Induction

Proof by induction

To prove P(*n*) is true for all natural numbers *n*: i.e. $(\forall n \in N)\ P(n)$ holds

1. State that we will give a proof by induction, define predicate P in terms of our variable, and state the base case and inductive step.
2. **Base case(or basis step):** Prove P(0) holds
3. **Inductive step:**
   - **Induction hypothesis:** assume that P(*k*) holds
   - Use the induction hypothesis to prove that P(*k*+1) holds

   $P(k) \Rightarrow P(k+1)$
4. Conclude that we have proved our statement by induction for all natural numbers *n*

if $P(0)$ and $(\forall k \in \mathbb{N})(P(k) \Rightarrow P(k+1))$, then $(\forall n \in \mathbb{N})\ P(n)$

Example 1

Prove $(\forall n \in \mathbb{N})\ P(n)$ holds.

Prove: (∀ *n* ∈ **N**) 0·0! + 1·1! + 2·2! + ... + *n*·*n*! = (*n* + 1)! - 1 (Note: by definition 0! = 1)

Proof by induction: Let P(*n*) : $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$

Base case: Prove P(0) holds

$P(0) = \sum_{i=0}^{k} i \cdot i! = 0 \cdot 0! = (n+1)! - 1$

$0 \cdot 0! = 0 \cdot 0 = 0$

$(n+1)! - 1 = 1! - 1 = 1 - 1 = 0$ $\quad P(0)$ holds.

Inductive step: Assume P(*k*) holds and prove P(*k*+1) holds

Induction hypothesis: P(*k*) holds, i.e., $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$ (IH) induction hypothesis.

want to prove $\sum_{i=0}^{k+1} i \cdot i! = ((k+1)+1)! - 1 \Leftarrow P(k+1)$

$\sum_{i=0}^{k+1} i \cdot i! = \left(\sum_{i=0}^{k} i \cdot i!\right) + (k+1)(k+1)!$ pull off last term

$= (k+1)! - 1 + (k+1)(k+1)!$ by IH.

$= (k+1+1)(k+1)! - 1 = (k+2)(k+1)! - 1 = (k+2)! - 1$

$= ((k+1)+1)! - 1$

So, $P(k+1)$ holds.

$\therefore$ by induction $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Example 2

Prove: $(\forall\ n \in \mathbf{N}^+)\ 2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$

In binary

$$\begin{array}{r} 1 \\ 10 \\ 100 \\ 1000 \\ +\,10000 \\ \hline 11111 \\ +\quad\ 1 \\ \hline 100000 \end{array}$$

i.e. $\overbrace{100000 - 1}^{2^n - 1} = \overbrace{11111}^{2^0 + 2^1 + \cdots 2^{n-1}}$

Proof by induction: Let P(n) : $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

Base case: Prove P(1) holds   needs to prove $2^0 = 2^1 - 1$

left side: $2^0 = 1$

right side: $2^1 - 1 = 2 - 1 = 1$ P(1) holds.

Inductive step: Assume P(k) holds and prove P(k+1) holds

Induction hypothesis: P(k) holds, i.e.,   P(k): $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$.

want to prove $\sum_{i=1}^{k+1} 2^{i-1} = 2^{k+1} - 1 \Leftarrow P(k+1)$

$$\begin{aligned} \sum_{i=1}^{k+1} 2^{i-1} &= \sum_{i=1}^{k} 2^{i-1} + 2^k \\ &= (2^k - 1) + 2^k \quad \text{by IH} \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

So. P(k+1) holds

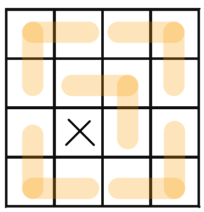
$\therefore$ by induction, $\sum_{i=1}^{n} i^{n-1} = 2^n - 1$ holds for all $n \in \mathbb{N}$.

More Induction Examples

Example 3

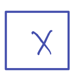
Prove: For every $n \in \mathbf{N}$, it is possible to tile a $2^n \times 2^n$ grid with L-shaped triominoes of the remaining $4^n - 1$ squares, leaving one (arbitrary) square uncovered.

Example: 4×4 grid



Proof by induction: Let P(n) : It is possible to tile $2^n \times 2^n$ grid such triominoes leaving an arbitrary uncovered.

Base case: Prove P(0) holds Suppose we have $2^0 \times 2^0$ grid, i.e. 1×1 grid. It is already tiled with 1 square (the only square) uncovered



□

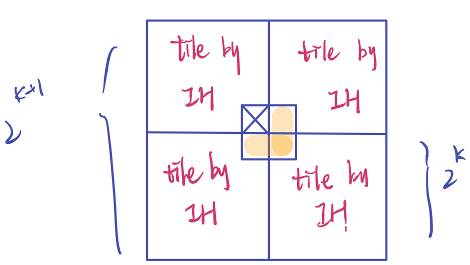
To convince ourselves, show P(1) holds

□

Inductive step: Prove P(k) ⇒ P(k+1)

Induction hypothesis: P(k) holds, i.e., It is possible to tile on $2^k \times 2^k$ grid with triominoes leaving an arbitrary square uncoverd (IH)

Consider a $2^{k+1} \times 2^{k+1}$ grid, divide grid into 4 squares of equal size ($2^k \times 2^k$)



- By IH, we can tile an $2^k \cdot 2^k$ grid leaving an arbitrary square open.
- So we can tile it leaving 4 center square open.

Then add 1 triomino

- To make sure we can do an arbitrary square left open: pick triomino so that open square in center is in desired quadrant

Given a square leave open, pick triomino so that open square in center is in desired quadrant of $2^{k+1} \times 2^{k+1}$ grid.

Then we (IH) to get the desired location in the $2^k \times 2^k$ quadrant to be left uncovered.

- So we can tile a $2^{k+1} \times 2^{k+1}$ grid leaving an arbitrary square open, so P(k+1) holds □

By induction, P(n) holds $\forall n \in \mathbb{N}$ ■

Motivating Strong Induction

<u>Prove</u>: Every positive natural number ($n \in \mathbb{N}^+$) can be expressed as a sum of distinct powers of 2.
For example, $21 = 2^4 + 2^2 + 2^0$, $10 = 2^3 + 2^1$, $32 = 2^5$

<u>Proof by induction</u>: Let P(n) : n can be expressed as a sum of distinct power of 2.

Goal: show that $(\forall n \in \mathbb{N}^+)\, P(n)$.

Base case: Prove P(1) holds

$1 = 2^0$ ← sum of distinct power of 2, so $P(1)$ holds □

Inductive step: Prove P(k) ⇒ P(k+1)

Induction hypothesis: P(k) holds, i.e., k can be expressed as a sum of distinct power of 2.

$k = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_i}$ for some distinct $a_1, a_2, \ldots, a_i \in \mathbb{N}$

Then $k+1 = 2^{a_1} + 2^{a_2} + \cdots 2^{a_i} + 1 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_i} + a^0$

but we can not gurantee $a_1, a_2 \cdots a_i, a^0$ are distinct.

If $2^{a_1} + 2^{a_2} + \cdots + 2^{a_i}$ is even, then $0 \notin \{a_1, a_2 \cdots a_i\}$.

If $2^{a_1} + 2^{a_2} + \cdots + 2^{a_i}$ is odd, then it is of the for $2 \cdot j + 1$ for some $j \in \mathbb{N}$ where $j < 2^{a_1} + 2^{a_2} + \cdots 2^{a_i}$, we don't know if $P(j)$ holds.

Strong Induction

Proof by strong induction

To prove P(*n*) is true for all natural numbers *n*:

1. State that we will give a proof by strong induction, define predicate P in terms of our variable, and state the base case and inductive step.
2. **Base case (or basis step):** Prove P(0) holds
3. **Inductive step:**
   - **Induction hypothesis:** assume that P(*j*) holds for all $j \in \mathbf{N}$ where $0 \le j \le k$.
   - Use the induction hypothesis to prove that P(*k*+1) holds
4. Conclude that we have proved our statement by strong induction for all natural numbers *n*

Example 1

Prove: Every positive natural number can be expressed as a sum of distinct powers of 2.
For example, $21 = 2^4 + 2^2 + 2^0$, $10 = 2^3 + 2^1$, $32 = 2^5$

Proof by strong induction: Let P(*n*) : n can be expressed as a sum of distinct power of 2

Base case: Prove P(1) holds

$1 = 2^0$ is sum of distinct power of 2, so P(1) holds □

Inductive step: Assume P(*j*) holds for all $j \in \mathbf{N}$ such that $1 \le j \le k$. Prove P(*k*+1) holds.

Induction hypothesis: Assume P(*j*) holds for all $j \in \mathbf{N}$ such that $1 \le j \le k$ (IH)
i.e. assume P(1), P(2), P(3) .. P(k) all hold.

Consider *k*+1

Case: *k*+1 is even: Then $\exists m \in \mathbb{N}$ s.t. $k+1 = 2\cdot m$
Since $k+1 \geq 2$, $m \geq 1$ Also $m \leq k$, i.e. $1 \leq m \leq k$. So (IH) holds for P(m)
So $m = 2^{a_1} + 2^{a_2} + \dots + 2^{a_i}$ for distinct $a_1, a_2 \dots a_i \in \mathbb{N}$
Then $k+1 = 2\cdot m$
$= 2\cdot(2^{a_1} + 2^{a_2} + \dots + 2^{a_i})$
$= 2^{a_1+1} + 2^{a_2+1} + \dots 2^{a_i+1}$
Since a_1, a_2 ___ a_i are distinct,
$a_1+1, a_2+1 \dots a_i+1$ are also distinct.
The k+1 can be expressed as a sum of distinct power of 2.

Case: k+1 is odd: The $\exists m \in \mathbb{N}$ s.t. $k+1 = 2 \cdot m + 1$

Since $k \geq 1$, $m \geq 1$, since $k = 2 \cdot m$, $m \leq k$, so $1 \leq m \leq k$, so (IH) holds for $P(m)$

By same argument above, $2 \cdot m$ can be expressed as a sum of distinct power of 2

moreover, $a_1 + 1$, $a_2 + 1$, ..., $a_i + 1$ are all ≥ 1

So $1 = 2^0$ is power of distinct from those in $2 \cdot m$.

Thus $k+1 = 2 \cdot m + 1$ can be expressed as a sum of distinct power of 2.

Since $k+1$ is either odd or even, $P(k+1)$ holds □

$\therefore$ by strong induction, $P(n)$ holds $\forall n \in \mathbb{N}^+$ ■

Example 2

Consider a game in which two players take turns removing any (positive) number of stones they want from one of two piles of stones. The player who removes the last stone wins the game. Prove that if the two piles contain the same number of stones initially, the second player can always guarantee a win.

Proof by strong induction: Let P(n) : Player 2 can always wins whenever there are n stones of each pile at the begining of the game.

Base case: Prove P(1) holds

if there is one stone in each pile, Player 1 takes 1 stone from a pile, Player 2 wins by take the same stone in the other pile □

Inductive step: Assume P(j) holds for all $j \in$ **N** such that $1 \leq j \leq k$. Prove P(k+1) holds.

Induction hypothesis: Assume P(j) holds for all $j \in$ **N** such that $1 \leq j \leq k$ (IH)

Suppose we start with k + 1 stones in each pile.

Case: Player 1 takes **all** k + 1 stones from one pile

The Player 2 takes all $k+1$ stones from the other pile.

Case: Player 1 takes m stones from one pile where $1 \leq m \leq k$, leaving k + 1 - m stones in that pile

$1 \leq k+1-m \leq k$, so (IH) holds.

By (IH) Player 2 can win if both piles start with $k+1-m$ stones.

So Player 2 takes m stones from the other pile. Now its pile one's turn, each pile has $k+1-m$ stones, and by (IH). Player 2 can win.

$\therefore$ by strong induction, player 2 can win.

Think about, what if a unequivent # stones pile ?

Invariants

Application of induction to systems that evolve in discrete time steps

- state of system, state at step t+1 only depend on state at step t.

Goal: Analyze these systems to determine properties about them

Where induction comes in:

Let P(t): The system has a property at time t.

If we can show

1. the system has property at beginning, $t=0$
2. if the system has property at time t, then it has the property at time $t+1$

then, by induction, the property always holds.

↳ so this is an invariant.
does not vary over time

Example

Consider the following game:

Given: an arbitrary number of "flippable" items (e.g., cards, coins, bits) with one orientation designated "up" (e.g., face up, heads, 1) and the other designated "down"

Start: arrange the items in a row, with some of them up and some of them down (at random)

Allowable moves: two consecutive items (i.e., two items side-by-side) can be simultaneously flipped (so up becomes down and down becomes up)

Goal: all items are up

Which of these games are winnable?

1. Starting arrangement: `1 1 0 1 0 0 1 1 0 1` yes
2. Starting arrangement: `0 1 1 0 0 1 0 1 0` no
3. Starting arrangement: `0 0 1 0 0 1 0 0 1 0` no
4. Starting arrangement: `1 0 0 1 1 0 1 0 1` yes

Is there a way to determine if a game is winnable by just seeing the starting arrangement?

Invariants (continued)

Proving an invariant

parity: *n* and *m* have the same **parity** iff either they are both odd, or they are both even.

Proposed invariant: The parity of the number of down items doesn't change.

$\forall n \in \mathbb{N}$ **Theorem 1:** After *n* moves, the number of down items has the same parity as the number of down items in the starting arrangement.

Proof by induction: Let **P(n):** After n moves, # down items has the same parity as #down items in starting arrangement.

Base case: show $P(0)$ holds. After 0 moves, we are still in the starting arrangement, so #down items = initial #down items.

Inductive step: show $P(k) \Rightarrow P(k+1)$ Let z_i = #down items after i moves.

Induction hypothesis: z_k and z_0 have same parity. (IH).

Consider move k+1

Case 1 ...00... → ...11... Then $z_{k+1} = z_k - 2$

Case 2 ...01... → ...10... $z_{k+1} = z_k$

Case 3 ...10... → ...01... $z_{k+1} = z_k$

Case 4 ...11... → ...00... $z_{k+1} = z_k + 2$.

So, overall change in # down item item is -2, 0, or 2.

Adding -2, 0, 2 to an integer doesn't chang its parity.

So, z_{k+1} and z_k have same parity.

So z_{k+1}, z_k, and z_0 have the same parity.

Thus $P(k+1)$ holds. $\square$

$\therefore$ by induction, $P(n)$ holds $\forall n \in \mathbb{N}$. $\blacksquare$

Can we determine if a game is winnable by just seeing the starting arrangement?

- If initial number of down items is odd, then since the goal is to have 0 down (even #), by invariant we know that we can't get an even number down starting with #odd down.
- If initial number of down items is even can we guarantee a win? Prove it.

Theorem 2: Any starting arrangement that has an even number of down items is winnable.

100 110 101 → 0000 111111

Theorem 3: Any starting arrangment that begins with an even number of down items followed by any number of up items is winnable.

Theorem 3: prove by induction.

P(n): if starting arrangement consists of 2n down items, followed by any # of up items, then the game is winable.

Base case: P(0) holds:

If 0 items are down, all items are up. Then game is already won.

Induction Hypothesis: A starting arrangement of 2k down items followed by any # of up items is winable.

Consider starting arrangement of 2(k+1) down items. followed by any # of odd items

$$\overbrace{0\underbrace{0\cdots0}_{2k}}^{2(k+1)}11\cdots1$$

By IH, the arrangement of items 3, 4 .... end has 2k down items and is thus winable.

can turn into 00 11 - - - 11

To win game, just flip 1 & 2. → 11 ...... 11 ⇒ Thus P(k+1) holds.

By induction, P(n) holds $\forall n \in \mathbb{N}$ ∎

Theorem 2.

- First sort the arbitrary starting arrangement to get all down items on left.
  - ↳ must play by the rules of game
- Then apply theorem 3.