STAT 525 — Spring 2020 — Section A1

Homework 2 Solution

1. We choose Uniform$[0,1]$ as instrumental distribution with density function $g(x) = 1(0 \le x \le 1)$, and use the fact that $\pi(x) \le cg(x)$ with $c = \frac{25}{8}$. The rejection sampling algorithm can be described as following:

   - Generate $x \sim$ Uniform$(0,1)$
   - Generate $u \sim$ Uniform$(0,1)$
   - Accept x if $u \le \frac{\pi(x)}{cg(x)}$
   - Repeat the steps above until we get $n = 1000$ accepted samples, denoted by $x^{(1)}, \ldots, x^{(n)}$
   - Estimate $E\left(X^2\right)$ by $\frac{1}{n}\left\{(x^{(1)})^2 + \cdots + (x^{(n)})^2\right\}$
   - The standard error of the estimate is $\frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\sigma}^2$ is the sample variance

   By implementing the following R code, we find that the estimate of $E\left(X^2\right)$ is 0.5912, and the standard error of the estimate is 0.0088.

```
Pi <- function(x){
        return(3*x^3/2+11*x^2/8+x/6+1/12)
}

n = 1000; c = 25/8;
X = rep(0,n)
n_accept = 0
while(n_accept<1000){
        x = runif(1,0,1)
        U = runif(1,0,1)
        if(U <= Pi(x)/c){
                n_accept = n_accept + 1
                X[n_accept] = x
        }
}
mean(X^2)
sd(X^2)/sqrt(n)
```

2. We choose Exp(1) as instrumental distribution with density function $g(x) = e^{-x}(x \ge 0)$, and use the fact that $\pi(x) \le cg(x)$ with $c = 4$.

   The rejection sampling algorithm can be described as following:

- Generate $x \sim \text{Exp}(1)$
- Generate $u \sim \text{Uniform}(0, 1)$
- Accept x if $u \leq \frac{\pi(x)}{cg(x)}$
- Repeat the steps above until we get $n = 1000$ accepted samples, denoted by $x^{(1)}, \ldots, x^{(n)}$
- Estimate mean of $\pi(x)$ by $\frac{1}{n}\left\{x^{(1)} + \cdots + x^{(n)}\right\}$
- The standard error of the estimate is $\frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\sigma}^2$ is the sample variance

By implementing the following R code, we find that the estimated mean of this distribution is 0.9694, and the standard error of the estimate is 0.0308.

```
n = 1000; c = 4;
X = rep(0,n)
n_accept = 0
while(n_accept <1000){
        x = rexp(1)
        U = runif(1,0,1)
        if(U <= ((sin(8*x))^2+2*(cos(3*x))^4+1)/c){
                n_accept = n_accept + 1
                X[n_accept] = x
        }
}
mean(X)
sd(X)/sqrt(n)
```

3. The target posterior distribution is

$$
\begin{aligned}
\pi(\theta|x_1, \ldots, x_n) &\propto \frac{1}{\pi(1+\theta^2)} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\theta)^2/2} \\
&\propto \frac{1}{\pi(1+\theta^2)} e^{-\sum_{i=1}^{n} -(x_i-\theta)^2/2} \\
&\propto \frac{1}{\pi(1+\theta^2)} e^{-\sum_{i=1}^{n} -(x_i-\bar{x}+\bar{x}-\theta)^2/2} \\
&\propto \frac{1}{\pi(1+\theta^2)} e^{-n(\bar{x}-\theta)^2/2} \\
&\propto \frac{1}{1+\theta^2} \frac{1}{\sqrt{2\pi \cdot \frac{1}{n}}} e^{-\frac{(\theta-\bar{x})^2}{2/n}} \\
&\leq \frac{1}{\sqrt{2\pi \cdot \frac{1}{n}}} e^{-\frac{(\theta-\bar{x})^2}{2/n}} = \mathcal{N}(\bar{x}, \frac{1}{n}).
\end{aligned}
$$

Thus, we can consider $\mathcal{N}(\bar{x}, \frac{1}{n})$ as the instrumental distribution with $c = 1$. The rejection method can be described as following:

- Generate $\theta \sim \mathcal{N}(\bar{x}, \frac{1}{n})$
- Generate $u \sim \text{Uniform}(0, 1)$
- Accept x if $u \leq \frac{1}{1+\theta^2}$
- Repeat the steps above until we get $n = 1000$ accepted samples, denoted by $\theta^{(1)}, \dots, \theta^{(n)}$
- Estimate the mean of the posterior by $\frac{1}{n}\left\{\theta^{(1)} + \cdots + \theta^{(n)}\right\}$
- The standard error of the estimate is $\frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\sigma}^2$ is the sample variance

By implementing the following R code, we find that the estimated mean of this distribution is 0.5517, and the standard error of the estimate is 0.0128. In order to have 1000 accepted samples, we need approximately 1492 samples generated from the instrumental distribution. The acceptance rate is 67%.

```
xbar = mean(c(1.6,0.6,-0.7,1.1,0.8))
n = 1000; Theta = rep(0,n);
n_accept = 0; total = 0
while(n_accept<n){
        theta = rnorm(1,xbar,sqrt(1/5))
        u = runif(1,0,1)
        total = total + 1
        if(u<=1/(1+theta^2)) {
                n_accept = n_accept + 1
                Theta[n_accept] = theta
        }
}
c(total,n/total)
mean(Theta)
sd(Theta)/sqrt(n)
```