STAT 525 Homework 5
Spring 2020 Feb 27

Homework 5

Due in class on Thursday, Mar 12.

1. *Normal Mean Shift Model*: Let $\{Y_1, Y_2, \ldots\}$ be a sequence of independent $N(\mu_t, 1)$ random variables, where the μ_t's undergo occasional changes.

$$Y_t = \mu_t + \varepsilon_t, \qquad \mu_t = \begin{cases} \mu_{t-1}, & \text{with probability } 0.9, \\ Z_t, & \text{with probability } 0.1, \end{cases}$$

where ε_t are i.i.d. $N(0,1)$, and Z_t are i.i.d $N(0,1)$. Suppose $\mu_0 = 0$. Assume we observe

$$(Y_1, Y_2, \ldots, Y_{15}) = (-0.4, -1.4, 1.1, -0.3, 0.7, -0.1, 1.3, 0.3, 0.1, 1.2, 2.0, 0.9, -0.4, 0.9, 1.0).$$

Use sequential importance sampling with sample size $m = 10,000$ to implement the filtering problem, i.e., estimating $E(\mu_t \mid Y_1, \ldots, Y_t)$, for $t = 1, 2, \ldots, 15$. You may use either the state equation as the proposal or the adaptive proposal. Adding the resampling step to your sequential importance sampling algorithm is optional but not required. Attach your code and results.

2. Suppose $X \sim$Uniform(0,1), and we would like to estimate $\mu = E(\frac{1}{1+X^3})$. First use the naive Monte Carlo method to estimate μ based on 1000 i.i.d. samples from Uniform(0,1), and give the variance of the estimate. Now consider the random variable $1 + X^3$, whose mean $E(1+X^3) = \int_0^1 (1+x^3)dx = \frac{5}{4}$ is known. Since $\frac{1}{1+X^3}$ and $1 + X^3$ are correlated, we can use the control variates method to construct

$$X(b) = \frac{1}{1+X^3} - b\left[(1+X^3) - \frac{5}{4}\right],$$

which has the same mean as $\frac{1}{1+X^3}$. The best choice of b is

$$b = \frac{Cov(\frac{1}{1+X^3},\ 1+X^3)}{Var(1+X^3)}.$$

Use 1000 i.i.d. samples from Uniform(0,1) to give an estimate of b by replacing $Cov(\frac{1}{1+X^3},\ 1+X^3)$ and $Var(1+X^3)$ by the sample covariance and the sample variance. Suppose your estimate of b is b^*. Then estimate the mean of $X(b^*)$ by the naive Monte Carlo method based on 1000 i.i.d. samples from Uniform(0,1), and give the variance of your estimate. Is this variance smaller the variance you obtained earlier for estimating $E(\frac{1}{1+X^3})$ with naive Monte Carlo method? Attach your code and results.

3. In Problem 2 of Homework 1, we used the naive Monte Carlo method (1000 samples) to estimate the mean of the Pareto distribution with the following pdf:

$$f(x) = 160x^{-6}, \qquad 2 \leq x < \infty.$$

Now describe an antithetic variates method to estimate the mean of the above Pareto distribution, and implement your algorithm. Compare the variance of these two estimates based on the same number of samples. Attach your code and results.