Binorminal Distribution (2.2)

Def: Bernoulli Trails.

1. Each trail yeilds one of 2 possible outcomes, technically called success (S) and failure (F).
   "event of interest occurs".
2. For each trail, the probability of success P(S) is the same and is denoted by $P = P(S)$. The probability of Failure $P(F) = 1-p$ for each trail, and is denoted by q. So that $p+q=1$
3. Each are independent. The probability of success for one trail remains unchanged given the outcome of all the other trails.

Remark: if one of more condition are violated, then it's not a Bernoulli trail model.

Eg. 1. Determine if the stated experiment confirm to the model of Bernoulli Trails. If yes, determine P.

(a) Roll a dice and observe the number shows up.

No, violates C1.

(b) Roll a fair dice and observe whether number 6 shows up or not.

Yes $p = P(S) = \frac{1}{6}$.

(c) Roll 2 fair dices, and observe the sum of the points.

No, violates C1.

(d) Roll 2 fair dices, and observe whether the numbers show up are same.

Yes. $p = P(S) = \frac{6}{36} = \frac{1}{6}$.

(e) Roll a loaded dice, observe whether or not number 6 shows up or not.

$p = P(S) = \frac{?}{?}$ relative frequency $= \frac{\#(\ )}{\text{total } \# \text{trails}}$.

long run frequency

E.g. 2. the experiment, randomly draw 1 ball with replacement from a box containing 10 yellow balls and 20 black ones, observe if it's a yellow one.

Sol. BT? $p = P(S) = \frac{10}{30} = \frac{1}{3}$

if, without replacement, is it BT?

No. not independent.

1^{st} $P(S) = \frac{10}{30}$.

2^{nd} $P(S) = P(2nd\ Y \cap 1Y) + P(2nd\ Y \cap 1B)$

$= \frac{1}{3} \cdot \frac{9}{29} + \frac{2}{3} \cdot \frac{10}{29} = \frac{1}{3}$.

(C3) $\times$ $\underbrace{P(S \mid S)}_{\frac{9}{29}} \neq \underbrace{P(S)}_{\frac{1}{3}}$

E.g. 3. Suppose we have 4 Bernoulli trails, with $p = 0.7$, what is the probability that the first three yield success and the last fails.

Sol: $SSSF = S \cap S \cap S \cap F$

$P(SSSF) \underset{\text{by independent}}{=} P(S) \cdot P(S) \cdot P(S) \cdot P(F) = 0.7^3 \times 0.3 = 0.1029$.

Remark: if we have BT, we can use multiplication rule under independence to calculate probabilities.

$2^4 = 16$ possible outcomes.

SSSS	SSSF	SSFS	SSFF
SFSS	SFSF	SFFS	SFFF
FSSS	FSSF	FSFS	FSFF
FFSS	FFSF	FFFS	FFFF

If $P(S) = P(F) = 0.5$, then they are equally likely

E.g. Suppose we are interested in the number of students who prefers news from the Internet among a random sample of 4 students from a large university. If S: student prefer news from Internet, F: otherwise. $p = 0.6$, the proportion of students who prefer news from the Internet. Obtain all the probability of the possible number of students who prefers news from I among the 4 students.

Sol: Let's say k students prefer news from Internet.

$k = 0, 1, 2, 3, 4.$

$P(0) = P(FFFF) = q^4 = 0.4^4 = 0.0256.$

$P(1) = P(SFFF) = \binom{4}{1} \cdot q^3 \cdot p = 4 \times 0.4^3 \times 0.6$

FSFF
FFSF ← addition rule.
FFFS

$P(2) = P(SSFF) = \binom{4}{2} \cdot q^2 \cdot p^2 = 6 \times 0.4^2 \times 0.6^2.$

$P(3) = P(SSSF) = \binom{4}{3} \cdot q \cdot p^3 = 4 \times 0.4 \times 0.6^3$

$P(4) = P(SSSS) = 0.6^3$

Binomial Experiment.

a Binomial experiment consists of n Bernoulli trails.

- n trails with n fixed in advance
- each trail has 2 possible outcomes : S and F.
- $p = P(S)$, is constant from trail to trail.
- Trails are independent.

Binomial Distribution (n, p) parameters.

given a Binomial experiment, consisting of n trails, the probability of K successes among n trails is given by the binomial probability formula.

$$P(K \text{ successes in } n \text{ trails}) = \binom{n}{K} \cdot p^k \cdot q^{n-k}$$

$$\text{for } k = \underbrace{0, 1, \cdots, n}_{\text{for } n+1 \text{ possible \#S}}$$

The Binomial probabilities over integers $\{0, 1, \cdots n\}$, is called binomial (n, p) distribution

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e.g. ① what is the probability of getting 4 or more hands in 6 tosses of a fair coin?

Sol. the number of hands in 6 tosses follows a binomial distribution $(6, \frac{1}{2})$

$P(4 \text{ or more hands}) = P(4 \text{ or } 5 \text{ or } 6)$

$$= P(4) + P(5) + P(6)$$

$$= \binom{6}{4} \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 + \binom{6}{5} \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^1 + \binom{6}{6} \cdot \left(\frac{1}{2}\right)^6$$

$$= \frac{11}{38}$$

e.g. 2 what is the chance that among 5 families, each with 6 children, exactly 3 of the families have 4 or more boy?

Sol. "event of interest": number of 5 families (6 child) [have 4 or more boys]

$P(S) = P(\text{4 or more boys in 6-child family}) = \frac{11}{32}$

Binomial $(5, \frac{11}{32})$

$P(\text{exactly 3 families}) = \binom{5}{3} \cdot \left(\frac{11}{32}\right)^5 \cdot \left(\frac{21}{32}\right)^2 = .175$

e.g. 3. A fair dice is rolled 10 times, given that there were 4 sixes in the 10 rolls, what is the probability that there are 3 sixes in the first 4 rolls?

Sol. the number of sixes in 10 rolls follows a binormial distribution $(10, \frac{1}{6})$.

$$P(B|A) = \frac{P(AB)}{P(A)}$$

$AB = \{\text{4 sixes in 10 rolls}\} \cap \{\text{3 six in first 4 rolls}\}$

$$= \{ \underbrace{\overset{\text{3 sixes}}{\quad}}_{\text{first (4)}} \mid \underbrace{\overset{\text{1 six}}{\quad}}_{\text{(last 6)}} \}$$

$P(AB) = P[\text{3 sixes in the first 4 and 1 six in the last 6 rolls}]$

$$= \binom{4}{3} \times \left(\frac{1}{6}\right)^3 \times \frac{5}{6} \times \binom{6}{1} \times \frac{1}{6} \times \left(\frac{5}{6}\right)^6$$

$$P(A) = \binom{10}{4} \cdot \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right)^5$$

$$P(B|A) = \frac{6\ \cancel{24} \times \cancel{\left(\frac{1}{6}\right)^4} \times \left(\frac{5}{6}\right)^{\cancel{6}\,\cancel{7}\,2}}{5\ \cancel{\frac{10 \times 9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}} \times \cancel{\left(\frac{1}{6}\right)^4} \times \cancel{\left(\frac{5}{6}\right)^5}} = \frac{21}{240} = 0.1174.$$

e.g.4. A backpacking party carries 3 emergency signal flares, each of which light with a probability of 0.98. Assume the three flares are independent. What is the probability at least one flare lights?

Sol: P(at least one flare lights)

The # of flares light follows a binormial distribution (3, 0.98)

I: P(1, or 2, or 3)

II; or $1 - P(0) = 1 - \binom{3}{0} \times 0.98^0 \times 0.02^3 = 0.999992$

Most likely number of successes (mode of Binormial Distribution)

For $(0 < p < 1)$, the most likely number of successes in n independent trails with probability p of successes, is $m = [np + p]$, the largest integer less or equal to $np+p$

- If $np+p$ is integer, then there are 2 more likely numbers m and m-1
- If $np+p$ is not integer, then there is one unique most likely number m.

e.g. $[3.5] = 3$ $[3] = 3$.

$[3.7] = 3$

E.g.5(a) In 6-child families, each child is equally likely to be a boy or a girl, independent of the others. What number of boys would be the most likely number?

Sol: the number of boys in 6-child family follows binormial $(6, \frac{1}{2})$

$m = [np + p] = [3.5] = 3.$

(b) how about 5-child family?

Sol: $m = [5 \cdot \frac{1}{2} + \frac{1}{2}] = [3] = 3.$

since $np+p = 3$ is integer, 3 and 2 are most likely numbers.

Remark $n=6$, $p=0.5$

$P(0) = \binom{6}{0} \times .5^0 \times .5^6 = \binom{6}{0} \times 0.5^6 = 0.016$

$P(1) = \binom{6}{1} \times .5^1 \times .5^5 = \binom{6}{1} \times 0.5^6 = 0.093$

$P(2) \quad \vdots \quad \vdots \quad = 0.235$

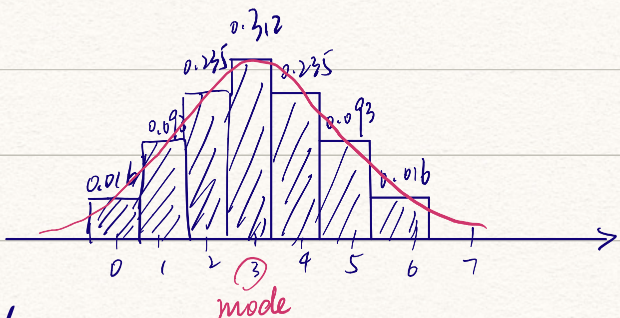
$P(3) \quad = 0.312 \qquad \binom{6}{3} = \frac{6\times5\times4}{3\times2\times1} = 20.$

$P(4) \quad = 0.235$

$P(5) \quad = 0.093$

$P(6) = \binom{6}{6} \times 0.5^6 \times 0.5^1 = \binom{6}{6} \times 0.5^6 = 0.16.$

Histogram: Binormial $(6, \frac{1}{2})$



Area of rectangle " probability

In general:

The probabilities are strictly increasing before they reach the maximum, and strictly decreasing after the maximum.

Expected number of success (mean of Binomial Distribution)

Def: The mean of binormial distribution (n, p), is np, also called the expected number of successes, usually denoted by μ, $\mu = np$.

Remark: ① μ is not necessarily an integer

$n = 5$, $p = \frac{1}{2}$, $\mu = 2.5$.

② If μ is an integer, then it is also the mode

$n = 6$, $p = \frac{1}{2} \Rightarrow \mu = np = 3$ is an integer

$[\underbrace{np}_{\text{Integer}} + \underbrace{p}_{\in(0,1)}] = np$

E.g. 6 A player bets on one number: 3 of the numbers 1 through 6. Three fair dice are then rolled independently. If the number bet, 3, appears k times, $k = 0, 1, 2, 3$, then the player must pay a fee \$1 for each round of the game. Is this a fair game?

(fair game: expected wins = expected loss)

Sol: the number of times 3 appears follows binominal $(3, \frac{1}{6})$

Expected wins = expected number of times 3 appears

$= np = 0.5$.

Expected loss $= ? = 1$

not a fair game.