1 Rotating Vectors

Rotating vectors is one of those operations I've been taught but never really understood it when using it, why exactly is it defined this way? Only when I became more and more interested in mathematics did I realized why (sadly most teacher just teach formulas, not reasoning). Now I want to share some of the derivations and intuition I gathered along the way.

Let's start with the common formulas used for rotating vectors in two dimensions, assuming they all start at origin point $(0;0)$. If you want to change the origin point all you have to do is add your desired origin point to that vector.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

$$x' = x\cos\theta - y\sin\theta \tag{2}$$
$$y' = x\sin\theta + y\cos\theta \tag{3}$$

This is pretty much all you need in order to rotate a vector, but where does it all come from? For that I'd like to use complex numbers, because they nicely join together $\cos x$ and $\sin x$ functions. You could also just use polar coordinates and you'd arrive at the same results, I just find complex numbers and this derivation really lovely for some reason so I'll use that method instead.

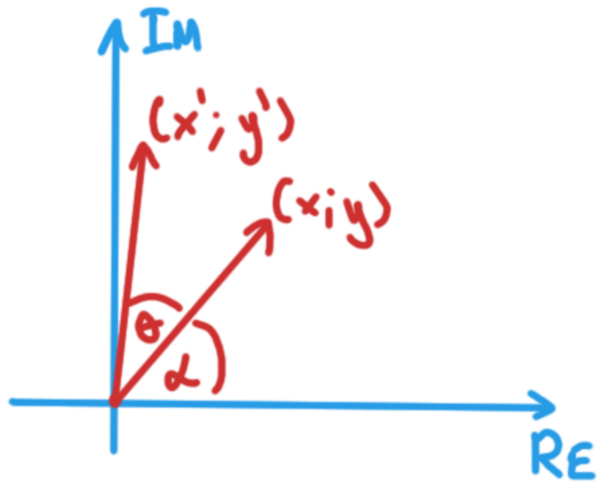

Figure 1: Visualisation of complex coordinates and rotation by an angle

You can tell that our original vector that we're starting with $(x; y)$ can be represented as such using Euler's notation:

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \tag{4}$$

Which also implies that our rotated vector can be represented in a similar way:

$$e^{i(\alpha+\theta)} = e^{i\alpha} \cdot (\cos\theta + i\sin\theta) \tag{5}$$

We can combine those two by multiplying them together:

$$e^{i(\alpha+\theta)} = (\cos\alpha + i\sin\alpha) \cdot (\cos\theta + i\sin\theta) \tag{6}$$

$$e^{i(\alpha+\theta)} = \cos\alpha\cos\theta + i\cos\alpha\sin\theta + i\sin\alpha\cos\theta - \sin\alpha\sin\theta \tag{7}$$

$$e^{i(\alpha+\theta)} = \cos\alpha\cos\theta - \sin\alpha\sin\theta + i(\cos\alpha\sin\theta + \sin\alpha\cos\theta) \tag{8}$$

This is most of the work done, now all we need to do is realize that we already know $\sin\alpha$ and $\cos\alpha$, they are equivalent to our $(x; y)$ coordinates if we were to translate those using polar coordinates on a regular XY plane.

$$e^{i(\alpha+\theta)} = x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta) \tag{9}$$

Last thing left to do is to also realize that imaginary unit i disappears when we translate this equation back into regular every-day XY plane, it is worth noting however, that the part with imaginary unit i describes what happens on Y plane when translated back. All of this basically means that our real part is x' rotation and imaginary part is y' rotation.

$$x' = x\cos\theta - y\sin\theta \tag{10}$$

$$y' = x\sin\theta + y\cos\theta \tag{11}$$

Thus we arrive at a general formula for vector rotation, as mentioned at the beginning this could also be derived using polar coordinates, however personally I prefer complex numbers because they are very powerful when it comes to computer graphics and different forms of rotations (or even Fast Fourier Transforms).