Geogebra.

Math 241 Calculus III
Prof. Honold

Final Examination

Jan 4, 2019
13.00–16.00

Question 1 (ca. 7 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := \big(x - \tfrac{1}{2}\big)\, y\, \mathrm{e}^{-x^2 - 2y^2}.$$

(a) local extrema $\Rightarrow$ critical point $\nabla f = 0$, Hess matrix $H(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$

c) $|(x,y)| \geq R$. $|f(x,y)| \leq P$.

a) Determine all local extrema of f and their types.
Hint: There are 5 critical points.

(b). limitation. ① 定义 先证不存在 找反例 $y = kx$, $y = x^k$ $\to y = mx^n$ 对不同k值, 极限若存在, 则下一步; 不相同, 则不存在. ↓ 先证存在 ② 路径要任意 $\lim_{(x,y)\to(a,y)} f(x) = c$ →

② $|(x,y)| \to +\infty$ 用极坐标 $r = \sqrt{x^2+y^2}$

b) Show that $\displaystyle\lim_{|(x,y)|\to+\infty} f(x,y) = 0.$

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$ (不能含 $\sin\theta$ 不行)

Hint: Recall from Calculus I that $\lim_{r\to+\infty} r\,\mathrm{e}^{-r^2} = \lim_{r\to+\infty} r^2\,\mathrm{e}^{-r^2} = 0$.

可用放缩 $\leq$

c) Determine all global extrema of f.

先算后选.

Question 2 (ca. 7 marks)

经验: 一次项. $x^2 + 2x + 2yz = 3$, $(x+1)^2 + 2yz = 2$

Let S be the quadratic surface in $\mathbb{R}^3$ with equation $x^2 + 2yz = 3$.

$\frac{1}{2}(y+z)^2 - \frac{1}{2}(y-z)^2$

$f(x,y,z)$ □
$g_1(x,y,z) = 0$ ①
$g_2(x,y,z) = 0$ ②
$f_x = \lambda_1 g_{1x} + \lambda_2 g_{2x}$
$f_y = \lambda_1 g_{1y} + \lambda_2 g_{2y}$
$f_z = \lambda_1 g_{1z} + \lambda_2 g_{2z}$
g_1
g_2

a) Determine the type of S. 双曲抛

b) Using the method of Lagrange multipliers, determine the (unique) point P on S that minimizes the distance to the point $(2,2,2)$.

c) Determine the tangent plane of S in $Q = \big(0, \frac{3}{2}, 1\big)$.

d) The plane through P, Q and $(0,0,0)$ intersects S in a conic C. Determine the type of C and the tangent line of C in Q.

Question 3 (ca. 6 marks)

Consider the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) := \int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,\mathrm{d}t \quad \text{for} \quad x \in \mathbb{R}.$$

a) Show that F is well-defined and that $F'(x)$, $F''(x)$ exist for all $x \in \mathbb{R}$.

= 可积 $\leq$ 可积函数 $= \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt$ → 可求值

b) Show that

$$x^2 F''(x) + x\, F'(x) + x^2 F(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

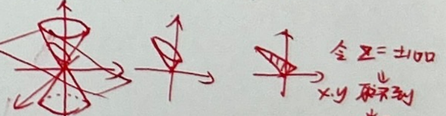
integral by part.

Hint: Partial integration of the expression obtained for $F'(x)$ may help.

Question 4 (ca. 8 marks)

Let K be the solid in $\mathbb{R}^3$ bounded by the cone $z^2 = x^2 + y^2$, the plane $z = 0$, and the plane $x + y + 2z = 1$.

smooth? $\to \nabla f \neq \vec{0}$



令 $z = \pm 100$ ⇓ x.y 限不到 有界

小一/

a) Show that K is compact? 有界的 = bounded.

b) Determine the volume $\mathrm{vol}(K)$.

c) Explain which parts of the boundary ∂K need to be considered when computing the surface area of ∂K, and why.

d) Determine the surface area of ∂K.

Hint: You may use without proof that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(2+\cos\theta+\sin\theta)^2} = \sqrt{2}\,\pi.$$

A proof will be honored by 2 bonus marks.

Question 5 (ca. 6 marks)

Evaluate the following integrals:

$\to \frac{1}{(1+x)(1+y)}$

a) $\displaystyle\int_\Delta \frac{1}{1+x+y+xy}\,\mathrm{d}^2(x,y)$, where Δ denotes the (solid) triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, and $(2,2)$;

$= x_2 x_3^2 \cdots x_n^{n-1}(1 + x_1x_2\cdots x_n + x_1^2x_2^2\cdots x_n^2 + \cdots)$

$= \sum_{k=0}^{\infty} x_1^k x_2^{k+1} \cdots x_n^{k+n-1}$

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b) $\displaystyle\int_{[0,1]^n} \frac{x_2x_3^2\cdots x_n^{n-1}}{1-x_1x_2\cdots x_n}\,\mathrm{d}^n(x_1,x_2,\dots,x_n)$ for integers $n\geq 2$. $=$

$\frac{1}{1-a} = 1+a+a^2+\cdots+a^n$

Question 6 (ca. 6 marks)

Let P, Q, R be real-valued C^1-functions on $D\subseteq\mathbb{R}^2$. Show:

a) If there exists (a C^3-function) $f\colon D\to\mathbb{R}$ with

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y)\end{pmatrix} = \begin{pmatrix} P(x,y) & Q(x,y) \\ Q(x,y) & R(x,y)\end{pmatrix} \qquad (\star)$$

for all $(x,y)\in D$ then $\nabla Q = (P_y, R_x)$.

b) Conversely, if D is simply connected and $\nabla Q = (P_y, R_x)$ then there exists $f\colon D\to\mathbb{R}$ satisfying $(\star)$.

c) For the case $D=\mathbb{R}^2$,

$$P(x,y) = 1+2xy, \quad Q(x,y) = x^2-y^2, \quad R(x,y) = 1-2xy$$

determine *all* functions $f\colon D\to\mathbb{R}$ satisfying $(\star)$.

$f_{xx} = 1+2xy$ $\qquad$ $f_{yy} = 1-2xy$

$f_x = x + x^2y + C_1(y)$ $\qquad$ $f_y = y - xy^2 + C_1(x)$

$f = \frac{1}{2}x^2 + \frac{1}{3}x^3y + C_1(y)x + C_2(y)$ $\qquad$ $f = \frac{1}{2}y^2 - \frac{1}{3}xy^3 + C_1(x)y + C_2(x)$

Solutions

1 a) Local extrema (if any) must be critical points.

$$
\begin{aligned}
f &= \left(x-\tfrac{1}{2}\right) y e^{-x^2-2y^2},\\
f_x &= \left[1+\left(x-\tfrac{1}{2}\right)(-2x)\right] y e^{-x^2-2y^2} = (1+x-2x^2) y e^{-x^2-2y^2}\\
&= -2(x-1)\left(x+\tfrac{1}{2}\right) y e^{-x^2-2y^2}\\
f_y &= \left(x-\tfrac{1}{2}\right)(1-4y^2) e^{-x^2-2y^2}\\
&= -4\left(x-\tfrac{1}{2}\right)\left(y-\tfrac{1}{2}\right)\left(y+\tfrac{1}{2}\right) e^{-x^2-2y^2}\\
\nabla f(x,y) = 0 &\iff y(x-1)\left(x+\tfrac{1}{2}\right) = 0 \wedge \left(x-\tfrac{1}{2}\right)\left(y-\tfrac{1}{2}\right)\left(y+\tfrac{1}{2}\right) = 0\\
&\iff (x,y) \in \left\{\left(1, \pm\tfrac{1}{2}\right), \left(-\tfrac{1}{2}, \pm\tfrac{1}{2}\right), \left(\tfrac{1}{2}, 0\right)\right\}
\end{aligned}
$$

Hence the critical points of f are $\mathbf{p}_1 = \left(1, \tfrac{1}{2}\right)$, $\mathbf{p}_2 = \left(1, -\tfrac{1}{2}\right)$, $\mathbf{p}_3 = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$, $\mathbf{p}_4 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right)$, $\mathbf{p}_5 = \left(\tfrac{1}{2}, 0\right)$. $\boxed{2\tfrac{1}{2}}$

Further we have

$$
\begin{aligned}
f_{xx} &= \left[1-4x+(1+x-2x^2)(-2x)\right] y\, e^{-x^2-2y^2} = (1-6x-2x^2+4x^3) y\, e^{-x^2-2y^2},\\
f_{xy} = f_{yx} &= (1+x-2x^2)(1-4y^2) e^{-x^2-2y^2},\\
f_{yy} &= \left(x-\tfrac{1}{2}\right)\left[-8y+(1-4y^2)(-4y)\right] e^{-x^2-2y^2} = \left(x-\tfrac{1}{2}\right)(-12y+16y^3) e^{-x^2-2y^2}
\end{aligned}
$$

and hence

$$
\mathbf{H}_f(x,y) = e^{-x^2-2y^2}\begin{pmatrix} (1-6x-2x^2+4x^3)y & (1+x-2x^2)(1-4y^2)\\ (1+x-2x^2)(1-4y^2) & \left(x-\tfrac{1}{2}\right)(-12y+16y^3)\end{pmatrix}.
$$

$$\mathbf{H}_f(\mathbf{p}_1) = e^{-3/2}\begin{pmatrix} -3/2 & 0\\ 0 & -2\end{pmatrix} \implies \mathbf{p}_1 \text{ is a (strict) local maximum,} \quad \boxed{\tfrac{1}{2}}$$

$$\mathbf{H}_f(\mathbf{p}_2) = e^{-3/2}\begin{pmatrix} 3/2 & 0\\ 0 & 2\end{pmatrix} \implies \mathbf{p}_2 \text{ is a local minimum,} \quad \boxed{\tfrac{1}{2}}$$

$$\mathbf{H}_f(\mathbf{p}_3) = e^{-3/4}\begin{pmatrix} 3/2 & 0\\ 0 & 4\end{pmatrix} \implies \mathbf{p}_3 \text{ is a local minimum,} \quad \boxed{\tfrac{1}{2}}$$

$$\mathbf{H}_f(\mathbf{p}_4) = e^{-3/4}\begin{pmatrix} -3/2 & 0\\ 0 & -4\end{pmatrix} \implies \mathbf{p}_4 \text{ is a local maximum,} \quad \boxed{\tfrac{1}{2}}$$

$$\mathbf{H}_f(\mathbf{p}_5) = e^{-1/4}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \implies \mathbf{p}_5 \text{ is a saddle point.} \quad \boxed{\tfrac{1}{2}}$$

Alternatively, since f is an odd function of y, a local minimum at (x,y) implies a local maximum at $(x,-y)$ and vice versa. Thus the types of $\mathbf{p}_2$, $\mathbf{p}_4$ can be inferred from the types of $\mathbf{p}_1$, $\mathbf{p}_3$. Moreover, it is clear that in every neighborhood of $\mathbf{p}_5$ the function f takes both positive and negative values. This already shows that $\mathbf{p}_5$ is not a local extremum. (A proof of the saddle point property was not required.)

b) With $r = |(x,y)| = \sqrt{x^2+y^2}$ we have, using $|x-1/2| \leq |x| + 1/2 \leq r + 1/2$ and $|y| \leq r$,

$$\begin{aligned}|f(x,y)| &= \left| xy\mathrm{e}^{-r^2-y^2} - (y/2)\mathrm{e}^{-r^2-y^2} \right| \leq \left(r+\tfrac{1}{2}\right) r\mathrm{e}^{-r^2} + \tfrac{r}{2}\,\mathrm{e}^{-r^2} \\ &= r^2\,\mathrm{e}^{-r^2} + r\,\mathrm{e}^{-r^2}.\end{aligned}$$

With the quoted result from Calculus I this implies $\lim_{|(x,y)|\to\infty} f(x,y) = 0$. $\boxed{1}$

c) Since global extrema are in particular local extrema, the only candidates are $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$. The corresponding values of f are

$$\begin{aligned} f(\mathbf{p}_1) &= (1/4)\mathrm{e}^{-3/2} \approx 0.056, & f(\mathbf{p}_2) &= -(1/4)\mathrm{e}^{-3/2} \approx -0.056,\\ f(\mathbf{p}_3) &= -(1/2)\mathrm{e}^{-3/4} \approx -0.24, & f(\mathbf{p}_4) &= (1/2)\mathrm{e}^{-3/4} \approx 0.24.\end{aligned}$$

This already excludes $\mathbf{p}_1$, $\mathbf{p}_2$. (The approximations above are not required; the inequality $(1/4)\mathrm{e}^{-3/2} < (1/2)\mathrm{e}^{-3/4}$, which is immediate, is sufficient for this.) If we can show that f has a global minimum and a global maximum, then $\mathbf{p}_3$ and $\mathbf{p}_4$ must be (the only) global extrema. $\boxed{1}$

Using $f(\mathbf{p}_3) < 0 < f(\mathbf{p}_4)$ and the result in b), there exists $R > 0$ such that

$$f(\mathbf{p}_3) < f(x,y) < f(\mathbf{p}_4) \quad \text{for } |(x,y)| > R. \tag{$\star$}$$

Since f is continuous, it attains on $\overline{\mathrm{B}_R(\mathbf{0})}$ a minimum, say in $\mathbf{p}$, and a maximum, say in $\mathbf{q}$.

From ($\star$) we have $\mathbf{p}_3 \in \overline{\mathrm{B}_R(\mathbf{0})}$ and hence

$$f(\mathbf{p}) \begin{cases} \leq f(\mathbf{p}_3) < f(x,y) & \text{for } |(x,y)| > R,\\ \leq f(x,y) & \text{for } |(x,y)| \leq R.\end{cases}$$

This shows that $\mathbf{p}$ is indeed a global minimum of f (and hence equal to $\mathbf{p}_3$). Similarly one proves that $\mathbf{q}$ is a global maximum of f (and equal to $\mathbf{p}_4$). $\boxed{+1}$

$$\sum_1 = 7+1$$

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2 a) The (linear) coordinate change $x = x'$, $y = y' + z'$, $z = y' - z'$ changes S into the surface with equation $x'^2 + 2y'^2 - 2z'^2 = 3$, which is a hyperboloid of one sheet. (The corresponding canonical form is $x^2 + y^2 - z^2 = 1$.) $\boxed{+1}$

Alternatively, we can transform the representing symmetric matrix into canonical form using the algorithm outlined in the lecture:

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \xrightarrow[R2=R2+R3]{C2=C2+C3} \begin{pmatrix} 1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 1 & 0\end{pmatrix} \xrightarrow{R3=R3-\frac{1}{2}R2} \begin{pmatrix} 1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & -\frac{1}{2}\end{pmatrix}$$
$$\xrightarrow{C3=C3-\frac{1}{2}C2} \begin{pmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -\frac{1}{2}\end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$$

Since the right-hand side of $x^2 + 2yz = 3$ is positive, the canonical form of the quadric must be $x^2 + y^2 - z^2 = 1$.

b) Define $f, g\colon \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = (x-2)^2 + (y-2)^2 + (z-2)^2$ and $g(x,y,z) = x^2 + 2yz - 3$. Then the task is to minimize f on $\mathbb{R}^3$ under the constraint $g(x,y,z) = 0$.

$$\nabla f(x,y,z) = 2(x-2, y-2, z-2), \qquad \nabla g(x,y,z) = (2x, 2z, 2y).$$

Since $x^2 + 2yz - 3$ implies $\nabla g(x,y,z) \neq (0,0,0)$, the theorem on Langrange multipliers is applicable everywhere and yields that an optimal solution must satisfy $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some $\lambda \in \mathbb{R}$. This gives the system of equations

$$\begin{aligned} x-2 &= \lambda x,\\ y-2 &= \lambda z,\\ z-2 &= \lambda y,\\ x^2 + 2yz &= 3. \end{aligned} \qquad \boxed{2}$$

Obviously any solution must have $x \neq 0$ (and $y, z \neq 0$). Hence we can use the first equation to eliminate λ and obtain the new system

$$\begin{aligned} x(y-2) &= (x-2)z,\\ x(z-2) &= (x-2)y,\\ x^2 + 2yz &= 3. \end{aligned}$$

From this we obtain

$$(x-2)(z-y) = x(y-z) = -x(z-y) \implies y = z \lor x = 1.$$

If $y = z$ then the 1st equation becomes $x(y-2) = (x-2)y$, which implies $x = y = z$ and further, using the 3rd equation, $x = y = z = \pm 1$.

If $x = 1$ then we are left with the system $y - 2 = -z \land yz = 1$, which has the unique solution $y = z = 1$, and hence doesn't yield anything new.

Since $f(1,1,1) = 3 < f(-1,-1,-1) = 27$, the only candidate for an optimal solution is $P = (1,1,1)$ (and the distance from P to S is $\sqrt{3}$). $\boxed{2}$

It remains to show that f actually attains a global minimum. For this consider a closed ball $B = \overline{B_R(2,2,2)}$ centered at $(2,2,2)$ and containing a point of S. (For example, we can take $R = \sqrt{3}$ and the point as P.) Since $S \cap B$ is nonempty, closed, and bounded and f is continuous, f attains a minimum on $S \cap B$, say in (x^*, y^*, z^*). This minimum must be global, since $f(x,y,z) = |(x,y,z) - (2,2,2)|^2 > R^2 \geq f(x^*, y^*, z^*)$ for all $(x,y,z) \in S \setminus B$. $\boxed{+1}$

c) Since S is a level set of the function g in b), a normal vector for the tangent plane E in Q (which is on S) is $\nabla g(Q) = (0,2,3)$ and an equation for E is $2y + 3z = 6$. $\boxed{1}$

d) The plane in question, name it F, has an equation of the form $ax + by + cz = 0$. From $P, Q \in F$ we have $a + b + c = 3b + 2c = 0$, which gives $b = 2a$, $c = -3a$, so that we can take the equation as $x + 2y - 3z = 0$.

On $F \cap S$ we have $2y = 3z - x$ and hence $x^2 + (3z - x)z = 3$, which is equivalent to $(x - z/2)^2 + \frac{11}{4} z^2 = 3$. Hence C is an ellipse (and has a tangent at every point). $\boxed{+1}$

The tangent to C in Q, name it T, must be contained in $E \cap F$ and hence equal to $E \cap F$, since E and F are not parallel. A direction vector for T must be orthogonal

平面内

该点切线既在 E: 2y+3z=6 中.
又在 F: x+2y-3z=0 中. → 与两个 normal vector 垂直

to any normal vectors for E, F and hence can be taken as $(-12, 3, -2)$ (from direct inspection, or compute the cross product of the two given normal vectors). Thus we have

$$T = \begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} -12 \\ 3 \\ -2 \end{pmatrix}. \qquad \boxed{1}$$

Alternatively, we can parametrize C locally at Q by two functions $y(x)$, $z(x)$ with $y(0) = 3/2$, $z(0) = 1$ and obtain a direction vector of T as $\big(1, y'(0), z'(0)\big)$. Since C is a level set of the vectorial function $F(x, y, z) = (x^2 + 2yz - 3, x + 2y - 3z)$, we can obtain $y'(x)$, $z'(x)$ by differentiating the identity $F\big(x, y(x), z(x)\big) = (0, 0)$ and setting $x = 0$. The chain rule gives

$$\begin{pmatrix} 2x & 2z & 2y \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 & 3 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ y'(0) \\ z'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solving this 2×2 linear system for $y'(0)$, $z'(0)$ gives $y'(0) = -\frac{1}{4}$, $z'(0) = \frac{1}{6}$, and the direction vector $\big(1, -\frac{1}{4}, \frac{1}{6}\big)$.

$$\sum_{2} = 6 + 3$$

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3 a) We have $F(x) = \displaystyle\int_{(-1,1)} f(x, t)\,\mathrm{d}t$ with

$$f\colon \mathbb{R} \times (-1, 1) \to \mathbb{R}, \quad (x, t) \mapsto \frac{\cos(xt)}{\sqrt{1 - t^2}}.$$

For fixed x the function $t \mapsto f(x, t)$ is integrable over $(-1, 1)$, since $|f(x, t)| \leq \frac{1}{\sqrt{1-t^2}}$ and $\Phi(t) = \frac{1}{\sqrt{1-t^2}}$ is independent of x and integrable over $(-1, 1)$. (Since $1 - t^2 = (1 - t)(1 + t)$ has a simple root at the endpoints $t = \pm 1$, the situation is the same as with $\int_0^1 \frac{\mathrm{d}t}{\sqrt{t}}$, which exists.) $\boxed{+1}$

The partial derivatives

$$\frac{\partial f}{\partial x}(x, t) = \frac{-t \sin(xt)}{\sqrt{1 - t^2}}, \quad \frac{\partial^2 f}{\partial x^2}(x, t) = \frac{-t^2 \cos(xt)}{\sqrt{1 - t^2}}$$

are bounded in absolute value by the same function $\Phi(t) = \frac{1}{\sqrt{1-t^2}}$, since $|t| \leq 1$ for $t \in [-1, 1]$. $\boxed{+1}$

$\Longrightarrow$ F is twice differentiable and $F'(x)$, $F''(x)$ are obtained by differentiating under the integral sign; cf. the discussion of parameter integrals in the lecture.

$$F'(x) = \int_{-1}^{1} \frac{\partial f}{\partial x}(t, x)\,\mathrm{d}t = \int_{-1}^{1} -\frac{t \sin(xt)}{\sqrt{1 - t^2}}\,\mathrm{d}t,$$
$$F''(x) = \int_{-1}^{1} \frac{\partial^2 f}{\partial x^2}(t, x)\,\mathrm{d}t = \int_{-1}^{1} -\frac{t^2 \cos(xt)}{\sqrt{1 - t^2}}\,\mathrm{d}t. \qquad \boxed{1}$$

b) Partial integration of the integral representing $F'(x)$ with $u'(t) = -\frac{t}{\sqrt{1-t^2}}$, $u(t) = \sqrt{1-t^2}$, $v(t) = \sin(xt)$, $v'(t) = x\cos(xt)$, gives

$$F'(x) = \Big[\sqrt{1-t^2}\sin(xt)\Big]_{-1}^{1} - \int_{-1}^{1}\sqrt{1-t^2}\,x\cos(xt)\,\mathrm{d}t = -x\int_{-1}^{1}\sqrt{1-t^2}\cos(xt)\,\mathrm{d}t \qquad \boxed{1}$$

It follows that

$$\begin{aligned}
x^2\,F''(x) + x^2\,F(x) &= -x^2\int_{-1}^{1}\frac{t^2\cos(xt)}{\sqrt{1-t^2}}\,\mathrm{d}t + x^2\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,\mathrm{d}t\\
&= x^2\int_{-1}^{1}\frac{(1-t^2)\cos(xt)}{\sqrt{1-t^2}}\,\mathrm{d}t = x^2\int_{-1}^{1}\sqrt{1-t^2}\cos(xt)\,\mathrm{d}t\\
&= -x\,F'(x), \qquad \boxed{2}
\end{aligned}$$

as desired.

$$\sum_{3} = 4 + 2$$

4 a) Analytically,

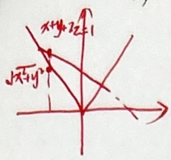
$$K = \big\{(x,y,z)\in\mathbb{R}^3;\ \sqrt{x^2+y^2} \le z \le (1-x-y)/2\big\}.$$

K is closed because it's defined using '$\le$' and the functions involved are continuous. K is bounded, since $(x,y,z)\in K$ implies

$$\frac{|x|+|y|}{\sqrt{2}} \le \sqrt{x^2+y^2} \le z \le \frac{1+|x|+|y|}{2}$$

and hence $|x|, |y| \le 1+\sqrt{2}$ and $|z| \le (3+2\sqrt{2})/2 = \frac{3}{2}+\sqrt{2}$. $\boxed{1+1}$
Alternatively, setting $x = r\cos\theta$, $y = r\sin\theta$ and using $\cos\theta + \sin\theta = \sqrt{2}\cos(\theta-\pi/4)$, we have $r \le (1 - r\cos\theta - r\sin\theta)/2 = \frac{1}{2} - \frac{\sqrt{2}}{2}r\cos(\theta-\pi/4) \le \frac{1}{2} + \frac{\sqrt{2}}{2}r$, hence $|x|, |y| \le r \le \frac{1/2}{1-\sqrt{2}/2} = \frac{1}{2-\sqrt{2}} = 1 + \frac{1}{2}\sqrt{2}$ and $0 \le z \le (1+2r)/2 \le \frac{3}{2} + \frac{1}{2}\sqrt{2}$.

b) 拆不出来



$$
\begin{aligned}
\operatorname{vol}(K) &= \int_K 1\,\mathrm{d}^3(x,y,z) = \int\limits_{\sqrt{x^2+y^2}\leq(1-x-y)/2} \left(\int\limits_{z=\sqrt{x^2+y^2}}^{(1-x-y)/2} 1\,\mathrm{d}z\right)\mathrm{d}^2(x,y)\\
&= \int\limits_{\sqrt{x^2+y^2}\leq(1-x-y)/2} (1-x-y)/2-\sqrt{x^2+y^2}\,\mathrm{d}^2(x,y)\\
&= \int\limits_{r\leq(1-r\cos\theta-r\sin\theta)/2} \big((1-r\cos\theta-r\sin\theta)/2-r\big)r\,\mathrm{d}^2(r,\theta)\\
&= \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{(2+\cos\theta+\sin\theta)^{-1}} r-r^2(2+\cos\theta+\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \frac{1}{2}\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}-\frac{r^3}{3}(2+\cos\theta+\sin\theta)\right]_{r=0}^{(2+\cos\theta+\sin\theta)^{-1}}\mathrm{d}\theta\\
&= \frac{1}{12}\int_0^{2\pi}\frac{\mathrm{d}\theta}{(2+\cos\theta+\sin\theta)^2} && \boxed{3}\\
&= \frac{1}{12}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{(2+\cos\theta+\sin\theta)^2}\\
&= \frac{1}{12}\int_{-\infty}^{+\infty}\frac{\frac{2}{1+t^2}}{\left(2+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}\right)^2}\,\mathrm{d}t && (\text{Subst. } t=\tan(\theta/2))\\
&= \frac{1}{6}\int_{-\infty}^{+\infty}\frac{t^2+1}{(t^2+2t+3)^2}\,\mathrm{d}t = \frac{1}{6}\int_{-\infty}^{+\infty}\frac{t^2+2t+3-(2t+2)}{(t^2+2t+3)^2}\,\mathrm{d}t\\
&= \frac{1}{6}\int_{-\infty}^{+\infty}\frac{1}{t^2+2t+3}-\frac{2t+2}{(t^2+2t+3)^2}\,\mathrm{d}t\\
&= \frac{1}{6}\left[\frac{1}{\sqrt{2}}\arctan\frac{t+1}{\sqrt{2}}+\frac{1}{t^2+2t+3}\right]_{-\infty}^{+\infty} && \left(\text{since } \frac{1}{t^2+2t+3}=\frac{1/2}{\left(\frac{t+1}{\sqrt{2}}\right)^2+1}\right)\\
&= \frac{\pi}{6\sqrt{2}}\approx 0.37 && \boxed{+2}
\end{aligned}
$$

Alternative solution: Using the affine coordinate change $x'=x$, $y'=y$, $z'=(1-x-$

The latter also follows from the observation that S_1 is the graph of the length function $f(x,y)=\sqrt{x^2+y^2}$, $(x,y)\in\Omega$, whose gradient has unit length, and the formula $g^\gamma(x,y)=1+|\nabla f(x,y)|^2$.

$(x,y)\mapsto(x,y,\sqrt{x^2+y^2})$

$$
\begin{aligned}
\Longrightarrow\ \mathrm{vol}_2(S_1)&=\int_\Omega\sqrt{2}\,\mathrm{d}^2(x,y)\\
&=\sqrt{2}\int\limits_{r\leq(1-r\cos\theta-r\sin\theta)/2} r\,\mathrm{d}^2(r,\theta)\\
&=\sqrt{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{(2+\cos\theta+\sin\theta)^{-1}} r\,\mathrm{d}r\,\mathrm{d}\theta\\
&=\frac{\sqrt{2}}{2}\int_0^{2\pi}\frac{\mathrm{d}\theta}{(2+\cos\theta+\sin\theta)^2}\\
&=\frac{\sqrt{2}}{2}\sqrt{2}\,\pi=\pi.
\end{aligned}
\qquad \boxed{2}
$$

Alternatively, using $\cos\theta+\sin\theta=\sqrt{2}\cos(\theta-\pi/4)$, we can rewrite the condition $r\leq(1-r\cos\theta-r\sin\theta)/2$ as $r\leq\frac{1/2}{1+(\sqrt{2}/2)\cos(\theta-\pi/4)}$, showing that $\partial\Omega$ is an ellipse with semilatus rectum $l=1/2$ and eccentricity $e=\sqrt{2}/2$. The semi-axes of such an ellipse are $a=\frac{l}{1-e^2}=1$, $b=\sqrt{al}=\frac{l}{\sqrt{1-e^2}}=\sqrt{2}/2$, which gives $\mathrm{vol}(\Omega)=ab\pi=\pi/\sqrt{2}$ and $\mathrm{vol}_2(S_1)=\sqrt{2}\,\mathrm{vol}(\Omega)=\pi$.

An immersion parametrizing S_2 is $\gamma(x,y)=(x,y,(1-x-y)/2)$, $(x,y)\in\Omega$ (with the same domain Ω as before).

$$
\begin{aligned}
\mathbf{J}_\gamma(x,y)&=\begin{pmatrix}1&0\\0&1\\-\frac{1}{2}&-\frac{1}{2}\end{pmatrix},\\
\mathbf{J}_\gamma(x,y)^\mathsf{T}\mathbf{J}_\gamma(x,y)&=\begin{pmatrix}\frac{5}{4}&\frac{1}{4}\\\frac{1}{4}&\frac{5}{4}\end{pmatrix},\\
g^\gamma(x,y)&=\frac{24}{16}=\frac{3}{2},\\
\sqrt{g^\gamma(x,y)}&=\frac{\sqrt{3}}{\sqrt{2}},\\
\Longrightarrow\ \mathrm{vol}_2(S_2)&=\int_\Omega\frac{\sqrt{3}}{\sqrt{2}}\,\mathrm{d}^2(x,y)=\frac{\sqrt{3}}{\sqrt{2}}\mathrm{vol}(\Omega)\\
&=\frac{\sqrt{3}}{2\sqrt{2}}\int_0^{2\pi}\frac{\mathrm{d}\theta}{(2+\cos\theta+\sin\theta)^2}\qquad\boxed{1\tfrac{1}{2}}\\
&=\frac{\sqrt{3}}{2}\pi.
\end{aligned}
$$

In all it follows that

$$
\mathrm{vol}_2(\partial K)=\mathrm{vol}_2(S_1)+\mathrm{vol}_2(S_2)=\pi+\frac{\sqrt{3}}{2}\pi=\left(1+\tfrac{1}{2}\sqrt{3}\right)\pi\approx 5.86\qquad\boxed{\tfrac{1}{2}}
$$

$$\sum_4=9+3$$

$y)/2 - z$, which has jacobian determinant equal to -1, we obtain

$$
\begin{aligned}
\operatorname{vol}(K) &= \int\limits_{\sqrt{x^2+y^2}\leq z\leq(1-x-y)/2} \mathrm{d}^3(x,y,z)\\
&= \int\limits_{0\leq z'\leq(1-x'-y')/2-\sqrt{x'^2+y'^2}} \mathrm{d}^3(x',y',z')\\
&= \int_{z'=0}^{1/2}\int_{2\sqrt{x'^2+y'^2}+x'+y'\leq 1-2z'} \mathrm{d}^2(x',y')\,\mathrm{d}z'\\
&= \int\limits_{z'=0}^{1/2} \int\limits_{\substack{0<r<\frac{1-2z'}{2+\cos\theta+\sin\theta}\\ 0<\theta<2\pi}} r\,\mathrm{d}^2(r,\theta)\,\mathrm{d}z'
\end{aligned}
$$

At this point one sees that the transformed solid K' is a cone with base B given in polar cordinates by $r(\theta) < (2+\cos\theta+\sin\theta)^{-1}$ and height $1/2$, and the volume of K therefore is

$$
\operatorname{vol}(K) = \operatorname{vol}(K') = \frac{1}{6}\operatorname{vol}(B) = \frac{1}{12}\int_0^{2\pi}\frac{\mathrm{d}\theta}{(2+\cos\theta+\sin\theta)^2} = \frac{\pi}{6\sqrt{2}},
$$

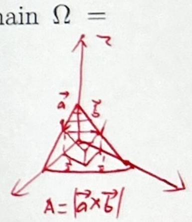
reflecting the fact that B is a (solid) ellipse with semi-axes $a = 1$ and $b = \frac{1}{2}\sqrt{2}$; cf. the solution of d).

c) $\partial K = S_1 \uplus S_2 \uplus S_3 \uplus S_4$ with

$$
\begin{aligned}
S_1 &= \bigl\{(x,y,z);\, \sqrt{x^2+y^2} = z < (1-x-y)/2\bigr\},\\
S_2 &= \bigl\{(x,y,z);\, \sqrt{x^2+y^2} < z = (1-x-y)/2\bigr\},\\
S_3 &= \bigl\{(x,y,z);\, \sqrt{x^2+y^2} = z = (1-x-y)/2\bigr\},\\
S_4 &= \bigl\{(0,0,0)\bigr\}.
\end{aligned}
$$

S_3 (a conic section, which in fact is an ellipse) and S_4 (a point) are smooth surfaces of dimension 1 and 0, respectively, and hence have 2-dimensional volume zero.
$\Longrightarrow$ Only S_1 and S_2 matter for the surface area computation of ∂K. $\boxed{1}$

d) An immersion parametrizing S_1 is $\gamma(x,y) = \left(x, y, \sqrt{x^2+y^2}\right)$ with domain $\Omega = \bigl\{(x,y)\in\mathbb{R}^2;\, \sqrt{x^2+y^2} < (1-x-y)/2\bigr\} \setminus \bigl\{(0,0)\bigr\}$. We compute



$A = |\vec{a}\times\vec{b}|$

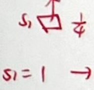
$(1, 0, -\frac{1}{2})\times(0, 1, -\frac{1}{2})$
$= (\frac{1}{2}, \frac{1}{2}, 1)$
$A = |(\frac{1}{2}, \frac{1}{2}, 1)| = \frac{\sqrt{6}}{2}$

$$
\mathbf{J}_\gamma(x,y) = \begin{pmatrix} 1 & 0\\ 0 & 1\\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}}\end{pmatrix},
$$

$$
\mathbf{J}_\gamma(x,y)^\top\mathbf{J}_\gamma(x,y) = \begin{pmatrix} 1+\frac{x^2}{x^2+y^2} & \frac{xy}{x^2+y^2}\\ \frac{xy}{x^2+y^2} & 1+\frac{y^2}{x^2+y^2}\end{pmatrix},
$$

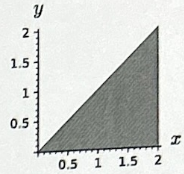
$$
g^\gamma(x,y) = \left(1+\frac{x^2}{x^2+y^2}\right)\left(1+\frac{y^2}{x^2+y^2}\right) - \frac{x^2y^2}{x^2+y^2} = 2,
$$

$$
\sqrt{g^\gamma(x,y)} = \sqrt{2}
$$



$S_1 = 1 \to S_2 = 2\sqrt{6}$

5 a) The triangle Δ is shown on the right. We evaluate the integral in the order $\mathrm{d}y\,\mathrm{d}x$, using that for fixed $x \in [0,1]$ we have $(x,y) \in \Delta \iff 0 \le y \le x$.



求 

再画一遍图

$= \int_\Delta \frac{1}{1+x} \cdot \frac{1}{1+y} = \frac{1}{2} \int_0^1 \frac{1}{1+x}\,dx \cdot \int_0^1 \frac{1}{1+y}\,dy$

$$\begin{aligned}
\int_\Delta \frac{1}{1+x+y+xy}\,\mathrm{d}^2(x,y) &= \int_{x=0}^{2} \int_{y=0}^{x} \frac{1}{1+x+(1+x)y}\,\mathrm{d}y\,\mathrm{d}x && \boxed{1}\\
&= \int_{x=0}^{2} \left[\frac{\ln(1+x+(1+x)y)}{1+x}\right]_{y=0}^{x} \mathrm{d}x\\
&= \int_{x=0}^{2} \frac{\ln((1+x)^2) - \ln(1+x)}{1+x}\,\mathrm{d}x\\
&= \int_{x=0}^{2} \frac{\ln(1+x)}{1+x}\,\mathrm{d}x && (\ln(a^2) = 2\ln a)\\
&= \left[\tfrac{1}{2}\bigl(\ln(1+x)\bigr)^2\right]_0^2 = \tfrac{1}{2}(\ln 3)^2 && \boxed{2}
\end{aligned}$$

Alternative solution: Let Δ' be the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$. The triangle Δ' is obtained by reflecting Δ at the line $y = x$. Since the integrand $f(x,y) = \frac{1}{1+x+y+xy}$ satisfies $f(x,y) = f(y,x)$, we have $\int_\Delta f(x,y)\,\mathrm{d}^2(x,y) = \int_{\Delta'} f(x,y)\,\mathrm{d}^2(x,y)$ and hence

$$\begin{aligned}
\int_\Delta f(x,y)\mathrm{d}^2(x,y) &= \frac{1}{2}\left(\int_\Delta f(x,y)\,\mathrm{d}^2(x,y) + \int_{\Delta'} f(x,y)\,\mathrm{d}^2(x,y)\right)\\
&= \frac{1}{2} \int_{[0,2]^2} f(x,y)\,\mathrm{d}^2(x,y)\\
&= \frac{1}{2} \int_{[0,2]^2} \frac{1}{(1+x)(1+y)}\,\mathrm{d}^2(x,y)\\
&= \frac{1}{2}\left(\int_0^2 \frac{\mathrm{d}x}{1+x}\right)^2 = \frac{1}{2}(\ln 3)^2.
\end{aligned}$$

b) Expanding the integrand into a geometric series, we obtain

$$\begin{aligned}
\int_{[0,1]^n} \frac{x_2 x_3^2 \cdots x_n^{n-1}}{1 - x_1 x_2 \dots x_n} \, \mathrm{d}^n\mathbf{x} &= \int_{[0,1]^n} \left(\sum_{k=0}^{\infty} x_1^k x_2^{k+1} \cdots x_n^{k+n-1} \right) \mathrm{d}^n\mathbf{x} \\
&= \sum_{k=0}^{\infty} \int_{[0,1]^n} x_1^k x_2^{k+1} \cdots x_n^{k+n-1} \, \mathrm{d}^n\mathbf{x} \\
&= \sum_{k=0}^{\infty} \left(\int_0^1 x_1^k \, \mathrm{d}x_1 \right) \left(\int_0^1 x_2^{k+1} \, \mathrm{d}x_2 \right) \cdots \left(\int_0^1 x_n^{k+n-1} \, \mathrm{d}x_n \right) \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)\cdots(k+n)} \qquad \boxed{2} \\
&= \frac{1}{n-1} \sum_{k=0}^{\infty} \frac{k+n-(k+1)}{(k+1)(k+2)\cdots(k+n)} \\
&= \frac{1}{n-1} \sum_{k=0}^{\infty} \left(\frac{1}{(k+1)\cdots(k+n-1)} - \frac{1}{(k+2)\cdots(k+n)} \right) \qquad \boxed{+1} \\
&= \frac{1}{n-1} \frac{1}{1 \cdot 2 \cdots (n-1)} = \frac{1}{(n-1)(n-1)!}
\end{aligned}$$

$\int_0^1 x^k dx = \frac{1}{k+1}$

$= \frac{[(k+n)-(k+1)]}{(k+1)(k+2)\cdots(k+n)} \cdot \frac{1}{n-1}$

裂项

$\frac{1}{(1-x)^2} = 1+2x+3x^2$

$\frac{1}{1-x} = 1+x+\dots+x^n+\dots$

Term-wise integration is justified, since the resulting series is non-negative and convergent (hence absolutely convergent).

Alternative solution: We use the change-of-variables $y_1 = x_1 x_2 \dots x_n$, $y_2 = x_2 x_3 \dots x_n$, $\dots$, $y_{n-1} = x_{n-1} x_n$, $y_n = x_n$. The inverse transformation is

$$\mathbf{x} = T(\mathbf{y}) = \begin{pmatrix} y_1/y_2 \\ y_2/y_3 \\ \vdots \\ y_{n-1}/y_n \\ y_n \end{pmatrix}.$$

T maps the region $S = \{\mathbf{y} \in \mathbb{R}^n; 0 < y_1 < y_2 < \dots < y_n < 1\}$ bijectively onto $(0,1)^n$ and has jacobian $\frac{\partial(x_1,\dots,x_n)}{\partial(y_1,\dots,y_n)} = (y_2 y_3 \cdots y_n)^{-1}$. The change-of-variables theorem gives

$$\begin{aligned}
\int_{[0,1]^n} \frac{x_2 x_3^2 \cdots x_n^{n-1}}{1 - x_1 x_2 \cdots x_n} \, \mathrm{d}^n(x_1, x_2, \dots, x_n) &= \int_{(0,1)^n} \frac{x_2 x_3^2 \cdots x_n^{n-1}}{1 - x_1 x_2 \cdots x_n} \, \mathrm{d}^n(x_1, x_2, \dots, x_n) \\
&= \int_S \frac{y_2 y_3 \cdots y_n}{1 - y_1} \left| \frac{1}{y_2 y_3 \cdots y_n} \right| \mathrm{d}^n(y_1, y_2, \dots, y_n) \\
&= \int_S \frac{1}{1 - y_1} \, \mathrm{d}^n(y_1, y_2, \dots, y_n) \\
&= \int_0^1 \frac{\mathrm{vol}(S_y)}{1 - y} \, \mathrm{d}y, \qquad \text{(writing } y_1 = y\text{)}
\end{aligned}$$

where $S_y = \{(y_2, \dots, y_n) \in \mathbb{R}^{n-1}; y < y_2 < y_3 < \dots < y_n\}$. The set S_y has volume $\frac{(1-y)^{n-1}}{(n-1)!}$, since the hypercube $(y,1)^{n-1}$ can be partitioned, except for a set of measure zero, into $(n-1)!$ copies of S_y, all of which have the same volume. For each permutation σ of $\{2, 3, \dots, n\}$ there is one copy

$$S_y^\sigma = \{(y_2, \dots, y_n) \in \mathbb{R}^{n-1}; y < y_{\sigma(2)} < y_{\sigma(3)} < \dots < y_{\sigma(n)}\}.$$

The set of points in $(y, 1)^{n-1}$ whose coordinates are pairwise distinct has the same volume as $(y, 1)^{n-1}$, viz. $(1-y)^{n-1}$, and is partitioned by the sets S_y^σ. The sets S_y^σ all have the same volume, because they can be transformed into each other using a coordinate permutation. (This observation generalizes the obvious fact that the unit square $[0, 1]^2$ is partitioned into two congruent triangles by the line $y = x$.)

With this obervation the proof is now easy to finish:

$$\int_0^1 \frac{\mathrm{vol}(S_y)}{1-y}\,\mathrm{d}y = \frac{1}{(n-1)!}\int_0^1 (1-y)^{n-2}\,\mathrm{d}y = \frac{1}{(n-1)(n-1)!}.$$

$$\sum_5 = 5 + 1$$

---

6 a) $(\star)$ is equivalent to $\nabla f_x = (P, Q)$, $\nabla f_y = (Q, R)$. Thus, under the given assumption, (P, Q) has an antiderivative, viz. f_x, and hence we have $P_y = Q_x$; similarly, $Q_y = R_x$. This shows $\nabla Q = (Q_x, Q_y) = (P_y, R_x)$. $\boxed{1}$

b) Assume $\nabla Q = (P_y, R_x)$, i.e., $P_y = Q_x$ and $Q_y = R_x$. Then (P, Q) satisfies the condition in Poincaré's Lemma and, since D is simply connected, has an antiderivative $g(x, y)$ on D. Similarly, (Q, R) has an antiderivative $h(x, y)$ on D. Moreover, we have $g_y = Q = h_x$, and hence by the same token there exists an antiderivative $f(x, y)$ of (g, h) on D. The function f then satisfies $(\star)$:

Region D connected
$f(P, Q)$
$P_y = Q_x \to$ exact
存在 F
$P = F_x$
$Q = F_y$

$$f_{xx} = g_x = P, \quad f_{xy} = g_y = Q, \quad f_{yy} = h_y = R,$$

$\nabla Q = (P_y, R_x)$
$\therefore Q_x = P_y$. $Q_y = R_x$
$E(P, Q)$ $E_x = P$. $E_y = Q$
$F(Q, R)$ $F_x = Q$. $F_y = R$.

and of course also $f_{yx} = f_{xy} = Q$. $\boxed{+2}$

c) $f_{xx} = P = 2xy + 1 \Longrightarrow f_x = x^2y + x + c_1(y) \Longrightarrow f_{xy} = x^2 + c_1'(y) = Q = x^2 - y^2 \Longrightarrow c_1'(y) = -y^2 \Longrightarrow c_1(y) = -\frac{1}{3}y^3 + C_1 \Longrightarrow f_x = x^2y - \frac{1}{3}y^3 + x + C_1 \Longrightarrow f = \frac{1}{3}x^3y - \frac{1}{3}xy^3 + \frac{1}{2}x^2 + C_1x + c_2(y) \Longrightarrow f_y = \frac{1}{3}x^3 - xy^2 + c_2'(y) \Longrightarrow f_{yy} = -2xy + c_2''(y) = R = 1 - 2xy \Longrightarrow c_2''(y) = 1 \Longrightarrow c_2(y) = \frac{1}{2}y^2 + C_2y + C_3$

$\partial(E, F)$
$E_y = F_x$
$f_x = E$
$f_y = F$.

$$\Longrightarrow f(x, y) = \tfrac{1}{3}(x^3y - xy^3) + \tfrac{1}{2}(x^2 + y^2) + C_1x + C_2y + C_3,$$

where $C_1, C_2, C_3 \in \mathbb{R}$ are arbitrary constants. $\boxed{3}$

$$\sum_6 = 4 + 2$$

$$\sum_{\text{Final Exam}} = 35 + 12$$
