1 Barycentric Coordinates

Barycentric coordinates can be thought of as 'normalized coordinates within a triangle'. Similarly to when we interpolate a value between x and y, we are now interpolating a value between three vertices of a triangle $(x_0; y_0)$, $(x_1; y_1)$ and $(x_2; y_2)$.

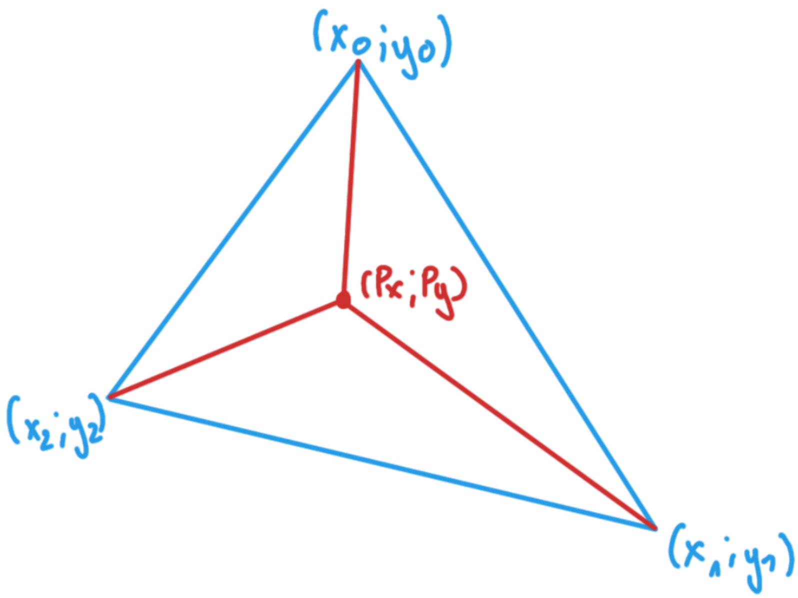

Figure 1: Visualisation of barycentric coordinates

Just like with regular 'linear interpolation' our interpolation values also add up to one:

$$u_0 + u_1 + u_2 = 1 \tag{1}$$

Based on the little visualisation above, along with the explanation, we can deduce the following:

$$u_0x_0 + u_1x_1 + u_2x_2 = p_x \tag{2}$$

$$u_0y_0 + u_1y_1 + u_2y_2 = p_y \tag{3}$$

$$u_0 + u_1 + u_2 = 1 \tag{4}$$

This tells us that given a point $(p_x; p_y)$ we can tell how close it is to each vertex and interpolate something like its colour accordingly. This equation can also be used to figure out whether or not our point is in a triangle.

Solving this system of equations is rather simple, albeit answers that we get are rather 'ugly' to solve by hand (that's why we're using a computer, we can just give it a bunch of formulas and it will calculate those for us). Solving this system of equations would look something like this:

$$u_0x_0 + u_1x_1 + u_2x_2 = p_x \tag{5}$$
$$u_0y_0 + u_1y_1 + u_2y_2 = p_y \tag{6}$$
$$u_2 = 1 - u_0 - u_1 \tag{7}$$

$$u_0x_0 + u_1x_1 + (1 - u_0 - u_1)x_2 = p_x \tag{8}$$
$$u_0y_0 + u_1y_1 + (1 - u_0 - u_1)y_2 = p_y \tag{9}$$

$$u_0x_0 + u_1x_1 + x_2 - u_0x_2 - u_1x_2 = p_x \tag{10}$$
$$u_0y_0 + u_1y_1 + y_2 - u_0y_2 - u_1y_2 = p_y \tag{11}$$

$$u_0(x_0 - x_2) + u_1(x_1 - x_2) = p_x - x_2 \tag{12}$$
$$u_0(y_0 - y_2) + u_1(y_1 - y_2) = p_y - y_2 \tag{13}$$

Having the equations in this form brings forward an interesting pattern, they can be rewritten in 'matrix' form:

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \begin{bmatrix} (x_0 - x_2) & (x_1 - x_2) \\ (y_0 - y_2) & (y_1 - y_2) \end{bmatrix} = \begin{bmatrix} p_x - x_2 \\ p_y - y_2 \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} p_x - x_2 \\ p_y - y_2 \end{bmatrix} \begin{bmatrix} (x_0 - x_2) & (x_1 - x_2) \\ (y_0 - y_2) & (y_1 - y_2) \end{bmatrix}^{-1} \tag{15}$$

2 Full Solution

Most math textbooks would say that the rest is obvious/trivial, but I know from experience that when something is obvious for one person it is extremely difficult for another. That being said – if you know how to follow the rest you can skip this section.

$$A = \begin{bmatrix} (x_0 - x_2) & (x_1 - x_2) \\ (y_0 - y_2) & (y_1 - y_2) \end{bmatrix} \qquad det(A) = [(x_0 - x_2)(y_1 - y_2)] - [(x_1 - x_2)(y_0 - y_2)]$$

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} p_x - x_2 \\ p_y - y_2 \end{bmatrix} \frac{1}{det(A)} \begin{bmatrix} (y_1 - y_2) & (x_2 - x_1) \\ (y_2 - y_1) & (x_0 - x_2) \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} (p_x - x_2)(y_1 - y_2) + (p_y - y_2)(x_2 - x_1) \\ (p_x - x_2)(y_2 - y_1) + (p_y - y_2)(x_0 - x_2) \end{bmatrix} \tag{17}$$

The solution is then:

$$u_0 = [(p_x - x_2)(y_1 - y_2) + (p_y - y_2)(x_2 - x_1)] \cdot \frac{1}{det(A)} \tag{18}$$

$$u_1 = [(p_x - x_2)(y_2 - y_1) + (p_y - y_2)(x_0 - x_2)] \cdot \frac{1}{det(A)} \tag{19}$$

$$u_2 = 1 - u_0 - u_1 \tag{20}$$