Qsys Mathematics

Evolution

The quantum system will evolve according to the Schrodinger equation in the absence of any user input:

$$\frac{\partial}{\partial t}\psi(x,t) = \frac{-i}{\hbar}\hat{H}(t)\psi(x,t) \tag{1}$$

Without explicit time dependence in the Hamiltonian, we can immediately find the general solution is:

$$\hat{U}(t) = exp\left(\frac{-i\hat{H}t}{\hbar}\right) \tag{2}$$

We call this matrix a "propagator". For the sake of computational simplicity, we partition the simulation time finely and evaluate the hamiltonian at each time step:

$$\begin{aligned}
&[t_0, t_f] \rightarrow t_0, t_1, t_2, ..., t_f \\
&\quad \hat{H}_0 := \hat{H}(t_0) && U_0 := exp\left(\frac{-i\hat{H}_0\delta t}{\hbar}\right) \\
&\quad \hat{H}_1 := \hat{H}(t_1) && U_1 := exp\left(\frac{-i\hat{H}_1\delta t}{\hbar}\right) \\
&\quad \hat{H}_2 := \hat{H}(t_2) && U_2 := exp\left(\frac{-i\hat{H}_2\delta t}{\hbar}\right) \\
&\qquad . \\
&\qquad . \\
&\qquad .
\end{aligned}$$

where $\delta t = t_{n+1} - t_n$ is the width of the time partition. Note that U_n does not have explicit time dependence but it does keep a factor of δt. This is because we are shifting all of the time dependence into the evaluation of the Hamiltonian. Each of these propagators should only advance a single time step before the next propagator is used.

Symbolically,

$$\psi(t_{n+1}) = U_{n+1}\psi(t_n) \tag{3}$$

Measurements occur when user input is detected. The measurement procedure takes several steps:

- Wavefunction Sampling
- Conditional Probabilities Sampling
- Bayesian Update

Step 1: Wavefunction Sampling:
The wavefunction vector is multiplied by its complex conjugate with a dot product to give a probability distribution:

$$P(x) = \psi(x) \cdot \psi^*(x) \tag{4}$$

This probability distribution is sampled to give the "hidden state" of the system, λ.
Step 2: Conditional Probabilities Sampling:
The user input, ϕ, and λ together index an $n \times n \times n$ collection of conditional probabilities (preset and arbitrary, you can think of this as a list of n, $n \times n$ matrices.) These probabilities are of the form

$$P(o|\lambda, \phi) \tag{5}$$

We call this collection a list of POVMs (this stands for Positive-Operator-Valued Measure). They control how measurements are performed on the system. This list of POVMs is pre-defined and arbitrary. Imagine it as a matrix of matrices, where the user selects a column from the pitch they pressed (ϕ) and the system selects a row from its internal state (λ). You can think of ϕ as being a measurement setting on our measuring device. Now, you have selected a single element (matrix) from this collection of matrices. We refer to this as M.

$$M = \begin{pmatrix} P(A|\lambda,\phi) & 0 & 0 & 0 & ... \\ 0 & P(B|\lambda,\phi) & 0 & 0 & ... \\ 0 & 0 & P(C|\lambda,\phi) & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix} \tag{6}$$

By selecting a λ and a ϕ, we can extract a conditioned probability distribution for the output pitches, o. This comes from the diagonal elements of M. This distribution is sampled to give the result of the measurement and thus the pitch that is played.

Step 3: Bayesian Update:
The state is updated to:

$$\psi(x)_{new} = \frac{\sqrt{M}\psi(x)_{old}}{\sqrt{\psi^*(x)_{old} M \psi(x)_{old}}} \tag{7}$$

where $\psi(x)_{old}$ was the state immediately before the measurement occurred.

Sample Hamiltonians

Labeling the rows/columns by [G, A, B, D, F#]:

$$\frac{\omega}{2}\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{8}$$

This hamiltonian should cause oscillations between two ”chords”. The state should oscilate from G, B, D to D, F#, A at some frequency determined by ω (higher value for ω will yield faster oscillations).

We can immediately come up with a time dependent Hamiltonian by introducing functions of t somewhere in the matrix (maintaining the Hermetian symmetry condition):

$$\frac{\omega}{2}\begin{pmatrix} 0 & e^t & 0 & i & 1 \\ e^t & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & i\cos(t) \\ -i & 1 & 1 & 2\sin(t) & 1 \\ 1 & 0 & -i\cos(t) & 1 & 0 \end{pmatrix} \tag{9}$$

- Wavefunction
  - Mathematical Type: complex-valued probability amplitude
  - Computational Type: complex array (used here as a column vector)
  - Parameters:
    * Dimension - 2-12 complex dimensions, 4-24 real
    * Time Dependent
- Symbol: $\psi(x,t)$
- Hamiltonian

- Mathematical Type: complex-valued Hermetian matrix
- Computational Type: complex array (always square, symmetric under conjugate transpose)
- Parameters:
  * Dimension - 2-12 cplx, 4-24 real
  * Time Dependent

- Symbol: $\hat{H}(t)$