labor inputs):

$$x_j = k_j$$

- Monopolistic capital good producers choose the quantity supplied (or the price charged) to maximize profits:

$$\max_{x_j} \pi_j = p_j(x_j) \cdot x_j - r x_j$$

$\uparrow$ idea $\to \uparrow$ people produce idea

- A unit of capital good is produced with a unit of "raw" capital good, with rental cost r (notice that we are ignoring the depreciation cost δ).
- Capital good producers are monopolists because they own patents for the products developed by innovators in the research sector (more details in a moment).

Demand Function for Capital Goods

The (inverse) demand function comes from the optimal input demand decision of competitive producers of final goods

$$p_j(x_j) = \alpha L_Y^{1-\alpha} x_j^{\alpha-1}$$

which is the inverse of the function that gives the quantity demanded as a function of the price,

$$x_j(p_j) = \left(\alpha L_Y^{1-\alpha}/p_j\right)^{\frac{1}{1-\alpha}} = \left(\alpha L_Y^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \cdot p_j^{\frac{-1}{1-\alpha}}$$

The price elasticity equals ... $1/(1-\alpha)$ $\Leftarrow \alpha \to 1$ elasticity $\to \infty$ $\Rightarrow$ perfect competition

$$p_j \uparrow 1\% \Rightarrow x_j \downarrow \tfrac{1}{1-\alpha}\%$$

Optimal Supply of Capital Goods

$$\text{elasticity} = \frac{\partial \ln(P_j(x_j))}{\partial \ln(x_j)} = \frac{\partial P_j(x_j)}{\partial x_j} \cdot \frac{x_j}{P_j}$$

Monopolist supply goods up to the point where marginal revenue equals marginal cost, i.e.,

$$MR_j = MC_j$$

$$\frac{\partial(P(x_j)\cdot x_j)}{\partial x_j} = x_j \cdot \frac{\partial P_j(x_j)}{\partial x_j} + P_j(x_j) = r$$

Multiply and divide leftmost part by p_j and rearrange:

$$p_j = \frac{1}{1 + \underbrace{\frac{\partial P_j(x_j)}{\partial x_j} \cdot \frac{x_j}{P_j(x_j)}}_{\text{inverse elasticity price } = \alpha - 1}} \cdot r$$

$$= \frac{1}{1+(\alpha-1)} \cdot r = \frac{r}{\alpha}$$

? what r plays here ?

Aggregate Quantity of Capital Goods

Aggregating over all intermediate (capital) goods,

$$\int_0^A x_j dj = K,$$

Since they are used in the same amounts ($x_j = x$) this simplifies to:

$$Ax = K$$

Since $x_j = x_i \ \forall i, j$, then $x_j = x$.

$\int_0^A x_j dj = \int_0^A x dj = x \int_0^A 1 dj = Ax$

Aggregate Output

Substituting $x = K/A$ into the production function of the final good

$$
\begin{aligned}
Y &= L_Y^{1-\alpha} A x^{\alpha} = \int_0^A x^{\alpha} dj \\
&= L_Y^{1-\alpha} A \left(\frac{K}{A}\right)^{\alpha} \\
&= (L_Y A)^{1-\alpha} K^{\alpha}
\end{aligned}
$$

Profits of Capital Good Producers

$p_j = \frac{r}{\alpha}$

$$\pi_j = p_j(x_j)x_j - rx_j$$

According to the inverse elasticity rule . . . profits π_j are

$$\begin{aligned} \pi_j &= p_j(x_j)\, x_j - p_j(x_j)\,\alpha\, x_j \\ &= (1-\alpha)\, p_j(x_j)\, x_j . \end{aligned}$$

Since $p_j(x_j) = \alpha L_Y^{1-\alpha} x_j^{\alpha-1}$ (see the inverse demand function four slides ago):

$$\pi_j = (1-\alpha)\alpha\, L_Y^{1-\alpha}\, x_j^{\alpha}$$

Profits of Capital Good Producers

Since all capital good producers are identical, that is, $\pi_j = \pi$, $x_j = x$, and substituting $x = K/A$:

$$\pi = (1-\alpha)\alpha L_Y^{1-\alpha} \underbrace{\left(\frac{K}{A}\right)}_{x_j}^{\alpha}$$

$$= \alpha(1-\alpha)\frac{(AL_Y)^{1-\alpha}K^{\alpha}}{A}$$

$$= \alpha(1-\alpha)\frac{Y}{A}$$

Research Sector

- Individuals can devote time to developing new products. New varieties x_j are added to the current measure A of goods.
- After developing a new capital good, they are granted a (perpetual) patent to produce this new variety.
- Innovator can sell this patent at price P_A.

Research Sector

- Recall that new ideas are produced with the technology $\dot{A} = \bar{\theta} L_A$.
- Researchers will enter until gains from innovating equal opportunity cost:

$$\bar{\theta} P_A - w$$

- With free entry of innovators:

think of prob of success

$\bar{\theta}$ decreasing in L_A.

$$P_A \bar{\theta}(L_A, A) = w$$

- How is the price of an innovation determined?

Price of an Idea, P_A

An idea is like a stock. We value it because it give us some dividends, π, and because we might expect its price to increase in the future,

$$\underbrace{\pi}_{\text{competition}} + \underbrace{\dot{P}_A}_{\text{capital gain.}}$$

If there is competition between resources to fund ideas and capital (with bonds), then different investment strategies must provide the same return:

$$\underbrace{rP_A}_{\text{return on investment}} = \pi + \dot{P}_A$$

This is what economists call a *no arbitrage* condition.

Price of an Idea, P_A

- Divide the previous equation by P_A to obtain:

$$r = \frac{\pi}{P_A} + \frac{\dot{P}_A}{P_A}$$

- Along a balanced growth path, r is constant.
- P_A and $\pi = \alpha(1-\alpha)Y/A$ ought to grow at the same rate.
- Recall that $\frac{\dot{A}}{A} = \frac{\dot{y}}{y} = n$ and $\frac{\dot{Y}}{Y} = 2n$ when $\lambda = 1 - \phi$.
- If π grows at rate n, then balanced growth requires that P_A grows at rate n too;

$$r = \frac{\pi}{P_A} + n$$

$$\Rightarrow P_A = \frac{\pi}{r-n}$$

How is s_R determined?

Innovators must be indifferent between drawing ideas or working to produce final goods,

$$\bar{\theta} P_A = w$$

must be indifferent between being researcher / worker.

Substituting $P_A = \frac{\pi}{r-n}$ and $w = \frac{(1-\alpha)Y}{L_Y}$,

$$\bar{\theta}\frac{\pi}{r-n} = \frac{(1-\alpha)Y}{L_Y}$$

if indifferent → x

game theory

all do → x.

random in research / production.

Substituting $\pi = \alpha(1-\alpha)Y/A$

$$\bar{\theta}\frac{\alpha(1-\alpha)}{(r-n)}\frac{Y}{A} = \frac{(1-\alpha)Y}{L_Y}.$$

or

$$\frac{\alpha}{r-n}\cdot\frac{\bar{\theta}}{A} = \frac{1}{L_Y}$$

How is s_R determined?

Using that in a balance growth path $\dot{A} = \bar{\theta} L_A$ or $\bar{\theta} = \dot{A}/L_A$

$$\frac{\alpha}{r-n}\frac{\dot{A}}{A}\frac{1}{L_A} = \frac{1}{L_Y}$$
$$\frac{\alpha}{r-n} g_A \frac{1}{L_A} = \frac{1}{L_Y}.$$

Using $s_R = L_A/L$ and $1 - s_R = L_Y/L$

$$\frac{\alpha}{r-n} g_A \frac{1}{s_R} = \frac{1}{1-s_R}$$

or

$$s_R = \frac{1}{1 + \frac{r-n}{\alpha g_A}}.$$

How is s_R determined?

When $\lambda = 1 - \phi$ the optimal share of researchers is given by:

$$s_R = \frac{1}{1 + \frac{r-n}{\alpha n}}. \quad = \frac{1}{1 + \frac{r}{\alpha n} - \frac{1}{\alpha}}$$

$n \uparrow \rightarrow s_R \uparrow$.