STAT 525 — Spring 2020 — Section A1

Homework 7 Solution

1. We use the following Gibbs sampling algorithm to sample from the target distribution $\pi(x) \propto \exp\left\{\mu \sum_{i=1}^{d-1} x_i x_{i+1}\right\}$:

   - Draw initial value $x^{(0)} = \left(x_1^{(0)}, \ldots, x_d^{(0)}\right)$, with $P(x_i^{(0)} = +1) = P(x_i^{(0)} = -1) = \frac{1}{2}, \quad i = 1, \ldots, d$
   - Suppose we get $x^{(t)} = \left(x_1^{(t)}, \ldots, x_d^{(t)}\right)$ in the t-th iteration, then at the $(t+1)$-th iteration, for $i = 1, \ldots, d$, we draw $x_i^{(t+1)}$ from the conditional distribution $\pi\left(x_i | x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, x_{i+1}^{(t)}, \ldots, x_d^{(t)}\right)$, with

$$P\left(x_i^{(t+1)} = +1\right) = \frac{\pi\left(x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, +1, x_{i+1}^{(t)}, \ldots, x_d^{(t)}\right)}{\pi\left(x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, +1, x_{i+1}^{(t)}, \ldots, x_d^{(t)}\right) + \pi\left(x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, -1, x_{i+1}^{(t)}, \ldots, x_d^{(t)}\right)},$$

$$P\left(x_i^{(t+1)} = -1\right) = \frac{\pi\left(x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, -1, x_{i+1}^{(t)}, \ldots, x_d^{(t)}\right)}{\pi\left(x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, +1, x_{i+1}^{(t)}, \ldots, x_d^{(t)}\right) + \pi\left(x_1^{(t+1)}, \ldots, x_{i-1}^{(t+1)}, -1, x_{i+1}^{(t)}, \ldots, x_d^{(t)}\right)}$$

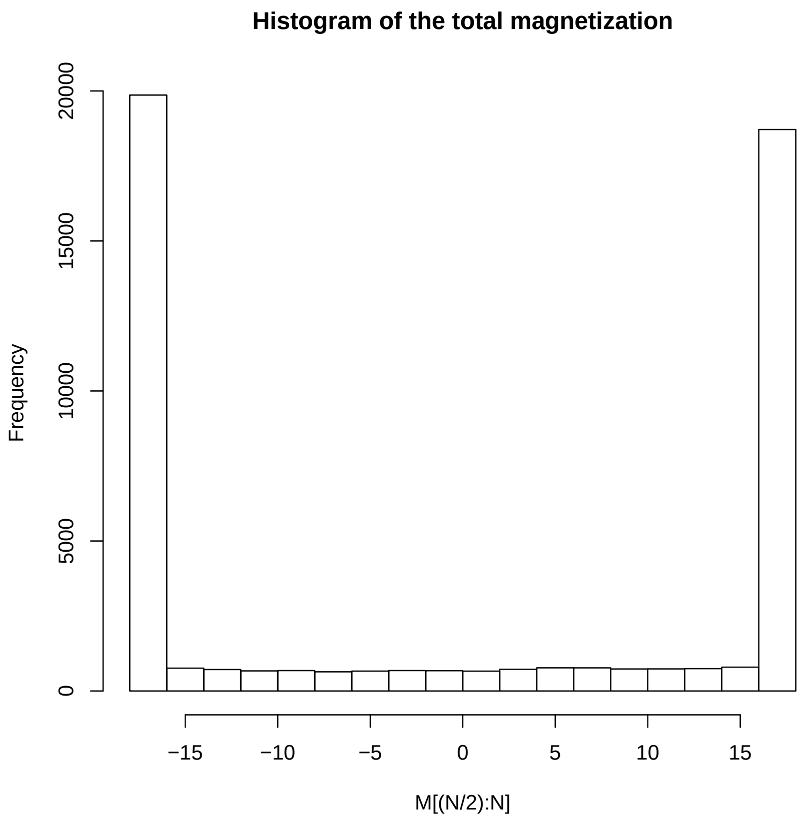
   - After updating $x^{(t)}$, we calculate the total magnetization $M^{(t)} = \sum_{i=1}^{d} x_i^{(t)}$
   - Repeat the above steps for $t = 1, \ldots, N$

   We let $N = 100000$. By implementing the following code, we get the histogram of the total magnetization in Figure 1.

```
mu=2; d=18; N = 100000
A = rbinom(d,1,0.5)*2 - 1
M = rep(0,N)
pi_x <- function(A){
        s = sum(A[1:(d-1)]*A[-1])
        return(exp(mu*s))
}

for(t in 1:N){
        for(i in 1:d){
                tmp = A
                tmp[i] = 1; a = pi_x(tmp);
                tmp[i] = -1; b = pi_x(tmp);
                A[i] = rbinom(1,1,a/(a+b))*2-1
```

```
        }
        M[t] = sum(A)
}
hist(M[(N/2):N])
```



2. In order to design the Gibbs sampling algorithm, we have to calculate the conditional posterior distribution as follows.

$$
\begin{aligned}
\pi(\mu|\tau, y_1, \ldots, y_n) =& \frac{\pi(\mu, \tau|y_1, \ldots, y_n)}{\pi(\tau|y_1, \ldots, y_n)} = \frac{e^{-\frac{\tau}{2}\sum_{i=1}^n (y_i-\mu)^2} e^{-\frac{\omega}{2}\mu^2}}{\int e^{-\frac{\tau}{2}\sum_{i=1}^n (y_i-\mu)^2} e^{-\frac{\omega}{2}\mu^2} d\mu} \\
&\propto \exp\left(-\frac{1}{2\cdot\frac{1}{n\tau+\omega}}\left(\mu - \frac{\tau\sum_{i=1}^n y_i}{n\tau+\omega}\right)^2\right), \\
\pi(\tau|\mu, y_1, \ldots, y_n) =& \frac{\pi(\mu, \tau|y_1, \ldots, y_n)}{\pi(\mu|y_1, \ldots, y_n)} = \frac{\tau^{\frac{n}{2}} e^{-\frac{\tau}{2}\sum_{i=1}^n (y_i-\mu)^2} \tau^{\alpha-1} e^{-\frac{\tau}{\beta}}}{\int \tau^{\frac{n}{2}} e^{-\frac{\tau}{2}\sum_{i=1}^n (y_i-\mu)^2} \tau^{\alpha-1} e^{-\frac{\tau}{\beta}} d\tau} \\
&\propto \tau^{\frac{n}{2}+\alpha-1} \exp\left(-\tau\left(\frac{1}{2}\sum_{i=1}^n (y_i-\mu)^2 + \frac{1}{\beta}\right)\right).
\end{aligned}
$$

We use the following Gibbs sampling algorithm to sample from the posterior distribution:

- Set initial values $\mu^{(0)} = 2$, $\tau^{(0)} = 1$

- Draw the parameters from the following distributions

$$\mu^{(t+1)} \sim \mathcal{N}\left(\frac{\tau^{(t)}\sum_{i=1}^{n} y_i}{n\tau^{(t)}+\omega},\ \frac{1}{n\tau^{(t)}+\omega}\right)$$
$$\tau^{(t+1)} \sim \text{Gamma}\left(\frac{n}{2}+\alpha,\ \frac{1}{2}\sum_{i=1}^{n}(y_i-\mu^{(t+1)})^2+\frac{1}{\beta}\right)$$

- Repeat the above step for $t = 1, \ldots, N$, and estimate the posterior means by

$$\mathbb{E}[\mu|y_1,\ldots,y_n] = \frac{2}{N}\sum_{t=\frac{N}{2}+1}^{N}\mu^{(t)}, \quad \mathbb{E}[\tau|y_1,\ldots,y_n] = \frac{2}{N}\sum_{t=\frac{N}{2}+1}^{N}\tau^{(t)}$$

By implementing the following R code, we find that the estimated posterior mean of μ is 2.1320, and the posterior mean of τ is 1.0174.

```
N = 100000; n = 6
omega = 0.04; alpha <- 2; beta <- 0.5
mu = rep(2, N); tau = rep(1, N)
y = c(1.8,3.3,0.4,2.5,2.6,2.3)
for (i in 1:(N-1)){
        mu[i+1] <- rnorm(1, (tau[i]*sum(y))/(n*tau[i]+omega),
                sqrt(1/(n*tau[i]+omega)))
        tau[i+1] <- rgamma(1, shape = n/2+alpha,
                scale = 1/(1/2*sum((y-mu[i])^2) + 1/beta))
}
mean(mu)
mean(tau)
```