Cooperate Game

Before: individual player makes decision individually.

assume no enforcing agreement between players

Now: binding agreement

Coalitions / groups

Coalitional game

- players
- set of actions for each coalition
- perferences.

E.g. Two-player unanimity game

individual coalition

coalitions: $S = \{1\}, \{2\}, \{1,2\}$ - grand coalition.

Output: $V(S)$: $V(\{1\}) = 0$ $\quad V(\{2\}) = 0$ $\quad V(\{1,2\}) = 1$

Action: $a_N = (x_1, x_2)$ s.t. $x_1 + x_2 = 1$ $\quad x_1 \geq 0$ $\quad x_2 \geq 0$

E.g. Land owner and workers

m workers

owner and k workers produce $f(k+1)$

workers without owner produce 0

action of coalition = allocation to members.

Transferable payoffs

sum of payoffs for coalition S is $V(S)$

E.g. House allocation.

n players

each player has one house

Coalition can reallocate houses among members

actions = allocations

$\{1\} = h_1$ to 1 — actions

$\{1,3\}$ $(h_1 \to 1,\ h_3 \to 3)$ or $(h_1 \to 3,\ h_3 \to 1)$

Players may have different payoffs for different houses

no transferable payoffs

E.g. Merriage Market.

Group of men and group of women.

Coalitions actions:

pairs between men and women in group

and possibly some unpaired.

preferences over potential partners or being single.

no transferable payoffs.

Above games are cohesive

for any coliations and actions, the grand coliation (all players involved) always has an action that is at least as good for all players

E.g. players { Andy: microchip set sells for \$900
Bill: software sells for \$100

Combination: sells for \$3000, \$2000 for surplus

Divide: \$3000 between X_A, X_B
for Andy, for Bill

BATNA: best alternative to a negotiated agreement

$d_A = 900$, $d_B = 100$, $X_A \geq d_A$, $X_B \geq d_B$.

Nash Bargaining:

no extra action to make to have one player better off w/o make another worse off.

Axiomatic: (EFF) Pareto efficiency

(SYM) Symmetry

If players have equal bargaining power, they get the same amount.

(INV) Invariant to linear transformation

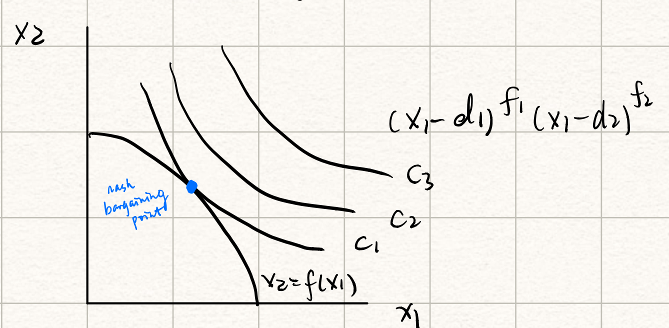
if transform payoff linearly, no change in preference.

(IND) Independent of irrelevant alternatives

Solution will maximize $(u(x_1) - u(d_1))^{f_1} (u(x_2) - u(d_2))^{f_2}$

subject to $x_2 = f(x_1)$ ($x_1 + x_2 = V$ here)

f_1, f_2: bargaining power, $f_1 + f_2 = 1$



$$\max\ (x_1-d_1)^{f_1}(v-x_1-d_2)^{f_2} \qquad (x_1+x_2=v)$$

$$\Rightarrow f_1\ln(x_1-d_1)+f_2\ln(v-x_1-d_2)$$

$$\Rightarrow \frac{f_1}{x_1-d_1}-\frac{f_2}{v-x_1-d_2} \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{f_1}{f_2}=\frac{x_1-d_1}{x_2-d_2}$$

E.g. Andy & Bill

$$\text{Max}\,(u(x_1)-u(d_1))^{f_1}(u(x_2)-u(d_2))^{f_2} \qquad \text{(when cooperate)}$$

$$\Rightarrow \frac{x_1-d_1}{x_2-d_2}=\frac{f_2}{f_1}$$

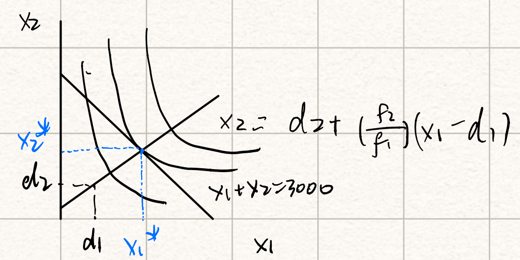
① outside option
② smart, persuasion

Suppose $\frac{x_1-900}{x_2-100}=\frac{0.8}{0.2}$

$0.2x_1 = 100+0.8x_2$

Also, $x_1+x_2=3000$

$x_1 = 2500$
$x_2 = 500$



The Core

players coalition/actions, payoffs

Core of a game: set of all stable actions by grand coal

is set of actions, a_N, such that no coalition (including N)

has an action that's all its members prefer to action in a_N.

E.g. two player unanimity

$v(\{1\})=0=v(\{2\})$

$v(\{1,2\})=1 \qquad x_1+x_2=1 \quad x_1\geq 0,\ x_2\geq 0$

all allocations are in core

E.g. Landowner - Workers game

k people, including landowner, generate $f(k)$

any coliation without landowner generates 0

Assume 2 workers (core)

$x_1 + x_2 + x_3 = f(3)$ $x_1 \geq f(1)$, $x_2, x_3 \geq 0$

$x_1 + x_2 \geq f(2)$

$x_1 + x_3 \geq f(2)$

E.g. $f(1) = 2$, $f(2) = 5$, $f(3) = 8$

$x_1 \geq 2$ $x_1 + x_2 \geq 5$, $x_1 + x_3 \geq 5$, $x_1 + x_2 + x_3 = 8$

$x_1 = 4$, $x_2 = x_3 = 2$

E.g. Indivisible good

Owner starts with good

2 buyers with money

owner values good at 0 no transferable payoff

buyers value good at 1

coalition rearrange goods and money of its members

actions are S - allocations

2 actions in core: owners sells good to one buyer for \$1

↑

if <1, the one not have the good will coalition with owner at higher price (≤1)

If one buyer values the object at 2

Core: ↓ buys the good for between \$1 and \$2

E.g. Matching Market.

Two sided one-to-one

Each X can be matched with at most one Y. and vice versa

$M(i)$: partner of i

strict preference ranking, including being single.

x_1	x_2	x_3	y_1	y_2	y_3
y_2	y_1	y_1	x_1	x_2	x_1
y_1	y_2	y_2	x_3	x_1	x_3
	y_3		x_2	x_3	x_2

Core: Pairing s.t. no rearrangement of pairings can make those involved better off.

Ex. $M(x_1)=y_1$, $M(x_2)=y_2$, $M(x_3)=x_3$, $M(y_3)=y_3$, is in core.

E.g. Matching

deferred acceptance

procedure with proposals

1. One side (Say, X)

   makes proposal to most preferred.

2. Other side (Say, Y)

   chooses most preferred proposal or reject all (if all unacceptable)

3. If all proporsal are accepted,
process is over and matches are set.

4. If some proposal is rejected, rejected Xs make proposal to next favorite.

5. Ys receiving propersals
choose best option available.

6. Repeat until no more propersals are possible.

x_1	x_2	x_3	y_1	y_2	y_3
y_2	y_1	y_1	x_1	x_2	x_1
y_1	y_2	y_2	x_3	x_1	x_3
	y_3		x_2	x_3	x_2

	1	2	3	4
x_1	$\to y_2$	R	$\to y_1$	
x_2	$\to y_1$ R	$\to y_2$		
x_3	$\to y_1$		R	$\to y_2$ R

Ends

X_3 does not have anymore preference

$M(x_1) = y_1$ $M(x_2) = y_2$, $M(x_3) = x_3$, $M(y_3) = y_3$.

E.g. Matching (another preference)

x_1	x_2	x_3	y_1	y_2	y_3
y_1	y_1	y_1	x_1	x_1	x_1
y_2	y_2	y_3	x_2	x_3	x_2
y_3	y_3	y_2	x_3	x_2	x_3

X's proposal

	1	2
x_1	$\to y_1$	
x_2	$\to y_1$ R	$\to y_2$
x_3	$\to y_1$ R	$\to y_3$

$M(x_1) = y_1 \quad M(x_2) = y_2, \quad M(x_3) = y_3$

Y's proposal

	1	2
y_1	$\to x_1$	
y_2	$\to x_1$ R	$\to x_3$
y_3	$\to x_1$ R	$\to x_2$

$M(x_1) = y_1, \quad M(x_2) = y_3, \quad M(x_3) = y_2.$

E.g Exchanging homogenous horses

Identical horses

At most 1 horse / person

At least 2 owners, 2 nonowners

Person i values horse at v_i

Some trade is mutually desirable.

Payoff: $v_i + r$ if ends up with a horse & \$r

r if ends up w/o a horse & \$r

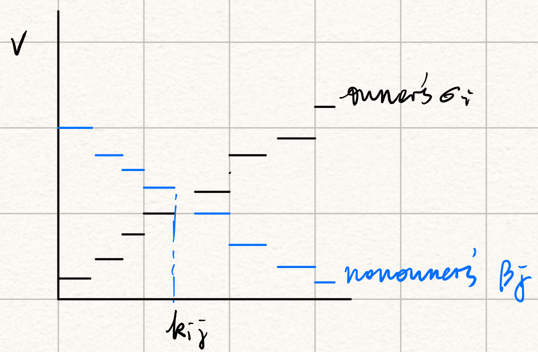
Coalition action is S-allocation of horses and money.

No transferable payoff (coalition does not have a fixed value)

Core: sort owners by value (call it σ_i) and nonowner by value (β_i)

so that σ_i is increasing, β_j is decreasing in j

(σ_i smallest, β_i highest)



k is the largest number s.t. $\beta_k > \sigma_k$

L is the set of sellers

B is the set of buyers

r_i is money received by owner i

p_j is money paid by nonowner j

Core: $p_j = 0$ $\forall$ nonowners not in B (not buy)

$r_i = 0$ $\forall$ owners not in L (not sell)

$p^* = p_j = r_i$ for every owner in L and nonowner j in B.

If $r_i < p_i$ for some i, j.

then they could exchange at $\frac{r_i + p_i}{2}$

So $r_i \geq p_j$ for all i in L and j in B

conservation of money $\Rightarrow$ $p^* = r_i = p_j$

$\sigma_{k+1} \geq p^* \geq \sigma_k$, $\beta_k \geq p^* \geq \beta_{k+1}$

owners $1 \dots k$ are in L

nonowners $1 \dots k$ are in B.

The core can be empty

E.g. Roomate Problem

i	j	k
j	k	i
k	i	j
l	l	l

(j, l) (i, k)

not stable