

Qsys Mathematics

Evolution

The quantum system will evolve according to the Schrodinger equation in the absence of any user input:

$$\frac{\partial}{\partial t}\psi(x, t) = \frac{-i}{\hbar}\hat{H}(t)\psi(x, t) \quad (1)$$

Without explicit time dependence in the Hamiltonian, we can immediately find the general solution is:

$$\hat{U}(t) = \exp\left(\frac{-i\hat{H}t}{\hbar}\right) \quad (2)$$

We call this matrix a "propagator". For the sake of computational simplicity, we partition the simulation time finely and evaluate the hamiltonian at each time step:

$$\begin{aligned} [t_0, t_f] &\rightarrow t_0, t_1, t_2, \dots, t_f \\ \hat{H}_0 &:= \hat{H}(t_0) & U_0 &:= \exp\left(\frac{-i\hat{H}_0\delta t}{\hbar}\right) \\ \hat{H}_1 &:= \hat{H}(t_1) & U_1 &:= \exp\left(\frac{-i\hat{H}_1\delta t}{\hbar}\right) \\ \hat{H}_2 &:= \hat{H}(t_2) & U_2 &:= \exp\left(\frac{-i\hat{H}_2\delta t}{\hbar}\right) \\ &\vdots & & \\ &\vdots & & \\ &\vdots & & \end{aligned}$$

where $\delta t = t_{n+1} - t_n$ is the width of the time partition. Note that U_n does not have explicit time dependence but it does keep a factor of δt . This is because we are shifting all of the time dependence into the evaluation of the Hamiltonian. Each of these propagators should only advance a single time step before the next propagator is used.

Symbolically,

$$\psi(t_{n+1}) = U_{n+1}\psi(t_n) \quad (3)$$

Measurements occur when user input is detected. The measurement procedure takes several steps:

- Wavefunction Sampling
- Conditional Probabilities Sampling
- Bayesian Update

Step 1: Wavefunction Sampling:

The wavefunction vector is multiplied by its complex conjugate with a dot product to give a probability distribution:

$$P(x) = \psi(x) \cdot \psi^*(x) \quad (4)$$

This probability distribution is sampled to give the "hidden state" of the system, λ .

Step 2: Conditional Probabilities Sampling:

The user input, E_ϕ , and λ together index an $n \times n \times n$ collection of conditional probabilities (preset and arbitrary, you can think of this as a list of n , $n \times n$ matrices.) These probabilities are of the form

$$P(o|\lambda, E_\phi) \quad (5)$$

We call this collection a list of POVMs (this stands for Positive-Operator-Valued Measure). They control how measurements are performed on the system. This list of POVMs is pre-defined and arbitrary. Imagine it as a matrix of matrices, where the user selects a column from the pitch they pressed (E_ϕ) and the system selects a row from its internal state (λ). Now, you have selected a single element (matrix) from this collection of matrices. We refer to this as M .

$$M = \begin{pmatrix} P(A|\lambda, E_\phi) & 0 & 0 & 0 & \dots \\ 0 & P(B|\lambda, E_\phi) & 0 & 0 & \dots \\ 0 & 0 & P(C|\lambda, E_\phi) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (6)$$

By selecting a λ and a E_ϕ , we can extract a conditioned probability distribution for the output pitches, o . This comes from the diagonal elements of M . This distribution is sampled to give the result of the measurement and thus the pitch that is played.

Step 3: Bayesian Update:

The state is updated to:

$$\psi(x)_{new} = \frac{M\psi(x)_{old}M}{\psi^*(x)_{old}M\psi(x)_{old}} \quad (7)$$

where $\psi(x)_{old}$ was the state immediately before the measurement occurred.

Sample Hamiltonians

Labeling the rows/columns by [G, A, B, D, F#]:

$$\frac{\omega}{2} \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \quad (8)$$

This hamiltonian should cause oscillations between two "chords". The state should oscilate from G, B, D to D, F#, A at some frequency determined by ω (higher value for ω will yield faster oscillations).

We can immediately come up with a time dependent Hamiltonian by introducing functions of t somewhere in the matrix (maintaining the Hermetian symmetry condition):

$$\frac{\omega}{2} \begin{pmatrix} 0 & e^t & 0 & i & 1 \\ e^t & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & i \cos(t) \\ -i & 1 & 1 & 2 \sin(t) & 1 \\ 1 & 0 & -i \cos(t) & 1 & 0 \end{pmatrix} \quad (9)$$

- Wavefunction
 - Mathematical Type: complex-valued probability amplitude
 - Computational Type: complex array (used here as a column vector)
 - Parameters:
 - * Dimension - 2-12 complex dimensions, 4-24 real
 - * Time Dependent
- Symbol: $\psi(x, t)$
- Hamiltonian
 - Mathematical Type: complex-valued Hermetian matrix

- Computational Type: complex array (always square, symmetric under conjugate transpose)
 - Parameters:
 - * Dimension - 2-12 cplx, 4-24 real
 - * Time Dependent
- Symbol: $\hat{H}(t)$