Limits

Def: $\lim_{x\to a} f(x) = l$ if $\forall \varepsilon>0$, $\exists \delta>0$ s.t. if $0<|x-a|<\delta$, then $|f(x)-l|<\varepsilon$.

$f(a)$ is irrelavent

$\lim_{x\to a} c = c$, $\lim_{y\to x} y^2 = x^2$

Def. $\lim_{x\to\infty} f(x) = l$ if $\forall \varepsilon>0$, $\exists M>0$ s.t. $x>M$, then $|f(x)-l|<\varepsilon$.

$\lim_{x\to a} f(x) = -\infty$ if $\forall M<0$, $\exists \delta>0$ s.t. if $0<|x-a|<\delta$ s.t. $f(x)<M$

Lemma: Sum, product, quotient

if $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = k$, then $\lim_{x\to a} (f(x)+g(x)) = l+k$

$\lim_{x\to a} f(x)g(x) = lk$

if $k\neq 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{k}$.

Prove limits of poly, rational, etc.

Ex. 1. Let $f(x) = \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases}$ then $\forall a\in\mathbb{R}$, $\lim_{x\to a} f(x) = 0$.

Proof: let $a\in\mathbb{R}$, $\varepsilon>0$.

if $a\neq 0$, let $0<\delta<|a|$, then if $0<|x-a|<\delta$, we have $|x|>0$, $f(x)=0$, $f(a)=0$.

so $|f(x)| = 0 < \varepsilon$.

if $a=0$, let $\delta>0$, then if $0<|x-a|<\delta$, then $0<|x|<\delta$, $x\neq 0$, $f(x)=0$.

$|f(x)| = 0 < \varepsilon$. ∎

2. Let $g(x) = \begin{cases} 0 & x\le 0 \\ 1 & x>0 \end{cases}$ then $\lim_{x\to 0} g(x)$ does not exist.

Proof: use one-sided limits. $\lim_{x\to 0^-} g(x) = 0 \neq 1 = \lim_{x\to 0^+} g(x)$

proof ? Def limit not exist

3. Let $h(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$

Then $\lim_{x\to 0^+} h(x)$ does not exist.

Thm (sandwich) Let $f(x) \leq g(x) \leq h(x)$ for $x \in (a-\delta, a+\delta)$

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$, then $\lim_{x\to a} g(x) = l$.

Lemma: Let $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a} (f(x) - l) = 0$.

$\lim_{x\to a} g(x) = 0$ iff $\lim_{x\to a} |g(x)| = 0$.

So to prove $\lim_{x\to a} f(x) = l$, it's equivalent to show $\lim_{x\to a} |f(x) - l| = 0$.

Ex. $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

Proof: we show $\lim_{x\to 0} |x\sin(\frac{1}{x})| = 0$

Clearly, $|x\sin(\frac{1}{x})| \geq 0$ as $x \to 0$

as $|\sin(y)| \leq 1$ $\forall y$, we have $\forall x \neq 0$, $|x\sin(\frac{1}{x})| \leq |x|$

So, $0 \leq |x\sin(\frac{1}{x})| \leq |x|$ $\forall x \neq 0$.

Hence as $\lim_{x\to 0} 0 = \lim_{x\to 0} |x| = 0$, we are done. ∎

Note: $x\sin(\frac{1}{x}) \leq x$ it's true but not enough

Also need $x\sin(\frac{1}{x}) \geq -x$.

Continuity

Def ① $f(x)$ is cts at a if $\lim_{x\to a} f(x) = f(a)$

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

② f is cts on $\mathbb{R}$ if it's cts on $\forall a \in \mathbb{R}$.

③ f is cts on $[a,b]$ if f is cts on x $\forall x \in (a,b)$ & $\lim_{x\to a^+} f(x) = f(a)$, $\lim_{x\to b^-} f(x) = f(b)$

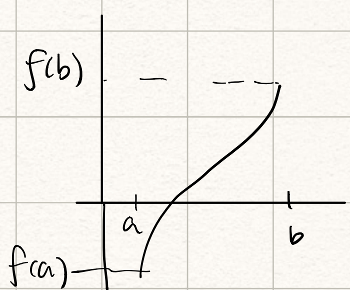
Lemma, Sum, Product, quotient rules.

Composition:

If f is cts at a and g cts at $f(a)$, the $g \circ f$ is cts at a.

IVT Suppose $f: [a,b] \to \mathbb{R}$ is cts on $[a,b]$

$f(a) < v < f(b)$. Then $\exists c \in (a,b)$ s.t. $f(c) = v$.



Ex 1. $\forall x > 0$, $\exists y > 0$ s.t. $y^2 = x$.

Proof. Let $x > 0$, $f(y) = y^2$

Then $f(0) = 0 < x$, $f(x+1) = x^2 + 2x + 1 > x$

So, as f is cts on $[0, x+1]$.

$\exists y \in (0, x+1)$ s.t. $f(y) = x$

IVT relies on completeness thm

Every non-empty set $A \subseteq \mathbb{R}$ which is bdd above has a least upper bound.

Exercise on §8

EVT Let $f : [a,b] \to \mathbb{R}$ be cts

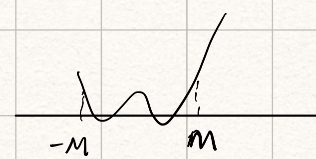
i) $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in [a,b]$.

ii) $\exists x_-, x_+ \in [a,b]$ s.t. $f(x_-) \leq f(x) \leq f(x_+) \quad \forall x \in [a,b]$.

Remember $\sup A$ may not be in A

EVT: if f is cts, then $\sup \{f(x) \mid x \in [a,b]\}$ is in $\{f(x) \mid x \in [a,b]\}$.

Ex. Every even degree polynomial attain its limits.



Idea: show that $\lim_{x \to \pm\infty} p(x) = \infty$.

So, $\exists M > 0$ s.t. if $|x| > M$, $p(x) > p(0)$

By EVT, $\exists x_* \in [-M, M]$ s.t. $p(x) \geq p(x_*) \quad \forall x \in [-M, M]$

So if $|x| > M$, we still have $p(x) > p(0) \geq p(x_*)$

leaves x^* is the min pt.

△ Use IVT, EVT only work on bdd interval

To apply them to specific function, always work out which interval to apply them on

Differentiation

Def: f is diff'ble at a if $\underbrace{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}_{f'(a)}$ exists.

For $f'(a)$ to exist, f must be defined on an interval of the form $(a-\delta, a+\delta)$

Lemma, f diff'ble at $a \Rightarrow f$ cts at a
$\not\Leftarrow$

Ex: $f(x) = |x|$ is diff'ble at $x \neq 0$, not diff'ble at 0, cts on $\mathbb{R}$

Sum, Product, Quotient Rule as usual.

$g \circ f$ is diff'ble at a & $(g \circ f)'(a) = g'(f(a)) f'(a)$

Ex 1) if $f(x) = c \ \forall x$, $f'(x) = 0 \ \forall x$

2) $\frac{d}{dx}(x^n) = nx^{n-1} \quad \forall n \in \mathbb{Z}$

3) $\frac{d}{dx}(x^2 \sin \frac{1}{x}) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} (-\frac{1}{x^2}) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$

(if $x \neq 0$, by the chain rule.)

Lemma: if $f: (a,b) \to \mathbb{R}$ has a max at $c \in (a,b)$ & f diff'ble at a, then $f'(c) = 0$

Idea, if c is a max pt. $f(c+h) - f(c) \leq 0 \ \forall h \neq 0$

So, if $h > 0$ $\quad \frac{f(c+h) - f(c)}{h} \leq 0$

$h < 0 \quad \frac{f(c+h) - f(c)}{h} \geq 0$

Hence, $f'(c) = 0$.

MVT Let $f: [a,b] \to \mathbb{R}$ be cts on $[a,b]$ & diff'ble on (a,b)

Then $\exists x \in (a,b)$ s.t. $f'(x)(b-a) = f(b) - f(a)$

Cor if $f'=0$ on (a,b), the f is constant on (a,b)

if $f'>0$ on (a,b), then f is strictly inc on (a,b)

Ex. $f: (0,2) \to \mathbb{R}$ $f(x) = \begin{cases} 0 & 0<x\leq 1 \\ 1 & 1<x<2 \end{cases}$

f is not constant $f'=0$ on $(0,1) \cup (1,2)$ but is not defined on 1.

L'Hôpital: Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = l$

Integration

Def: Let $[a,b]$ be a closed interval. A partition P of $[a,b]$ is a set $P = \{t_j\}_{j=0}^{n}$ s.t.

$t_0 = a < t_1 \cdots < t_n = b$

Suppose f is bdd on $[a,b]$, then $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$,

$M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. $i=1 \cdots n$.

We define the lower sum $L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$

Upper Sum $U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$

Lemma: for any partitions P, Q, $L(f,P) \leq U(f,Q)$

Def: f is int'ble on $[a,b]$ is $\sup\{L(f,P) \mid P \text{ partition}\} = \inf\{U(f,P) \mid P \text{ partition}\}$.

Then, this number is called the integral of f on $[a,b]$, write $\int_a^b f$.

Lemma, f is int'ble iff $\forall \varepsilon > 0$ $\exists$ a partition P s.t. $0 \le U(f,P) - L(f,P) < \varepsilon$.

Ex ① Let $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

Then f is int'ble on $[0,1]$ & $\int_0^1 f = 0$

Idea, define partition as $0 = t_0 < t_1 = \varepsilon < t_2 = 1$

② Let $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ g is not int'ble on $[0,1]$.

Proof: $\forall P = \{t_j\}_{j=1}^{n}$ $m_j = 0$, $M_j = 1$

So, $L(f,P) = 0$ $U(f,P) = 1$.

Thm i) f cts $\Rightarrow$ f int'ble.

ii) f increasing $\Rightarrow$ f int'ble.

Let f be int'ble on $[a,b]$ Let $F(x) = \int_a^x f$

Thm F is cts on $[a,b]$

FTC I If f is cts at $c \in [a,b]$, then F is diff'ble at c & $F'(c) = f(c)$

FTC II Let F be diff'ble & s.t. F' is int'ble

Then $F(b) - F(a) = \int_a^b F'(x)\,dx$

Ex. ① $H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$ is int'ble, but integral is not diff'ble at 0

$\int_{-1}^{x} H(y)\,dy = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$

② $g(x) = \begin{cases} \frac{x^2}{2}\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is diff'ble on $\mathbb{R}$, but g' is not int'ble.

Use FTC to find $\int_0^x t^n \, dt = \frac{x^{n+1}}{n+1}$ if $n \neq 1$ as $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, int'ble.

Let $F(x) = \int_0^x f(t)(x-t)\,dt$ $\quad G(x) = \int_0^x \left(\int_0^t f(a)\,da\right) dt$

f is cts. So $t f(t)$ is also cts.

Then by FTC I, F is diff'ble.

$F(x) = x\int_0^x f(t)\,dt - \int_0^x t f(t)\,dt$

$F'(x) = \int_0^x f(t)\,dt + x f(x) - x f(x) = \int_0^x f(t)\,dt.$

$G(x) = \int_0^x \left(\int_0^t f(a)\,da\right) dt$

as f cts, $\int_0^t f$ is cts

So by FTC I. $G'(x) = \int_0^x f(a)\,da = \int_0^x f(t)\,dt = F'(x)$

So, $F(x) = G(x) + C.$

But $F(0) = \int_0^0 f(t)(x-t)\,dt = 0$

$G(0) = \int_0^0 \left(\int_0^t f(a)\,da\right) dt = 0$

So $C = 0$ $\quad F(x) = G(x)$

$\int_0^x f(t)(x-t)\,dt = \int_0^x \left(\int_0^t f(a)\,da\right) dt.$