Data structure and algorithm in Python

Tree

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General Trees

Tree is one of the most important nonlinear data structures.

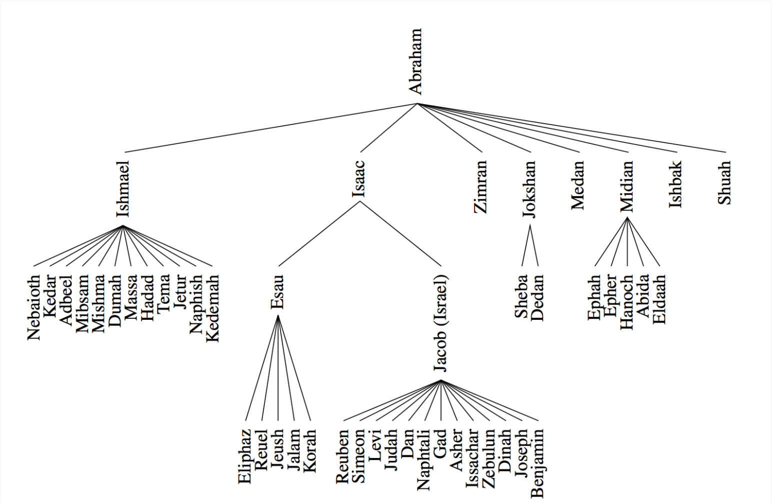
- Tree structures are indeed a breakthrough in data organization, for they allow us to implement a host of algorithms much faster than when using linear data structures, such as array-based lists or linked lists.
- Trees also provide a natural organization for data, and consequently have become ubiquitous structures in file systems, graphical user interfaces, databases, Web sites, and other computer systems.

General Trees

When we say that trees are "nonlinear", we are referring to an organizational relationship that is richer than the simple "before" and "after" relationships between objects in sequences. The relationships in a tree are hierarchical, with some objects being "above" and some "below" others.

General Trees

Actually, the main terminology for tree data structures comes from family trees, with the terms “parent”, “child”, “ancestor” and “descendant” being the most common words used to describe rela- tionships.



General Trees

Tree Definitions and Properties

Tree Definitions and Properties

A tree is an abstract data type that stores elements hierarchically.

With the exception of the top element, each element in a tree has a parent element and zero or more children elements.

We typically call the top element the root of the tree, but it is drawn as the highest element, with the other elements being connected below.

Tree Definitions and Properties

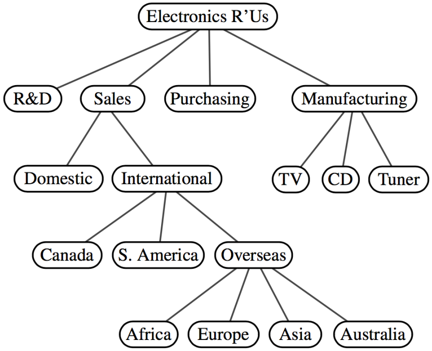


Figure 8.2: A tree with 17 nodes representing the organization of a fictitious corporation. The root stores *Electronics R'Us*. The children of the root store *R&D*, *Sales*, *Purchasing*, and *Manufacturing*. The internal nodes store *Sales*, *International*, *Overseas*, *Electronics R'Us*, and *Manufacturing*.

Tree Definitions and Properties

Definition : Formal Tree Definition

A tree T is a set of nodes storing elements such that the nodes have a parent-child relationship that satisfies the following properties:

- If T is nonempty, it has a special node, called the root of T , that has no parent.
- Each node v of T different from the root has a unique parent node w; every node with parent w is a child of w.

Tree Definitions and Properties

According to the definition, a tree can be empty, meaning that it does not have any nodes. This convention also allows us to define a tree recursively such that a tree T is either empty or consists of a node r, called the root of T, and a (possibly empty) set of subtrees whose roots are the children of r.

Tree Definitions and Properties

- Two nodes that are children of the same parent are siblings.
- A node v is external if v has no children.
- A node v is internal if it has one or more children. External nodes are also known as leaves.
- A node u is an ancestor of a node v if $u = v$ or u is an ancestor of the parent of v.

  Conversely, we say that a node v is a descendant of a node u if u is an ancestor of v.
- The subtree of T rooted at a node v is the tree consisting of all the descendants of v in T (including v itself).

Tree Definitions and Properties

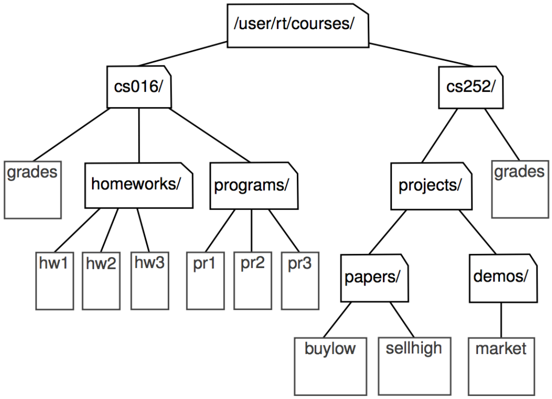


Figure 8.3: Tree representing a portion of a file system.

General Trees

- An edge of tree T is a pair of nodes (u, v) such that u is the parent of v, or vice versa.
- A path of T is a sequence of nodes such that any two consecutive nodes in the sequence form an edge.

Tree Definitions and Properties

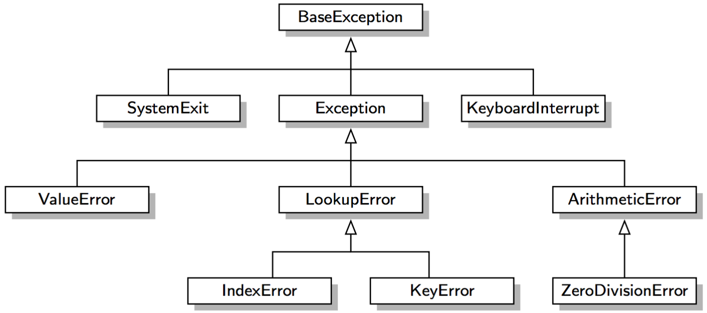


Figure 8.4: A portion of Python's hierarchy of exception types.

Tree Definitions and Properties

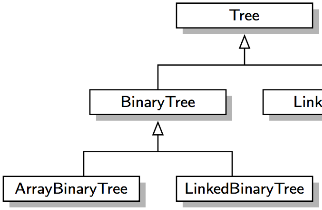


Figure 8.5: Our own inheritance hierarchy for modeling various abstractions and implementations of tree data structures. In the remainder of this chapter, we provide implementations of Tree, BinaryTree, and LinkedBinaryTree classes, and high-level sketches for how LinkedTree and ArrayBinaryTree might be designed.

Tree Definitions and Properties

Definition : Ordered Tree

A tree is ordered if there is a meaningful linear order among the children of each node; that is, we purposefully identify the children of a node as being the first, second, third, and so on. Such an order is usually visualized by arranging siblings left to right, according to their order.

Tree Definitions and Properties

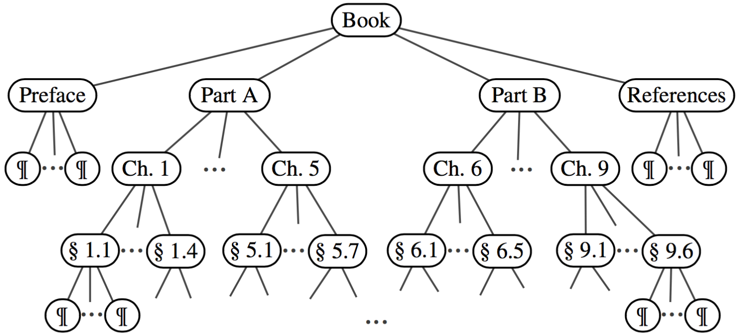


Figure 8.6: An ordered tree associated with a book.

General Trees

The Tree Abstract Data Type

The Tree Abstract Data Type

We define a tree ADT using the concept of a position as an abstraction for a node of a tree. An element is stored at each position, and positions satisfy parent-child relationships that define the tree structure.

A position object for a tree supports the method:

- `p.element()`: Return the element stored at position p.

The Tree Abstract Data Type

The tree ADT then supports the following accessor methods, allowing a user to navigate the various positions of a tree:

- `T.root()`: Return the position of the root of tree T or None if T is empty.
- `T.is_root(p)`: Return True if position p is the root of tree T.
- `T.parent(p)`: Return the position of the parent of position p, or None if p is the root of T.
- `T.num_chidren(p)`: Return the number of children of position p.
- `T.children(p)`: Generate an iteration of the children of position p.
- `T.is_leaf()`: Return True if position p does not have any chidlren.

The Tree Abstract Data Type

- `len(T)`: Return the number of positions (and hence elements) that are contained in tree T.
- `T.is_empty()`: Return True if tree T does not contain any positions.
- `T.positions()`: Generate an iteration of all positions of T.
- `iter(T)`: Generate an iteration of all elements stored within T.

The Tree Abstract Data Type

- `len(T)`: Return the number of positions (and hence elements) that are contained in tree T.
- `T.is_empty()`: Return True if tree T does not contain any positions.
- `T.positions()`: Generate an iteration of all positions of T.
- `iter(T)`: Generate an iteration of all elements stored within T.

Any of the above methods that accepts a position as an argument should generate a **ValueError** if that position is invalid for T.

The Tree Abstract Data Type

- If T is ordered, then `T.chidren(p)` reports the chidren of p in the natural order.
- If p is a leaf, then `T.chidren(p)` generates an empty iteration.
- If T is empty, then both `T.positions()` and and `iter(T)` generate empty iterations.

The Tree Abstract Data Type

A formal mechanism to designate the relationships between different implementations of the same abstraction is through the definition of one class that serves as an abstract base class, via inheritance, for one or more concrete classes.

The Tree Abstract Data Type

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We choose to define a Tree class that serves as an abstract base class corresponding to the tree ADT.

The Tree Abstract Data Type

However, our `Tree` class does not define any internal representation for storing a tree, and five of the methods given in that code fragment remain abstract (`root`, `parent`, `num_children`, `children`, and `__len__`); each of these methods raises a **NotImplementedError**.

The Tree Abstract Data Type

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The subclasses are responsible for overriding abstract methods, such as children, to provide a working implementation for each behavior, based on their chosen internal representation.

The Tree Abstract Data Type

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The subclasses are responsible for overriding abstract methods, such as children, to provide a working implementation for each behavior, based on their chosen internal representation.

With the `Tree` class being abstract, there is no reason to create a direct instance of it, nor would such an instance be useful. The class exists to serve as a base for inheritance, and users will create instances of concrete subclasses.

The Tree Abstract Data Type

```
class Tree:
  class Position:
    def element(self):
      raise NotImplementedError('must be
      implemented by subclass')

    def __eq__(self, other):
      raise NotImplementedError('must be
      implemented by subclass')

    def __ne__(self, other):
      return not (self == other)
```

The Tree Abstract Data Type

```
def root(self):
  raise NotImplementedError('must be
  implemented by subclass')

def parent(self, p):
  raise NotImplementedError('must be
  implemented by subclass')

def num_children(self, p):
  raise NotImplementedError('must be
  implemented by subclass')

def children(self, p):
  raise NotImplementedError('must be
  implemented by subclass')
```

The Tree Abstract Data Type

```
def __len__(self):
  raise NotImplementedError('must be
  implemented by subclass')

def is_root(self, p):
  return self.root() == p

def is_leaf(self, p):
  return self.num_children(p) == 0

def is_empty(self):
  return len(self) == 0
```

General Trees

Computing Depth and Height

Computing Depth and Height

Definition : Depth

Let p be the position of a node of T. The depth of p is the number of ancestors of p, excluding p itself.

Note that this definition implies that the depth of the root of T is 0.

Computing Depth and Height

The depth of p can also be recursively defined as follows:

- If p is the root, then the depth of p is 0.
- Otherwise, the depth of p is one plus the depth of the pararent of p.

Computing Depth and Height

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- If *p* is the root, then the depth of *p* is 0.
- Otherwise, the depth of *p* is one plus the depth of the pararent of *p*.

```
def depth(self, p):
  if self.is_root(p):
    return 0
  else:
    return 1 + self.depth(self.parent(p))
```

Computing Depth and Height

- The running time of `T.depth(p)` for a position p is $O(d_p + 1)$, where d_p denotes the depth of p in T.
- `T.depth(p)` runs in $O(n)$ worst-case time, where n is the total number of positions of T, because a position of T may have depth $n-1$ if all nodes form a single branch.

Computing Depth and Height

Definition : Height

The height of a position p in a tree T is also defined recursively:

- If p is a leaf,then the height of p is 0.
- Otherwise, the height of p is one more than the maximum of the heights of p’ s children.

The height of a nonempty tree T is the height of the root of T.

Computing Depth and Height

Property

The height of a nonempty tree T is equal to the maximum of the depths of its leaf positions.

Computing Depth and Height

Property

The height of a nonempty tree T is equal to the maximum of the depths of its leaf positions.

```
def _height1(self): # O(n^2) worst-case time
  return max(self.depth(p)
             for p in self.positions()
             if self.is_leaf(p))
```

Computing Depth and Height

We can compute the height of a tree more efficiently, in $O(n)$ worst-case time, by relying instead on the original recursive definition. To do this, we will parameterize a function based on a position within the tree, and calculate the height of the subtree rooted at that position.

Computing Depth and Height

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```python
def _height2(self, p):  # time is linear in
size of subtree
  if self.is_leaf(p):
    return 0
  else:
    return 1 + max(self._height2(c)
                   for c in self.children(p))
```

Computing Depth and Height

```
def height(self, p=None):
  if p is None:
    p = self.root()
  return self._height2(p)
```

Binary Tree

Binary Tree

Definition : Binary Tree

A binary tree is an ordered tree with following properties:

1. Every node has at most two children.
2. Each child node is labled as being either a left child or a right child.
3. A left child precedes a right child in the order of children of a node.

Binary Tree

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2. Each child node is labled as being either a left child or a right child.
3. A left child precedes a right child in the order of children of a node.

- The subtree rooted at a left or right child of an internal node v is called a left subtree or right subtree, respectively, of v.
- A binary tree is proper if each node has either zero or two children. Some people also refer to such trees as being full binary trees. Thus, in a proper binary tree, every internal node has exactly two children. A binary tree that is not proper is improper.

Binary Tree

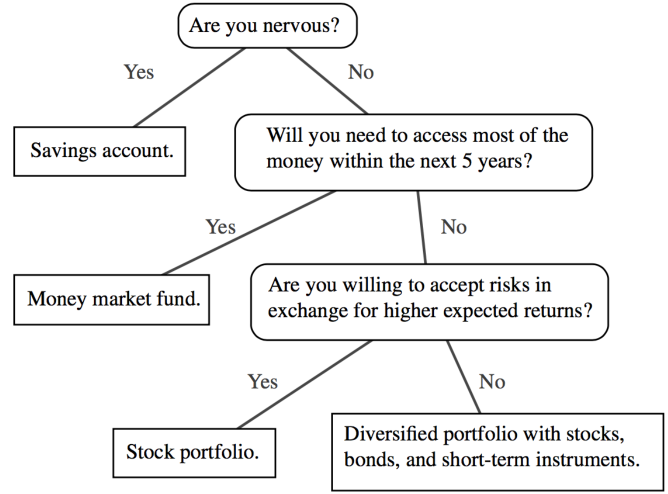

Are you nervous?
Yes
No
Savings account.
Will you need to access most of the money within the next 5 years?
Yes
No
Money market fund.
Are you willing to accept risks in exchange for higher expected returns?
Yes
No
Stock portfolio.
Diversified portfolio with stocks, bonds, and short-term instruments.

Figure 8.7: A decision tree providing investment advice.

Binary Tree

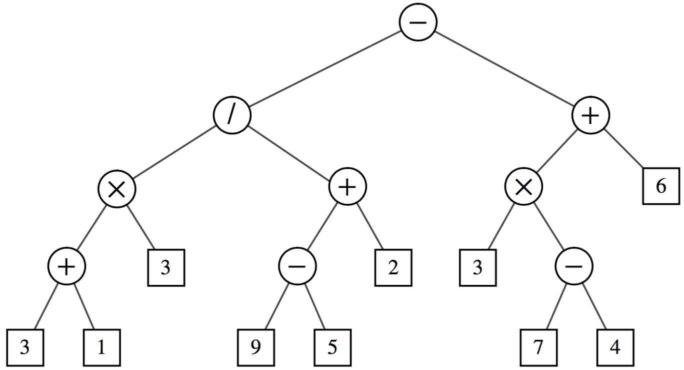


Figure 8.8: A binary tree representing an arithmetic expression. This tree represents the expression $((((3+1)\times 3)/((9-5)+2))-((3\times(7-4))+6))$. The value associated with the internal node labeled "/" is 2.

Binary Tree

Definition : A Recursive Binary Tree Definition

A binary tree is either empty or consists of

1. A node r, called the root of T, that stores an element
2. A binary tree (possibly empty), called the left subtree of T
3. A binary tree (possibly empty), called the right subtree of T

Binary Tree

The Binary Tree Abstract Data Type

The Binary Tree Abstract Data Type

As an abstract data type, a binary tree is a specialization of a tree that supports three additional accessor methods:

- `T.left(p)`: Return the position that represents the left child of *p*, or **None** if *p* has no left child.
- `T.right(p)`: Return the position that represents the right child of *p*, or **None** if *p* has no right child.
- `T.sibling(p)`: Return the position that represents the sibling of *p*, or **None** if *p* has no sibling.

The Binary Tree Abstract Data Type

```
from tree import Tree
class BinaryTree(Tree):

  def left(self, p):
    raise NotImplementedError('must be
    implemented by subclass')

  def right(self, p):
    raise NotImplementedError('must be
    implemented by subclass')
```

The Binary Tree Abstract Data Type

```
def sibling(self, p):
  parent = self.parent(p)
  if parent is None:
    return None
  else:
    if p == self.left(parent):
      return self.right(parent)
    else:
      return self.left(parent)
```

The Binary Tree Abstract Data Type

```
  def children(self, p):
    if self.left(p) is not None:
      yield self.left(p)
    if self.right(p) is not None:
      yield self.right(p)
```

Binary Tree

Properties of Binary Trees

Properties of Binary Trees

Binary trees have several interesting properties dealing with relationships between their heights and number of nodes.

We denote the set of all nodes of a tree T at the same depth d as level d of T. In a binary tree,

- level 0 has at most one node (the root),
- level 1 has at most two nodes (the children of the root),
- level 2 has at most four nodes,
- ……

In general, level d has at most 2^d nodes.

Properties of Binary Trees

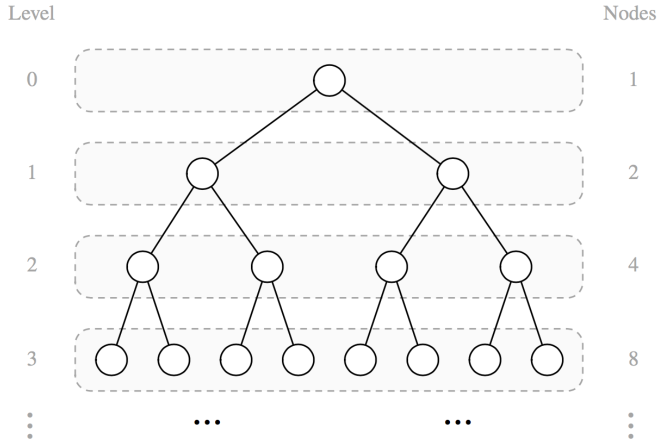


Figure 8.9: Maximum number of nodes in the levels of a binary tree.

Properties of Binary Trees

Property

Let T be a nonempty binary tree, and let n, n_E , n_I and h denote the number of nodes, number of external nodes, number of internal nodes, and height of T , respectively. Then T has the following properties:

- $h+1 \leq n \leq 2^{h+1}-1$
- $1 \leq n_E \leq 2^h$
- $h \leq n_I \leq 2^h-1$
- $\log(n+1)-1 \leq h \leq n-1$

Properties of Binary Trees

Property : Continue

If T is proper, then T has the following properties:

- $h+1 \leq n \leq 2^{h+1}-1$
- $1 \leq n_E \leq 2^h$
- $h \leq n_I \leq 2^h - 1$
- $\log(n+1) - 1 \leq h \leq n-1$

Properties of Binary Trees

Property

In a nonempty proper binary tree T, with n_E external nodes and n_I internal nodes, we have $n_E = n_I + 1$.

Implementing Trees

Implementing Trees

The Tree and BinaryTree classes that we have defined are both formally abstract base classes.

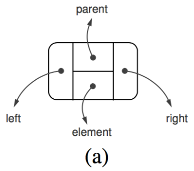
- Neither of them can be directly instantiated.
- Have not yet defined key implementation details for how a tree will be represented internally, and how we can effectively navigate between parents and children.
- A concrete implementation of a tree must provide methods
  - `root`
  - `parent`
  - `num_children`
  - `children`
  - `__len__`

  and in the case of BinaryTree, the additional accessors
  - `left`
  - `right`

Implementing Trees

Linked Structure for Binary Trees

Linked Structure for Binary Trees



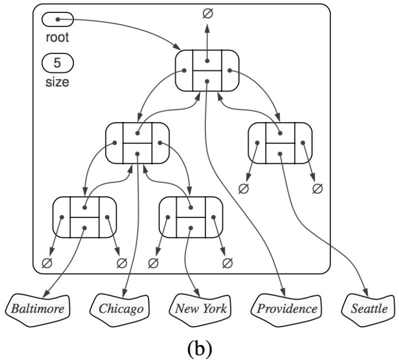


Figure 8.11: A linked structure for representing: (a) a single node; (b) a binary tree.

Linked Structure for Binary Trees

A natural way to realize a binary tree T is to use a linked structure, with a node that maintains references

- to the element stored at a position p
- to the nodes associated with the children and parent of p.

Linked Structure for Binary Trees

- If p is the root of T, then the parent field of p is `None`.
- If p does not have a left child (respectively, right child), the associated field is `None`.

Linked Structure for Binary Trees

The tree itself maintains

- an instance variable storing a reference to the root node (if any),
- a variable size that represents the overall number of nodes of T.

Operations for Updating a Linked Binary Tree

- `T.add_root(e)`:

  Create a root for an empty tree, storing e as the element, and return the position of that root; an error occurs if the tree is not empty.

- `T.add_left(p, e)`:

  Create a new node storing element e, link the node as the left child of position p, and return the resulting position; an error occurs if p already has a left child.

- `T.add_right(p, e)`:

  Create a new node storing element e, link the node as the right child of position p, and return the resulting position; an error occurs if p already has a right child.

Operations for Updating a Linked Binary Tree

- `T.replace(p, e)`:

  Replace the element stored at position p with element e, and return the previously stored element.

- `T.delete(p)`:

  Remove the node at position p, replacing it with its child, if any, and return the element that had been stored at p; an error occurs if p has two children.

- `T.attach(p, T1, T2)`:

  Attach the internal structure of trees T1 and T2, respec- tively, as the left and right subtrees of leaf position p of T, and reset T1 and T2 to empty trees; an error condition occurs if p is not a leaf.

Linked Structure for Binary Trees

```
from binary_tree import BinaryTree
class LinkedBinaryTree(BinaryTree):
  class _Node:
    __slots__ = '_element', '_parent', '_left',
    '_right'
    def __init__(self, element, parent=None,
    left=None, right=None):
      self._element = element
      self._parent = parent
      self._left = left
      self._right = right
```

Linked Structure for Binary Trees

```
class Position(BinaryTree.Position):
  def __init__(self, container, node):
    self._container = container
    self._node = node

  def element(self):
    return self._node._element

  def __eq__(self, other):
    return type(other) is type(self) and other
    ._node is self._node
```

Linked Structure for Binary Trees

```
def _validate(self, p):
  if not isinstance(p, self.Position):
    raise TypeError('p must be proper Position
     type')
  if p._container is not self:
    raise ValueError('p does not belong to
    this container')
  if p._node._parent is p._node:
    raise ValueError('p is no longer valid')
  return p._node

def _make_position(self, node):
  return self.Position(self, node) if node is
  not None else None
```

Linked Structure for Binary Trees

```
def __init__(self):
  self._root = None
  self._size = 0

def __len__(self):
  return self._size

def root(self):
  return self._make_position(self._root)

def parent(self, p):
  node = self._validate(p)
  return self._make_position(node._parent)
```

Linked Structure for Binary Trees

```
  def left(self, p):
    node = self._validate(p)
    return self._make_position(node._left)

  def right(self, p):
    node = self._validate(p)
    return self._make_position(node._right)
```

Linked Structure for Binary Trees

```
  def num_children(self, p):
    node = self._validate(p)
    count = 0
    if node._left is not None:
      count += 1
    if node._right is not None:
      count += 1
    return count
```

Linked Structure for Binary Trees

```
def _add_root(self, e):
  if self._root is not None:
    raise ValueError('Root exists')
  self._size = 1
  self._root = self._Node(e)
  return self._make_position(self._root)
```

Linked Structure for Binary Trees

```
  def _add_left(self, p, e):
    node = self._validate(p)
    if node._left is not None:
      raise ValueError('Left child exists')
    self._size += 1
    node._left = self._Node(e, node)
    return self._make_position(node._left)
```

Linked Structure for Binary Trees

```
def _add_right(self, p, e):
  node = self._validate(p)
  if node._right is not None:
    raise ValueError('Right child exists')
  self._size += 1
  node._right = self._Node(e, node)
  return self._make_position(node._right)
```

Linked Structure for Binary Trees

```
def _replace(self, p, e):
  node = self._validate(p)
  old = node._element
  node._element = e
  return old
```

Linked Structure for Binary Trees

```
def _delete(self, p):
  node = self._validate(p)
  if self.num_children(p) == 2:
    raise ValueError('Position has two
    children')
  child = node._left if node._left else node.
  _right
  if child is not None:
    child._parent = node._parent
```

Linked Structure for Binary Trees

```
    if node is self._root:
      self._root = child
    else:
      parent = node._parent
      if node is parent._left:
        parent._left = child
      else:
        parent._right = child
```

Linked Structure for Binary Trees

```
def _attach(self, p, t1, t2):
  node = self._validate(p)
  if not self.is_leaf(p):
    raise ValueError('position must be leaf')
  if not type(self) is type(t1) is type(t2):
    raise TypeError('Tree types must match')
  self._size += len(t1) + len(t2)
  if not t1.is_empty():
    t1._root._parent = node
    node._left = t1._root
    t1._root = None
    t1._size = 0
  if not t2.is_empty():
    t2._root._parent = node
    node._right = t2._root
    t2._root = None
    t2._size = 0
```

Implementing Trees

Linked Structure for General Trees

Linked Structure for General Trees

When representing a binary tree with a linked structure, each node explicitly maintains fields left and right as references to individual children.

Linked Structure for General Trees

When representing a binary tree with a linked structure, each node explicitly maintains fields left and right as references to individual children.

For a general tree, there is no a priori limit on the number of children that a node may have.

A natural way to realize a general tree T as a linked structure is to have each node store a single container of references to its children.

Linked Structure for General Trees

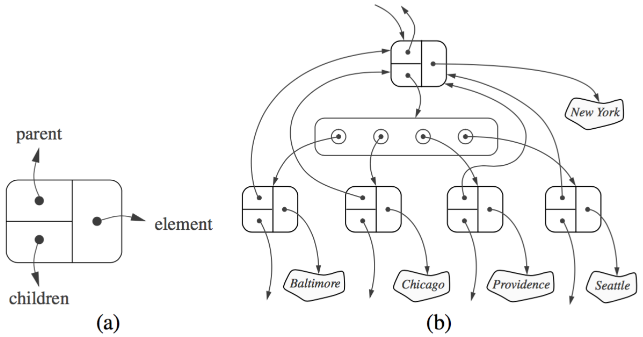
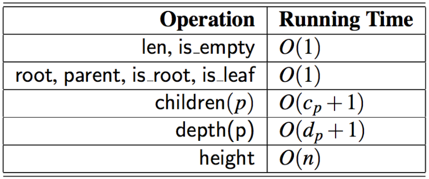


Figure 8.14: The linked structure for a general tree: (a) the structure of a node; (b) a larger portion of the data structure associated with a node and its children.

Linked Structure for General Trees



Operation	Running Time
len, is_empty	$O(1)$
root, parent, is_root, is_leaf	$O(1)$
children(p)	$O(c_p+1)$
depth(p)	$O(d_p+1)$
height	$O(n)$

Table 8.2: Running times of the accessor methods of an n-node general tree implemented with a linked structure. We let c_p denote the number of children of a position p. The space usage is $O(n)$.

Tree Traversal Algorithms

Tree Traversal Algorithms

A traversal of a tree T is a systematic way of accessing, or “visiting”, all the positions of T .

The specific action associated with the “visit” of a position p depends on the application of this traversal, and could involve anything from increment- ing a counter to performing some complex computation for p.

Tree Traversal Algorithms

Preorder and Postorder Traversals of General Trees

Preorder and Postorder Traversals of General Trees

In a preorder traversal of a tree T, the root of T is visited first and then the subtrees rooted at its children are traversed recursively. If the tree is ordered, then the subtrees are traversed according to the order of the children.

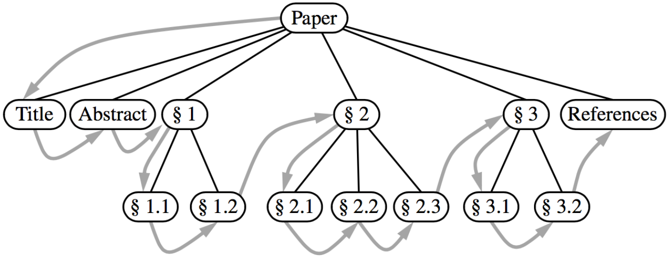


Figure 8.15: Preorder traversal of an ordered tree, where the children of each position are ordered from left to right.

Preorder and Postorder Traversals of General Trees

```
class Tree:

  def preorder(self):
    if not self.is_empty():
      for p in self._subtree_preorder(self.root
      ()):
        yield p

  def _subtree_preorder(self, p):
    yield p
    for c in self.children(p):
      for other in self._subtree_preorder(c):
        yield other
```

Preorder and Postorder Traversals of General Trees

```
class Tree:

  def postorder(self):
    if not self.is_empty():
      for p in self._subtree_postorder(self.root
      ()):
        yield p

  def _subtree_postorder(self, p):
    for c in self.children(p):
      for other in self._subtree_postorder(c):
        yield other
    yield p
```

Preorder and Postorder Traversals of General Trees

Both preorder and postorder traversal algorithms are efficient ways to access all the positions of a tree.

Preorder and Postorder Traversals of General Trees

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At each position p, the nonrecursive part of the traversal algorithm requires time $O(c_p + 1)$, where c_p is the number of children of p, under the assumption that the “visit” itself takes $O(1)$ time.

Preorder and Postorder Traversals of General Trees

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At each position p, the nonrecursive part of the traversal algorithm requires time $O(c_p + 1)$, where c_p is the number of children of p, under the assumption that the “visit” itself takes $O(1)$ time.

The overall running time for the traversal of tree T is $O(n)$, where n is the number of positions in the tree. This running time is asymptotically optimal since the traversal must visit all the n positions of the tree.

Tree Traversal Algorithms

Breadth-First Tree Traversal

Breadth-First Tree Traversal

Definition : Breadth-First Tree Traversal

Although the preorder and postorder traversals are common ways of visiting the positions of a tree, another common approach is to traverse a tree so that we visit all the positions at depth d before we visit the positions at depth $d+1$. Such an algorithm is known as a breadth-first traversal.

Breadth-First Tree Traversal

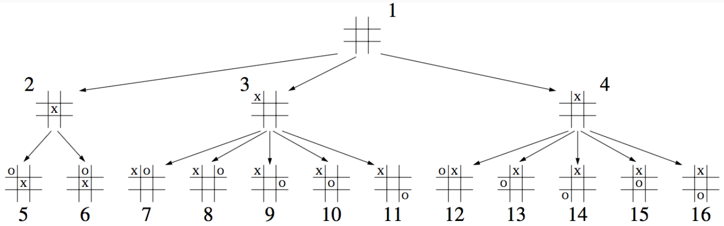

Figure 8.17: Partial game tree for Tic-Tac-Toe, with annotations displaying the order in which positions are visited in a breadth-first traversal.

Breadth-First Tree Traversal

```
class Tree:

  def breadthfirst(self):
    if not self.is_empty():
      fringe = LinkedQueue()
      fringe.enqueue(self.root())
      while not fringe.is_empty():
        p = fringe.dequeue()
        yield p
        for c in self.children(p):
          fringe.enqueue(c)
```

Tree Traversal Algorithms

Inorder Traversal of a Binary Tree

Inorder Traversal of a Binary Tree

During an inorder traversal, we visit a position between the recursive traversals of its left and right subtrees. The inorder traversal of a binary tree T can be informally viewed as visiting the nodes of T “from left to right”.

Inorder Traversal of a Binary Tree

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Indeed, for every position p, the inorder traversal visits p after all the positions in the left subtree of p and before all the positions in the right subtree of p.

Inorder Traversal of a Binary Tree

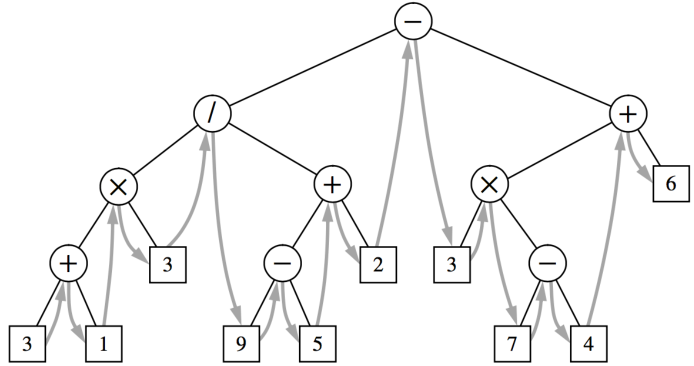

Figure 8.18: Inorder traversal of a binary tree.

Inorder Traversal of a Binary Tree

```
class BinaryTree(Tree):
```

```
  def inorder(self):
    if not self.is_empty():
      for p in self._subtree_inorder(self.root()
      ):
        yield p

  def _subtree_inorder(self, p):
    if self.left(p) is not None:
      for other in self._subtree_inorder(self.
      left(p)):
        yield other
    yield p
    if self.right(p) is not None:
      for other in self._subtree_inorder(self.
      right(p)):
        yield other
```

Expression Tree

Expression Tree

```
from linked_binary_tree import LinkedBinaryTree

class ExpressionTree(LinkedBinaryTree):
  def __init__(self, token, left=None, right=
  None):
    super().__init__()
    if not isinstance(token, str):
      raise TypeError('Token must be a string')
    self._add_root(token)
    if left is not None:
      if token not in '+-*x/':
        raise ValueError('token must be valid
        operator')
      self._attach(self.root(), left, right)
```

Expression Tree

```
def _parenthesize_recur(self, p, result):
  if self.is_leaf(p):
    result.append(str(p.element()))
  else:
    result.append('(')
    self._parenthesize_recur(self.left(p),
    result)
    result.append(p.element())
    self._parenthesize_recur(self.right(p),
    result)
    result.append(')')
```

Expression Tree

```
  def __str__(self):
    pieces = []
    self._parenthesize_recur(self.root(), pieces
    )
    return ''.join(pieces)
```

Expression Tree

```
  def evaluate(self):
    return self._evaluate_recur(self.root())
```

Expression Tree

```
def _evaluate_recur(self, p):
  if self.is_leaf(p):
    return float(p.element())
  else:
    op = p.element()
    left_val = self._evaluate_recur(self.left(
    p))
    right_val = self._evaluate_recur(self.
    right(p))
    if op == '+':
      return left_val + right_val
    elif op == '-':
      return left_val - right_val
    elif op == '/':
      return left_val / right_val
    else:
      return left_val * right_val
```

Expression Tree

```
def tokenize(raw):
  SYMBOLS = set('+-x*/() ')
  mark = 0
  tokens = []
  n = len(raw)
  for j in range(n):
    if raw[j] in SYMBOLS:
      if mark != j:
        tokens.append(raw[mark:j])
      if raw[j] != ' ':
        tokens.append(raw[j])
      mark = j+1
  if mark != n:
    tokens.append(raw[mark:n])
  return tokens
```

Expression Tree

```
def build_expression_tree(tokens):
  S = []
  for t in tokens:
    if t in '+-x*/':
      S.append(t)
    elif t not in '()':
      S.append(ExpressionTree(t))
    elif t == ')':
      right = S.pop()
      op = S.pop()
      left = S.pop()
      S.append(ExpressionTree(op, left, right))
  return S.pop()
```

Expression Tree

```
if __name__ == '__main__':
  big = build_expression_tree(tokenize('((((3 +
  1) * 3)/((9-5)+2))-((3x(7-4))+6))'))
  print(big, '=', big.evaluate())
```