L28 04.03

§13 Integration.

Idea: - Area of piecewise constant approximation is close to $\int_a^b f$ and we can calculate it as a sum of area of rectangle.

- $\inf\{\text{upper sums}\} = \sup\{\text{lower sums}\} = \int_a^b f$.
- if f is not piecewise constant, we will not get a upper sum s.t. $\int_a^b f$ = upper sum.

Def 13.1 - Let $a < b$, a **partition** of the interval $[a,b]$ is a finite collection of points in $[a,b]$, within $t_0 \to t_n$ s.t. $t_0 = a < t_1 < t_2 \cdots < t_{n-1} < t_n = b$.

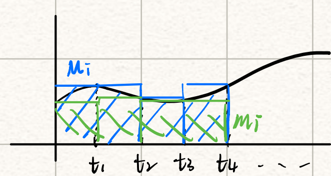
- A **step function**, $S: [a,b] \to \mathbb{R}$ is a function s.t. $\exists$ a partition $\{t_j\}_{j=0}^{n}$ s.t. for each $j \in \{0,\dots,n-1\}$, S is constant on the interval $[t_j, t_{j+1})$.

Def 13.2 Let $f: [a,b] \to \mathbb{R}$ be bounded. Let $P = \{t_0 \dots t_n\}$ be a partition of closed interval $[a,b]$.

Let $m_i = \inf\{f(x) \mid t_{i-1} \le x \le t_i\}$, $M_i = \sup\{f(x) \mid t_{i-1} \le x \le t_i\}$, $i = 1 \dots n$.

We define the **lower sum** of f for P as $L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$

We define the **upper sum** of f for P as $U(f,P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1})$



Fact: $\forall$ partition P, we have $L(f,P) \le U(f,P)$

Proof: Let P be a partition, $P = \{t_0 \dots t_n\}$.

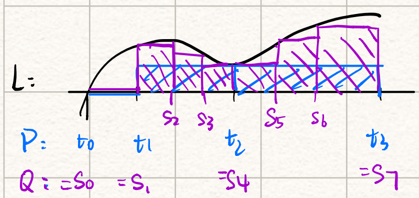
Let $m_i = \inf\{f \mid t_{i-1} \le x \le t_i\}$ $M_i = \sup\{f \mid t_{i-1} \le x \le t_i\}$.

Clearly, $m_i \le M_i$

So, $L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}) \le \sum_{i=1}^{n} M_i (t_i - t_{i-1}) = U(f,P)$

L30 04.08

Lemma 13.3 Let P, Q be partitions of $[a,b]$ s.t. $P \subset Q$. Then $L(f,P) \leq L(f,Q)$, $U(f,P) \geq U(f,Q)$



Proof: Proof of $L(f,P) \leq L(f,Q)$:

Case (i) $Q = P \cup \{t_*\}$ (one extra point)

So, $P = \{t_0 \dots t_n\}$ $\quad Q = \{t_0 \dots t_k, t_*, t_{k+1}, \dots t_n\}$.

Define: $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. $\quad M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$

$m' = \inf\{f(x) \mid t_k \leq x \leq t_*\}$ $\quad m'' = \inf\{f(x) \mid t_* \leq x \leq t_{k+1}\}$.

As $[t_k, t_*]$, $[t_*, t_{k+1}] \subseteq [t_k, t_{k+1}]$

So, $m', m'' \geq m_{k+1}$

$$\begin{aligned}
\text{Now, } L(f,P) &= \sum_{i=1}^{n} m_i (t_i - t_{i-1}) = \sum_{i=1}^{k} m_i (t_i - t_{i-1}) + m_{k+1}(t_{k+1} - t_k) + \sum_{i=k+2}^{n} m_i (t_i - t_{i-1}) \\
&= \sum_{i=1}^{k} m_i (t_i - t_{i-1}) + m_{k+1}(t_{k+1} - t_*) + m_{k+1}(t_* - k) + \sum_{i=k+2}^{n} m_i (t_i - t_{i-1}) \\
&\leq \sum_{i=1}^{k} m_i (t_i - t_{i-1}) + m'(t_{k+1} - t_*) + m''(t_* - k) + \sum_{i=k+2}^{n} m_i (t_i - t_{i-1}) \\
&= L(f,Q).
\end{aligned}$$

Case (ii) General Case $P \subset Q$

Note: $Q \setminus P$ is a finite set of points, $Q \setminus P = \{s_1 \dots s_n\}$

Define a sequence of partitions $P_0 = P \subset P_1 = P_0 \cup \{s_1\}$

$\subset P_2 = P_1 \cup \{s_2\}$

$\subset \cdots \subset P_n = P_{n-1} \cup \{s_n\} = Q$.

By part one, we have $L(f,P_0) \leq L(f,P_1) \leq \dots \leq L(f,P_n) = L(f,Q)$ as required.

Proof of $U(f,Q) \leq U(f,L)$ is exercise, very similar.

Thm 13.4: Let P_1, P_2 be partition of $[a,b]$. Let $f: [a,b] \to \mathbb{R}$ be bdd. Then $L(f, P_1) \le U(f, P_2)$

Lower Sum always below curve, Upper Sum always above curve

Proof: Let $P = P_1 \cup P_2$ Then P is a partition s.t. $P_1 \subset P$, $P_2 \subset P$

Then by Lemma 13.3, We have that $L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_1)$

& $L(f, P_2) \le L(f, P) \le U(f, P) \le U(f, P_2)$

So, $L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2)$ as required ∎

Ex. 1. Let $c \in \mathbb{R}$, Let $f: [a,b] \to \mathbb{R}$ be $f(x) = c$

Claim: $\forall$ partition P, $L(f, P) = U(f, P) = c(b-a)$

Proof: Let $P = \{t_0 \dots t_n\}$ be a partition

Then define $m_i = \inf\{f(x) \mid t_{i-1} \le x \le t_i\}$, $M_i = \sup\{f(x) \mid t_{i-1} \le x \le t_i\}$.

As f is a constant, we have $m_i = c = M_i$

Therefore, $L(f, P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}) = c \sum_{i=1}^{n} (t_i - t_{i-1}) = c(t_n - t_0) = c(b-a)$

Similarly, $U(f, P) = c(b-a)$ ∎

2. Let $f(x) = x$ on $[0,1]$ and let $P = \{0, 1/4, 1/2, 3/4, 1\} = \{t_0, t_1, t_2, t_3, t_4\}$.

Now, $m_1 = 0$, $m_2 = 1/4$, $m_3 = 1/2$, $m_4 = 3/4$

$M_1 = 1/4$, $M_2 = 1/2$, $M_3 = 3/4$, $M_4 = 1$

Then, $L(f, P) = \sum_{i=1}^{4} m_i (t_i - t_{i-1}) = \frac{1}{4}(0 + 1/4 + 1/2 + 3/4) = 3/8$

$U(f, P) = \sum_{i=1}^{4} M_i (t_i - t_{i-1}) = 1/4 (1/4 + 1/2 + 3/4 + 1) = 5/8$.

Note: $\int_0^1 x = 1/2 \in [L(f, P) + U(f, P)]$

3. Let $f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \notin \mathbb{Q} \end{cases}$

Let $P = \{t_0 \dots t_n\}$ be a partition

Now for each $i = 1 \dots n$, $\exists q_i \in [t_{i-1}, t_i] \cap \mathbb{Q}$, $\exists r_i \in [t_{i-1}, t_i] \setminus \mathbb{Q}$

So, $m_i < f(r_i) = 0$ $\quad M_i > f(q_i) = 1$

In fact, $m_i = 0$, $M_i = 1$

So, $L(f, P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}) = 0$

$U(f, P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}) = b - a$

So, for any partition, $L(f, P) = 0 < b - a = U(f, P)$

Problem: we want to approximate $\int f$ with $L(f, P)$, $U(f, P)$

$\Rightarrow$ restrict the class of functions that we integrate on.

L31 04.10

$\forall$ partition P, we have $L(f, P) \le U(f, P)$

Let Q be a partition, then $\forall P$, $L(f, P) \le U(f, Q)$

So, $U(f, Q)$ is a UB for $\{L(f, P) \mid P \text{ is a partition}\}$

So, $\sup \{L(f, P) \mid P\} \le U(f, Q)$.

So, $\sup \{L(f, P)\} \le \inf \{U(f, Q)\}$

Idea is to get $L(f, P)$, $U(f, P)$ close to $\int_a^b f$.

Def 13.5. Let $f : [a, b] \to \mathbb{R}$ be bdd.

We say f is integrable on $[a, b]$ if $\sup \{L(f, P) \mid P \text{ partition of } [a, b]\}$

$= \inf \{U(f, P) \mid P \text{ partition of } [a, b]\}$.

In this case, we define integral of f on $[a, b]$ to be this number and write it $\int_a^b f$

When $f \ge 0$ on $[a, b]$, we call $\int_a^b f$ the area under the graph of f.

Notes: $\forall P$, $L(f,P) \le \int_a^b f \le U(f,P)$. $\int_a^b f$ is the unique number with this property.

Ex 1. If $f(x) = c$ $\forall x \in [a,b]$, then $\forall P$ $L(f,P) = U(f,P) = c(b-a)$.

So, f is int'ble & $\int_a^b c = c(b-a)$

2. $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$

Then $\forall P$ $L(g,P) = 0$ $\quad U(g,P) = b-a$

So, $\sup\{L(g,P)\} = 0 < b-a = \inf\{U(g,P)\}$

Hence g is not int'ble.

Thm 13.6. Let $f: [a,b] \to \mathbb{R}$ be bdd. Then f is int'ble iff $\forall \varepsilon > 0$, $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Proof. Suppose f is int'ble. Then $\sup\{L(f,P)\} = \inf\{U(f,P)\}$

Let $\varepsilon > 0$, By property of sup $\exists P_1$ s.t. $\int_a^b f - \varepsilon/2 = \sup\{L(f,P)\} - \varepsilon/2 < L(f,P_1)$

Similarly, $\exists P_2$ s.t. $U(f,P_2) < \inf\{U(f,P)\} + \varepsilon/2 = \int_a^b + \frac{\varepsilon}{2}$.

Let $P = P_1 \cup P_2$, then $\int_a^b f - \varepsilon/2 < L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2) < \int_a^b + \varepsilon/2$.

So, $U(f,P) - L(f,P) < \varepsilon$.

Suppose $\forall \varepsilon > 0$, $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Recall $\sup\{L(f,P)\} \le \inf\{U(f,P)\}$, it's enought to show $\sup\{L(f,P)\} \ge \inf\{U(f,P)\}$.

Let $\varepsilon > 0$, $\exists P$ s.t. $U(f,P) < L(f,P) + \varepsilon \le \sup\{L(f,P)\} + \varepsilon$.

So, $\inf\{U(f,P)\} \le \sup\{L(f,P)\} + \varepsilon$

As $\varepsilon > 0$ is arbitary, this gives $\inf\{U(f,P)\} \le \sup\{L(f,P)\}$ as required $\blacksquare$

Ex 1. Let $f: [0,2] \to \mathbb{R}$ be $f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$

Let $P = \{t_0 \dots t_3\}$ be $t_0 = 0$, $t_1 = 1 - 1/n$, $t_2 = 1 + 1/n$, $t_3 = 2$ $\forall n \in \mathbb{N}$.

Let $m_i = \inf\{f(x) \mid t_{i-1} \le x \le t_i\}$, $M_i = \sup\{f(x) \mid t_{i-1} \le x \le t_i\}$.

So, $m_i = 0 \; \forall i = 1 \cdots 3$

$M_i = \begin{cases} 0 & i = 1,3 \\ 1 & i = 2 \end{cases}$

Then $L(f,P) = \sum_{i=1}^{3} m_i(t_i - t_{i-1}) = 0$

$U(f,P) = \sum_{i=1}^{3} M_i(t_i - t_{i-1}) = 1(1 + 1/n - 1 + 1/n) = 2/n$.

So as all $L(f,P) = 0$ and $U(f,P) > 0$, we have $0 = \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le 2/n$ $n \in \mathbb{N}$

Hence $\inf\{U(f,P)\} = 0$, so f is int'ble & $\int_a^b f = 0$

2. Let $g(x) = x$ on $[0,b]$.

Let $n \in \mathbb{N}$ Define P by $t_0 = 0$, $t_i = i\,b/n$ for $i = 1 \dots n$

Let $m_i = \inf\{g(x) \mid t_{i-1} \le x \le t_i\} = t_{i-1} = (i-1)\frac{b}{n}$

$M_i = \sup\{g(x) \mid t_{i-1} \le x \le t_i\} = t_i = i\frac{b}{n}$

Then $L(g,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} (i-1)\frac{b}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2}\sum_{j=1}^{n-1} j = \frac{(n-1)n}{2}\frac{b^2}{n^2} = \frac{b^2}{2}\frac{(n-1)}{n}$

$U(g,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = \sum_{i=1}^{n} i\left(\frac{b}{n}\right) \cdot \frac{b}{n} = \frac{b^2}{n^2}\sum_{i=1}^{n} i = \frac{b^2}{n^2}\frac{n(n+1)}{2} = \frac{b^2}{2}\frac{n+1}{n}$

So, $U(g,P) - L(g,P) = \frac{b^2}{2}\left(\frac{n+1}{n} - \frac{n-1}{n}\right) = \frac{b^2}{2} \cdot \frac{2}{n}$

Hence, $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{b^2}{2} \cdot \frac{2}{n} < \varepsilon$ So, $U(g,P) - L(g,P) < \varepsilon$.

So, g is int'ble

Note: $\frac{b^2}{2}$ is the unique number s.t. $\frac{b^2}{2} \cdot \frac{n-1}{n} \le \frac{b^2}{2} \le \frac{b^2}{2}\frac{n+1}{n}$

Hence, $\int_0^b g = \frac{b^2}{2}$

Exercise on $\int_a^b x$, $\int_a^b x^2$ $\leftarrow$ Hint $\sum_{j=1}^{k} j^2 = \frac{1}{6}k(k+1)(2k+1)$

L32 04.13

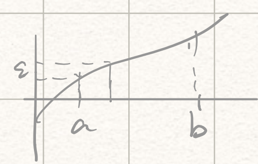
Notation : $\int_a^b f = \int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \cdots$

Thm 13.7 : Let $f:[a,b]\to\mathbb{R}$ be cts, then f is int'ble

Note: f int'ble $\not\Rightarrow$ f cts   Ex. $f(x) = \begin{cases} 1 & , x>0 \\ -1 & , x\le 0 \end{cases}$ is int'ble on $[-1,1]$, but not cts.

Rough Draft :

Goal: show that $\forall \varepsilon>0$, $\exists P$ s.t. $U(f,P)-L(f,P)<\varepsilon$.



1. f is bdd by EVT

2. Suppose $M_i - m_i < \frac{\varepsilon}{b-a}$ $\forall i$

$$\begin{aligned} \text{Then } U(f,P)-L(f,P) &= \sum_{i=1}^{n} M_i(t_i-t_{i-1}) - \sum_{i=1}^{n} m_i(t_i-t_{i-1}) \\ &= \sum_{i=1}^{n}(M_i-m_i)(t_i-t_{i-1}) \\ &< \frac{\varepsilon}{b-a}\sum_{i=1}^{n}(t_i-t_{i-1}) = \frac{\varepsilon}{b-a}(t_n-t_0) \\ &= \frac{\varepsilon}{(b-a)}(b-a) = \varepsilon \end{aligned}$$

3. Need: choose P s.t. $\sup\{f(x) \mid t_{i-1}\le x\le t_i\} - \inf\{f(x)\mid t_{i-1}\le x \le t_i\} < \frac{\varepsilon}{b-a}$

Recall cts at x: $\exists \delta>0$ (depend on ε, x) s.t. if $|y-x|<\delta$,

then $|f(y)-f(x)| < \frac{\varepsilon}{b-a}$

Need: δ independent of x.

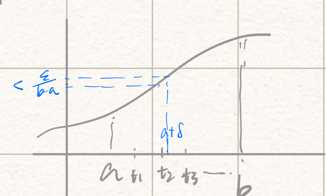
Require: uniform continuity.

Recall: Let $f:[a,b]\to\mathbb{R}$ be cts on $[a,b]$, then f is uniformly cts.

So, $\exists\delta>0$ (depend on ε) s.t. if $|y-x|<\delta$, then $|f(y)-f(x)|<\frac{\varepsilon}{b-a}$

4. Choose a uniform partition s.t. $t_i - t_{i-1} < \delta_i$ i.e. $t_i = \frac{i}{n}(b-a)$ with $n\in\mathbb{N}$

s.t. $\frac{b-a}{n} < \delta$



Let $\varepsilon>0$, $\exists\delta$ Let $n\in\mathbb{N}$ s.t. $\frac{b-a}{n}<\delta$.

Let $t_i = i\,\frac{b-a}{n}$.

Proof: By EVT, f is bdd

Let $\varepsilon > 0$, by a thm from §8, we know f is uniformly cts

So $\exists \delta > 0$ s.t. if $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Let $n \in \mathbb{N}$ be s.t. $\frac{b-a}{n} < \delta$.

Define a uniform partition of $[a,b]$ by setting $t_i = i\frac{b-a}{n}$ for each $i = 0 \cdots n$

So, $t_0 = a < t_1 < t_2 < \cdots < t_n = b$.

Then, $t_i - t_{i-1} = \frac{b-a}{n} < \delta \quad \forall i = 1 \cdots n$

Let $m_i = \inf\{f(x) \mid t_{i-1} \le x \le t_i\}$, $M_i = \sup\{f(x) \mid t_{i-1} \le x \le t_i\}$.

As f is cts on each $[t_{i-1}, t_i]$, there exist (by EVT), $\underline{x_i}, \overline{x_i} \in [t_{i-1}, t_i]$

s.t. $f(\underline{x_i}) \le f(x) \le f(\overline{x_i}) \quad \forall x \in [t_{i-1}, t_i]$

So, $M_i = f(\overline{x_i})$, $m_i = f(\underline{x_i})$

As $\underline{x_i}, \overline{x_i} \in [t_{i-1}, t_i]$, we have $|\underline{x_i} - \overline{x_i}| \le t_i - t_{i-1} = \frac{b-a}{n} < \delta$

So, by uniform cty, $M_i - m_i = f(\overline{x_i}) - f(\underline{x_i}) < \frac{\varepsilon}{b-a}$

$$\begin{aligned} \text{So, } U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) - \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1}) < \frac{\varepsilon}{b-a}\sum_{i=1}^{n}(t_i - t_{i-1}) \\ &= \frac{\varepsilon}{b-a}(t_n - t_0) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon \end{aligned}$$

Hence, f is int'ble. ☒

Note: All we know is int'ble, not what integral is

In general, this is the best we can do.

If f is actually a derivative, we can say more.

See §14

L33 04.15

Thm 13.8: Let $a<c<b$. Then f is int'ble on $[a,b]$ iff f is int'ble on $[a,c]$ & $[c,b]$

Then, $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof: (i) Suppose f is int'ble on $[a,b]$

Let $\varepsilon>0$, then $\exists P$ of $[a,b]$ s.t. $U(f,P)-L(f,P)<\varepsilon$.

Let $P=\{t_0 \dots t_n\}$

If $c\notin P$, consider $\bar{P}=P\cup\{c\}$ & relabeled so that $\bar{P}=\{t_0,\dots,t_n\}$, $c=t_j$

Then $P_1=\{t_0 \dots t_j\}$ is a partition of $[a,c]$

$P_2=\{t_j,\dots,t_n\}$ is a partition of $[c,b]$

By definition, $L(f,\bar{P})=L(f,P_1)+L(f,P_2)$

& similarly, $U(f,\bar{P})=U(f,P_1)+U(f,P_2)$

As $U(f,\bar{P})-L(f,\bar{P})\le U(S,P)-L(S,P)<\varepsilon$

We have $U(S,P_1)+U(S,P_2)-L(S,P_1)-L(S,P_2)<\varepsilon$

i.e. $U(f,P_1)-L(f,P_1)<\varepsilon$ & $U(f,P_2)-L(f,P_2)<\varepsilon$

as $U(f,P_1)-L(f,P_1)\ge 0$ & $U(f,P_2)-L(f,P_2)\ge 0$.

Hence, we have partitions P_1, P_2 of $[a,c]$, $[c,b]$

s.t. $U(f,P_1)-L(f,P_1)<\varepsilon$ So, f int'ble on $[a,c]$,

$U(f,P_2)-L(f,P_2)<\varepsilon$ So, f int'ble on $[c,b]$

Moreover, $L(f,P_1)\le\int_a^c f\le U(f,P_1)$

$L(f,P_2)\le\int_c^b f\le U(f,P_2)$

So, $L(f,P)\le L(f,\bar{P})\le L(f,P_1)+L(f,P_2)\le\int_a^c f+\int_c^b f\le U(f,P_1)+U(f,P_2)\le U(f,\bar{P})\le U(f,P)$

Hence, $L(f,P)\le\int_a^c f+\int_c^b f\le U(f,P)$

As this is true $\forall P$, we have $\int_a^c f+\int_c^b f=\int_a^b f$.

(ii) Suppose that f is int'ble on $[a,c]$ & $[c,b]$

Let $\varepsilon > 0$, then we get partitions P_1 of $[a,c]$ & P_2 of $[c,b]$

s.t. $U(f,P_1) - L(f,P_1) < \varepsilon/2$, $U(f,P_2) - L(f,P_2) < \varepsilon/2$.

Defining $P = P_1 \cup P_2$, a partition of $[a,b]$, we have

$$U(f,P) - L(f,P) = U(f+P_1) + U(f+P_2) - L(f,P_1) - L(f,P_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

So, f is int'ble on $[a,b]$.

Also, $L(f,P) \leq \int_a^b f \leq U(f,P)$

As we know f is int'ble on $[a,b]$, $[a,c]$, $[c,b]$, we can repeat the previous argument to get $\int_a^b f = \int_a^c f + \int_c^b f$ ∎

Thm 13.9 i) Let f, g be int'ble on $[a,b]$, then $f+g$ is int'ble and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

ii) Let f be int'ble on $[a,b]$, then cf is int'ble and $\int_a^b cf = c\int_a^b f$

Proof (i): Let P be any partition of $[a,b]$, write $P = \{t_0 \dots t_n\}$

Define, for $i = 1 \cdots n$, $m_i = \inf\{(f+g)x \mid t_{i-1} \leq x \leq t_i\}$

$m_i' = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$, $m_i'' = \inf\{g(x) \mid t_{i-1} \leq x \leq t_i\}$.

Note, $\forall x \in [t_{i-1}, t_i]$, $(f+g)(x) = f(x) + g(x) \geq m_i' + m_i''$

So, $m_i' + m_i''$ is a lower bound for $(f+g)(x)$ on $[t_{i-1}, t_i]$

Hence, $m_i \geq m_i'' + m_i'$

Now, let $M_i = \sup\{(f+g)(x) \mid t_{i-1} \leq x \leq t_i\}$

$M_i' = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$, $M_i'' = \sup\{g(x) \mid t_{i-1} \leq x \leq t_i\}$

For any $x \in [t_{i-1}, t_i]$, we have $(f+g)(x) = f(x) + g(x) \leq M_i' + M_i''$

So, $M_i \leq M_i' + M_i''$

So, $\forall P$, we have $L(f,P) + L(g,P) = \sum_{i=1}^n m_i'(t_i - t_{i-1}) + \sum_{i=1}^n m_i''(t_i - t_{i-1})$

$= \sum_{i=1}^n (m_i' + m_i'')(t_i - t_{i-1})$

$$\leq \sum_{i=1}^{n} M_i (t_i - t_{i-1}) = L(f+g, t)$$

So, $L(f,P) + L(g,P) \leq L(f+g, P) \leq U(f+g, P) \leq U(f,P) + U(g,P) \quad \forall P \quad (*)$

As f is int'ble, $\exists P'$ s.t. $U(f,P') - L(f,P') < \varepsilon/2$

As g is int'ble, $\exists P'$ s.t. $U(g,P') - L(g,P') < \varepsilon/2$

Let $P = P' \cup P''$.

Then by $(*)$, $U(f+g, P) - L(f+g, P) \leq U(f,P) + U(g,P) - L(f,P) - U(f,P)$

$$\leq U(f,P') - L(f,P') + U(g,P'') - L(f,P'')$$

$$< \varepsilon/2 + \varepsilon/2 = P$$

Moreover, we can put $\int_a^b (f+g)$ in the middle of $(*)$ to get

$$L(f,P) + L(g,P) \leq L(f+g, P) \leq \int_a^b (f+g) \leq U(f+g,P) \leq U(f,P) + U(g,P) \quad \forall P$$

Hence, we get $L(f,P) + L(g,P) \leq \int_a^b f + \int_a^b g \leq U(f,P) + U(f,P)$,

So taking P as above s.t. $U(f,P) + U(g,P) - L(f,P) - L(g,P) < \varepsilon$,

we have that $\int_a^b (f+g)$ & $\int_a^b f + \int_a^b g$ lie in the same interval length ε

So $\left| \int_a^b (f+g) - \left(\int_a^b f + \int_a^b g \right) \right| < \varepsilon$

As $\varepsilon > 0$ is arbitrary, we obtain $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ ∎

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Thm 13.10. Let $g: [a,b] \to \mathbb{R}$ be int'ble, $m \leq f(x) \leq M$ $\forall x \in [a,b]$, then $m(b-a) \leq \int_a^b f \leq M(b-a)$

Proof. Let P be any partition of $[a,b]$, $P = \{t_0, \ldots, t_n\}$

Let $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \geq m$

$M_i = \sup\{f(x) \mid t_{i-1} \leq x \in t_i\} \leq M$.

So, $\int_a^b f \geq L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}) \geq m \sum_{i=1}^{n} (t_i - t_{i-1}) = m(b-a)$

Also, $\int_a^b f = U(f,P) \leq \sum_{i=1}^{n} M_i (t_i - t_{i-1}) \leq M \sum_{i=1}^{n} (t_i - t_{i-1}) = M(b-a)$ ∎

Whenever f is int'ble on $[a,b]$, $x \in [a,b]$, we know f is also int'ble on $[a,x]$ by Thm 13.8.

So, we can define its indefinite integral as $F(x) = \int_a^x f = \int_a^x f(y)\,dy$

Thm 13.11: If f is int'ble on $[a,b]$, then $F(x) = \int_a^x f$ is cts on $[a,b]$. (use F is bdd)

Proof: Let $c \in (a,b)$. Let $h > 0$

Then if h is small enough so that $c+h \le b$,

we have $F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f$

$= \int_a^c f + \int_c^{c+h} f - \int_a^c f$ by Thm 13.8

$= \int_c^{c+h} f$

As f is int'ble, $\exists M > 0$ s.t. $-M \le f(x) \le M \quad \forall x \in [a,b]$

So, by previous thm, $-Mh \le \int_c^{c+h} f \le Mh$

Hence, $\lim_{h \to 0^+} (F(c+h) - F(c)) = \lim_{h \to 0^+} \int_c^{c+h} f = 0$.

Similarly, if $h < 0$ and $c+h \ge a$, we have $F(c) - F(c+h) = \int_a^c f - \int_a^{c+h} f$

$= \int_a^{c+h} f + \int_{c+h}^c f - \int_a^{c+h} f$

$= \int_{c+h}^c f$

So, $-M|h| \le F(c) - F(c+h) \le M|h|$

Hence, $\lim_{h \to 0^-} (F(c) - F(c+h)) = 0$.

So, $\lim_{h \to 0} F(c+h) = F(c)$ & hence F is cts at c.

cty at a, b is similar ∎

Thm 13.12: Suppose $f : [a,b] \to \mathbb{R}$ is increasing. Then f is int'ble.

Proof: As f is increasing, f is bdd: $f(a) \le f(x) \le f(b) \quad \forall x \in [a,b]$

If f is constant, it is int'ble.

Suppose f is not constant, then $f(b) > f(a)$

Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be s.t. $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$

Define a partition $P = \{t_0 \dots t_n\}$ by $t_i = a + i\frac{b-a}{n}$

Then, let $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$, $M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$

As f inc, we have $m_i = f(t_{i-1})$, $M_i = f(t_i)$

Then also $t_i - t_{i-1} = \frac{b-a}{n}$ $\forall i$

So $U(f,P) - L(f,P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}) - \sum_{i=1}^{n} m_i (t_i - t_{i-1}) = \frac{b-a}{n} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))$

$= \frac{b-a}{n}(f(b) - f(a)) < \varepsilon$ by construction.

So, $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$

Hence f is int'ble. $\blacksquare$