Digital Electronics and Systems

Question: Use Karnaugh map to find the minimum cost SOP and POS expressions for the function.

$$f(x_1, \ldots, x_4) = \overline{x}_1\overline{x}_3\overline{x}_4 + x_3x_4 + \overline{x}_1\overline{x}_2x_4 + x_1x_2\overline{x}_3x_4$$

Assuming that there are also don't cares defined as $D = \sum(9, 12, 14)$.

- Standard SOP

$$f(x_1, x_2, x_3, x_4) = \bar{x}_1 (x_2 + \bar{x}_2)\bar{x}_3 \bar{x}_4 + (x_1 + \bar{x}_1)(x_2 + \bar{x}_2)x_3 x_4 +$$

$$\bar{x}_1 \bar{x}_2 (x_3 + \bar{x}_3)x_4 + x_1 x_2 \bar{x}_3 x_4$$

$$f(x_1, x_2, x_3, x_4) = \bar{x_1}\,(x_2 + \bar{x_2})\,\bar{x_3}\,\bar{x_4} +$$

$$(x_1 + \bar{x_1})(x_2 + \bar{x_2})(x_3 x_4) + \bar{x_1}\,\bar{x_2}\,(x_3 + \bar{x_3})\,x_4$$

$$+ x_1 x_2 \bar{x_3} x_4 .$$

$$= \bar{x_1} x_2 \bar{x_3} \bar{x_4} + \bar{x_1} \bar{x_2} \bar{x_3} \bar{x_4} +$$

$$x_1 x_2 x_3 x_4 + x_1 \bar{x_2} x_3 x_4 + \bar{x_1} x_2 x_3 x_4 +$$

$$\underline{\bar{x_1}\,\bar{x_2}\,x_3\,x_4} + \underline{\bar{x_1}\,\bar{x_2}\,x_3\,x_4} + \bar{x_1}\,\bar{x_2}\,\bar{x_3}\,x_4$$

$$+ x_1 x_2 \bar{x_3} x_4$$

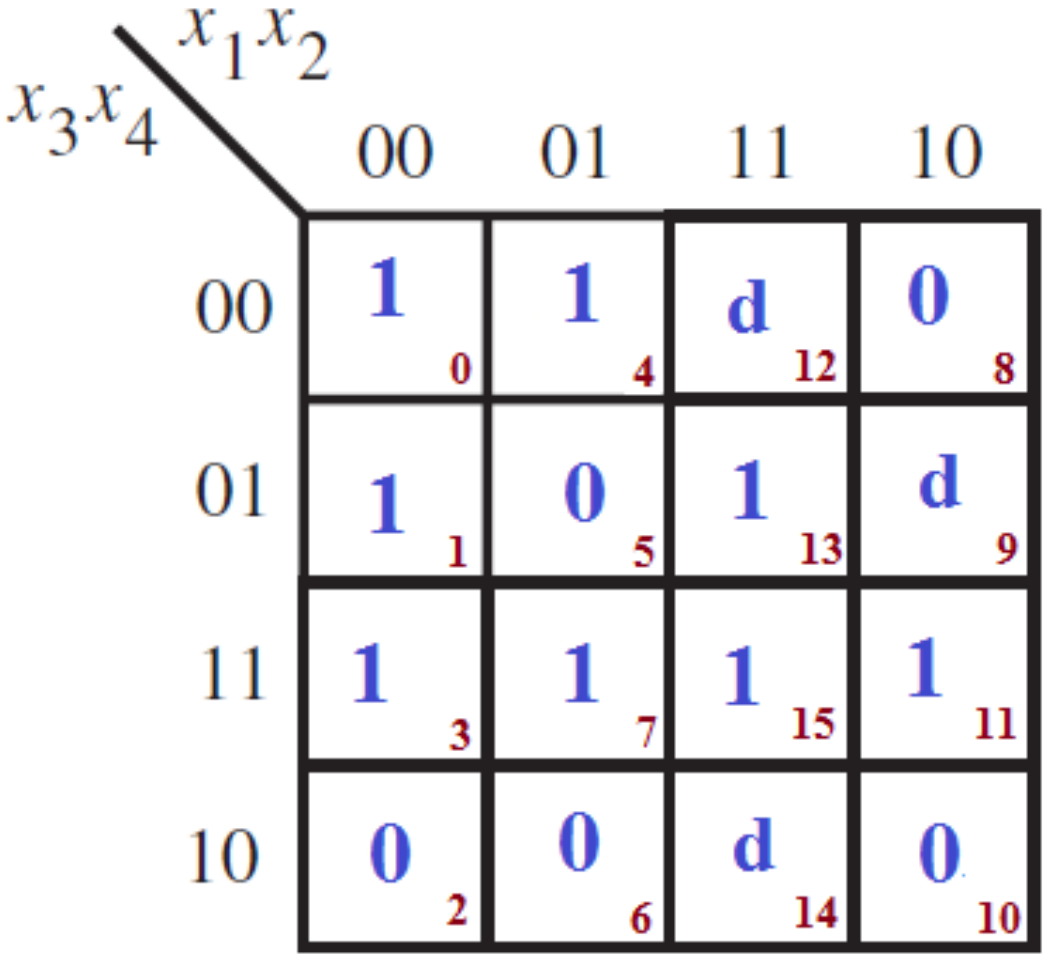
$$= \bar{x_1} x_2 \bar{x_3} \bar{x_4} + \bar{x_1} \bar{x_2} \bar{x_3} \bar{x_4} + x_1 x_2 x_3 x_4$$

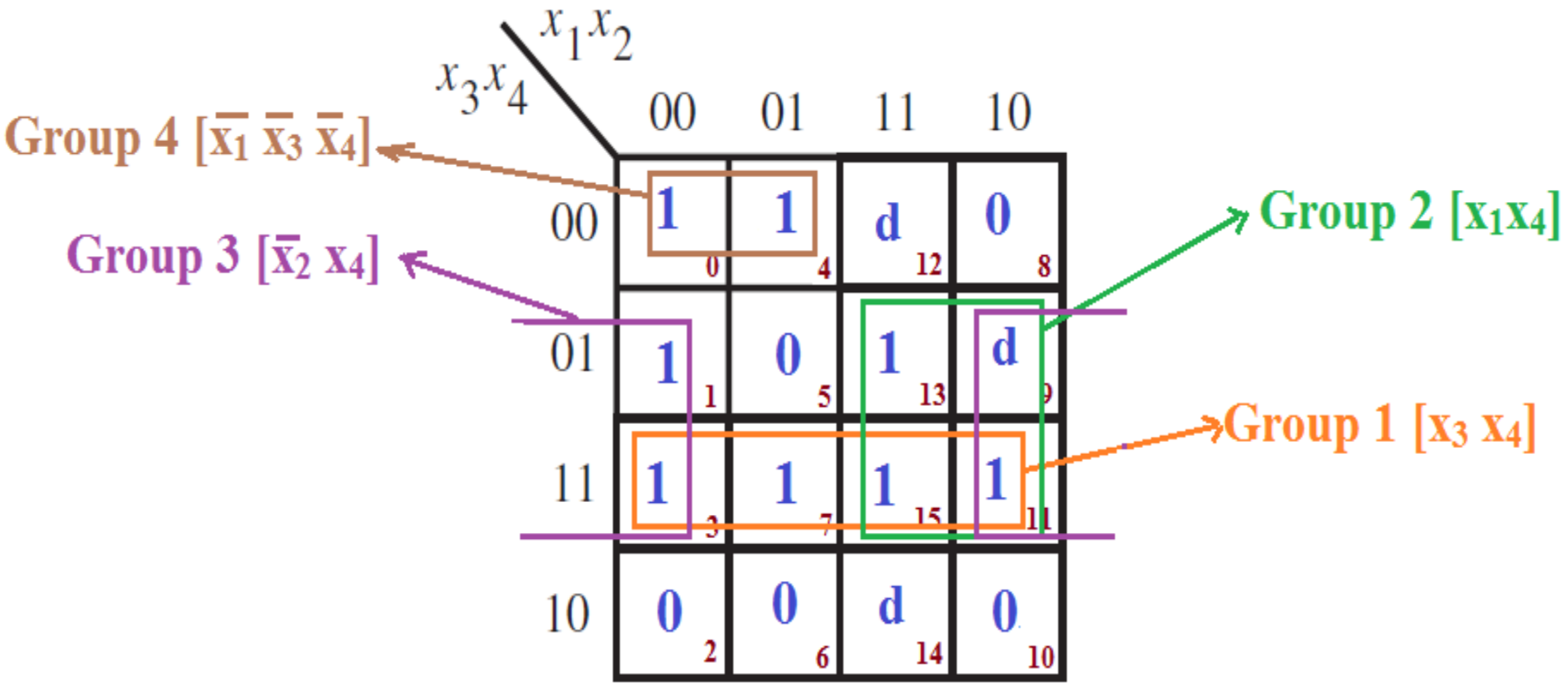
$$+ x_1 \bar{x_2} x_3 x_4 + \bar{x_1} x_2 x_3 x_4 + \bar{x_1} \bar{x_2} x_3 x_4$$

$$+ \bar{x_1} \bar{x_2} \bar{x_3} x_4 + x_1 x_2 \bar{x_3} x_4$$

Product Term	Binary Value	Decimal value	Minterm No:
$\bar{x_1}\ x_2\ \bar{x_3}\ \bar{x_4}$	0 1 0 0	4	m_4
$\bar{x_1}\ \bar{x_2}\ \bar{x_3}\ \bar{x_4}$	0 0 0 0	0	m_0
$x_1\ x_2\ x_3\ x_4$	1 1 1 1	15	m_{15}
$x_1\ \bar{x_2}\ x_3\ x_4$	1 0 1 1	11	m_{11}
$\bar{x_1}\ x_2\ x_3\ x_4$	0 1 1 1	7	m_7
$\bar{x_1}\ \bar{x_2}\ x_3\ x_4$	0 0 1 1	3	m_3
$\bar{x_1}\ \bar{x_2}\ \bar{x_3}\ x_4$	0 0 0 1	1	m_1
$x_1\ x_2\ \bar{x_3}\ x_4$	1 1 0 1	13	m_{13}

$f(x_1, x_2, x_3, x_4) = \Sigma m\,(0, 1, 3, 4, 7, 11, 13, 15) + D\,(9, 12, 14)$

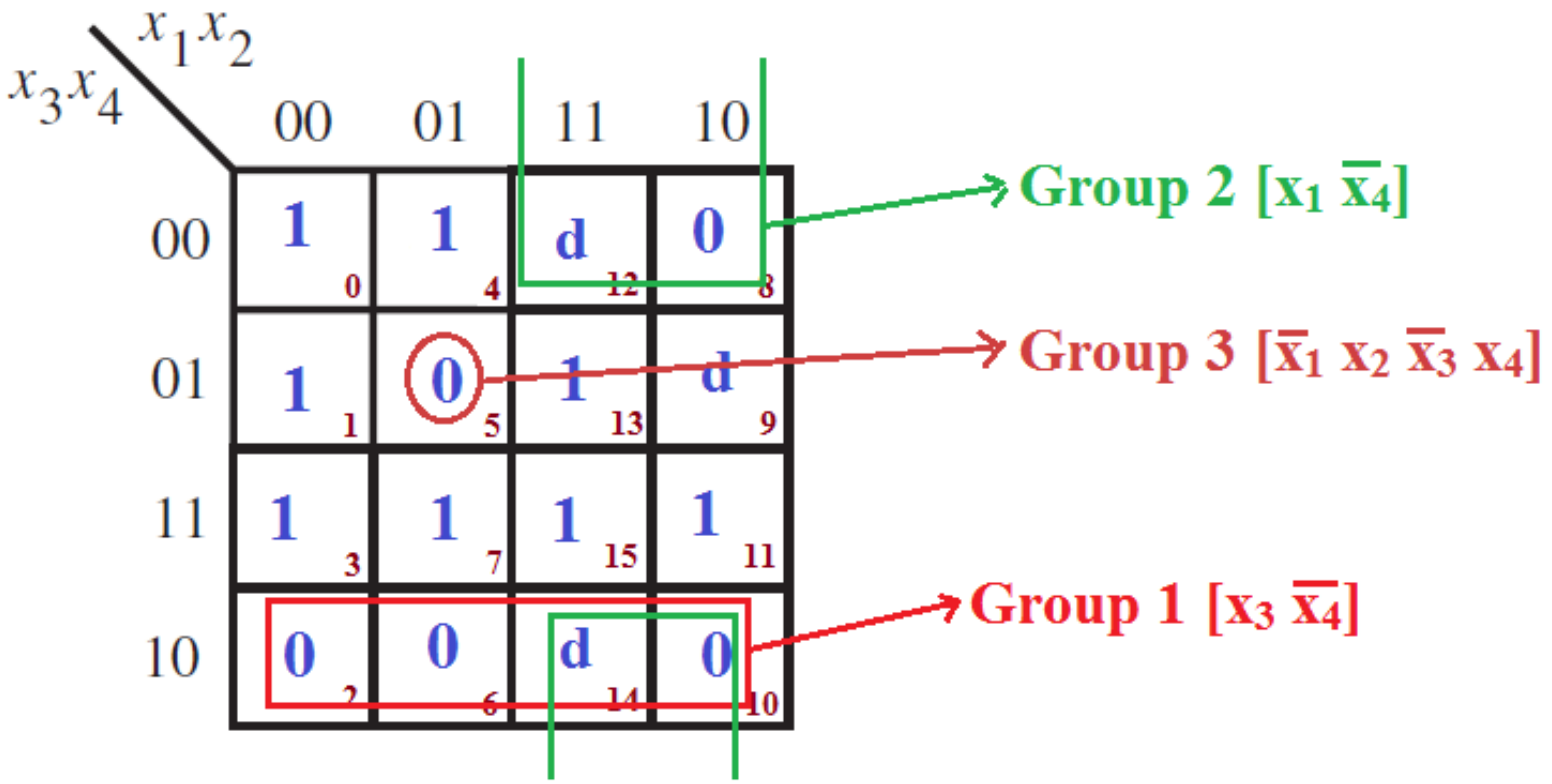




Minimum cost SOP expression is

$$f = x_3x_4 + \bar{x}_1\bar{x}_3\bar{x}_4 + \bar{x}_2x_4 + x_1x_4$$

Standard POS expression



$$\overline{f} = x_3\overline{x_4} + x_1\overline{x_4} + \overline{x_1}x_2\overline{x_3}x_4$$

$$f = \overline{\overline{f}} = \overline{x_3\overline{x_4} + x_1\overline{x_4} + \overline{x_1}x_2\overline{x_3}x_4}$$

$$f = (\overline{x_3}+x_4)(\overline{x_1}+x_4)(x_1+\overline{x_2}+x_3+\overline{x_4})$$

This is the minimum cost POS expression.

- Practice Problem:

Problem: Determine the minimum-cost SOP and POS expressions for the function $f(x_1, x_2, x_3, x_4) = \sum m(4, 6, 8, 10, 11, 12, 15) + D(3, 5, 7, 9)$.

SOP expression

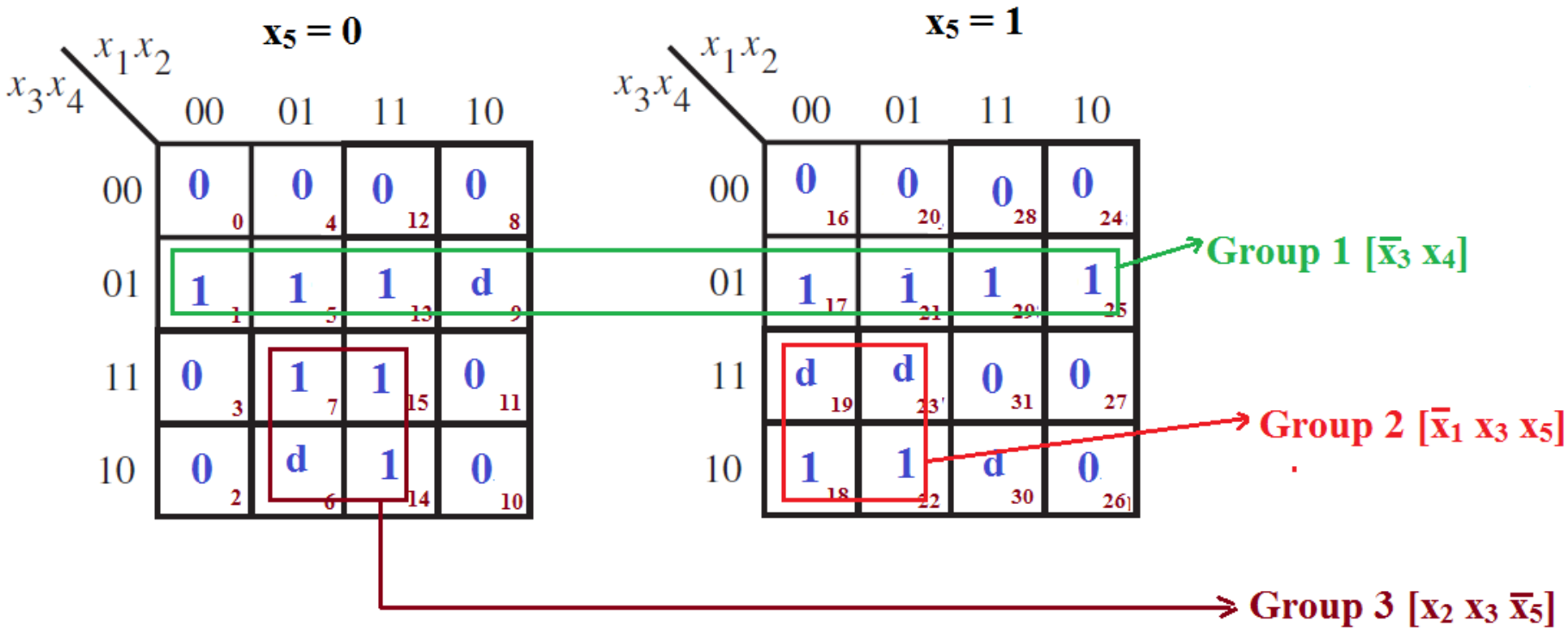
$$f = \bar{x}_1 x_2 + x_1 \bar{x}_2 + x_3 x_4 + x_1 \bar{x}_3 \bar{x}_4$$

POS expression

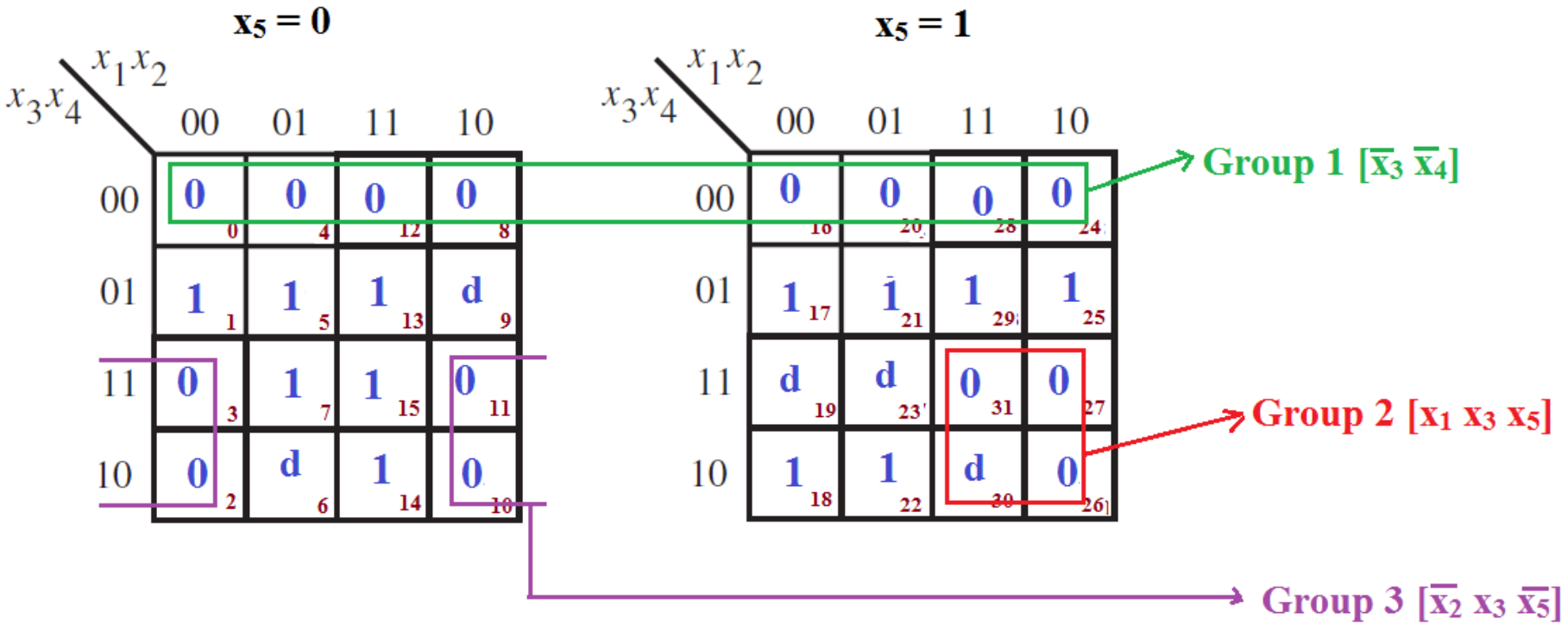
$$f = (x_1 + x_2)(x_3 + \bar{x}_4)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + x_4)$$

- Question: Obtain the minimal SOP expression and minimal POS expression for the function

$$f = \sum_m (1,5,7,13,14,15,17,18,21,22,25,29) + D(6,9,19,23,30)$$



Minimal SOP expression. $f = \bar{x}_3 x_4 + \bar{x}_1 x_3 x_5 + x_2 x_3 \bar{x}_5$



$$\overline{f} = (\overline{x_3}\ \overline{x_4}) + x_1 x_3 x_5 + \overline{x_2}\ x_3\ \overline{x_5}$$

$$\overline{\overline{f}} = \overline{\overline{x_3}\ \overline{x_4} + x_1 x_3 x_5 + \overline{x_2}\ x_3\ \overline{x_5}}$$

$$f = (x_3 + x_4)(\overline{x_1} + \overline{x_3} + \overline{x_5})(x_2 + \overline{x_3} + x_5)$$

This is the minimal POS expression.

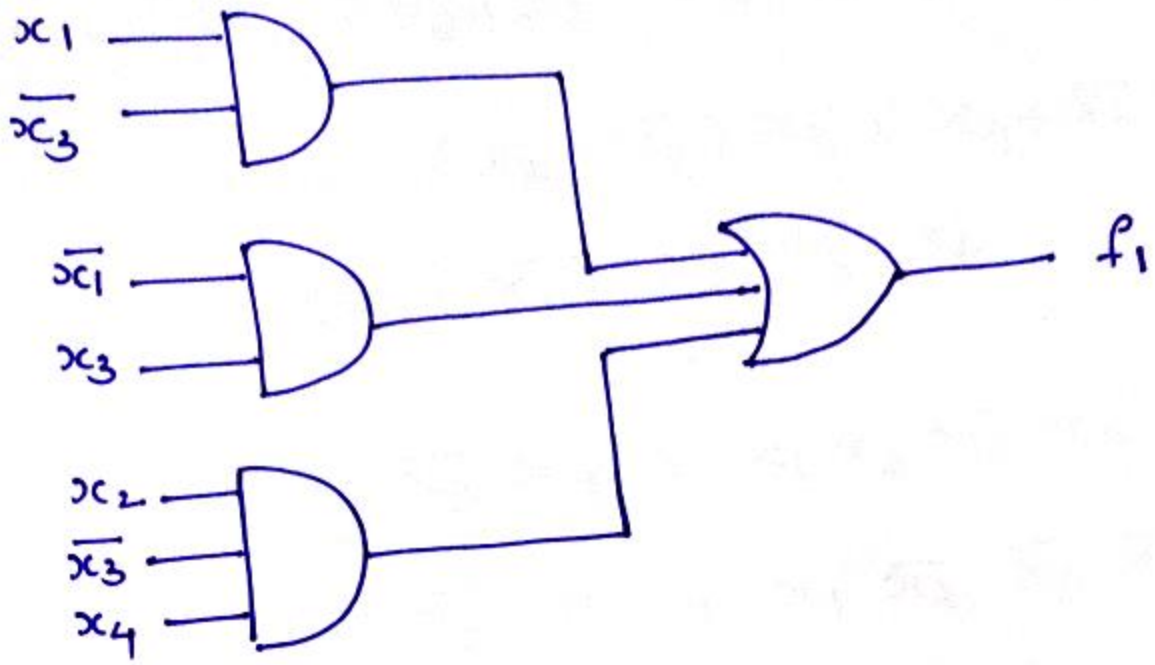
MULTIPLE OUTPUT CIRCUITS

- Implement the given functions, f_1 and f_2 with minimum cost.

$$f_1 = x_1\bar{x}_3 + \bar{x}_1 x_3 + x_2\bar{x}_3 x_4$$

$$f_2 = x_1\bar{x}_3 + \bar{x}_1 x_3 + x_2 x_3 x_4$$

$$f_1 = x_1\bar{x}_3 + \bar{x}_1 x_3 + x_2\bar{x}_3 x_4$$

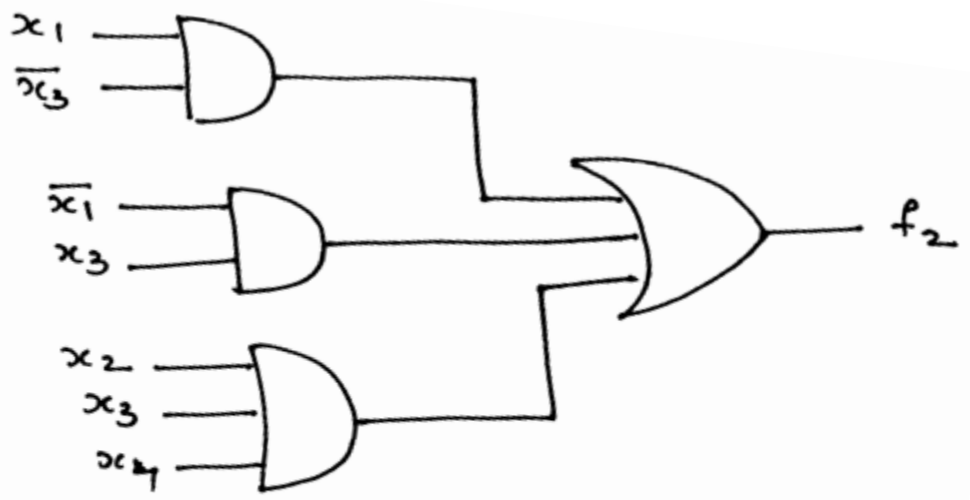


No: of gates = 4

No: of inputs = 10

∴ Total cost = 14

$$f_2 = x_1\bar{x}_3 + \bar{x}_1 x_3 + x_2 x_3 x_4$$



No: of gates = 4

No: of inputs = 10

$\therefore$ Total cost = 14.

$$f_1 = x_1 (x_2 + \overline{x_2}) \overline{x_3} (x_4 + \overline{x_4})$$

$$+ \overline{x_1} (x_2 + \overline{x_2}) x_3 (x_4 + \overline{x_4})$$

$$+ (x_1 + \overline{x_1}) x_2 \overline{x_3} x_4$$

$$= x_1 x_2 \overline{x_3} x_4 + x_1 x_2 \overline{x_3} \overline{x_4} +$$

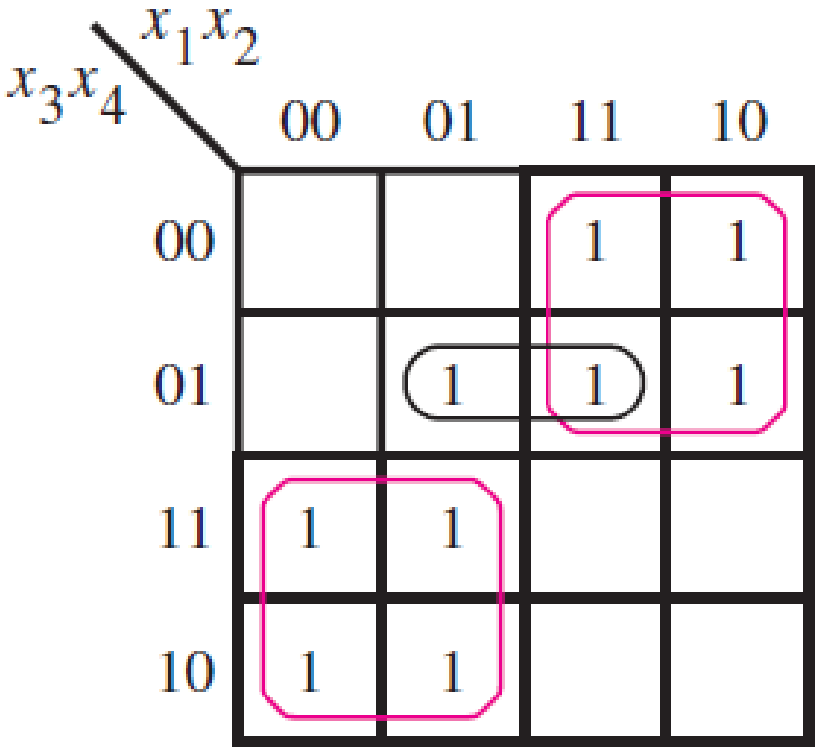
$$x_1 \overline{x_2} \overline{x_3} x_4 + x_1 \overline{x_2} \overline{x_3} \overline{x_4} +$$

$$\overline{x_1} x_2 x_3 x_4 + \overline{x_1} x_2 x_3 \overline{x_4} +$$

$$\overline{x_1} \overline{x_2} x_3 x_4 + \overline{x_1} \overline{x_2} x_3 \overline{x_4} +$$

$$x_1 x_2 \overline{x_3} x_4 + \overline{x_1} x_2 \overline{x_3} x_4$$

Product Term	Binary	Decimal	Minterm No:
$x_1\ x_2\ \bar{x_3}\ x_4$	1101	13	m_{13}
$x_1\ x_2\ \bar{x_3}\ \bar{x_4}$	1100	12	m_{12}
$x_1\ \bar{x_2}\ \bar{x_3}\ x_4$	1001	9	m_9
$x_1\ \bar{x_2}\ \bar{x_3}\ \bar{x_4}$	1000	8	m_8
$\bar{x_1}\ x_2\ x_3\ x_4$	0111	7	m_7
$\bar{x_1}\ x_2\ x_3\ \bar{x_4}$	0110	6	m_6
$\bar{x_1}\ \bar{x_2}\ x_3\ x_4$	0011	3	m_3
$\bar{x_1}\ \bar{x_2}\ x_3\ \bar{x_4}$	0010	2	m_2
$\bar{x_1}\ x_2\ \bar{x_3}\ x_4$	0101	5	m_5



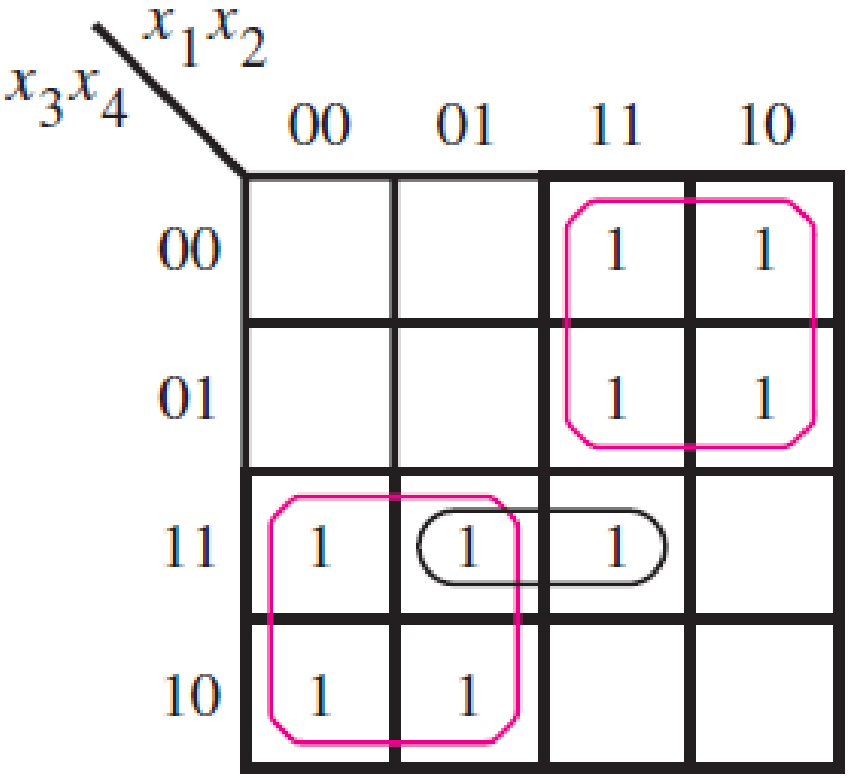
(a) Function f_1

$$f_2 = x_1 (x_2 + \overline{x_2})\, \overline{x_3}\, (x_4 + \overline{x_4}) +$$
$$\overline{x_1}\, (x_2 + \overline{x_2})\, x_3\, (x_4 + \overline{x_4}) +$$
$$(x_1 + \overline{x_1})\, x_2\, x_3\, x_4$$

$$= x_1 x_2 \overline{x_3} x_4 + x_1 x_2 \overline{x_3}\, \overline{x_4} +$$
$$x_1 \overline{x_2}\, \overline{x_3} x_4 + x_1 \overline{x_2}\, \overline{x_3}\, \overline{x_4} +$$
$$\overline{x_1}\, x_2\, x_3\, x_4 + \overline{x_1}\, x_2\, x_3\, \overline{x_4} +$$
$$\overline{x_1}\, \overline{x_2}\, x_3\, x_4 + \overline{x_1}\, \overline{x_2}\, x_3\, \overline{x_4} +$$
$$x_1 x_2 x_3 x_4 + \overline{x_1}\, x_2 x_3 x_4$$

Product term	Binary	Decimal	Minterm no:
$x_1\ x_2\ \bar{x_3}\ x_4$	1 1 0 1	13	m_{13}
$x_1\ x_2\ \bar{x_3}\ \bar{x_4}$	1 1 0 0	12	m_{12}
$x_1\ \bar{x_2}\ \bar{x_3}\ x_4$	1 0 0 1	9	m_9
$x_1\ \bar{x_2}\ \bar{x_3}\ \bar{x_4}$	1 0 0 0	8	m_8
$\bar{x_1}\ x_2\ x_3\ x_4$	0 1 1 1	7	m_7
$\bar{x_1}\ x_2\ x_3\ \bar{x_4}$	0 1 1 0	6	m_6
$\bar{x_1}\ \bar{x_2}\ x_3\ x_4$	0 0 1 1	3	m_3
$\bar{x_1}\ \bar{x_2}\ x_3\ \bar{x_4}$	0 0 1 0	2	m_2
$x_1\ x_2\ x_3\ x_4$	1 1 1 1	15	m_{15}

$$f_2 = \Sigma m\ (2, 3, 6, 7, 8, 9, 12, 13, 15)$$

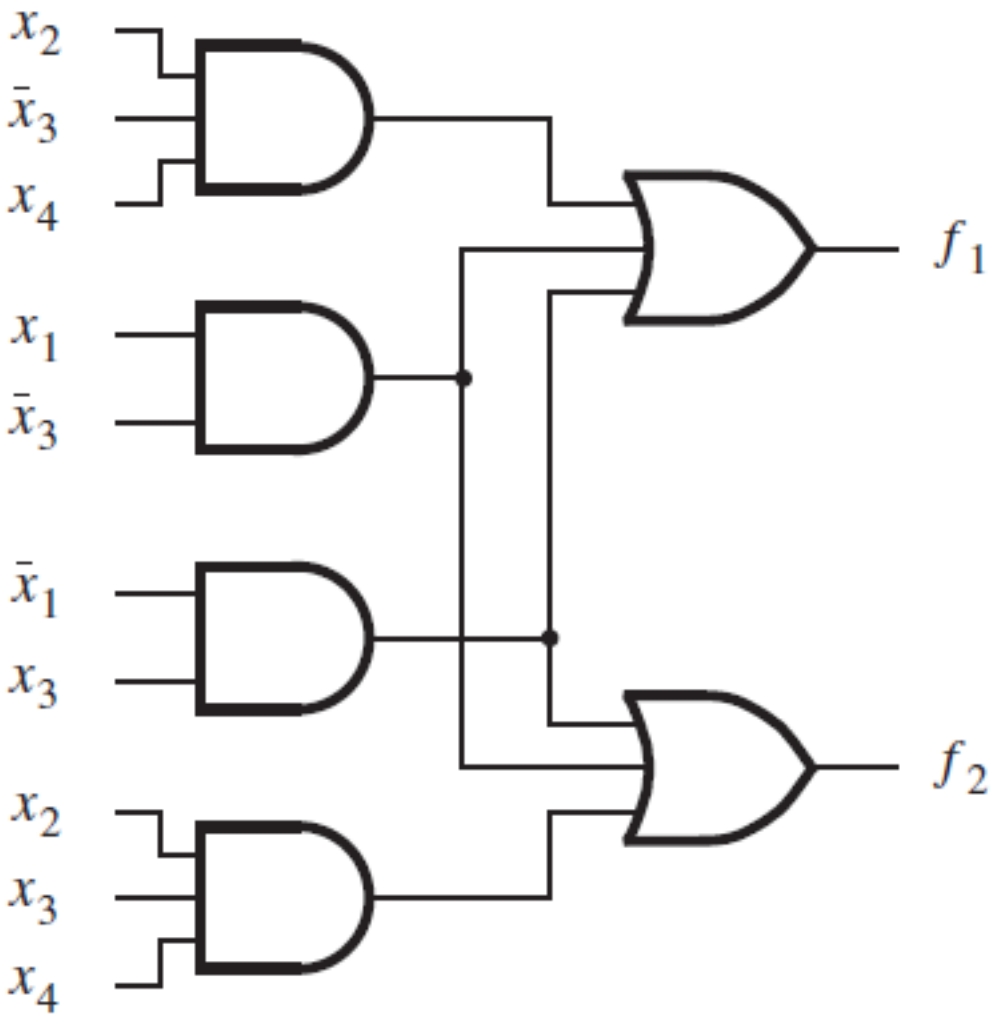


(b) Function f_2

- If both functions are implemented separately, then total cost = 28

A less expensive realization is possible if the two circuits are combined into a single circuit with two outputs.

Because first two product terms are identical in both expressions, the AND gate that implement them need not be duplicated.



Combined circuit for f_1 and f_2

Total cost = 22

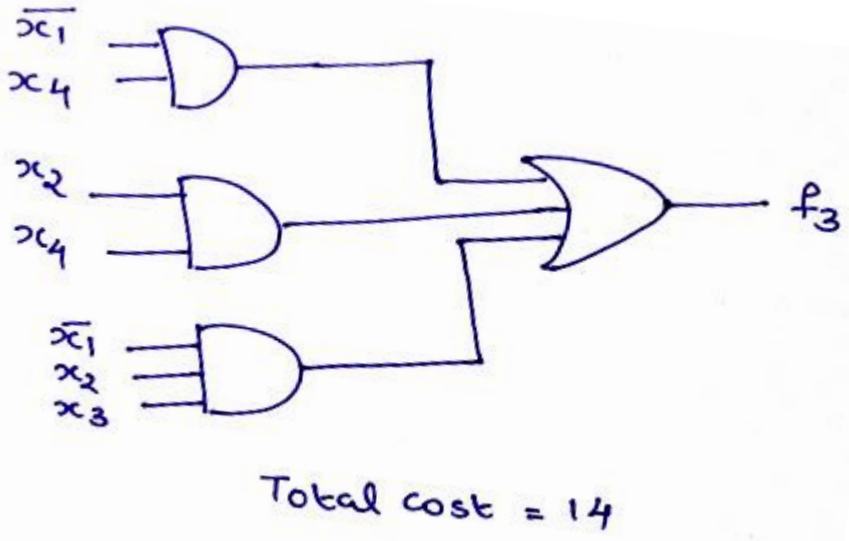
- Implement the given functions, f_3 and f_4 with minimum cost.

$$f_3 = \bar{x}_1 x_4 + x_2 x_4 + \bar{x}_1 x_2 x_3$$

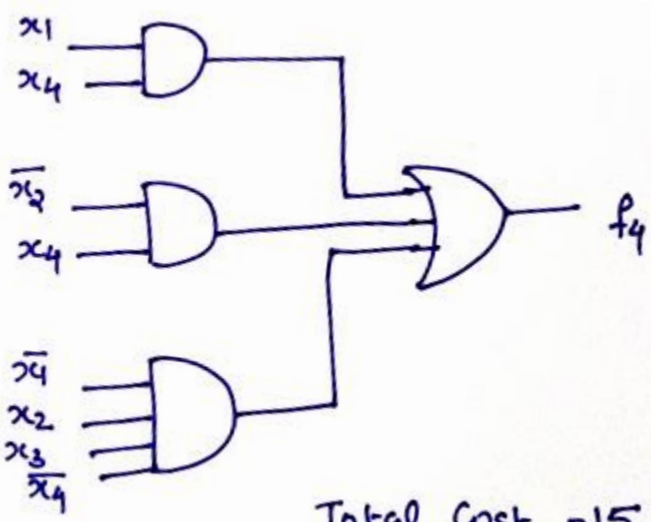
$$f_4 = x_1 x_4 + \bar{x}_2 x_4 + \bar{x}_1 x_2 x_3 \bar{x}_4$$

$$f_3 = \bar{x}_1 x_4 + x_2 x_4 + \bar{x}_1 x_2 x_3$$

$$f_4 = x_1 x_4 + \bar{x}_2 x_4 + \bar{x}_1 x_2 x_3 \bar{x}_4$$



Total cost = 14



Total Cost =15

$$f_3 = \bar{x_1}\, x_4 + x_2\, x_4 + \bar{x_1}\, x_2\, x_3$$

Convert it into standard SOP.

$$\therefore f_3 = \bar{x_1}\,(x_2 + \bar{x_2})(x_3 + \bar{x_3})\,x_4 +$$
$$(x_1 + \bar{x_1})(x_2)(x_3 + \bar{x_3})\,x_4 +$$
$$\bar{x_1}\, x_2\, x_3\,(x_4 + \bar{x_4})$$

$$= \bar{x_1}\, x_2\, x_3\, x_4 + \bar{x_1}\, \bar{x_2}\, x_3\, x_4 +$$
$$\bar{x_1}\, \bar{x_2}\, x_3\, x_4 + \bar{x_1}\, \bar{x_2}\, \bar{x_3}\, x_4 +$$
$$x_1\, x_2\, x_3\, x_4 + x_1\, x_2\, \bar{x_3}\, x_4 +$$
$$\bar{x_1}\, x_2\, x_3\, x_4 + \bar{x_1}\, x_2\, \bar{x_3}\, x_4 +$$
$$\bar{x_1}\, x_2\, x_3\, x_4 + \bar{x_1}\, x_2\, x_3\, \bar{x_4}$$

Product term	Binary	Decimal	Minterm No:
$\bar{x_1}\ x_2\ x_3\ x_4$	0111	7	m_7
$\bar{x_1}\ \bar{x_2}\ x_3\ x_4$	0011	3	m_3
$\bar{x_1}\ \bar{x_2}\ \bar{x_3}\ x_4$	0001	1	m_1
$x_1\ x_2\ x_3\ x_4$	1111	15	m_{15}
$x_1\ x_2\ \bar{x_3}\ x_4$	1101	13	m_{13}
$\bar{x_1}\ x_2\ \bar{x_3}\ x_4$	0101	5	m_5
$\bar{x_1}\ x_2\ x_3\ \bar{x_4}$	0110	6	m_6

$$f_3 = \Sigma_m (m_1, m_3, m_5, m_6, m_7, m_{13}, m_{15})$$

$$f_4 = x_1 x_4 + \overline{x_2}\, x_4 + \overline{x_1}\, x_2\, x_3\, \overline{x_4}$$

Convert it into standard SOP

$$\therefore f_4 = x_1 (x_2 + \overline{x_2})(x_3 + \overline{x_3})\, x_4 +$$
$$(x_1 + \overline{x_1})\, \overline{x_2}\, (x_3 + \overline{x_3})\, x_4 +$$
$$\overline{x_1}\, x_2\, x_3\, \overline{x_4}$$

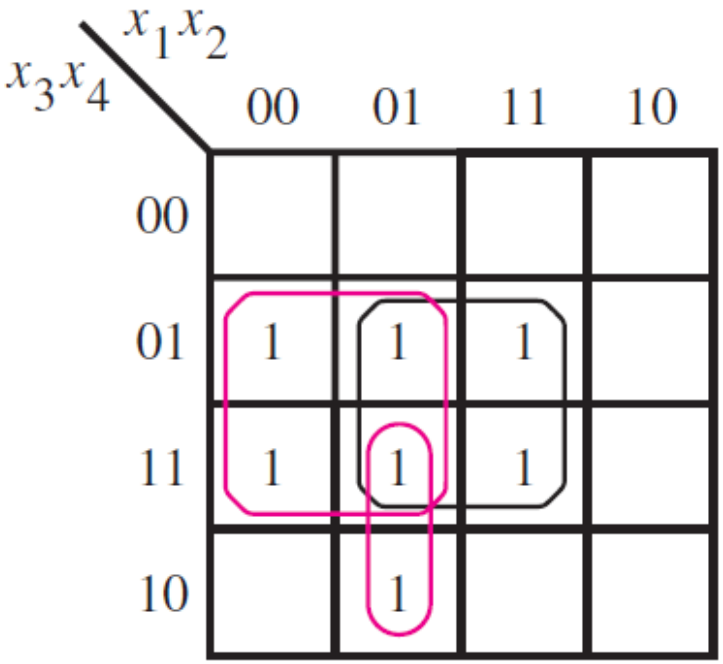
$$= x_1 x_2 x_3 x_4 + x_1 x_2 \overline{x_3} x_4 +$$
$$x_1 \overline{x_2} x_3 x_4 + x_1 \overline{x_2}\, \overline{x_3}\, x_4 +$$
$$x_1 \overline{x_2} x_3 x_4 + x_1 \overline{x_2}\, \overline{x_3}\, x_4 +$$
$$\overline{x_1}\, \overline{x_2}\, x_3 x_4 + \overline{x_1}\, \overline{x_2}\, \overline{x_3}\, x_4 +$$
$$\overline{x_1}\, x_2\, x_3\, \overline{x_4}$$

Product term	Binary	Decimal	Minterm No:
$x_1\ x_2\ x_3\ x_4$	1111	15	m_{15}
$x_1\ x_2\ \overline{x_3}\ x_4$	1101	13	m_{13}
$x_1\ \overline{x_2}\ x_3\ x_4$	1011	11	m_{11}
$x_1\ \overline{x_2}\ \overline{x_3}\ x_4$	1001	9	m_9
$\overline{x_1}\ \overline{x_2}\ x_3\ x_4$	0011	3	m_3
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ x_4$	0001	1	m_1
$\overline{x_1}\ x_2\ x_3\ \overline{x_4}$	0110	6	m_6

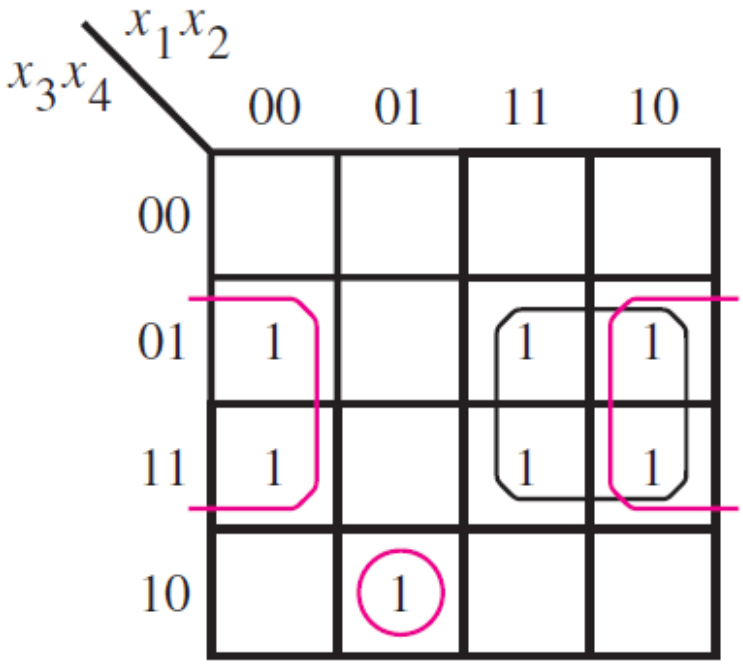
$$f_4 = \Sigma m\,(1, 3, 6, 9, 11, 13, 15)$$

$$f_3 = \bar{x}_1 x_4 + x_2 x_4 + \bar{x}_1 x_2 x_3$$

$$f_4 = x_1 x_4 + \bar{x}_2 x_4 + \bar{x}_1 x_2 x_3 \bar{x}_4$$



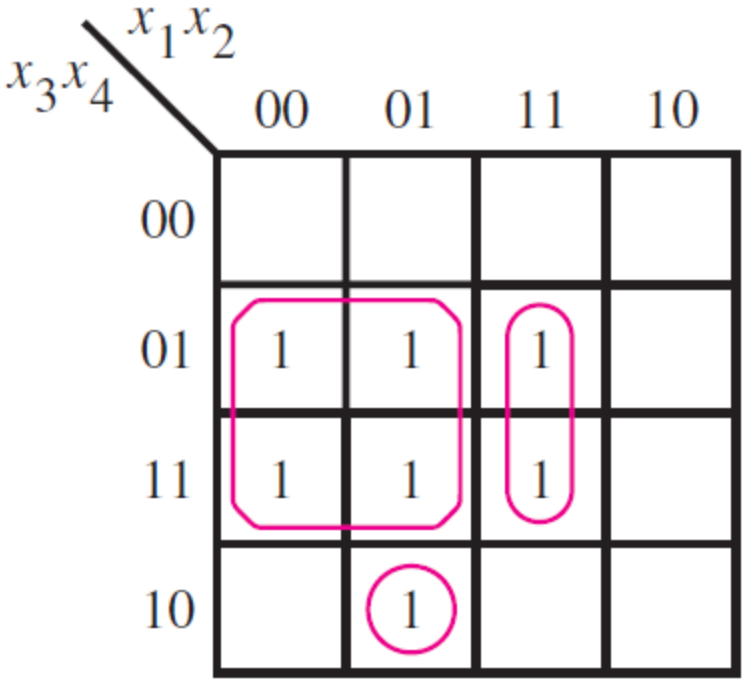
(a) Optimal realization of f_3

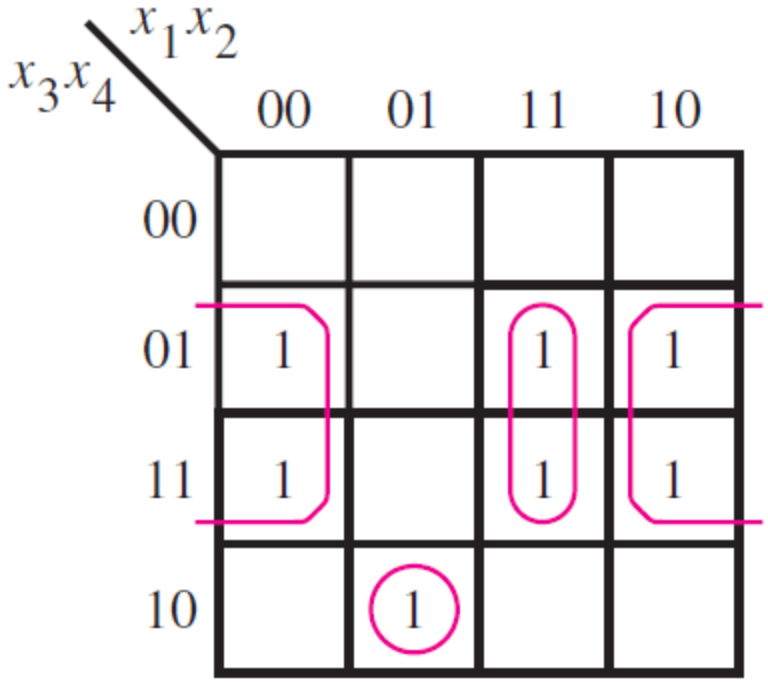


(b) Optimal realization of f_4

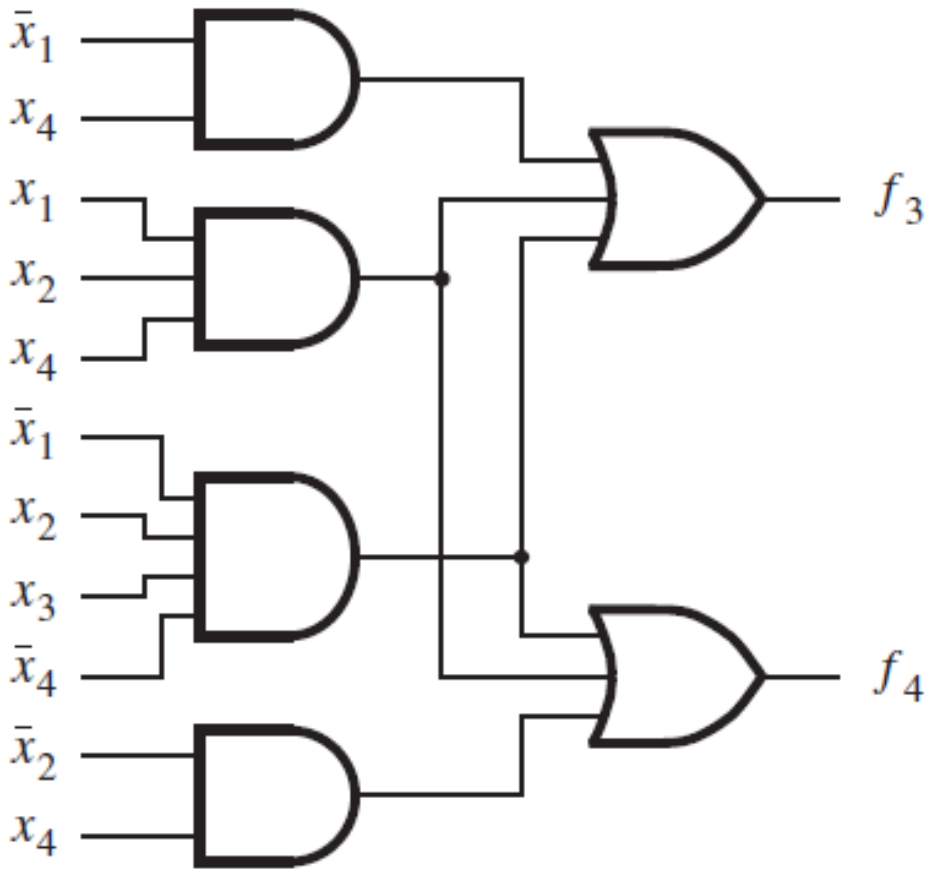
$$f_3 = x_1x_2x_4 + \bar{x}_1x_2x_3\bar{x}_4 + \bar{x}_1x_4$$

$$f_4 = x_1x_2x_4 + \bar{x}_1x_2x_3\bar{x}_4 + \bar{x}_2x_4$$





Optimal realization of f_3 and f_4 together



Combined circuit for f_3 and f_4

Total cost = 23

Practice problem:

Text book : Fundamentals of Digital Logic with Verilog Design

Authors : Stephen Brown, Zvonko Vranesic

Chapter title: Optimized Implementation of Logic functions

Question No: **4.5, 4.6,4.10**

Solve these problems.