The Normal Distribution

Def: the normal distribution with mean μ and standard deviation σ is the distribution over the x-axis defined by areas under the normal curve with these parameters.

The equation of the normal curve with parameters μ and σ, can be written as,

$$y = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{1}{2}\cdot\frac{(x-\mu)^2}{\sigma^2}}, \quad -\infty < x < \infty, \quad \text{"}x\text{-scale"}$$

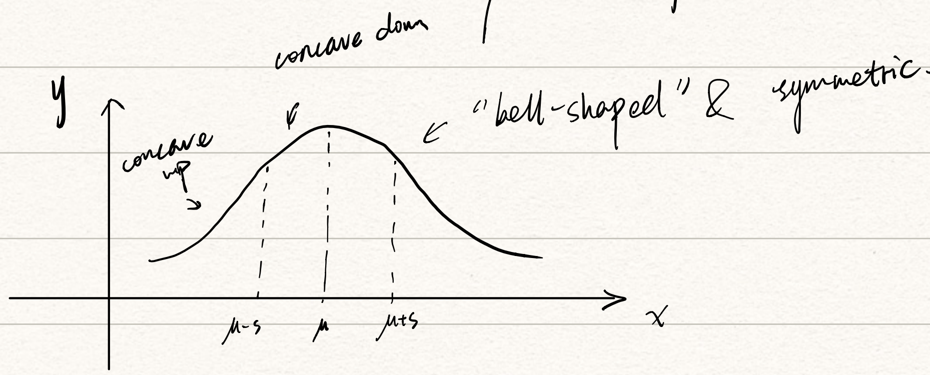
or

$$y = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{1}{2}\cdot z^2}, \text{ with } z = \frac{x-\mu}{\sigma}, \quad -\infty < x < \infty$$

Remark: z is x in standard units.

z measures the number of sd's from μ to x

Graph:



Remark: ① π, e – two fundamental constants.

$\pi \approx 3.14$, $e \approx 2.72$

② two parameter

mean: μ "location parameter".

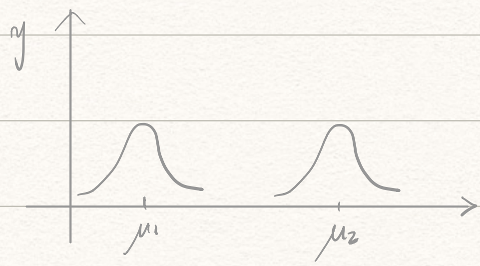
s.d: σ "shape parameter".

③ "bell-shaped" & symmetric (about μ)

eg. ① Two normal curve different means μ_1, μ_2 $(\mu_1 < \mu_2)$

and same standard deviations, s.

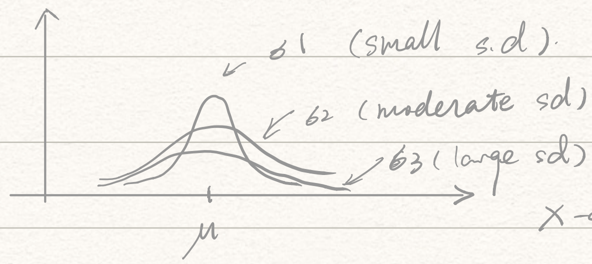
↳ same shape, different location



x-scale.

② Three normal curves with different standard devietions $\sigma_1, \sigma_2, \sigma_3$.

$(\sigma_1 < \sigma_2 < \sigma_3)$, and same means μ.



x-scale.

Total area under any normal curve $= 1$ (μ, s)

μ: can be any real number (+, −, or 0)

σ : $\sigma > 0$

The standard normal distribution

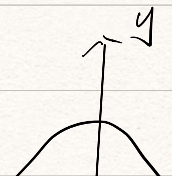
Def: The particular normal distribution that has a mean $\mu = 0$ and sd $\sigma = 1$ is called standard normal distribution.

$$z = \frac{x-\sigma}{\sigma}$$

The standard normal curve $y = \phi(z)$, where $\phi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot z^2}$, $-\infty < z < \infty$

is called the standard normal density function.

Graph:



← normal with $\mu = 0$, $\sigma = 1$

x-scale

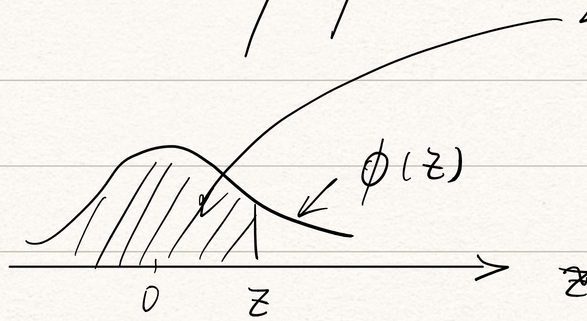
The standard normal distribution is defined by areas under the standard normal curve.

Recall: the area under standard normal curve $=1$

Q: How to find probabilities for standard normal distribution?
$(\mu=0,\ \sigma=1)$

(1) the probability to the left of a specified value of z in standard normal distribution.

$\Phi(z)=P$ (to the left of z) $=$ Area under the standard normal curve to left of z.



z-scale.

[Mathematically,

$$\Phi(z)=\int_{-\infty}^{z}\phi(y)\,dy,\quad -\infty<z<\infty$$

$\Phi(z)$ is called the standard normal commulative distribution function or standard normal c.d.f for short.]

[$\Phi(z)$ has been calculated numerically, value of $\Phi(z)$ are tabulated in Appendix 5 for $z\geq 0$.]

e.g. $P(\text{to left of } 1.23) = \Phi(1.23) = 0.8907$ (P537)

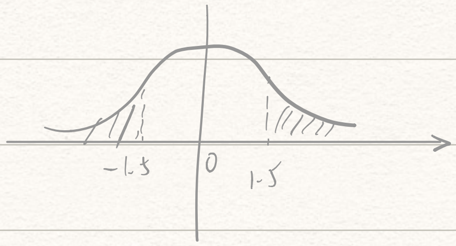
$P(\text{to the left of } 1.5) = \Phi 1.5 = 0.9332$.

$P(\text{to the left of } -1.5) = \text{Area to left of } -1.5$

$\Phi(-1.5)$

$= \text{Area to the right of } 1.5$

$= 1 - \Phi(1.5) = 0.0668$.

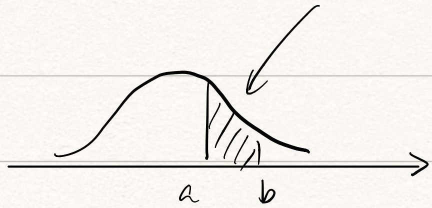


z-scale

In general, for negative z $\quad \Phi(z) = 1 - \Phi(-z)$

(2) Probability of an interval $[a, b]$

$P(\text{between } a \text{ and } b) = \text{Area under curve between } a \text{ and } b.$



$= [\text{Area to the left of } b]$

$- [\text{Area to the left of } a]$

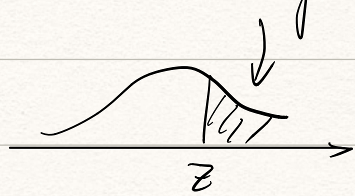
$= \Phi(b) - \Phi(a)$

E.g. $P(\text{between } 1.12 \text{ and } 1.6) = \Phi(1.6) - \Phi(1.12) = 0.9452 - 0.8686$

$= 0.0766$

(3) Probability to the right of a specified value of z.

$P(\text{to the right of } z) = \text{Area under curve to the right of } z$



$= 1 - \text{Area under curve to the left of } z$

$= 1 - \Phi(z)$

e.g. ① P(to the right of 1.37) = ?

Sol. $\Phi(1.37, \infty) = 1 - \Phi(1.37) = 1 - 0.9147 = 0.0853$

② P(to the left of -1.9 or to the right of 2.1)

$= P(\text{to the left of } -1.9) + P(\text{to right of } 2.1)$

$= \Phi(-1.9) + \Phi(2.1, \infty) = (1 - \Phi(1.9)) + (1 - \Phi(2.1))$

$= 0.0466$

Q: How to find probability of normal distribution (μ, σ)?

E.g. for a normal distribution $\mu = 200$, $\sigma = 5$, Find probability between 190 and 200.

Sol: X $\Phi(190, 200)$ ← refer to standard c.d.f.

step 1: convert to z scale. (standardization)

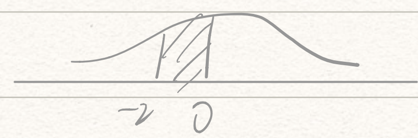
$$\frac{190-\mu}{\sigma} = \frac{190-200}{5} = -2.$$

$$\frac{200-\mu}{\sigma} = 0.$$

step 2: $\Phi(-2, 0) = \Phi(0) - \Phi(-2)$

$= 0.5 - (1 - 0.9772)$

$= 0.4772$



Normal Approximation to Binomial Distribution (n, p).

E.g. Binomial$(100, \frac{1}{2})$. Find probability between 45 and 55.

Sol P(between 45 and 55)

$= P(45) + P(46) + \cdots + P(55)$

$= \binom{100}{45}\left(\frac{1}{2}\right)^{45}\left(\frac{1}{2}\right)^{55} - - - - - + \binom{100}{55}\cdot\left(\frac{1}{2}\right)^{55}\cdot\left(\frac{1}{2}\right)^{45}$

= tedious.

w/o computation

Normal curve → Binomial (n, p)

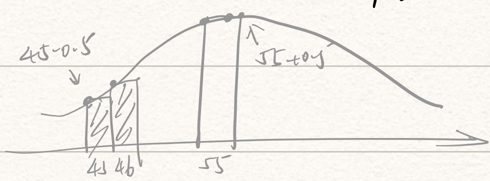
μ, σ? ⇒ $\mu = np$, $\sigma = \sqrt{npq}$

P(a to b success) $\approx \Phi\left(\frac{b+0.5-\mu}{\sigma}\right) - \Phi\left(\frac{a-0.5-\mu}{\sigma}\right)$

inclusive

Remark: $+/-\ 0.5$: continuity correction.

essential $\sqrt{npq} = \sigma$ is small



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E.g. (continue) Binomial $(100, \frac{1}{2})$

P(45 to 55 success) $\approx \Phi\left(\frac{55+0.5-50}{5}\right) - \Phi\left(\frac{45-0.5-50}{5}\right)$

$\approx \Phi(1.1) - \Phi(-1.1)$

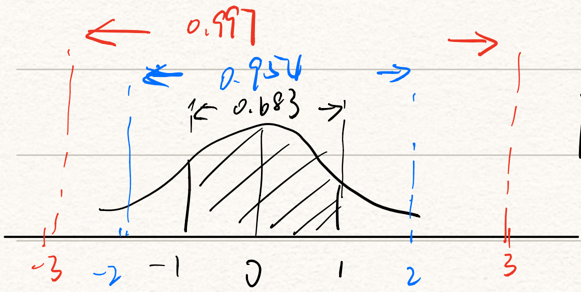
$= 2\times\Phi(1.1) - 1 \approx 0.0796$ (exact. 0.0728--)

P(50 success) ⇒ use "a = b = 50"

$\approx \Phi\left(\frac{50+0.5-50}{5}\right) - \Phi\left(\frac{50-0.5-50}{5}\right) = 2\times\Phi(0.5)-1$

$= 2\times 0.5398 - 1 = 0.0796$ (exact: .079589)

For standard normal distribution ($\mu = 0$, $\sigma = 1$)



$P(-1$ to $1) = \Phi(-1, 1) = 0.683$

[note: $\Phi(1) - \Phi(-1) = 2\Phi(1) - 1$].

$\Phi(-2, 2) = 0.954$

$\Phi(-3, 3) = 0.997$.

$\Phi(3.59) = 0.9998 \Rightarrow n > 3.59,\ \Phi(n) = 1$

$\Phi(100) = 1$

E.g. $P(\text{at least } 35) = P(\geq 35) = P(35 \text{ to } 100)$

$P(\text{greater than } 35) = P(>35) = P(36 \text{ to } 100)$

Law of complement $\quad P(\geq 35) = 1 - P(<35) = 1 - P(0 \text{ to } 34)$

$= 1 - P(\leq 34)$