Conditional distribution $P(A|B) = \frac{P(A\cap B)}{P(B)}$

Discrete case

Def. for two random variables X and Y (discrete), the conditional pmf of X given that $Y=y$ by $P_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$.

$$= \frac{P(x,y)}{P_Y(y)}$$

- $P_{X|Y}(x|y) \geq 0$
- $\sum_x P_{X|Y}(x|y) = 1$

Def (conditional cdf)

$$F_{X|Y}(x|y) = P(X \leq x \mid Y=y) = \sum_{a \leq x} P_{X|Y}(a|y)$$

If X and Y are independent $P_{X|Y}(x|y) = P_X(x)$

(if $X \perp\!\!\!\perp Y$, then $p(x,y) = P_X(x)\cdot P_Y(y)$

$$P_{X|Y}(x|y) = \frac{P(x,y)}{P_Y(y)} = P_X(x)$$

Continuous case

Def. Let X and Y be 2 continuous r.v. with joint pdf $f(x,y)$. For any value x value which $f_X(x) > 0$, the conditional pdf of Y given that $X=x$, is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}, \quad -\infty < y < \infty$$

Probability:

For any set A $P(Y \in A \mid X=x) = \int_A f_{Y|X}(y|x)\,dy$.

if $X \perp\!\!\!\perp Y$, then

$$F_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f_X(x)\cdot f_Y(y)}{f_X(x)} = f_Y(y).$$

Expetation

The conditional expetitional of Y given that $X=x$ is

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy, \quad \text{if } X \,\&\, Y \text{ continuous.}$$

and $$E(Y|X=x) = \sum_y y \cdot P_{Y|X}(y|x), \quad \text{if } X \,\&\, Y \text{ discrete.}$$

Properties: $E(X) = E[E(X|Y)]$.

Remarks: (X,Y) - joint distributed r.v.s. Then for a function h of X and Y.

$$E(h(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\cdot f(x,y)\,dx\,dy. \quad \text{if } x,y \text{ ctn.}$$

$$= \sum_x \sum_y h(x,y)\cdot P(x,y) \quad \text{if } x,y \text{ discrete.}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad x\text{ - ctn}$$

$$= \sum_x x\,p(x) \qquad x\text{ - discrete.}$$

E.g. If X and Y are independent Poisson r.v.s with parameter λ_1, λ_2 respectively.

a) Find the conditional pmf of X given that $X+Y=n$

b) Find $E(X|X+Y=n)$.

Sol: a) $$P_{X|X+Y}(x|X+Y=n) \overset{def}{=} \frac{P(X=x,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=x,\ Y=n-x)}{P(X+Y=n)}$$

$$= \frac{P(X=x)\cdot P(Y=n-x)}{P(X+Y=n)}$$

$$= \frac{\dfrac{\cancel{e^{-\lambda_1}}\cdot \lambda_1^x}{x!}\cdot\dfrac{\cancel{e^{-\lambda_2}}\cdot\lambda_2^{n-x}}{(n-x)!}}{\dfrac{\cancel{e^{-(\lambda_1+\lambda_2)}}\cdot(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{x!\,(n-x)!}\cdot\frac{\lambda_1^x\cdot\lambda_2^{n-x}}{(\lambda_1+\lambda_2)^n}$$

$$= \binom{n}{x} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-x}$$

$$X \mid X+Y=n \sim \text{Binormial}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

b) $E(X \mid X+Y=n) = n\frac{\lambda_1}{\lambda_1+\lambda_2}$

Rule of Average Conditional Probabilities

If X and Y are discrete r.v.s., then $P(Y=y) = \sum_x \underbrace{P(Y=y \mid X=x) \cdot P(X=x)}_{P(Y=y,\, X=x)}$

E.g. the joint pdf of X and Y is given by.

$$f(x,y) = \begin{cases} \frac{15}{2}y(2-x-y) & , \quad 0<x<1 \ , \ 0<y<1 \\ 0 & , \quad o/w. \end{cases}$$

Find the conditional density of Y given that $X=x$, where $0<x<1$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}.$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy. = \int_0^1 \frac{15}{2}y(2-x-y)\,dy.$$

$$= \frac{15}{2}y^2 - \frac{15x}{4}y^2 - \frac{15}{6}y^3 \Big|_0^1$$

$$= 5 - \frac{15}{4}x \ , \quad 0<x<1.$$

$$f_{Y|X}(y|x) = \frac{\frac{15}{2}y(2-x-y)}{5-\frac{15}{4}x} = \frac{y(2-x-y)}{\frac{2}{3}-\frac{x}{2}} \ , \quad 0<y<1$$

E.g. Suppose X and Y has the joint pdf.

$$f(x,y) = \begin{cases} e^{-x/y} \cdot e^{-y}/y & , \quad \text{if } x>0,\ y>0 \\ 0 & \quad o/w. \end{cases}$$

Find $E(X \mid Y=y)$ where $y>0$.

Sol. $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} e^{-x/y} \cdot e^{-y}/y \, dx$

$$= e^{-y} \cdot (-e^{-x/y}) \Big|_0^{\infty}$$

$$= e^{-y}. \qquad y>0.$$

$$f_{X|Y}(x|y) = \frac{1}{y} e^{-x/y} \quad x > 0.$$

(i) $E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx$

$= \int_0^{\infty} \overset{u}{x} \cdot \overset{dv}{\frac{1}{y} e^{-x/y}}\, dx$

$= y.$

$e^{-x/y}$

$= -x \cdot e^{-x/y} - \int -e^{-x/y}\, dx.$

$= -x \cdot e^{-x/y} \cancel{+} y e^{-x/y} \Big|_0^{\infty}$

$= 0 + 0 - 0 + y.$

(ii) $X|Y=y \sim \exp(\frac{1}{y})$.

$E(X|Y=y) = \frac{1}{1/y} = y.$

Covariance & Correlation, Variance of Sums

Def. $Cov(X, Y) = E[(X-\mu_X)(Y-\mu_X)]$ $\leftarrow h(x, y)$

$= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y) \cdot p(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\, dx\, dy & \text{ctn} \end{cases}$

Alternate formula: $Cov(X, Y) = E(XY) - E(X)E(Y)$

$= \begin{cases} \sum_x \sum_y xy\, p(x,y) - [\sum_x x P_X(x)][\sum_y y P_Y(x)] & \text{discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y)\, dx\, dy - \int_{-\infty}^{\infty} x f_X(x)\, dx \cdot \int_{-\infty}^{\infty} y f_Y(y)\, dy & \text{(ctn)} \end{cases}$

If X and Y are independent, then

$Cov(X, Y) = 0$, $Cov(h(X), g(Y)) = 0$

$(E(XY) = EX\,EY)$

Properties: $Cov(X, Y) = Cov(Y, X)$

$Cov(X, X) = Var(X)$

$Cov(X, aY) = a\, Cov(X, Y)$

$Cov(\sum_i X_i, \sum_j Y_j) = \sum_i \sum_j Cov(X_i, X_j)$

Variance of Sums

If X and Y are 2 r.v.s, then

$$(*)\ Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Recall: If X, Y independent, then $Var(X+Y) = Var(X) + Var(Y)$.

→ more generally $Var(X_1+X_2+\cdots+X_n) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$

If $X_1, X_2 \cdots X_n$ independent $Var(X_1+\cdots+X_n) = \sum_{i=1}^{n} Var(X_i)$

Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{SD(X)\cdot SD(Y)}$$

↓

measure how strong the linear relationship between X and Y.

Properties $-1 \le \rho(X,Y) \le 1$

$\rho(X,Y) = 0$ if $Cov(X,Y) = 0$, X, Y uncorrelated.

If $\rho(X,Y) = \pm 1$, the $Y = aX + b$.

$\rho(aX+b, c+dY) = \rho(X,Y)$ if $bd > 0$.

X, Y independent $\Rightarrow$ $Cov(X,Y) = 0$ or $\rho(X,Y) = 0$

no relationship $\not\Leftarrow$ uncorrelated

↓

no linear relationship. $\underline{Y = X^2}$.