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§8 Least Upper Bounds

Def 8.1 Let $A \subseteq \mathbb{R}$, A upper bound for A is a number $\beta \in \mathbb{R}$ s.t. $\alpha \leq \beta$ $\forall \alpha \in A$.

If A has a upper bound, we say A is <u>bounded above</u>

We say A has a <u>least upper bound</u>, $\alpha = \sup A$ ← suprime.

if $\exists \alpha \in \mathbb{R}$ s.t. i) α is an upper bound for A

ii) if β is an upper bound for A, then $\alpha \leq \beta$

Ex. $[0,1]$ is bounded above, $\sup [0,1] = 1$

$[0,1)$ - - - - — - clearly 1 is an UB

if $\beta < 1$ $\exists x \in (\beta, 1) \cap [0,1)$ so β is not UB.

Hence $\sup [0,1) = 1$.

Def 8.2 Let $A \subseteq \mathbb{R}$, We say A is bounded below if $\exists \beta \in \mathbb{R}$ s.t. $a \geq \beta$ $\forall a \in A$

We call β a <u>lower bound</u> for A

A has a <u>greatest lower bound</u>, $\alpha = \inf A$

if i) α is a lower bd for A

ii) if β is a lower bd, $\beta \leq \alpha$

A is <u>bounded</u> if it is both bdd above and bdd below.

Ex. $\inf [0,1] = 0$

$\inf (0,1] = 0$

$(0, \infty)$ is not bdd above, $\inf (0, \infty) = 0$

$\mathbb{N}$ is not bdd, $\inf \mathbb{N} = 1$

ϕ : if $\beta \in \mathbb{R}$, $\nexists a \in \phi$ s.t. $a > \beta$. Hence β is an upper bound. So ϕ is bdd above.

ϕ has no sup as if β is an UB, so is $\beta - 1$.

Completeness Axiom

Every non-empty set $A \subseteq \mathbb{R}$ s.t. A is bounded above has a least upper bound.

Thm 8.3. Let $A \subseteq \mathbb{R}$, $A \neq \phi$, A bdd below. Then $\alpha = \inf A$ exists in A.

Proof: Let $B = \{\beta \mid \beta \text{ is a lower LB for } A\}$

As A is bdd below, $B \neq \phi$

Let $\beta \in B$. As A is non-empty, $\exists a \in A$

As β is a lower bd for a, $\beta \leq a$

So, a is an upper bound for B

By the completeness axiom, $\exists \alpha = \sup B$.

If β is a lower bd for A. $\beta \in B$, so $\beta \leq \alpha$.

To show α is a lower bound. suppose $a \in A$.

Then $\exists \beta \in B$, $\beta \leq a$. Hence a is a upper bd for B.

So $a \geq \sup B = \alpha$

So, α is a lower bd for A.

Hence $\alpha = \inf A$ ∎

RMK 8.4 $\sup A$, $\inf A$ may not be in A or they may be.

Ex. $\sup [0, \sqrt{2}) = \sqrt{2}$.

So $\sup ([0, \sqrt{2}) \cap \mathbb{Q}) = \sqrt{2} \notin \mathbb{Q}$.

Prop 8.5 $\exists x \in \mathbb{R}$ s.t. $x^2 = 2$, $x > 0$

Proof: Let $A = \{ y \in \mathbb{R} \mid y \geq 0 \text{ and } y^2 < 2 \}$.

Clearly $1 \in A$

Also, if $y \in A$, $y^2 < 2 < 4 = 2^2$

Hence, as $y \geq 0$, $y < 2$.

So, 2 is an upper bd for A.

By completeness axiom, let $x = \sup A \in \mathbb{R}$.

Claim: $x^2 = 2$

Assume, for a contradiction, $x^2 \neq 2$.

Case 1. $x^2 < 2$

Case 2 $x^2 > 2$.

Note $1 \in A$, so $x \geq 1 > 0$

Case 1: Suppose $x^2 < 2$. We find $\delta \in (0,1)$ s.t. $x + \delta \in A$.

Let $\delta \in \left(0, \min\left(1, \frac{2-x^2}{2x+1}\right)\right)$

Then $(x+\delta)^2 = x^2 + 2x\delta + \delta^2$

$< x^2 + \delta(2x+1)$

$< x^2 + 2 - x^2 = 2$ as $\delta < \frac{2-x^2}{2x+1}$

So $x + \delta \in A$, $x + \delta > x = \sup A$ ※

Case 2: Suppose $x^2 > 2$.

Let $\delta \in \left(0, \frac{x^2-2}{2x}\right)$

Then $(x-\delta)^2 = x^2 - 2\delta x + \delta^2 > x^2 - 2x\left(\frac{x^2-2}{2x}\right) = 2$

So, $x - \delta$ is an upper bd for A $x - \delta < x = \sup A$ ※

So, $x^2 = 2$.

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Completeness Axiom

Every non-empty set $A \subseteq \mathbb{R}$ that is bounded above has a least upper bound.

$\alpha \in \mathbb{R}$ is LUB for A if

i) α is an UB for A ii) if β is an UB for A, $\alpha \leq \beta$

Lemma 8.6

Let $A \neq \emptyset$, A bounded above. Let $\alpha = \sup A$ $\forall \varepsilon > 0$, $\exists x \in A$ s.t. $\alpha - \varepsilon < x \leq \alpha$

Proof: assume for a contradiction that $\exists \varepsilon > 0$ s.t. $\nexists x \in A$ with $\alpha - \varepsilon < x \leq \alpha$

So $\forall x \in A$, either $x \leq \alpha - \varepsilon$ or $x > \alpha$

As α is an UB for A, if $x \in A$, $x \leq \alpha$

So $\forall x \in A$, $x \leq \alpha - \varepsilon$

Hence, $\alpha - \varepsilon$ is an UB for A

※ to α is least UB ☒

Thm 7.1 (IVT v.1)

Let $f: [a,b] \to \mathbb{R}$ be cts on $[a,b]$, $f(a) < 0 < f(b)$

Then $\exists c \in (a,b)$ s.t. $f(c) = 0$

Proof: Let $A = \{x \in [a,b] \mid f(x) < 0\}$

If $x \in A$, $x \in [a,b]$, so $x \leq b$. So A is bdd above.

As $f(a) < 0$, $a \in A$, so $A \neq \emptyset$

By completeness, A has a least UB $c = \sup A$

As f cts at a and $f(a) < 0$, $\exists \delta_1 > 0$ s.t. $f < 0$ on $[a, a+\delta_1]$

So, $c \geq a + \delta_1 > a$

As f cts at b and $f(b)>0$, $\exists \delta_2>0$, s.t. $f>0$ on $[b-\delta_2, b]$

So, $c \le b-\delta_2$ as $b-\delta_2$ is an UB for A.

Claim: $f(c)=0$.

Suppose not. Either $f(c)<0$ or $f(c)>0$

Case 1: Let $f(c)<0$

As f is cts at c and $f(c)<0$, $\exists \delta>0$ s.t. $f<0$ on $(c-\delta, c+\delta)$

So $f(c+\frac{\delta}{2})<0$, hence $c+\frac{\delta}{2} \in A$ -X- to $c=\sup A$.

Case 2: Let $f(c)>0$ As f is cts at c, $\exists \delta>0$

By Lemma 8.6, $\exists x \in A$ s.t. $c-\delta \le x < c$

So $f(x)<0$ as $x \in A$, but $f(x)>0$ as $c \in (c-\delta, c]$

-X- to $f(c)=0$.

Lemma 8.7 Let f be cts at a. Then $\exists \delta>0$ s.t. f is bdd on $(a-\delta, a+\delta)$

Proof: Let $\varepsilon=1>0$ So $\exists \delta>0$ s.t. if $x \in (a-\delta, a+\delta)$, then $|f(x)-f(a)|<1$

So, if $x \in (a-\delta, a+\delta)$, $|f(x)| \le |f(a)|+1$ ∎

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Thm 7.2 (EVT): Let $f:[a,b] \to \mathbb{R}$ be cts on $[a,b]$. Then f is bounded and $\exists x_+, x_- \in [a,b]$ s.t. $f(x_-) \le f(x) \le f(x_+)$ $\forall x \in [a,b]$.

Proof: i) we show f bounded

Let $A=\{x \in [a,b] \mid x$ bdd on $[a,x]\}$

As $[a,a]=\{a\}$, f is bdd on $[a,a]$ by $f(a)$.

So $a \in A$, $A \ne \emptyset$.

As $A \subset [a,b]$, A is bdd by b.

Hence, $\exists \alpha = \sup A$ in $\mathbb{R}$, $\alpha \le b$.

Assume $\alpha < b$ for a contridiction.

By Lemma 8.7, $\exists \delta > 0$ s.t. f is bdd on $(\alpha-\delta, \alpha+\delta)$

(assuming $\alpha \neq a$, $\exists \delta > 0$ s.t. f bdd on $[a, a+\tilde{\delta})$ hence $a + \frac{\delta}{2} \in A$).

As $\alpha = \sup A$, $\exists x \in A$ s.t. $\alpha - \delta < x \le \alpha$.

So, f is bdd on $[a, x]$ as $x \in A$ and f is bdd on $(\alpha-\delta, \alpha+\delta)$.

i.e. $\exists M_1 > 0$ s.t. $|f(y)| \le M_1$, $\forall y \in [a, x]$.

& $\exists M_2 > 0$ s.t. $|f(y)| \le M_2$, $\forall y \in (\alpha-\delta, \alpha+\delta)$.

So $|f(y)| \le M := \max(M_1, M_2)$ $\forall y \in [a, \alpha + \frac{\delta}{2}]$.

Thus, $\alpha + \frac{\delta}{2} \in A$. $\times$ to $\alpha = \sup A$ as $\alpha + \frac{\delta}{2} > x$.

So, $\alpha = b$.

By cty on b, $\exists \delta' > 0$ s.t. f is bdd on $(b - \delta', b]$

As $b = \sup A$, $\exists x \in A$ s.t. $b - \delta' < x \le b$

So, f is bdd on $[a, x]$ & $(b-\delta', b]$, so f bdd on $[a, b]$.

ii) We show $\exists x_+$ s.t. $f(x) \le f(x_+)$ $\forall x \in [a, b]$.

Let $B = \{f(x) \mid x \in [a, b]\}$.

By (i), $\exists M > 0$ s.t. $|f(x)| \le M$ $\forall x \in [a, b]$. i.e. B is bdd.

Also $f(a) \in B$, So B is not empty.

Hence, $\alpha = \sup B$ exists.

Suppose $\nexists x_+$ s.t. $f(x_+) = \alpha$.

Then $\alpha - f(x) > 0$ $\forall x \in [a, b]$.

Let $g = \frac{1}{\alpha - f}$. As $\alpha - f > 0$, g is cts on $[a, b]$.

So, by (i), g is bdd. i.e. $\exists M > 0$ s.t. $|g(x)| \le M$ $\forall x \in [A, B]$.

Hence, $\frac{1}{M} \le \alpha - f(x)$ $\forall x \in [a, b]$

So, $f(x) \le \alpha - \frac{1}{M}$ $\forall x \in [a,b]$ ※ to $\alpha = \sup B$.

So, $\exists x_+$ s.t. $f(x_+) = \alpha$, i.e. $f(x) \le f(x_+)$ $\forall x \in [a,b]$.

Note $-f$ is cts on $[a,b]$ so $\exists x_-$ on $[a,b]$ s.t. $-f(x) \le -f(x_-)$ $\forall x \in [a,b]$

So $f(x_-) \le f(x)$ $\forall x \in [a,b]$ ∎

Thm 8.8 $\mathbb{N}$ is not bounded above.

Proof: Suppose $\mathbb{N}$ is bdd above.

As $1 \in \mathbb{N}$, $\mathbb{N} \neq \phi$. So $\alpha = \sup \mathbb{N}$ exists in $\mathbb{R}$.

Thus $\exists n$ in $\mathbb{N}$ s.t. $\alpha - 1 \le n \le \alpha$.

Then $\alpha \le n+1 \in \mathbb{N}$. ※ to $\alpha = \sup \mathbb{N}$ ∎

Thm 8.9. $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$

Proof: Let $\varepsilon > 0$ Suppose $\nexists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$.

Then $\forall n \in \mathbb{N}$, $\frac{1}{n} \le \varepsilon$, hence $n \le \frac{1}{\varepsilon}$ $\forall n \in \mathbb{N}$.

※ as $\mathbb{N}$ is not bdd.

This implies. $\forall x > 0$, $y \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t $nx > y$.