Fractional Knapsack

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Introduction

- Classical problems in algorithmic optimization
- In Knapsack problems, a knapsack with capacity W and n items with weights $w_1, w_2, , w_n$ and values $v_1, v_2, \ldots, v_n$ are given. The problem is to add items to the knapsack such that the total weight of the added items is $\leq W$ and the total value is maximized.
- Two variants
  1. Fractional Knapsack
     Fractions of items can be taken. Greedy algorithm possible.
  2. 0-1 Knapsack
     Take item fully or none. Greedy algorithm not possible.

Fractional Knapsack Greedy solution

We are given an instance of fractional Knapsack with capacity W = 30 and the weights and values of $n = 4$ items, as follows $(a, 140, 20)$, $(b, 50, 5)$, $(c, 60, 10)$, $(d, 50, 14)$

Greedy Algorithm:
Step 1. Compute *value to weight* $\frac{v_i}{w_i}$ ratio of each items.

Item	Value	Weight	Ratio
a	140	20	7
b	50	5	10
c	60	10	6
d	50	14	3.5

Fractional Knapsack example ...

Step 2. Sort all items in the decreasing order of *value to weight* ratio.

Item	Value	Weight	Ratio
b	50	5	10
a	140	20	7
c	60	10	6
d	50	14	3.5

Step 3. Fill the Knapsack with the (*fraction of the*) item in the decreasing order.

#	Fraction of item added	Remaining capacity	Value
0	...	30	0
1	(1) b	25	50
2	(1) a	5	190
3	(1/2) c	0	220

Optimality of the greedy algorithm

Greedy Choice property. We need to show that there exists an optimal solution that selects the fraction f of an item i, as the greedy choice did.

Let T^* be a another optimal solution that does not select the fraction f of the item i , as the greedy choice did.

This means T^* has selected only a lesser fraction of i, because the fraction f is a greedy choice.

From T^* we can remove some items with total weight $f * weight(i)$ and can replace with $f * i$.
Since i has better *value to weight* ratio, this gives another optimal solution.

Optimality of the greedy algorithm

Optimal substructure property. Let $T = \{i_1, i_2, \ldots, i_k\}$ be an optimal solution of the fractional Knapsack problem S with weight W. We need to show that the solution set T^* we obtain by removing an item i_j from T is an optimal solution for the the problem $S* = S - \{i_j\}$, with weight $W - weight(i_j)$.

Assume the contrary that the solution T^* of the problem is not optimal. Then it means that there exists an optimal solution T^{**} such that $value(T^{**}) \geq value(T^*)$.

If so, then $T^{**} \cup \{i_j\}$ is an optimal solution to the original problem S, which is a contradiction.