§14. Fundamental Theorem of Calculus

L35 04.20

Idea: make rigorous the sense in which derivative & integral are inverse operations.

Q1. suppose $F(x) = \int_a^b f$, is $F' = f$?

Recall: we know if f is int'ble, then $F(x) = \int_a^b f(x)$ is cts

Proof: as f is int'ble, it is bdd.

$\exists M > 0$ s.t. $|f(x)| \leq M$.

So, if $h \neq 0$, $|F(x+h) - F(x)| = |\int_x^{x+h} f| \leq \int_x^{x+h} |f| \leq Mh$

So, as $h \to 0$ $F(x+h) - F(x) \to 0$ also ■

Thm 14.1 (FTC I): Let $f : [a,b] \to \mathbb{R}$ be int'ble. Define $F(x) = \int_a^b f$ for $x \in [a,b]$

If f is cts at $c \in [a,b]$, then F is diff'ble at c and $F'(c) = f(c)$

Note: if $c = a$ or b, we intepret this as one sided derivative

Proof: Let f be cts at $c \in (a,b)$.

Let $\varepsilon > 0$, As f is cts at c, $\exists \delta > 0$ s.t. if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$

Now let $|h| < \delta$ s.t. $c+h \in (a,b)$

So, $F(c+h)$ is well defined

Then
$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \left|\frac{1}{h}\left(\int_a^{c+h} f - \int_a^c f\right) - f(c)\right|$$
$$= \left|\frac{1}{h}\int_c^{c+h} f - f(c)\right|$$
$$\leq \left|\frac{1}{h}\int_c^{c+h} (f(x) - f(c))\, dx\right|$$
$$< \frac{1}{|h|}\int_c^{c+h} \varepsilon\, dx \quad \text{by cty as } |h| < \delta$$
$$= \frac{1}{|h|}\varepsilon \cdot |h|$$
$$= \varepsilon$$

So if $x \in (c, c+h)$, then $|x - c| \leq |h| < \delta$

Hence $\lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$ as required ■

Notation: if $x<a$ and f is int'ble on $[x,a]$, we write $\int_a^x f = -\int_x^a f$

Consider $G(x) = \int_x^b f$. Then suppose f cts at c

Note $G(x) = \int_x^b f = \int_a^b f - \int_a^x f$

So, $G'(c) = 0 - f(c) = f(c)$

So, we can differentiate W.R.T. either limit of the integral.

Now for $x<a$, assuming f cts, $F(x) = \int_a^x f$, then $F(x) = -\int_x^a f$. So $F'(x) = -(-f(x)) = f(x)$

So, this is consistent

Corollary 14.2 Let f be cts on $[a,b]$ and suppose $\exists g$ s.t. $f = g'$

Then $\int_a^b f = g(b) - g(a)$

Proof: Let $F(x) = \int_a^x f$. As f is cts, $F'(x) = f(x) = g'(x)$ on (a,b)

So, $(F-g)'(x) = 0 \;\forall x \in (a,b)$

Hence $(F-g)(x) = C \in \mathbb{R}$ for some C

Note $F(a) = \int_a^a f = 0$, so $-g(a) = C$

Thus $g(b) = F(b) - C = \int_a^b f + g(a)$

Hence $g(b) - g(a) = \int_a^b f$ ∎

Ex. Let $n \in \mathbb{N}$, Let $f(x) = x^n$. Then f is cts on $\mathbb{R}$ and $f(x) = g'(x)$ where $g(x) = \frac{x^{n+1}}{n+1}$

Hence, by Cor 14.2 $\int_a^b x^n = g(b) - g(a) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$

Now consider x^{-n}. To integrate, we need a bdd function.

So, to study $\int_a^b x^{-n}$, need either $0<a<b$, or $a<b<0$

Provided $n \neq 1$, x^{-n} is derivative of $\frac{x^{-n+1}}{-n+1}$,

So as x^{-n} is cts away from 0, if a,b lie on the same side of zero,

we have $\int_a^b x^{-n} = \frac{b^{-n+1}}{-n+1} - \frac{a^{-n+1}}{-n+1}$

So, we can integrate x^k $\forall k \in \mathbb{Z}$ except $k=-1$

because x^{-1} is not derivate of x^0

L36 04.22

Thm 14.3 (FTC II): Let $f: [a,b] \to \mathbb{R}$ be int'ble s.t. $f = g'$

Then $\int_a^b f = g(b) - g(a)$

Note: we not assume f is cts.

Proof: Let $P = \{t_0 \ldots t_n\}$ be a partition of $[a,b]$.

Let $m_i = \inf\{f(x) \mid t_{i-1} \le x \le t_i\}$, $M_i = \sup\{f(x) \mid t_{i-1} \le x \le t_i\}$

So, for any $x \in [t_{i-1}, t_i]$, $m_i(t_i - t_{i-1}) \le f(x)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})$ $(*)$

By MVT, $\forall i = 1 \ldots n$, as g is diff'ble on (t_{i-1}, t_i) & cts on $[t_{i-1}, t_i]$,

$\exists x_i \in (t_{i-1}, t_i)$ s.t. $g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1})$

$= f(x_i)(t_i - t_{i-1})$

By $(*)$, we therefore obtain $m_i(t_i - t_{i-1}) \le g(t_i) - g(t_{i-1}) \le M_i(t_i - t_{i-1})$

So $L(f,P) = \sum_{i=1}^n m_i(t_i - t_{i-1}) \le \sum_{i=1}^n g(t_i) - g(t_{i-1}) = g(b) - g(a) \le \sum_{i=1}^n M_i(t_i - t_{i-1}) = U(f,P)$.

As $L(f,P) \le g(b) - g(a) \le U(f,P)$ $\forall$ Partition P of $[a,b]$.

we must have that $\int_a^b f = g(b) - g(a)$ as required ∎

Examples where FTC fails

i) non-int'ble derivative Ex. $F(x) = \begin{cases} \frac{x^2}{2}\sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

By chain rule, F is diff'ble $\forall x \neq 0$

Also $\left|\frac{F(h) - F(0)}{h}\right| \le \frac{|h|}{2} \xrightarrow{h \to 0} 0$ so $F'(0) = 0$

So, $F'(x) = \begin{cases} 0 & x = 0 \\ x\sin\left(\frac{1}{x^2}\right) - \frac{1}{x}\cos\left(\frac{1}{x^2}\right) & x \neq 0 \end{cases}$

which is not bdd near 0, so not int'ble

So $\int_{-1}^{1} F'$ does not exist

Thus $\int_{-1}^{1} F'(x)\,dx \neq F(1) - F(-1)$ as LHS does not exist.

ii) non-diff'ble integral Ex. Let $f: [-1,1] \to \mathbb{R}$ be $f(x) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases}$

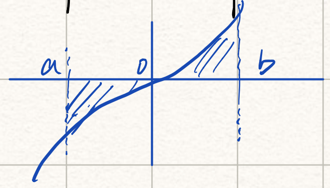
Then if $x \leq 0$, $F(x) = \int_{-1}^{x} f = \int_{-1}^{x} -1 = -1 + |x|$

which is not diff'ble at 0

(i.e. F is not diff'ble where f is not cts)

Note: integral gives a signed area

Ex (i)



$\int_a^b x^3\,dx = \int_a^0 x^3\,dx + \int_0^b x^3\,dx$

$= -\frac{a^4}{4} + \frac{a^4}{4}$

Area $= -\left(\int_a^0 x^3\,dx\right) + \left(\int_0^b x^3\,dx\right) = \frac{a^4}{4} + \frac{b^4}{4}$

(ii) area between 2 graphs, f and g $= \int_a^b |f-g|$

integrals of absolute value are not always easy to calculate

L37 04.24

Chain Rule: Let $\alpha: (c,d) \to (a,b)$ be diff'ble, $f: [a,b] \to \mathbb{R}$ be int'ble.

Suppose f is cts at $\alpha(x)$

Define $F: [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f$ & $G: (c,d) \to \mathbb{R}$ by $G(x) = \int_a^{\alpha(x)} f$

As f cts at $\alpha(x)$, we have, by FTC I, F is diff'ble at $\alpha(x)$,

so $F'(\alpha(x)) = f(\alpha(x))$

Note: $G = F \circ \alpha$.

By chain Rule, $G'(x) = F'(\alpha(x))\alpha'(x) = f(\alpha(x))\alpha'(x)$

i.e. $\frac{d}{dx}\int_a^{\alpha(x)} f(y)\,dy = f(\alpha(x))\alpha'(x)$

Ex. (i) Let $g(x) = \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}}\,dt$

$= \cancel{\frac{1}{\cos x}}$?

Then $g'(x) = \frac{1}{\sqrt{1-\sin^2 x}}\cdot\cos x = \frac{1}{\cos x}\cos x = 1$

So $g(x) = x + C$ for some $C \in \mathbb{R}$.

Clearly, $g(0) = \int_0^0 \frac{1}{\sqrt{1-t^2}}\,dt = 0$ So, $C = 0$.

Hence, $g(x) = x$.

$g = F\circ(\sin)$ where $F(y) = \int_0^y \frac{1}{\sqrt{1-t^2}}\,dt$.

Then F is $\sin^{-1}(y)$

(ii) $f(x) = \sin\left(\int_{x^2}^1 \frac{1}{1+t^2}\,dt\right)$

$f'(x) = \cos\left(\int_{x^2}^1 \frac{1}{1+t^2}\,dt\right) + \left(-\frac{1}{1+x^4}2x\right)$

Def: Let $f: [a,b] \to \mathbb{R}$ be bdd.

We know $\sup\{L(f,P) \mid P \text{ partition}\}$ & $\inf\{U(f,P) \mid P \text{ partitions}\}$ exist.

Call them the lower integral $L\int_a^b f$ & upper integral $U\int_a^b f$

If $L\int_a^b f = U\int_a^b f$, then f is int'ble & $\int_a^b f = L\int_a^b f = U\int_a^b f$

but this may not be true.

Ex. (i) if $c \in (a,b)$, then $L\int_a^c f + L\int_c^b f = L\int_a^b f$

$U\int_a^c f + U\int_c^b f = U\int_a^b f$

(ii) if $m \le f \le M$, then $m(b-a) \le L\int_a^b f \le U\int_a^b f \le M(b-a)$

Thm 14.4: Let $f:[a,b]\to\mathbb{R}$ be cts, then f is int'ble.

Proof: we will show $L\int_a^x f = U\int_a^x f \quad \forall x\in[a,b]$

define $L(x) = L\int_a^x f$, $U(x) = U\int_a^x f$

Let $x\in(a,b)$. If $h>0$, let $m_h = \inf\{f(y) \mid y\in[x,x+h]\}$.

$M_h = \sup\{f(y) \mid y\in[x,x+h]\}$.

We know $m_h \le f \le M_h$ on $[x,x+h]$

& as f is cts at x, $\lim_{h\to0} m_h = \lim_{h\to0} M_h = f(x)$

Then, $m_h\cdot h \le L\int_x^{x+h} f \le U\int_x^{x+h} f \le M_h\cdot h$

We rewrite this as $m_h \le \frac{L(x+h)-L(x)}{h} \le \frac{U(x+h)-U(x)}{h} \le M_h$

So, $\lim_{h\to0^+}\frac{L(x+h)-L(x)}{h} = f(x)$, $\lim_{h\to0^+}\frac{U(x+h)-U(x)}{h} = f(x)$

Similarly, $\lim_{h\to0^-}\frac{L(x+h)-L(x)}{h} = f(x)$, $\lim_{h\to0^-}\frac{U(x+h)-U(x)}{h} = f(x)$

Hence, $L'(x) = U'(x) = f(x)$

So, $(U-L)'(x) = 0 \quad \forall x\in(a,b)$

As $U(a) = L(a) = 0$, we have $U(x) = L(x) \quad \forall x\in[a,b]$, so f int'ble ∎

Ex. we believe $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, Near 0, $\frac{1}{2\sqrt{x}}$ is not bdd.

But, let $\varepsilon>0$, $a>0$, then $\frac{1}{2\sqrt{x}}$ is bdd and cts on $[\varepsilon, a]$

So, we can apply Thm 14.4 to see $\int_\varepsilon^a \frac{1}{2\sqrt{x}}\,dx$ exists.

Moreover, by FTC II, if $x>$, $\int_x^a \frac{1}{2\sqrt{x}}\,dx = \sqrt{a} - \sqrt{x}$

So we checked $\lim_{\varepsilon\to0^+}\int_\varepsilon^a \frac{1}{2\sqrt{x}}\,dx = \sqrt{\varepsilon}$, i.e. the limit exists

We define, for $a>0$, $\int_0^a \frac{1}{2\sqrt{x}}\,dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^a \frac{1}{2\sqrt{x}}\,dx \;(=\sqrt{a})$

This gives a integration for this unbdd function