Conditional Probability

The conditional probability of event A occurs, given that event B has occured, denoted by $P(A|B)$, read "P of A given B".

Counting formula for $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

if it is equally likely outcomes situation

$$P(A|B) = \frac{\#(A \cap B)}{\#(B)}$$

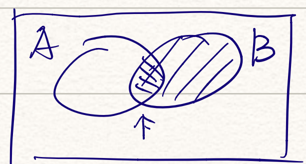
L3:

E.g. A family has 2 children. Assume all four possible outcomes (Boy, girls), (Boy, Boy), (Girl, Boy), (Girl, Girl) are equally likely. What is the probability that both are boys given that at least one is a boy?

Sol. A: both are boys

B: at least one is boy

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\#(AB)}{\#(B)} = \frac{1}{3}$$



$P(A|B)$

E.g. Prediction record of a TV weather forecaster over the past several years:

		Forecast Sunny	Cloudy	Rainy
Actually	Sunny	0.50	0.5	0.04
	Cloudy	0.04	0.10	0.02

Rainy 0.10 0.05 0.10

What is the prob of rain when forcast is Sunny.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.10}{(0.50+0.04+0.10)} = \frac{5}{32}$$

Multiplication Rule.

Overall $P(A|B) = \frac{P(AB)}{P(B)}$

$\rightarrow P(AB) = P(A|B) \cdot P(B)$ for any 2 events A, B.

If there are three events A, B, C, then

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|AB)$$

$$\text{or} = P(C) \cdot P(B|C) \cdot P(A|BC)$$

More generally, $P(A_1 \cap A_2 \cap \cdots A_n)$

$$= P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1A_2) \cdots P(A_n|A_1 \cdots A_{n-1})$$

E.g. A box contains 6 white ball and 5 blue balls.
Two balls are drawn sequently without replacement.
What is the prob of obtaining (W, B)?

Sol: A: the 1st white
B: the 2nd blue.

$$P(AB) = P(A) \cdot P(B|A)$$

$$= \frac{6}{11} \cdot \frac{5}{10} = \frac{3}{11}$$

The Law of total Probability.
or rule of coverage conditional probabilities.

Suppose $B_1, B_2 \cdots\cdots B_n$ is a partition of Ω, Then for any event A

$$P(A) = P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2) \cdots\cdots$$
$$= \sum_{i=1}^{n} P(A|B_i)\cdot P(B_i)$$

(Right hand side)

Proof: $RHS = P(A|B_1)\cdot P(B_1) + \cdots + P(A|B_n)\cdot P(B_n)$

multiplication rule: $= P(A\cap B_1) \cdots + P(A\cap B_n)$.

$A, B_1, AB_2 \cdots, AB_n$ disjoint $= P(AB_1 \cup AB_2 \cup \cdots \cup AB_n)$

$= P(A\cap(B_1 \cup B_2 \cup B_3 \cdots \cup B_n))$

$= P(A\Omega)$

$= P(A)$

$=$ LHS (Left Hand Side).

E.g Suppose two kids draw one card from a well shuffled deck of 52 cards. whenever get Diamond Jack get a cookie. (without replacement). They both want to draw firstly since they think if the Diamond Jack is drawn by the first kid, the the later one would have no chance. Is this right?

Sol: $A = \{$the first draw is Diamond Jack$\}$

$B = \{$the second draw is Diamond Jack$\}$

$P(A) = \frac{1}{52}$

$P(B) = \frac{51}{52} \cdot \frac{1}{51} = \frac{1}{52}$

the first is not diamond Jack.

E.g. There are 3 identical boxes. Box i contains i white balls and 1 black ball, $i = 1, 2, 3$. The boxes are well mixed and one box is selected randomly. After the box is selected, one ball is drawn from this box. Given that the ball drawn is white, what is the prob. that the box is box 2?

Sol: Box 1: 1W 1B | Box 2: 2W 1B | Box 3: 3W 1B.

A: the selected ball is white.

B_1: the selected box is box 1

B_2: - - - - - - - - 2

B_3: - - - - - - - - 3.

$$P(B_2|A) = \frac{P(B_2 A)}{P(A)}$$

$$P(B_2 A) = P(A|B_2)P(B_2) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$$

$$= \frac{1}{3}\left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4}\right) = \frac{1}{3} \cdot \frac{23}{12} = \frac{23}{36}$$

$$P(B_2|A) = \frac{P(B_2 A)}{P(A)} = \frac{2}{\cancel{9}1} \cdot \frac{\cancel{36}4}{23} = \frac{8}{23} \approx 35\%.$$

Independence

Two events A and B are independent if $P(A|B) = P(A)$.

or $P(B|A) = P(B)$

i.e the chance of B does not affect the chance of A

$P(AB) = P(A)P(B|A)$

Multiplication Rule: $P(AB) = P(A) \cdot P(B)$

formal mathematical definition of independence

If A, B indep, then $P(AB) = P(A) \cdot P(B)$

If $P(AB) \neq P(A) \cdot P(B)$, then A, B are not independent.

Check A, B independent:

$P(AB) \stackrel{?}{=} P(A) \cdot P(B)$ or $P(A|B) \stackrel{?}{=} P(A)$ or $P(B|A) \stackrel{?}{=} P(B)$

Remark: If A, B are independent,

then A and B^c are independent

- - - . A^c - - - B - - - - - - -

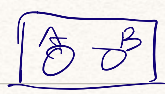
- - - A^c . . . B^c . - - - - - -

Proof: A^c and B are independent ⟸ Bonus

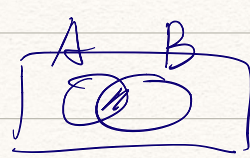
$P(A^cB) = P(A^c)P(B)$. 下周之前

Difference between disjoint & independent events A, B

- Disjoint $A \cap B = \phi$



- Independent $P(AB) = P(A) \cdot P(B)$



if $P(A) > 0$, $P(B) > 0$, $P(AB) > 0$

E.g.: $P(A)=0.5$, $P(B)=0.3$

(1) For what value of $P(AB)$ would A, B disjoint?

(2) — — — — — — — — independent?

Sol: (1) A, B is disjoint $A \cap B = \phi \Rightarrow P(AB) = 0$

(2) A, B independent, $P(AB) = P(A)P(B) = 0.15$