Positions, Iterators, Tree Structures and Tree Traversals

Readings - Chapter 7.3, 7.4 and 8

Positional Lists

- A position acts as a marker or token within the broader positional list.
- A position p is unaffected by changes elsewhere in a list; the only way in which a position becomes invalid is if an explicit command is issued to delete it.
- A position instance is a simple object, supporting only the following method:
  - P.getElement( ): Return the element stored at position p.
- To provide for a general abstraction of a sequence of elements with the ability to identify the location of an element, we define a **positional list** ADT.

Positional List ADT

- Accessor methods:

first( ): Returns the position of the first element of L (or null if empty).

last( ): Returns the position of the last element of L (or null if empty).

before(p): Returns the position of L immediately before position p (or null if p is the first position).

after(p): Returns the position of L immediately after position p (or null if p is the last position).

isEmpty( ): Returns true if list L does not contain any elements.

size( ): Returns the number of elements in list L.

Positional List ADT (2)

- Update methods:

addFirst(e): Inserts a new element e at the front of the list, returning the position of the new element.

addLast(e): Inserts a new element e at the back of the list, returning the position of the new element.

addBefore(p, e): Inserts a new element e in the list, just before position p, returning the position of the new element.

addAfter(p, e): Inserts a new element e in the list, just after position p, returning the position of the new element.

set(p, e): Replaces the element at position p with element e, returning the element formerly at position p.

remove(p): Removes and returns the element at position p in the list, invalidating the position.

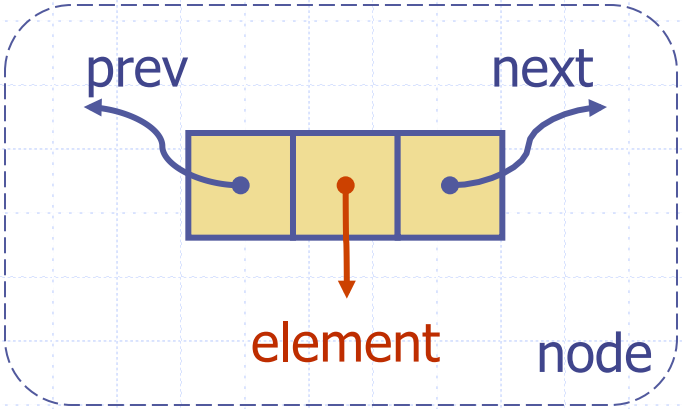
Example

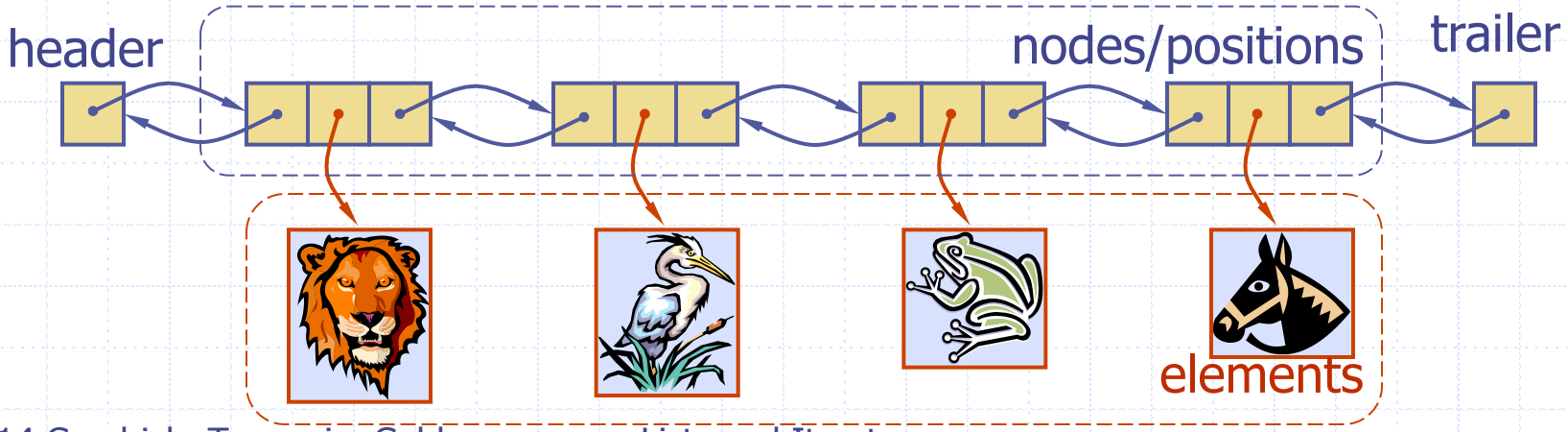
- A sequence of Positional List operations:

Method	Return Value	List Contents
addLast(8)	p	(8_p)
first( )	p	(8_p)
addAfter(p, 5)	q	$(8_p, 5_q)$
before(q)	p	$(8_p, 5_q)$
addBefore(q, 3)	r	$(8_p, 3_r, 5_q)$
r.getElement( )	3	$(8_p, 3_r, 5_q)$
after(p)	r	$(8_p, 3_r, 5_q)$
before(p)	null	$(8_p, 3_r, 5_q)$
addFirst(9)	s	$(9_s, 8_p, 3_r, 5_q)$
remove(last( ))	5	$(9_s, 8_p, 3_r)$
set(p, 7)	8	$(9_s, 7_p, 3_r)$
remove(q)	"error"	$(9_s, 7_p, 3_r)$

Positional List Implementation

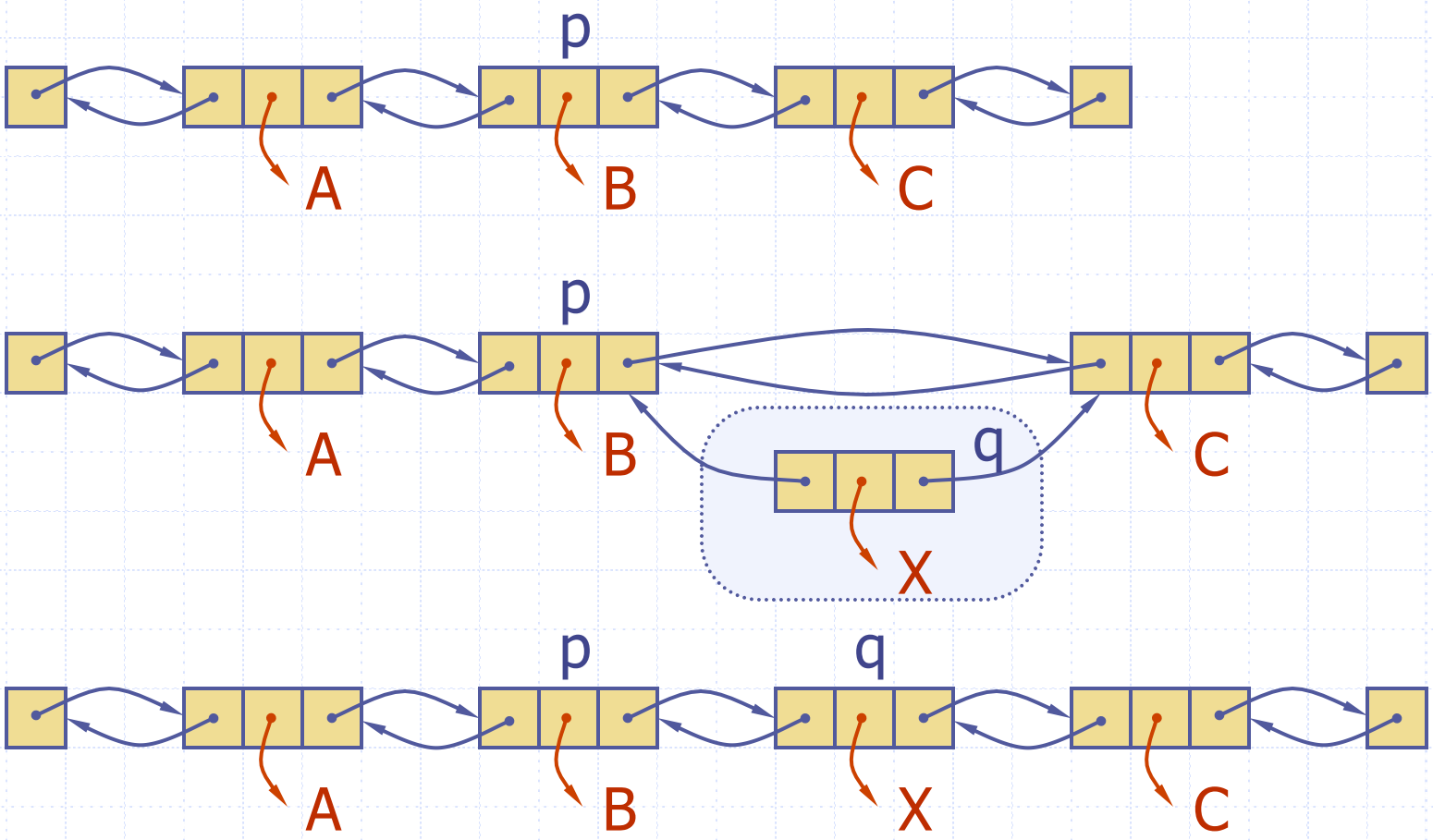
- The most natural way to implement a positional list is with a doubly-linked list.





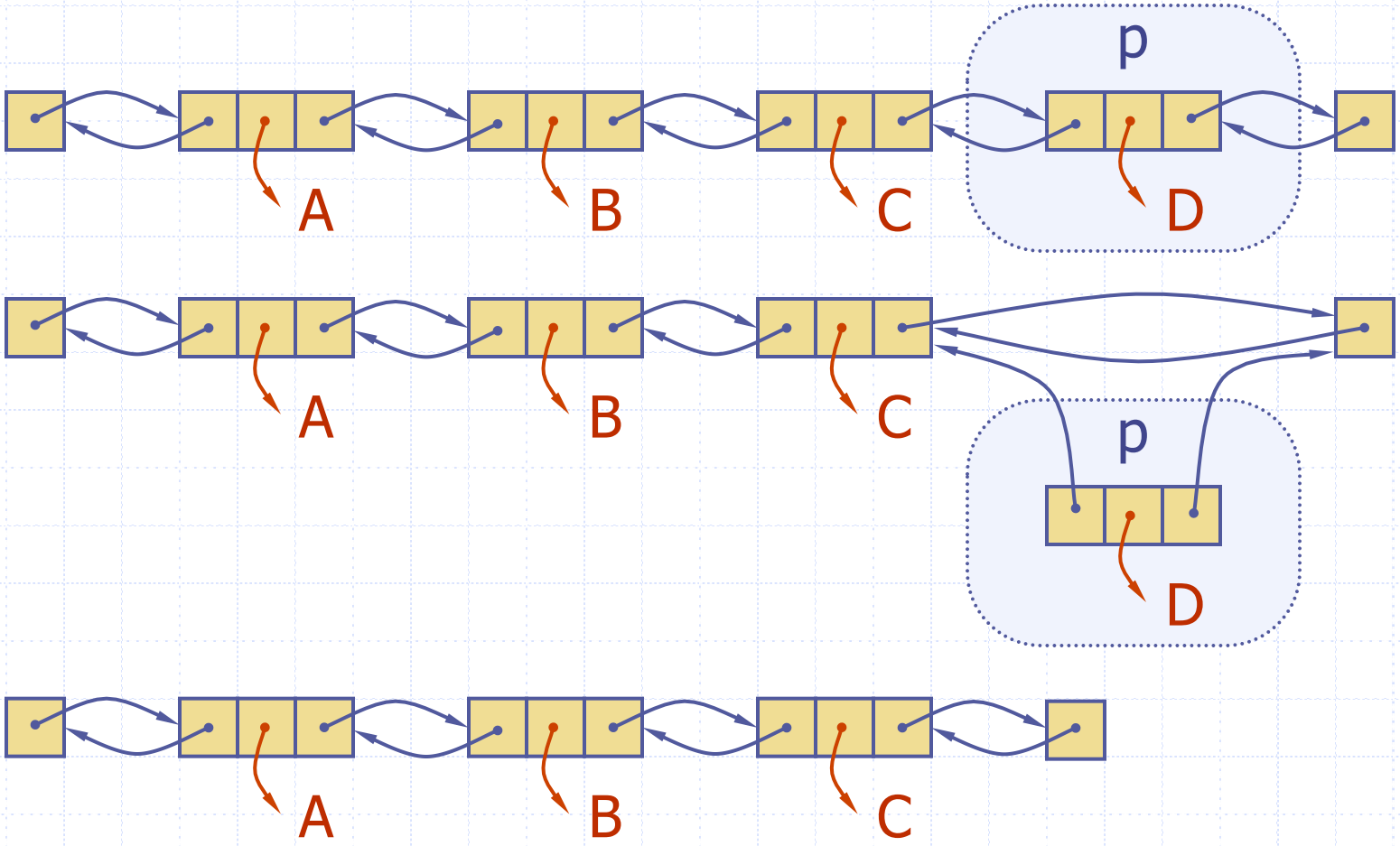
Insertion

- Insert a new node, q, between p and its successor.



Deletion

- Remove a node, p, from a doubly-linked list.



Iterators

- An iterator is a software design pattern that abstracts the process of scanning through a sequence of elements, one element at a time.

hasNext( ):	Returns true if there is at least one additional element in the sequence, and false otherwise.
next( ):	Returns the next element in the sequence.

The Iterable Interface

- Java defines a parameterized interface, named Iterable, that includes the following single method:
  - iterator( ): Returns an iterator of the elements in the collection.
- An instance of a typical collection class in Java, such as an ArrayList, is iterable (but not itself an iterator); it produces an iterator for its collection as the return value of the iterator( ) method.
- Each call to iterator( ) returns a new iterator instance, thereby allowing multiple (even simultaneous) traversals of a collection.

The for-each Loop

- Java's Iterable class also plays a fundamental role in support of the "for-each" loop syntax:

```
for (ElementType variable : collection) {
  loopBody                                   // may refer to "variable"
}
```

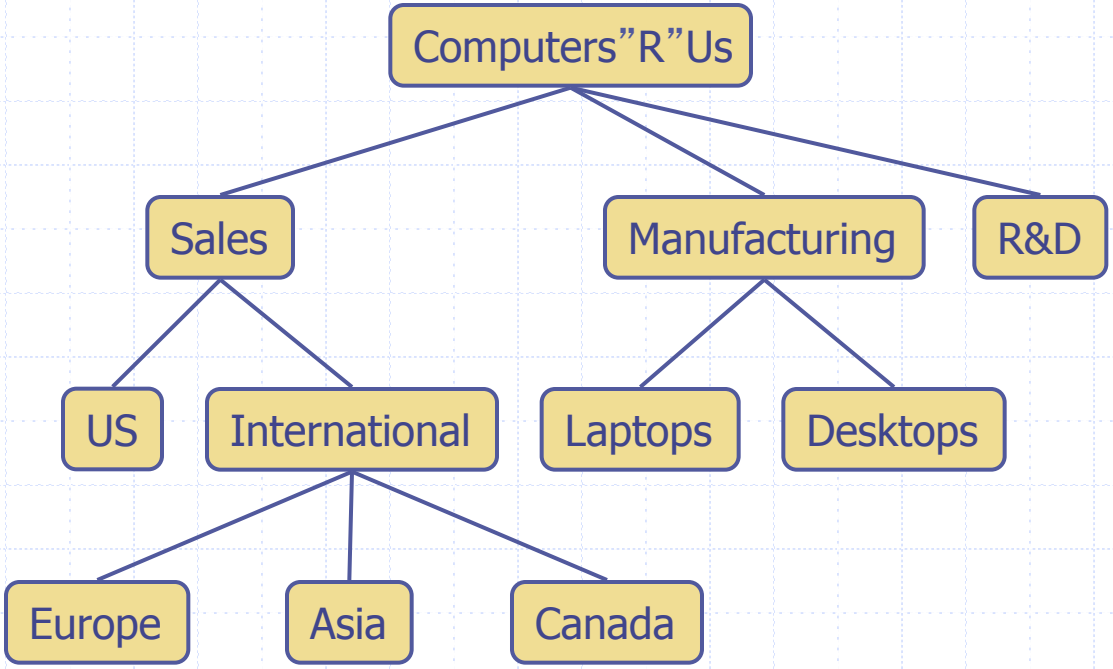
```
Iterator<ElementType> iter = collection.iterator( );
while (iter.hasNext( )) {
  ElementType variable = iter.next( );
  loopBody                                   // may refer to "variable"
}
```

Trees

- Abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation

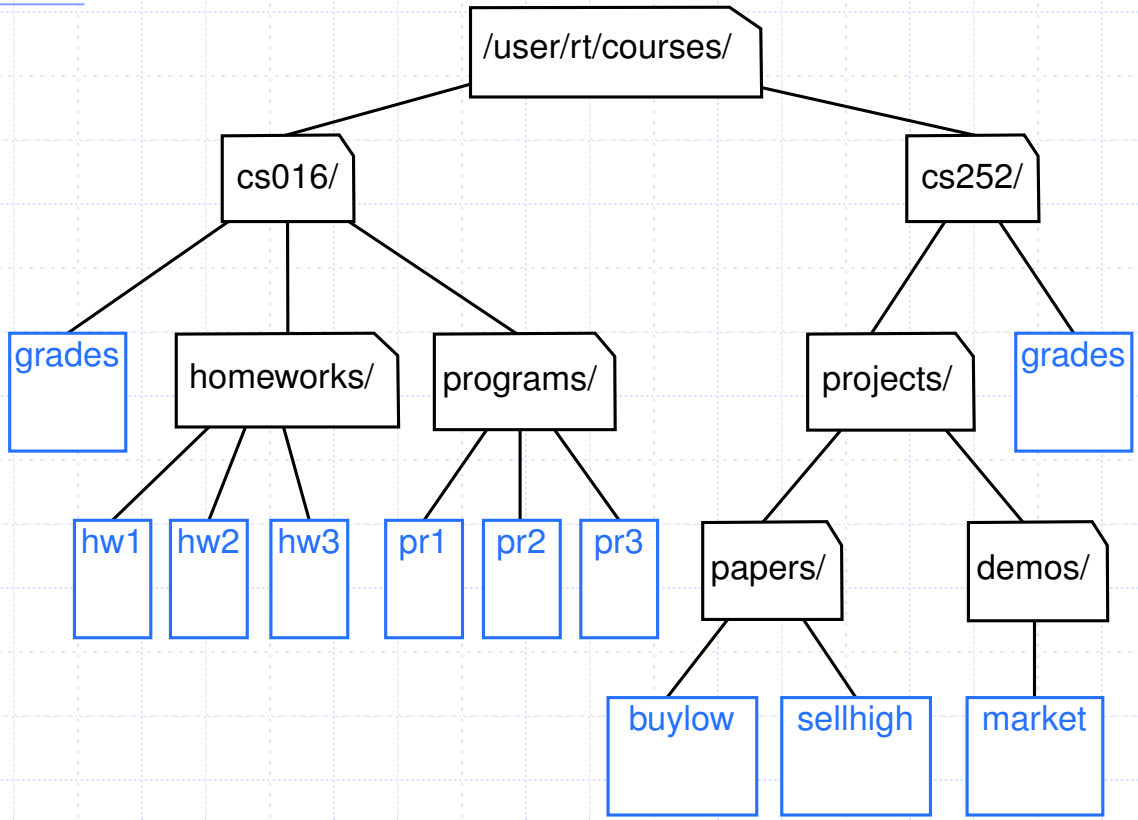
google images

Trees - Examples



organization structure of a corporation

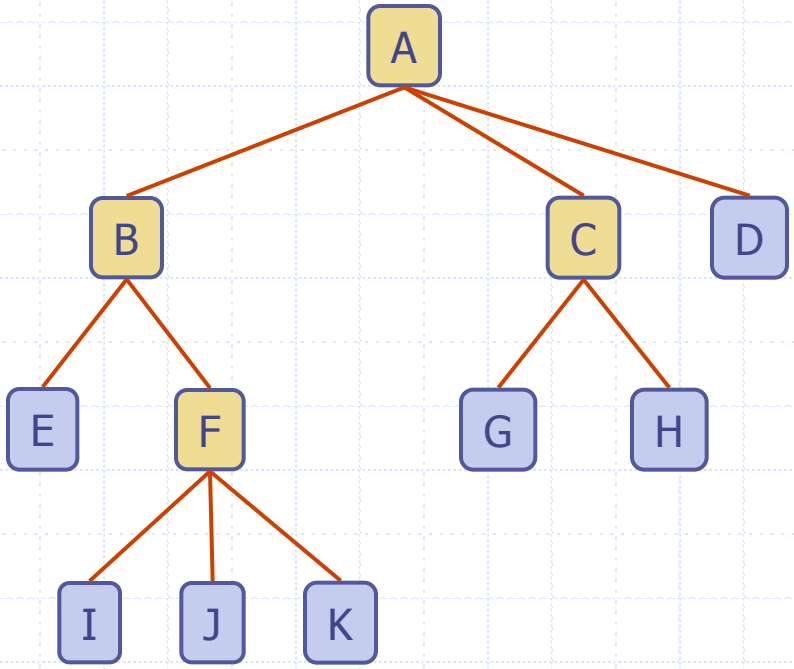
Trees - Examples (2)



Portion of a file system

Trees - Terminology

- *A* is the *root* node
- *B* is *parent* of *E* and *F*
- *A* is *ancestor* of *E* and *F*
- *E* and *F* are *descendants* of *A*
- *C* is the *sibling* of *B*
- *E* and *F* are *children* of *B*
- *E*, *I*, *J*, *K*, *G*, *H*, and *D* are *leaves*
- *A*, *B*, *C*, and *F* are *internal nodes*



Trees - Terminology (2)

- *A* is the *root* node
- *B* is *parent* of *E* and *F*
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- *E* and *F* are *children* of *B*
- *E*, *I*, *J*, *K*, *G*, *H*, and *D* are *leaves*
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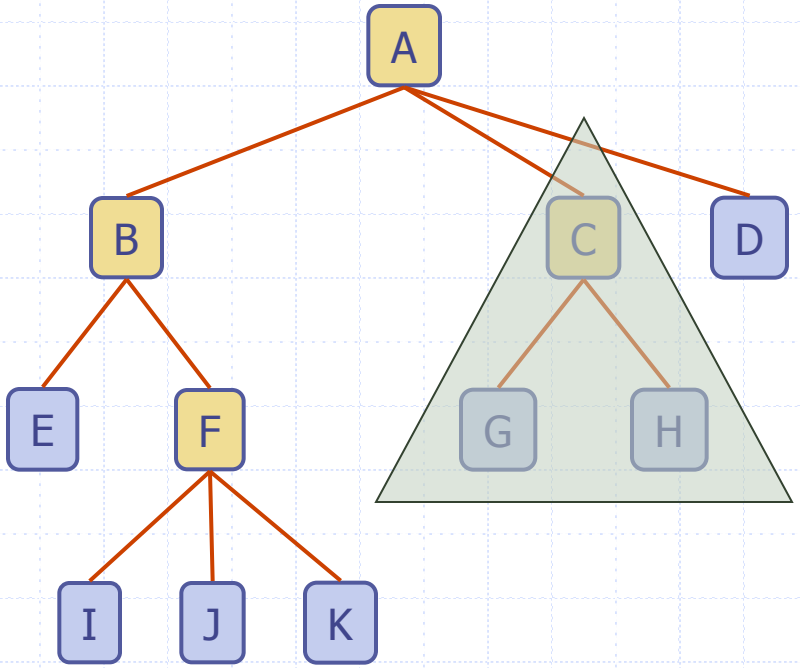
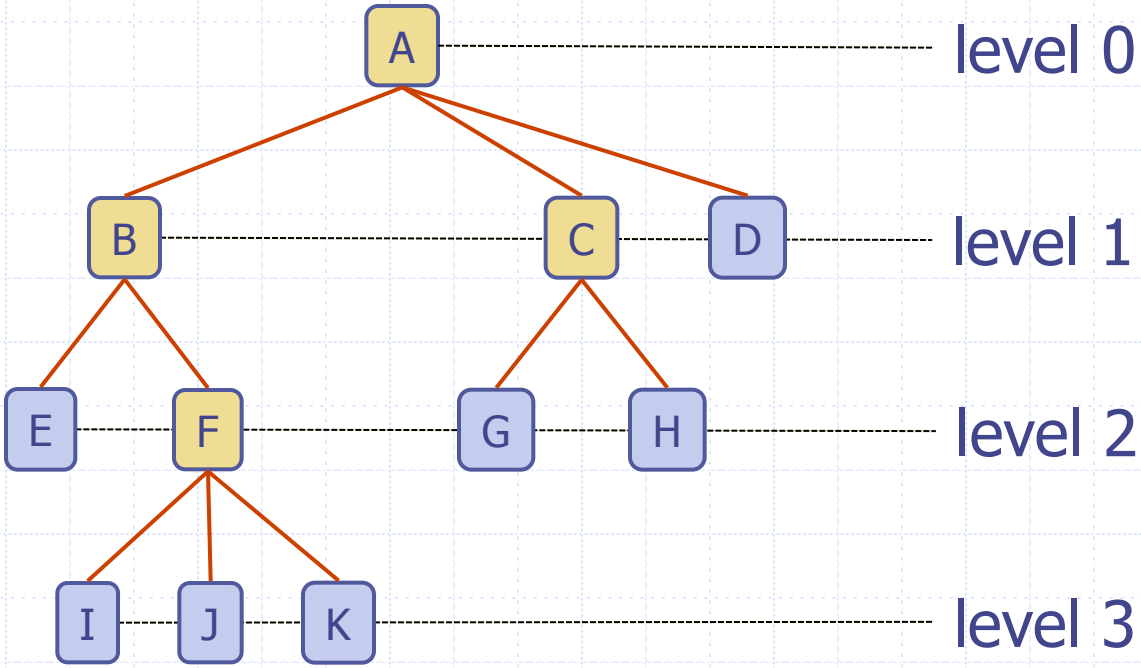
- *Subtree:* tree consisting of node and its descendants



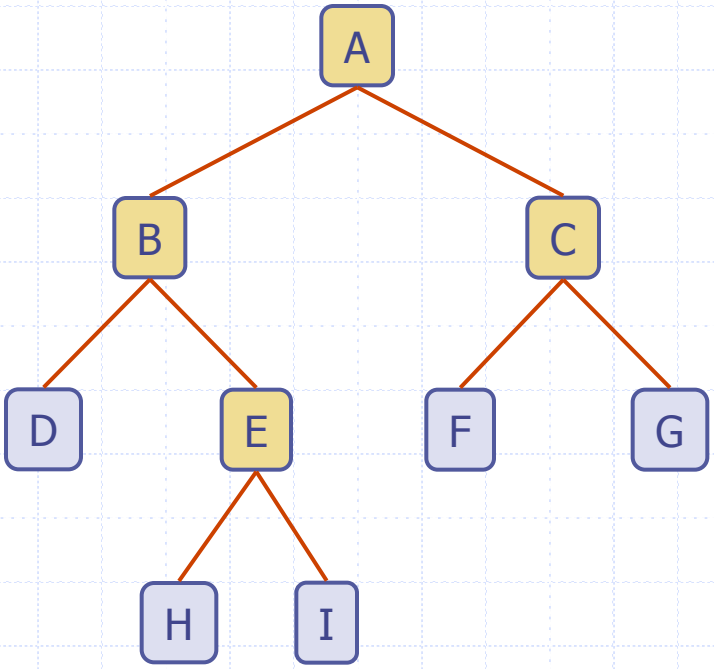
Trees - Terminology (3)

- The *depth* (*level*) of *E* is 2
- The *height* of the tree is 3
- The *degree* of node F is 3



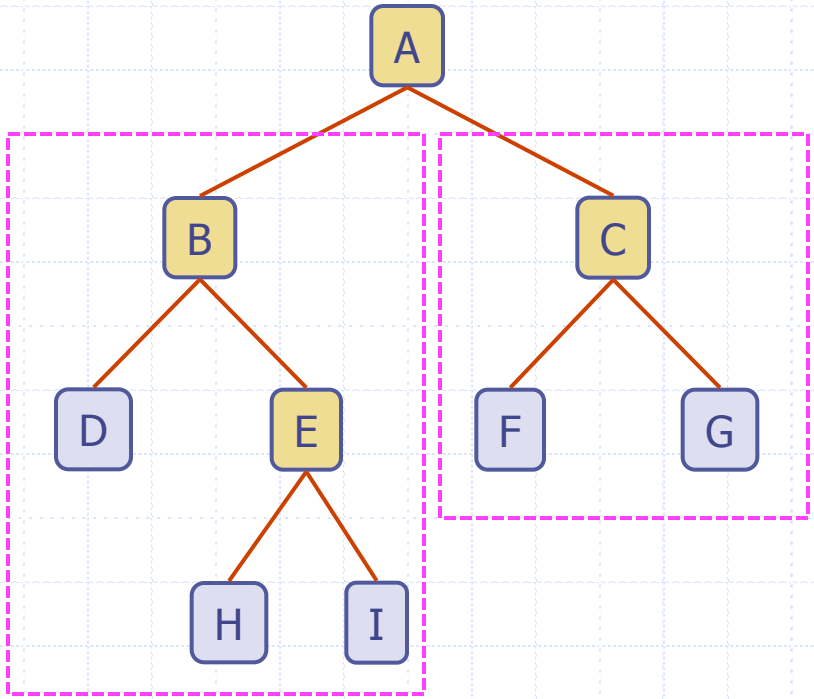
Binary Trees

- An **ordered tree** is one in which the children of each node are ordered
- **Binary tree**: ordered tree with all nodes having at most 2 children
  - left child and right child



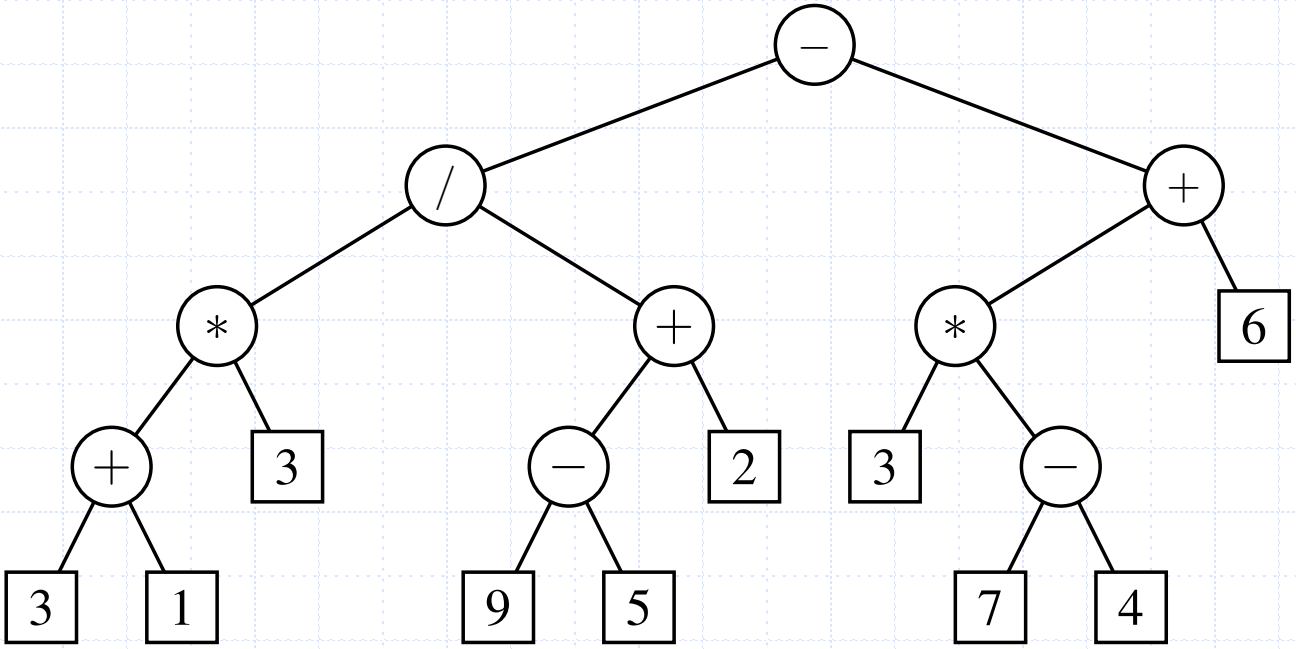
Binary Trees

- Recursive definition of binary tree
  - either a leaf or
  - an internal node (the root) and one/two binary trees (left subtree and/or right subtree)



Example of Binary Trees - Arithmetic Expression Tree

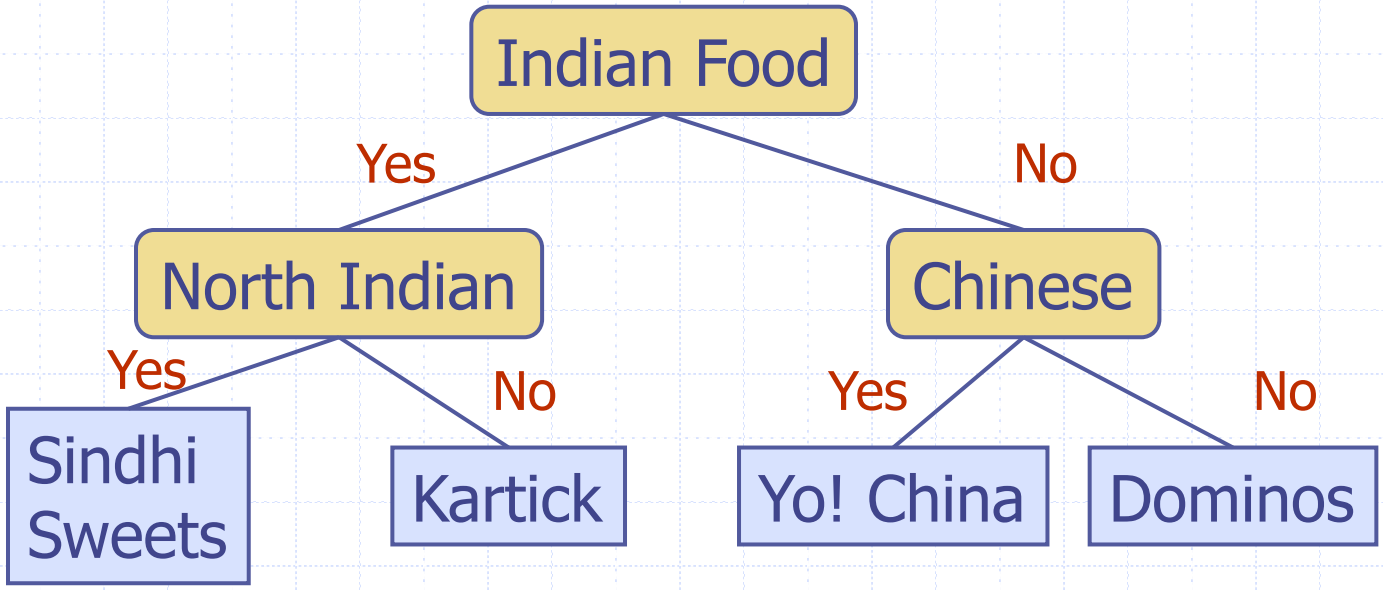
- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands



((((3 + 1) *3) / (9 - 5)+2))-((3*(7 - 4)) +6))

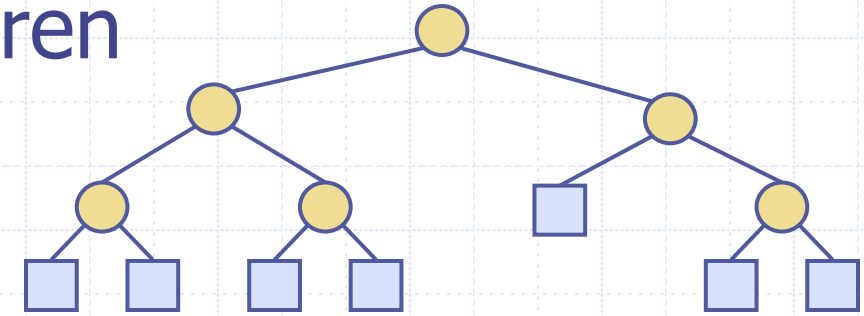
Example of Binary Trees - Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision

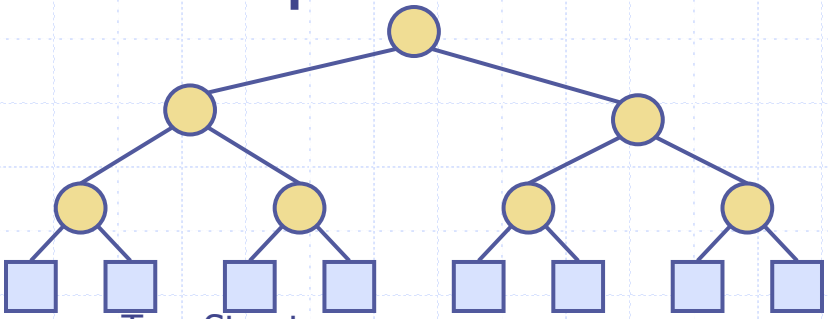


Proper, Full, Complete Binary Trees

- Proper/Full - Every node has either zero or two children

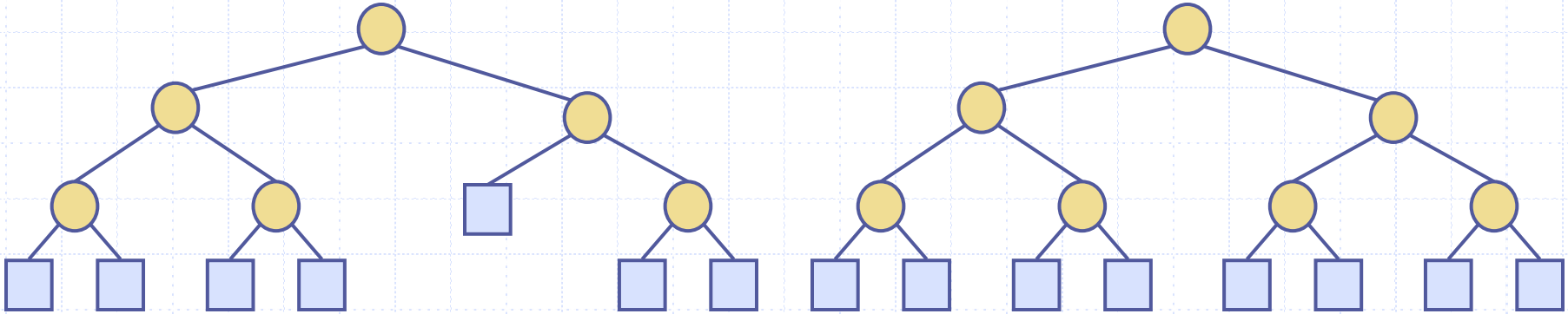


- Complete - every level except possibly the last is completely filled and all leaf nodes are as left as possible.



Binary tree from a complete binary tree

- A binary tree can be obtained from appropriate complete binary tree by pruning.



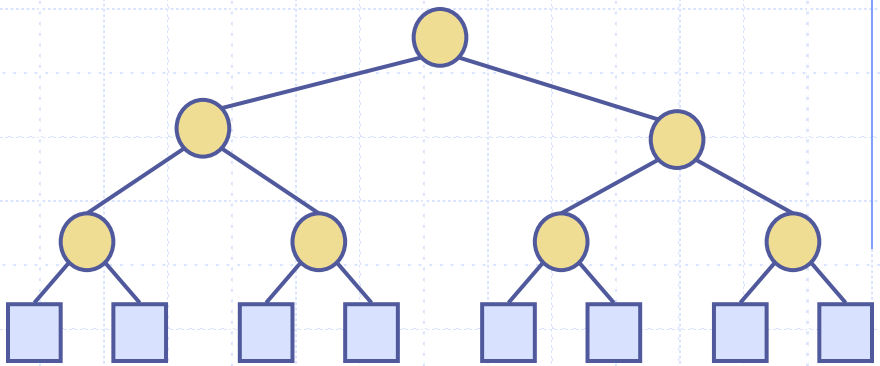
Properties of a Binary Tree

- Notations
  - n - number of nodes
  - n_E - number of leaves (external nodes)
  - n_I - number of internal nodes
  - h - height of the tree
- $h+1 \leq n \leq 2^{h+1} - 1$
- $1 \leq n_E \leq 2^h$
- $h \leq n_I \leq 2^h - 1$
- $\log(n+1) - 1 \leq h \leq n-1$

Properties of Binary Trees (2)

- $n_E \leq n_I + 1$
- proof by induction on n_I
  - Tree with 1 node has a leaf but no internal node
  - Assume $n_E \leq n_I + 1$ for tree with k-1 internal nodes
  - A tree with k internal nodes has k_1 internal nodes in the left subtree and k-k_1-1 internal nodes in the right subtree
  - By induction $n_E \leq (k_1+1) + (k-k_1-1+1) = k+1$

Complete Binary Tree

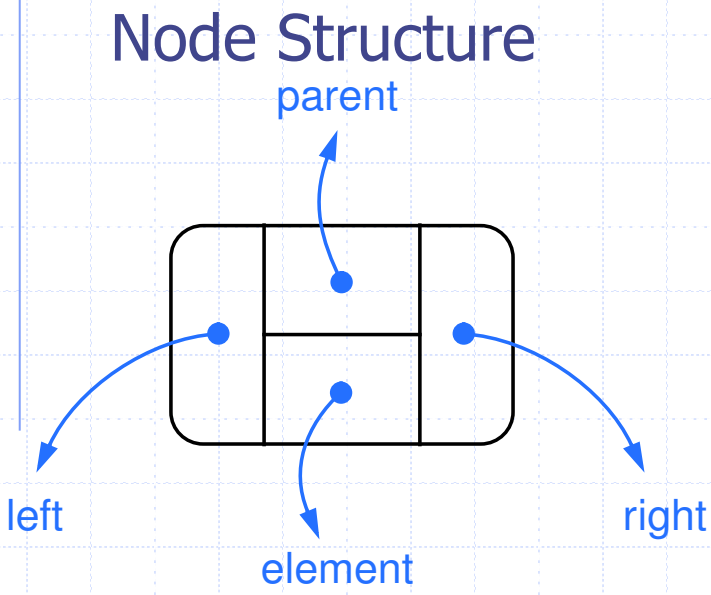


- level i has 2^i nodes
- In a tree of height h
  - leaves are at level h
  - $n_E = 2^h$
  - $n_I = 1 + 2 + 2^2 + \ldots + 2^{h-1} = 2^h - 1$
  - $n_I = n_E - 1$
  - $n = 2^{h+1} - 1$
- In a tree of n nodes
  - n_E is $(n+1)/2$
  - $h = \log_2 (n_E)$

Tree ADT

- We use positions to abstract nodes
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - Iterator iterator()
  - Iterable positions()
- Accessor methods:
  - position root()
  - position parent(p)
  - Iterable children(p)
  - Integer numChildren(p)
- Query methods:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)
- Additional update methods may be defined by data structures implementing the Tree ADT

Linked Structure for Binary Trees



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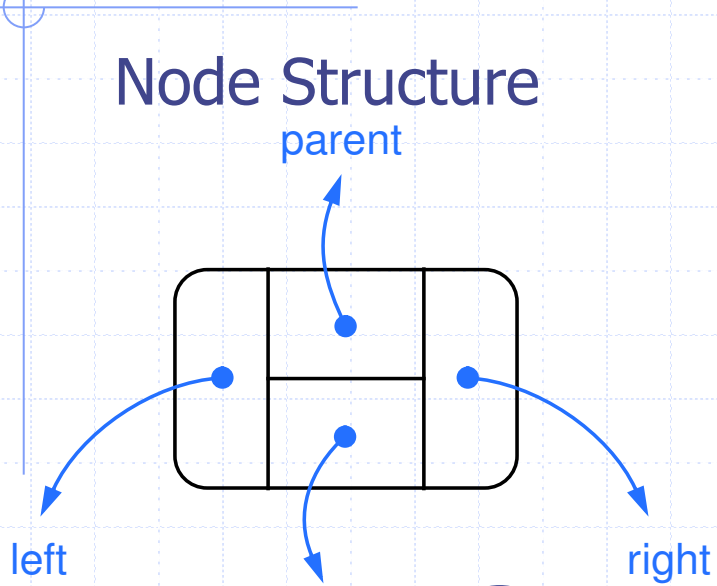
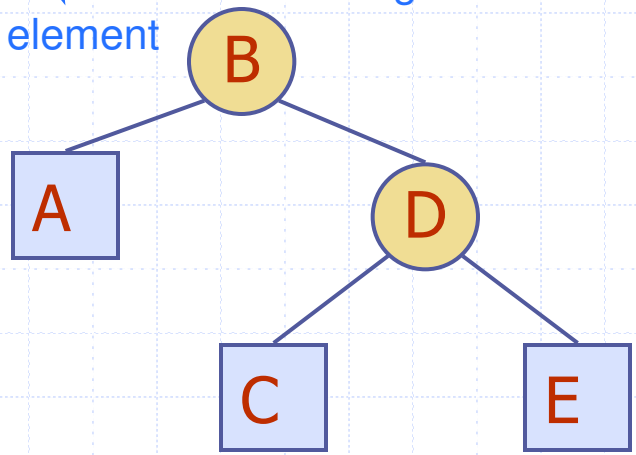
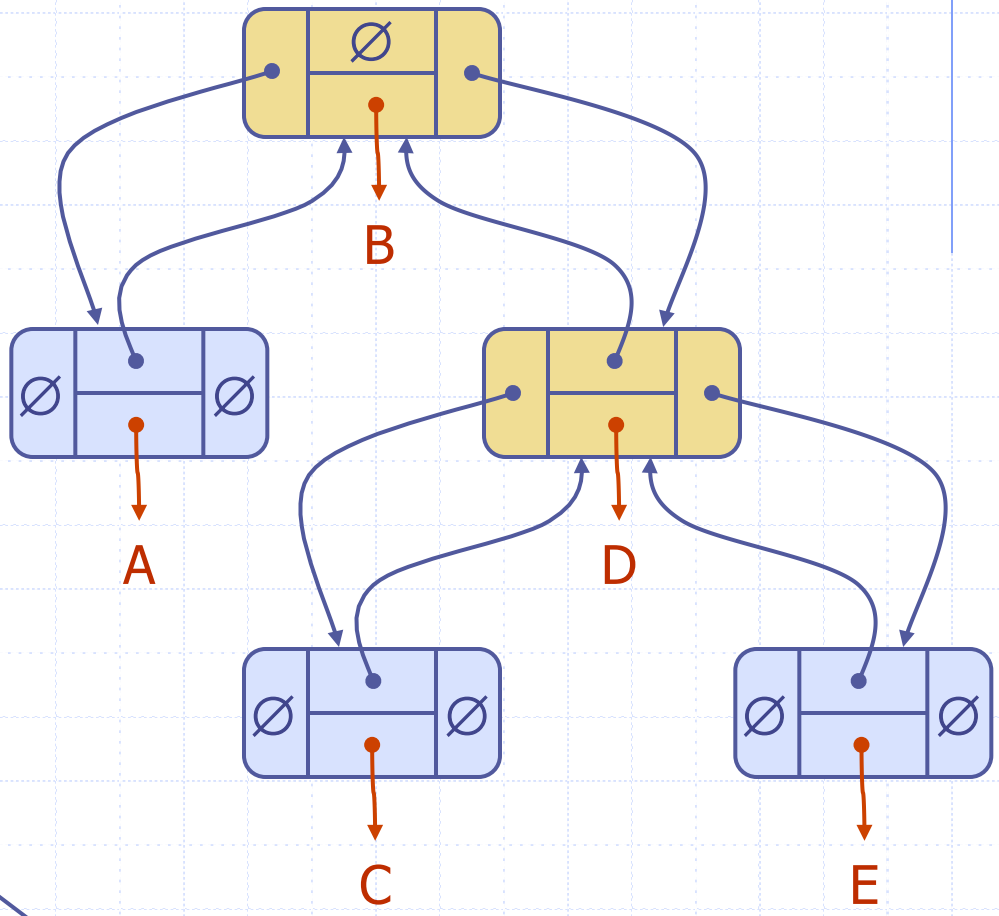


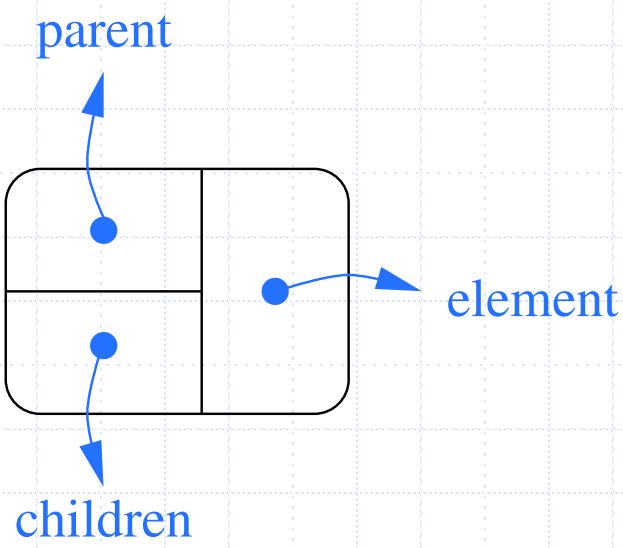




Code Fragments
8.8, 8.9, 8.10, 8.11

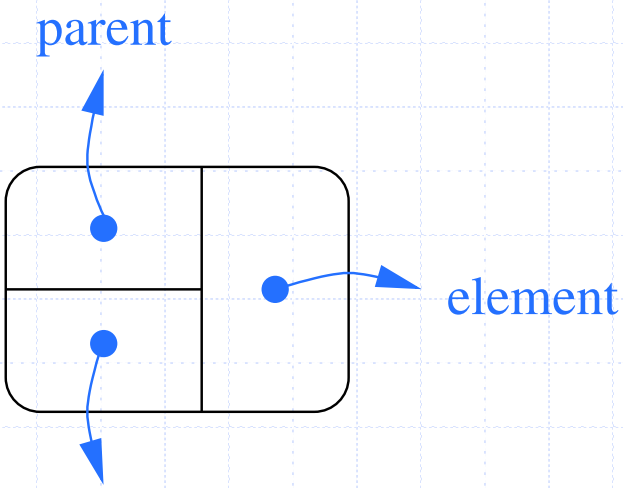
Linked Structure for General Trees

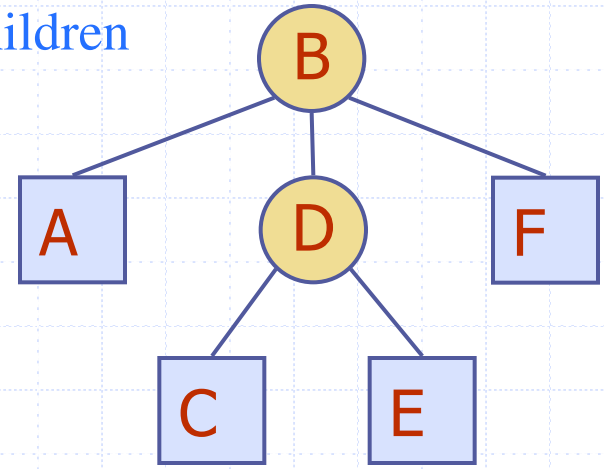
Node Structure

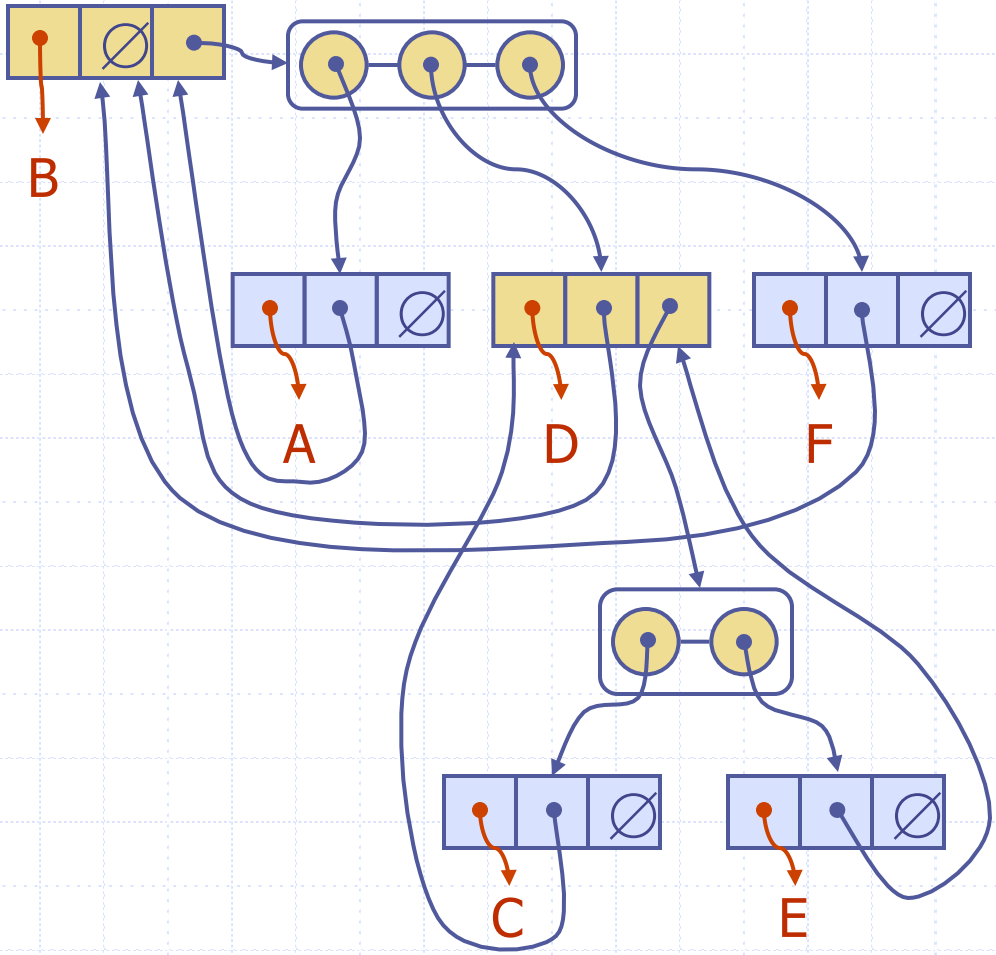


Linked Structure for General Trees

Node Structure







Computing Depth

- p be a position within the tree T
- calculate depth(p)

```
  public int depth(Position<E> p) throws
IllegalArgumentException {
      if (isRoot(p))
        return 0;
      else
        return 1 + depth(parent(p));
   }
```

Computing Height

```
 private int heightBad() { // works, but quadratic
worst-case time
    int h = 0;
    for (Position<E> p : positions())
      if (isExternal(p))// only consider leaf positions
        h = Math.max(h, depth(p));
    return h;
  }
```

Analysis:

$O(n + \Sigma_{p \varepsilon L}(d_p+1))$ is $O(n^2)$

Computing Height

```
 public int height(Position<E> p) throws
IllegalArgumentException {
    int h = 0;              // base case if p is external
    for (Position<E> c : children(p))
      h = Math.max(h, 1 + height(c));
    return h;
  }
```

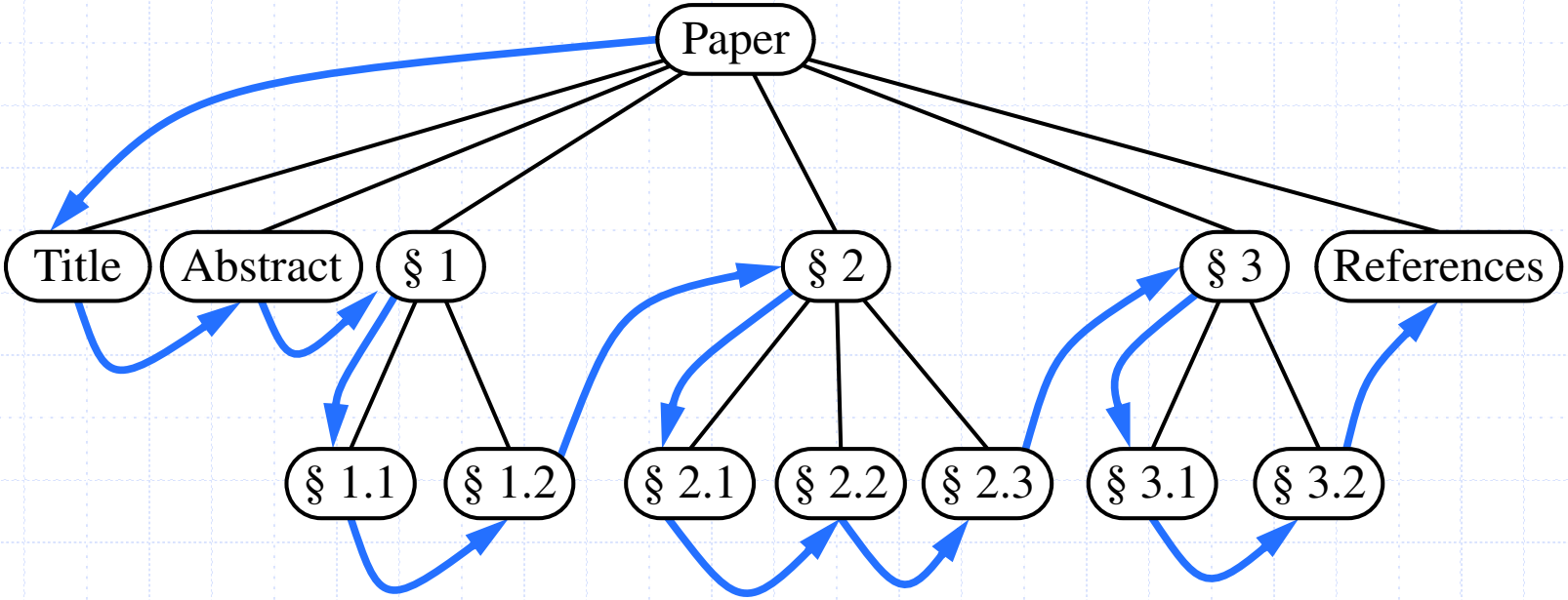
Analysis

$O(\Sigma_p(c_p+1))$ is $O(n)$

Tree Traversals

- Systematic way of visiting all nodes in a tree in a specified order
  - preorder - processes each node before processing its children
  - postorder - processes each node after processing its children

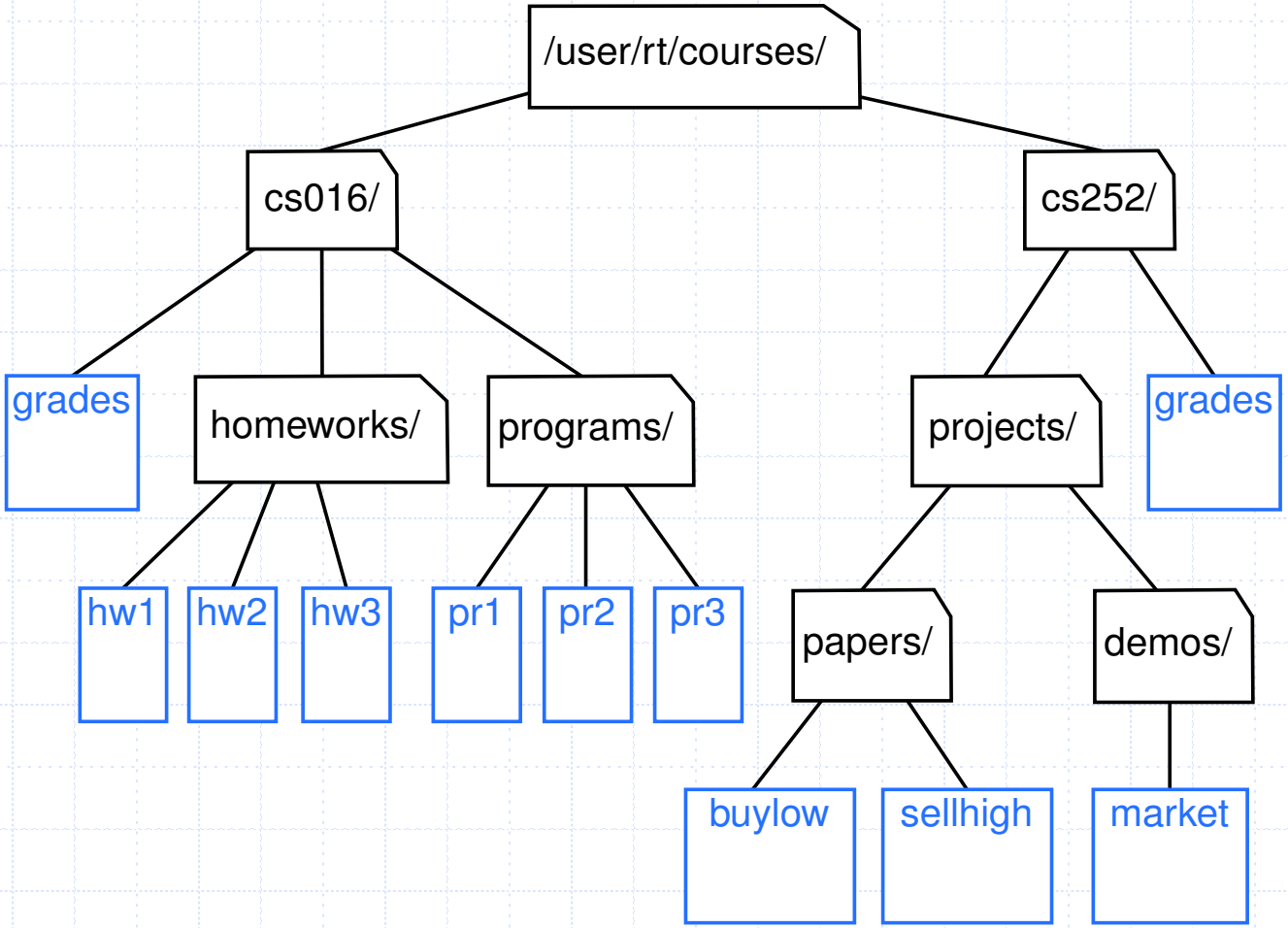
Preorder Traversal



Preorder Traversal - Algorithm

- Algorithm preorder(p)
  - perform the "visit" action for position p
  - for each child c in children(p) do
    - preorder(c)
- Example:
  - reading a document from beginning to end

Postorder Traversal



Postorder Traversal - Algorithm

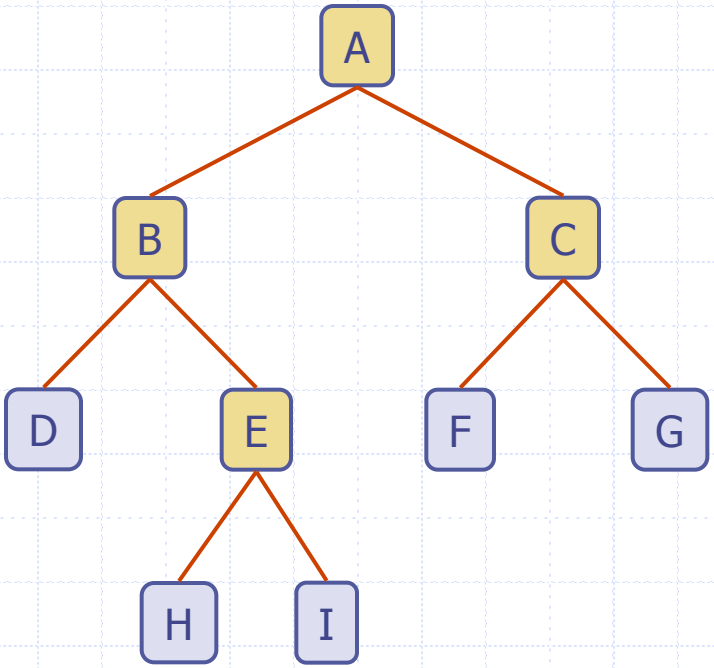
- Algorithm postorder(p)
  - for each child c in children(p) do
    - postorder(c)
  - perform the "visit" action for position p
- Example
  - du - disk usage command in Unix

Traversals of Binary Trees

- preorder(v)
  - visit(v)
  - preorder(v.leftchild())
  - preorder(v.rightchild())
- postorder(v)
  - postorder(v.leftchild())
  - postorder(v.rightchild())
  - visit(v)

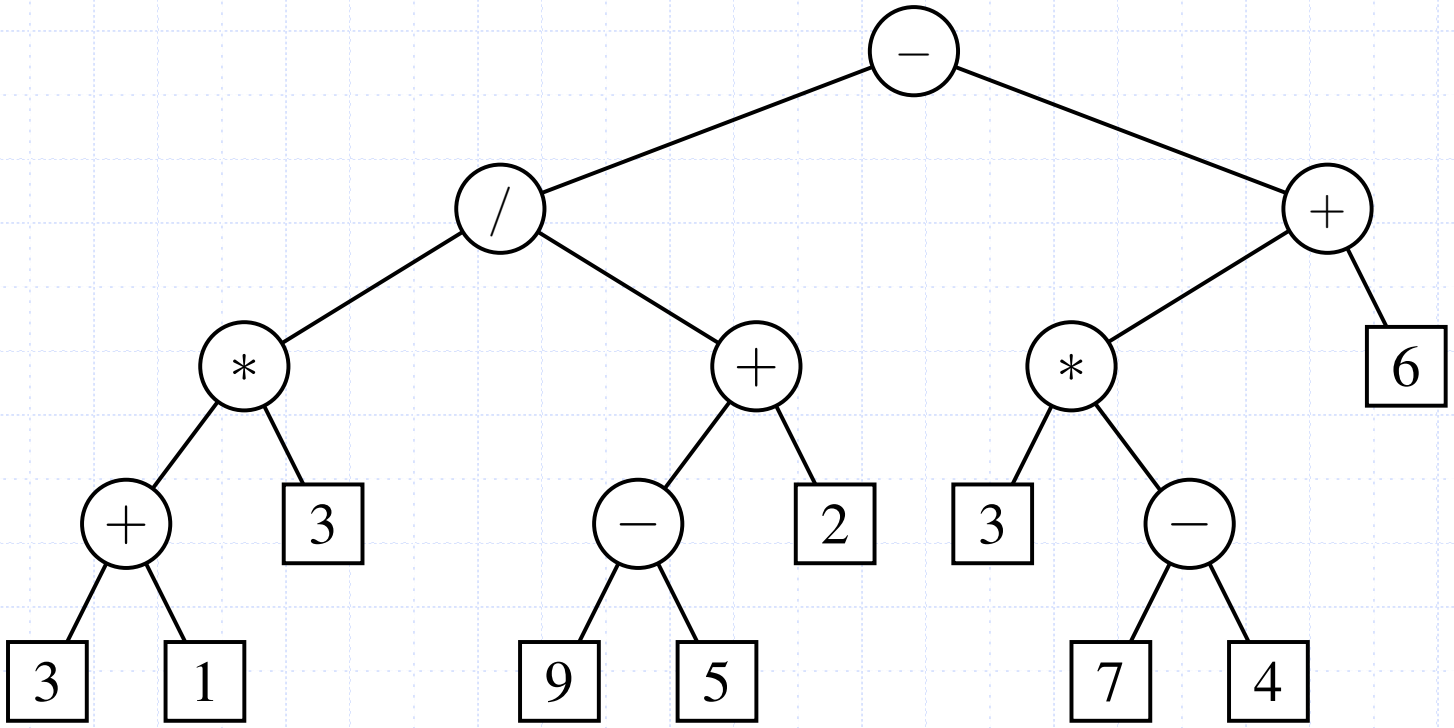
More Example of Traversals

- Visit - printing the data in the node
- Preorder traversal
  - a b d e h i c f g
- Postorder traversal
  - d h i e b f g c a



Application of Postorder Traversal

- Evaluating Arithmetic Expressions



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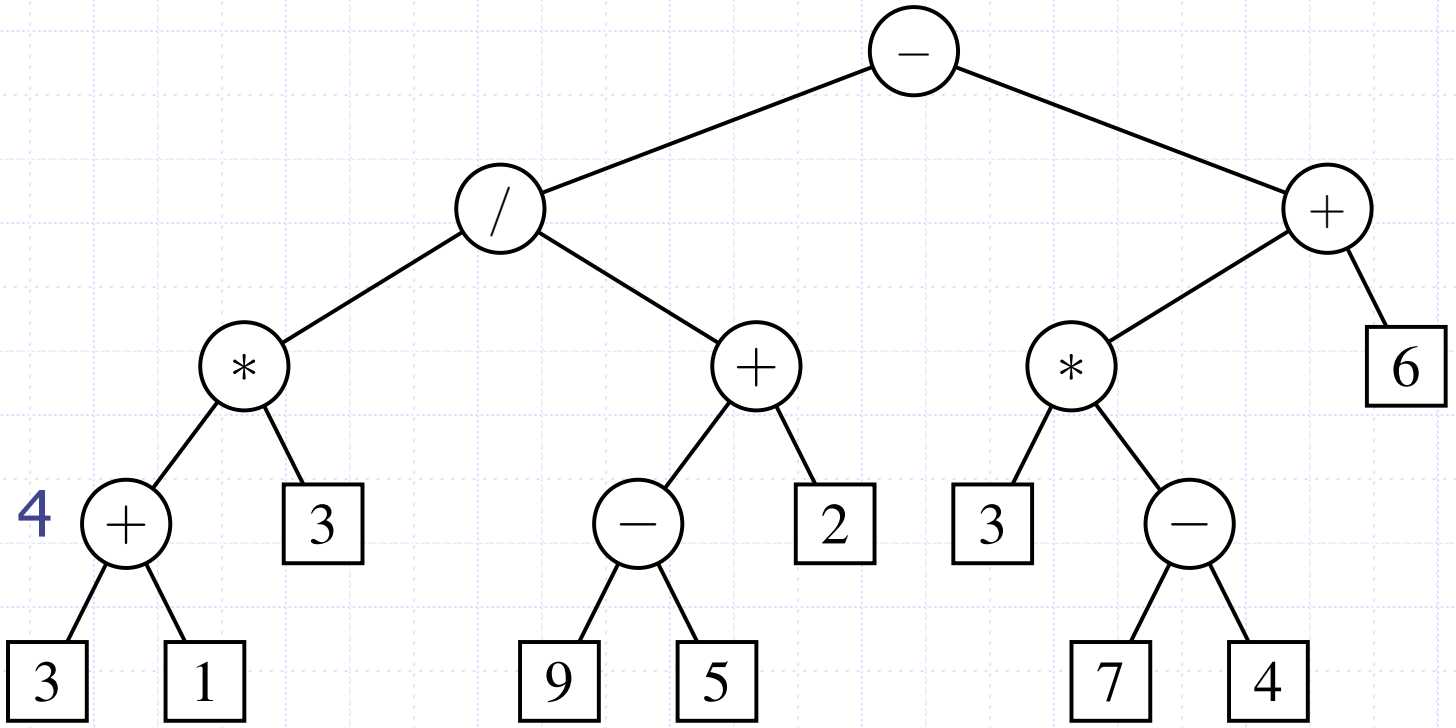

Application of Postorder Traversal

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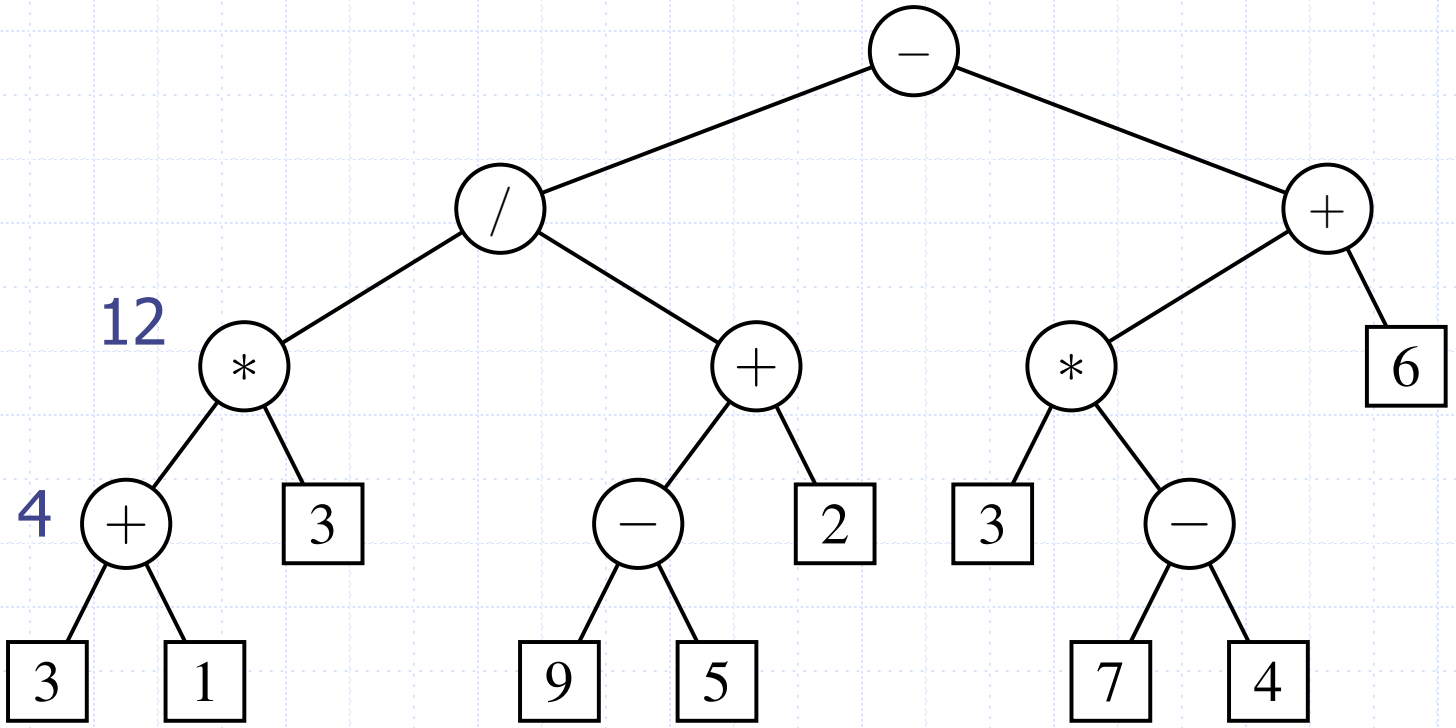

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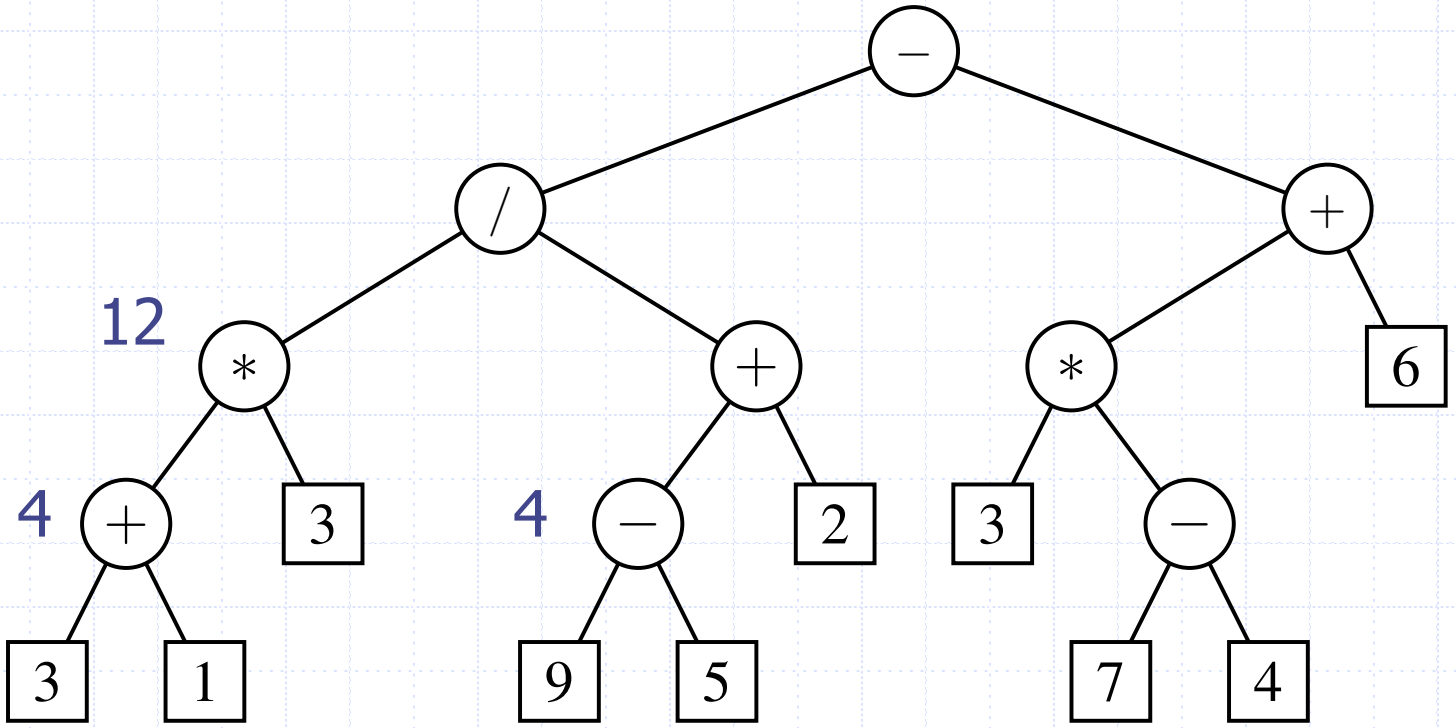
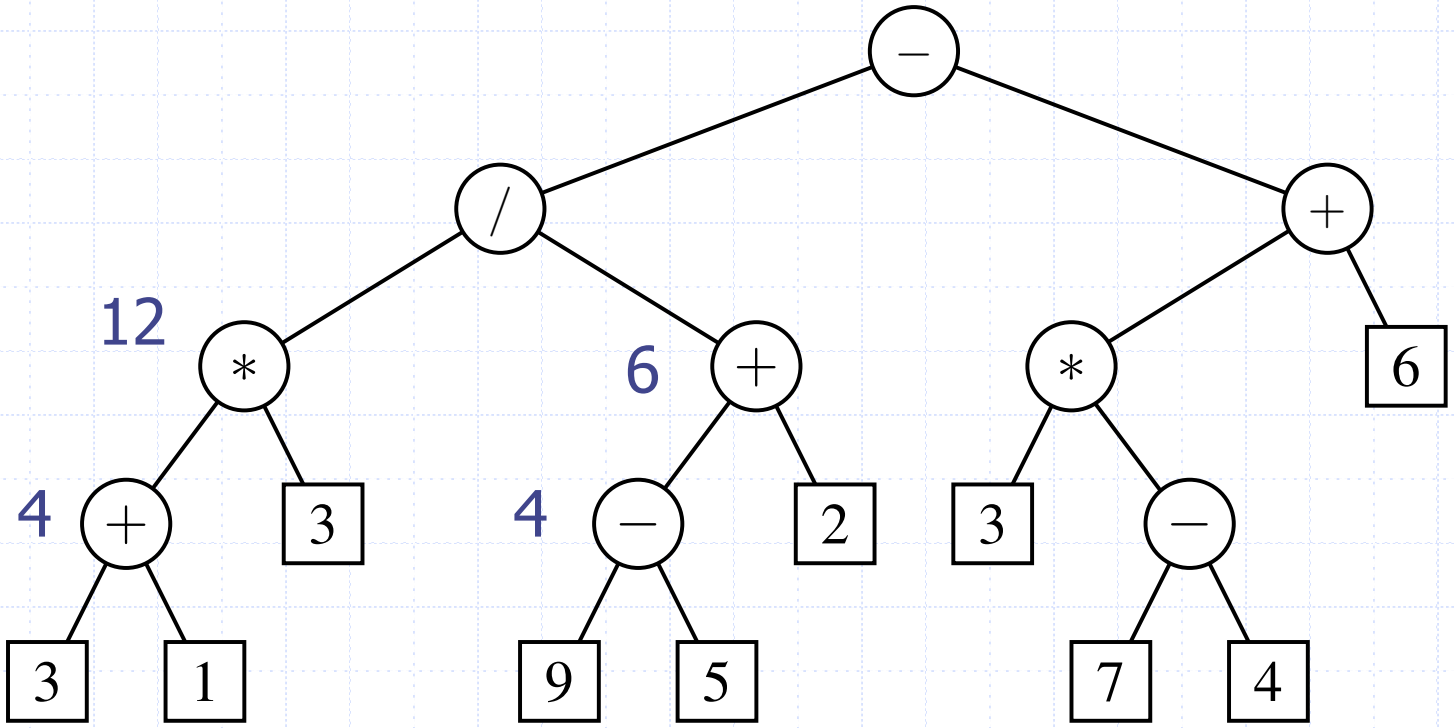


Application of Postorder Traversal

- Evaluating Arithmetic Expressions



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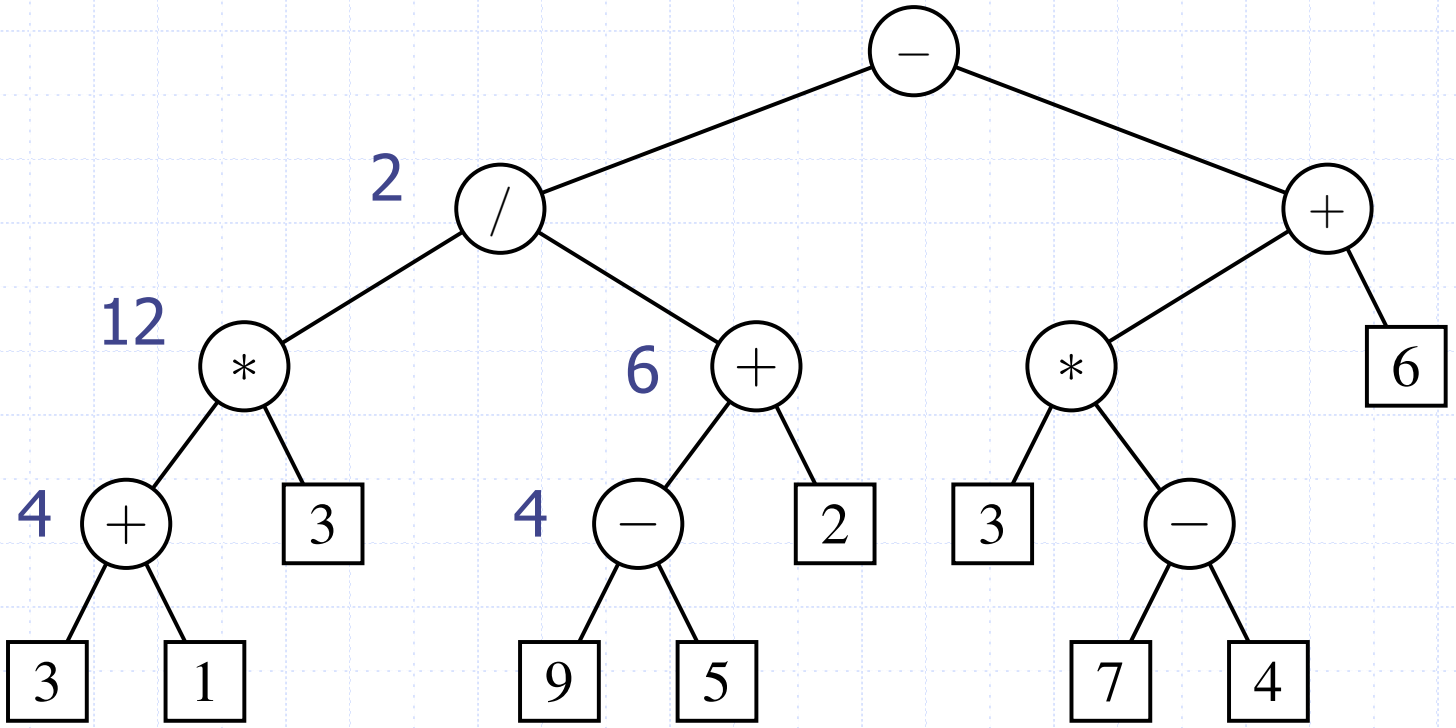

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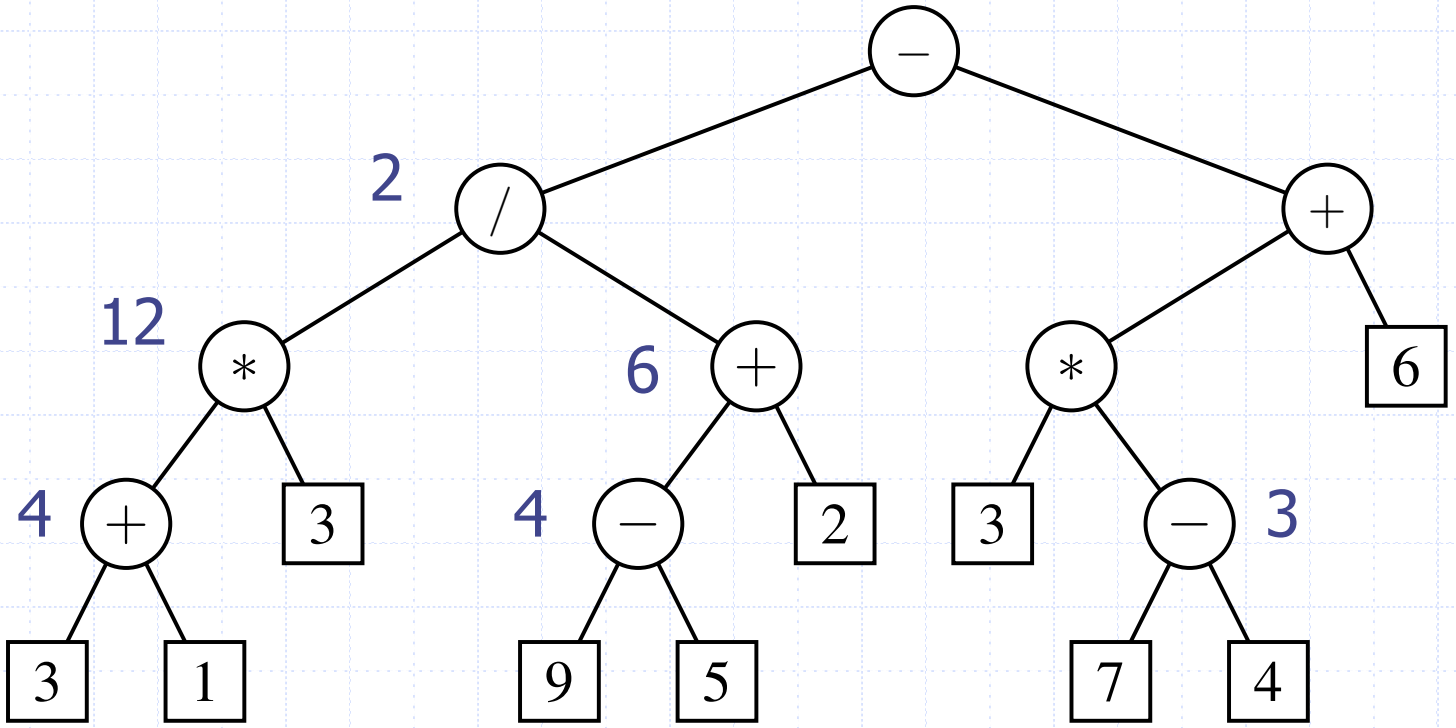

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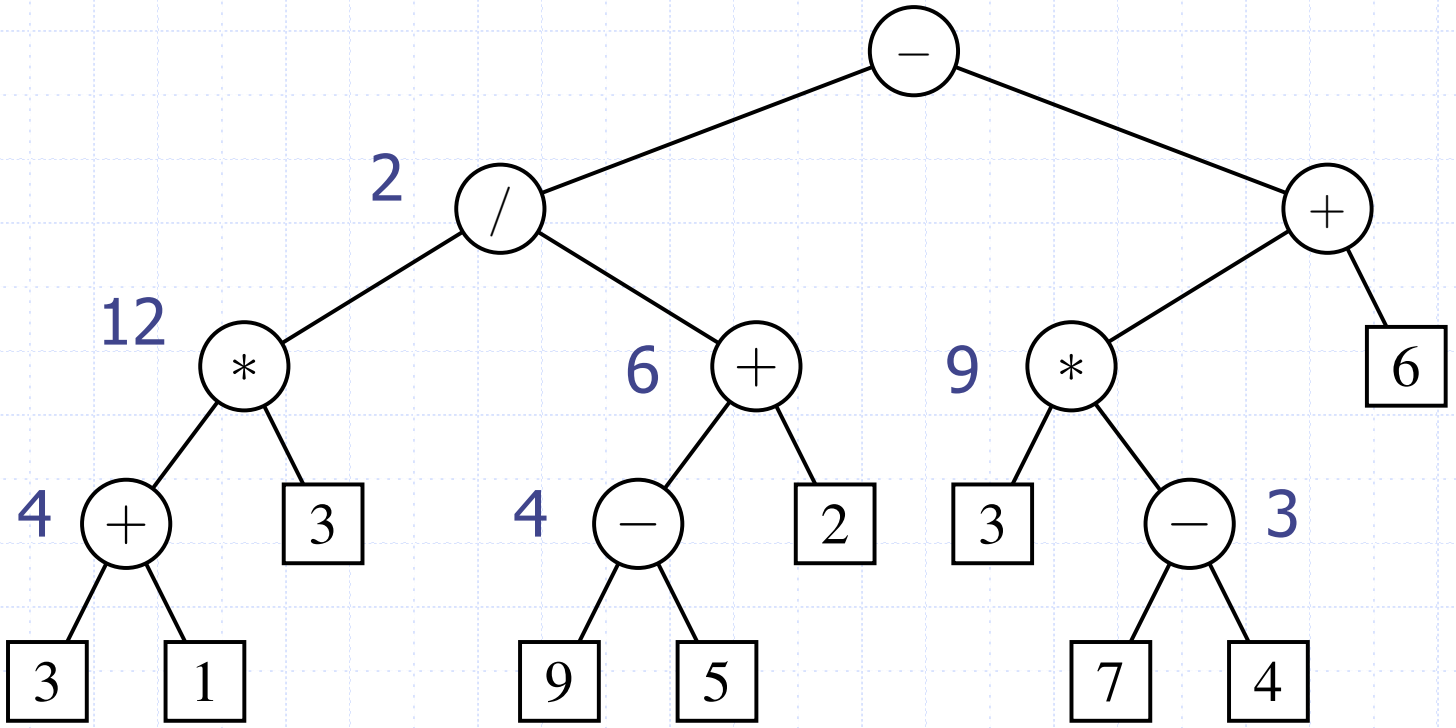

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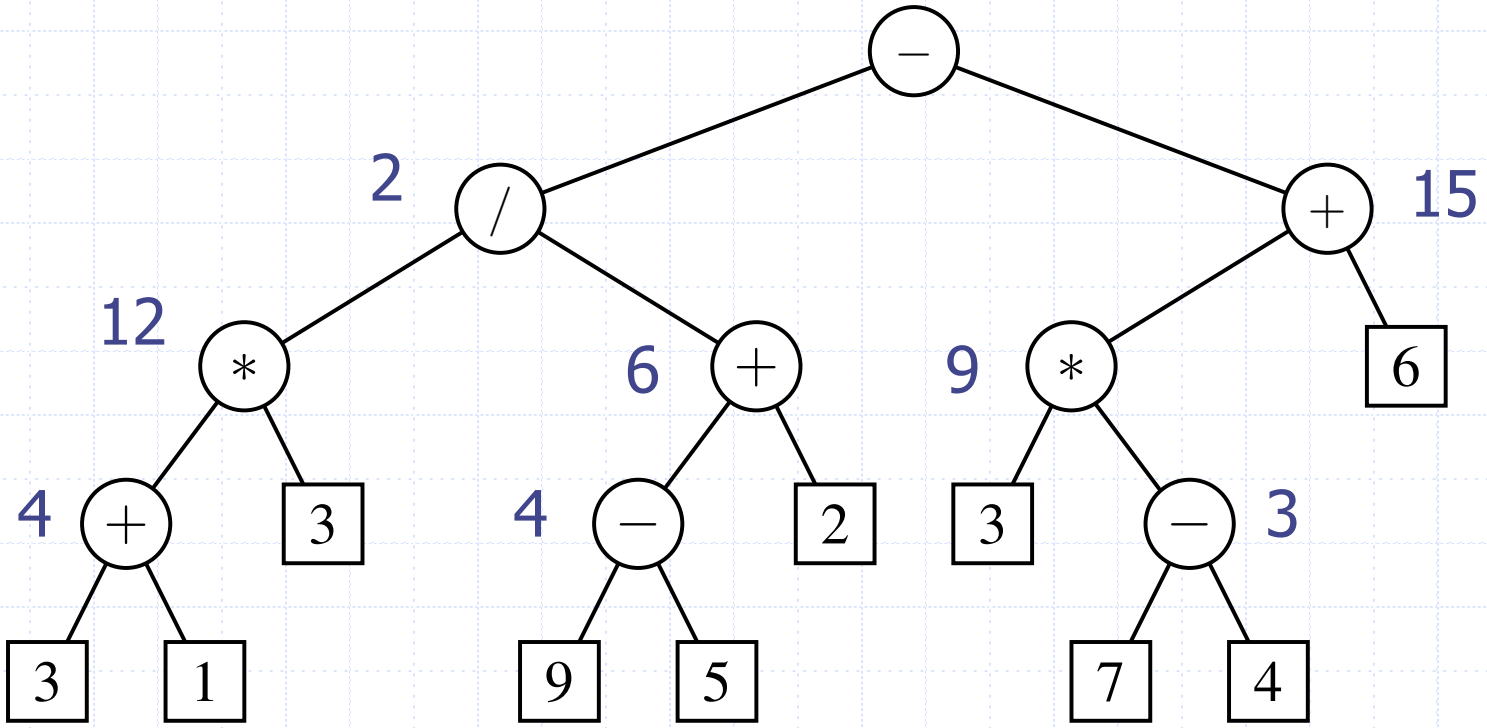

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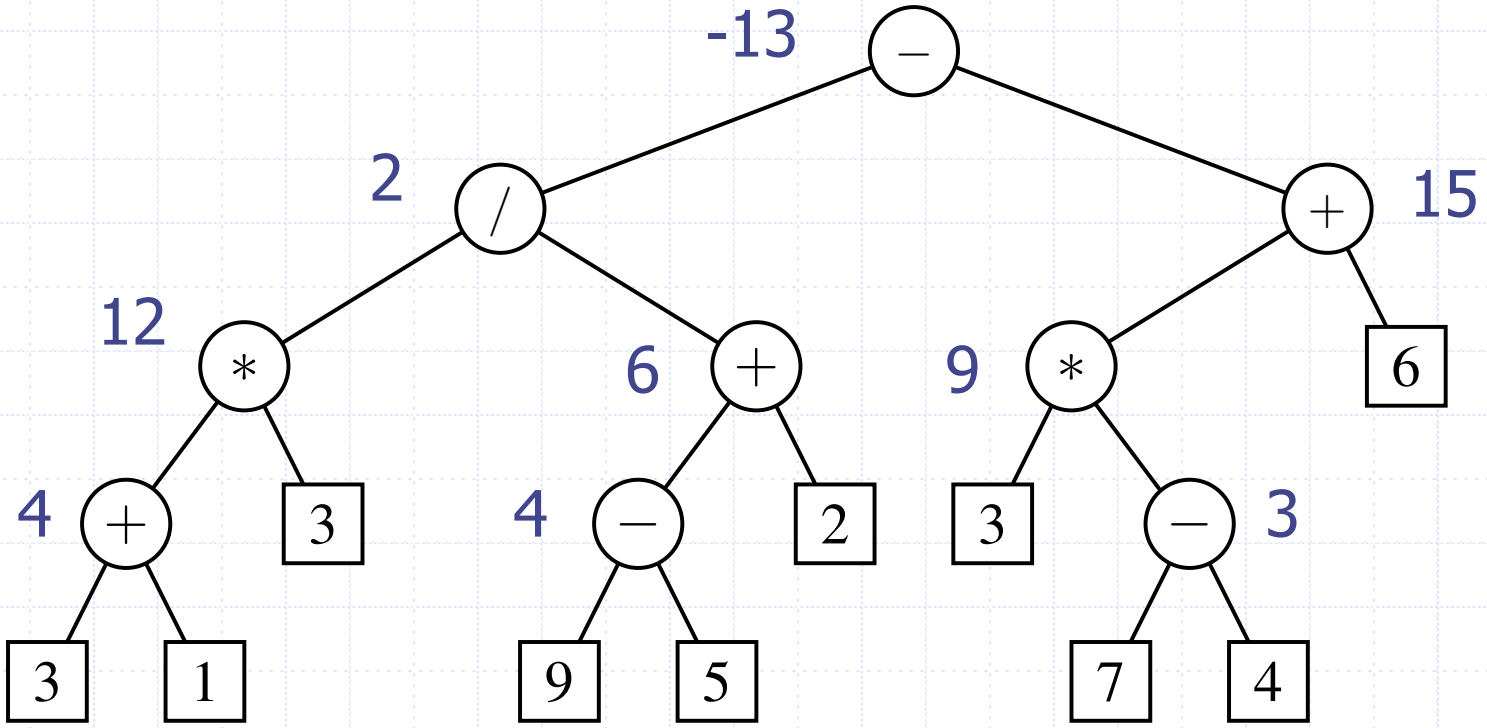
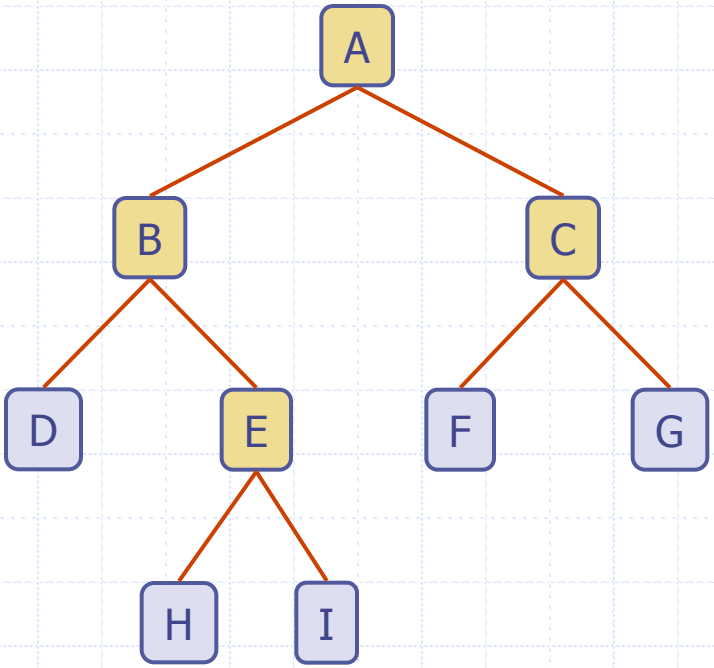


Inorder traversals

- Visit the node between the visit to the left and right subtree
- Algorithm inorder(p)
  - If p has a left child lc then
    - inorder(lc)
  - perform "visit" action for position p
  - If p has a right child rc then
    - inorder(rc)

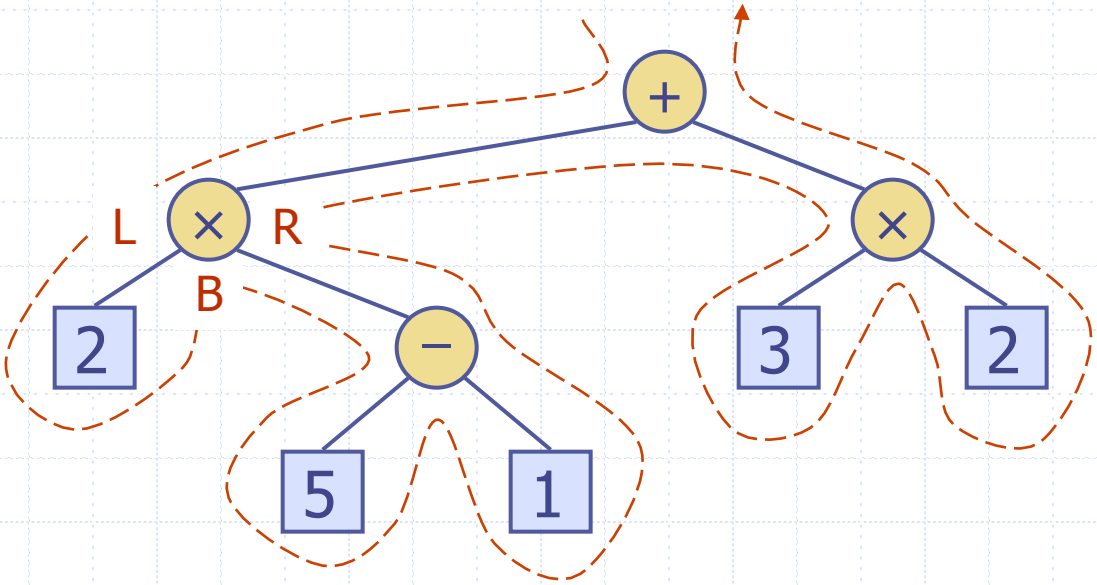
Example - Inorder Traversal

- Inorder
  - d b h e i a f c g



Euler Tour Traversal

- Generic traversal of a binary tree
- Includes as special cases the preorder, postorder and inorder traversals
- Walk around the tree and visit each node three times:
  - on the left (preorder)
  - from below (inorder)
  - on the right (postorder)



Building Tree from Preorder Traversal

- Given the preorder traversal, can we uniquely determine the binary tree?

Preorder

a b d e h i c f g

Building Tree from Postorder Traversal

- Given the postorder traversal, can we uniquely determine the binary tree?

Postorder

d h i e b f g c a

Building Tree from Pre- and In-Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder	Inorder
a b d e h i c f g	d b h e i a f c g

Building Tree from Pre- and In-Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a b d e h i c f g

Inorder

d b h e i a f c g

A

Building Tree from Pre- and In-Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a	b	d	e	h	i	c	f	g

Inorder

d	b	h	e	i	a	f	c	g

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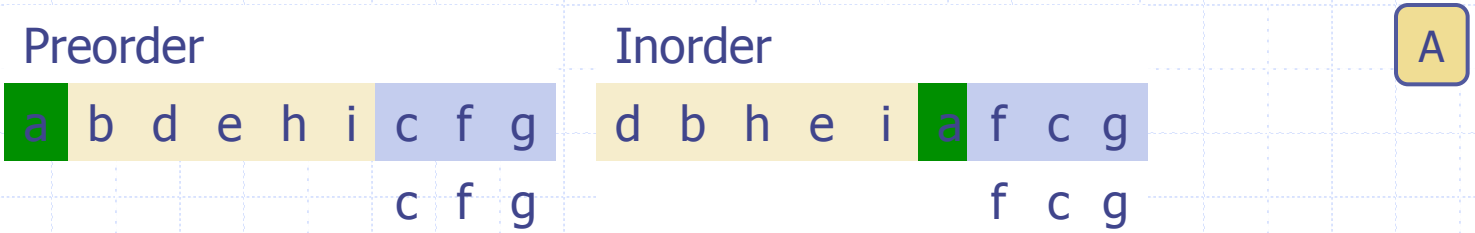

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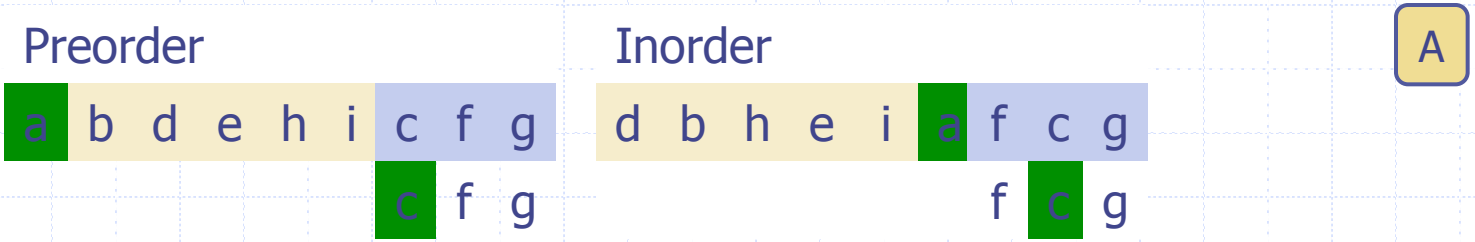

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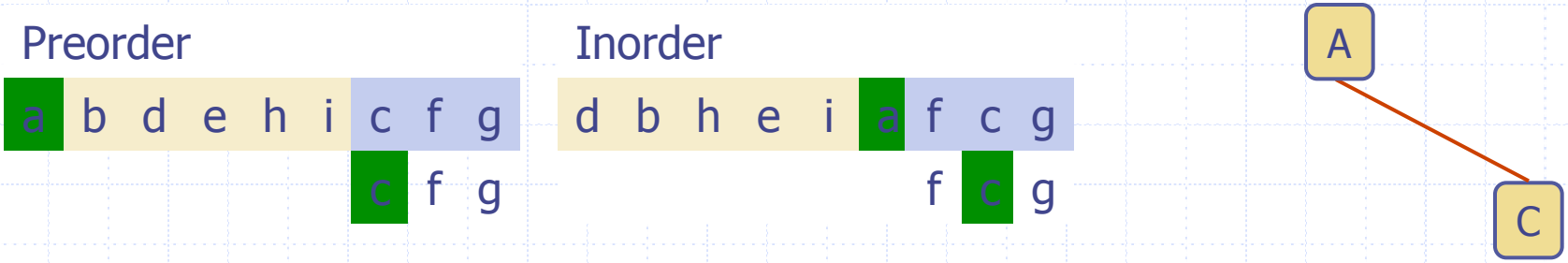

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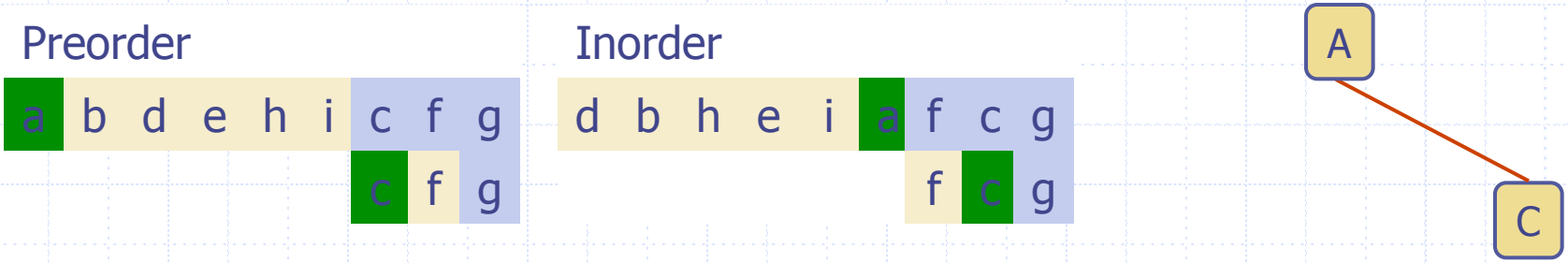

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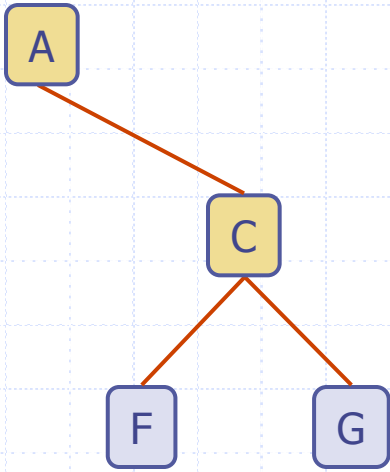
Preorder

a	b	d	e	h	i	c	f	g
						c	f	g

Inorder

d	b	h	e	i	a	f	c	g
						f	c	g



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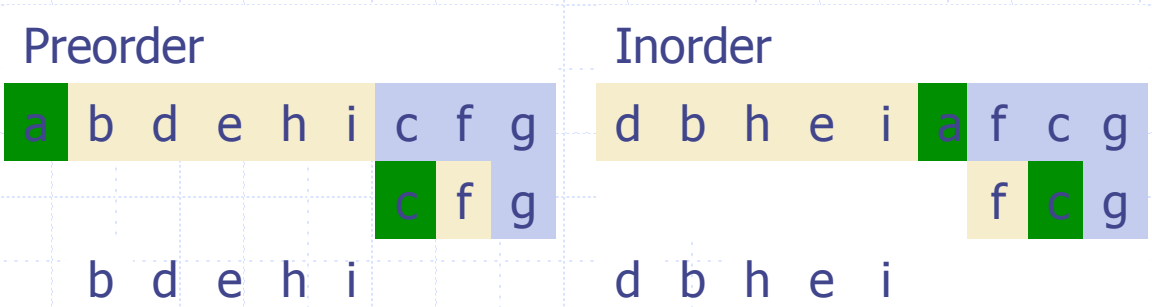

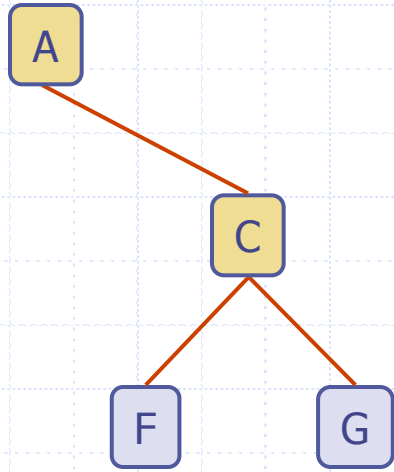

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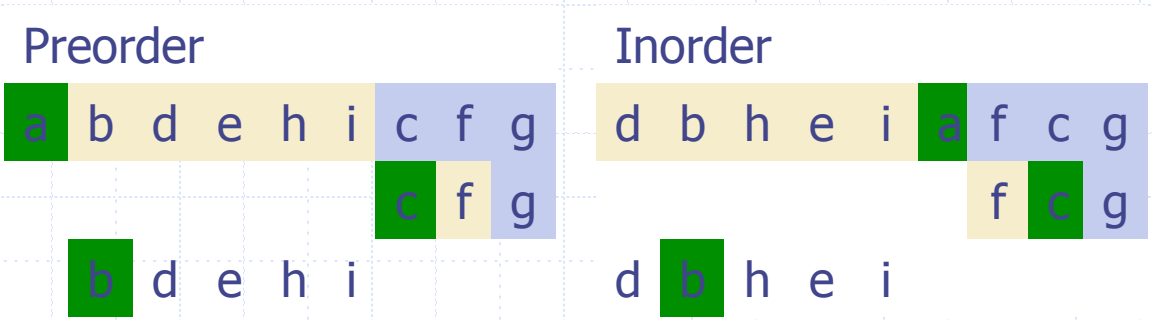

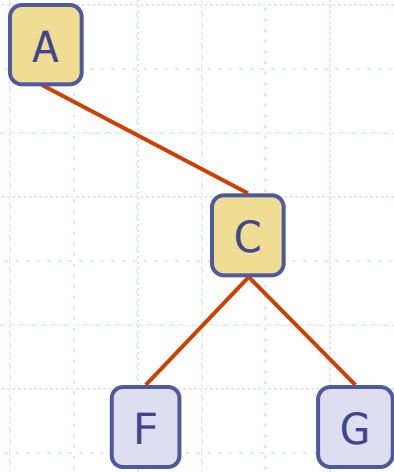

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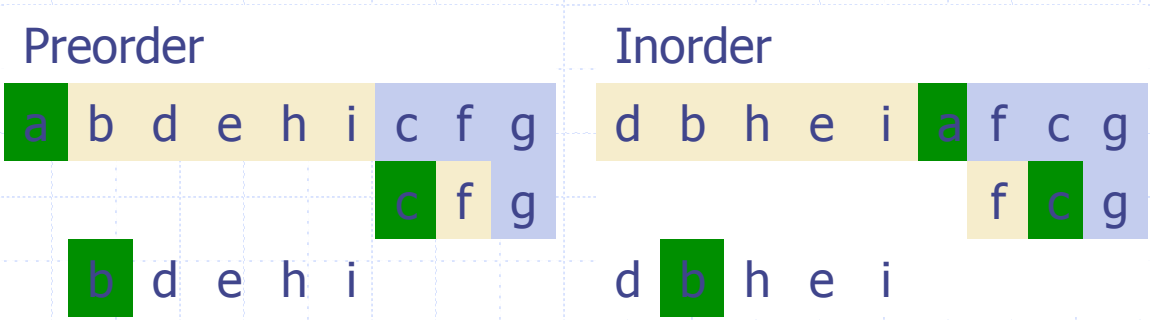

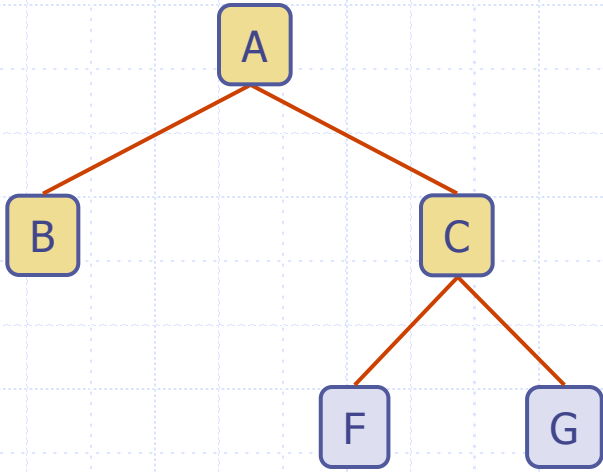

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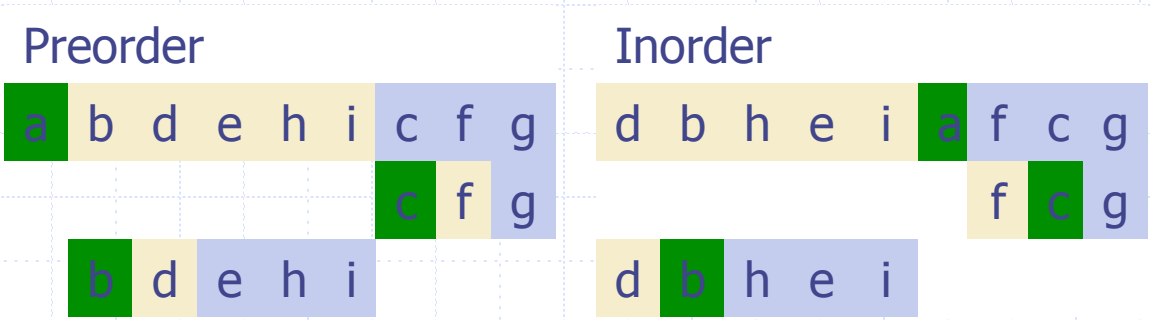

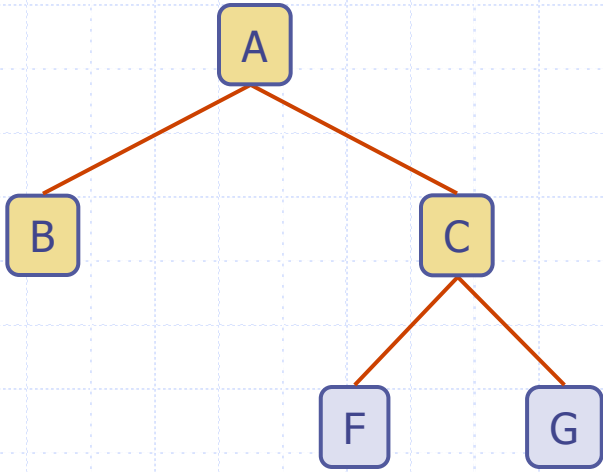

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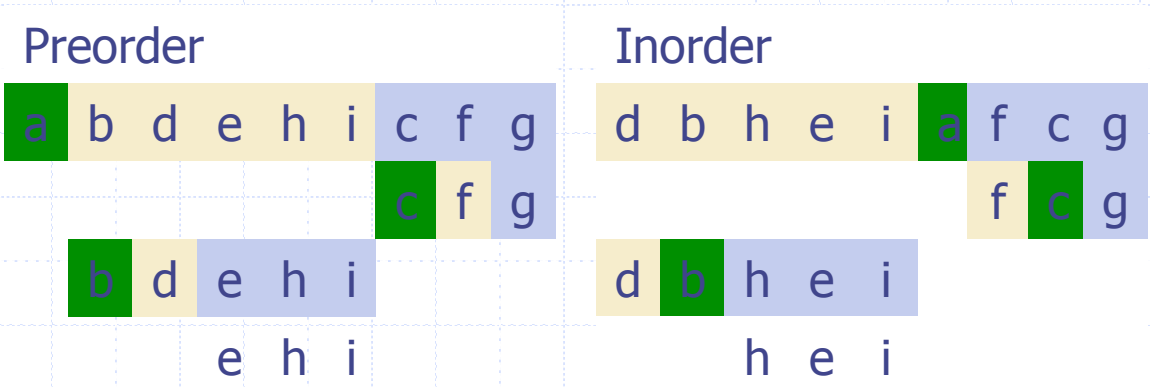

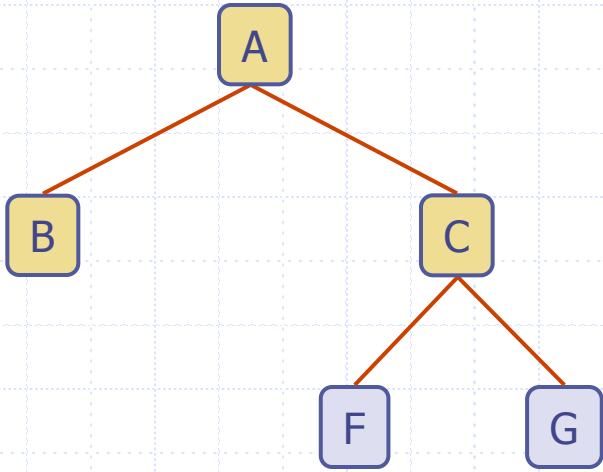

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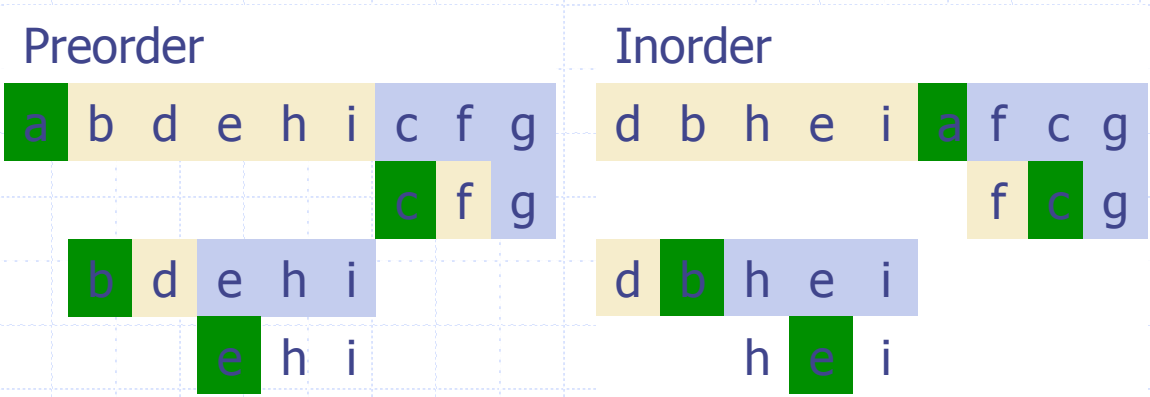

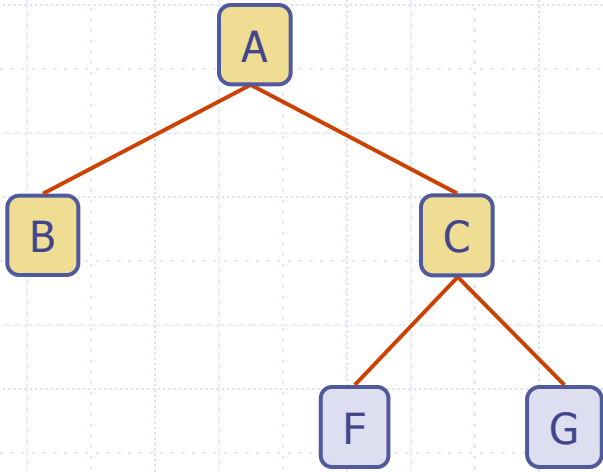

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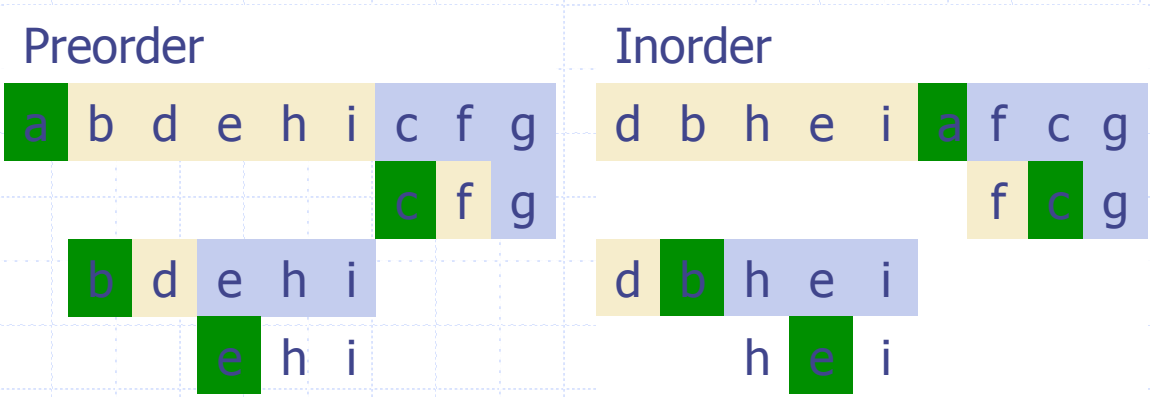

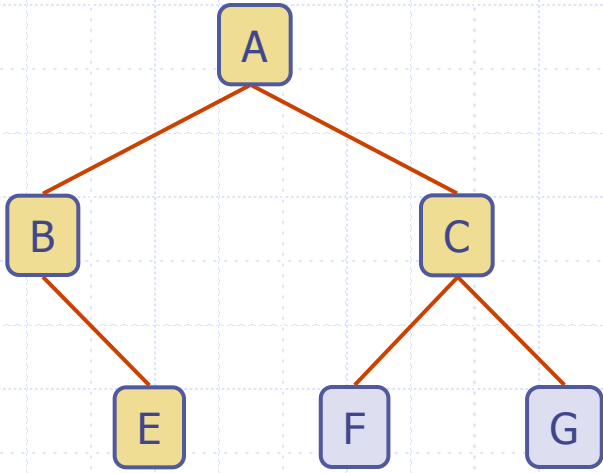

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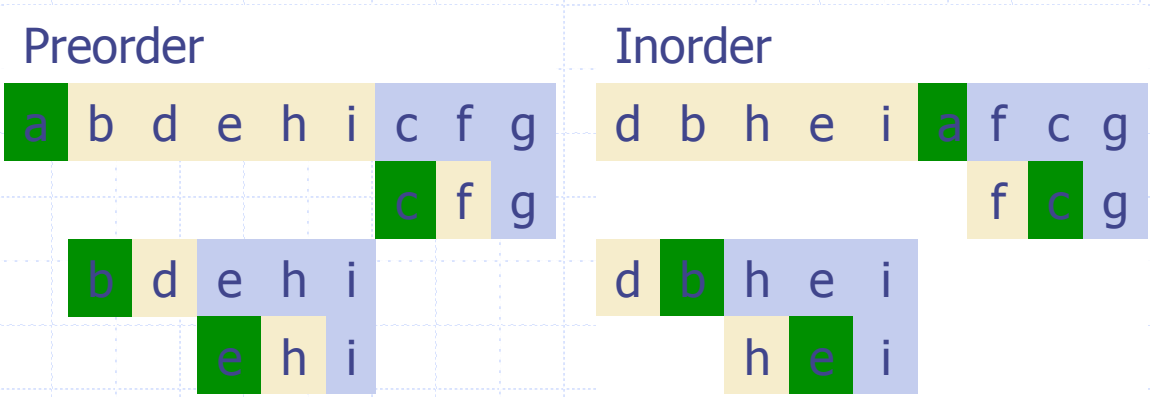

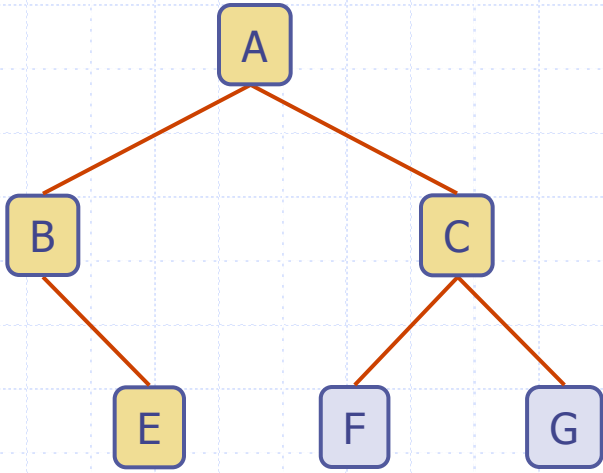

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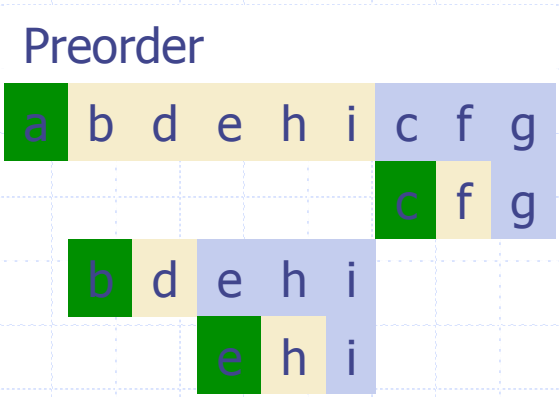

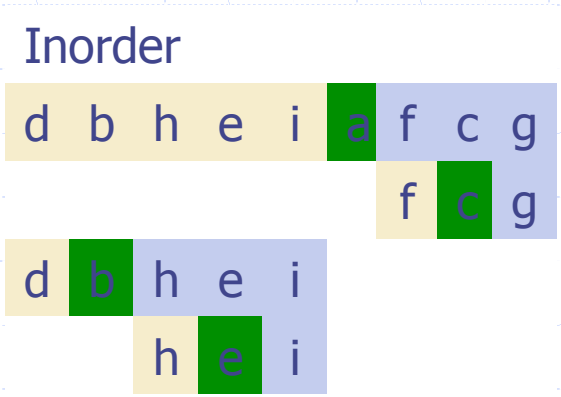

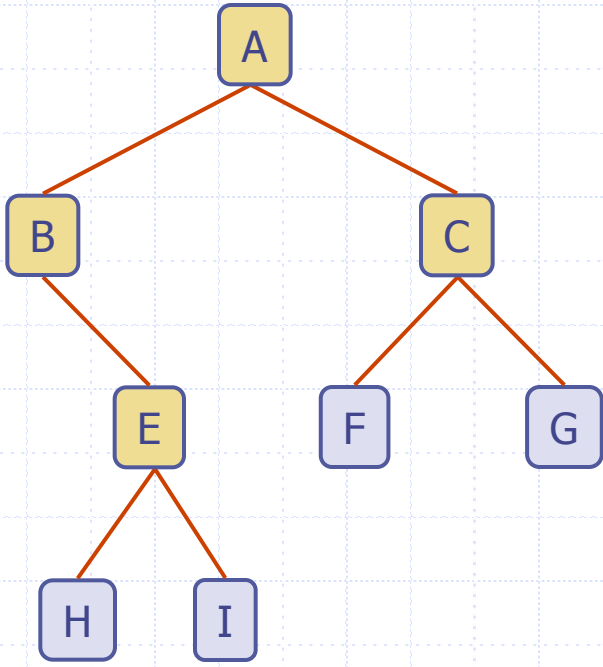

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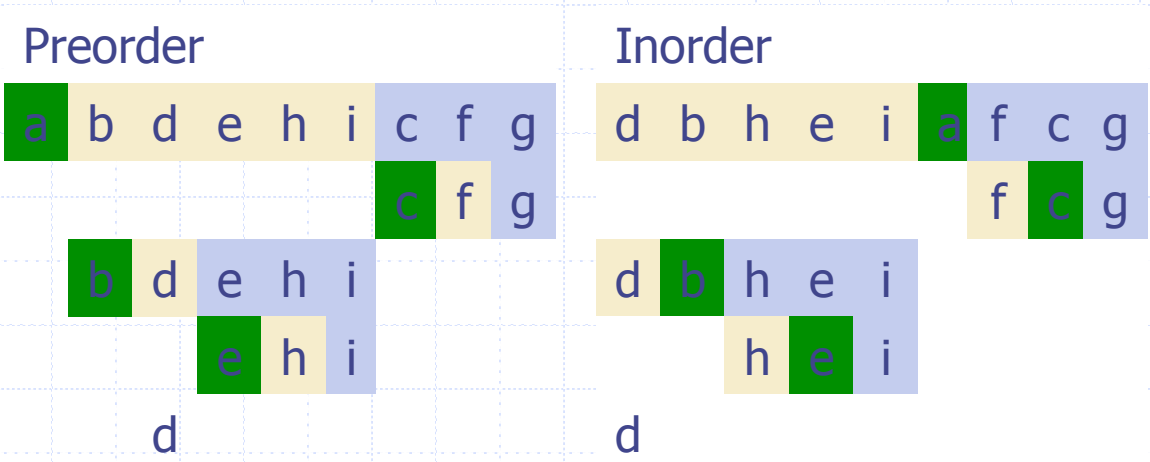

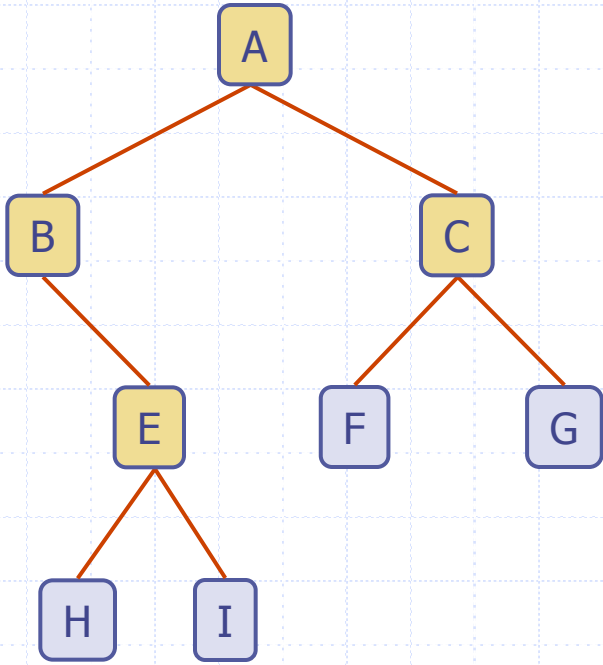

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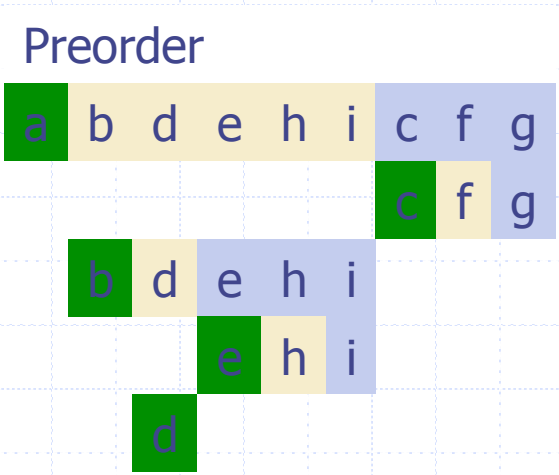

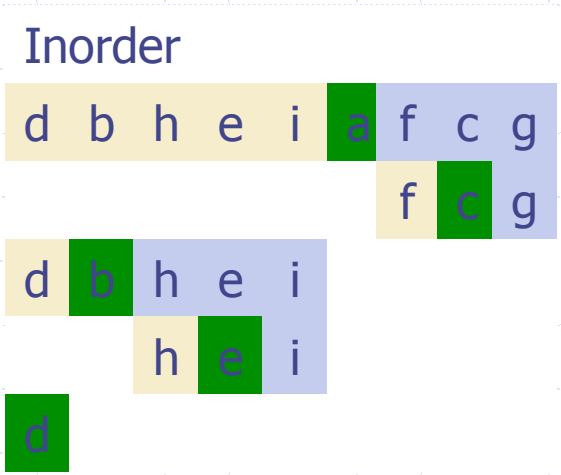

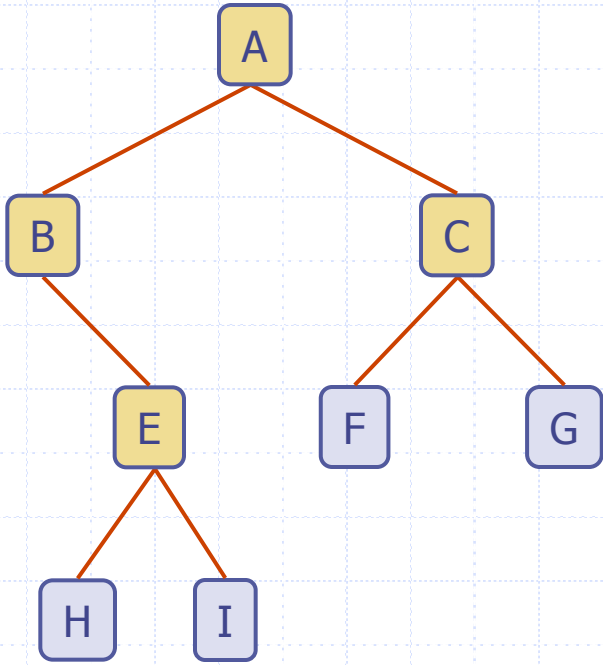

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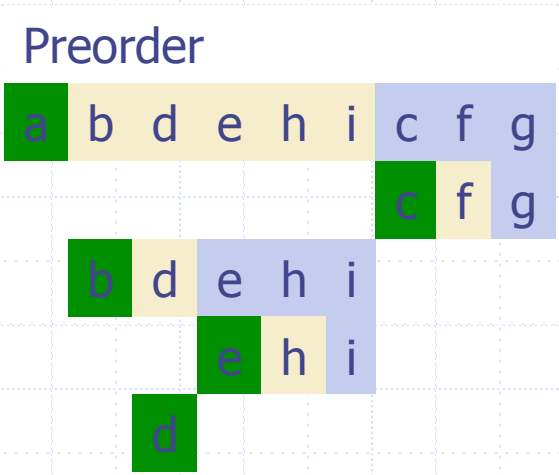

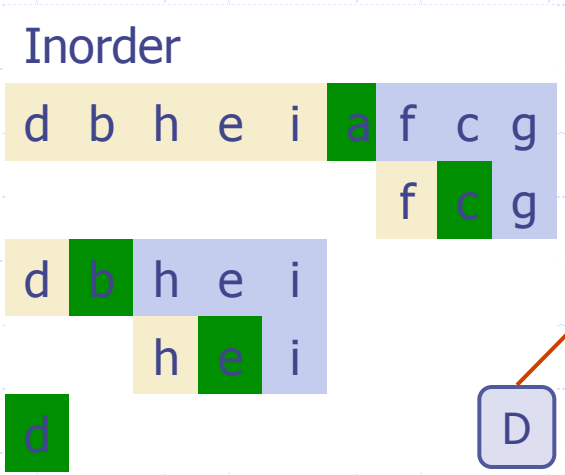

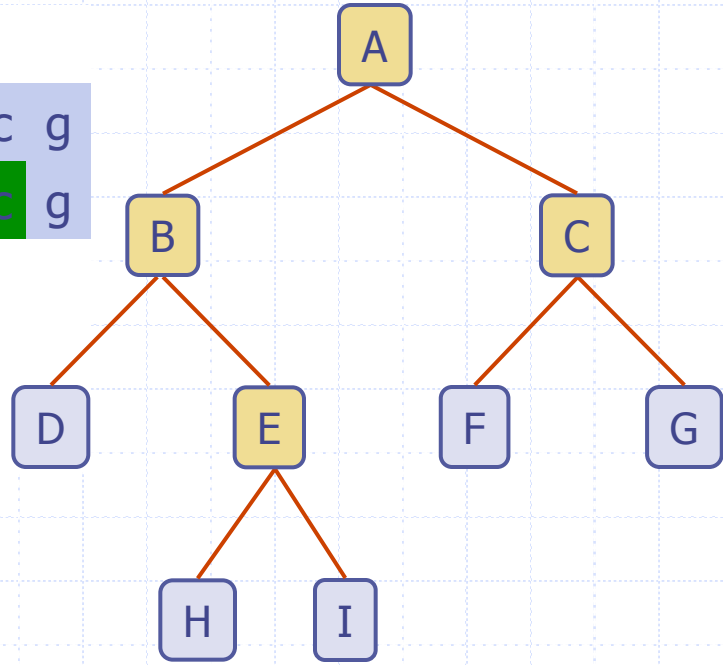

Building Tree from Post- and In-Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
- The last node visited in the postorder traversal is the root of the binary tree

Postorder	Inorder
d h i e b f g c a	d b h e i a f c g

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Postorder

d h i e b f g c a

Inorder

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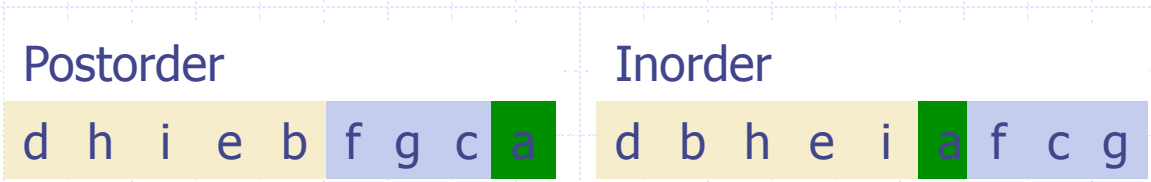

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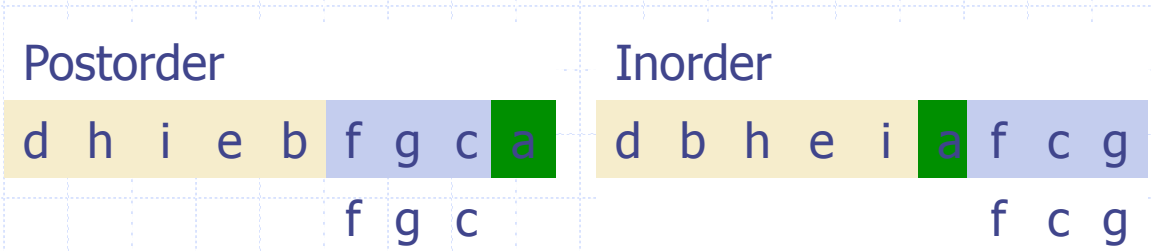

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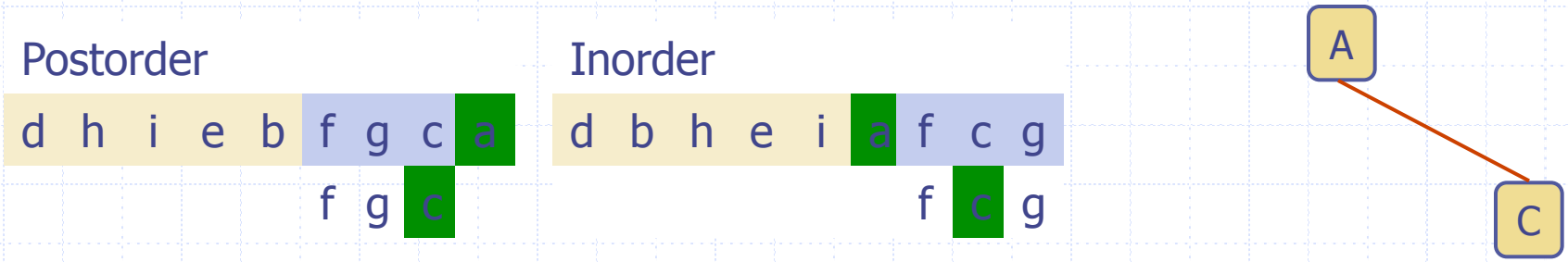

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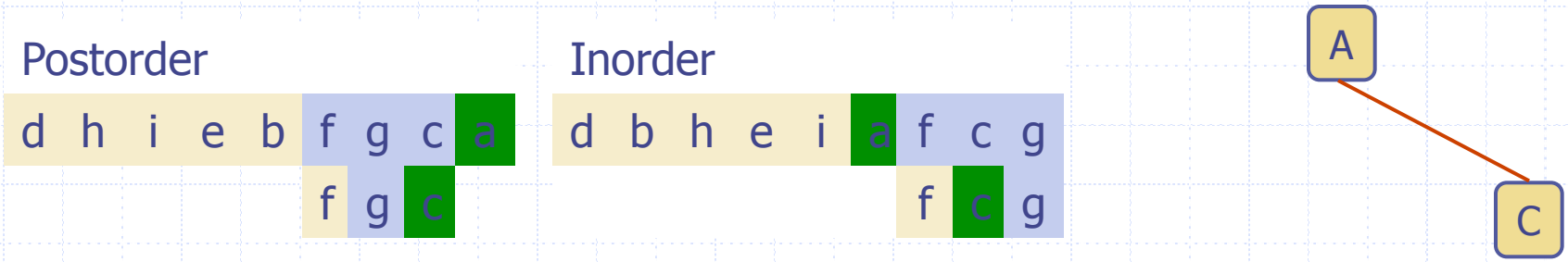


Building Tree from Post- and In-Order Traversals

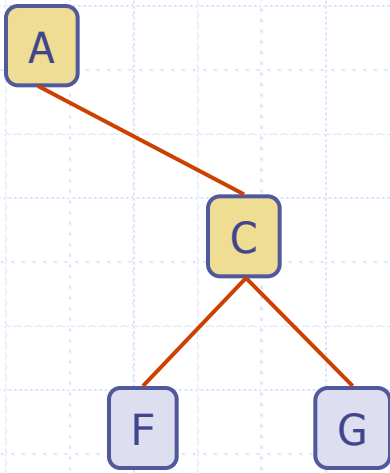
- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
- The last node visited in the postorder traversal is the root of the binary tree

Postorder

d	h	i	e	b	f	g	c	a
					f	g	c	

Inorder

d	b	h	e	i	a	f	c	g
						f	c	g



Building Tree from Post- and In-Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
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Postorder

d h i e b f g c a

f g c

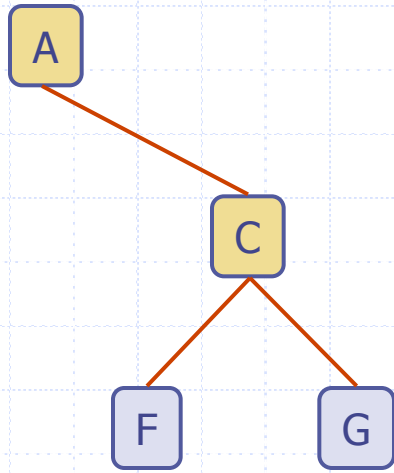
d h i e b

Inorder

d b h e i a f c g

f c g

d b h e i



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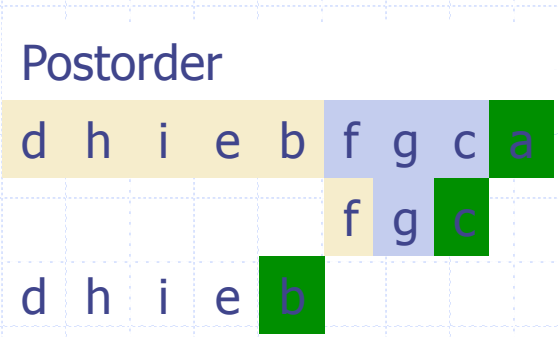
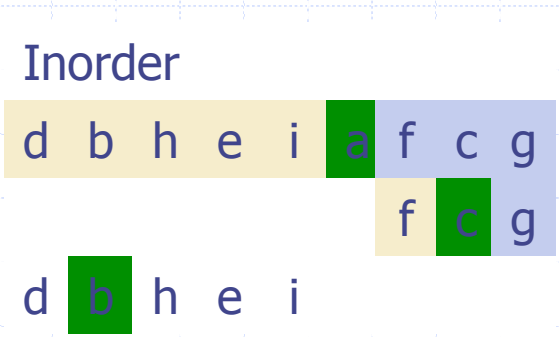
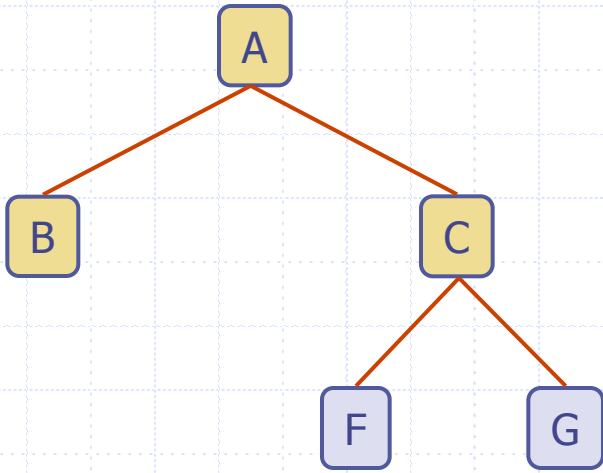






Building Tree from Post- and In-Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
- The last node visited in the postorder traversal is the root of the binary tree

Postorder

d h i e b f g c a

f g c

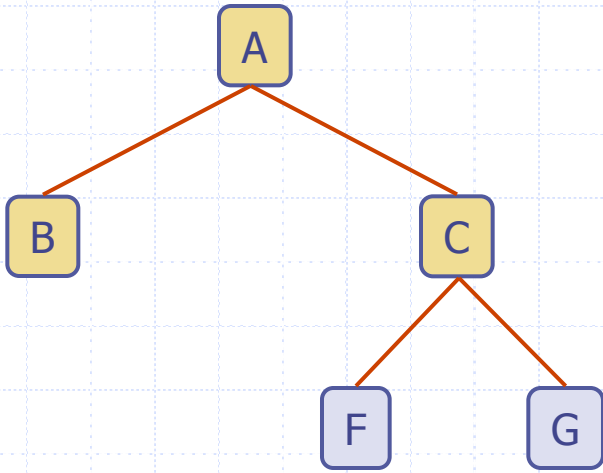
d h i e b

Inorder

d b h e i a f c g

f c g

d b h e i



- and so on..

Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder	Postorder
a b d e h i c f g	d h i e b f g c a

Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b d e h i c f g

Postorder

d h i e b f g c a

A

Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b d e h i c f g

Postorder

d h i e b f g c a

A

Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

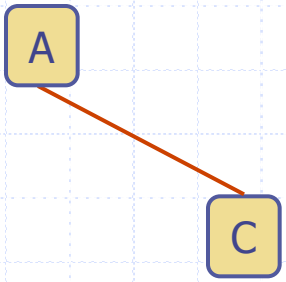
a b d e h i c f g

c f g

Postorder

d h i e b f g c a

f g c



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Preorder

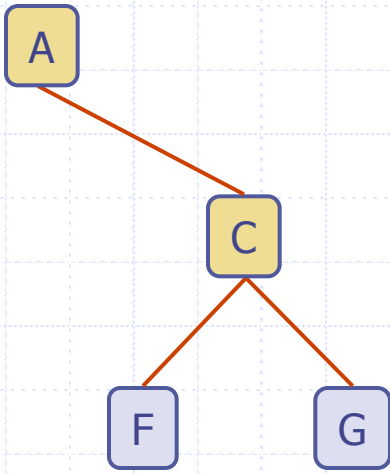
a b d e h i c f g

c f g

Postorder

d h i e b f g c a

f g c



Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b d e h i c f g

c f g

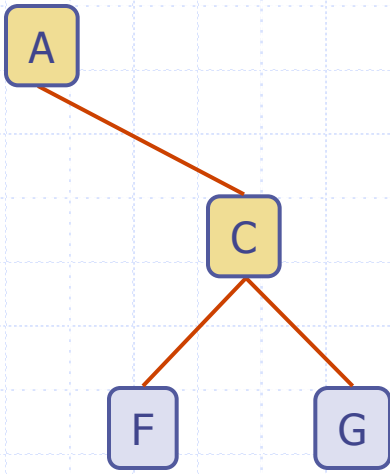
b d e h i

Postorder

d h i e b f g c a

f g c

d h i e b



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Preorder

a b d e h i c f g

c f g

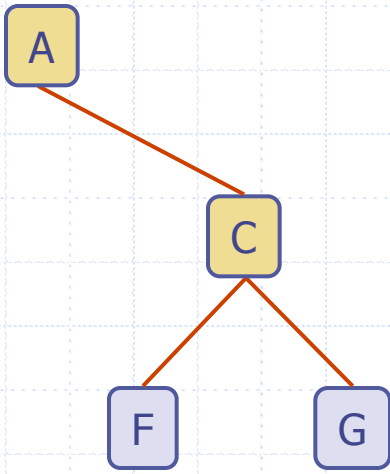
b d e h i

Postorder

d h i e b f g c a

f g c

d h i e b



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Preorder

a b d e h i c f g

c f g

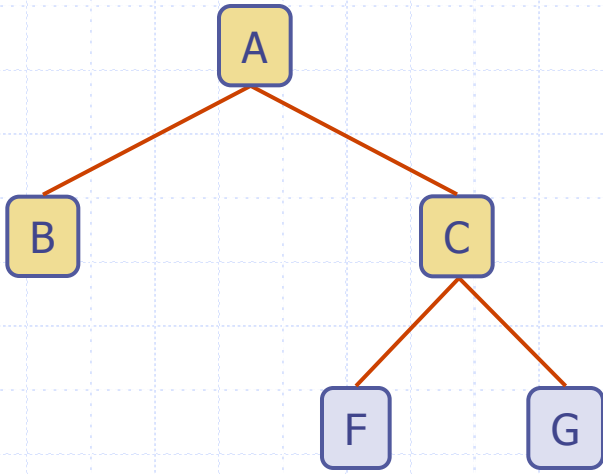
b d e h i

Postorder

d h i e b f g c a

f g c

d h i e b



Building Tree from Pre- and Post-Order Traversals

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Preorder

a b d e h i c f g

c f g

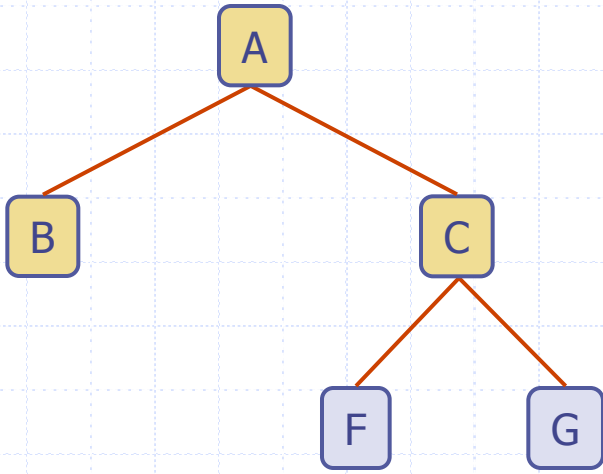
b d e h i

Postorder

d h i e b f g c a

f g c

d h i e b



Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b d e h i c f g

c f g

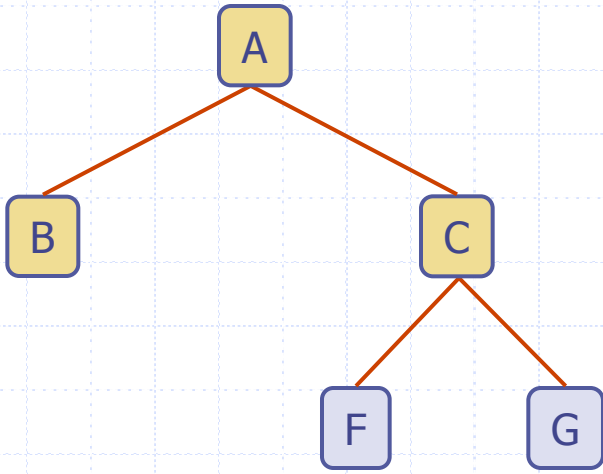
b d e h i

Postorder

d h i e b f g c a

f g c

d h i e b



Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b d e h i c f g

c f g

b d e h i

e h i

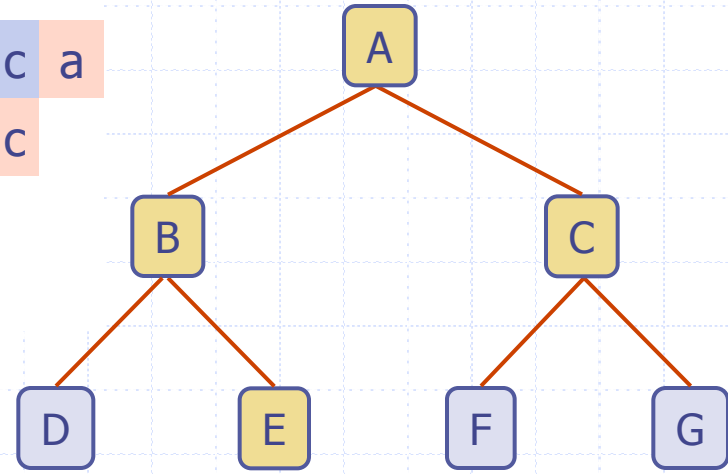
Postorder

d h i e b f g c a

f g c

d h i e b

h i e



Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b d e h i c f g

c f g

b d e h i

e h i

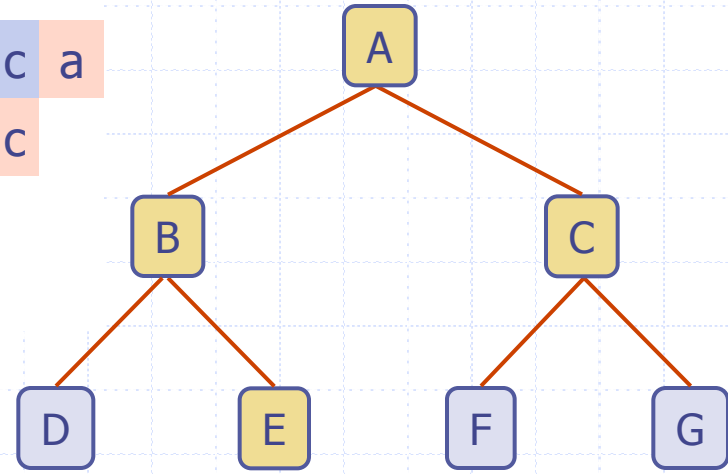
Postorder

d h i e b f g c a

f g c

d h i e b

h i e



Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b d e h i c f g

c f g

b d e h i

e h i

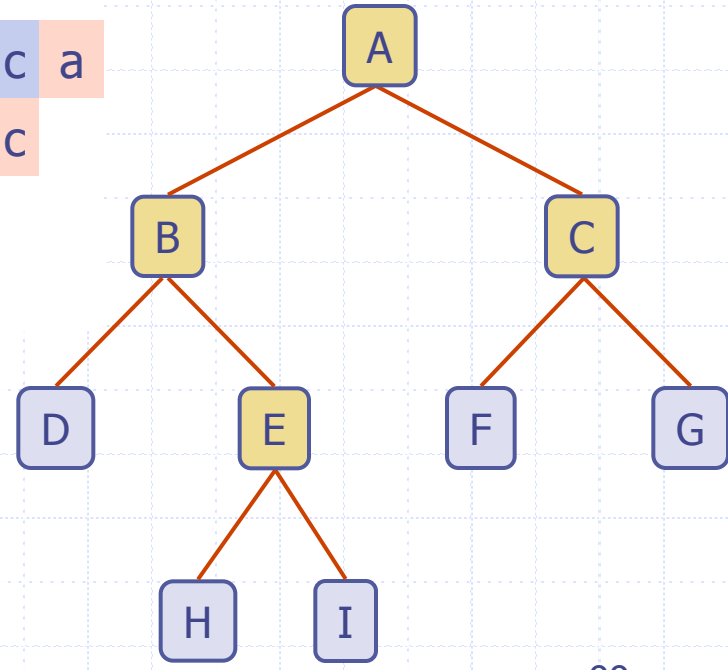
Postorder

d h i e b f g c a

f g c

d h i e b

h i e



Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder	Postorder
a b e i c	i e b c a

Building Tree from Pre- and Post-Order Traversals

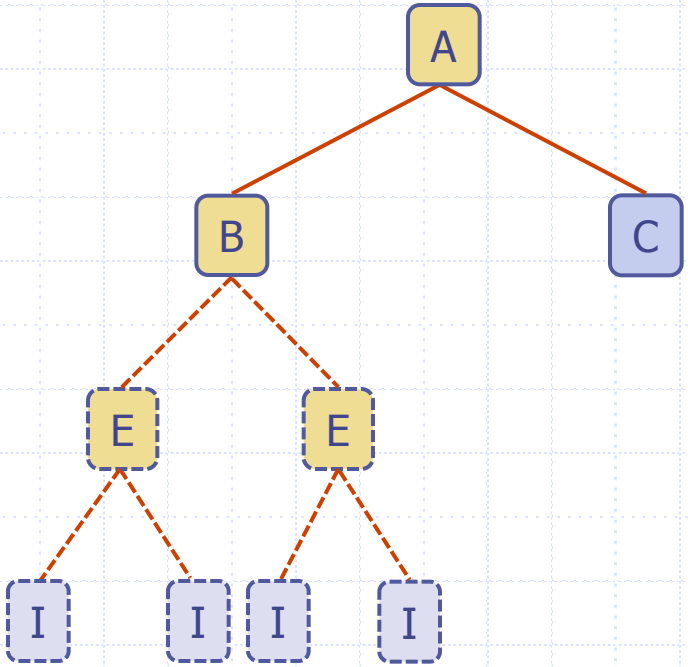
- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b e i c

Postorder

i e b c a

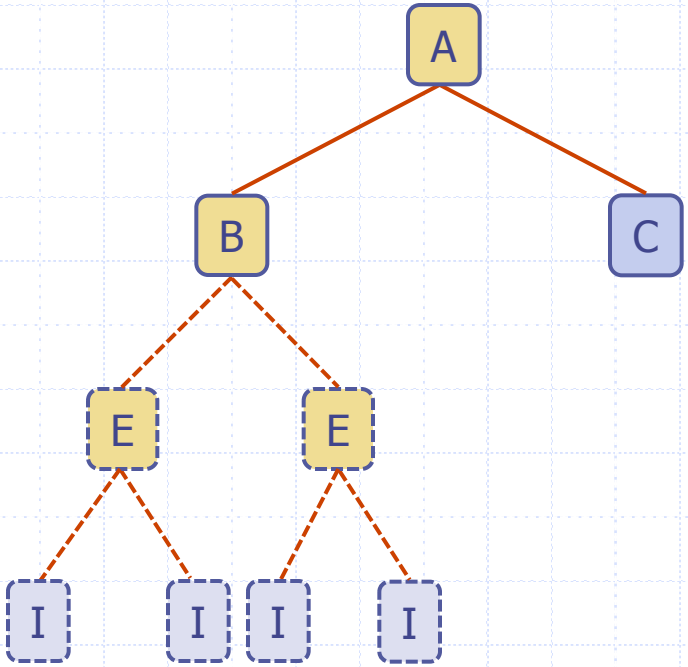


Building Tree from Pre- and Post-Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

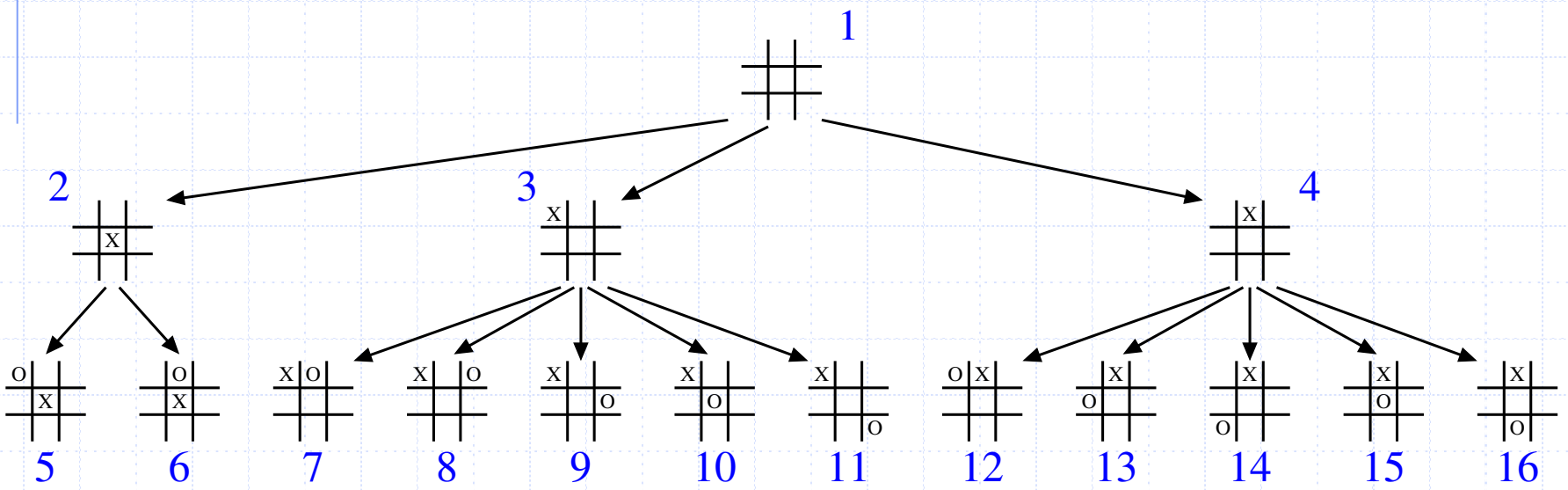
Preorder	Postorder
a b e i c	i e b c a

Only if the internal nodes in a binary tree have exactly two children



Breadth First Traversal

- Visit all positions at depth d, before visiting the positions at depth d+1



Let T be a binary tree with n nodes and let D be the sum of the depths of all the external nodes of T. Show that if T has the minimum number of external nodes possible, then D is O(n) and if T has the maximum number of external nodes possible, then D is O(nlogn)

Let T be an ordered tree with n >1. Is it possible that the preorder traversal of T visits the nodes in the same order as the postorder traversal of T? If so given an example, otherwise explain why. Is it possible that the preorder traversal of T visits the nodes in the reverse order of postorder traversal?

Give the pseudocode for a nonrecursive method for performing an inorder traversal of a binary tree in linear time.

Let T be a tree with n positions. Define the lowest common ancestor(LCA) between two positions p and q as the lowest point in T that has both p and q as descendants. Given two positions p and q, describe an efficient algorithm for finding the LCA of p and q. What is the running time of your algorithm?