Neoclassical Growth Model

Neoclassical Model

(household)

Preferences

$\beta \in (0,1)$

discount factor

exponential (assume)

$$\sum_{t=0}^{+\infty} \beta^t u(c_t)$$

consumption

value get from consumption.

try to maximize.

Neoclassical Model

Endowments

- initial capital stock: K_0
- household have one unit of labor each period;
  since leisure is not valued, labor is supplied inelastically

Neoclassical Model

Technology

- Production function:

$$Y = f(K, L) = K^{\alpha} L^{1-\alpha}$$

- Resource constraint:

$$Y_t = C_t + I_t$$

- Law of motion for capital:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

Competitive Equilibrium

Markets

- For output at time t. Price $\underline{P_t}$

  Price for output at time 0: $p_0 = 1$

- For labor at time t. Price $w_t P_t$ ← relative to output in t_0.

  w_t ↓ in terms of output

- For capital services at time t. Price $r_t P_t$.

  P_t expressed in p_0 as unit.

Competitive Equilibrium

rental fee.

technology —

Firm

household own
k capital.

- Household own the capital stock and firms rent the capital services period by period.
  - pay rental rate, r_t
  - return $(1-\delta) K_t$ back to households at the end of period

- Households own the firms) ← constant return to scale.
- Firm (act as monopolist) takes prices P_t, w_t, r_t, as given; doesn't own capital.

assume one firm

Competitive Equilibrium

Firm's Problem

$$\Pi = \max_{\{y_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t [y_t - w_t l_t - r_t k_t].$$

$$y_t = f(k_t, l_t)$$

Competitive Equilibrium

Household's Problem

$$\max_{\{c_t, i_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

subject to $k_{t+1} = (1-\delta) k_t + i_t$

let $l_t = 1$

$$\sum_{t=0}^{\infty} p_t (c_t + i_t) \le \sum_{t=0}^{\infty} p_t (w_t + r_t k_t) + \pi_t.$$

$l_t = 1$

Competitive Equilibrium

Definition

A competitive equilibrium is given by an allocation

$\{y_t, c_t, k_t, l_t\}_{t=0}^{\infty}$ and prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ s.t.

1. y_t, k_t, l_t solve the firm's optimization problem for given prices

2. c_t, i_t, k_{t+1} solve the household problem for given prices

3. market clear

   (a) $y_t = c_t + i_t$

   (b) $l_t = 1$. ← assuming household have 1 unit of labor.

Competitive Equilibrium

Firm's Problem: FOC

$$\max_{k_t, c_t} p_t f(k_t, l_t) - p_t r_t k_t - p_t w_t l_t .$$

p_t cancel out.

F.O.C

$$f_k(k_t, l_t) = r_t \qquad (MPK = r)$$

$$f_l(k_t, l_t) = w_t \qquad (MPL = w)$$

Competitive Equilibrium

Household's Problem: Lagrangean → find the lower bound.

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

(zero profit)

subject to: $\sum_{t=0}^{\infty} p_t (c_t + i_t) = \sum_{t=0}^{\infty} p_t [w_t + r_t k_t]$. $[\lambda]$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left\{ \sum_{t=0}^{\infty} p_t [w_t + r_t k_t] - p_t (c_t + i_t) \right\}$$

$l_t = 1$

Competitive Equilibrium

Household's Problem: FOC

$$\beta^t \underbrace{u'(c_t)}_{\frac{\partial u(c_t)}{\partial c_t}} = \lambda p_t .$$

$$\frac{\cancel{\beta^t}\, u'(c_t)}{\beta^{t+1}\, u'(c_{t+1})} = \frac{\lambda p_t}{\lambda p_{t+1}} = \frac{p_t}{p_{t+1}}$$

$$\rightarrow \quad u'(c_t) = \beta \frac{p_t}{p_{t+1}} u'(c_{t+1}) \qquad \text{Euler equation.}$$

No arbitrage: $\dfrac{p_t}{p_{t+1}} = r_{t+1} + (1-\delta)$

Competitive Equilibrium

Combining FOCs

$$u'(c_t) = \beta[r_{t+1} + 1 - \delta]\, u'(c_{t+1})$$

$$f_k(k_{t+1}, 1) = r_{t+1}$$

$$u'(c_t) = \beta[f_k(k_{t+1}, 1) + 1 - \delta]\, u'(c_{t+1})$$

Resource constraint: $k_{t+1} = \underbrace{f(k_t, 1) - c_t}_{i_t} + (1-\delta)k_t$

$\rightarrow$ first order difference equation in 2 unknowns: k_{t+1}, c_t

Competitive Equilibrium

Steady State

2 initial / boundary condition

(1) k_0

(2) ?

Assuming a steady state exists, then we have

$$1 = \beta [f_k(k_{ss}, 1) + 1 - \delta]$$

$$C_{ss} = f(k_{ss}, 1) - \underbrace{\delta k_{ss}}_{= \text{investment}}$$

Sequential Problem

$u'(c_t) = u'(c_{t+1})[f_k(k_{t+1}, l) + 1-\delta]$ transform output today → tomorrow.

bace on each other.

$y_t = C_t + I_t$

$k_{t+1} = (1-\delta)k_t + I_t$

(unlimited years after.)

Alternative: recursive formulation ← one period before / after is not different

bellman equation

$$\underset{k_t}{V(k)} = \max_{k'} u(c) + \beta \underset{k_{t+1}}{V(k')}$$

$$C = f(\underset{\text{output}}{k, l}) - i = f(k, l) - (k' - (1-\delta)k)$$

arbitary "guess" for the value of any k'

$$V(k) = \max_{k'} u[f(k, l) - (k' - (1-\delta)k)] + \beta V(k')$$

utility get consuming some capital + transferred to tomorrow.

find V to max Utility

describe the state of economy

Dynamic Programing k: state variable ← given, can't change.

k': control variable. ← can choose.

V: can used for any k.

$$V(k') = \max_{k''} u[\underbrace{f(k', l) - (k'' - (1-\delta)k')}_{u(c')}] + \beta V(k'')$$

Method: Value Function Iteration

Assume in the last period: $V_1(k) = \max_{k'} u(c) + \beta \underbrace{V(k')}_{0}$ → no investment

$$= \max_{k'} u(k^\alpha l^{1-\alpha} - (k' - (1-\delta)k)) + \beta 0$$

$$= u(k^\alpha l^{1-\alpha} + (1-\delta)k)$$

↳ negative investment

utility of all output and captial for this period.

Assume in the second last period.

$$V_2(k) = \max_{k'} u(c) + \beta V_1(k')$$

$$= u(k^\alpha l^{1-\alpha} - (k' - (1-\delta)k)) + \beta V_1(k')$$

trade off $k' \uparrow \Rightarrow u_k(c) \downarrow, \beta V_1(k) \uparrow$

⋮

$$V_T(k) = \max_{k'} u(c) + \beta V_{T-1}(k')$$

with $c = k^\alpha l^{1-\alpha} - (k' - (1-\delta)k)$

⋮

until : $V_T(k) = V_{T-1}(k)$. $\equiv V(\ )$

For spreadsheet

Params / Functional Forms

$f(k, l) = \alpha k^{\alpha-1} l^{1-\alpha}$

$u(c) = \ln c$

$\alpha = 1/3$

$\delta = 0.06$

$\beta = 0.96$.

$\varphi = 0.25$

ε

↳ $|V_t(\ ) - V_{t+1}(\ )| < \varepsilon$ threshold

Grid for state variables.

$\beta [\alpha k_{ss}^{\alpha-1} l^{1-\alpha} + (1-\delta)] = 1.$

$$k_{ss} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha l^{1-\alpha}} \right)^{\frac{1}{\alpha-1}}$$

$\underline{k} = k_{ss}(1-\varphi)$

$\bar{k} = k_{ss}(1+\varphi)$

Blackwell Sufficient Condition

① monotinicity

② discounting