STAT 525 — Spring 2020 — Section A1

Homework 3 Solution

1. We choose Beta$(1.5, 1)$ as the proposal distribution. The pdf is $g(x) = \frac{3}{2}\sqrt{x}$, $0 \le x \le 1$.

   The importance sampling algorithm can be described as following:

   - Generate i.i.d. samples $x^{(1)}, \ldots, x^{(n)} \sim$ Beta$(1.5, 1)$
   - Calculate the importance weights: $w^{(i)} = \frac{\sin(\sqrt{x^{(i)}})}{g(x^{(i)})}$ for $i = 1, \ldots, n$
   - Estimate $E\left[\sin(\sqrt{X})\right]$ by $\frac{1}{n}\left\{w^{(i)}\sin(\sqrt{x^{(1)}}) + \cdots + w^{(n)}\sin(\sqrt{x^{(n)}})\right\}$
   - The standard error of the estimate is $\frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\sigma}^2$ is the sample variance of $\left\{w^{(i)}\sin(\sqrt{x^{(i)}})\right\}_{i=1}^{n}$

   By implementing the following R code, we find that the estimate of $E\left(\sin(\sqrt{X})\right)$ is 0.5944. The standard error of the naive Monte Carlo estimate is 0.0063, and the standard error of the importance sampling estimate is 0.0009.

```
n = 1000
X = runif(n,0,1) #Naive Monte Carlo
mean.mc = mean(sin(X^0.5))
se.mc = sqrt(var(sin(X^0.5))/n)

Y = rbeta(n,1.5,1) #Importance Sampling
w = 1/(3*Y^0.5/2)
mean.is = mean(sin(Y^0.5)*w)
se.is = sqrt(var(sin(Y^0.5)*w)/n)
```

2. From Homework 2, we have

$$\pi(\theta|x_1, \ldots, x_n) \propto \frac{1}{\pi(1+\theta^2)} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\theta)^2/2}$$
$$\propto \frac{1}{1+\theta^2} \frac{1}{\sqrt{2\pi \cdot \frac{1}{n}}} e^{-\frac{(\theta-\bar{x})^2}{2/n}} = l(\theta).$$

   We choose $\mathcal{N}(\bar{x}, 1/n)$ as proposal distribution with density function $g(\theta)$. The importance weight can be calculated by

$$w = \frac{l(\theta)}{g(\theta)} = \frac{1}{1+\theta^2}.$$

   The importance sampling algorithm can be described as following:

- Generate i.i.d samples $\theta^{(1)}, \ldots, \theta^{(N)} \sim \mathcal{N}(\bar{x}, \frac{1}{n})$
- Calculate importance weights $w^{(i)} = \frac{1}{1+\theta^{(i)2}}$
- Estimate the mean of the posterior by $\frac{w^{(1)}\theta^{(1)}+\cdots+w^{(n)}\theta^{(N)}}{w^{(1)}+\cdots+w^{(N)}}$
- The standard error of the estimate is given by

$$\sqrt{\frac{Var\left[\theta w\right] + \mu^2 Var[w] - 2\mu Cov\left[\theta w, w\right]}{NE^2[w]}}$$

By implementing the following R code, we find that the estimated mean of this distribution is 0.5459. The standard error of the estimate is 0.0106, which is smaller than the standard error based on rejection method.

R code:

```
xbar = mean(c(1.6,0.6,-0.7,1.1,0.8))
N = 1492;
theta = rnorm(N,xbar,sqrt(1/5))
w = 1/(1+theta^2)
mu = sum(theta*w)/sum(w)
sqrt((var(theta*w) + mu^2*var(w) - 2*mu*cov(theta*w, w))/(N*mean(w)^2))
```

3. We choose $\mathcal{N}(0,1)$ as proposal distribution with density function $g(x)$. Let $l(x) = e^{-x^8/2}$, and $h(x) = x^2$. The importance sampling algorithm can be described as following:

- Generate i.i.d. samples $x^{(1)}, \ldots, x^{(n)} \sim \mathcal{N}(0,1)$
- Calculate the importance weights

$$w^{(i)} = \frac{l(x^{(i)})}{g(x^{(i)})}$$

- Estimate $\mathbb{E}\left[X^2\right]$ by $\hat{\mu} = \frac{\sum_{i=1}^n w^{(i)} h(x^{(i)})}{\sum_{i=1}^n w^{(i)}}$
- The standard error of the estimate is

$$\sqrt{\frac{Var\left[h(x)w\right] + \mu^2 Var\left[w\right] - 2\mu Cov\left[h(x)w, w\right]}{nE^2\left[w\right]}}$$

By implementing the R code, we find that the estimate of $\mathbb{E}\left[X^2\right]$ is 0.3700, and the standard error is 0.0114.

R code:

```
n = 1000; x = rnorm(n)
w = exp(-x^8/2)/dnorm(x)
mu=sum(w*x^2)/sum(w)
sqrt((var(x^2*w) + mu^2*var(w) - 2*mu*cov(x^2*w, w))/(n*mean(w)^2))
```