STAT525 HW2

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1

Design a rejection sampling algorithm for generating samples from a distribution with the following pdf:

$$\pi(x) = \frac{3}{2}x^3 + \frac{11}{8}x^2 + \frac{1}{6}x + \frac{1}{12}, 0 \leq x \leq 1$$

$$\pi(x) \leq cg(x)$$

$$g(x) = Uniform[0,1], c = \max_{[0,1]} \pi(x) = \pi(1) = \frac{25}{8}$$

```
pi = function(x) {3/2*x^3 + 11/8*x^2 + 1/6*x + 1/12}
c = 25/8
i = 1
sample = c(rep(0,1000))
while (i <= 1000) {
  x = runif(1)
  u = runif(1)
  if (u <= pi(x)/c) {
    sample[i] = x
    i = i + 1
  }
}
# Estimate of E(x^2)
mean(sample^2)
```

```
## [1] 0.6050957
```

```
# standard error of the estimate
sqrt(var(sample^2)/1000)
```

```
## [1] 0.008358333
```

2

$$\pi(x) \propto \{[sin(8x)]^2 + 2[cos(3x)]^4 + 1]\}e^{-x}, 0 \leq x \leq \infty$$

$$\pi(x) \propto l(x), \quad l(x) \leq cg(x)$$

$$g(x) = e^{-x}, 0 \leq x \leq \infty; \quad c = \max([sin(8x)]^2 + 2[cos(3x)]^4 + 1) = 4$$

$$u \leq \frac{l(x)}{cg(x)} = \frac{[sin(8x)]^2 + 2[cos(3x)]^4 + 1}{4}$$

```
pi = function(x) {(sin(x)^2 + 2*cos(3*x)^4 + 1)}
c = 4
i = 1
sample = c(rep(0,1000))
while (i <= 1000) {
```

```
  x = rexp(1)
  u = runif(1)
  if (u <= pi(x)/c) {
    sample[i] = x
    i = i + 1
  }
}
# Estimate of the mean of \pi(x)
mean(sample)
```

```
## [1] 1.06376
```

```
# standard error of the estimate
sqrt(var(sample)/1000)
```

```
## [1] 0.03144219
```

3

Let $X_1, ..., X_n$ be i.i.d from $N(0,1)$ where θ is the unknown parameter. In Bayesian inference, we may put a Cauchy prior distribution on θ and Bayes estimate of θ is the mean of posterior distribution

$$\pi(\theta|x_1, ...x_n) \propto \frac{1}{\pi(1+\theta^2)} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\theta)^2/2}$$

$$\pi(x) \propto l(x), \quad l(x) \leq cg(x)$$

$$\begin{aligned}
l(\theta) &= \frac{1}{\pi(1+\theta^2)}(2\pi)^{-5/2} exp(-\frac{1}{2}[\sum(x_i-\theta)^2]) \\
&= \frac{1}{\pi(1+\theta^2)}(2\pi)^{-5/2} exp(-\frac{1}{2}[\sum x_i^2 + 5\theta^2 - 2\theta\sum x_i]) \\
&= \frac{1}{\pi(1+\theta^2)}(2\pi)^{-5/2} exp(-\frac{1}{2}[5(\theta^2 - \frac{2\sum x_i}{5}\theta + \frac{\sum x_i^2}{5})]) \\
&= \frac{1}{\pi(1+\theta^2)}(2\pi)^{-5/2} exp(-\frac{5}{2}[(\theta - \frac{\sum x_i}{5})^2 + \frac{\sum x_i^2}{5} - (\frac{\sum x_i}{5})^2]) \\
&\leq \frac{1}{\pi}\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{2\pi}} exp(-\frac{5}{2}(\theta - \frac{\sum x_i}{5})^2)\frac{1}{4\pi^2} exp(-\frac{\sum x_i^2}{2} + \frac{(\sum x_i)^2}{10}) \\
&= \frac{1}{\sqrt{80}\pi^3} exp(-\frac{\sum x_i^2}{2} + \frac{(\sum x_i)^2}{10}) N(\frac{\sum x_i}{5}, \frac{1}{5}) \\
&= cg(\theta)
\end{aligned} \tag{1}$$

Instrucmental distribution is $N(\frac{\sum x_i}{5}, \frac{1}{5})$

$$\frac{\pi(\theta)}{cg(\theta)} = \frac{\frac{1}{\pi(1+\theta^2)}}{\frac{1}{\sqrt{80}\pi^3} exp(-\frac{\sum x_i^2}{2} + \frac{(\sum x_i)^2}{10})} = \frac{\pi^2\sqrt{80}}{(1+\theta^2) exp(-\frac{\sum x_i^2}{2} + \frac{(\sum x_i)^2}{10})}$$

```
x = c(1.6, 0.6, -0.7, 1.1, 0.8)
pi <- function(theta,x) {
  return (3.141593^2*sqrt(80)/((1+theta^2)*exp(-sum(x^2)/2+(sum(x)^2)/10)))
}
i = 1
```

```r
N = 0
sample = c(rep(0,1000))
while (i <= 1000) {
  N = N + 1
  theta = rnorm(1, mean=sum(x)/5, sd=sqrt(1/5))
  if(runif(1) <= pi(theta, x)){
    sample[i] = theta
    i=i+1
  }
}
# the number of samples I need to generate from the instrumental distribution in order to have 1000 acc
N
```

```
## [1] 1000
```

```r
# estimated acceptance of rate
1000/N
```

```
## [1] 1
```

```r
# the mean of pi(theta|X)
mean(sample)
```

```
## [1] 0.6940899
```

```r
# standard error of my estimate
sqrt(var(sample)/1000)
```

```
## [1] 0.01415574
```