MML Book Solutions

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Contents

Chapter 2: Linear Algebra

Question 2.1.

We consider $(\mathbb{R} \setminus \{-1\}, \star)$, where

$$a \star b := ab + a + b \quad a, b \in \mathbb{R} \setminus \{-1\}$$

Subquestion a.

Show that $(\mathbb{R} \setminus \{-1\}, \star)$ is an Abelian group.

Insight

Properties of an Abelian group: commutativity, closure, associativity, existence of a neutral element, and existence of an inverse for every element. I show commutativity first so I can use it in later segments of the proof.

Solution

Commutative:

$$\begin{aligned} a \star b &= ab + a + b \\ &= ba + b + a \\ &= b \star a \end{aligned} \qquad \square$$

Closure:

$$\begin{aligned} \textit{Assuming} \quad a \star b &= -1 \\ ab + a + b &= -1 \\ (a+1)(b+1) &= 0 \\ a = -1 \text{ OR } b &= -1 \qquad \perp \\ a \star b &\in \mathbb{R} \setminus \{-1\} \qquad \square \end{aligned}$$

Associativity:

$$\begin{aligned} (a \star b) \star c &= (ab + a + b)c + (ab + a + b) + c \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= a(b \star c) + a + (b \star c) \\ &= a \star (b \star c) \end{aligned} \qquad \square$$

Existence of e:

$$\begin{aligned} a \star e &= a \\ ae + a + e &= a \\ e(a+1) &= 0 \\ e &= 0 \quad (a \neq -1) \end{aligned}$$

$$\exists e : \forall a \in R \setminus \{-1\} : a \star e = a = e \star a \quad (\textit{commutative}) \qquad \square$$

Existence of a^{-1}:

$$\begin{aligned}
a \star a^{-1} &= 0 \\
aa^{-1} + a + a^{-1} &= 0 \\
a^{-1}(a+1) &= -a \\
a^{-1} &= \frac{-a}{a+1} \quad (a \neq -1)
\end{aligned}$$

$$\forall a \in R \setminus \{-1\} : \exists a^{-1} \in R \setminus \{-1\} : a \star a^{-1} = e = a^{-1} \star a \quad (commutative) \qquad \square$$

Thus $(\mathbb{R} \setminus \{-1\}, \star)$ is an Albelian group. $\blacksquare$

Subquestion b.

Solve

$$3 \star x \star x = 15$$

Solution

$$\begin{aligned}
3 \star x \star x &= 15 \\
3 \star (x^2 + 2x) &= 15 \\
3(x^2 + 2x) + 3 + (x^2 + 2x) &= 15 \\
4x^2 + 8x - 12 &= 0 \\
x^2 + 2x - 3 &= 0 \\
(x+3)(x-1) &= 0 \\
x &\in \{1, -3\}
\end{aligned}$$

Question 2.2.

Let n be in $\mathbb{Z} \setminus \{0\}$. Let k, x be in $\mathbb{Z}$. For all $\overline{a}, \overline{b} \in \mathbb{Z}_n$, we define

$$\overline{a} \oplus \overline{b} \coloneqq \overline{a+b}$$

Subquestion a.

Show that $(\mathbb{Z}_n, \oplus)$ is a group. Is it Albelian?

Insight

Properties of an Abelian group: commutativity, closure, associativity, existence of a neutral element, and existence of an inverse for every element. I test commutativity first so I can use it in later segments of the proof if it is indeed Albelian.

Solution

Commutative:

$$\begin{aligned} \overline{a} \oplus \overline{b} &= \overline{a+b} \\ &= \overline{b+a} \\ &= \overline{b} \oplus \overline{a} \end{aligned} \qquad \square$$

Closure:

$$\begin{aligned} \overline{a} \oplus \overline{b} &= \overline{a+b} \\ &= \{x \in \mathbb{Z} \mid \exists c \in \mathbb{Z} : x - (a+b) = nc\} \\ &= \{x \in \mathbb{Z} \mid \exists c \in \mathbb{Z} : x - (kn+r) = nc\} && (k, r \in \mathbb{Z},\ 0 \leq r < n) \\ &= \{x \in \mathbb{Z} \mid \exists c' \in \mathbb{Z} : x - r = nc'\} && (c' = c + k) \\ &= \overline{r} \\ &\in \mathbb{Z}_n \end{aligned} \qquad \square$$

Associativity:

$$\begin{aligned} (\overline{a} \oplus \overline{b}) \oplus \overline{c} &= \overline{a+b} \oplus \overline{c} \\ &= \overline{a+b+c} \\ &= \overline{a} \oplus \overline{b+c} \\ &= \overline{a} \oplus (\overline{b} \oplus \overline{c}) \end{aligned} \qquad \square$$

Existence of $\overline{e}$:

$$\begin{aligned} \overline{a} \oplus \overline{e} &= \overline{a} \\ \overline{a+e} &= \overline{a} \\ \overline{e} &= \overline{0} \end{aligned} \qquad \square$$

Existence of $\overline{a}^{-1}$: letting $\overline{a}^{-1} = \overline{b}$

$$
\begin{aligned}
\overline{a} \oplus \overline{a}^{-1} &= \overline{0} \\
\overline{a+b} &= \overline{0} \\
b &\in \{kn - a | k \in \mathbb{Z}\} \\
\overline{b} &= \overline{n-a} && (0 \leq b < n) \\
\overline{a}^{-1} &= \overline{n-a} && \square
\end{aligned}
$$

Thus $(\mathbb{R} \setminus \{-1\}, \star)$ is an Albelian group. ■

Subquestion b.

We now define

$$\overline{a} \otimes \overline{b} = \overline{a \times b}$$

Let $n = 5$. Draw the times table of the elements of $\mathbb{Z}_5 \setminus \left\{\overline{0}\right\}$ under $\otimes$.

$\otimes$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Observing the table, $\mathbb{Z}_5 \setminus \left\{\overline{0}\right\}$ is closed under $\otimes$. The neutral element is $\overline{1}$. The inverses are as follows:

$\overline{a}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{a}^{-1}$	$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{4}$

Associativity:

$$
\begin{aligned}
(\overline{a} \otimes \overline{b}) \otimes \overline{c} &= \overline{a \times b} \oplus \overline{c} \\
&= \overline{a \times b \times c} \\
&= \overline{a} \otimes \overline{b \times c} \\
&= \overline{a} \otimes (\overline{b} \otimes \overline{c}) && \square
\end{aligned}
$$

Commmutative:

$$
\begin{aligned}
\overline{a} \otimes \overline{b} &= \overline{a \times b} \\
&= \overline{b \times a} \\
&= \overline{b} \otimes \overline{a} && \square
\end{aligned}
$$

Since $(\mathbb{Z}_5 \setminus \left\{\overline{0}\right\}, \otimes)$ is closed, associative, possesses a neutral element, posesses an inverse element for every element, and is commutative, therefore it is an Albelian group. ■

Subquestion c.

Show that $(\mathbb{Z}_8 \setminus \{\overline{0}\}, \otimes)$ is not a group.

Insight

From the previous part, we observe that the only properties that may not always hold are closure and existence of the inverse element for every member. Thus, we need to find a counterexample for either of them.

Solution

There is no inverse for $\overline{2}$, thus $(\mathbb{Z}_8 \setminus \{\overline{0}\}, \otimes)$ is not a group.

Subquestion d.

Show that $(\mathbb{Z}_n \setminus \{\overline{0}\}, \otimes)$ is a group IFF $n \in \mathbb{N} \setminus \{0\}$ is prime.

Insight

To prove IFF, we need to prove the statement both ways. Same as the previous subquestion, we are only concerned with the properties of closure and existence of inverse element for every member, as the proofs of others are invariant to the choice of n.
It turns out that to fully prove the existence of the inverse element, it needs part of the proof towards closure, thus I've placed that first.

Solution

Let $\mathcal{A}_n = \{x \in \mathbb{N}, 1 \leq x < n\}$.

Lemma 1. Given $n \in \mathbb{N} \setminus \{0\}$ is prime, then $(\mathbb{Z}_n \setminus \{\overline{0}\}, \otimes)$ is a group.

Sublemma 1.a We show that $\forall m \in \mathbb{Z}, m \neq 0 \pmod{n} : \overline{m} \in (\mathbb{Z}_n \setminus \{\overline{0}\})$.

$$\begin{aligned}
\forall m \in \mathbb{Z}, m \neq 0 \pmod{n} : \overline{m} &\coloneqq \{x \in \mathbb{Z} \mid x - m = 0 \pmod{n}\} \\
&= \{x \in \mathbb{Z} \mid x - r = 0 \pmod{n}\} && (r \in \mathcal{A}_n) \\
&= \overline{r} \\
&\in \mathbb{Z}_n \setminus \{0\} && \square
\end{aligned}$$

Existence of Inverse Element:

$$\begin{aligned}
&\forall m \in \mathcal{A}_n : gcd(m, n) = 1 && \text{(def. of prime)} \\
&\forall m \in \mathcal{A}_n : \exists u, v \in \mathbb{Z} : mu + nv = 1 && \text{(Bézout theorem)} \\
&\forall m \in \mathcal{A}_n : \exists u \in \mathbb{Z} : mu = 1 \pmod{n} && (1) \\
\therefore\ &\forall m \in \mathcal{A}_n : \exists \overline{u} \in \mathbb{Z}_n : \overline{m} \otimes \overline{u} = \overline{mu} \\
&\quad = \{x \in \mathbb{Z} \mid x - mu = 0 \pmod{n}\} \\
&\quad = \{x \in \mathbb{Z} \mid x - 1 = 0 \pmod{n}\} && \text{(using 1)} \\
&\quad = \overline{1} && \square
\end{aligned}$$

Furthermore, the inverse element $\overline{u}$ is a member of the set, evident as (1) shows that $u \neq 0 \pmod{n}$, allowing us to apply sublemma 1.a.

Closure:

For $x, y \in \mathcal{A}_n$, $x \neq 0 \pmod{n}$, $y \neq 0 \pmod{n}$.

$$\begin{aligned} Let\ xy\ mod\ n &= (an+b)(cn+d)\ mod\ n \\ &= acn^2 + adn + bcn + bd\ mod\ n \\ &= bd\ mod\ n \\ &\neq 0 \end{aligned} \tag{2}$$

Therefore

$$\begin{aligned} \forall \overline{x}, \overline{y} \in \mathbb{Z}_n \setminus \{0\} : xy &\neq 0 \bmod (n) && \text{(using 2)} \\ \overline{xy} &\in \mathbb{Z}_n \setminus \{0\} && \text{(using sublemma 1.a)} \ \square \end{aligned}$$

End of Lemma 1. $\square$

Lemma 2. Given $(\mathbb{Z}_n \setminus \{\overline{0}\}, \otimes)$ is a group, then $n \in \mathbb{N} \setminus \{0\}$ is prime.

By the property of the inverse element:

$$\begin{aligned} \forall m \in \mathcal{A}_n : \exists u \in \mathcal{A}_n : \overline{m} \otimes \overline{u} &= \overline{1} \\ mu &= 1 \pmod{n} \\ mu + nv &= 1 && (v \in \mathbb{Z}) \end{aligned}$$

Therefore

$$\begin{aligned} &\forall m \in \mathcal{A}_n : \exists u, v \in \mathbb{Z} : mu + nv = 1 \\ &\forall m \in \mathcal{A}_n : gcd(m, n) = 1 && \text{(Bézout theorem)} \ \square \end{aligned}$$

End of Lemma 2. $\square$

Conclusion Given both lemma 1 and 2, we conclude that $(\mathbb{Z}_n \setminus \{\overline{0}\}, \otimes)$ is a group IFF $n \in \mathbb{N} \setminus \{0\}$ is prime. $\blacksquare$

Question 2.3.

Is $(\mathcal{G}, \cdot)$ a group? Is it albelian?

Insight

Properties of a group: closure, associativity, existence of a neutral element, and existence of an inverse for every element. Albelian = commutative.

Solution

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{G}$,

$$\boldsymbol{A} := \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{B} := \begin{bmatrix} 1 & j & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Closure:

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= \begin{bmatrix} 1 & j + x & l + kx + z \\ 0 & 1 & ky \\ 0 & 0 & 1 \end{bmatrix} \\ &\in \mathcal{G} \end{aligned} \qquad \square$$

Associativity: By def. of matrix multiplication.
Neutral Element: $\mathbb{I}_3$.
Inverse Element: Members of $\mathcal{G}$ are full rank and thus invertible.

Not Commutative:

$$\begin{aligned} (\boldsymbol{A} \cdot \boldsymbol{B})_{3,3} &= l + kx + z \\ &\neq (\boldsymbol{B} \cdot A)_{3,3} = z + yj + l \end{aligned} \qquad \square$$

Conclusion $\quad (\mathcal{G}, \cdot)$ is a group but is not albelian. ■

Question 2.4-2.8.

See Jupyter Notebook

Question 2.9.

Which of the following sets are subspaces of $\mathbb{R}^3$?

Insight

Definition of vector space:

1. $(\mathcal{V}, +)$ is an albelian group.
2. $\cdot$ is distributive over $+$ in both the left and right arguments.
3. $\cdot$ is associative.
4. $\cdot$ has a neutral element in $\mathcal{V}$.

Vector addition $+$ is commutative and associative by definition. Vector-scalar multiplication is distributive and associative by definition. These properties do not need proof.

Subquestion a.

Is $(\mathcal{A}, +)$ an albelian group?
Closure:

$$\begin{aligned}
\forall x, y \in \mathcal{A} : x + y &= \begin{bmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_2 + \mu_1^3 + \mu_2^3 \\ \lambda_1 + \lambda_2 - \mu_1^3 - \mu_2^3 \end{bmatrix} \\
&= \begin{bmatrix} \lambda \\ \lambda + \mu^3 \\ \lambda - \mu^3 \end{bmatrix} && (\lambda = \lambda_1 + \lambda_2, \mu = (\mu_1^3 + \mu_2^3)^{\frac{1}{3}}) \\
&\in \mathcal{A} && \square
\end{aligned}$$

Neutral Element: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{A}$ $\square$

Inverse Element: $\begin{bmatrix} -\lambda \\ -\lambda + (-\mu)^3 \\ -\lambda - (-\mu)^3 \end{bmatrix} \in \mathcal{A}$ $\square$

Does $(\mathcal{V}, \cdot)$ have a neutral element?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathcal{A}$$

$\square$

Conclusion $\mathcal{A}$ is a subspace of $\mathbb{R}^3$. $\blacksquare$

Subquestion b.

Is $(\mathcal{B}, +)$ an albelian group?
Closure:

$$\begin{aligned}\forall x, y \in \mathcal{B} : x + y &= \begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ -(\lambda_1^2 + \lambda_2^2) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda^2 \\ -\lambda^2 \\ 0 \end{bmatrix} && (\lambda = (\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}) \\ &\in \mathcal{B} && \square\end{aligned}$$

Neutral Element: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{A}$ $\square$

Inverse Element: There's no inverse element for $\lambda \neq 0$.

Conclusion $\mathcal{B}$ is not a subspace of $\mathbb{R}^3$. $\blacksquare$

Subquestion c.

Is $(\mathcal{C}, +)$ an albelian group?
Closure:

$$\begin{aligned}\forall x, y \in \mathcal{C} : x + y &= \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} && (\xi_1 + \xi_2 + \xi_3 = 2r) \\ &\notin \mathcal{C}\end{aligned}$$

Conclusion $\mathcal{C}$ is not a subspace of $\mathbb{R}^3$. $\blacksquare$

Subquestion d.

Closure of $(\mathcal{D}, \cdot)$

$$\forall x \in \mathcal{D}, \lambda \in \mathbb{R} \setminus \mathbb{Z} : \lambda \cdot x \notin \mathcal{D}$$

Conclusion $\mathcal{D}$ is not a subspace of $\mathbb{R}^3$. $\blacksquare$

Question 2.10-2.14.

See Jupyter Notebook

Question 15.

Let $\mathcal{F} = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$ and $\mathcal{G} = \{(a - b, a + b, a - 3b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\}$

Subquestion a.

Show that $\mathcal{F}$ and $\mathcal{G}$ are subspaces of $\mathbb{R}^3$.

Insight

Definition of vector space:

1. $(\mathcal{V}, +)$ is an albelian group.
2. $\cdot$ is distributive over $+$ in both the left and right arguments.
3. $\cdot$ is associative.
4. $\cdot$ has a neutral element in $\mathcal{V}$.

Vector addition $+$ is commutative and associative by definition. Vector-scalar multiplication is distributive and associative by definition, it also has a neutral element of 1. These properties do not need proof.

Solution

Let $m, n \in \mathcal{F}$.

Closure $(+)$:

$$\begin{aligned} m + n &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x, y, z) && (x = x_1 + x_2, y = y_1 + y_2, z = z_1 + z_2) \\ &\in \mathcal{F} && \square \end{aligned}$$

Neutral Element $(+)$: $(0, 0, 0) \in \mathcal{F}$.
Inverse Element $(+)$:

$$\begin{aligned} m + m^{-1} &= 0 \\ m^{-1} &= (-x_1, -y_1, -x_1) \\ &\in \mathcal{F} && \square \end{aligned}$$

Conclusion $\mathcal{F}$ is a subspace of $\mathbb{R}^3$. ■

Subquestion b.

Calculate $\mathcal{F} \cap \mathcal{G}$ without resorting to any basis vectors.

Insight

The intersection between the sets is the subset of $\mathbb{R}^3$ satisfying the constraints of both sets.

Solution

$$\begin{aligned}
\mathcal{F} \cap \mathcal{G} &= \{(a-b, a+b, a-3b) \in \mathbb{R}^3 \mid (a-b)+(a+b)-(a-3b) = 0\} \\
&= \{(a-b, a+b, a-3b) \in \mathbb{R}^3 \mid a+3b = 0\} \\
&= \{(-4b, -2b, -6b) \in \mathbb{R}^3 \mid b \in \mathbb{R}\} \qquad \blacksquare
\end{aligned}$$

Subquestion c.

Find one basis for $\mathcal{F}$ and one for $\mathcal{G}$, calculate $\mathcal{F} \cap \mathcal{G}$ using the basis vectors.

Insight

For $\mathcal{F}$, the basis is found by solving for $\boldsymbol{A}\vec{x} = \vec{0}$ for all $\vec{x} \in \mathcal{F}$, i.e. finding the kernel space of $\boldsymbol{A}$ which represents the constraints of the set.
For $\mathcal{G}$, the basis can be found directly by breaking up the two component vectors making up each element of the set.

Solution

Set $\mathcal{F}$ is equivalent to $ker(\boldsymbol{A})$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Sovling for $ker(\boldsymbol{A})$, we find that the basis vectors spanning it are:

$$\boldsymbol{B}_{\mathcal{F}} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

From the definition of set $\mathcal{G}$, we observe that each element can be written as:

$$(a, a, a) + (-b, b, -3b) = a(1, 1, 1) + b(-1, 1, -3)$$

Therefore its basis is:

$$\boldsymbol{B}_{\mathcal{G}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

Finally, solving for $\mathcal{F} \cap \mathcal{G}$:

$$\begin{aligned}
\boldsymbol{B}_{\mathcal{F}} \begin{bmatrix} a \\ b \end{bmatrix} &= \boldsymbol{B}_{\mathcal{G}} \begin{bmatrix} c \\ d \end{bmatrix} && (a, b, c, d \in \mathbb{R}) \\
\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &= \vec{0} \\
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &= \begin{bmatrix} 2 \\ 6 \\ 3 \\ -1 \end{bmatrix} \lambda && (\lambda \in \mathbb{R})
\end{aligned}$$

Hence:

$$\begin{aligned}
\mathcal{F} \cap \mathcal{G} &= \left\{ \boldsymbol{B}_{\mathcal{F}} \begin{bmatrix} a \\ b \end{bmatrix} \lambda \mid \lambda \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \lambda \mid \lambda \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix} \lambda \mid \lambda \in \mathbb{R} \right\} && \blacksquare
\end{aligned}$$

Question 2.16.

Are the following mappings linear?

Insight

Criterion for linearity of $\Phi : \mathcal{X}, \mathcal{Y} \mapsto \Phi(x, y)$:

$$\forall x \in \mathcal{X}, y \in \mathcal{Y} : \forall \lambda, \phi \in \mathbb{R} : \Phi(\lambda x, \phi y) = \lambda\Phi(x) + \phi\Phi(y)$$

Subquestion a.

$$\begin{aligned} \Phi : L^1([a, b]) &\to \mathbb{R} \\ f &\mapsto \int_a^b f(x) \; \mathrm{d}x \end{aligned}$$

Where L^1 denotes the set of integratable functions on $[a, b]$.

Solution

$$\begin{aligned} \forall f, g \in L^1 : \forall \lambda, \phi \in \mathbb{R} : \Phi(\lambda f + \phi g) &= \int_a^b \lambda f(x) + \phi g(x) \; \mathrm{d}x \\ &= \lambda \int_a^b f(x)\,\mathrm{d}x + \phi \int_a^b g(x)\,\mathrm{d}x \\ &= \lambda\Phi(f) + \phi\Phi(g) \end{aligned} \qquad \blacksquare$$

Subquestion b.

$$\begin{aligned} \Phi : \mathcal{C}^1 &\to \mathcal{C}^0 \\ f &\mapsto f' \end{aligned}$$

Where C^1 denotes the set of 1 time continuously differentiable functions.

Solution

$$\begin{aligned} \forall f, g \in \mathcal{C}^1 : \forall \lambda, \phi \in \mathbb{R} : \mathcal{C}(\lambda f + \phi g) &= \frac{\mathrm{d}}{\mathrm{d}x}(\lambda f(x) + \phi g(x) \;) \\ &= \lambda \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \phi \frac{\mathrm{d}}{\mathrm{d}x} g(x) \\ &= \lambda\Phi(f) + \phi\Phi(g) \end{aligned} \qquad \blacksquare$$

Subquestion c.

$$\begin{aligned} \Phi : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto cos(x) \end{aligned}$$

Solution

$$\exists x, y \in \mathbb{R} : cos(x) + cos(y) \neq cos(x + y) \qquad \perp$$

Subquestion d.

$$\begin{aligned} \Phi : \mathbb{R}^3 &\to \mathbb{R}^2 \\ \vec{x} &\mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \vec{x} \end{aligned}$$

Solution

Matrix-vector multiplication is linear by definition.

Subquestion e.

Solution

Matrix-vector multiplication is linear by definition.

Question 2.17.

Consider the given linear mapping

Subquestion a.

Find the transformation matrix $\boldsymbol{A}_\Phi$.

Solution

From inspecting the given mapping,

$$\boldsymbol{A}_\phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Subquestion b.

Determine $\text{rk}(\boldsymbol{A}_\Phi)$.

Solution

From the python code: $\text{rk}(\boldsymbol{A}_\Phi) = 3$.

Subquestion c.

Compute $\ker(\Phi)$ and $\text{Im}(\Phi)$.

Solution

From the python code:

$$\begin{aligned} \ker(\Phi) &= \{0\} \\ \dim(\ker(\Phi)) &= 0 \end{aligned}$$

Hence, the image is the full column space:

$$\begin{aligned} \text{Im}(\Phi) &= \left\{ \boldsymbol{A}_\Phi \vec{x} \mid \vec{x} \in \mathbb{R}^3 \right\} \\ \dim(\text{Im}(\Phi)) &= \text{rk}(\boldsymbol{A}_\Phi) \\ &= 3 \end{aligned}$$

Question 2.18.

Let E be a vector space. Let f and g be two automorphisms on E such that $f \circ g = \mathrm{id}_E$.

Insight

Automorphism: bijective (invertible) linear map from $E \to E$.

Subquestion $\ker(f) = \ker(g \circ f)$.

Solution

$$\begin{aligned}
\forall \vec{v} \in \ker(f) : f(\vec{v}) &= \vec{0}_E \\
(g \circ f)(\vec{v}) &= \vec{0}_E \\
\therefore \vec{v} &\in \ker(g \circ f) \\
\ker(f) &\subseteq \ker(g \circ f) && \square
\end{aligned}$$

$$\begin{aligned}
\forall \vec{w} \in \ker(g \circ f) : (g \circ f)(\vec{w}) &= \vec{0}_E \\
(f \circ g \circ f)(\vec{w}) &= \vec{0}_E \\
f(\vec{w}) &= \vec{0}_E && (f \circ g = \mathrm{id}_E) \\
\therefore \vec{w} &\in \ker(f) \\
\ker(g \circ f) &\subseteq \ker(f) && \square
\end{aligned}$$

Therefore, $\ker(f) = \ker(g \circ f)$. ■

Subquestion $\mathbf{Im}(f) = \mathbf{Im}(g \circ f)$.

$$\begin{aligned}
\mathrm{Im}(g) = \mathrm{Im}(f) &= E && \text{(surjectivity)} \\
\therefore \mathrm{Im}(g \circ f) = g(E) &= E && \blacksquare
\end{aligned}$$

Subquestion $\ker(f) \cap \mathbf{Im}(g) = \left\{\vec{0}_E\right\}$.

$$\begin{aligned}
\ker(f) &= \left\{\vec{0}_E\right\} && \text{(injectivity)} \\
\mathrm{Im}(g) &= E && \text{(surjectivity)} \\
\therefore \ker(f) \cap \mathrm{Im}(g) &= \left\{\vec{0}_E\right\} && \blacksquare
\end{aligned}$$

Question 2.19.

For the given endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$:

Subquestion a.

Determine $\ker(\Phi)$ and $\text{Im}(\Phi)$.

Solution

From the python code:

$$\begin{aligned}\ker(\Phi) &= \left\{\vec{0}\right\} \\ \text{Im}(\Phi) &= \mathbb{R}^3\end{aligned} \qquad \blacksquare$$

Subquestion b.

Determine the transformation matrix $\tilde{\boldsymbol{A}}_\Phi$ wrt the basis $\boldsymbol{B}$.

Insight

The solution is to find the composed transform:

$$B \leftarrow \mathbb{R}^3 \xleftarrow{\Phi} \mathbb{R}^3 \leftarrow B$$

Solution

$$\boldsymbol{B}^{-1}\boldsymbol{A}_\Phi\boldsymbol{B} = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix} \qquad \blacksquare$$

Question 2.20.

For vectors $\vec{b}_1, \vec{b}_2, \vec{b}'_1, \vec{b}'_2$ and ordered bases $B = (\vec{b}_1, \vec{b}_2)$ and $B' = (\vec{b}'_1, \vec{b}'_2)$ of $\mathbb{R}^2$.

Subquestion a.

Show that B and B' are two bases of $\mathbb{R}^2$ and draw those basis vectors.

Insight

Since there's only two dimensions, we can easily show that the two basis vectors are linearly independent.

Solution

$$\begin{aligned} \forall x, y \in \mathbb{R} : x = y = 0 &\iff x\vec{b}_1 + y\vec{b}_2 \\ \therefore \operatorname{span}\left[\vec{b}_1, \vec{b}_2\right] &= \mathbb{R}^2 \end{aligned} \qquad \blacksquare$$

The same argument applies to the basis B'.

Subquestion b.

Comptue the matrix $\boldsymbol{P}_1$ that performs a basis change from B' to B.

$$\begin{aligned} \boldsymbol{P}_1 &= \boldsymbol{B}^{-1}\boldsymbol{B}' \\ &= \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} \end{aligned} \qquad \blacksquare$$

Subquestion c.i.

Show that C is a basis of $\mathbb{R}^3$.

Solution

Let $\boldsymbol{C} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{bmatrix}$.

$$\begin{aligned} \det(\boldsymbol{C}) &\neq 0 \\ \operatorname{rk}(\boldsymbol{C}) &= 3 \\ \operatorname{span}(\boldsymbol{C}) &= \mathbb{R}^3 \end{aligned} \qquad \blacksquare$$

Subquestion c.ii.

Determine the matrix of basis change from C to C'.

Solution

$$\boldsymbol{P}_2 = \boldsymbol{C} \qquad \blacksquare$$

Subquestion d.

Find the transformation matrix $\boldsymbol{A}_\Phi$.

Insight

The transformation matrix describes how each component of a vector $\vec{b}$ w.r.t basis B is mapped to a new vector as a sum of and w.r.t. to basis C. Thus, first figure out what Φ does to $\vec{b}_1, \vec{b}_2$ individually. Hint: homomorphism $\equiv$ linear mapping.

Solution

Find $\Phi\left(\vec{b}_1\right)$:

$$\begin{aligned}\Phi\left(\vec{b}_1\right) &= \frac{1}{2}\left[\Phi\left(\vec{b}_1 + \vec{b}_2\right) + \Phi\left(\vec{b}_1 - \vec{b}_2\right)\right] && \text{(linearity)} \\ &= \vec{c}_1 + \vec{c}_3\end{aligned}$$

Similarly, find $\Phi\left(\vec{b}_2\right)$:

$$\begin{aligned}\Phi\left(\vec{b}_2\right) &= \frac{1}{2}\left[\Phi\left(\vec{b}_1 + \vec{b}_2\right) - \Phi\left(\vec{b}_1 - \vec{b}_2\right)\right] && \text{(linearity)} \\ &= -\vec{c}_1 + \vec{c}_2 - \vec{c}_3\end{aligned}$$

Arranging into matrix:

$$\boldsymbol{A}_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \qquad \blacksquare$$

Subquestion e.

Determine $\boldsymbol{A}'$, the transformation matrix of Φ wrt the bases $\boldsymbol{B}'$ and $\boldsymbol{C}'$.

Insight

We're essentially looking for a matrix representing the composite mapping: $\boldsymbol{B}' \to \boldsymbol{B} \xrightarrow{\Phi} \boldsymbol{C} \to \boldsymbol{C}'$.

Solution

$$
\begin{aligned}
\boldsymbol{A}' &= \boldsymbol{P}_2 \boldsymbol{A}_\Phi \boldsymbol{P}_1 \\
&= \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}
\end{aligned}
\qquad \blacksquare
$$

Subquestion f.

Let us consider the vector $\vec{x} \in \mathbb{R}^2$ whose coordinates in B' are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. In other words, $\vec{x} = 2\vec{b}_1' + 3\vec{b}_2'$.

(i) Calculate the coordinates of $\vec{x}$ in B.

$$
\begin{aligned}
\vec{x}_B &= \boldsymbol{P}_1 \vec{x} \\
&= \begin{bmatrix} 8 \\ 9 \end{bmatrix}
\end{aligned}
\qquad \blacksquare
$$

(ii) Based on that, compute the coordinates of $\Phi(\vec{x})$ expressed in C.

$$
\begin{aligned}
\Phi(x)_C &= \boldsymbol{A}_\Phi \vec{x}_B \\
&= \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}
\end{aligned}
\qquad \blacksquare
$$

(iii) Then, write $\Phi(x)$ in terms of $\vec{c}_1', \vec{c}_2', \vec{c}_3'$.

$$
\begin{aligned}
\Phi(x)_{C'} &= \boldsymbol{P}2\Phi(x)_C \\
&= \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix} \\
&= 6\vec{c}_1' - 11\vec{c}_2' + 12\vec{c}_3'
\end{aligned}
\qquad \blacksquare
$$

(iv) Use the representation of $\vec{x}$ in B' and the matrix $\boldsymbol{A}'$ to find this result directly.

$$
\begin{aligned}
\Phi(x)_{C'} &= \boldsymbol{A}' \vec{x} \\
&= \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}
\end{aligned}
\qquad \blacksquare
$$

Chapter 3: Analytic Geometry

Question 3.1.

Show that $\langle \vec{x}, \vec{y} \rangle \coloneqq x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$ is an inner product.

Insight

We need to test the 3 properties of an inner product:

- Symmetric
- Bilinear
- Positive Definite

Solution

Symmetric

$$\langle , \vec{x} \rangle \vec{y}$$