Joint distributed random variable L22 11/26

(§5.2) Some examples:

joint behavior of 2 r.v.s

2 discrete r.v.s
- Years in college v.s. number of credits takes
- number of cups of coffee one student drinks per day v.s. day of the week.

2 continuous r.v.s
- Hight v.s. weight
- study time v.s. sleep time.

Def: (Joint pmf)

If X and Y are 2 discrete random variables, the joint probability mass function is defined as $p(x,y) = P(X=x \text{ and } Y=y)$.

For a function $p(x,y)$ to be a valid joint pmf, it must satisfy

1. $p(x,y) \geq 0$. (non-neg)
2. $\sum_{\text{all } x,y} p(x,y) = 1$ (total-one)

Probability calculation

Let A be any set consisting of pairs of (x,y) values, then

$$P((x,y) \in A) = \sum_{(x,y)\in A} p(x,y).$$

Def: marginal pmf

The marginal pmf of X and Y, denoted by $P_X(x)$ and $P_Y(y)$, respectively are given by

$$P_X(x) = \sum_y p(x,y) \quad \text{for any } x$$

$$P_Y(y) = \sum_x p(x,y) \quad \text{for any } y.$$

E.g. consider 2 r.v.s. X and Y with joint pmf given.

	$Y=0$	1	2	X
$x=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{13}{24}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{11}{24}$
Y	$7/24$	$5/12$	$7/24$	1

(a) find $P(X=0, Y\leq 1)$ (b) find the marginal pmf of X and Y

(c) find $P(Y=1 \mid X=0)$

Sol (a) $P(X=0, Y\leq 1) = P(X=0, Y=0) + P(X=0, Y=1)$

$= P(0,0) + P(0,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$.

(b) marginal pmf of X $P_X(0) = P(X=0) = \sum_y P(0,y) = P(0,0) + P(0,1) + P(0,2)$

$= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$

(c) $P(Y=1 \mid X=0) = \frac{1/4}{13/24} = \frac{6}{13}$.

Joint pdf

For 2 discrete r.v.s., the joint cdf is defined for each pair of num (x,y) by $F(x,y) = P(X\leq x \text{ and } Y\leq y)$.

- The cdf of X and Y can be obtained from the joint cdf $F(x,y)$

$F_X(x) = F(x, \infty)$

$F_Y(y) = F(\infty, y)$

Def: If X, Y are 2 continuous r.v.s, then $f(x,y)$ is the joint pdf for X and Y if for any 2-dimensional set A.

$$P((x,y)\in A) = \iint_{x\,y} f(x,y)\,dx\,dy.$$

- If A is a rectangle, then

$$P(a\le x\le b,\ c\le Y\le d) = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

and the joint cdf

$$F(a,b) = P(X\le a, Y\le b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy.$$

L23 12/3

Marginal PDF

The marginal pdfs of X and Y, denoted by $f_X(x)$ and $f_Y(y)$, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad -\infty < y < \infty$$

Marginal distribution

$$F_X(x) = P(X\le x) = P(X\le x, Y<\infty) = \underbrace{F(x,\infty)}_{\text{joint cdf}}$$

$$F_Y(y) = P(Y\le y) = P(X<\infty, Y\le y) = F(\infty, y)$$

Independence [$P(AB) = P(A)\cdot P(B)$]

Two random variables X and Y are independent if for all choices of 2 sets A and B.

(*) $P(X\in A, Y\in B) = P(X\in A)\cdot P(X\in B)$

If X and Y are continuous, the condition of independence is equivalent to

(**) $f(x,y) = f_X(x) \cdot f_Y(y)$

If X and Y are discrete, then X and Y are independent

$P(x,y) = P_X(x) \cdot P_Y(y)$

i.e. joint pdf (pmf) = product of marginal pdf (pmf)

If X and Y are not independent, then we say they are dependent.

E.g. A study shows the number of hours, X, a student watches TV and the number of hours, Y, a student works on his homework are approximated by the joint pdf.

$$f(x,y) = e^{-(x+y)}, \quad x>0,\ y>0$$

a) Find the probability that a student chosen at random spends at least twice as much time watching TV as he does working on his homework.

b) Are X and Y independent?

Sol: a) $P(Y \geq 2X) = \iint_{y \geq 2x} f(x,y)\,dx\,dy.$

limits of x and y $(y \geq 2x,\ x>0,\ y>0)$

$$= \int_0^\infty \int_{2x}^\infty e^{-(x+y)}\,dy\,dx$$

$$= \int_0^\infty \left[e^{-x} \left[\int_{2x}^\infty e^{-y}\,dy \right] \right] dx$$

$$= \int_0^\infty e^{-x} e^{-2x}\,dx$$

$$= -\frac{1}{3} e^{-3x} \Big|_0^\infty = \frac{1}{3}$$

b) $f(x,y) \stackrel{?}{=} f_X(x) \cdot f_Y(y).$

$f_X(x) = \int_{-\infty}^\infty f(x,y)\,dy = \int_0^\infty e^{-(x+y)}\,dy = e^{-x}. \quad x>0$

$f_Y(x) = \int_{-\infty}^\infty f(x,y)\,dx = \int_0^\infty e^{-(x+y)}\,dx = e^{-y} \quad y>0$

$f_X(x) \cdot f_Y(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} = f(x,y)$ independent.