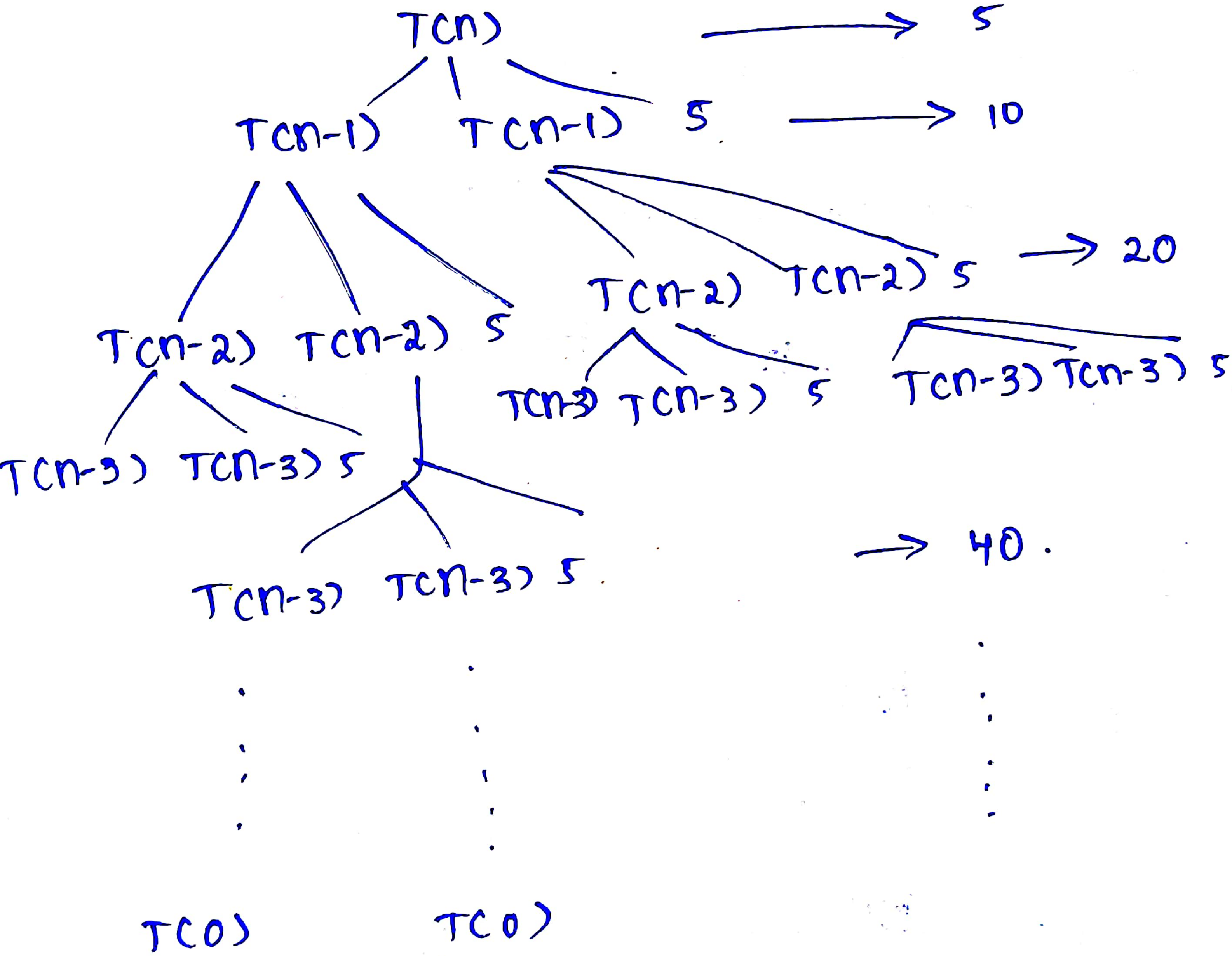
(1) $T(n) = 2(T(n-1)) + 5.$ Recursion Tree



$$T(n) = \begin{cases} 1 & n = 0 \\ 2T(n-1)+1 & n > 0 \end{cases}$$

$\rightarrow 5 + 10 + 20 + 40 \cdots$

$5 \times 2^0 + 5 \times 2^1 + 5 \times 2^2 + \cdots + 5 \times 2^k$

$5(2^0 + 2^1 + 2^2 + \cdots + 2^k)$

$5(1 + 2 + 4 + \cdots + 2^k)$

$$1 + x + x^2 + \cdots + x^k = \frac{x^{k+1} - 1}{x - 1} \quad (x \neq 1)$$

$\boxed{x = 2}$

$$5\left(\frac{2^{k+1} - 1}{2 - 1}\right)$$

$n - k = 0$

$\boxed{n = k}$

$= 5(2^{k+1} - 1)$

$= 5(2^{n+1} - 1)$

$\Rightarrow = 2^{n+1}(5) - 5$

$O(2^n)$ (or) $\boxed{O(2^n)}$

(2) $T(n) = 3T(n-1)$

$$T(n) = \begin{cases} 0 & n=1 \\ 3T(n-1) & n>1 \end{cases}$$

$T(n-1) = 3(3T(n-2))$

$= 3^2 T(n-2)$

$T(n-2) = 3^2(3T(n-3))$

$= 3^3 T(n-3)$

$T(k) = 3^k T(n-k)$

$n-k=0$

$\boxed{n=k}$

$T(n) = 3^n T(0)$

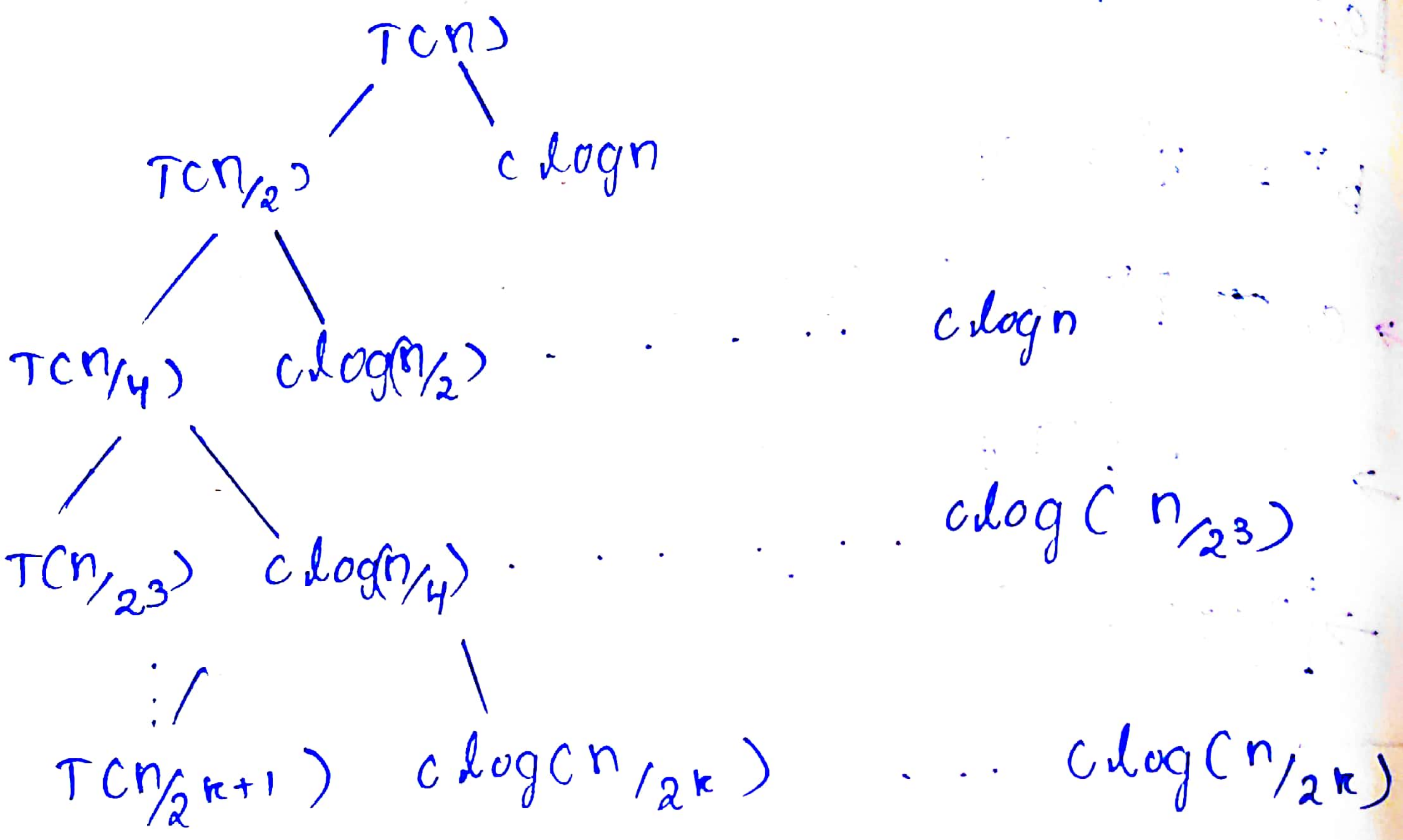
$T(n) = 3^n$

$\boxed{T(n) = \Theta(3^n)}$

(or)

$T(n) = \Theta(\exp(n))$

(3) $T(n) = T(n/2) + c\log n.$



$$T(n) = c\log n + \cdots + c\log(n/2^k)$$

$$= c(\log n + \cdots + \log(n/2^k))$$

$$= c(\log n + \cdots + 1)$$

$$= \log(n)^k \log(n) - \log(n) * \frac{(\log(n)+1)}{2}$$

$$= \log^2(n) - \frac{\log^2(n) - \log(2)}{2}$$

$= \theta(\log^2 n)$

---

(4) $T(n) = T(n/2) + cn^2$

$T(n)$

$T(n/2) \quad cn^2 \quad \cdots \quad cn^2$

$T(n/4) \quad c(n/2)^2 \qquad \frac{cn^2}{4}$

$T(n/8) \quad \frac{cn^2}{16} \qquad \frac{cn^2}{16}$

$\vdots$

$T(n/2^k) \quad \cdots \quad \frac{cn}{2^{2k}}$

$$T(n) = cn^2 + \frac{cn^2}{4} + \cdots \frac{cn}{2^{2k}}$$

$$= cn^2 \left(1 + \frac{1}{4} + \frac{1}{16} + \cdots \frac{1}{4^k}\right)$$

$\boxed{k = n}$

$$T(n) = cn^2 \left(\frac{1}{1 - \frac{1}{4}}\right)$$

$$= cn^2 \left(\frac{1}{\frac{3}{4}} \right)$$

$$= cn^2 \left(\frac{4}{3} \right)$$

$$T(n) = \frac{4}{3} cn^2$$

$$\boxed{T(n) = O(n^2)}$$

⑤ $T(n) = 2T(n/2) + \log n.$

$\log n \rightarrow \log n$

$\log \frac{n}{2} \quad \log n/2 \rightarrow 2\log(\frac{n}{2})$

$\log \frac{n}{4} \quad \log \frac{n}{4} \quad \log \frac{n}{4} \quad \log \frac{n}{4} \rightarrow 4\log \frac{n}{4}$

$\vdots$

$$T(n) = \log n + 2\log(\frac{n}{2}) + \cdots$$

$$T(n) = \log n + \sum_{k=1}^{\log n} 2^k \log(\frac{n}{2^k})$$

$$T(n) = \sum_{k=1}^{\log n} 2^k (\log n - k)$$

$$T(n) = \log n \sum_{k=1}^{\log n} 2^k - \sum_{k=1}^{\log n} k 2^k$$

$$T(n) = \log n (2^{\log n+1} - 2) - (\log n \, 2^{\log n+1} -$$

$$2^{\log n+1} + 2)$$

$$T(n) = 2n - \log n - 2 = O(n)$$

$$\rightarrow \boxed{O(n)}$$

---

(6) $T(n) = 8T(n/2) + n^2 \quad T(1) = 1.$

$$T(n) = \begin{cases} 1 & n = 1 \\ 8T(n/2) + n^2 & n > 1 \end{cases}$$

$$n^2 \longrightarrow n^2$$

$$\left(\frac{n}{2}\right)^2 \quad \left(\frac{n}{2}\right)^2 \longrightarrow 8\left(\frac{n}{2}\right)^2 = 2n^2$$

$$\left(\frac{n}{4}\right)^2 \quad \left(\frac{n}{4}\right)^2 \quad \left(\frac{n}{4}\right)^2 \quad \left(\frac{n}{4}\right)^2 \longrightarrow 8^2\left(\frac{n}{4}\right)^2 = 2^2 n^2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \longrightarrow 2^{\log n - 1} n^2$$

$$T(1) \quad T(1) \quad T(1) \longrightarrow 8^{\log n} T(1)$$

$$\Rightarrow n^2 + 2n^2 + 2^2 n^2 + \cdots + 2^{\log n - 1} n^2 + 8^{\log n}$$

$$\Rightarrow \sum_{k=0}^{\log n - 1} 2^k n^2 + 8^{\log n}$$

$$\Rightarrow n^2 \sum_{k=0}^{\log n - 1} 2^k + (2^3)^{\log n}$$

$$\Rightarrow n^2\left(\theta(2^{\log n - 1})\right) + (2^{\log n})^3$$

$$\Rightarrow n^2(\theta(n)) + n^3$$

$$\rightarrow \boxed{\theta(n^3)}$$

7 $T(n) = 2T(\frac{n}{2}) + n^3$

$n^3 \longrightarrow n^2$

$(\frac{n}{2})^3 \quad (\frac{n}{2})^3 \longrightarrow (\frac{1}{2}) n^3$

$(\frac{n}{4})^3 \quad (\frac{n}{4})^3 \quad (\frac{n}{4})^3 \quad (\frac{n}{4})^3 \longrightarrow \frac{1}{4} n^3$

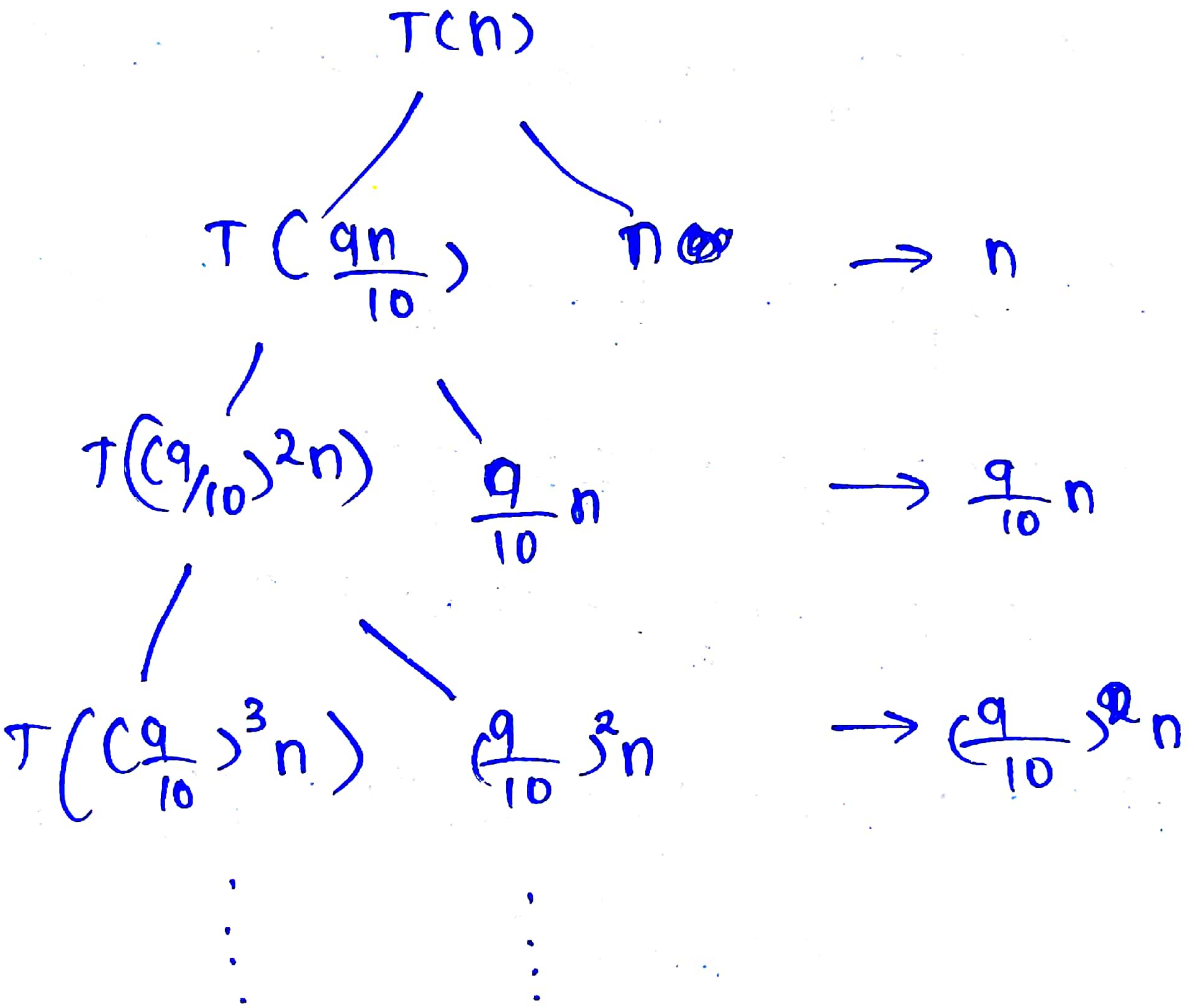
$T(\cancel{n}) = 1$

$n^3 + \frac{1}{2^k} n^3 + \frac{1}{2^2} * n^3 + \frac{1}{8} n^3 + \cdots$

$\frac{1}{2^0} n^3 + \frac{1}{2^1} n^3 + \frac{1}{2^2} n^3 + \frac{1}{2^3} n^3 + \cdots$

$$\sum_{k=0}^{\log n} \frac{n^3}{2^k} \rightarrow \boxed{O(n^3)}$$

(8) $T(n) = 2T\left(\frac{9n}{10}\right) + n.$



$$T(n) = n + \frac{9}{10}n + \left(\frac{9}{10}\right)^2 n + \left(\frac{9}{10}\right)^3 n + \cdots$$

$$\boxed{T(n) = \Theta(n)}$$

$n = 2\log_{10/9} n$

$n^{\log_{10/9} 2}$

$$= n + \frac{18}{10}n + \frac{364}{100}n + \cdots \Rightarrow n\left[\frac{1\left(9/5^{n} - 1\right)}{\left(9/5 - 1\right)}\right]$$

$$= nc. \quad \boxed{\Theta(n)}$$

(9) $T(n) = 16T(n/2) + (n\log n)^4$

$T(\frac{n}{2}) = 16(16T(\frac{n}{4}) + (\frac{n}{2}\log\frac{n}{2})^4) + (n\log n)^4$

$T(n/2) = 16^2 T(\frac{n}{4}) + 16(\frac{n}{2}\log\frac{n}{2})^4 + (n\log n)^4$

$T(n/4) = 16^2 (16T(\frac{n}{8}) + (\frac{n}{4}\log\frac{n}{4})^4) +$
$16(\frac{n}{2}\log\frac{n}{2}) + (n\log n)^4$

$T(n/4) = 16^3 T(\frac{n}{8}) + 16^2(\frac{n}{4}\log\frac{n}{4})^4 + 16(\frac{n}{2}\log\frac{n}{2})$
$+ (n\log n)^4$

$= 16^k T(\frac{n}{2^k}) + \sum_{k=0}^{k=\log n} 16^k (\frac{n}{2^k}\log\frac{n}{2^k})$

$= 16^k (T(\frac{n}{2^k}) + \sum_{k=0}^{k=\log n} (\frac{n}{2^k}\log\frac{n}{2^k}))$

$T(n) = n^4 (\log n)^5$

$\boxed{T(n) = O(n^4(\log n)^5)}$

(1) $T(n) = 7T(n/3) + n.$

$T(n) = 7\left(7T\left(\frac{n/3}{3}\right) + n/3\right) + n$

$T(n) = 7(7T(n/9) + n/3) + n.$

$T(n) = 49(T(n/9) + n/3) + n.$

$T(n) = 49T(n/9) + 7(n/3) + n$

$T(n) = 49\left(7T(n/27) + \frac{n}{9}\right) + 7(n/3) + n$

$T(n) = 7^3 T\left(\frac{n}{3^3}\right) + 7^2\left(\frac{n}{3^2}\right) + 7\left(\frac{n}{3}\right) + n$

$T(n) = n + 7\left(\frac{n}{3}\right) + 7^2\left(\frac{n}{3^2}\right) + 7^3 T\left(\frac{n}{3^3}\right) + \cdots$

$\cancel{\sum_{k=0} 7^k \left(\frac{n}{3^k}\right)} = n^{\log_3 7} + \frac{3}{4} n^{\log_3 7} - \frac{3}{4} n.$

$T(n) = \frac{7}{3} n^{\log 7}$

$\boxed{T(n) = O(n^{\log_3 7})}$

(11) $T(n) = 9T(n/3) + n^3 \log n$

$= 9(9T(n/3^2) + (\frac{n}{3})^3 \log(\frac{n}{3})) + n^3 \log n$

$= 9^2(T(n/3^2)) + 9(\frac{n}{3})^3 \log(\frac{n}{3}) + n^3 \log n$

$= 9^2 T(n/3^2) + 9(\frac{n^3}{\cancel{27}\,3}) \log n - \log 3 + n^3 \log n$

$= 9^2 T(n/3^2) + \frac{n^3}{3} \log(n/3) + n^3 \log n.$

$\vdots$

$= 9^k T(n/3^k) + \sum_{k=0}^{\log n} \frac{n^3}{3^k} \log \frac{n}{3^k}$

$\boxed{n = 3^k}$ $T(1) = 1$

$n = 3^k$

$\boxed{k = \log_3 n}$ $\boxed{T(n) = O(n^3 \log n)}$

$$= cn^2 \left(\frac{1}{\frac{3}{4}}\right)$$

$$= cn^2 \left(\frac{4}{3}\right)$$

$$T(n) = \frac{4cn^2}{3}$$

$$\boxed{T(n) = O(n^2)}$$

---

(12) $T(n) = 2T(n/4) + \sqrt{n}$

$$= 2\left[2T(n/16) + \sqrt{\frac{n}{4}}\right] + \sqrt{n}$$

$$= 2^2 T(n/16) + \sqrt{n} + \sqrt{n}$$

$$= 2^2\left[2T\left(\frac{n}{64}\right) + \frac{\sqrt{n}}{4}\right] + \sqrt{n} + \sqrt{n}$$

$$= 2^3 T\left(\frac{n}{64}\right) + \sqrt{n} + \sqrt{n} + \sqrt{n}$$

$$= 2^i T\left(\frac{n}{4^i}\right) + i\sqrt{n}$$

$1 = \frac{n}{4^i}$

$\boxed{i = \log_4 n}$

$T(1) = a.$

$2^{\log_4 n} \times a + \log_4 n \times \sqrt{n}$

$= a\sqrt{n} + \sqrt{n}\log_4 n.$

Solution: $O(\sqrt{n}\log_4 n)$

---

(13) $T(n) = 3T\left(\frac{n}{2}\right) + n\log n$

$= 3\left(3T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)\log\left(\frac{n}{2}\right)\right) + n\log n.$

$= 3^2 T\left(\frac{n}{4}\right) + 3\left(\frac{n}{2}\right)\log\left(\frac{n}{2}\right) + n\log n.$

$= 3^2\left(3T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)\log\left(\frac{n}{4}\right)\right) + \frac{3n}{2}\log\left(\frac{n}{2}\right)$

$+ n\log n.$

$$= 3^i T\left(\frac{n}{2^i}\right) + n\left(\log n + \frac{3}{2}\log\left(\frac{n}{2}\right) + \frac{9}{4}\left(\log\left(\frac{n}{4}\right)\right) + \cdots \frac{3^{k-1}}{2^{k-1}}\log\left(\frac{n}{2^{k-1}}\right)\right]$$

$$\boxed{n = 2^k}$$

$$= 3^k T(1) + 2^k \log^2 \left[\frac{k}{1-3/2} + \frac{(-1)(1-(\frac{3}{2})^{k-1})}{(1-3/2)^2} - \frac{k(k-1)(-1)}{(1-3/2)}\right]$$

$$= 3^k T(1) + 2^k\left[-2k + 4\left(\frac{3}{2}\right)^{k+1} - 1\right] + \left(\frac{3}{2}\right)^k \times 2\right]$$

$$1 = \frac{n}{2^k}$$

$$k = \log_2 n$$

$$= 3^{\log_2 n} + n\left[-2\log_2 n + 4\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right) + n^{\log_2 3/2} \times 2\right]$$

$$\boxed{\theta(n^{\log_2 3})}$$

(14) $5T(n/5) + \frac{\log n}{n}$ $\quad T(1) = 1$ $\quad \begin{cases} 1 & n=1 \\ 5T(n/5) + \frac{\log n}{n} & n>1 \end{cases}$

$$T(n) = 5^2 T(n/5^2) + \frac{5\log(n/5)}{n/5} + \frac{\log n}{n}$$

$$T(n) = 5^3 T\left(\frac{n}{5^3}\right) + \frac{5^2 \log(n/5^2)}{(n/5^2)} + \frac{5\log(n/5)}{(n/5)} + \frac{\log n}{n}$$

$\vdots$

$$= 5^3 T(n/5^k) + \frac{\log(n/5^{k-1})}{n} + \frac{\log(n/5^{k-2})}{n} + \cdots + \frac{\log(n/5)}{n} + \frac{\log n}{n}$$

$$= 5^k T(n/5^k) + \frac{1}{n}\left[(\log n - \log 5^{k-1}) + (\log n - \log 5^{k-2}) + \cdots + (\log n - \log 5) + \log n\right]$$

$$= 5^k T\left(\frac{n}{5^k}\right) + \frac{1}{n}\left[k\log n - \log 5^k + \log 5 - \log 5^k + \log 5^2 + \cdots - \log 5^2 - \log 5\right]$$

$$T(n) = 5^k T(n/5^k) + \frac{1}{n}[k \log n - k \log_5 k]$$

$$\boxed{\frac{n}{5^k} = 1} \quad \boxed{n = 5^k}$$

$$T(n) = n + \frac{1}{n}[k(\log n)(\log n) - (\log n)(\log n)]$$

$$T(n) = n$$

$$T(n) = \Theta(n)$$

---

(15) $4T(n/2) + n^2\sqrt{n}$

$$T(n) = 4T(n/2) + n^2\sqrt{n}.$$

$$= 4\left(4T(n/4) + (\tfrac{n}{2})^2\sqrt{\tfrac{n}{2}}\right) + n^2\sqrt{n}$$

$$= 4^2 T(\tfrac{n}{4}) + 4(\tfrac{n}{2})^2(\sqrt{\tfrac{n}{2}}) + n^2\sqrt{n}$$

$$= 4^2 T(n/4) + \frac{\cancel{4} \times n^2 \times \sqrt{n}}{\cancel{4} \times 2} + n^2\sqrt{n}$$

$$= 4^2 T(n/4) + \frac{\cancel{n}^2 \times \cancel{n}^{1/2}}{2} + n^2\sqrt{n}$$

$$= 4^2 T(n/4) + \frac{\cancel{4} n}{\cancel{8}} + n^2\sqrt{n}$$

$$T(n) = 4^2 T(n/4) + n^2\sqrt{n} + \frac{n}{2}$$

$$= 4^2 \left(4T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2 \left(\sqrt{\frac{n}{4}}\right) \right) + n^2\sqrt{n} + \frac{n}{2}$$

$$= 4^3 T\left(\frac{n}{8}\right) + \frac{\cancel{4^2} \times n^2 \times \sqrt{n}}{\cancel{4^2} \times 4} + n^2\sqrt{n} + \frac{n}{2}$$

$$= 4^3 T(n/8) + \frac{n}{4} + n^2\sqrt{n} + \frac{n}{2}$$

$$\vdots$$

$$= 4^k T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + n^2\sqrt{n} + \frac{n}{2^1}$$

$$\boxed{T(n) = \theta(n^{5/2})}$$

(16) $T(n) = 2T(n/2) + \frac{n}{\log n}$

$= 2\left[2T(n/2) + \frac{n/2}{\log(n/2)}\right] + \frac{n}{\log n}.$

$= 2^k T\left(\frac{n}{2^k}\right) + \left(\frac{n}{\log(n)} + \frac{n}{\log n/2} + \cdots\right)$

Base case $n = 2^k$

$= n \times T(1) + n\left[\frac{1}{\log n} + \frac{1}{\log \frac{n}{2}} \cdots + \frac{1}{\log \frac{n}{2^k}}\right]$

$= n \times T(1) + n\left[\frac{1}{\log n} + \cdots + 1\right]$

$= n \times T(1) + n\left[\frac{1}{k \log 2} + \frac{1}{(k-1)\log 2} + \cdots + \frac{1}{\log 2}\right]$

$= n \times (T(1)) + \frac{n}{\log 2}\left[\frac{1}{k} + \frac{1}{k} + \cdots + 1\right]$

$= na + \frac{n}{\log 2}\left[\log k\right] \Rightarrow na + \frac{n}{\log 2}\left[\log(\log_2 n)\right]$

$= na + n\log(\log_2 n) - \frac{n \log \log_2}{\log_2}$

$= \Theta(n\log(\log n))$

---

(17) $T(n) = T(n-1) + \frac{1}{n}$

$T(n) = \left[T(n-2) + \frac{1}{n-1}\right] + \frac{1}{n}$

$T(n) = T(n-k) + \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k+1}$

$\boxed{n = k}$

$T(n) = T(0) + \frac{1}{n} + \cdots + 1$

$T(n) = a + \log n.$

$\boxed{\Theta(\log n)}$

(18) $T(n) = T(n-1) + \log n.$

$T(n) = T(n-2) + \log(n-1) + \log(n)$

$T(n) = T(n-k) + \log(n-k-1) + \cdots + \log n$

$\boxed{n = k}$ $\boxed{n-k = 0}$

$T(n) = T(0) + \log(k-1) + \cdots + \log(1)$

$T(n) = T(0) + n\log n$

$T(n) = a + n\log n.$

$\boxed{T(n) = O(n\log n)}$

---

(19) $T(n) = T(n-2) + 2\log(n)$

$T(n) = T(n-4) + 2\log(n-2) + 2\log n$

$T(n) = T(n-6) + 2\log(n-4) + 2\log(n-2)$
$+ 2\log n.$

$$T(n) = T(n-2k) + 2\log(n-(2k-2)) + 2\log(n-(2k-4)) + \dots + 2\log n.$$

$$\boxed{T(1) = 1}$$

$$\boxed{n-2k=1} \Rightarrow \boxed{n = 2k} \rightarrow \boxed{k = \frac{(n-1)}{2}}$$

$$T(n) = T(1) + 2[\log 3 + \log 5 + \dots + \log n]$$

$$T(n) = T(1) + 2(\log(3 * 5 * \dots n))$$

$$= T(1) + 2\left(\log\left(\frac{n!}{2 * 4 * n}\right)\right)$$

$$= T(1) + 2\left(\log\left(\frac{n!}{2(1 * \dots \lfloor \frac{n}{2} \rfloor}\right)\right)$$

$$= T(1) + 2\left(\log\left(\frac{n!}{2\lfloor \frac{n}{2} \rfloor!}\right)\right)$$

$$\boxed{T(n) = O(n \log n)}$$

20) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$

$$T(n) = \sqrt{n}\, T(\sqrt{n}) + n$$

$$= \sqrt{n}\left(n^{\frac{1}{2^2}}\, T(n^{\frac{1}{2^2}}) + n^{\frac{1}{2}}\right) + n$$

$$= n^{\frac{1}{2}+\frac{1}{2^2}}\, T(n^{\frac{1}{2^2}}) + n^{\frac{1}{2}+\frac{1}{2}} + n$$

$$= n^{\frac{1}{2}+\frac{1}{2^2}}\left(n^{\frac{1}{2^3}}\, T(n^{\frac{1}{2^3}}) + n^{\frac{1}{2^2}}\right) + 2n$$

$$= n^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}\, T(n^{\frac{1}{2^3}}) + 3n$$

$$\vdots$$

$$= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\, T(n^{\frac{1}{2^k}}) + kn.$$

$$n^{\frac{1}{2^k}} = 2$$

$$\frac{1}{2^k} \log_2(n) = \log_2(2)$$

$$\log_2(n) = 2^k$$

$$\log_2 \log_2(n) = k \log_2(2)$$

$$\log_2 \log_2(n) = k$$

$$T(n) = n \sum_{i=1}^{\log_2 \log_2(n)} \frac{1}{2^i} T\left(\frac{1}{n\, 2^{\log_2 \log_2(n)}}\right) +$$

$$n \log_2 \log_2(n)$$

$$\sum_{i=1}^{\log_2 \log_2(n)} \frac{1}{2^i} = 1 - \frac{1}{\log_2(n)}$$

$$\boxed{T(n) = O(n \log_2 \log_2 n)}$$