· Variance and standard deviation.

Def. Let X be a discrete random variable with possible values x_i and corresponding probability $p(x_i)$, and let $\mu = E(X)$

The variance of X, denoted $Var(X)$ or σ^2, is

$$Var(X) = \sigma^2 = E[(X-\mu)^2] := \sum_i (x_i-\mu)^2 p(x_i)$$

meaning: the variance is the weighted average of the squared deviations of the values of X from its mean μ, where the weights are the corresponding probabilities.

- $x_i - \mu$, the deviation of x_i from μ.
- $(x_i-\mu)^2$, the squared deviation (≥ 0)
- $Var(X) \geq 0$, and 0 iff X is a constant i.e $P(X=\mu)=1$

Def: the standard deviation of X, devoted as $SD(X)$ or σ, is the non-negative square root of the variance of X.

$$SD(X) = \sigma = \sqrt{E(X-E(X))^2}$$

- $SD(X)$ is a measure of how spread out the distribution of X is around its mean μ.
- $SD(X)$ has the same unit with X.

Important Properties:

① $Var(X) = \sigma^2 = EX^2 - (EX)^2$.

$$= \sum_i x^2 p(x_i) - [\sum_i x_i p(x_i)]^2.$$

Computation formula of Variance

proof: $LHS = Var(X) \overset{def}{=} E[(X-\mu)^2]$

$$= E(X^2 - 2\mu X + \mu^2)$$
$$= EX^2 - 2\mu E(X) + \mu^2$$
$$= EX^2 - 2\mu \cdot \mu + \mu^2.$$
$$= EX^2 - \mu^2$$
$$= E(X^2) - E(X)^2 = RHS \quad \square.$$

$EX^2 \geq (EX)^2$, w/ equality iff X is a constant $P(X=\mu)=1$.

②. For any constants $a, b (\in \mathbb{R})$ $Var(aX+b) = a^2 Var(X)$.

proof: $LHS = Var(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2$.

$$= E(a^2X^2 + 2abX + b^2) - (aEX + b)^2$$
$$= a^2EX^2 + \cancel{2abEX} + \cancel{b^2} - a^2(EX)^2 \cancel{-2abEX} \cancel{-b^2}$$
$$= a^2[E(X^2) + (EX)^2] = a^2 Var(X).$$

③ $SD(aX+b) = |a| \, SD(X)$

④ If X and Y are independent, then

$$Var(X+Y) = Var(X) + Var(Y).$$

In general, if $X_1, X_2, \ldots, X_n$ are independent.

$$Var(X_1 + X_2 + \cdots + X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n).$$

proof: $LHS = Var(X+Y) = E(X+Y)^2 - [E(X+Y)]^2$.

$$= E(X^2 + 2XY + Y^2) - (EX + EY)^2.$$
$$= EX^2 + 2E(XY) + EY^2 - [(EX)^2 + 2EX\,EY + (EY)^2].$$
$$= Var(X) + Var(Y) + \underbrace{2E(XY) - 2EX \cdot EY}_{=0}$$

$= Var(X) + Var(Y)$. independent.

E.g 1. Let X be the number on a fair dice. $Var(X)$, $SD(X) = ?$

Sol. $Var(X) = E(X^2) - (EX)^2$

$= 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + \cdots 36 \times \frac{1}{6} - 3.5^2$

$= \frac{35}{12}$.

$SD(X) = \sqrt{Var(X)} = \sqrt{\frac{35}{12}} = 1.71$

E.g 2. Let Y be the sum of # of four fair dice rolled independently. Find $Var(Y)$ and $SD(Y)$.

Sol: $Var(Y) = 4 Var(X) = 4 \times \frac{35}{12} = \frac{35}{3}$

$SD(Y) = \sqrt{\frac{35}{3}} = 3.42$.

E.g 3. Suppose we flip an unfair coin with probability p to be head. Let X be the indicator for getting a head. Find $Var(X)$, $SD(X)$.

Sol. $X = \begin{cases} 1 & \text{if head} & p \\ 0 & \text{otherwise} & 1-p. \end{cases}$

$E(X) = p$.

$Var(X) = EX^2 - (EX)^2$

$= 1^2 \cdot p + 0^2 (1-p) - p^2 = p(1-p)$.

$SD(X) = \sqrt{Var(X)} = \sqrt{p(1-p)} = \sqrt{pq}$. w/ $q = 1-p$.

Bernoulli Random Variable.

Def: any random variable whose only possible values are 0 and 1, is called a Bernoulli random variable. $X \sim Bern(p)$.

x	1	0
$p(x)$	p	$1-p$

$E(x) = p$

$Var(x) = p(1-p)$

$SD(x) = \sqrt{p(1-p)}$.

· Binomial (n, p).

A random variable, x, follows a binomial distribution, is denoted by $X \sim Binomial(n, p)$ or $Bin(n, p)$.

$E(x) = np$.

$Var(x) = npq$, $q = 1-p$.

$SD(x) = \sqrt{npq}$.

proof: $Var(x)$ X - # successes in n indep Bernoulli trails.

X_i - the Bernoulli r.v. for i^{th} trail.

$X = X_1 + X_2 + \cdots X_n$

$Var(x) = Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n)$

$= pq + \cdots + pq = npq$.

Important Property:

if $X_1 \sim Bin(n_1, p)$, $X_2 \sim Bin(n_2, p)$ and X_1, X_2 are independent.

then $x_1 + x_2 \sim Bin(n_1 + n_2, p)$.

Reason: Binormial Experiment (n0

- 2 outcomes each trail (s or F).
- $P(s) = p$, same for all trails
- independent trails.

$n_1 + n_2$ trails.