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(a). Use Jensen's Inequality to obtain a bound on the log-likelihood. and divide the bound into two parts. one of which is the KL divergence.

$KL(q(z|x), p(z))$

$$\log P_\theta(x) = \log \sum_z q(z|x) \frac{P_\theta(x,z)}{q(z|x)} \geq \sum_z q(z|x) \log \frac{P_\theta(x,z)}{q(z|x)}$$

$$= \sum_z q(z|x) \log \frac{P_\theta(x,z)\, p(z)}{q(z|x)\, p(z)}$$

$$= \sum_z q(z|x) \log \frac{p(z)}{q(z|x)} + \sum_z q(z|x) \log \frac{P_\theta(x,z)}{p(z)}$$

$$= \mathcal{L}(P_\theta(x,z), p(z)) - KL(q(z|x), p(z))$$

(b). State at least two properties of the KL divergence

① non-negative $\sum_z q(z|x) \log \frac{p(z)}{q(z|x)} \leq \log \sum_z q(z|x) \frac{p(z)}{q(z|x)} = \log 1 = 0.$

② non-symmetric $KL(P, Q) \neq KL(Q, P)$. $KL = -\sum_z q(z|x) \log \frac{p(z)}{q(z|x)} \geq 0$

(c) Let $q(z|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(z-\mu_q)^2)$

What is the value for KL-divergence $KL(q(z|x), q(z|x))$. Why?

$$KL(q(z|x), q(z|x)) = \sum_z q(z|x) \log \overbrace{\frac{q(z|x)}{q(z|x)}}^{=1} = 0$$

(d). let $p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(z-\mu_p)^2)$

What is the value for the KL-divergence. $KL(q(z|x), p(z))$ in terms of μ_p, μ_q, σ ?

$$KL(q(z|x), p(z)) = \sum_z q(z|x) \log \frac{q(z|x)}{p(z)}$$

$$= \sum_z \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(z-\mu_q)^2) \log \frac{\exp(-\frac{1}{2\sigma^2}(z-\mu_q)^2)}{\exp(-\frac{1}{2\sigma^2}(z-\mu_p)^2)}$$

$$= \sum_z \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(z-\mu_q)^2) \cdot (-\frac{1}{2\sigma^2}[(z-\mu_q)^2 - (z-\mu_p)^2]$$

$$= \sum_z (-\frac{1}{2\sigma^2})((\mu_q^2 - \mu_p^2) - 2(\mu_q - \mu_p) z) \underline{N(\cancel{\mu_p}\,\mu_q, \sigma^2)}$$

$$= (-\frac{1}{2\sigma^2})(\mu_q^2 - \mu_p^2 - 2(\mu_q - \mu_p) E(z)) = -\frac{1}{2\sigma^2}(\mu_q^2 - \mu_p^2 + 2\mu_p\mu_q - 2\mu_p^2) = \underline{\underline{+\frac{1}{2\sigma^2}(\mu_q - \mu_p)^2}}$$

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(e) Now, let $q(z|x)$ and $p(z)$ be arbitrary probability distns. We want to find that $q(z|x)$. which maximizes $\sum_z q(z|x) \log P_\theta(z|x) - KL(q(z|x), p(z))$ subject to $\sum_z q(z|x) = 1$. Ignore non-negativity constraints. State the Langrangian and compute its stationary point. i.e. solve for $q(z|x)$ which depends on $P_\theta(x|z)$ and. $p(z)$. Make sure to get rid of Langrange multiplier.

$$\min.\ KL(q(z|x).\ p(z)) - \sum_z q(z|x) \log P_\theta(\cancel{z|x}) + \lambda(1 - \sum_z q(z|x))$$

$$= \sum_z q(z|x)\left(\log \frac{q(z|x)}{p(z)} - \lambda - \log P_\theta(x|z)\right)$$

$$\frac{\partial \mathcal{L}(q(z|x,\lambda)}{\partial q(z|x)} = \log \frac{q(z|x)}{p(z)} - \lambda - \log P_\theta(x|z) + \cancel{q(z|x)\frac{p(z)}{q(z|x)}} \cdot \frac{1}{p(z)}$$

$$= \log \frac{q(z|x)}{p(z) P_\theta(x|z)} - \lambda + 1 = 0$$

$$q(z|x) = e^{\lambda - 1} \times p(z) P_\theta(x|z)$$

$$\sum_z q(z|x) = 1 \Rightarrow. e^{\lambda-1} \sum_z p(z) P_\theta(x|z) = 1$$

$$e^{\lambda-1} = \frac{1}{\sum_z p(z) P_\theta(x|z)}$$

$$\Rightarrow.\ q(z|x) = \frac{p(z) P_\theta(x|z)}{\sum_z p(z) P_\theta(x|z)}$$

(f): Which of the following terms should $q(z|x)$ be equal to:

(1) $p(z)$ (2) $p_\theta(x|z)$ (3) $P_\theta(z|x)$. (4) $P_\theta(x,z)$

$$q(z|x) = P_\theta(z|x)$$

(g) Provide the code for implementing the 'reparameterize' founction in AE/VAE.py

```
std = torch.exp(logvar * 0.5)
eps = torch.randn_like(std)
return mu + ~~std~~ eps * std
```

Sorry for the inconvenience. I don't have a printer at home so I have to write it at paper. But Gradescope requires to submit at least three pages file.