§10. differentiation

L19 03.04

Thm 10.1. Suppose f, g are diff'ble at a. Then $f+g$ is diff'ble at a and $(f+g)'(a) = f'(a) + g'(a)$

Proof: Let $h \neq 0$, then $\frac{(f+g)(a+h) - (f+g)(a)}{h}$

$$= \frac{(f(a+h) + g(a+h)) - (f(a) + g(a))}{h}$$

$$= \frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}$$

As f is diff'ble at a, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

As g is diff'ble at a, $\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

Then, by sum rule, $\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \right) = f'(a) + g'(a)$

i.e. $(f+g)(a)$ is diff'ble at a and $(g+f)'(a) = g'(a) + f'(a)$

L20 03.06

Thm 10.2 Suppose f, g are differentiable at a. The fg is diff'ble at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

Proof: Let $h \neq 0$, then $\frac{(fg)(h+a) - (fg)(a)}{h}$

$$= \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \frac{(f(a+h) - f(a))g(a+h)}{h} + \frac{f(a)(g(a+h) - g(a))}{h}$$

As f is diff'ble at a, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

As g is diff'ble at a, $\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$

As g is diff'ble at a, g is cts at a. So $\lim_{h \to 0} g(h+a) = g(a)$

Hence, by product rule for limits.

$$\lim_{h \to 0} \left(\frac{(f(a+h) - f(a))g(a+h)}{h} + \frac{f(a)(g(a+h) - g(a))}{h} \right)$$

$$= \left(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right) \left(\lim_{h \to 0} g(a+h) \right) + f(a) \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = f'(a)g(a) + f(a)g'(a)$$

$\blacksquare$

Cor 10.3: If $c \in \mathbb{R}$, f is diff'ble at a, then $(cf)'a = cf'(a)$

Proof: As derivative of c at a is 0, from the product rule, we have

$$(cf)'(a) = (c')(a)f(a) + cf'(a) = cf'(a) \quad ∎$$

Ex. Let $f(x) = x^n$, $n \in \mathbb{N}$. Then $f'(x) = nx^{n-1}$

Proof. By induction

Base case: We know if $n=1$. $\frac{d}{d(x)}(x)\Big|_a = 1 = 1 \cdot x^0$

Now suppose for $k \in \mathbb{N}$ that $\frac{d}{d(x)} x^k = kx^{k-1}$

Then by product rule $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k) \cdot x + x^k \frac{d}{dx} x$

$$= kx^{k-1} \cdot x + x^k = (k+1)x^k \quad ∎$$

Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$.

Then by previous example, Cor 10.3 & Thm 10.1

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

$$p''(x) = n(n-1)a_n x^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \cdots + 2a_2$$

Inductly (exercise)

$$p^{(k)}(x) = n(n-1)\cdots(n-(k-1))a_n x^{n-k} + (n-1)(n-2)\cdots(n-k)a_{n-1}x^{n-k-1} + \cdots + k!a_k \quad (1 \le k < n)$$

So $p^{(n)}(x) = n!a_n$, $p^{(k)}(x) = 0$ if $k > n$.

Thm 10.4 Suppose f, g are diff'ble at a and $g(a) \neq 0$

Then $(\frac{f}{g})$ is diff'ble at a and $(\frac{f}{g})'(a) = \frac{g(a)f'(a) - g'(a)f(a)}{g(a)^2}$

Proof: By product rule, it's enough to show $(\frac{1}{g})'(a) = -\frac{g'(a)}{g(a)^2}$

Let $h \neq 0$. As g cts at a, $g(a) \neq 0$, $\exists \delta > 0$ s.t. if $|h| < \delta$, then $g(a+h) \neq 0$.

So, Let $|h| < \delta$, also $\lim_{h \to 0} g(a+h) = g(a)$

Now $\dfrac{(\frac{1}{g})(a+h) - \frac{1}{g}(a)}{h} = \dfrac{g(a) - g(a+h)}{h\, g(a)\, g(a+h)}$

As $\lim_{h\to 0} \dfrac{g(a) - g(a+h)}{h} = -g'(a)$,

$\lim_{h\to 0} \dfrac{\frac{1}{g}(a+h) - \frac{1}{g}a}{h} = -\dfrac{g'(a)}{g(a)^2}$

Ex. Let $n \in \mathbb{Z}$. Then $(x^n)' = nx^{n-1}$ if $x \neq 0$

Proof: Suppose $n = 0$. Then $x^n = 1$ $\forall x$ So $(x^n)' = 0 = nx^{n-1}$

Suppose $n \leq -1$. Then $-n \in \mathbb{N}$, Hence $(x^{-n})' = (-n)x^{-n-1}$

So $(x^n)' = \left(\dfrac{1}{x^{-n}}\right)' = \dfrac{(-n)x^{-n-1}}{x^{-2n}} = nx^{n-1}$ by quotient rule.

Fact: $\dfrac{d}{dx}\sin(x) = \cos(x)$ $\qquad \dfrac{d}{dx}\cos(x) = -\sin(x)$

By product rule, $\dfrac{d}{dx}\sin^2(x) = 2\sin(x)\cos(x)$

$\dfrac{d}{dx}\cos^2(x) = -2\sin(x)\cos(x)$

Hence $\dfrac{d}{dx}(\sin^2(x) + \cos^2(x)) = 0$

Suppose f, g, h diff'ble at a.

Then $(fgh)'(a) = (fg)'(a)h(a) + (fg)(a)h'(a)$

$= f'(a)g(a)h(a) + f(a)g'(a)h(a) + f(a)g(a)h'(a)$

Inductively, Leibniz formula

$(f_1 \cdots f_n)'(a) = \sum_{k=1}^{n} f_1(a) \cdots f_k'(a) \cdots f_n(a)$

Ex. $f(x) = x^3 \sin(x)\cos(x)$

$f'(x) = 3x^2\sin(x)\cos(x) + x^3\cos^2(x) - x^3\sin^2(x)$

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Chain Rule: $(f\circ g)'(a) = f'(g(a))\,g'(a)$

make sense if f diff'ble at $g(a)$, g diff'ble at a

Ex. (i) Let $f(x) = \sin(x^2) = (\sin\circ q)(x)$ where $q = x^2$

Then $f'(x) = \cos(q(x))\,q'(x) = 2x\cos(x^2)$

(ii) Let $g(x) = (q\circ\sin)(x) = \sin^2 x$

Then $g'(x) = q'(\sin x)\cdot\sin' x = 2\sin x\cdot\cos x$

Agrees with product rule.

Note: $((f\circ g)\circ h)'(x) = (f\circ g)'(h(x))\,h'(x)$

$= f'(g(h(x)))\,g'(h(x))\,h'(x)$

$= f'(g(h(x)))\,(g\circ h)'(x)$

$= (f\circ(g\circ h))'(x)$

Associativity is preserved

Thm 10.6. (Chain rule): Let f be diff'ble at $g(a)$, g diff'ble at a.

Then $f\circ g$ is diff'ble at a and $(f\circ g)'(a) = f'(g(a))\,g'(a)$.

Not a proof. Consider $\dfrac{(f\circ g)(a+h) - (f\circ g)(a)}{h}$

$$= \frac{f(g(a+h)) - f(g(a))}{h}$$

$$\overset{(*)}{=} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \overbrace{\frac{g(a+h) - g(a)}{h}}^{g'(a)}$$

Let $k = g(a+h) - g(a)$

$$(*) = \frac{f(g(a)+k) - f(g(a))}{k} \cdot \frac{g(a+h) - g(a)}{h}$$

As g cts on a, $\lim_{h\to 0} k(h) = 0$

So, taking $h \to 0$, get $\lim_{h\to 0} \frac{f(g(a)+k) - f(g(a))}{k} \cdot \frac{g(a+h)-g(a)}{h}$

$$= f'(g(a))\, g'(a)$$

Proof: define $\varphi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & g(a+h) - g(a) \neq 0. \\ f'(g(a)) & g(a+h) - g(a) = 0 \end{cases}$

Claim: $\varphi(h)$ is cts at $h = 0$, So $\lim_{h\to 0} \varphi(h) = \underset{\substack{\uparrow \\ g(a+h) = g(a)}}{\varphi(0)} = f'(g(a))$

Assuming the claim, Let $h \neq 0$

Then $\frac{f(g(a+h)) - f(g(a))}{h} = \varphi(h) \frac{g(a+h) - g(a)}{h} \quad \forall h \neq 0$

As g is diff'ble at a and φ cts at $h=0$,

$$\lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h\to 0} \varphi(h) \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$$= f'(g(a))\, g'(a)$$

Proof of Claim:

Let $\varepsilon > 0$, need $\delta > 0$ s.t. if $|h| < \delta$, then $|\varphi(h) - \varphi(0)| < \varepsilon$.

As g is cts on a, $\exists \delta_1 > 0$ s.t. if $|h| < \delta_1$, then $|g(a+h) - g(a)| < \delta_2$

where δ_2 is defined as follows.

As f is diff'ble at $g(a)$, $\exists \delta_2 > 0$ s.t. if $|k| < \delta_2$, then

$$\left| \frac{f(g(a)+k) - f(g(a))}{k} \right| = |f'(g(a))| < \varepsilon.$$

Let $k(h) = g(a+h) - g(a)$, let $|h| < \delta_1$.

If $k(h) \neq 0$, $\varphi(h) = \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} = \frac{f(g(a)+k) - f(g(a))}{k}$

if $k(h) = 0$, then $\varphi(h) = f'(g(a)) = \varphi(0)$

So, if $|h| < \delta_1$, $|\varphi(h) - \varphi(0)| < \varepsilon$. Hence, φ is cts at 0. ∎

Ex. (i) Let $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Recall $f'(0) = 0$.

For $x \neq 0$, $f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$

So $f'(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Not cts on 0.

(ii) Let $g(x) = \begin{cases} x^3 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$g'(x) = \begin{cases} 3x^2 \sin(\frac{1}{x}) - x\cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

is cts on at 0.

$\lim_{h \to 0} \frac{h\cos(\frac{1}{h})}{h} = \lim_{h \to 0} \cos(\frac{1}{h})$ does not exist.

So g is not diff'ble.

Prop 10.7 Let $n \in \mathbb{N}$

i) if $f(x) = \begin{cases} x^{2n} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

then $f(0), f'(0), \dots, f^{(n)}(0)$ exists, but $f^{(n)}(0)$ is not cts at 0.

ii) if $f(x) = \begin{cases} x^{2n+1} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

then $f(x)$ is n time diff'ble, $f^{(n)}$ is cts at 0, but $f^{(n)}$ is not diff'ble at 0.

Proof: by induction HW8