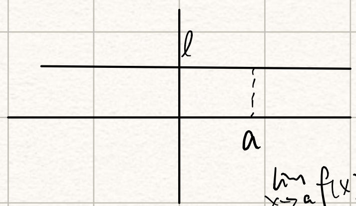
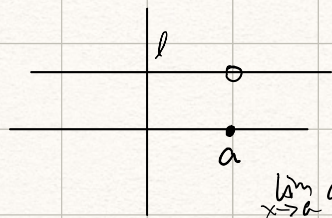
§6. Continuity.

L9/2.10



$\lim_{x\to a} f(x) = a$



$\lim_{x\to a} g(x) = a.$

Def 6.1. Let f be a real-valued function, $a \in \mathrm{dom}(g)$ We say f is continuous at a if $\lim_{x\to a} f(x) = f(x)$

Alternative: $\forall \varepsilon > 0$, $\exists > 0$ s.t. if $|x-a| < \delta$, $|f(x) - f(a)| < \varepsilon$

Ex: 1) Let $c \in \mathbb{R}$, let $f\ \mathbb{R} \to \mathbb{R}$ be $f(x) = c$. Then f is cts at all $a \in \mathbb{R}$

Let $\varepsilon > 0$, Let $\delta > 0$. Then if $|x-a| < \delta$, we have $|f(x) - f(a)| = |c - c| = 0 < \varepsilon$. ∎

2) Let $p(x) = a_n x^n + \cdots + a_1 n + a_0$, let $a \in \mathbb{R}$, the $p(x)$ is cts at a.

We know from last lecture that $\lim_{x\to a} p(x) = p(a)$, so p is cts at all $a \in \mathbb{R}$.

3) Let $g(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$, cts?

As $\lim_{x\to a} g(x)$ does not exist, g is not cts at 0

Cts at $x \neq 0$? Do not know composition interacts with limits.

4) Let $h(x) = \begin{cases} x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ The $\lim_{x\to 0} h(x) = 0 = h(0)$, so cts

Proofs: Let $\varepsilon > 0$, $\delta > 0$, $\delta = \varepsilon$. s.t. if $0 < |x - 0| = |x| < \delta$, then

$|h(x) - 0| = |h(x)| = |x| \cdot |\sin(\frac{1}{x})| \leq |x| < \delta = \varepsilon.$

5) Let $q(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ So $\lim_{x \to 0} q(x) = 0 = q(0)$

Let $\varepsilon > 0$, $\delta > 0$, $\delta = \varepsilon$ s.t. if $|x - a| < \delta$.

$|q(x) - q(0)| = |q(x) - 0| = |q(x)| \leq |x| < \delta = \varepsilon$.

Let $a \neq 0$, then $\lim_{x \to a} q(x) \neq q(a)$

Case 1: $a \in \mathbb{Q}$.

Let $\varepsilon = |a| > 0$. Then, Let $\delta > 0$, let $x \in (a - \delta, a + \delta)$ s.t. $x \notin \mathbb{Q}$.

Then $|q(x) - q(a)| = |q(a)| = |a| \geq \varepsilon$.

So, if $a \in \mathbb{Q}$, $\lim_{x \to a} q(x) \neq q(a)$

Case 2: $a \notin \mathbb{Q}$.

Let $\varepsilon = |a| > 0$. Then, Let $\delta > 0$ Let $x \in (a - \delta, a + \delta) \cap \mathbb{Q}$. s.t. $|x| > |a|$

Then $|q(x) - q(a)| = |q(x)| = |x| > |a| = \varepsilon$

q is cts at 0, discts at $a \neq 0$.

Def: f is not cts at a if $\exists\, \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x$ s.t. $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$

Thm 6.2. Let f, g be cts at a, then i) $f + g$ is cts at a

ii) fg is cts at a

iii) if $g(a)$, then $\frac{f}{g}$ is cts at a

Proof As f, g cts at a, $\lim_{x \to a} f(x) = f(a)$, $\lim_{x \to a} g(x) = g(a)$

i) By sum rule, for limits, $\lim_{x \to a} (f + g)(x) = (f + g)(a)$ as required.

ii), iii) similar ∎

Hence rational functions are cts where they are defined.

Thm 6.3 Let f be cts at a, g be cts at $f(a)$, the $g \circ f$ is cts at a.

Proof: Let $\varepsilon > 0$, As g is cts at $f(a)$, $\exists \delta_1 > 0$ s.t. if $|y - f(a)| < \delta_1$, then $|g(y) - g(f(a))| < \varepsilon$

As $\delta_1 > 0$ and f cts at a, $\exists \delta > 0$ s.t. if $|x-a| < \delta$, then $|f(x) - f(a)| < \delta_1$.

So, if $|x-a| < \delta$, then $|f(x) - f(a)| < \delta_1$, So $|g(f(x)) - g(f(a))| < \varepsilon$. ∎

Ex. now we can show $g(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is cts at $a \neq 0$

Let $h(x) = \sin(x)$, let $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $g(x) = h(f(x))$ $g = h \circ f$.

As $\frac{1}{x}$ is cts at a if $a \neq 0$, f is cts at $a \neq 0$.

As $\sin(x)$ is cts on $\mathbb{R}$, h is cts at $f(x)$

So $h \circ f$ is cts at $a \neq 0$. ∎

Def 6.4: Let f be a real-valued function. We say f is cts if it is cts at all $a \in \mathrm{dom}(f)$. We say f is cts on a set A if f is cts at all $a \in A$.

e.g. $A = (a, b)$.

We say f is cts on $[a,b]$ if f is cts on (a,b) and $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$.

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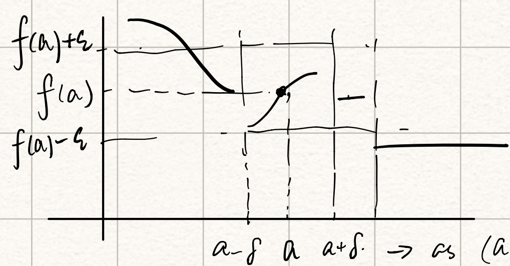

→ as $(a-\delta, a+\delta)$ shrink, $(f(a)-\varepsilon, f(a)+\varepsilon)$ shrink.

Thm 6.5: Let f be a real valued function s.t. $f(a)>0$, and f is cts at a. Then $\exists \delta>0$ s.t. $x\in(a-\delta, a+\delta)$, then $f(x)>0$.

Proof: Let $f(a)>0$, f cts at a.

Let $\varepsilon = \frac{f(a)}{2} > 0$. As the $\lim_{x\to a} f(x) = f(a)$ $\exists \delta>0$ s.t. if $|x-a|<\delta$,

then $|f(x)-f(a)|<\varepsilon$.

in particular, if $x\in(a-\delta, a+\delta)$, then $f(x) > f(a)-\varepsilon = \frac{f(a)}{2} > 0$.

Note: On $(a-\delta, a+\delta)$ $\frac{1}{f(x)} < \frac{2}{f(x)}$

If $f(a)<0$, f cts at a, $\exists \delta>0$, s.t. if $x\in(a-\delta, a+\delta)$, then $f(x) < \frac{f(a)}{2} < 0$.

Proof: Using $\varepsilon = \frac{|f(a)|}{2} > 0$

Or, let $g=-f$. then $g(a)>0$. g cts at a.

So, $\exists \delta>0$, s.t. $x\in(a-\delta, a+\delta)$, $g(x) > \frac{g(a)}{2}$, $f(x) < \frac{f(a)}{2}$

If $f(a)=0$, no info on the sign of $f(x)$ e.g. $g(x)=0$. $a=0$.

Thm 6.6. Suppose f is cts at ℓ, $\lim_{x\to a} g(x) = \ell$, then $\lim_{x\to a} f(g(x)) = g(\ell)$

Proof: Suppose $h(x) = \begin{cases} g(x) & x\neq a \\ \ell & x=a \end{cases}$

Claim: h is cts at a.

This is true as $\lim_{x\to a} g(x) = \ell = h(\ell)$

So, by Thm 6.3. $\lim_{x\to a} f(h(x)) = f(h(a)) = f(l)$

f cts at $h(a)$, h cts at a.

For $x \neq a$, note $f(g(x)) = f(h(x))$

Thus, $\lim_{x\to a} f(g(x)) = \lim_{x\to a} f(h(x)) = f(l)$ ▨

Extension of Function.

If a function f is defined on $A \subset \mathbb{R}$, we say that $g: \mathbb{R} \Rightarrow \mathbb{R}$ is an extension of f to $\mathbb{R}$ if $g(x) = f(x)$ $\forall x \in A$.

Thm 6.7 (Extension of cts function)

Suppose f is cts on $[a,b]$. Then $\exists\, g: \mathbb{R} \to \mathbb{R}$ s.t. g is cts and $g = f$ on $[a,b]$.

Proof: Let $g(x) = \begin{cases} f(x) & x \in [a,b]. \\ f(a) & x < a \\ f(b) & x > b \end{cases}$

Clearly, g is cts on (a,b), for $x<a$, $x>b$

Need to check g is cts at a, b.

To show $\lim_{x\to a} g(x) = g(a)$, we show $\lim_{x\to a^+} g(x) = \lim_{x\to a^-} g(x) = g(a)$.

Two cases: $a<b$, $a=b$

Case 1. $a=b$. The $g(x)$ is the constant function $g(x) = f(a)$.

Then $g(x)$ is cts on $\mathbb{R}$.

Case 2: $a<b$. Then $\exists\, \delta = b-a > 0$ s.t. if $a<x<\delta+a$,

then $g(x) = f(x)$. So $\lim_{x\to a^+} g(x) = \lim_{x\to a^+} f(x) = f(a) = g(a)$.

Also, $\lim_{x\to a^-} g(x) = \lim_{x\to a^-} f(a) = f(a) = g(a)$.

Hence, $\lim_{x\to a} g(x) = g(a)$. So, g is cts at a

Proof at b is similar ▨

Defn 6.8 Let f be a function s.t. $\lim_{x\to a} f(x) = \ell \neq f(a)$.

Then we say f has a removable discontinuity at a.

Ex. $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$ f has a removable discontinuity at 0.

Ex. Let $g(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Then, $\lim_{x\to 0} g(x)$ does not exist

So, 0 is not a removable discontinuity at 0.

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Thm 6.9: i) If f has a removable discts at a,

Let $g(x) = \begin{cases} f(x) & x \neq 0 \\ \lim_{x\to a} f(x) & x = a \end{cases}$

Then, $g(x)$ is cts at a

ii) Suppose every discts of f is removable.

Define $g(x) = \lim_{y\to x} f(y)$. Then g is cts.

Proofs: i) $\forall \delta > 0$, if $0 < |x-a| < \delta$, then $g(x) = f(x)$

So, $\lim_{x\to a} g(x) = \lim_{x \to a} f(x) = g(a)$.

So, g is cts at a.

ii) Let $a \in dom(g)$, we need to show $\lim_{x\to a} g(x) = g(a)$.

Let $\varepsilon > 0$. As $\lim_{y\to a} f(y) = g(a)$,

$\exists \delta_1 > 0$, s.t. if $|y-a| < \delta_1$, then $|f(y) - g(a)| < \frac{\varepsilon}{2}$.

Let $|x-a| < \delta_1$. Then $\exists \delta_2 > 0$ s.t. if $0 < |y-x| < \delta_2$, then $|g(x) - f(y)| < \frac{\varepsilon}{2}$.

Choose y s.t. $0 < |x-y| < \delta_2$ and $0 < |y-a| < \delta_1$,

this is possible as x lies in $(a-\delta_1, a+\delta_1)$

So, $\exists \delta_3 > 0$, s.t. $(x-\delta_3, x+\delta_3) \subset (a-\delta_1, a+\delta_1)$ by choosing

$\delta_3 = \min(|x-(a-\delta_1)|, |x+(a+\delta_1)|)$.

So, $|g(x) - f(y)| < \frac{\varepsilon}{2}$, $|f(y) - g(a)| < \frac{\varepsilon}{2}$.

thus, $|g(x) - g(a)| \leq |g(x) - f(y)| + |f(y) - g(a)|$

Hence, g is cts at a $\blacksquare$