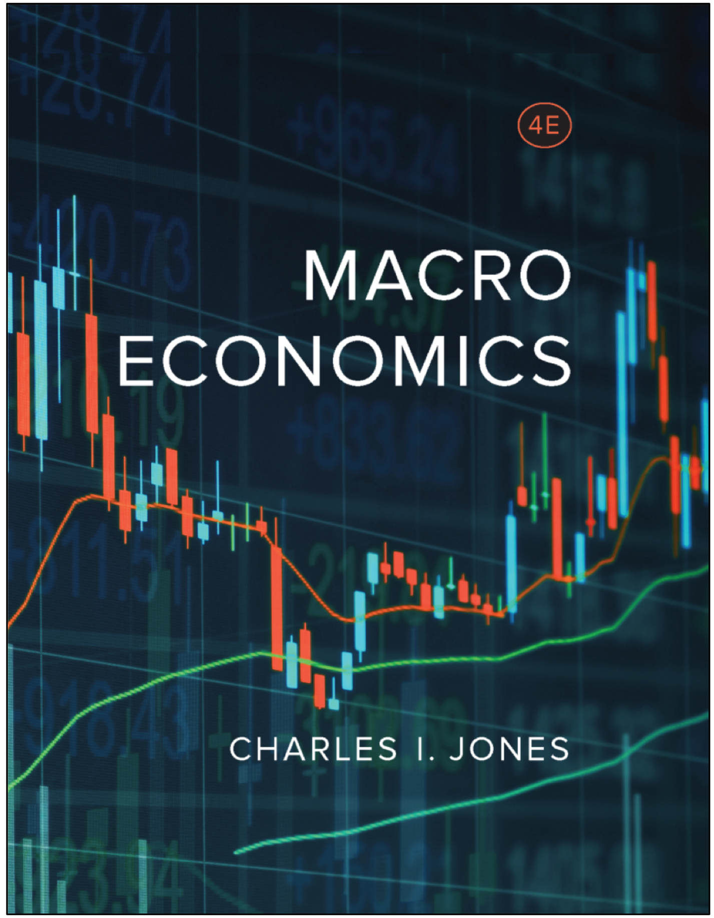
Chapter 5

The Solow Growth Model



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5.1 Introduction

- In this chapter, we learn:
  - how capital accumulates over time,
  - how diminishing MPK explains differences in growth rates across countries,
  - the principle of transition dynamics, and
  - the limitations of capital accumulation.

Changes in the Model

- The Solow Growth model:
  - Augments the production model with capital accumulation
    - capital stock is no longer exogenous
    - capital stock is now endogenized
- The accumulation of capital
is a possible engine of long-run economic growth.

5.2 Model Set-up (Production)

- Begin with the previous production model.
  - Add an equation for the accumulation of capital over time.
- The production function:
  - Cobb-Douglas
  - Constant returns to scale in capital and labor
  - Assume exponent of one-third on K ($\alpha = {}^{1}/_{3}$)
- Variables are time subscripted (t)

$$Y_t = F(K_t, L_t) = \underset{\text{constant}}{\bar{A}} K_t^{\alpha} L_t^{1-\alpha}$$

Model Set-up (Resources)

- Output can be used for consumption or investment: $C_t + I_t = Y_t$
  - C_t consumption
  - I_t Investment
- This equation is a "resource constraint".

一直会在宏观的

- Note that:
  - No imports / outports
  - no government spending (and taxes)

Capital Accumulation—1

- Goods invested for the future determine the accumulation of capital

  Law of motion (for K)

- **Capital accumulation equation:**

  economic value of capital scale

  $$K_{t+1} = K_t + I_t - dK_t$$

  - d: depreciation rate ($0 \le d \le 1$) usually $d = 0.07$ or 0.1
  - K_{t+1} = next year's capital
  - K_t: this year's capital
  - I_t: this year's investment.

Capital Accumulation—2

- Change in capital stock defined as:

$$\Delta K_{t+1} = K_{t+1} - K_t$$

- Thus:

$$\Delta K_{t+1} = I_t - \delta K_t$$

- Future capital depends on (net) investment of today.

Case Study: An Example of Capital Accumulation

- The initial amount of capital is 1,000 bushels of corn (can represent investment + consumption)
- The depreciation rate is 0.10 (= 10%)
- Investment = 200 bushels each period

$K_{t+1} = K_t + I_t - \bar{d}K_t$

Time, t	Capital, K_t	Investment, I_t	Depreciation, $\bar{d}K_t$	Change in capital, ΔK_{t+1}
0	1000	200	100	100
1	1100	200	110	90
2	1190	200	119	81
4	1271	200	127	73
5	1344	200	134	66
6	1410	200	141	59

↑ capital → ↑ depreciation → ΔK ↓

Model Set-up (Labor)

(exogenous)

- For simplicity, labor supply is not included.
- The amount of labor in the economy is given exogenously at a constant level.

(population boost economic growth?)

rich country have ↓ fertility rate (mostly?)

$$L_t = \bar{L}$$

Model Set-up (Investment)

- The economy consumes a fraction of output and invests the rest:

$$I_t = \bar{s} Y_t$$

  - I_t = Investment
  - $\bar{s}$: fraction of total output invested

- Therefore:

$$C_t = (1-\bar{s}) Y_t.$$

Consumption is the share of output not invested

Five Equations and Five Unknowns

Unknowns/endogenous variables: Y_t, K_t, L_t, C_t, I_t

Production function $Y_t = \bar{A} K_t^{\frac{2}{3}} L_t^{\frac{1}{3}}$

Capital accumilation $\Delta K_{t+1} = I_t - \bar{d}K$

Labor force $L_t = \bar{L}$

Resourse constraint $C_t + I_t = Y_t$

Allocation of Resources $I_t = \bar{s}Y$

Parameters: $\overline{A}, \bar{s}, \bar{d}, \overline{L}, \overline{K}_0$

no price level.

Case Study: Some Questions about the Solow Model

- Differences between the Solow model and production model:
  - Added dynamic of capital accumulation → how economy evolve.
  - capital and labor supplied inelastically.
- Why include the investment share but not the consumption share? 可推导
  - It would be redundent
  - Preserve 5 equations and 5 unknowns

Variables – Stock vs. Flow

- Stock variable

"Durable" good or service (home, airplane).

stock net flow

$$K_{t+1} = K_t + \underbrace{\underbrace{I_t}_{\text{inflow}} - \underbrace{dK_t}_{\text{outflow}}}_{\text{net flow}}$$

(consumption is also flow)

- Flow variable

a quality

- A change in the stock of capital is investment.

5.3 Prices and the Real Interest Rate

- Adding the wage and rental price:
  - Two more equations, two more unknowns
    - w=MPL and r=MPK
  - Omitting them does not change the model

trade off of giving today and investment for tomorrow.

- The real interest rate (in units of output):
  - amount earned by saving one unit of output for a year
  - cost to borrow one unit of output for a year
  - reflects opportunity cost of forgone consumption

Saving

- Saving = differences between income and consumption.

- Saving is equal to investment

$$Y_t - C_t = I_t$$

- where $Y_t - C_t$ is saving

$\Downarrow$

saving = investment

government $\left\{ \begin{array}{l} \text{private saving} \\ + \\ \text{public saving} \end{array} \right.$

$\| $

investment.

real interest rate = rental price of K = MPK

5.4 Solving the Solow Model

- The model needs to be solved at every point in time, which cannot be done algebraically.
- Two ways to make progress:
  - show a graphical sollution
  - solve for long-run equilibrium in the model
- Begin by combining equations algebraically

Solving the Solow Model

- Combine the investment allocation and capital accumulation equation:

$$\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_t$$

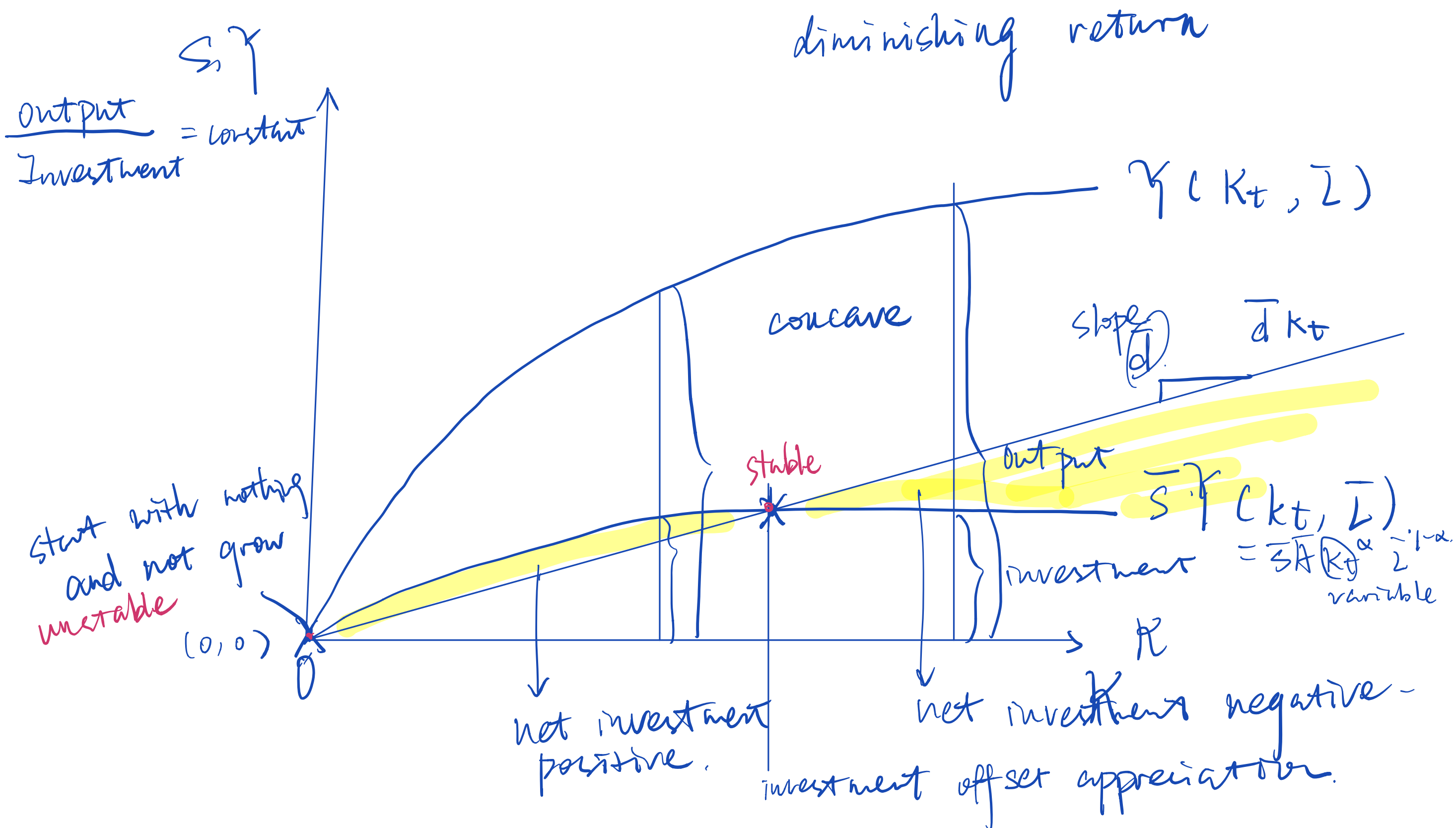
change in capital   net investment

- Substitute the fixed amount of labor into the production function:

$$Y_t = \bar{A}K_t^{\alpha}\bar{L}^{1-\alpha}$$

The Solow Diagram

assume $\bar{L}$ abor is constant



Using the Solow Diagram

- If the amount of investment > depreciation
  - capital stock will increase until $\bar{s}Y_t = \bar{d}K_t$.
    - Here, the change in capital is equal to 0.
    - The capital stock will stay at this value of capital forever.
    - This is called the steady state.
- If depreciation is greater than investment
  - the economy converges to the same steady state as above.

Model Dynamics

- When not in the steady state,

  the economy exhibits a change in capital toward the steady state

- As K moves to its steady state,

  output will move to its steady state

- At the rest point of the economy,

  all the endogenous variables are constant (or grow at a constant rates, depending on model assumptions

- Transition dynamics

  take the economy from its initial level of capital to the steady state.

The Solow Diagram with Output

Solving Mathematically for the Steady State

$\Delta K_{t+1} = \bar{s} Y_t - I_t$

$0 = \bar{s}\bar{A}K_t^{\alpha}\bar{L}^{1-\alpha} - \bar{d}K_t \Rightarrow$ can solve K_t $\Uparrow$ K^* not depend on time.

$\bar{s}\bar{A}K^{*\alpha}\bar{L}^{1-\alpha} = \bar{d}K^*$ $\quad K^* = \left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}\bar{L}$

- In the *steady state*, investment equals depreciation:

- Substitute the production function for Y_t^* :

Solving for the Steady State—1

- Solve for K^*

$$K_t^* = \bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}$$

- The steady state level of capital is:
  - Positively related to the
    - investment rate
    - the size of workforce
    - the productivity of the economy
  - Negatively correlated with
    - the depreciation rate

Solving for the Steady State—2

- Plug K^* into the production function to get Y^*:

$$Y_t^* = \bar{A} K_t^{*\alpha} \bar{L}^{1-\alpha}.$$

- Plug in our solved value of K^*:

$$Y = \bar{A}\bar{L}^{1-\alpha}\left(\bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}$$

$$= \bar{A}^{\frac{1}{1-\alpha}} \bar{L} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}$$

- Higher steady state production due to higher
  - productivity, investment rate
- Lower steady state production due to faster
  - depreciation

Solving for the Steady State—3

- Divide both sides by labor to get output per person in the steady state:

$$y^* = \frac{Y_t^*}{\bar{L}} = \bar{A}^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}$$

$$= \bar{A}^{\frac{1}{1-\alpha}} \cdot f(\bar{s}, \bar{d}) \Leftarrow \text{tfp matter more} \ ?$$

- Note the exponent on productivity is different here than in the production model.
  - Higher productivity has direct effect on y^*
  - Addition (indirect) effect via capital accumulation.

5.5 Looking at Data through the Lens of the Solow Model

- Recall the steady state: investment = depreciation

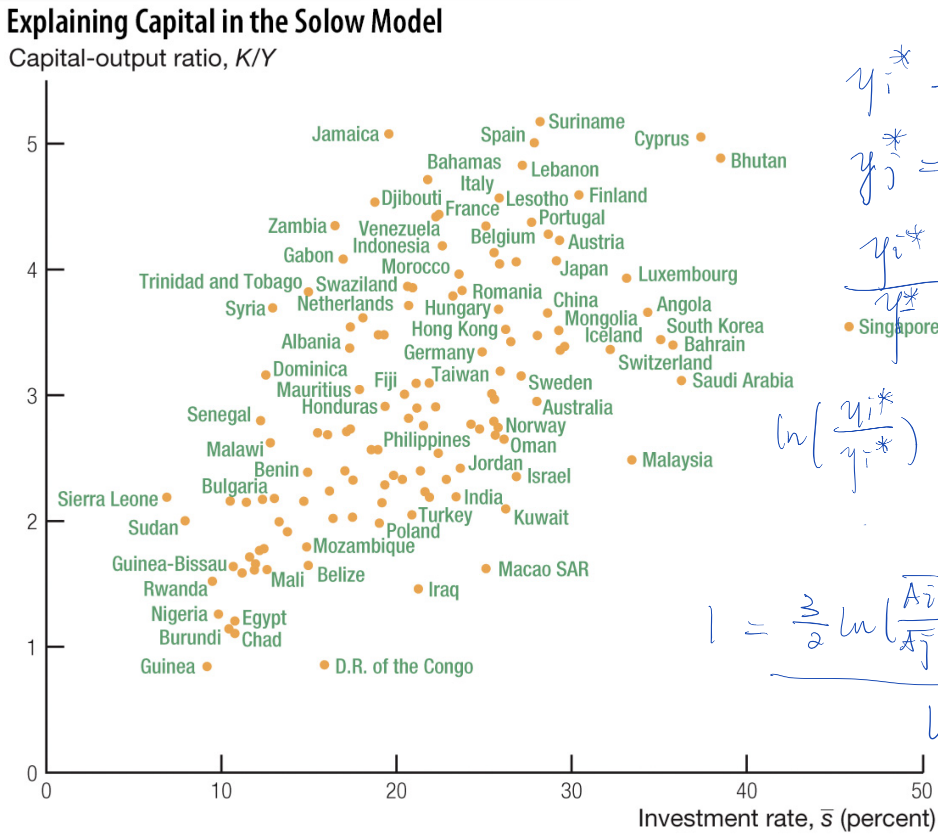
$$\bar{s}Y^* = \bar{d}K^*$$

- The capital to output ratio is the ratio of the investment rate to the depreciation rate:

$$\frac{K^*}{Y^*} = \frac{\bar{s}}{\bar{d}}$$

- Investment rates vary across countries
- It is assumed that the depreciation rate is relatively constant

Explaining Capital in the Solow Model



$$y_i^* = \bar{A}_i^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}_i}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}$$

$$y_j^* = \bar{A}_j^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}_j}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{y_i^*}{y_j^*} = \left(\frac{\bar{A}_i}{\bar{A}_j}\right)^{\underset{\Uparrow\,\frac{3}{2}}{\frac{1}{1-\alpha}}} \left(\frac{\bar{s}_i}{\bar{s}_j}\right)^{\underset{\Uparrow\,\frac{1}{2}}{\frac{\alpha}{1-\alpha}}}$$

$$\ln\left(\frac{y_i^*}{y_j^*}\right) = \frac{1}{1-\alpha}\ln\left(\frac{\bar{A}_i}{\bar{A}_j}\right) + \frac{\alpha}{1-\alpha}\ln\left(\frac{\bar{s}_i}{\bar{s}_j}\right)$$

$$1 = \frac{\frac{3}{2}\ln\left(\frac{\bar{A}_i}{\bar{A}_j}\right) + \frac{1}{2}\ln\left(\frac{\bar{s}_i}{\bar{s}_j}\right)}{\ln\left(\frac{y_i^*}{y_j^*}\right)}$$

Differences in *Y/L*

- The Solow model shows TFP is more important in explaining per capita output
  - Can be used to understand differences in *y*
- Take the ratio of *y** for two countries, assuming the depreciation rate is the same:

counts more.

$$\underbrace{\frac{y^*_{\text{rich}}}{y^*_{\text{poor}}}}_{70} = \underbrace{\left(\frac{\bar{A}_{\text{rich}}}{\bar{A}_{\text{poor}}}\right)^{3/2}}_{35} \times \underbrace{\left(\frac{\bar{s}_{\text{rich}}}{\bar{s}_{\text{poor}}}\right)^{1/2}}_{2}$$

investment rate.

$\frac{\ln(35)}{\ln(70)}$ $\frac{\ln(2)}{\ln(70)}$

contribution of tfp. investment

From Chapter 4 See figure 5.3 (previous slide) slide)

5.6 Understanding the Steady State

GDP not depends on time

- The economy reaches a steady state because:
  - investment has diminishing returns
  - the rate at which production and investment rise is smaller as the capital stock is larger
- Also, a constant fraction of the capital stock depreciates every period.
  - Depreciation is not diminishing as capital increases.
- Eventually, net investment is zero.
  - The economy rests in steady state.

5.7 Economic Growth in the Solow Model

- Important result: driven by capital accumulation ⇐ that grow
  - There is no long-run growth in the Solow model.
- In the steady state, growth stops, and
  - output,
  - capital,
  - output per person, and
  - consumption per person

  are **constant**.

Economic Growth in the Solow Model

- Empirically, however, economies appear to continue to grow over time.
  - Thus, we see a drawback of the model.
- According to the model:
  - Capital accumulation is not the engine of long-run economic growth.
  - After we reach the steady state, there is no long-run growth in output.
  - Saving and investment
    - are beneficial in the short run
    - do not sustain long-run growth due to diminishing returns

Case Study: Population Growth in the Solow Model

should have $\bar{A}$ changed overtime

$$Y_t = \bar{A} K_t^{\alpha} L_t^{1-\alpha}$$

$$K_{t+1} = (1-\delta) K_t + \bar{s} Y_t$$

$$L_{t+1} = (1+n) L \Rightarrow \text{new}$$

$$y_t = \frac{Y_t}{L_t} = \bar{A} \frac{K_t^{\alpha}}{L_t^{\alpha}} = \bar{A} k_t^{\alpha}$$

$$\frac{K_{t+1}}{L_t} = (1-\delta) \frac{K_t}{L_t} + \bar{s} \frac{Y_t}{L_t}$$

$$\Downarrow$$

$$\neq k_{t+1} \quad = (1-\delta) k_t + \bar{s} \bar{A} k_t^{\alpha}$$

$$\Downarrow$$

$$L_t (1+n) = L_{t+1}$$

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \left[(1-\delta) k_t + \bar{s} \bar{A} k_t^{\alpha}\right] (n+1) = k_t (1+n)$$

$$k^* = k_t = k_{t+1}$$

$$(1-\delta) k^* + \bar{s} \bar{A} k^{*\alpha} = k^* (1+n)$$

$$-\delta k^* + \bar{s} \bar{A} k^{*\alpha} = k^* n$$

$$\bar{s} \bar{A} k^{*\alpha} = (n+\delta) k^*$$

$$k^{*1-\alpha} = \frac{\bar{s} \bar{A}}{n+\delta}$$

$$k^* = \left(\frac{\bar{s} \bar{A}}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y^* = \bar{A} k^{*\alpha} = \bar{A}^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

constant in steady state

$$Y_t = y_t L_t = y^* L_t$$

$$\frac{Y_{t+1} - Y_t}{Y_t} = \frac{Y_{t+1}}{Y_t} - 1$$

$$= \frac{y^* L_{t+1}}{y^* L_t} - 1 = n \Rightarrow \text{GDP grow as population}$$

GDP per capita constant

$\Downarrow$

$\Uparrow$

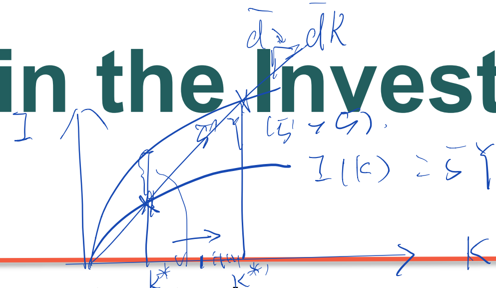
1fw

Comparison: Model of Production vs. Solow Model

5.8 Some Economic Experiments

- The Solow model:
  - Does not explain long-run economic growth
  - Does help explain some differences across countries
- Economists can experiment with the model by changing parameter values.

An Increase in the Investment Rate—1



- Suppose the investment rate increases permanently for exogenous reasons.
  - The investment curve rotates upward.
  - The depreciation curve remains unchanged.
  - The capital stock
    - *increases* by transition dynamics to reach the new steady state
    - this happens because investment exceeds depreciation
  - The new steady state
    - is located to the right
    - investment exceeds depreciation

An Increase in the Investment Rate—2

consumption $(1-s)y = (1-s)A(k\ldots)$

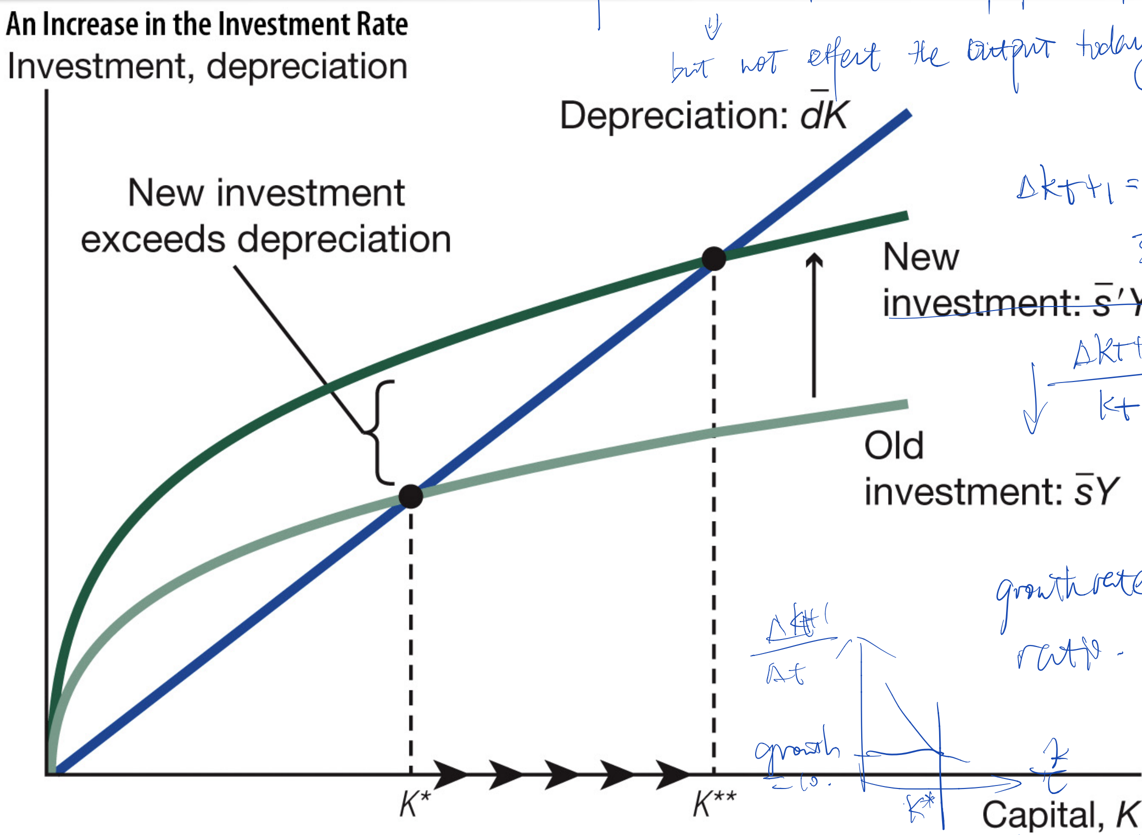
$= (1-s)\left(\frac{sA}{d}\right)^{\frac{\alpha}{1-\alpha}}$

$\frac{\partial y}{\partial s} = 0$

increasing investment ↑ -> steady point ↑.

⇓

but not effect the output today.



$\Delta k_{t+1} = \bar{s} Y_t - \bar{d} k_t$

$\bar{s} A k_t^{\alpha} L^{1-\alpha}$

$\downarrow \frac{\Delta k_{t+1}}{k_t} = \bar{s} A k^{\alpha-1} L^{1-\alpha}$

$= \bar{s} A \left(\frac{k}{L}\right)^{\alpha-1}$

growth rate of capital ratio.

$\frac{\Delta k_{t+1}}{\Delta t}$

growth $=0$

K^*

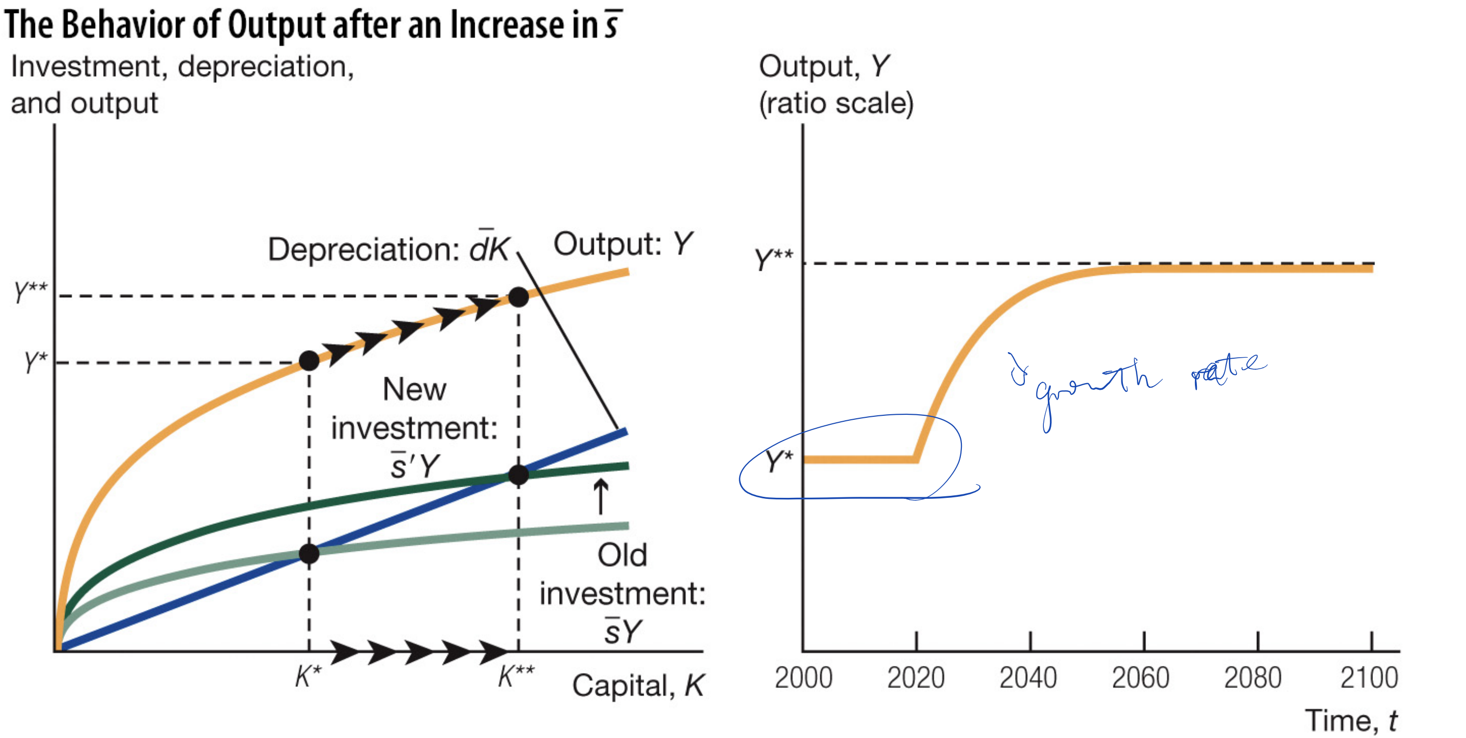
t

steady state.

An Increase in the Investment Rate—3

- What happens to output in response to this increase in the investment rate?
  - The rise in investment leads capital to accumulate over time.
  - This higher capital causes output to rise as well.
  - Output increases from its initial steady state level Y^* to the new steady state Y^{**}.

The Behavior of Output after an Increase in $\bar{s}$



(a) The Solow diagram with output.

(b) Output over time.

Experiments on Your Own

- Try experimenting with all the parameters in the model:
  - Figure out which curve (if either) shifts
  - Follow the transition dynamics of the Solow model
  - Analyze steady state values of:
    - capital (K^*)
    - output (Y^*)
    - output per person (y^*)

5.9 The Principle of Transition Dynamics

- If an economy is below steady state
  - It will grow
- If an economy is above steady state
  - It will shrink
- When graphing this, a ratio scale is used.
  - Output changes more rapidly if we are further from the steady state.
  - As the steady state is approached, growth shrinks to zero.

The Principle of Transition Dynamics

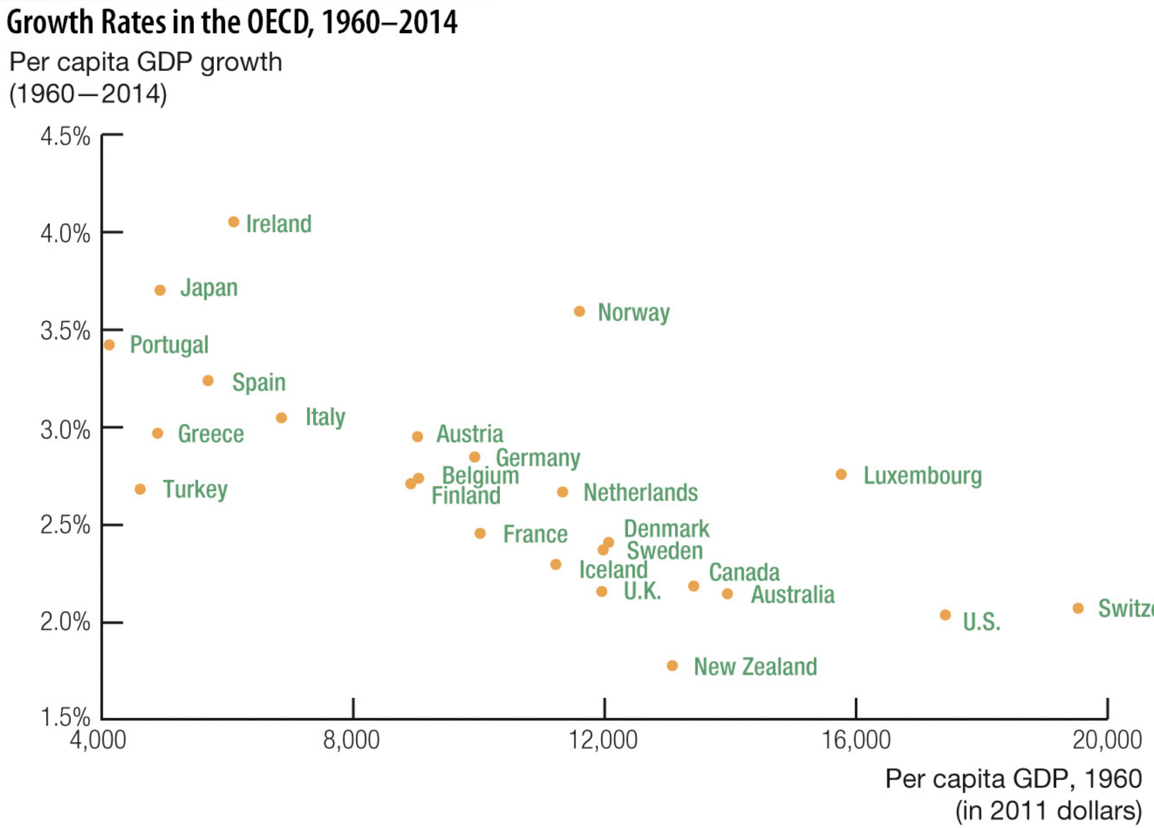
- The farther below its steady state an economy is, (in percentage terms)
  - the faster the economy will grow.
- The closer to its steady state,
  - the slower the economy will grow.
- Allows us to understand why economies grow at different rates

Understanding Differences in Growth Rates

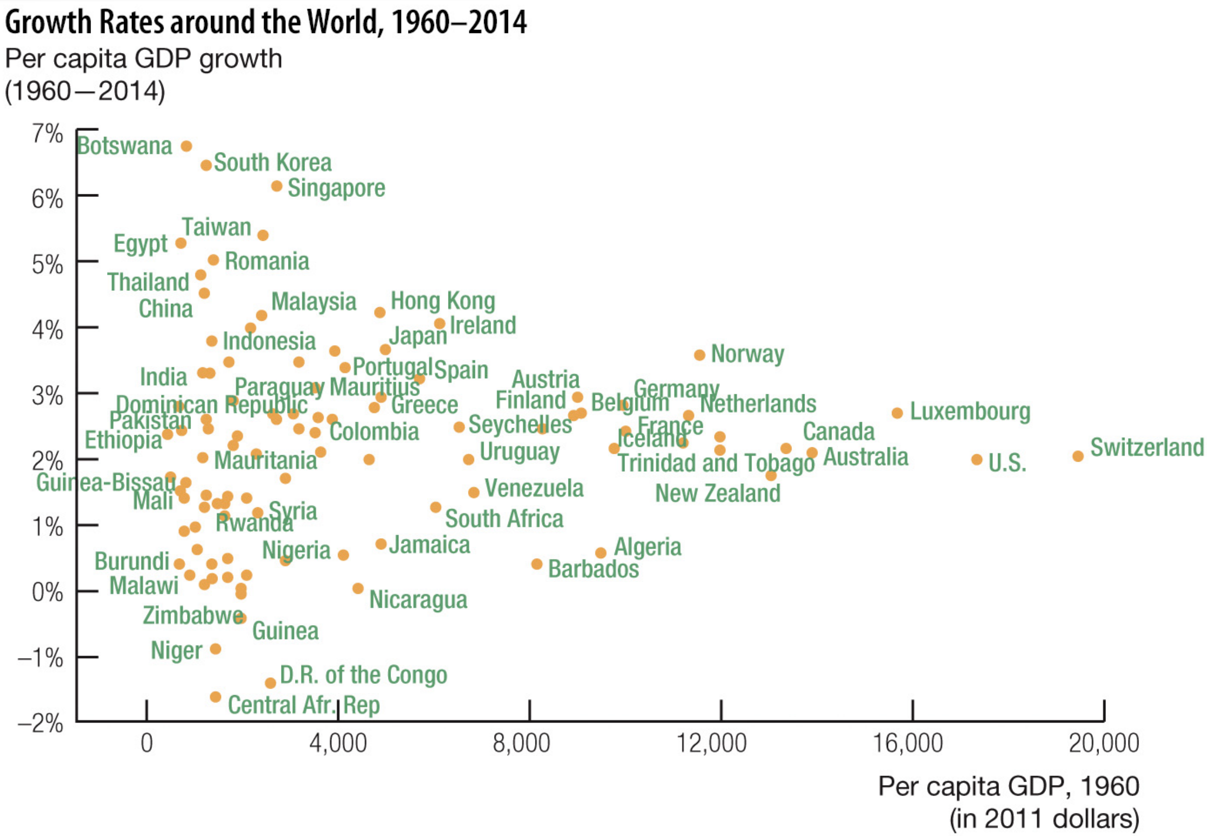
predict, provide a country for [illegible] [illegible]

- Empirically, for OECD countries, transition dynamics holds:
  - Countries that were poor in 1960 grew quickly
  - Countries that were relatively rich grew slower
- For the world as a whole, on average, rich and poor countries grow at the same rate.
  - Two implications of this:
    - most countries (rich and poor) have already reached their steady states
    - countries are poor not because of a bad shock, but because they have parameters that yield a lower steady state (determinants of the steady state invest rates and *A*)

Growth Rates in the OECD, 1960–2014



Growth Rates around the World, 1960–2014



5.10 Strengths and Weaknesses of the Solow Model

- The strengths of the Solow Model:
  - It provides a theory that determines how rich a country is in the long run.
    - long run = steady state
  - The principle of transition dynamics
    - allows for an understanding of differences in growth rates across countries
    - a country further from the steady state will grow faster

Strengths and Weaknesses of the Solow Model

- The weaknesses of the Solow Model:
  - It focuses on investment and capital
    - the much more important factor of TFP is still unexplained
  - It does not explain why different countries have different investment and productivity rates.
    - a more complicated model could endogenize the investment rate
  - The model does not provide a theory of sustained long-run economic growth.