Permutation & Combination

Fundamental principle of Counting

The Sum Rule

Suppose a person has **m** shirts and **n** different T-shirts, then in how many ways can he dress up for the picnic? The answer for the question will be **m + n** ways. This is because either he can wear shirt or he will wear T – shirt but not both together. This rule is identified by the word "or".

> ***Do you know?***
>
> Whenever there is '**OR**' we have to **add**, & whenever there is '**AND**', we have to **multiply**.

The Product Rule

Suppose this time a person is having **m** shirts and **n** jeans, then in how many ways can he dress up for the picnic? Here on every jean he can wear any of the **m** shirts and have **n** number of jeans, so total number of ways will be **m x n** ways. In this type of problems there is a word "and" which distinguish it from sum rule. Such as in the above example a person will wear Shirt **and** jean that is why both has been multiplied.

Event A	$\xrightarrow[can\ occur\ in]{}$	m ways.
Event B	$\xrightarrow[can\ occur\ in]{}$	n ways.
Then either (A **or** B)	$\xrightarrow[can\ occur\ in]{}$	m + n ways (Sum Rule)
A **and** B (both simultaneously)	$\xrightarrow[can\ occur\ in]{}$	$m \times n$ ways (Product Rule)

Meaning of Permutation and Combination

Let us understand the concept of permutation and combination with the help of an example. Suppose a person has 3 alphabets A, B and C. He is asked to arrange any two alphabets, and then AB, BA, BC, CB, CA or AC should be the required arrangements. But if asked to choose any two from the group of three alphabets, then his selection will be like this AB, BC or CA. In both the cases what difference we have? Here the difference is of **order**. In arrangement order of the objects arranged is important and plays a very important role as AB and BA are two different arrangements. While in case of selection order is not important and both AB and BA both are same and consider identical.

So, permutation deals with the arrangements of the objects in the given conditions, while combination deals with the selection of the objects.

> ***Permutations***
> for **arranging** the things like sitting arrangement, alphabets arrangement (making words), forming numbers etc.
>
> **&**
>
> ***Combinations***
> **For selection/grouping** like making a team

Concept of ^nP_r *and* ^nC_r

$^nP_r \rightarrow$ Number of arrangement of n different objects taken r at a time.

e.g.: If 5 persons have to sit on 7 chairs in a row. Then total number of ways they can sit will be equal to $^{7}P_{5} = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520$ ways.

$^{n}C_{r} \rightarrow$ Number of selection of n different objects taken r at a time.

e.g.: If a team of 11 players has to be selected from the group of 14 players in the training camp, then number of ways will be

$^{14}C_{11} = \frac{14!}{11! \times (14-11)!} = 364$ ways.

Do you know?

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$^{n}C_{r} = \frac{n!}{r! \times (n-r)!}$$

Ex.1 ***How many numbers are between 100 and 1000 such that***

(i) ***All the digits are distinct***

(ii) ***Even numbers with distinct digits.***

Sol. (i) ***All the digits are distinct***

A number between 100 and 1000 has three digits. So we have to form all possible 3 digit numbers with distinct digits.

We cannot have 0 at the hundred's place.

So, the hundred's place can be filled with any of the 9 digits 1, 2, 3….., 9.

So, ten's place can be filled with any of the remaining 9 digits in 9 ways.

And, there are 8 ways of filling the unit's place.

Total numbers are equal to

Hundred	Ten's	Once
↑	↑	↑
9 ×	9 ×	8 = 648

(ii) ***<u>Even numbers with distinct digits.</u>***

If unit's digit is 0, then hundred's place can be filled in 9 ways. If unit's digit is a number other than 0 then hundred's place can be filled in 8 ways. So you have to divide into two cases.

CASE I: If unit's digit is 0. Then

Hundred	Ten's	Once
↑	↑	↑
9 ×	8 ×	1 = 72

CASE II: Unit digit is digit other than 0. (i.e. 2, 4, 6, 8)

Hundred	Ten's	Once
↑	↑	↑
8 ×	8 ×	4 = 256

Total numbers = 72 + 256 = 328.

Formulae of Permutation

i. Permutations of n different things taken 'r' at a time is denoted by $^{n}P_{r}$ and is given by

$$\mathbf{^{n}P_{r} = \frac{n!}{(n-r)!}}$$

ii. The total number of arrangements of n things taken r at a time, in which a particular things always occurs = $^{n-1}\mathbf{P}_{r-1}$.

E.g. *The number of ways in a basketball game in which 5 players out of 8 players selected can plays at different positions such that captain always play in centre position* = $^{8-1}P_{5-1} = {}^{7}P_4 = 210$.

iii. The total number of permutations of n different things taken r at a time in which a particular thing never occurs = $^{n-1}P_r$.

E.g. *The number of ways in which we can form a 4 letter word from the letters of the word COMBINE such that the word never contain B* = $^{7-1}P_4 = {}^{6}P_4 = 30$

iv. The number of arrangements when things are not all different such as arrangement of n things, when p of them of one kind, q of another kind, r of still another kind and so on, then the total number of permutations is given by $\frac{n!}{(p!\,q!\,r!.......)}$.

E.g. *The total arrangements of letters of the word "M A T H E M A T I C S" in which M, A and T are repeated twice respectively* **=** $\frac{11!}{2!2!2!}$

v. The number of permutations of n different things taking r at a time when each thing may be repeated any number of times in any permutations is given by ($n \times n \times n \times n \times n$……………..r times) i.e. $\mathbf{n^r}$ ways.

E.g. *The total numbers of ways in which 7 balls can be distributed amongst 9 persons (when any man can get any number of balls*) = $\mathbf{9^7}$ *ways.*

Ex.2 ***How many different words can be formed with the letters of word ORDINATE'?***

(i) ***So that the vowels occupy odd places***

(ii) ***Beginning with 'O'***

(iii) ***Beginning with 'O' and ending with 'E'***

Sol. (i) ORDINATE contains 8 alphabets in which 4 vowels and 4 consonants. These 4 vowels has to be arranged on 4 odd places.

Number of arrangements of the vowels 4! Also number of arranging consonants is 4!

$\Rightarrow$ Number of words = $4! \times 4! = (4 \times 3 \times 2 \times 1)^2 = 576$.

(ii) When O is fixed we have been left with 7 alphabets to arrange

$\Rightarrow$ Number of words = 7! = 5040

(iii) When we have only six letters at our disposal, leaving 'O' and 'E' which are fixed.

Number of words = 6! = 720

Circular Permutations

In linear permutation we fill first place by n ways and next in (n – 1) ways and so on, but in circular arrangement we don't have any first place. So fix any object as a first place and arrange rest (n – 1) objects around it. Hence we have to arrange 1 less than the total number of things.

i. Number of circular permutations of n things taken all at a time = $(n-1)!$

ii. Number of circular permutations of n different things taking r at a time $= \frac{^{n}P_r}{r}$.

iii. If there be no difference between clockwise and anticlockwise arrangements, the total number of circular permutations of n things taking all at a time is $\frac{(n-1)!}{2}$ & the total number when taking r at a time all will be $\frac{{}^nP_r}{2r}$.

Formulae of Combination

1. Number of combinations of n dissimilar things taken 'r' at a time is denoted by ${}^nC_r$ & is given by

$$\mathbf{{}^nC_r} = \frac{n!}{(n-r)!r!}$$

2. Number of combinations of n different things taken 'r' at a time in which p particular things will always occur is $\mathbf{{}^{n-p}C_{r-p}}$

E.g. *The number of ways a basketball team of 5 players chosen from 8 players, so that the captain should be included in the team* = ${}^{8-1}C_{5-1} = {}^7C_4$ = 35

Do you know?

1. If ${}^nC_x = {}^nC_y$ then, x = y **or** x + y = n.
2. ${}^nC_r = {}^nC_{n-r}$
3. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

3. Number of combinations of n dissimilar things taken 'r' at a time in which 'p' particular things will never occur is $\mathbf{{}^{n-p}C_r}$

E.g. *The number of ways a basketball team of 5 players chosen from 10 players, such that the player named Saurav should not be included in the team* = ${}^{10-1}C_5 = {}^9C_5$ = 126.

4. The number of ways in which (m + n) things can be divided into two groups containing m & n things respectively

$$\mathbf{{}^{(m+n)}C_n} = \frac{(m+n)!}{(m!\,n!)} = \mathbf{{}^{(m+n)}C_m}.$$

5. If 2m things are to be divided into two groups, each containing m things, the number of ways = $\frac{(2m)!}{[2(m!)^2]}$.

6. The number of ways to divide n things into different groups, one containing p things, another q things & so on is equal to

$$\frac{(p+q+r+.....)!}{[p!\,q!\,r!......)} \text{ where } \{n = p + q + r + \ldots\}$$

Distribution of Identical Objects

The total number of ways of dividing n identical items among r persons, each whom can receive 0, 1, 2, or more items (≤ n) is $\mathbf{{}^{n+r-1}C_{r-1}}$.

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is $\mathbf{{}^{n+r-1}C_{r-1}}$.

Ex.3 ***How many non – negative integral solutions are possible for the given equation?***

$$\mathbf{x + y + z = 16}$$

Sol. Here if we look out to the problem we will find 16 objects have to be distributed among 3 different persons (i.e. x, y, z). Hence n = 16, r = 3 and Total number of non – negative solutions = ${}^{16+3-1}C_{3-1}$ = ${}^{18}C_2$ = 153.

Ex.4 ***The total number of ways in which; 30 mangoes can be distributed among 5 persons.***

Sol. Required number is ${}^{30+5-1}C_{5-1} = {}^{34}C_4$

Ex.5 ***Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all balls.***

Sol. Since each box should get atleast one ball, after keeping one ball in each box, there will be 2 balls left. They can be arranged in ${}^{2+3-1}C_{3-1} = {}^{4}C_2 = 6$ ways.

Some important Results

i. Number of lines with n points = $\mathbf{{}^{n}C_2}$.

$\because$ For making a line exactly two points are required. So no. of ways in which we can choose two points out of n point is $\mathbf{{}^{n}C_2}$.

Combination is used here because line from A to B in same as B to A. So AB & BA are same.

(i) n lines can intersect at a maximum of $\mathbf{{}^{n}C_2}$ points.

(ii) Number of triangles with n points = $\mathbf{{}^{n}C_3}$.

(iii) Number of diagonals in n sided polygon = $\mathbf{{}^{n}C_2 - n}$

ii. The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the groups is *not important*, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}$$

iii. The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the groups is *important*, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}\times \mathbf{m! =} \frac{(mn)!}{(n!)^m}$$

iv. Total number of rectangles formed by **n** horizontal and **m** vertical lines in a plane = ${}^{m}C_2 \times {}^{n}C_2$.

Ex.6 ***In how many ways can a pack of 52 cards be divided equally among four players in order?***

Sol. Here 52 cards are to be divided into four equal groups and the order of the groups is important. So, required number of ways is

$$\left(\frac{52!}{(13!)^4\,4!}\right)4! = \frac{52!}{(13!)^4}$$

Ex.7 There are 10 points in a plane out of which 5 are collinear. The number of straight lines that can be drawn by joining these points will be:

Sol. Since we can draw one straight line through any 2 non – collinear points, so if all the 10 points under consideration were non – collinear we would have drawn $^{10}C_2$ straight lines through them.

But as 5 of them are collinear they will have only one straight line passing through them & not $^{5}C_2$.

Thus, no. of straight lines that can be drawn = $^{10}C_2 - {}^{5}C_2 + 1 = 36$.

Ex.8 In the above problem the number of triangles that can be drawn will be:

Sol. Again, for drawing one triangle we require 3 points (not all collinear).

But as 5 points are collinear we cannot draw any triangle out of those five.

So, the required no. of triangles = $^{10}C_3 - {}^{5}C_3 = 110$.

Ex.9 A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry part II, unless Chemistry part I is also borrowed. In how many ways can he choose the three books to be borrowed?

Sol. We have the following three possibilities:

(i) **When only chemistry part I is borrowed.**

In this case the boy will not borrow Chemistry part II.

So, now he has to select two books out of the remaining 6 books of his interest.

This can be done in $^{6}C_2$ ways.

(ii) **When both chemistry part I and chemistry part II are borrowed.**

Now he has to select one book out of the remaining 6 books of his interest.

This can be done in $^{6}C_1$ ways.

(iii) **When chemistry part I is not borrowed.**

In this case the boy does not want to borrow Chemistry part II.

So, he has to select three books from the remaining 6 books.

This can be done in $^{6}C_3$ ways.

Hence, the required number of ways = $^{6}C_2 + {}^{6}C_1 + {}^{6}C_3 = 15 + 6 + 20 = 41$.