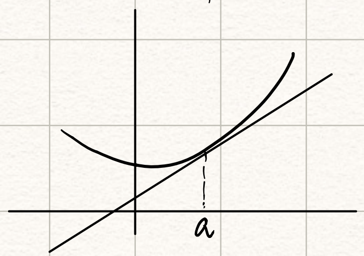
L18 03.02

# Chap 9: Differentiation




Idea: $f'(a)$ is the slope of the tangent line at a

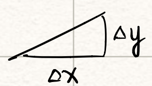
Tangent line: should intersect with $G_f$ at $(a, f(a))$.

straight line

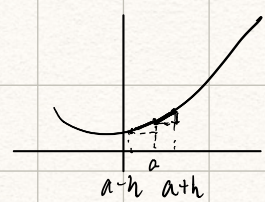
should only intersect with $G_f$ once in a neighborhood of a.

should be unique

slope:




$\frac{\Delta y}{\Delta x}$




$$\frac{f(a+h)-f(a)}{h}$$

**Def 9.1** $f$ is differentiable at $a$ if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{a}$ exist.

Then we write $f'(a)$ for this limit and say $f'(a)$ is the derivative of $f$ at a.

$f \quad (a,b) \to \mathbb{R}$ is differentiable if $f$ is differentiable at all $x \in (a,b)$.

**Remark 9.2** $f'$ is now a function with $\operatorname{dom} f' = \{x \mid f \text{ is differentiable at } x\}$.

Ex. i) $f(x) = c \in \mathbb{R} \ \forall x$.

Let $a \in \mathbb{R}$ $\quad \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$. So $f'(a) = 0$

ii) $g(x) = mx + c$

Let $a \in \mathbb{R}$. $\quad \lim_{h\to 0} \frac{g(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{m(a+h)+c-(ma+c)}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$. So, $f'(a) = m$.

iii) $h(x) = x^2$

Let $a \in \mathbb{R}$, $k \neq 0$ $\quad \dfrac{h(a+k)-h(a)}{k} = \dfrac{(a+k)^2 - a^2}{k} = \dfrac{a^2+2ak+k^2-a^2}{k} = 2a+k.$

So $h'(a) = \lim\limits_{k\to 0} \dfrac{h(a+k)-h(a)}{k} = \lim\limits_{k\to 0} (2a+k) = 2a.$

Tangent line is line of slope $f'(a)$ passing $(a, f(a))$

i.e. $\ell(x) = f'(a)(x-a) + f(a)$

So tangent to constant or linear function is the graph of the function.

Ex. i) tangent to $f(x) = x^2$ is $2a(x-a) + a^2$

So $\ell(x) - x^2 = 2a(x-a) + a^2 - x^2 = 0$ iff $x^2 - 2ax + a^2 = 0$.

solving, we must have $x = a$.

So, $\ell$ meets $h$ only at $a$.

ii) $g(x) = x^3$

tangent is $\ell(x) = 3a^2(x-a) + a^3$

If $a=2$. for example, $\ell(x) = g(x)$ for $x > 0$.

But $\dfrac{\ell(x)}{g(x)} \to \infty$ as $x \to -\infty$

So, $\exists M < 0$ s.t. if $x < M$, then $\left|\dfrac{\ell(x)}{g(x)}\right| < \frac{1}{2}$. So $|\ell(x)| < \dfrac{|g(x)|}{2}$

As for $x < -a$ $\ell(x) < 0$, $g(x) < 0$, $-\ell(x) < -g(x)$, so $\ell(x) > \dfrac{g(x)}{2} > g(x)$

Hence, by IVT. $\exists y \in (\min(M-a), 0)$ s.t. $\ell(x) = g(x)$.

Notation. People also write $\dfrac{df}{dx}$ for derivative. $\dfrac{df}{dx}(x) := f'(x)$

How to distinguish between $\dfrac{df}{dx}$ as a function and a value is unclear.

$d$ here has no independent meaning.

Q: is $\frac{dx^2}{dx}$ the function or the value at $x$ ?

We always use $f'(a) \left(= \frac{df(x)}{dx}\Big|_{x=a}\right)$

Ex. i) Let $f(x) = |x|$

$f$ is not differentiable at $x=0$. Need to show $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ does not exist.

Let $h \neq 0$ If $h>0$, $\frac{f(h)-f(0)}{h} = \frac{h}{h} = 1$. So $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = 1$

If $h<0$, $\frac{f(h)-f(0)}{h} = \frac{-h}{h} = -1$. So $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = -1 \neq \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$

So, $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ does not exist, i.e, $f$ is not differentiable at $0$ $\blacksquare$

ii) Let $g(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0. \end{cases}$

$g$ is not differentiable at $0$.

$\lim_{h\to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = \lim_{h\to 0^-} h = 0.$

$\lim_{h\to 0^+} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 \neq \lim_{h\to 0^-} \frac{g(h)-g(0)}{h}$

So, $\lim_{h\to 0} \frac{g(h)-g(0)}{h}$ does not exist.

iii) Let $h(x) = \begin{cases} x^2 & x \leq 0 \\ 0 & x > 0 \end{cases}$ $h$ is differentiable at $0$, exercise.

iv) Let $r(x) = x^{\frac{1}{3}}$, $x \in \mathbb{R}$.

The $\lim_{h\to 0^+} \frac{r(h)-r(0)}{h} = \lim_{h\to 0^+} \frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0^+} h^{-\frac{2}{3}} = \infty$

So, $\lim_{h\to 0} \frac{r(h)-r(0)}{h}$ does not exist.

So, $r(x)$ is not differentiable at $0$.

L19 03.04

Equivalently, $f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$ (Exercise)

**Thm 9.3** Suppose $f$ is differentiable at $a$, then $f$ is cts at $a$.

Proof: As $f$ is differentiable at $a$, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$

For $x \neq a$, $f(x) - f(a) = \frac{f(x) - f(a)}{(x-a)} (x-a) \to f'(a) \cdot 0 = 0$ as $x \to a$

by product rule for limits

**Rmk 9.4** The converse is not true

Ex. i) $f = |x|$ is cts but not diff'ble at 0

ii) Let $f(x) = |x|$ for $|x| < 1$ and extend periodically with period 2.

$f$ is cts on $\mathbb{R}$, not diff'ble at any $z \in \mathbb{Z}$

**Fact** $\exists g: \mathbb{R} \to \mathbb{R}$ s.t. $g$ is cts on $\mathbb{R}$, but $g$ is not diff'ble at any $x \in \mathbb{R}$.

Iterative construction, start with $f$ from Ex (ii)

Note $|f| \leq 1$ on $\mathbb{R}$, $f(4x)$ is not diff'ble at any $x \in \mathbb{Z}/4$

So, $f_1(x) = f(x) + \frac{1}{2} f(4x)$ is cts but not diff'ble at $x \in \frac{\mathbb{Z}}{4}$

$f_2(x) = f(x) + \frac{1}{2} f(4x) + \frac{1}{4} f(16x)$

$f_n(x) = \sum_{k=0}^{n} 2^{-k} f(4^k x)$

Claim: $f_n$ is not differentiable.

Note: $|f_n(x)| \leq \sum_{k=0}^{n} 2^{-k} |f(4^k x)| \leq \sum_{k=0}^{n} 2^{-k} \leq 2$

Suppose $f_n$ is diff'ble at $x$, so $x \notin \frac{\mathbb{Z}}{4^n}$

$f_n'(x) = f'(x) + \frac{1}{2} \cdot 4 f'(4x) + \cdots + \frac{1}{2^n} \cdot 4^n \cdot f'(4^k x)$

Recall $|f'(x)| = 1$

So $|f_n'(x)| \geq |\frac{1}{2^n}\cdot 4^n \cdot f'(4^k x)| - |f'(x) + 2f'(4x) + \cdots \frac{1}{2^{n-1}} 4^{n-1} f_{n-1}'(x)|$

$$\geq 2^n - \sum_{k=0}^{n-1} 2^k \to \infty$$

Ex i) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Consider, for $h \neq 0$, $\frac{f(h)-f(0)}{h} = \sin(\frac{1}{h})$

So, $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ does not exist.

ii) $g(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Let $h \neq 0$ then $\frac{g(h)-g(0)}{h} = h\sin(\frac{1}{h})$

Hence $0 = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = g'(0)$

Def 9.5: Suppose $f$ is diff'ble on $(a,b)$, so $f'\ (a,b) \to \mathbb{R}$

If $f'$ is differentiable at $x$, we call $(f')'(x)$ the <u>$2^{nd}$ derivative</u> of $f$ at $x$ and write $(f')'(x) = f''(x)$

Inductively, if $f$ is $n-1$ times diff'ble on $(a,b)$, we define the $n^{th}$ derivative of $f$ at $x$ as $f^{(n)}(x) = (f^{(n-1)})'(x)$ where $f^{(0)}(x) = f(x)$ and $f^{(1)}(x) = f'(x)$

Note: $f^{(n)} \neq f^n$, $f^n$ is either $(f(x))^n$ or $(\underbrace{f\circ f \cdots \circ f}_{n \text{ times}})(x)$

Ex. $f(x) = \begin{cases} \frac{1}{6}x^3 & x \geq 0 \\ -\frac{1}{6}x^3 & x \leq 0 \end{cases}$

Note: $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = 0$

So, $f'(x) = \begin{cases} \frac{1}{2}x^2 & x > 0 \\ 0 & x = 0 \\ -\frac{1}{2}x^2 & x < 0 \end{cases} = \begin{cases} \frac{1}{2}x^2 & x \geq 0 \\ -\frac{1}{2}x^2 & x < 0 \end{cases}$

Then $f''(x) = \begin{cases} x & \geq 0 \\ -x & x < 0 \end{cases} = |x|$

So, $f$ is twice diff'ble, but not 3 times.