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§12 Inverse Function.

Def 12.1 A function f is **injective** if $f(a) \neq f(b)$ wherever $a \neq b$

i.e. if $f(a) = f(b)$, then $a=b$.

Also called one to one

Ex. $f(x) = x$ is injective

$g(x) = x^n$ is not injective on $\mathbb{R}$.

However, g is injective on $[0, \infty)$

Recall a function f is a set of ortheral pair (a,b) s.t. if if (a,b) & (a,c) are in f,

then $b=c$ (Think $b = f(a)$)

Def 12.2: For any function f, we define the inverse f^{-1} as the set of pairs (b,a)

s.t. (a,b) is in f. provided this defines a function.

Thm 12.3: f^{-1} is defined iff f is injective.

Proof: Suppose that f is injective. Need to show that if (a,b), (a,c) are in f^{-1}, then $b=c$.

Suppose (a,b), (a,c) in f^{-1}. So (b,a), (c,a) are in f i.e. $f(b) = a = f(c)$

As f injective, $b=c$.

Now suppose f^{-1} well-defined, i.e. whenever (a,b), (a,c) in f^{-1}, then $b=c$.

Suppose $f(b) = f(c)$, then $(b, f(b))$, $(c, f(c))$ are in f,

So $(f(b), b)$ and $(f(c), c)$ are in f^{-1}. Hence $b=c$, so f is injective. ∎

To draw f^{-1}, reflect in line $y=x$.

Fact: $(f^{-1})^{-1} = f$ when this is well-defined (f, f^{-1} be injected)

$f(f^{-1}(x)) = x \quad \forall x \in \operatorname{dom} f^{-1}$

$f^{-1}(f(x)) = x \quad \forall x \in \operatorname{dom} f$

Thm 12.4: If f is continuous on an interval & injective, then f is either increasing or decreasing.

Proof: (i) Let $a<b<c$ in the interval.

Need to show that either $f(a)<f(b)<f(c)$ or $f(c)<f(b)<f(a)$.

Suppose $f(a)<f(c)$ (note, $f(a) \neq f(c)$ as injective)

Suppose, for a contradiction, that $f(b)<f(a)$. Then $f(b)<f(a)<f(c)$.

So, by IVT on $[b,c]$, $\exists x \in (b,c)$ s.t. $f(x)=f(a)$

But $x>b$, so $x>a$ $\quad$ ※ as f is inj

So, if $f(a)<f(c)$, then $f(a)<f(b)<f(c)$ by a similar argument.

ii) Let $a<b<c<d$, then either $f(a)<f(b)<f(c)<f(d)$ or $f(a)>f(b)>f(c)>f(d)$

Apply (i) to get either $f(a)<f(b)<f(c)$ or $f(a)>f(b)>f(c)$

either $f(b)<f(c)<f(d)$ or $f(b)>f(c)>f(d)$

In any cases, we get either $f(a)<f(b)<f(c)<f(d)$

or $f(a)>f(b)>f(c)>f(d)$.

iii) Let $a<b$ in the interval. either $f(a)<f(b)$ or $f(a)>f(b)$.

Let $c<d$, ordering $\{a,b,c,d\}$, we get that, by (ii), $f(c)<f(d)$

if $f(a)<f(b)$ or $f(c)>f(d)$ if $f(a)>f(b)$.

So, f is either inc or dec. ∎

Note: if $f:[a,b]\to\mathbb{R}$ is cts & strictly increasing, then $\operatorname{range}(f) = [f(a), f(b)]$ by IVT

Hence $\operatorname{dom} f^{-1} = [f(a), f(b)]$.

On a open interval, more complicated.

Let $c \in (a,b)$

Q: what value $v > f(c)$ does f attain? Exercise.

If $\{f(x) \mid x \in [c,b)\}$ is unbounded, then, by IVT, $f([c,b)) = [f(c), \infty)$

If $\{f(x) \mid x \in [c,b)\}$ is bounded above, it has a least upper bd, α.

Get $f([c,b)) = [f(c), \alpha)$ (f strictly inc)

So, if f is cts, injective, then $\mathrm{dom} f^{-1}$ is of the form $[a,b]$, (a,b), $(a,b]$, $[a,b)$

$(-\infty,b)$, $(-\infty,b]$, $[a,\infty)$, (a,∞)

All possible, exercise to get examples of all of them.

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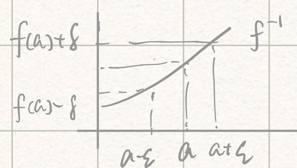
Thm 12.5: Suppose f is cts & inj on an interval. Then f^{-1} is also cts.

f^{-1} cts at $b=f(a)$ if $\forall \varepsilon>0$, $\exists \delta>0$ s.t. if $b-\delta<x<b+\delta$, then $|f^{-1}(b)-f^{-1}(x)|<\varepsilon$.

i.e. if $f(a)-\delta<x<f(a)+\delta$, then $|a-f^{-1}(x)|<\varepsilon$

Try $\delta = \min(f(a)-f(a-\varepsilon), f(a+\varepsilon)-f(a))$.



Proof: By Thm 12.4, f is either increasing or decreasing.

WLOG, suppose f is increasing (If not, consider $-f$).

Let $b \in \mathrm{dom}(f^{-1})$. Then $\exists a \in \mathrm{dom}(f)$ s.t. $b=f(a)$

Let $\varepsilon>0$, let $\delta = \min(f(a)-f(a-\varepsilon), f(a+\varepsilon)-f(a))$

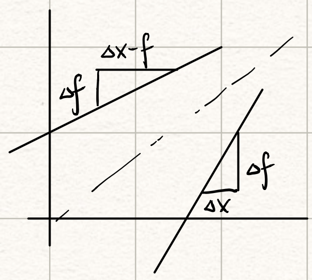
So, if $|x-b|<\delta$, as $b=f(a)$, we have $f(a-\varepsilon) \le f(a)-\delta < x < f(a)+\delta \le f(a+\varepsilon)$

i.e. $f(a-\varepsilon)<x<f(a+\varepsilon)$

Hence, as f increasing, f^{-1} is also increasing.

So, if $|x-b|<\delta$, we have $a-\varepsilon = f^{-1}(f(a-\varepsilon)) < f^{-1}(x) < f^{-1}(f(a+\varepsilon)) = a+\varepsilon$

So, $|f^{-1}(x)-f^{-1}(b)|<\varepsilon$ ∎



Now, Δf corresponds to change in domain for f^{-1}

$$\frac{f^{-1}(y)-f^{-1}(x)}{y-x} = \frac{b-a}{f(b)-f(a)} \quad \text{where } f(b)=y,\ f(a)=x.$$

Suggests $(f^{-1})'(f(a)) = \dfrac{1}{f'(a)}$

Suppose we have f is diff'ble at a, f^{-1} diff'ble at $f(a)$. Then $f(f^{-1}(x))=x$.

By chain rule, $f'(f^{-1}(x))\,(f^{-1})'(x)=1$. So $(f^{-1})(x)=\dfrac{1}{f'(f^{-1}(x))}$

Lemma 12.6 Suppose f is cts, inj, $f'(f^{-1}(a))=0$. Then f^{-1} is not diff'ble at a.

Proof: by ※ as above

E.g. $f(x)=x^3$ $f^{-1}=x^{\frac{1}{3}}$ is not diff'ble at 0 where $f'(0)=0$. ← $(f^{-1})'=\infty$.

Thm 12.7 Suppose f is cts, inj on an interval, diff'ble at $f^{-1}(b)$ and $f'(f^{-1}(b))\neq 0$.

then f^{-1} is diff'ble at b & $(f^{-1})'(b)=\dfrac{1}{f'(f^{-1}(b))}$

Proof: Let $b=f(a)$. Then $\dfrac{f^{-1}(b+h)-f^{-1}(b)}{h} = \dfrac{f^{-1}(b+h)-a}{h}$ for h s.t. $b+h\in \mathrm{dom}(f^{-1})$

Then $\exists k(h)$ s.t. $b+h=f(a+k)$ i.e. $k=f^{-1}(b+h)-f^{-1}(b)$

Now $\dfrac{f^{-1}(b+h)-f^{-1}(b)}{h} = \dfrac{a+k(h)-a}{f(a+k(h))-b} = \dfrac{k(h)}{f(a+k(h))-f(a)}$

By Thm 12.5, f^{-1} is cts at b, so $\lim_{h\to 0} k(h)=0$.

As f is diff'ble at a, $\lim_{k\to 0}\dfrac{f(a+k)-f(a)}{k} = f'(a)\neq 0$

So, by quotiant rule for limit, $\lim_{h\to 0}\dfrac{f^{-1}(b+h)-f^{-1}(b)}{h} = \lim_{h\to 0}\dfrac{k(h)}{f(a+k(h))-f(a)}$

$$= \lim_{k\to 0}\frac{k}{f(a+k)-f(a)}$$

$$= \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

∎

Ex. i) Let $n\in\mathbb{N}$, Let $f(x)=\begin{cases} x^n & x\geq 0 \quad n \text{ is even} \\ x^n & x\in\mathbb{R} \quad n \text{ is odd}\end{cases}$

$f^{-1}(x) = g(x) = x^{\frac{1}{n}}$

So, for $x \neq 0$, $g'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{n(f^{-1}(x))^{n-1}} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}$

i.e. $\frac{d}{dx}x^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}$

ii) $f(x) = x^{\frac{m}{n}}$ then $f'(x) = \frac{m}{n}x^{\frac{m}{n}-1}$ by chain rule