MML Book Solutions

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Contents

Chapter 2: Linear Algebra

Question 2.1.

We consider $(\mathbb{R} \setminus \{-1\}, \star)$, where

$$a \star b := ab + a + b \quad a, b \in \mathbb{R} \setminus \{-1\}$$

Subquestion a.

Show that $(\mathbb{R} \setminus \{-1\}, \star)$ is an Abelian group.

Insight

Properties of an Abelian group: commutativity, closure, associativity, existence of a neutral element, and existence of an inverse for every element. I show commutativity first so I can use it in later segments of the proof.

Solution

Commutative:

$$\begin{aligned} a \star b &= ab + a + b \\ &= ba + b + a \\ &= b \star a \end{aligned} \qquad \square$$

Closure:

$$\begin{aligned} \textit{Assuming} \quad a \star b &= -1 \\ ab + a + b &= -1 \\ (a+1)(b+1) &= 0 \\ a = -1 \text{ OR } b &= -1 && \perp \\ a \star b &\in \mathbb{R} \setminus \{-1\} && \square \end{aligned}$$

Associativity:

$$\begin{aligned} (a \star b) \star c &= (ab + a + b)c + (ab + a + b) + c \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= a(b \star c) + a + (b \star c) \\ &= a \star (b \star c) \end{aligned} \qquad \square$$

Existence of e:

$$\begin{aligned} a \star e &= a \\ ae + a + e &= a \\ e(a+1) &= 0 \\ e &= 0 \quad (a \neq -1) \end{aligned}$$

$$\exists e : \forall a \in R \setminus \{-1\} : a \star e = a = e \star a \quad \textit{(commutative)} \qquad \square$$

Existence of a^{-1}:

$$\begin{aligned} a \star a^{-1} &= 0 \\ aa^{-1} + a + a^{-1} &= 0 \\ a^{-1}(a+1) &= -a \\ a^{-1} &= \frac{-a}{a+1} \quad (a \neq -1) \end{aligned}$$

$$\forall a \in R \setminus \{-1\} : \exists a^{-1} \in R \setminus \{-1\} : a \star a^{-1} = e = a^{-1} \star a \quad (commutative) \qquad \square$$

Thus $(\mathbb{R} \setminus \{-1\}, \star)$ is an Albelian group. $\blacksquare$

Subquestion b.

Solve

$$3 \star x \star x = 15$$

Solution

$$\begin{aligned} 3 \star x \star x &= 15 \\ 3 \star (x^2 + 2x) &= 15 \\ 3(x^2 + 2x) + 3 + (x^2 + 2x) &= 15 \\ 4x^2 + 8x - 12 &= 0 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x &\in \{1, -3\} \end{aligned}$$

Question 2.2.

Let n be in $\mathbb{Z} \setminus \{0\}$. Let k, x be in $\mathbb{Z}$. For all $\overline{a}, \overline{b} \in \mathbb{Z}_n$, we define

$$\overline{a} \oplus \overline{b} \coloneqq \overline{a+b}$$

Subquestion a.

Show that $(\mathbb{Z}_n, \oplus)$ is a group. Is it Albelian?

Insight

Properties of an Abelian group: commutativity, closure, associativity, existence of a neutral element, and existence of an inverse for every element. I test commutativity first so I can use it in later segments of the proof if it is indeed Albelian.

Solution

Commutative:

$$\begin{aligned} \overline{a} \oplus \overline{b} &= \overline{a+b} \\ &= \overline{b+a} \\ &= \overline{b} \oplus \overline{a} \end{aligned} \qquad \square$$

Closure:

$$\begin{aligned} \overline{a} \oplus \overline{b} &= \overline{a+b} \\ &= \{x \in \mathbb{Z} \mid \exists c \in \mathbb{Z} : x - (a+b) = nc\} \\ &= \{x \in \mathbb{Z} \mid \exists c \in \mathbb{Z} : x - (kn+r) = nc\} && (k, r \in \mathbb{Z},\, 0 \leq r < n) \\ &= \{x \in \mathbb{Z} \mid \exists c' \in \mathbb{Z} : x - r = nc'\} && (c' = c + k) \\ &= \overline{r} \\ &\in \mathbb{Z}_n \end{aligned} \qquad \square$$

Associativity:

$$\begin{aligned} (\overline{a} \oplus \overline{b}) \oplus \overline{c} &= \overline{a+b} \oplus \overline{c} \\ &= \overline{a+b+c} \\ &= \overline{a} \oplus \overline{b+c} \\ &= \overline{a} \oplus (\overline{b} \oplus \overline{c}) \end{aligned} \qquad \square$$

Existence of $\overline{e}$:

$$\begin{aligned} \overline{a} \oplus \overline{e} &= \overline{a} \\ \overline{a+e} &= \overline{a} \\ \overline{e} &= \overline{0} \end{aligned} \qquad \square$$

Existence of $\overline{a}^{-1}$: letting $\overline{a}^{-1} = \overline{b}$

$$\begin{aligned}
\overline{a} \oplus \overline{a}^{-1} &= \overline{0} \\
\overline{a + b} &= \overline{0} \\
b &\in \{kn - a | k \in \mathbb{Z}\} \\
\overline{b} &= \overline{n - a} && (0 \leq b < n) \\
\overline{a}^{-1} &= \overline{n - a} && \square
\end{aligned}$$

Thus $(\mathbb{R} \setminus \{-1\}, \star)$ is an Albelian group. $\blacksquare$

Subquestion b.

We now define

$$\overline{a} \otimes \overline{b} = \overline{a \times b}$$

Let $n = 5$. Draw the times table of the elements of $\mathbb{Z}_5 \setminus \left\{\overline{0}\right\}$ under $\otimes$.

$\otimes$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Observing the table, $\mathbb{Z}_5 \setminus \left\{\overline{0}\right\}$ is closed under $\otimes$. The neutral element is $\overline{1}$. The inverses are as follows:

$\overline{a}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{a}^{-1}$	$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{4}$

Associativity:

$$\begin{aligned}
(\overline{a} \otimes \overline{b}) \otimes \overline{c} &= \overline{a \times b} \oplus \overline{c} \\
&= \overline{a \times b \times c} \\
&= \overline{a} \otimes \overline{b \times c} \\
&= \overline{a} \otimes (\overline{b} \otimes \overline{c}) && \square
\end{aligned}$$

Commmutative:

$$\begin{aligned}
\overline{a} \otimes \overline{b} &= \overline{a \times b} \\
&= \overline{b \times a} \\
&= \overline{b} \otimes \overline{a} && \square
\end{aligned}$$

Since $(\mathbb{Z}_5 \setminus \left\{\overline{0}\right\}, \otimes)$ is closed, associative, possesses a neutral element, posesses an inverse element for every element, and is commutative, therefore it is an Albelian group. $\blacksquare$

Subquestion c.

Show that $(\mathbb{Z}_8 \setminus \left\{\overline{0}\right\}, \otimes)$ is not a group.

Insight

From the previous part, we observe that the only properties that may not always hold are closure and existence of the inverse element for every member. Thus, we need to find a counterexample for either of them.

Solution

There is no inverse for $\overline{2}$, thus $(\mathbb{Z}_8 \setminus \left\{\overline{0}\right\}, \otimes)$ is not a group.

Subquestion d.

Show that $(\mathbb{Z}_n \setminus \left\{\overline{0}\right\}, \otimes)$ is a group IFF $n \in \mathbb{N} \setminus \{0\}$ is prime.

Insight

To prove IFF, we need to prove the statement both ways. Same as the previous subquestion, we are only concerned with the properties of closure and existence of inverse element for every member, as the proofs of others are invariant to the choice of n.

Solution

Lemma 1. Given $n \in \mathbb{N} \setminus \{0\}$ is prime, then $(\mathbb{Z}_n \setminus \left\{\overline{0}\right\}, \otimes)$ is a group.
Existence of Inverse Element:

$$\begin{aligned} &\forall m \in \mathbb{N}, 1 \leq m < n : gcd(m, n) = 1 && \text{(def. of prime)} \\ &\forall m \in \mathbb{N}, 1 \leq m < n : \exists u, v \in \mathbb{Z} : mu + nv = 1 && \text{(Bézout theorem)} \\ \therefore\ &\forall m \in \mathbb{N}, 1 \leq m < n : \exists u \in \mathbb{Z} : mu = 1 \ (\text{mod} n) && (1) \end{aligned}$$

therefore

$$\begin{aligned} \exists u \in \mathbb{Z} : \overline{m} \otimes \overline{u} &= \overline{mu} \\ &= \{x \in \mathbb{Z} \mid x - mu = 0 \ (\text{mod} n)\} \\ &= \{x \in \mathbb{Z} \mid x - 1 = 0 \ (\text{mod} n)\} && \text{(using 1)} \\ &= \overline{1} && \square \end{aligned}$$

Lemma 2. Given $(\mathbb{Z}_n \setminus \left\{\overline{0}\right\}, \otimes)$ is a group, then $n \in \mathbb{N} \setminus \{0\}$ is prime.

Chapter 3: Analytic Geometry

Question 2.1.