CS 446/ECE 449: Machine Learning

A. G. Schwing

University of Illinois at Urbana-Champaign, 2020

Scribe

L3: Logistic Regression

Goals of this lecture

- Understand logistic regression
- Understand how it fixes classification issues with linear regression
- Contrast linear and logistic regression
- Get to know an application of logistic regression

Reading Material

- K. Murphy; Machine Learning: A Probabilistic Perspective; Chapter 8

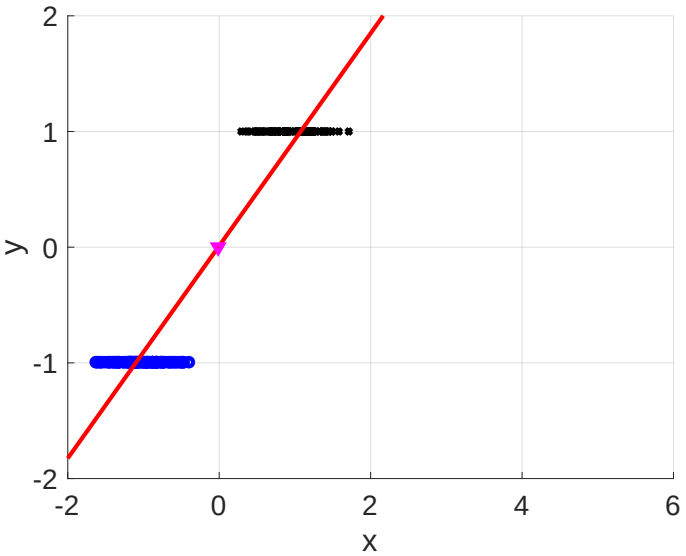
The Problem: Linear regression for classification

$$y^{(i)} \in \{-1, 1\}$$

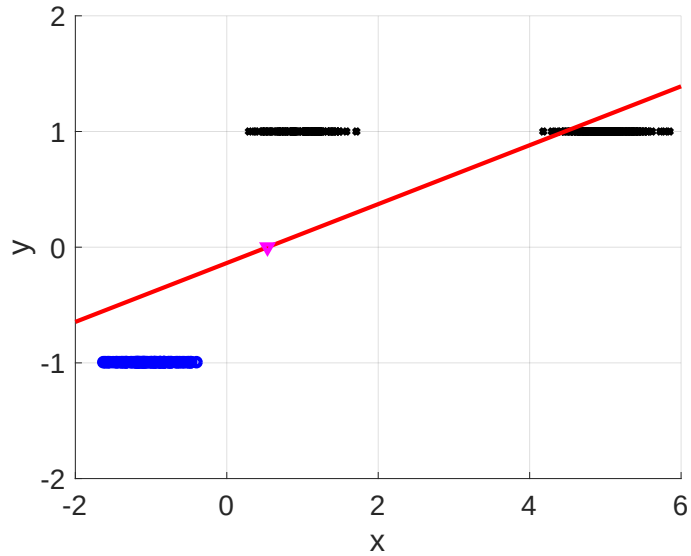
1D-Model:

$$y^{(i)} = \text{sign}(w_1 x^{(i)} + w_0)$$

pass the averager of groups



perfect classification



decision boundary shifted

Why is this?

Why is this?

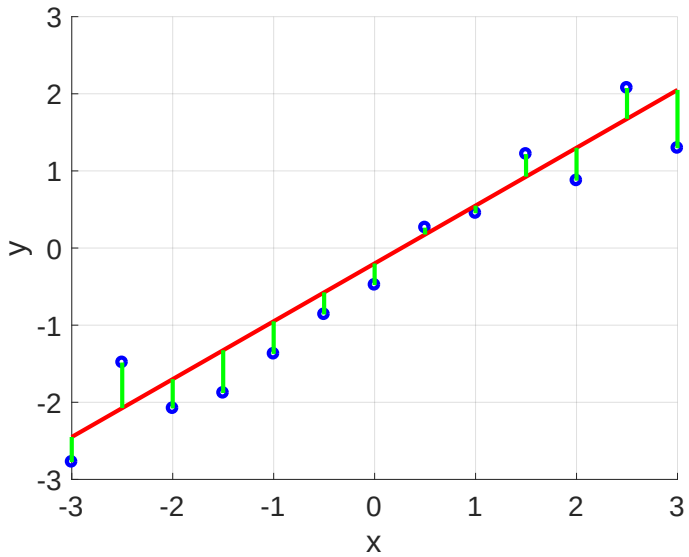
Assuming 1D-model

$$y = w_1 \cdot x + w_2$$

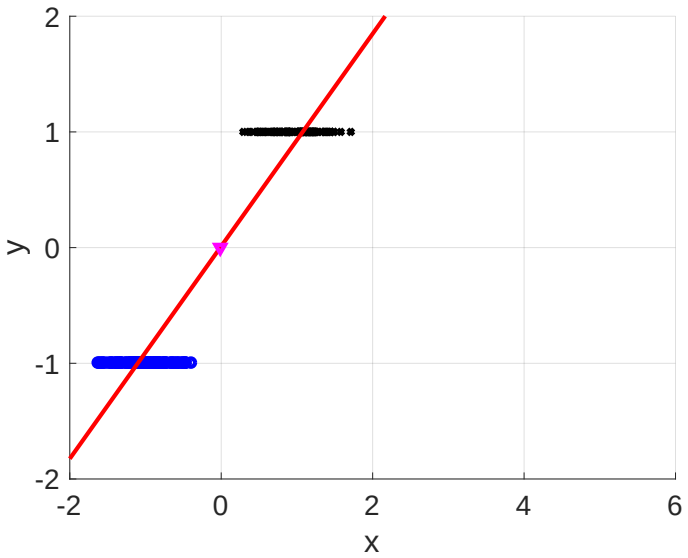
Linear regression finds parameters w_1, w_2 such that the squared error is small

$$\arg\min_{w_1, w_2} \frac{1}{2} \sum_{i=1}^{N} \left(y^{(i)} - w_1 \cdot x^{(i)} - w_2 \right)^2$$

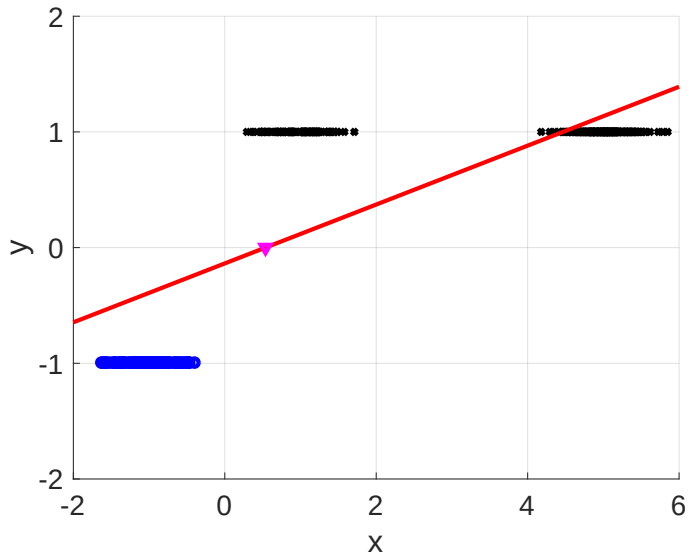
What exactly is the error?



In our case:



perfect classification

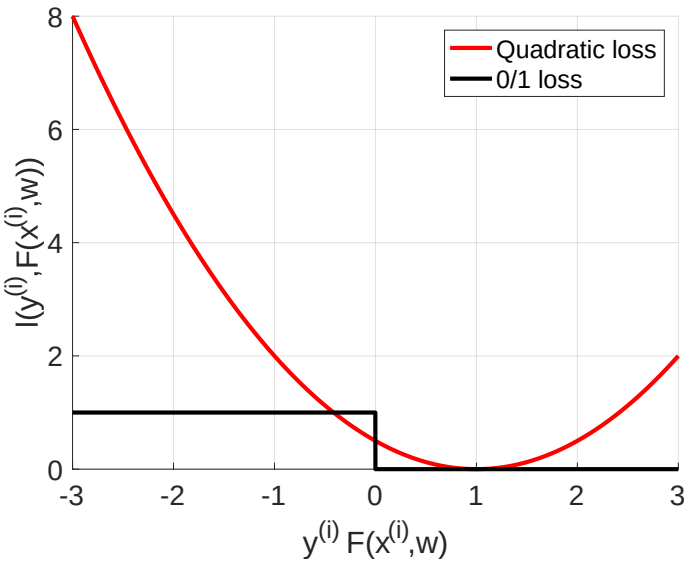


decision boundary shifted

Linear regression: Quadratic loss (recall $y^{(i)} \in \{-1, 1\}$)

$$\ell(y_i, \phi(x^{(i)})^\top \boldsymbol{w}) = \frac{1}{2}(y^{(i)} - \phi(x^{(i)})^\top \boldsymbol{w})^2$$

$$\overset{(y^{(i)})^2=1}{=} \underbrace{\frac{1}{2}(1 - y^{(i)} \underbrace{\phi(x^{(i)})^\top \boldsymbol{w}}_{F(x^{(i)}, \boldsymbol{w})})^2}_{F(x^{(i)}, \boldsymbol{w}, y^{(i)})}$$

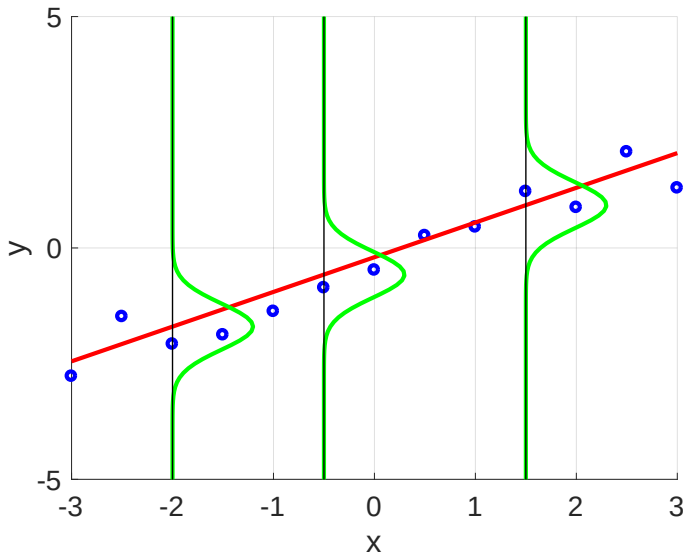


We penalize samples that are 'very easy to classify.'

How to fix this?

A probabilistic interpretation of linear regression ($y^{(i)} \in \mathbb{R}$):
Model: Gaussian distribution

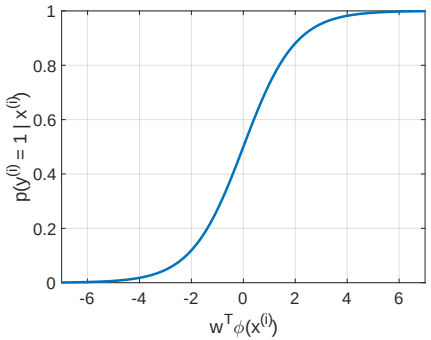
$$p(y^{(i)}|x^{(i)}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y^{(i)} - \boldsymbol{w}^\top \phi(x^{(i)}))^2\right)$$



maximize the likelihood of our data

Logistic Regression:

Another probabilistic formulation for classification ($y^{(i)} \in \{-1, 1\}$):
Model:



$$p(y^{(i)} = 1|x^{(i)}) = \frac{1}{1 + \exp(-\boldsymbol{w}^T \phi(x^{(i)}))}$$

$$p(y^{(i)} = -1|x^{(i)}) = 1 - p(y^{(i)} = 1|x^{(i)}) = \frac{1}{1 + \exp(\boldsymbol{w}^T \phi(x^{(i)}))}$$

Taken together:

$$p(y^{(i)}|x^{(i)}) = \frac{1}{1 + \exp(-y^{(i)} \boldsymbol{w}^T \phi(x^{(i)}))}$$

What to do with this model?

$$p(y^{(i)}|x^{(i)}) = \frac{1}{1+\exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))}$$

Recall that we are given a dataset $\mathcal{D} = \{(x^{(i)}, y^{(i)})\}$. How about we choose $\boldsymbol{w}$ which maximizes the likelihood/probability of this dataset?

Assumption:

Samples/Data points are i.i.d.

$$p(\mathcal{D}) = \prod_{(x^{(i)},y^{(i)})\in\mathcal{D}} p(y^{(i)}|x^{(i)})$$

Choose $\boldsymbol{w}$ to maximize probability:

$$\max_{\boldsymbol{w}} \prod_{(x^{(i)},y^{(i)})\in\mathcal{D}} p(y^{(i)}|x^{(i)})$$

Model:

$$p(y^{(i)}|x^{(i)}) = \frac{1}{1+\exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))}$$

Task:

$$\arg\max_{\boldsymbol{w}} \prod_{(x^{(i)},y^{(i)})\in\mathcal{D}} p(y^{(i)}|x^{(i)}) = \arg\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} -\log p(y^{(i)}|x^{(i)})$$

Combined:

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1+\exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))\right)$$

Comparison

Linear regression

Program:

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \underbrace{\frac{1}{2}(1-y^{(i)} \underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})^2}_{F(x^{(i)},w,y^{(i)})}$$

Logistic regression

Program:

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\Big(1+\exp(\underbrace{-y^{(i)} \underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)}}_{F(x^{(i)},w,y^{(i)})})\Big)$$

Empirical risk minimization:

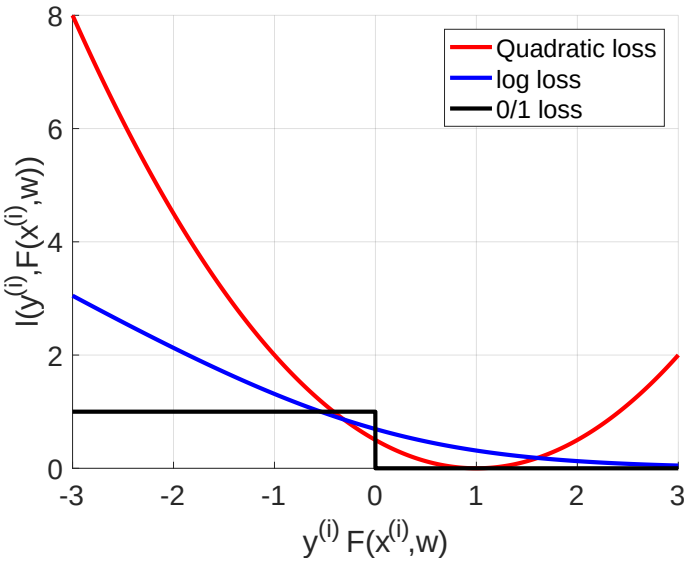
$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \ell(y^{(i)}, F(x^{(i)}, w))$$

Linear regression:

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \frac{1}{2}(1 - y^{(i)} \underbrace{\boldsymbol{w}^T \phi(x^{(i)})}_{F(x^{(i)},w)})^2$$

Logistic regression:

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1 + \exp(-y^{(i)} \underbrace{\boldsymbol{w}^T \phi(x^{(i)})}_{F(x^{(i)},w)})\right)$$



the less price you pay if you classify it correctly.

How to optimize

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1+\exp(-y^{(i)}\underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})\right)$$

Can we set the gradient to zero and solve for $\boldsymbol{w}$?

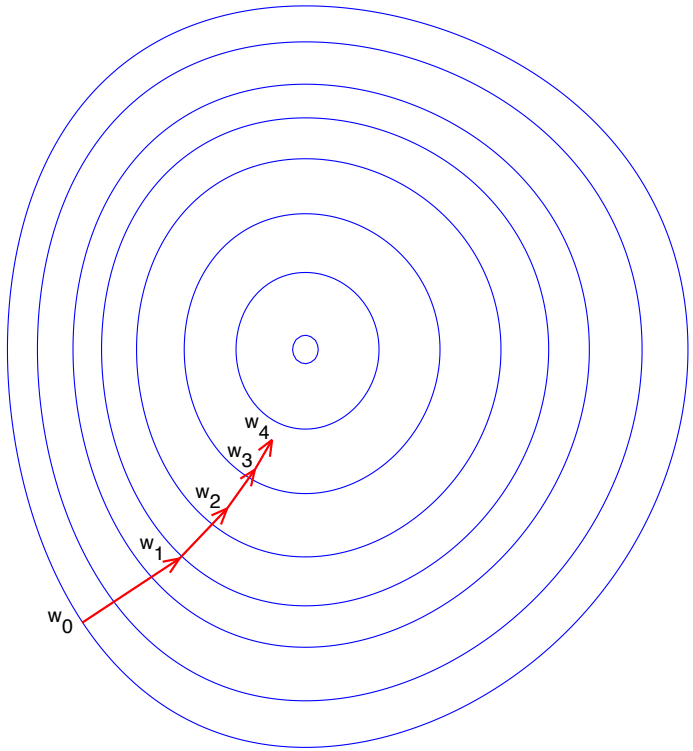
$$\sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \frac{-y^{(i)}\exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))}{1+\exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))}\phi(x^{(i)}) = 0$$

No analytic solution for $\boldsymbol{w}$ in general

How to optimize

$$\min_{\boldsymbol{w}} \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1+\exp(-y^{(i)}\underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})\right)$$

Gradient descent: (walking down a mountain)



To solve

$$\min_{\boldsymbol{w}} f(\boldsymbol{w}) := \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \log\left(1+\exp(-y^{(i)}\underbrace{\boldsymbol{w}^T\phi(x^{(i)})}_{F(x^{(i)},w)})\right)$$

we can use its gradient:

$$\nabla_{\boldsymbol{w}} f(\boldsymbol{w}) = \sum_{(x^{(i)},y^{(i)})\in\mathcal{D}} \frac{-y^{(i)}\exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))}{1+\exp(-y^{(i)}\boldsymbol{w}^T\phi(x^{(i)}))}\phi(x^{(i)})$$

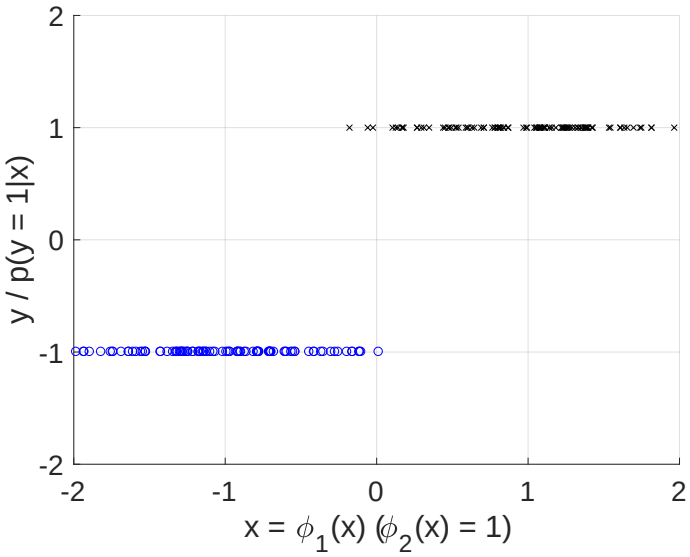
Simple algorithm: Initialize $t=0$, $\boldsymbol{w}_t$, and stepsize α

- Compute gradient $\mathbf{g}_t = \nabla_{\boldsymbol{w}} f(\boldsymbol{w}_t)$
- Update parameters $\boldsymbol{w}_{t+1} \leftarrow \boldsymbol{w}_t - \alpha\mathbf{g}_t$
- Update $t \leftarrow t+1$

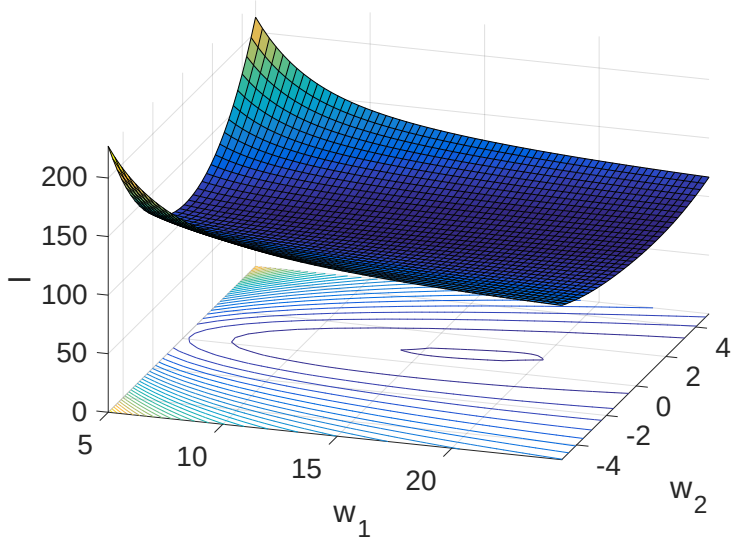
More complex algorithms may be 'better.'

Example:

Data



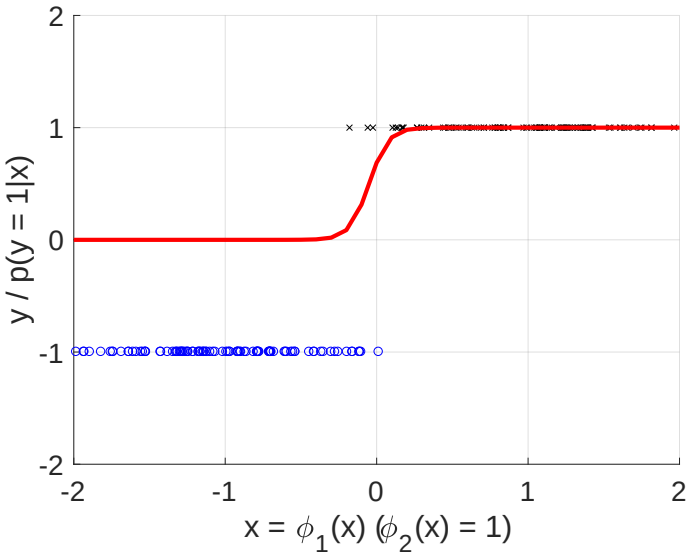
Loss



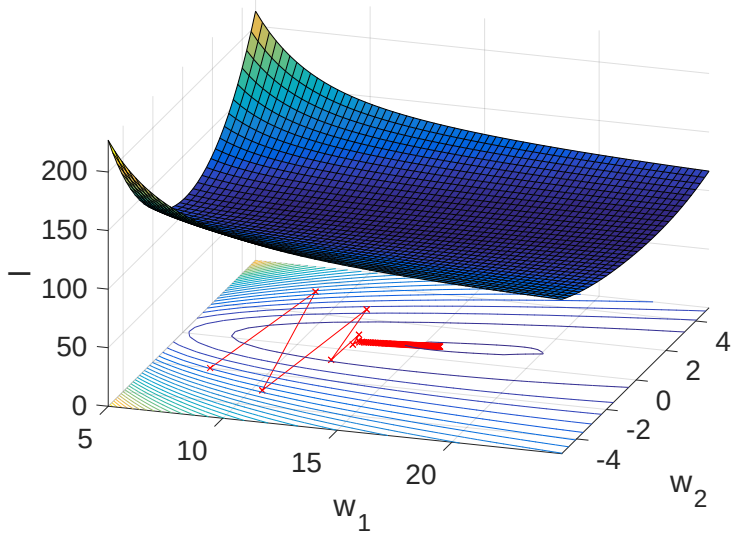
Movie

Example:

Data

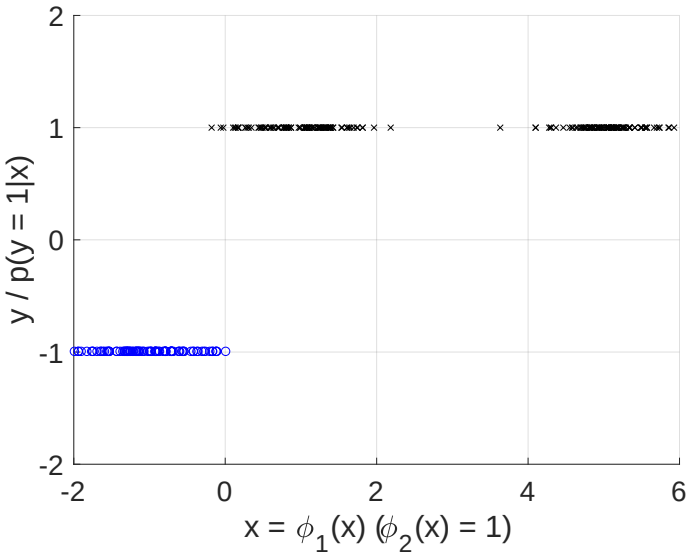


Loss

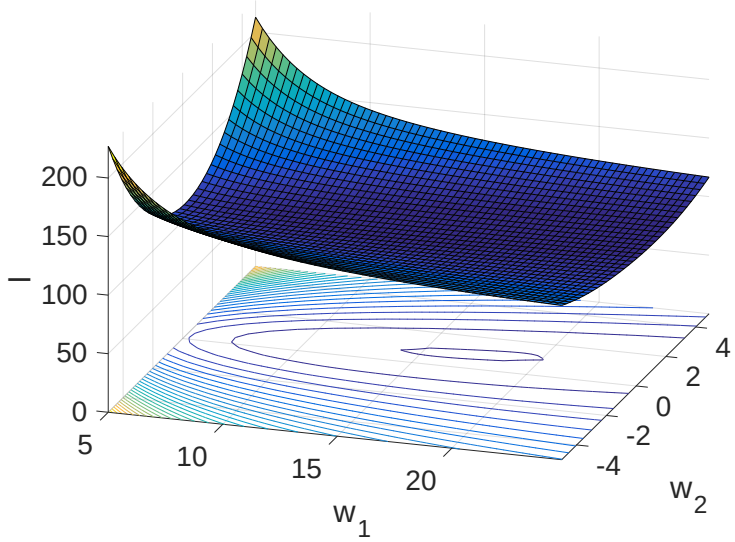


Example:

Data



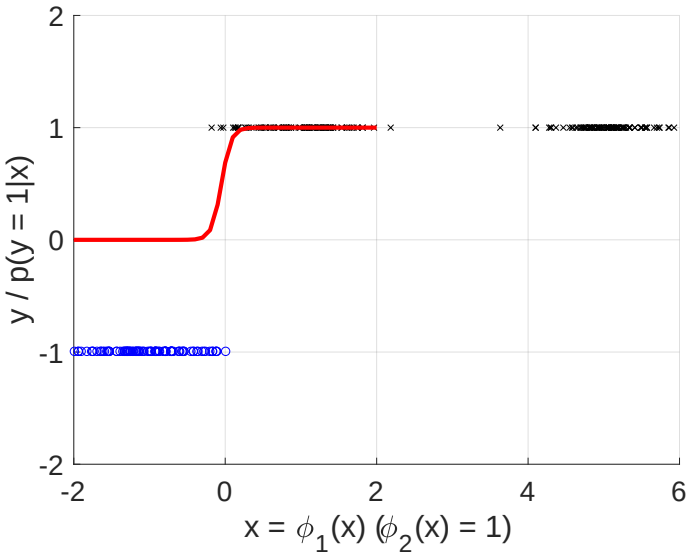
Loss



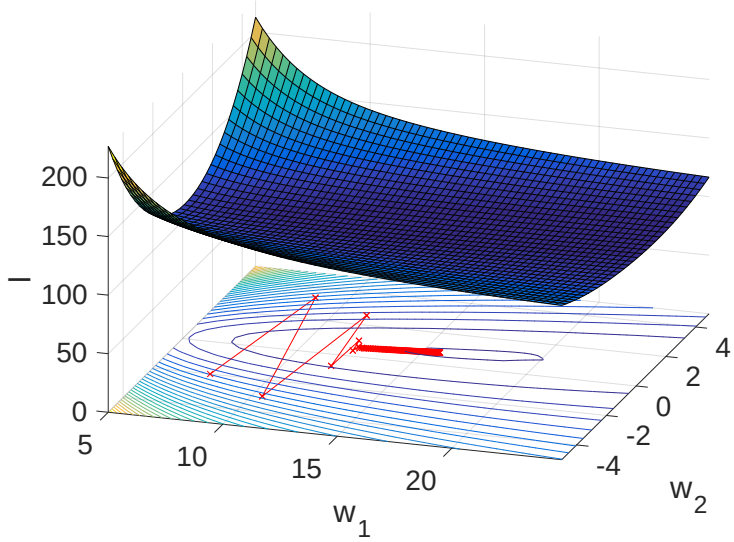
Movie

Example:

Data



Loss



Comparison:

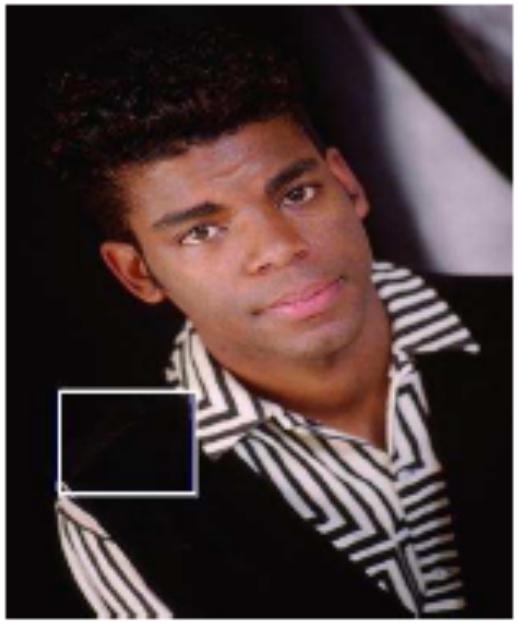
Linear regression:

- Closed form solution
- Gaussian probability model
- Not too well suited for classification

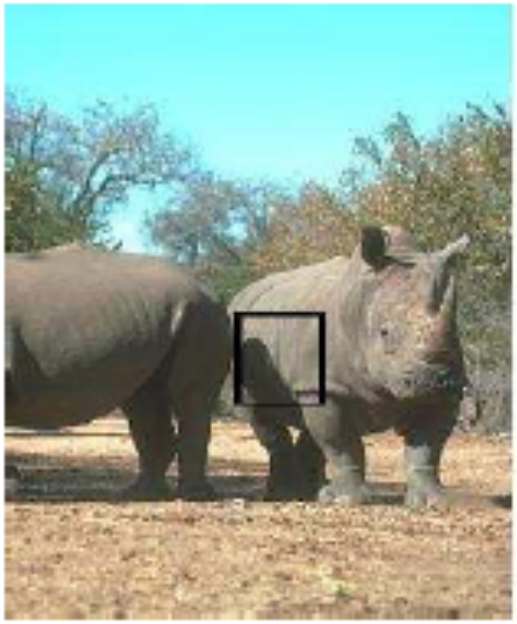
Logistic regression:

- Well suited for binary classification
- Logistic probability model
- No closed form solution

Application: **Edge/Boundary detection:** Issues?



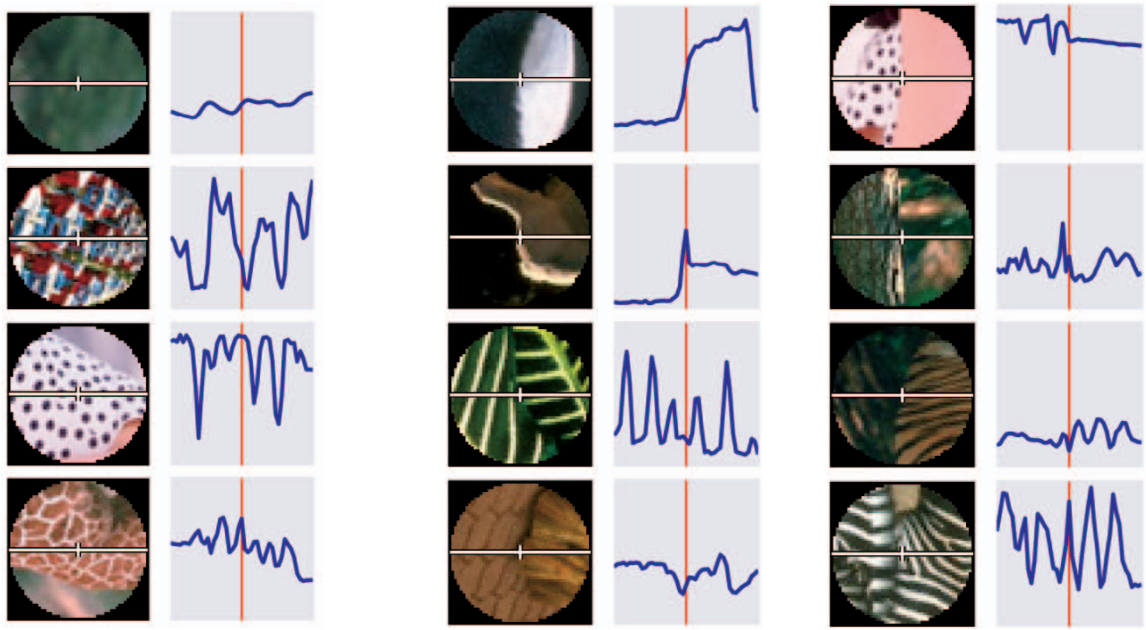
Poor contrast



Shadow

Texture

Edge/Boundary detection: Why is it so difficult?
Let's look at a local image region and the corresponding intensities:



Non-Boundaries Boundaries

Intensity cue is not necessarily a good indicator for boundaries.

Martin et al., *Learning to detect natural Image Boundaries ...*, PAMI 2004

Edge/Boundary detection: What other image cues could be useful?

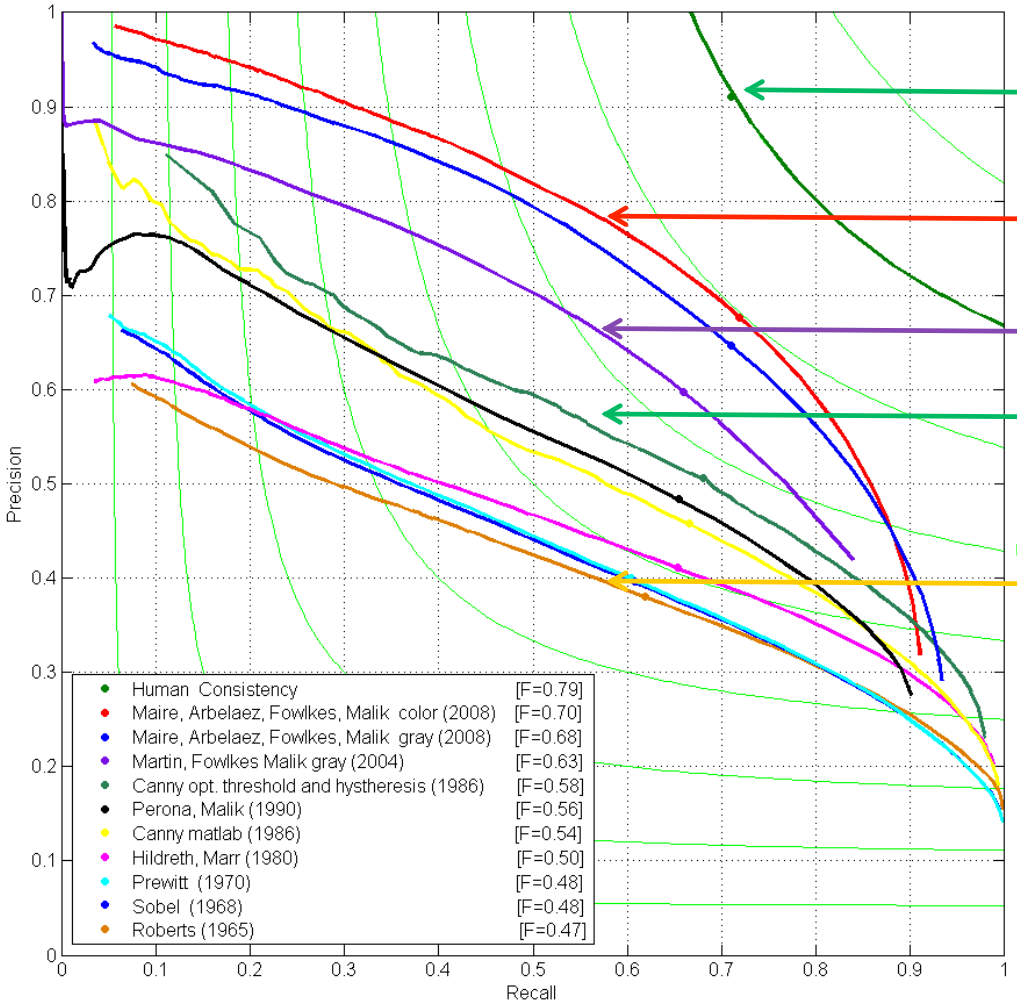
- Boundary gradient (BG)
- Color gradient (CG)
- Texture gradient (TG)
- ...

Martin et al., *Learning to detect natural Image Boundaries ...*, PAMI 2004

How to combine all those cues? Learn a linear combination of cues

- What is $y^{(i)}$? Annotated pixel label
- What is $x^{(i)}$? Image
- What is $\phi(x^{(i)})$? Vector of features computed in the **neighborhood** of pixel i, e.g., intensity, texture gradient, oriented gradient etc.

Boundary detection performance:



Humans

Berkeley gPb, ‘08

Berkeley PB, ‘04

Canny+ Hysteresis (’85)

Prewitt, 1965

Quiz:

- Which loss is used for logistic regression?
- What is the difference between logistic and linear regression?
- How to optimize linear and logistic regression?

Important topics of this lecture:

- Linear regression
- Logistic regression

Up next:

- Basics about optimization techniques