STAT525 HW1

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1

Use the navie Monte Carlo method to estimate $E(e^{X^2})$, where X has a Uniform[0,1] distribution. Describe and implement your algorithm. Give your estimate based on 1000 samples, and estimate the standard error of your estimate. Attach your code and results.

The formula to compute the mean is

$$E(e^{X^2}) = \int_0^1 xe^{x^2} dx$$

Based on Naive Monte Carlo, we can draw identical indenpendent distributed samples $x_1, x_2 ... x_m$ from $\pi(x) = Uniform[0, 1]$. The theoretical expection which is the mean can be computed as following formular.

$$\mu = E_\pi(h(X)) = E_\pi(e^{X^2})$$

Then estimated μ, sample mean is derived by

$$\hat{\mu} = \frac{1}{m}(h(x_1) + h(x_2) + ... + h(x_m)) = \frac{1}{m}\sum_{i=1}^{m}(e^{x_i^2}) \tag{1}$$

The variance can be computed by the following fomular and the standard error is the square root of variance.

$$\begin{aligned} Var(\mu) &= Var(\frac{1}{m}\sum_{i=1}^{m} h(x_i)) \\ &= \frac{1}{m^2} mVar(h(x_i)) \\ &= \frac{1}{m} Var(e^{x_i^2}) \end{aligned} \tag{2}$$

```
x = runif(1000)
mean(exp(x^2))
```

```
## [1] 1.465435
```

```
sqrt(var(exp(x^2))/1000)
```

```
## [1] 0.01493155
```

2

Suppose $U \sim Uniform(0, 1)$. Describe how to use the inversion method to generatea sample from the Pareto distribution with the following pdf:

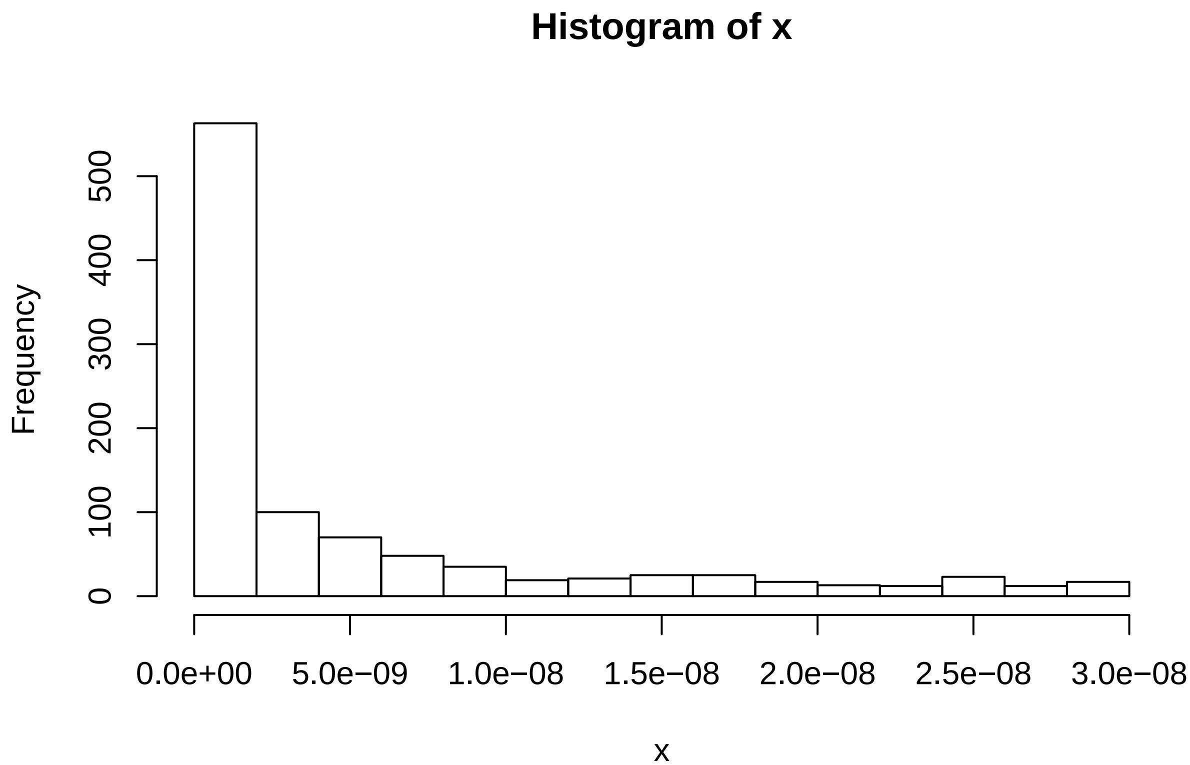
$$f(x) = 160x^{-6}, 2 \leq x \leq \infty \tag{3}$$

$$F(x) = P(t \leq x) = \int_2^x 160t^{-6} dt = 160(-\frac{1}{5}t^{-5})|_2^x = -32(x^{-5} - 2^{-5}) = -32x^{-5} + 1$$

$$y = F(x) = -32x^{-5} + 1$$
$$x = (-\frac{y-1}{32})^5$$
$$X = F_X^{-1}(u) = (-\frac{u-1}{32})^5$$

Use the inversion method to generate 1000 samples from the above Pareto distribution. Based on this sample, estimate the mean and the variance of this distribution. Attach your code and results.

```
u = runif(1000)
x = (-1/32*(u-1))^5
hist(x)
```




```
mean(x)
```

```
## [1] 4.997377e-09
```

```
var(x)
```

```
## [1] 5.692838e-17
```

3

Suppose $U \sim Uniform(0,1)$. Describe how to use the inversion method to generatea sample from a distribution with the following probability denstiy function:

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases} \tag{4}$$

When $0 \le x \le 1$,

$$F(x) = \int_0^x t dt = 0.5t^2|_0^x = 0.5x^2$$

$$X = F_X^{-1}(u) = \sqrt{2u}$$

When $1 \le x \le 2$,

$$F(x) = \frac{1}{2} + \int_1^x (2-t)dt = 0.5 + (2t - 0.5t^2)|_1^x = 0.5 + 2x - 2 - 0.5x^2 + 0.5 = 1 - 0.5(2-x)^2$$

$$X = F_X^{-1}(u) = 2 - \sqrt{2(1-u)}$$

Use the inversion method to generate 10 samples from the above distribution. Attach your code and results.

```r
u = runif(10)
x = c(rep(0,10))
for (i in 1:10){
  if (u[i] <= 0.5){
    x[i] = sqrt(2*u[i])
  } else {
    x[i] = 2 - sqrt(2*(1-u[i]))
  }
}
x
```

```
##  [1] 0.8629334 0.4068404 1.2481415 1.7068180 0.4014496 1.3442958 1.5135736
##  [8] 0.9114372 1.3254464 1.0659009
```