Solutions

Exercise 1

$$\begin{aligned} U \cdot S \cdot V &= A \\ U^T \cdot U \cdot S \cdot V &= U^T \cdot A \\ S \cdot V &= U^T \cdot A \\ V &= S^{-1} \cdot U^T \cdot A \end{aligned}$$

Exercise 2

Part 1

$$A \cdot A^T = \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} \cdot \begin{pmatrix} 13 & 9 & 0 \\ 5 & 15 & 0 \\ 5 & 15 & 0 \\ 0 & 0 & 20 \\ 13 & 9 & 0 \end{pmatrix} = \begin{pmatrix} 388 & 384 & 0 \\ 384 & 612 & 0 \\ 0 & 0 & 400 \end{pmatrix}$$

Part 2

1. Guess the first Eigenvector and the corresponding Eigenvalue

$$\begin{pmatrix} 388 & 384 & 0 \\ 384 & 612 & 0 \\ 0 & 0 & 400 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 400 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2. Compute the rest Eigenvalues

$$\begin{gathered} \begin{vmatrix} 388 - \lambda & 384 \\ 384 & 612 - \lambda \end{vmatrix} = 0 \\ \lambda^2 - 1000\lambda + 90000 = 0 \\ \lambda_1 = 900, \lambda_2 = 100 \end{gathered}$$

3. Figure out the corresponding Eigenvectors

$$\begin{pmatrix} 388 & 384 \\ 384 & 612 \end{pmatrix} \cdot \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} = 100 \cdot \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$\begin{pmatrix} 388 & 384 \\ 384 & 612 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = 900 \cdot \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

The EVD of $A \cdot A^T$ is

$$A \cdot A^T = \begin{pmatrix} 388 & 384 & 0 \\ 384 & 612 & 0 \\ 0 & 0 & 400 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 900 & 0 & 0 \\ 0 & 400 & 0 \\ 0 & 0 & 100 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}$$

Part 3

$$U = \begin{pmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix}, U \cdot U^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

U is indeed a 3×3 column-orthonormal matrix.

$$S = \begin{pmatrix} 30 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

Part 4

$$\begin{aligned} V &= S^{-1} \cdot U^T \cdot A \\ &= \begin{pmatrix} \frac{1}{30} & 0 & 0 \\ 0 & \frac{1}{20} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix} \cdot \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

Part 5

$$U \cdot S \cdot V = \begin{pmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 30 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 10 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} = A$$

Exercise 3

Variant 1

$$A_k = U_k \cdot S_k \cdot V_k = \begin{pmatrix} \frac{3}{5} & 0 \\ \frac{4}{5} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 9 & 9 & 0 & 9 \\ 12 & 12 & 12 & 0 & 12 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix}$$

$$q^T \cdot A_k = \begin{pmatrix} 4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 9 & 9 & 9 & 0 & 9 \\ 12 & 12 & 12 & 0 & 12 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} = \begin{pmatrix} 48 & 48 & 48 & 40 & 48 \end{pmatrix}$$

Variant 2

$$q_k^T = q^T \cdot U_k \cdot S_k = \begin{pmatrix} 4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} & 0 \\ \frac{4}{5} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} 96 & 40 \end{pmatrix}$$

$$q_k^T \cdot V_k = \begin{pmatrix} 96 & 40 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 48 & 48 & 48 & 40 & 48 \end{pmatrix}$$

Variant 3

$$T_k = U_k \cdot U_k^T = \begin{pmatrix} \frac{3}{5} & 0 \\ \frac{4}{5} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{9}{25} & \frac{12}{25} & 0 \\ \frac{12}{25} & \frac{16}{25} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

use $\frac{12}{25}$ as the 0-1 threshold：

$$T_k^{'} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$q^T \cdot T_k^{'} \cdot A = \begin{pmatrix} 4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} = \begin{pmatrix} 58 & 80 & 80 & 40 & 58 \end{pmatrix}$$