Economics of Ideas:
Romer's (1990) "Variety" Model

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ECON 312
Intermediate Macro Theory – Advanced Treatment

Variety Model

Based on Romer (1990)

Model with **two** sectors:

1. production sector
2. research sector.

good and services, resourse constraint. rival, non rival.

produce the new idea. excludibility. ⇒ recipes ideas.

high fixed cost

Production Sector

- Goods produced with the following technology:

$$Y_t = K_t^{\alpha} (A_t L_{Y,t})^{1-\alpha}$$

ideas. (research sector).

- L_Y is the labor employed to produce goods.
- As before, the stock of capital evolves according to:

$$\Delta K_t = s_K Y_t - \delta K_t.$$

Production Sector

- In per-capita terms,

$$y_t = k_t^{\alpha} \left(A_t \frac{L_{Y,t}}{L_t} \right)^{1-\alpha}$$

and

$$\Delta k_t = s_K y_t - (\delta + n)\, k_t$$

- As before, we assume that population grows at the constant rate n:

$$\frac{\Delta L_t}{L_t} = n$$

What Drives Technological Progress?

$$\frac{\Delta A_t}{A_t} = ?$$

Research Sector

- Individuals can spend time searching for ideas. Let L_A be the number of people employed in R&D.
- New ideas increase the effective units of labor for *all* workers:

$$A_{t+1} - A_t \equiv \Delta A_t.$$

- Each researcher generates new ideas at rate $\bar{\theta}$.

Research Sector (cont'd)

- The stock of ideas evolves according to: constant return to scale.

$$\Delta A_t = \bar{\theta} L_{A,t}$$

- What properties should $\bar{\theta}$ have?

Arrival Rate of Ideas

Properties of $\bar{\theta}$

$$\bar{\theta}(A, L_{a,t})$$

1. old ideas are inputs in the production of new ideas
2. Researchers are inputs in production of idea duplication

Old Ideas as Inputs

get harder to come up with new idea

$$\bar{\theta}(A, \ldots) = \theta \times \underbrace{A^{1}}_{\text{positive externality}} \times \underbrace{A^{\phi-1}}_{\text{negative externality}}$$

Two forces:

1. Newton's standing on the shoulders of giants before
2. Fishing out effect when $\phi < 1$

The parameter θ captures overall productivity of researchers.

Old Ideas are Inputs to Produce New Ideas

Net effect is:

$$\bar{\theta}\left(A,\ldots\right) = \theta A^{\phi}$$

discover of new research depends on old knowledge

ϕ measures the importance of the 'standing on the shoulders of giants' relative to the 'fishing out' effect.

Duplication of Ideas

$\downarrow\lambda \Rightarrow \uparrow$ duplication.

$$\bar{\theta}\left(A, L_A, \ldots\right) = \theta A^{\phi} \qquad \underbrace{L_A^{\lambda-1}}$$

$0 < \lambda < 1$ implies $\lambda - 1 < 0$.

Intuition for negative externality:

- Only number of *original* ideas matters.
- If research is done simultaneously by many researchers, and ideas are nonrival, then it is likely that there will be *duplication* of ideas.

Aggregate Evolution of Ideas

Adding up these three effects:

function $\bar{\theta}(\ )$, parameter θ.

$\bar{\theta} = \theta A_t^{\phi} L_{A,t}^{\lambda-1}$

number of new ideas

$$
\begin{aligned}
\Delta A_t &= \bar{\theta}(A_t, L_{A,t}) \times L_{A,t} \\
&= \theta A_t^{\phi} L_{A,t}^{\lambda-1} L_{A,t} \\
&= \theta A_t^{\phi} L_{A,t}^{\lambda}
\end{aligned}
$$

Externalities

negative:
competition ↑ → duplicate ↑
no one is resarcher
↳ strong motivation → research

- Individual's decision (worker vs. researcher):

$$\Delta A_t = \bar{\theta}\left(A_t, L_{A,t}\right) \overbrace{\left[L_{A,t}^1 + \ldots + L_{A,t}^i + \ldots + L_{A,t}^N\right]}^{L_{A,t}}$$

- Individual i takes occupational choice of others as given and ignores effect of *own* choice on the others' productivity:
  - direct negative effect on todays arrival of ideas.
    $\bar{\theta}(A_t, \ldots L_{At}^i)$
  - indirect effect on tomorrow's level of technology.

$A_{t+1} = A_t + \bar{\theta}(A_t, \ldots L_{A,t}^i)[\cdots L_A^i \cdots]$

Externalities

Society's choice:

$$\Delta A_t = \bar{\theta}(A_t, L_{A,t})\, L_{A,t}$$

$$= \bar{\theta} A_t^{\phi} L_{A_t}^{\lambda-1} L_A(t).$$

$$= \theta A_t^{\phi} L_{A,t}^{\lambda}$$

cares about the sum of the individual effects.

Simple Case with Constant Research Effort

Constant fraction s_R of the population works as researchers:

$$L_A = s_R L \rightarrow \text{researchers}$$

not choosing, just tell

assume constant

What is the Rate of Technological Progress?

$$\Delta A_t = \theta A_t^{\phi} L_{A,t}^{\lambda}$$
$$= \theta A_t^{\phi} (s_R L_t)^{\lambda}$$

Dividing by A_t : growth rate.

$$\frac{\Delta A_t}{A_t} = \theta s_R^{\lambda} \frac{L_t^{\lambda}}{A_t^{1-\phi}}$$

Balanced Growth Path

economy reaches a state where all endogenous variable grows at constant rate. (not same rate for all variables)

In a balanced growth path (i.e., when endogenous variables grow at a constant rate), the presumed growth rate of technology growth is:

assume economic grow to constant rate
⇓
solve the model

$$g_A = \frac{\Delta A_t}{A_t} = \theta s_R^{\lambda} \frac{L_t^{\lambda}}{A_t^{1-\phi}}$$

$\ln(x(t))$

$\frac{\partial \ln(x(t))}{\partial t} = \frac{1}{x(t)} \frac{\partial x(t)}{\partial t} = \frac{\dot{x}(t)}{x(t)} = \frac{\Delta x}{x}$ ⇒

$\ln g_A = \lambda \ln s_R + \lambda \ln L_t - (1-\phi)\ln A_t + \lambda \ln \theta$

$\frac{\dot{g}_A}{g_A} = \frac{\Delta \theta}{\theta} + \lambda \frac{\Delta s_R}{s_R} + \lambda \frac{\Delta L_t}{L_t} - (1-\phi)\frac{\Delta A_t}{A_t}$

$\frac{\dot{g}_A}{g_A} = 0$, $\frac{\Delta \theta}{\theta} = 0$, $\frac{\Delta s_R}{s_R} = 0$, $\frac{\Delta L_t}{L_t} = \lambda n$, $\frac{\Delta A_t}{A_t} = (1-\phi) g_A$

$(g_{L_t} = n)$

Balanced Growth Path

$$\lambda n = (1-\phi) g_A .$$

$$g_A = \frac{\lambda n}{1-\phi}$$

Taking logs:

Differentiating with respect to time t:

And rearranging:

$$\frac{\Delta A_t}{A_t} =$$

Intuition

1. $\frac{\Delta A_t}{A_t}$ is increasing in λ: lamuda
   larger λ implies less duplication of ideas (lower *negative* externalities)
2. $\frac{\Delta A_t}{A_t}$ is increasing in ϕ: fi
   larger ϕ implies stronger 'standing in the shoulder of giants' effect (higher *positive* externalities of past research) and/or weaker 'fishing-out effect'
3. $\frac{\Delta A_t}{A_t}$ is increasing in n:
   faster population growth implies more growth in the number of researchers

Growth Rate of Income Along Balanced Growth Path

$\ln y_t = \alpha \ln k_t + (1-\alpha)(\ln A_t + \ln(1-s_R))$

$$y_t = k_t^{\alpha} \left(A_t \left(1 - s_R\right)\right)^{(1-\alpha)} \quad \leftarrow \text{take log.}$$

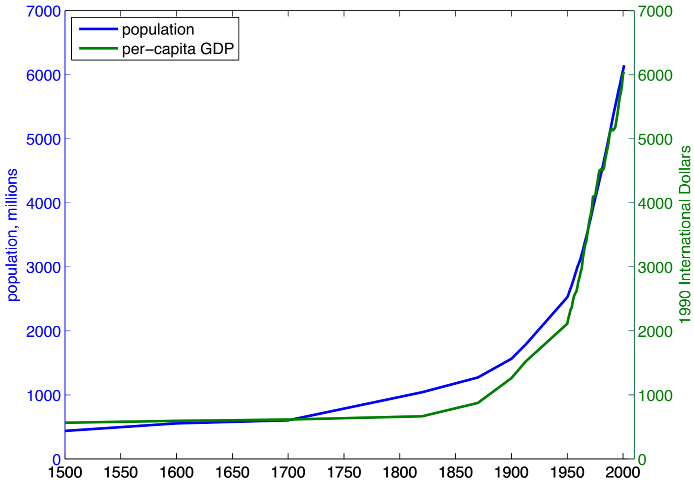
$$\frac{\Delta y_t}{y_t} = \alpha \frac{\Delta k_t}{k_t} + (1-\alpha)\frac{\Delta A_t}{A_t}$$

$$= \alpha \frac{\Delta A_t}{A_t} + (1-\alpha)\frac{\Delta A_t}{A_t}.$$

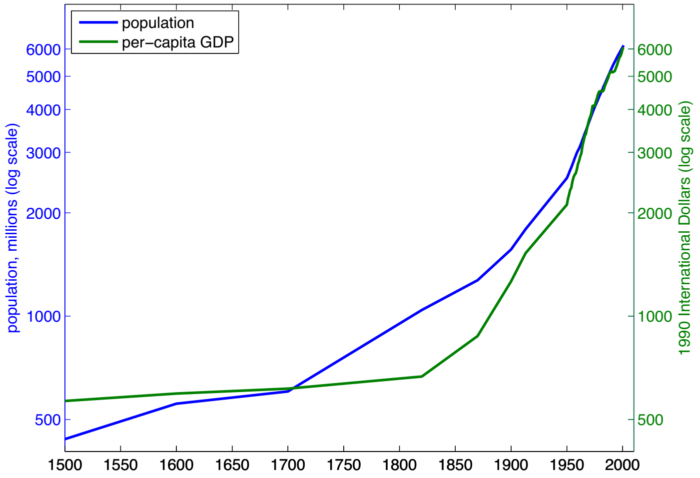
$$= \frac{\Delta A_t}{A_t} = g_A = \frac{\lambda n}{1-\phi}$$

population $\uparrow$, $y \uparrow$.

Population and Per-Capita Income Growth: Historical Perspective



Population and Per-Capita Income Growth: Historical Perspective



Basic Equation of Technological Progress

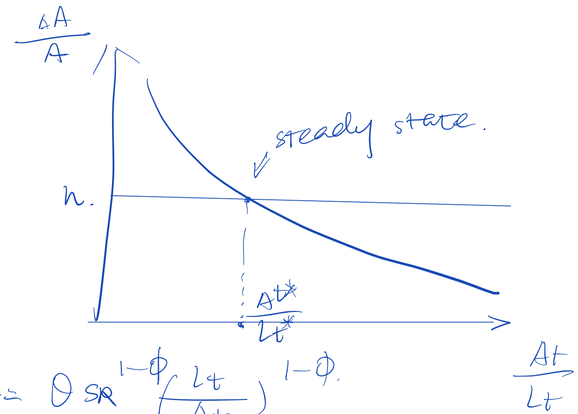
Off the balanced growth path the rate of technological progress is given by:

assume $\lambda = 1 - \phi$.

$$\frac{\Delta A_t}{A_t} = \theta s_R^{\lambda} \frac{L_t^{\lambda}}{A_t^{1-\phi}}$$

In the *special case* $1 - \phi = \lambda$ this simplifies to:

$$\frac{\Delta A_t}{A_t} = \theta s_R^{1-\phi} \left(\frac{L_t}{A_t} \right)^{1-\phi}$$



$$n = \theta s_R^{1-\phi} \left(\frac{L_t}{A_t} \right)^{1-\phi}.$$

Dynamics of Technology

$$\left(\frac{n}{\theta} s_R^{1-\phi}\right)^{\frac{1}{1-\phi}} = \frac{L_t}{A_t}$$

$$\frac{A_t^*}{L_t^*} = \left(\frac{n}{\theta} s_R^{1-\phi}\right)^{\frac{1}{\phi-1}}$$

Law of Motion for Physical Capital per Person

Physical capital accumulation:

$$\frac{\Delta k_t}{k_t} = s_K \left(\frac{k_t}{A_t}\right)^{\alpha-1} (1 - s_R)^{1-\alpha} - (\delta + n)$$

Dynamics of Physical Capital per Person

Dynamics of the Model with Endogenous Technology $(\lambda = 1 - \phi)$

k_t, A_t, y_t grow at same rate.

Technological progress:

$$\frac{\Delta A_t}{A_t} = \theta s_R^{1-\phi} \left(\frac{A_t}{L_t} \right)^{\phi - 1}$$

Physical capital accumulation:

$$\frac{\Delta k_t}{k_t} = s_K \left(\frac{k_t}{A_t} \right)^{\alpha - 1} (1 - s_R)^{1-\alpha} - (\delta + n)$$

Balanced Growth with Endogenous Technology ($\lambda = 1 - \phi$)

Technological progress:

$$g_A = \theta s_R^{1-\phi} \left(\frac{A_t}{L_t} \right)^{\phi - 1}$$

Physical capital accumulation:

$$g_A = s_K \left(\frac{k_t}{A_t} \right)^{\alpha - 1} (1 - s_R)^{1-\alpha} - (\delta + n)$$

Balanced Growth with Endogenous Technology ($\lambda = 1 - \phi$)

Technological progress:

$$A_t = \left(\frac{\theta}{g_A}\right)^{\frac{1}{1-\phi}} s_R L_t$$

Physical capital accumulation:

$$k_t = \left(\frac{s_K}{\delta + n + g_A}\right)^{\frac{1}{1-\alpha}} (1 - s_R) A_t$$

$$= \left(\frac{s_K}{\delta + n + g_A}\right)^{\frac{1}{1-\alpha}} (1 - s_R) \left(\frac{\theta}{g_A}\right)^{\frac{1}{1-\phi}} s_R L_t$$

Balanced Growth with Endogenous Technology ($\lambda = 1 - \phi$)

$$
\begin{aligned}
y_t &= k_t^{\alpha}(A_t(1-s_R))^{1-\alpha} \\
&= \left[\left(\frac{s_K}{\delta+n+g_A}\right)^{\frac{1}{1-\alpha}}(1-s_R)A_t\right]^{\alpha}[A_t(1-s_R)]^{1-\alpha} \\
&= \left(\frac{s_K}{\delta+n+g_A}\right)^{\frac{\alpha}{1-\alpha}}A_t(1-s_R) \\
&= \underbrace{\left(\frac{s_K}{\delta+n+g_A}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\theta}{g_A}\right)^{\frac{1}{1-\phi}}s_R(1-s_R)}_{\text{same.}}L_t.
\end{aligned}
$$

trade off.
produce more now
↓
less new ideas
↓
tomorrow ↓ produce

given s_R

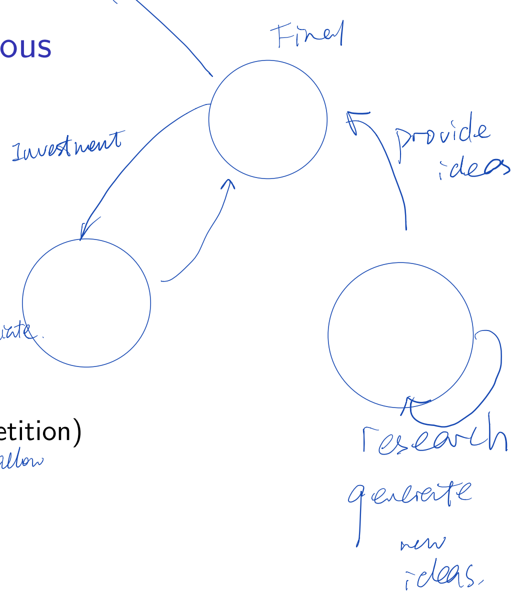
↑ proportional to population

Micro-Foundations of the Model with Endogenous Technological Progress

- So far, model "predicts" growth rate of GDP per capita for a given allocation of workers to research (s_R) and production ($1 - s_R$).
- How does the market allocate workers?

Consumption

Micro-Foundations of the Model with Endogenous Technological Progress



Three Productive Sectors:

1. Final Good Producers (Competitive)   Capital / Intermediate.
2. Intermediate Good Producers (Monopolistic Competition)
   ↑ government allow
3. Research Sector (Free entry→Competitive)

Flow of Factors of Production Across Sectors

goes to ideas

Final Good Producer(s)

K_t X_1 X_A

Aggregate production function

$$Y = L_Y^{1-\alpha} \sum_{j=0}^{A} x_j^{\alpha}$$

A is the number of different capital goods used in the production of the final-good, x_j is the quantity of the capital good j used in the production of the final good.

Final Good Producer(s)

measure of capital good

Aggregate production function

$$Y = L_Y^{1-\alpha} \int_0^A x_j^\alpha dj$$

where A is the *measure* of different capital goods used in the production of the final-good, x_j is the quantity of the capital good j used in the production of the final good.

Final Good Producer(s): Input Demands

Competitive firms choose the labor input L_Y and the quantity of each capital good to maximize profit:

capital.

price.

$$\max_{L_Y, x_j} L_Y^{1-\alpha} \int_0^A x_j^{\alpha} dj - wL_Y - \int_0^A p_j x_j dj$$

final good producer

price taker

cost of labor input.

Final Good Producer(s): Input Demands

Competitive firms choose labor and capital goods to maximize profits, i.e.,

$$
\begin{aligned}
MPL &= w \\
(1-\alpha)\,\frac{Y}{L_Y} &= w
\end{aligned}
$$

$$\frac{\partial Y}{\partial x_j} = \frac{\partial L_Y^{1-\alpha} \int_0^A x_j^{\alpha}\, dj}{\partial x_j}$$

and

$$
\begin{aligned}
MPx_j &= p_j \\
\alpha L_Y^{1-\alpha} x_j^{\alpha-1} &= p_j
\end{aligned}
$$

i.e., we obtain a demand for labor and capital goods x_j, $j \in [0, A]$.

Capital Good Producers

Monopolistic capital good producers choose the quantity supplied (or the price charged) to maximize profits:

$$\max_{x_j} \pi_j = p_j(x_j)\, x_j - r x_j$$

differentiated. undeferentiated unit.

- a unit of capital good is produced with a unit of raw capital good, with the rental cost r (ignoring depreciation δ).
- Capital good producers are monopolists because they own patents for the products developed by inovators in the research sector

Demand Function for Capital Goods

The (inverse) demand function comes from the optimal input demand decision of competitive producers of final goods

$$P_j(x_j) = \alpha L_Y^{1-\alpha} \underline{x_j}^{\underline{\alpha-1}} < 0$$

negative relative.

which is the inverse of the function that gives the quantity demanded as a function of price.

$$x_j(P_j) = (\alpha L_Y^{1-\alpha}/P_j)^{\frac{1}{1-\alpha}}.$$

price elasticity $= \frac{1}{1-\alpha}$

Optimal Supply of Capital Goods

Monopolist supply goods up to the point where marginal revenue equals marginal cost, i.e.,

$$MR_j = MC_j$$

$$\frac{\partial p(x_j)}{\partial x_j} x_j + p_j(x_j) = r.$$

$$\frac{\partial p_j(x_j) x_j}{\partial x_j}.$$

$$p_j(x_j) + \partial p_j(x_j) x_j.$$

Multiply and divide leftmost part by p_j and rearrange:

$$p_j = \frac{1}{1 + \frac{\partial p_j}{\partial x_j} \frac{x_j}{p_j}} r.$$

$$= \frac{1}{1-(1-\alpha)} r = \frac{r}{\alpha}$$

(inverse elasticity rule).

mark up depends on the α.

perfect competition $\alpha = 1$ $p_j = r$.

Aggregate Quantity of Capital Goods

Aggregating over all intermediate (capital) goods,

$$\int_0^A x_j dj = K,$$

Since they are used in the same amounts ($x_j = x$) this simplifies to:

$$Ax = K$$

$$x = \frac{K}{A}$$

Input use.

$x_j = x_{j'} = x$

$K = \int_0^A x dj = x \int_0^A dj$

$= xA.$

Aggregate Output

Substituting $x = K/A$ into the production function of the final good

$$\begin{aligned} Y &= L_Y^{1-\alpha} A x^{\alpha} \\ &= L_Y^{1-\alpha} A \left(\frac{K}{A}\right)^{\alpha} \\ &= K^{\alpha} (A L_Y)^{1-\alpha}. \end{aligned}$$

Profits of Capital Good Producers

$$\frac{r}{\alpha} x_j - r x_j$$

$$\pi_j = p_j(x_j)x_j - rx_j$$

According to the inverse elasticity rule $r = \alpha p_j(x_j)$ profits π_j are

$$\begin{aligned} \pi_j &= p_j(x_j)x_j - \alpha p_j(x_j)x_j \\ &= (1-\alpha)p_j(x_j)x_j \end{aligned}$$

Since $p_j(x_j) = \alpha L_Y^{1-\alpha} x_j^{\alpha-1}$ (see the inverse demand function four slides ago):

$$\pi_j = (1-\alpha)\alpha L_Y^{1-\alpha} x_j^{\alpha}$$

Profits of Capital Good Producers

Since all capital good producers are identical, that is, $\pi_j = \pi$, $x_j = x$, and substituting $x = K/A$:

$$\pi = (1-\alpha)\alpha L_Y^{1-\alpha} \left(\frac{K}{A}\right)^{\alpha}$$

$$= \alpha(1-\alpha)\frac{(AL_Y)^{1-\alpha}K^{\alpha}}{A}$$

$$= \alpha(1-\alpha)\frac{Y}{A}$$

Research Sector

- Individuals can devote time to developing new products. New varieties x_j are added to the current measure A of goods.

- After developing a new capital good, they are granted a (perpetual) patent to produce this new variety.

- Innovator can sell this patent at price P_A.

Research Sector

- Recall that new ideas are produced with the technology $\Delta A = \bar{\theta} L_A$.
- Researchers will enter until gains from innovating equal opportunity cost:

$$\underbrace{\bar{\theta} P_A}_{\text{return}} - \underbrace{w}_{\text{opportunity cost}}$$

- With free entry of innovators:

productability

$$\bar{\theta} P_A = w$$

sell new idea

- How is the price of an innovation determined?

Price of an Idea, P_A

An idea is like a stock. We value it because it give us some dividends, π, and because we might expect its price to increase in the future,

$\pi + \Delta P_A$

If there is competition between resources to fund ideas and capital, then ideas and capital must provide the same return:

indifferent: buying the capital, new idea.

$rP_A = \pi + \Delta P_A$ → potential change in price.

no arbitrage condition.

Price of an Idea, P_A

- Divide the previous equation by P_A to obtain:

$$r = \frac{\pi}{P_A} + \frac{\Delta P_A}{P_A}$$

- Along a balanced growth path, r is constant.
- P_A and $\pi = \alpha(1-\alpha)Y/A$ ought to grow at the same rate.
- $\frac{\Delta A}{A} = \frac{\Delta y}{y} = n$. $\frac{\Delta Y}{Y} = 2n$ when $\lambda = 1-\phi$
- if π grows at rate n, then balanced growth requires that P_A grow at rate n, too,

$$r = \frac{\pi}{P_A} + n$$

$$\Rightarrow P_A = \frac{\pi}{r-n}.$$

How is s_R determined?

Innovators must be indifferent between drawing ideas or working to produce final goods,

$$\bar{\theta} P_A = w$$

Substituting $P_A =$ $\pi / (r - n)$ $\quad w = (1-\alpha) Y / L_Y$

$$\bar{\theta} \frac{\pi}{r-n} = (1-\alpha) \frac{Y}{L_Y}$$

Substituting $\pi =$ $\alpha (1-\alpha) \frac{Y}{A}$

$$\bar{\theta} \frac{\alpha(1-\alpha)}{r-n} \frac{Y}{A} = (1-\alpha) \frac{Y}{L_Y}$$

$$\frac{\alpha}{r-n} \frac{\bar{\theta}}{A} = \frac{1}{L_Y}.$$

How is s_R determined?

Using that in a balance growth path $\Delta A = \bar{\theta} L_A$ or $\bar{\theta} = \Delta A / L_A$

$$\frac{\alpha}{r-n}\frac{\Delta A}{A}\frac{1}{L_A} = \frac{1}{L_Y}$$
$$\frac{\alpha}{r-n} g_A \frac{1}{L_A} = \frac{1}{L_Y}.$$

Using $s_R = L_A/L$ and $1 - s_R = L_Y/L$

$$\frac{\alpha}{r-n} g_A \frac{1}{s_R} = \frac{1}{1-s_R}$$

or

$$s_R = \frac{1}{1 + \frac{r-n}{\alpha g_A}}.$$

How is s_R determined?

When $\lambda = 1 - \phi$ the optimal share of researchers is given by:

$$s_R = \frac{1}{1 + \frac{r-n}{\alpha n}}.$$

optimal split.

depend on n.

↑ speed of knowledge ↑ as n↑.

↑ n -> more research

↓

↓ marginal return

↓ more valuable

workers

$y = \frac{AL_{y,t}}{L_t} = \frac{A_t(1-SR_{\phi,t})L_t}{L_t}$ practice exam 4

→ black death. $\underbrace{L_{a,t}}_{\text{live}} + L_{y,t} = L_t$.

$L_{a,t+1} = L_{a,t}(1+n)$

$L_{y,t} = \hat{L}_{y,t}(1+n)$. SR changes

$y \downarrow$. $\Delta A_t = \theta L_{A,t}^{\alpha} A_t^{\phi}$

$\frac{\Delta A_t}{A_t} = \theta L_{a,t} \Rightarrow$ × change.

$\Delta K_{t+1} = s_K A_t k_t^{\alpha} - \delta k_t$.

$\frac{\Delta k_{t+1}}{k_t} = s_K A_t k_t^{\alpha-1} - \delta = g_k$ costant in BGP

$s_K \frac{A_t}{K_t} k_t^{\alpha} - \delta$ costant

costant.

3/5 Review.

$Y_t = K_t^{\alpha}(A_t L_{y,t})^{1-\alpha}$.

$\Delta K_{t+1} = s_t Y_t - \delta K_t$

more know, hard new

$\Delta A_{t+1} = \bar{\theta}(A_t, L_{a,t}) L_{a,t} = \theta A_t \underbrace{A_t^{\phi-1}} L_{a,t}^{\lambda-1} \cdot L_{a,t}$.

$= \theta A_t^{\phi} L_{a,t}^{\lambda}$

$\frac{\Delta A_{t+1}}{A_t} = \theta A_t^{\phi-1} L_{a,t}^{\lambda}$ (generally, BGP or not). DEF BGP: constant growth rate of endougenous variables

assume BGP exists. $g_A = \frac{\Delta A_{t+1}}{A_t} = \theta A_t^{\phi-1} L_{a,t}^{\lambda}$

log: $\ln g_A = \ln\theta + (\phi-1)\ln A_t + \lambda \ln L_{a,t}$. ($SR L_t$)

differentiate: $\frac{g_A'}{g_A} = \frac{\theta'}{\theta} + (\phi-1)\frac{A_t'}{A_t} + \lambda\frac{SR'}{SR} + \lambda\frac{L_t'}{L_t}$.

$\frac{g_A'}{g_A} = 0$, $\frac{\theta'}{\theta} = 0$, $(\phi-1)\frac{A_t'}{A_t} = (\phi-1)g_A$, $\lambda\frac{L_t'}{L_t} = \lambda n$.

$g_A = \frac{\lambda n}{\phi-1}$

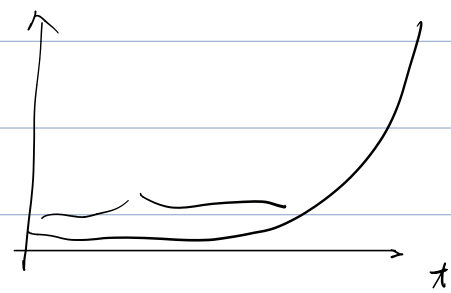
Assume $1-\phi = \lambda$

$g_A = \theta \left(\frac{A_t}{L_{a,t}}\right)^{1-\phi}$ ⇐ constant ⇒ A_t grow related to number of researchers.

$y_t = k_t^{\alpha} (A_t (1-s_R))^{1-\alpha}$ — endougenous

along BGP. $y_t = (\quad\quad) L_t$ — depend on size of population.

Romo Model. track quite well.



↑ people

↑ idea

↑ worker valuable