STAT 525 — Spring 2020 — Section A1

Homework 1 Solution

1. Naive Monte Carlo method can be described as following:

   - Generate i.i.d. samples $x^(1)$, ..., $x^{(n)}$ from Uniform$(0, 1)$
   - Estimate $E\left(e^{X^2}\right)$ by $\frac{1}{n}\left\{e^{(x^{(1)})^2} + \cdots + e^{(x^{(n)})^2}\right\}$
   - The standard error of the estimate is $\frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\sigma}^2$ is the sample variance

   By implementing the following R code, we find that the estimated mean is 1.4406, and the standard error of the estimate is 0.0144.

```
n = 1000
X = runif(n,0,1)
mean(exp(X^2))
sd(exp(X^2))/sqrt(n)
```

2. The CDF of the Pareto distribution is

$$F(x) = \int_2^x 160s^{-6}ds = 1 - 32x^{-5}, \quad x \geq 2.$$

   The inverse CDF is given by

$$F^{-1}(u) = 2(1-u)^{-1/5}, \quad 0 \leq u \leq 1.$$

   The inversion method can be described as following:

   - Generate i.i.d. samples $u^{(1)}, \ldots, u^{(n)}$ from Uniform$(0, 1)$
   - Calculate $x^{(i)} = F^{-1}(u^{(i)})$

   By implementing the following R code, we find that the estimated mean of this distribution is 2.4933, and the estimated variance of this distribution is 0.3925.

```
n = 1000
U = runif(n,0,1)
X = 2*(1-U)^{-1/5}
mean(X)
var(X)
```

3. The CDF of this distribution is

$$F(x) = \begin{cases} \int_0^x s ds = \frac{1}{2}x^2, & 0 \leq x \leq 1 \\ \int_0^1 s ds + \int_1^x (2-s) ds = -\frac{1}{2}x^2 + 2x - 1, & 1 < x \leq 2 \end{cases}.$$

Inverse CDF of this distribution is

$$F^{-1}(y) = \begin{cases} \sqrt{2y}, & 0 \leq y \leq \frac{1}{2} \\ 2 - \sqrt{2-2y}, & \frac{1}{2} < y \leq 1 \end{cases}.$$

The inversion method can be described as following:

- Generate i.i.d. samples $u^{(1)}, \ldots, u^{(n)}$ from Uniform$(0,1)$
- Calculate $x^{(i)} = F^{-1}(u^{(i)})$ for $i = 1, \ldots, n$

R code:

```
F_inv <- function(x){
    k = length(x)
    return(sqrt(2*x)*(x<=rep(1/2,k))+(2-sqrt(2-2*x))*(x>rep(1/2,k)))
}
n = 10
U = runif(n,0,1)
X = F_inv(U)
```