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Exponential Distribution ← simplest model for random time with no upper bound.

Applications: random time (≥ 0)

- time it takes to serve a customer
- lifetime of a component
- the time a patient services after a operation

Def: a random time T has an exponential distribution if the pdf ~~of~~ T is

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

mean, variance $\quad E(T) = \frac{1}{\lambda}$

$T \sim Exp(\lambda)$ $\quad Var(T) = \frac{1}{\lambda^2}$

$SD(T) = \frac{1}{\lambda}$

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Probability $P(a < x < b) = \int_a^b f(x)dx = \int_a^b \lambda e^{-\lambda x} dx$; where $a, b \geq 0$.

cdf: $F(x) = \int_0^x f(t)\,dt = 1 - e^{-\lambda x}, \quad x \geq 0$.

Survival Function:

T ~ random time: $exp(\lambda)$

$P(T > s) = \int_s^\infty f(t)\,dt = e^{-\lambda s}, \quad s \geq 0$

$F(x) = P(X \leq x)$

$P(T > s) = 1 - F(s)$ i.e. the cdf and survival function has sum $= 1$

Memoryless Property.

The exponential distribution is "memoryless". That is

$$P(\underbrace{T \geq t+s}_{\text{lifetime} \geq t+s} \mid \underbrace{T \geq t}_{\text{lifetime} \geq t}) = P(\underbrace{T \geq s}_{\text{lifetime} \geq s}), \text{ for all } s, t \geq 0$$

has an additional lifetime $\geq s$.

No aging effect (true for all exponentially distributed random time).

↳ future probability does not depend on the past info.

E.g. Suppose the lifetime of an electronic component is an exponentially distributed r.v. with the pdf. $f(t) = e^{-t}$, $t \geq 0$ (i.e $\lambda = 1$)

(a) Find the probability that a component survives over 2 years.

(b) Find the lifetime L which a typical component of this type is 50% certain to exceed.

(c) Find $E(T)$ and $SD(T)$

(d) Given that the component has functioned for 1 year, what is the probability that the component will work for at least an additional 2 years?

Sol. (a) $P(T>2) = e^{-t}\big|_{t=2} = e^{-2} = 0.135$

(b) 50th percentile

$P(T > L) = 0.5$

$e^{-L} = 0.5 \Rightarrow L = \ln 2 = 0.693$ years

(c) $E(T) = \frac{1}{\lambda} = 1 \quad SD(T) = \frac{1}{\lambda} = 1.$

(d) $P(T \geq 1+2 \mid T \geq 1) = P(T \geq 2) = e^{-2} = 0.135$

Poisson Arrival Process (when exp dist used to model the waiting time) 2 equivalent descriptions

I. Counts of arrivals

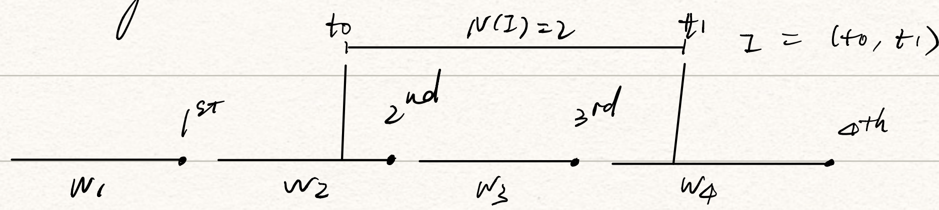
The distribution of the number of arrivals $N(I)$ in a fixed time interval I of length t is Poisson(λt), λ: rate of arrivals per unit time, and the number of arrivals in disjoint time intervals are independent.

II. time between intervals.

The distribution of the waiting time until the first arrival W_1 is exponential(λ), and W_1 and subsequent waiting time $W_2, W_3 \ldots$ between each arrival and the next are independent, all with the same exponential distribution $\exp(\lambda)$.

(W_2: the waiting time between 1st and 2nd arrival.

Graph



$I = (t_0, t_1)$

$N(I) \sim$ Poisson(λt)

length $= t_1 - t_0$.

Poisson dist, exp dist, gamma dist.

E.g. (Poisson Dist)

The number of customers arriving at a bus station follows a Poisson Process with the rate of 2 per minute.

(a) what is the probability that the time between 2 consecutive arrivals is at least 5 mins?

(b) what is the probability that no person arrives between $t=0$ and $t=5$ mins

(c) what is the probability that the third person takes more than 2 mins to arrive?

Sol: (a) $P(T \geq 5) = e^{-2 \cdot 5} = e^{-10} \simeq 4.5 \times 10^{-5}$

(b) $\sim$ Poisson (λt)

$P(N=0) = \frac{e^{-10} \cdot (10)^0}{0!} = e^{-10} \approx 4.5 \times 10^{-5}$.

i 1^{st} — 5^t — 2^{nd}

ii $\updownarrow$ no arrivals

$t=0$ $t=5$

(c) within 2 mins, # of arrivals ≤ 2.

N – # of arrivals within a time interval of 2 min

$N \sim$ Poisson $(\lambda \cdot 2)$ = Poisson (4)

$P(N \leq 3) = P(N=0) + P(N=1) + P(N=2)$

$= e^{-4} + e^{-4} \cdot 4 + e^{-4} \cdot \frac{4^2}{2!} = 13e^{-4} = 0.238$.

$P(W_1 + W_2 + W_3 > 2)$ sum of $W_1 + W_2 + W_3$ W = time until the 3^{rd} arrival

$\sim$ Gamma distribution

W_1 – time until 1^{st} arrival $\exp(\lambda)$ "sunen)

$\downarrow$

continuous analog to geometric dist.

Gamma $\Longleftrightarrow$ neg distribution.

$\downarrow$ $\downarrow$

time until r^{th} arrival # of trail until r^{th} success.

Gamma Distribution

For $\alpha > 0$, the Gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(x) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \quad \alpha > 0.$$

Properties

1) for $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

2) if $\alpha = n$ (n is positive integer), $\Gamma(n) = (n-1)!$

3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (i.e. $\int_0^\infty x^{-\frac{1}{2}} e^{-x} dx = \sqrt{\pi}$)

Def: a continuous r.v. X is said to be a Gamma Distribution with parameters (α, λ), if the pdf of X is

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0. \quad \text{where } \alpha > 0, \lambda > 0$$

Remarks: Gamma dist is defined by any $\alpha > 0$

if $\alpha = 1$, then Gamma $(\alpha = 1, \lambda)$ is $\exp(\lambda)$ (pdf w/ $\alpha = 1$)

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mean and Variance

- if $X \sim \text{Gamma}(\alpha, \lambda)$, then
  - $E(X) = \frac{\alpha}{\lambda}$
  - $\text{Var}(X) = \frac{\alpha}{\lambda^2}$
- if $\lambda = 1$, then Gamma $(\alpha, \lambda = 1)$ is called a standard Gamma Distribution and the mean & Variance are both α.

- If $X \sim \text{Gamma}(\alpha, 1)$, then the cdf

$$F(x; \alpha) = \int_0^x f(t)\, dt = \int_0^x \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t} dt > 0.$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^x \underbrace{t^{\alpha-1} e^{-t} dt}_{\text{incomplete Gamma Function}}, \quad x > 0$$

$$\underbrace{\Gamma(\alpha)}_{\text{Gamma function}} = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$F(\alpha; x) = 1$ ✓

- The cdf of Gamma (α, λ) $F(x; \alpha, \lambda) = P(x)$

$$\underset{\substack{\uparrow \\ \text{cdf of Gamma}}}{F(x; \alpha, \lambda)} = P(X \le x) = \underbrace{F(\lambda x; \alpha)}_{\substack{\uparrow \\ \text{cdf of standard gamma.}}}$$

- Gamma distribution probability

If $X_1, X_2, \cdots, X_n$ are iid r.v.s $\sim \exp(\lambda)$, (independent identically distribution.)

then $X_1 + X_2 + \cdots, X_n) \sim \text{Gamma}(\alpha = n, \lambda)$

E.g. A component with lifetime that is exponential distribution with failure rate 1 per 24 hours is put into service with a replacement component of the same kind which is substituted for the first one when it fails. What is the probability that the total time to failure is less than 40 hrs.

$T = T_1 + T_2$

life time of 1st comp

lifetime of 2nd comp.

$P(T \le 40)$

$T_1 \sim \exp(\frac{1}{24})$

lifetime (in hrs) of 1st componen.

$T_2 \sim \exp(\frac{1}{24})$

$T = T_1 + T_2 \sim \text{Gamma}(\alpha = 2, \frac{1}{24})$

$$P(T \le 4\ ) = \int_0^{40} f(x)\,dt$$

$$= \int_0^{40} \frac{(\frac{1}{24})^2}{\Gamma(2)} \cdot t^{2-1} \cdot e^{-\frac{1}{24}t}\,dt$$

$\Gamma(2) = (2-1)! = 1$

b/c $\Gamma(n) = (n-1)!$

$$= \frac{1}{24^2} \cdot \int_0^{40} \underset{u}{t} \cdot \underset{dv.}{e^{-\frac{1}{24}t}\,dt}$$

using integration parts

$\int u\,dv = uv - \int v\,du.$

$$= 0.496.$$

· other distribution.

- Let X follows a Gamma distribution with $\alpha = \frac{v}{2}$ (v is a positive integer), $\lambda = \frac{1}{2}$, the pdf of x

$$f(x; v) = \frac{1}{2^{\frac{v}{2}} \Gamma(\frac{v}{2})} x^{\frac{v}{2}-1} e^{-\frac{v}{2}}, \quad x \geq 0$$

we say x follows a chi-square distribution with v degrees of freedom, denoted $\chi^2(v)$ and read "chi-square-v".

$$\chi^2(v) \Downarrow \chi^2_v$$

If $X \sim \chi^2_v$, then $E(X) = v$, $Var(X) = 2v$

Recall. $E(X) = \frac{\alpha}{\lambda} = \frac{\frac{v}{2}}{\frac{1}{2}} = v$

$Var(X) = \frac{\alpha}{\lambda^2} = \frac{\frac{v}{2}}{(\frac{1}{2})^2} = 2v.$

Facts:

1. If $Z \sim N(0,1)$, then $Z \sim \chi^2(1)$
2. If $Z_1, \cdots Z_n$ iid $N(0,1)$, then

$$Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi^2_{(n)}$$

- Beta(α, β) Beta Distribution

Def: A r.v. X has a beta distribution if its pdf is

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1$$

when $\underbrace{\alpha > 0, \beta > 0}_{\text{parameters}}$

$0 < x < 1$ ↓ "range" of random variable.

- The uniform $(0,1)$ is a beta distribution with $\alpha=\beta=1$

$$f(x;\alpha=1,\beta=1) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \cdot x^{1-1}(1-x)^{1-1}, \quad 0<x<1$$
$$= 1.$$

- If $X \sim Beta(\alpha, \beta)$, then $E(X) = \frac{\alpha}{\alpha+\beta}$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- Cauchy Distribution

A r.v. X has a cauchy distribution with parameter θ, $-\infty<\theta<\infty$, if the pdf of X is

$$f(x;\theta) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}, \quad -\infty<x<\infty.$$

The expectation & variance DNE (does not exist).