An Investigation into HOGWILD!

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Contents

1 Introduction

Despite being the subject of much modern excitement, the ideas of gradient descent date all the way back to Cauchy in 1847. And although it's main application has been in the solution of recent big-data optimization problems, stochastic gradient has been around since the 1940s, formally by Robbins and Monro in 1951. In the last two decades, however, modern hardware has begun to see a tapering off of Moore's law, and has begun to expand out in a distributed fashion with multicore processors and GPUs; naturally the question becomes: In order to take advantage of the strengths of modern hardware, how can we parallelize a stochastic gradient method?

2 Hogwild!

Prior to 2011, parallel stochastic gradient methods had been introduced, but most suffered from poor scaling due to the necessity of locks. A naive implementation could look like:

Algorithm 1 Very Naive Parallel Stochastic Gradient

```
Require: Number of data points $N$, seperable loss function $f = \sum_{e \in E} f_e(x_e)$, Initial $x$.
for epoch = 1 → MAX_EPOCHS do
    #pragma omp parallel for
    for $k = 1 \rightarrow N$ do
        Choose $i$ uniformly from $\{1, \ldots, |E|\}$.
        #pragma omp critical
            Read current parameters $x$.
            Compute $\nabla f_i(x)$.
            $x \leftarrow x - \eta \nabla f_i(x)$.
    end for
end for
```

Note that the version presented above is one with a fixed number of iterations, as the discussion of stopping criteria seems to be similar to that of the stochastic gradient method, and for extremely large data sets, is often heuristic. But it's clear here that such an algorithm would only effectively be parallelizing the unform sample of i in $\{1, \ldots, |E|\}$, and it's overall parallel efficiency would likely be poor. Technically, you can improve the above by replacing the critical section with selective locks on components of x based on the sparsity pattern of $\nabla f_i(x)$, but because the process of acquiring locks is much more expensive than floating point arithmetic, this helps little.

However, in 2011, the article ”Hogwild!: A Lock-Free Approach to Parallelizing Stochastic Gradient Descent” by Niu et al. [NRRW11] proposed a very simple solution to this problem. Remove the locks![1]

Algorithm 2 Hogwild!: Asynchronous Stochastic Gradient with replacement

```
Require: Number of data points $N$, seperable loss function $f = \sum_{e \in E} f_e(x_e)$, Initial $x$.
```

[1] Apparently this was discovered by accident by Feng Niu, one of the original paper's authors, when he was debugging stochastic gradient method code. I wish my troubleshooting was nearly as effective... [Rec14]

```
for epoch = $1 \to$ MAX_EPOCHS do
    #pragma omp parallel for
    for $k = 1 \to N$ do
        Choose $i$ uniformly from $\{1, \ldots, |E|\}$.
        Read current parameters $x$.
        Compute $\nabla f_i(x)$.
        $x \leftarrow x - \eta \nabla f_i(x)$.                ▷ Must be done atomically
    end for
end for
```

and should we want to sample without replacement the algorithm is easily modified to:

Algorithm 3 Hogwild!: Asynchronous Stochastic Gradient without replacement

```
Require: Number of data points $N$, seperable loss function $f = \sum_{e \in E} f_e(x_e)$, Initial $x$.
for epoch = $1 \to$ MAX_EPOCHS do
    Let $P$ be a random permutation of $\{1, \ldots, |E|\}$.      ▷ i.e. a Fisher-Yates Shuffle.
    #pragma omp parallel for
    for $k = 1 \to N$ do
        $i \leftarrow P[k]$.
        Read current parameters $x$.
        Compute $\nabla f_i(x)$.
        $x \leftarrow x - \eta \nabla f_i(x)$.                ▷ Must be done atomically
    end for
end for
```

It should be noted that although the formal OMP locks have been removed, atomic operations are still required in order to prevent mutual exclusion. However, no guards have been placed to prevent a thread from overwriting another's computation midway through, and it's not obvious as to why such a race condition wouldn't destroy the performance of the Stochastic Gradient method. However, with certain assumptions one can show that Hogwild! behaves roughly like a noisy stochastic gradient method, and thus shares its convergence properties.

3 Convergence Analysis

Given that the result of Hogwild!, in the absence of noise generated by the asynchronous updates of x, henceforth denoted *asynchronous noise*, is equivalent to a stochastic gradient method, we should expect convergence rates similar to those of stochastic gradient, should noise be small. Take for example the typical linear least squares loss function $f(x) = \frac{1}{2n}\|Dx - b\|_2^2$, where $x \in \mathbb{R}^n$ and $D \in \mathbb{R}^{n \times n}$ a diagonal matrix. Writing this as a sum of each data entry:

$$f(x) = \frac{1}{2n}\sum_{k=1}^{n}(D_{ii}x_i - b_i)^2 = \frac{1}{n}\sum_{k=1}^{n} f_i(x) \implies (\nabla f_i(x))_k = \begin{cases} D_{ii}(D_{ii}x_i - b_i), & k = i \\ 0, & k = 0 \end{cases}$$

Because our $\nabla f_i(x)$'s only have a single entry in their own component, we can see that as long as no other thread is working on component i simultaneously, no asynchronous noise will be generated; should we use HOGWILD! without replacement then this is guaranteed. In the original paper [NRRW11], sparsity of the vector $\nabla f_i(x)$ was required in order to guarantee convergence, but as we'll see later, this isn't always necessary.

As a baseline of comparison, we first state a result on the convergence of the stochastic gradient method:

Theorem 3.1. *(Convergence of the Stochastic Gradient Method [BCN16]) Let $F : \mathbb{R}^d \to \mathbb{R}$ be an objective function we're seeking to minimize. We can write this as either an expected risk $F(x) = \mathbb{E}_\xi f(x, \xi)$, or an empirical risk $F(x) = \frac{1}{n}\sum_{k=1}^{n} f_k(x)$. Under the assumptions:*

(1) F is continuously differentiable and ∇F is Lipschitz continuous with Lipshitz constant L, i.e.

$$\left\|\nabla F(x) - \nabla F(y)\right\| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^d$$

(2) F is strongly convex with constant c, i.e.

$$F(x) \geq F(y) + \nabla F(y)^T(x - y) + \frac{1}{2}c\|x - y\|^2, \forall x, y \in \mathbb{R}^d$$

(3) F is bounded below over the region explored by stochastic gradient method.

(4) In expectation, the vector $-\nabla f(x_k, \xi_k)$ is a descent direction for F with norm bounded by it's own norm. That is: $\exists \mu_G \geq \mu > 0$ such that $\forall k \in \mathbb{N}$:

$$\nabla F(x_k)^T \mathbb{E}\left[\nabla f(x_k, \xi_k)\right] \geq \mu \left\|\nabla F(x_k)\right\|^2 \text{ and } \left\|\mathbb{E}\left[\nabla f(x_k, \xi_k)\right]\right\| \geq \mu_G \left\|\nabla F(x_k)\right\|$$

(5) $\exists M, M_V \geq 0$ such that $\forall k \in \mathbb{N}$:

$$\operatorname{Var}\left[\nabla f(x_k, \xi_k)\right] \leq M + M_V \left\|\nabla F(x_k)\right\|^2$$

Then, assuming a fixed stepsize α (a.k.a. learning rate), satisfying $\alpha \in (0, \mu/LM_g]$, we have:

$$\mathbb{E}\left[F(x_k) - F_*\right] \leq \frac{\alpha LM}{2c\mu} + (1 - \alpha c\mu)^{k-1}\left(F(x_1) - F_* - \frac{\alpha LM}{2c\mu}\right)$$

and if we instead choose α diminishing, i.e. let $\alpha_k = \frac{\beta}{\gamma + k}$ where $\beta > 1/c\mu, \gamma > 0$ are chose such that $\alpha_1 \leq \mu/LM_G$, then:

$$\mathbb{E}\left[F(x_k) - F_*\right] \leq \frac{1}{\gamma + k}\max\left\{\frac{\beta^2 LM}{2(\beta c\mu - 1)}, (\gamma + 1)(F(x) - F_*)\right\}$$

The result is technical, and very long to prove, so I just refer to the article [BCN16] for it. Regardless, the last result in the above theorem is what we strive for in *Hogwild*!: convergence in $\mathcal{O}(1/k)$ time.

3.1 Theoretical Results

Hogwild!'s original article [NRRW11] manages to prove that the asynchronous noise generated by Hogwild!, under certain sparsity assumptions of the f_i's, converged at the same rate as seen in the stochastic gradient method. However, a newer article from 2015 by De Sa et al. [SZOR15] presented a new framework in the context of martingale theory, which drops said sparsity assumptions and generalizes to certain non-convex formualations. Given that I just learned about martingale theory essentially two weeks ago in Basic Probability, and that much of my synthetic tests are on dense datasets, I feel compelled to present this argument instead.

The following is a cleaned up summary of the arguments presented in the article, except I focus on just demonstrating how martingale theory can be used to generalize convergence arguments for a sequential stochastic gradient method to the asynchronous case, and cut out the generalization required for the non-convex situation.

3.1.1 Initial Machinery

First, recall the definition of a martingale:

Definition 3.1. *[JP04] A sequence of random variables $(X_n)_{n\geq 0}$ is called a martingale, or an $(\mathcal{F}_n)$-martingale, if*

(i) $\mathbb{E}\left[|X_n|\right] < \infty, \forall n.$

(ii) X_n *is* $\mathcal{F}_n$ *measurable,* $\forall n.$

(iii) $\mathbb{E}\left[X_n \mid \mathcal{F}_m\right] = X_m$ *a.s.,* $\forall m \leq n.$

furthermore a supermartingale (submartingale) is one satisfying (i), (ii) exactly, and (iii) with $\leq$ $(\geq)$ *instead.*

Using this, the idea is to model our convergence with a non-negative supermartingale $W_t(x_t, \dots, x_0)$ which is a function of the previous stochastic gradient iterates. These W_t, when used in the theory later, will be associated with specific stochastic algorithms, as an example below we'll see it applied to serial stochastic gradient. However, given such a supermartingale, and given a bounded stopping time B (in literature known as a horizon), if our stochastic gradient iterates are written as $x_{t+1} = x_t - \eta\nabla f_t(x_t)$, where η is the learning rate, and f_t denotes the random function chosen at time step t, then we see that condition (iii) in the above definition implies that:

$$W_{t+1}(x_t - \eta\nabla f_t(x_t), x_t, \dots, x_0) \leq W_t(x_t, \dots, x_0), \forall t \leq B$$

which certainly makes sense if our $-\nabla f_t(x_t)$ is a sufficient search direction. Furthermore, letting our success region be denoted as $S = B_\varepsilon(x^*)$, where x^* is the minimizer of our optimization problem, if we impose that if $x_t \notin S, \forall t \leq T$ then:

$$W_T(x_T, \dots, x_0) \geq T$$

then we call W_t a *rate supermartingale.* To simplify notation, let F_t be the event where $\nexists t \leq T$ such that $x_t \in S$.

A good example of the power of this machinery is proving a convergence bound on the serial version of stochastic gradient: using (iii) of definition 3.1, one can see that considering F_T:

$$\mathbb{E}\left[W_0(x_0)\right] \underbrace{\geq}_{} \mathbb{E}\left[W_T\right] = \mathbb{P}\left[F_t\right] \underbrace{\mathbb{P}\left[F_T\right] \mathbb{E}\left[W_T | F_T\right]}_{W_T \geq T} + \underbrace{\mathbb{P}\left[F_T^c\right] \mathbb{E}\left[W_T | F_T^c\right]}_{W_T \geq 0} \geq \mathbb{P}\left[F_T\right] T$$

where the first inequality is by Doob's optional sampling theorem. Thus for a simple serial stochastic gradient method: $\mathbb{P}\left[F_t\right] = \mathbb{E}\left[W_0\right] / T$.

Now that we can characterize serial stochastic gradient in this model, we need a method to analyze asynchronous noise. Recall that since we've guaranteed in the description of Hogwild! that writes to the iteration variable x_t are done atomically, the only race condition possible is when updates to entries of x_t are interleaved with either the other thread's updates or its reads on x_t.

Indeed, when going to update the ith component of x_{t+1}, the variable used to compute the gradient may have long since changed, making our iteration look more like $x_{t+1} = x_t - \nabla f_t(v_t)$ where v_t's entries were the entries of some previous iterate x. Let $\tau_{i,t}$ denote the lag for the update of the ith component of x_{t+1}, i.e. $(v_t)_i = (x_{t-\tau_{i,t}})_i$. Then we recognize that this lag for each component i (supposing the computer hasn't crashed) must be bounded; let τ' be the the maximum over i of such bounds and let $\tau = \mathbb{E}\left[\tau'\right]$. This is known in literature as the *worst-case expected delay.*

Finally, we need one last definition, mainly one of convinience, to proceed onward. This describes the main conditions upon a rate supermartingale necessary to prove that the asynchronous noise error is irrelevant to the convergence rate.

Definition 3.2. *An algorithm with associated rate supermartingale W is (H, R, ξ)-bounded if the following conditions hold.*

(1) W must be Lipschitz continuous in the current iterate with parameter H, i.e.

$$\left\|W_t(u, x_{t-1}, \ldots, x_0) - W_t(v, x_{t-1}, \ldots, x_0)\right\| \leq H\|u - v\|, \forall t, u, v, x_t, \ldots, x_0.$$

(2) ∇f must be Lipschitz continuous in expectation with parameter R, i.e.

$$\mathbb{E}\left[||\nabla f(x) - \nabla f(y)||\right] \leq R\|u - v\|$$

(3) The expected magnitude of the update must be bounded by ξ, i.e.

$$\mathbb{E}\left[||\nabla f(x)||\right] \leq \xi$$

Note that these look very familiar to the conditions in the stochastic gradient method convergence theorem (theorem 3.1).

3.1.2 Convergence of Hogwild!

Finally, now that all of the machinery has been defined, we can get to the main result:

Theorem 3.2. *Suppose we have an asynchronous stochastic algorithm with associated rate supermartingale W which is (H, R, ξ) bounded with horizon B. Furthermore, assume that $HR\xi\tau < 1$; then $\forall T \leq B$:*

$$\mathbb{P}\left[F_T\right] \leq \frac{\mathbb{E}\left[W(0, x_0)\right]}{(1 - HR\xi\tau)T}$$

Before we prove it, we briefly discuss how to use this theorem in practice. Suppose we have a loss function f which we want to use Hogwild! on to minimize. Then first we need to obtain a rate supermartingale proving the problem[2], determine H, R, ξ such that W is (H, R, ξ)-bounded, and then apply this theorem to get a proper rate of convergence. This is a very powerful theorem, as we'll be able to state later, given strongly convex f and with the other required bounds, we can blanket prove that Hogwild! works on them. Furthermore, for nicer non-convex problems, such as the low-rank least-squared matrix completion problem presented in the paper, it's easy to derive a proper rate supermartingale as well.

Now, with respect to the proof, we only outline it because it's mainly just a repeated application of the above bounds we have in order to lower bound away noise terms, and then it follows the path outlined in the serial stochastic gradient method proof.

Proof. (i) With W defined exactly as in the serial case, it's not a rate supermartingale. Instead, from it we construct rate supermartingale V_t, where $\forall t, x$ where x_u not converged $\forall u < t$:

$$V_t(x_t, \ldots, x_0) = W_t(x_t, \ldots, x_0) - \underbrace{HR\xi\tau t}_{(1)} + \underbrace{HR \sum_{k=1}^{\infty} \|x_{t-k+1} - x_{t-k}\| \sum_{m=k}^{\infty} \mathbb{P}\left[\tau' \geq m\right]}_{(2)}$$

where (1) allows for longer iteration counts (as HOGWILD needs to allow for given noise corruption), and (2) measures distance between recent iterates. Otherwise if x_u is converged, then we let $V_t(x_t, \ldots, x_0) = V_u(x_u, \ldots, x_0)$. Basically, we've defined V a stopped process.

(ii) Show V_t is a rate supermartingale for Hogwild!

(iii) Using a similar process to the serial stochastic gradient proof, show that $\mathbb{E}\left[V_T\right] \leq \mathbb{E}\left[V_0\right] = \mathbb{E}\left[W_0\right]$, then using the law of total expectation on this with F_T, and recalling that $\mathbb{E}\left[W_T | F_T\right] \geq T$, we recieve the desired result.

□

Given this, the general theorem for the convex case is:

Theorem 3.3. *Consider trying to minimize f, which is:*

[2] In the article, De Sa et al. describes this as being no more difficult than proving serial convergence, but the proof (in the convex case) doesn't seem to have any similarity between it and serial convergence, so I can't validate this claim.

- *Strongly convex with parameter* c*.*
- ∇f_k *Continuously differentiable in* $||\cdot||_1$ *with Lipschitz constant* L*.*
- *Upper bounded second moment of the gradient by* M^2
- *Success criteria* $\|x - x^*\|^2 \leq \varepsilon$*, for some* $\varepsilon > 0$*.*

Then we can construct rate supermartingale W_t *such that it's* (H, R, ξ) *bounded with* $H = 2\sqrt{\varepsilon}(2\eta c\varepsilon - \eta^2 M^2)^{-1}, R = \eta L, \xi = \eta M$*. Then choosing step size* η *(for some* $\nu \in (0,1)$*):*

$$\eta = \frac{c\varepsilon\nu}{M^2 + 2LM\tau\sqrt{\varepsilon}}$$

then we recieve:

$$\mathbb{P}\left[F_T\right] \leq \frac{M^2 + 2LM\tau\sqrt{\varepsilon}}{c^2\varepsilon\nu T}\log\left(e\|x_0 - x^*\|^2\varepsilon^{-1}\right)$$

3.2 Numerical Experiments

Now that we've finally slogged through the presentation of the theory, we can produce a few numerical experiments. Take, for example, the linear least squares regression problem with $A \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$:

$$f(x) = \frac{1}{2}\|Ax - b\|_2^2 = \sum_{k=1}^{n}\underbrace{\frac{1}{2}(A_i x - b_i)^2}_{f_i(x)}$$

This is a particularly interesting example, as when we examing the stochastic descent direction of this loss function:

$$\nabla f_i(x) = A_{i*}^T(A_{i*}x - b_i), \text{ where } A = \begin{bmatrix} A_{1*} \\ \vdots \\ A_{n*} \end{bmatrix}$$

we notice that the sparsity pattern of $\nabla f_i(x)$ is entirely controlled by by the sparsity pattern of the rows of A. This makes this problem particularily convinient when trying to understand the effect of sparsity on the asynchronous noise generated by HOGWILD!. The original paper [NRRW11]'s results required strict assumptions on the sparsity pattern of $\nabla f_i(x)$ in order to guarantee convergence, but as we saw in Theorem 3.3, as long as certain regularity properties are satisfied, there's no need for such an assumption. Indeed f does satisfy the above, assuming that A is of full rank[3]. Therefore, when A is both sparse and dense, we should see a $\mathcal{O}(1/k)$ convergence rate, and indeed see figure 1 for the validation of that.

[3] One subtle note is that the second condition is actually equivalent to having upper bounded maximal eigenvalue. I think we proved this back when we were discussing Nesterov's method, and also can be found at this stackexchange https://math.stackexchange.com/a/1699082/245618.

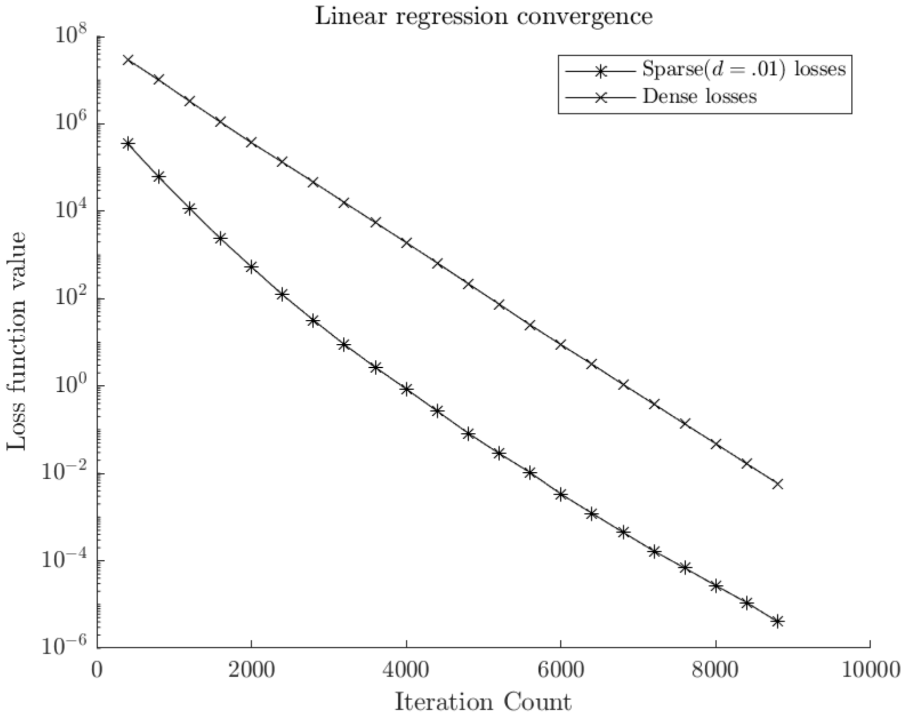


Figure 1: The matrix Sparse($d = 0.1$) is generated by the command `sprand(n,m,.01)`. Playing with the density parameter doesn't change the shape of the convergence line, but note that denser matrices (since the entries are Gaussian) produce a higher initial loss and suffer from more asynchronous noise, and therefore require more iterations.

3.2.1 Asynchronous Noise as a Function of Bandedness

So now that we've seen that, indeed, dense updates enjoy the same convergence properties as sparser ones, we can also investigate the relationship of asynchronous noise as a function of sparsity. Since the sampling of the next data point to be used in a stochastic iterate is done uniformly and independently, we note that in the interest of minimizing asynchronous noise, the exact pattern matters less than the number of non-zero elements. Therefore, an interesting way to represent different sparsity patterns in our linear regression, is in choosing A banded, and then varying the size of that band.

Definition 3.3. *Let $k \geq 1$ and $A \in \mathbb{R}^{n \times n}$. We denote A as a $(k-1)$-banded matrix if k is the maximal integer greater than zero such that either the kth or the $-k$th diagonal are not-identically zero.*

Now consider the operation of Hogwild! with two simultaneous threads on the loss function defined above, with A k-banded. Then an lower-bound to the probability that these two threads do not share an component of x_i, which they need to read/write from/to, is (assuming sampling with replacement) the probability of picking i_1, i_2 uniformly from $[|k+1, \ldots, n-k|]$ such that $[|i_1 - k, i_1 + k|] \cap [|i_2 - k, i_2 + k|] = \varnothing$. Via a simple counting argument, we find that this is:

$$\mathbb{P}\left[[|i_1 - k, i_1 + k|] \cap [|i_2 - k, i_2 + k|] = \varnothing\right] = \frac{\big((n-2k)-(2k+1)\big)_+}{n-2k} = \frac{(n-4k-1)_+}{n-2k}$$

where $(\cdot)_+ = \max(0, \cdot)$. This lower-bound gives us an upperbound on the probability that they do share a necessary component, one minus the above. This confirms something we already know, that if $k << n$, then the two threads are very unlikely to have asynchronous noise. However, with even just $k = n/4$, then the above probability is zero, and this is just for $P = 2$. I had wanted to calculate the k taking the above to zero as a function of P, but the analysis quickly becomes intractible for $P \geq 3$, barring a nice counting argument I don't see. Regardless, as long as we can measure asynchronous noise, we can get an idea of how it increases as a function of k.

To construct an experiment to see this, let x_k (k not a component, but an parameter) be the final iterate of Hogwild!, as applied to the linear regression problem with A k-banded. Then supposing we hold some solution x^* constant among all k, then we can view $f(x_k)$ as a random-variable, whose variance characterizes the amount of asynchronous noise experienced throughout computation, as the sequential algorithm (assuming the random choice of stochastic data points is seeded between intervals) will have $\text{Var}\left[f(x_k)\right] = 0, \forall k$. See figure 2 for the results of such an experiment.

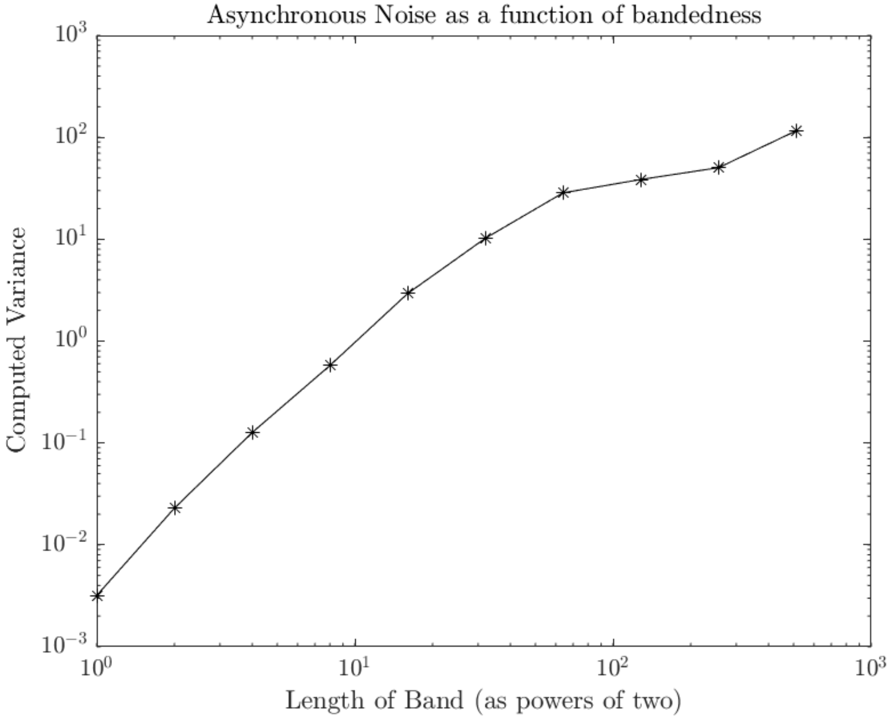
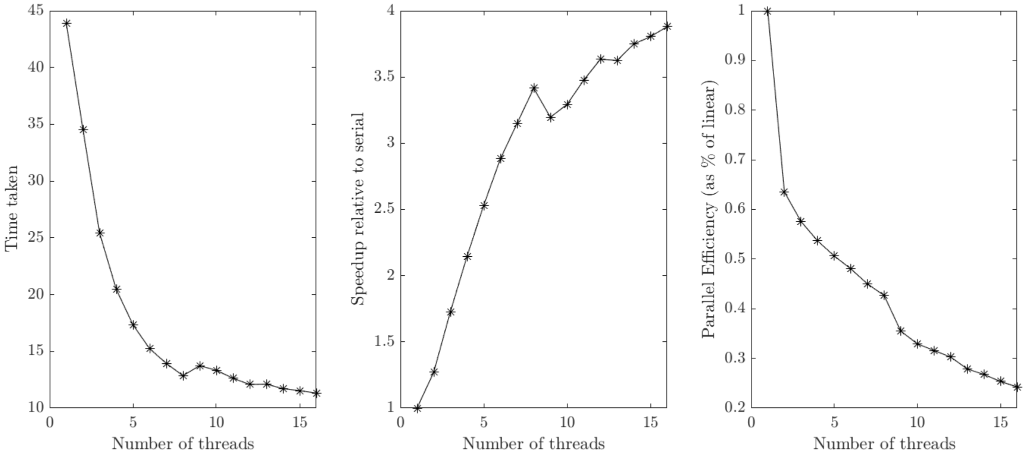
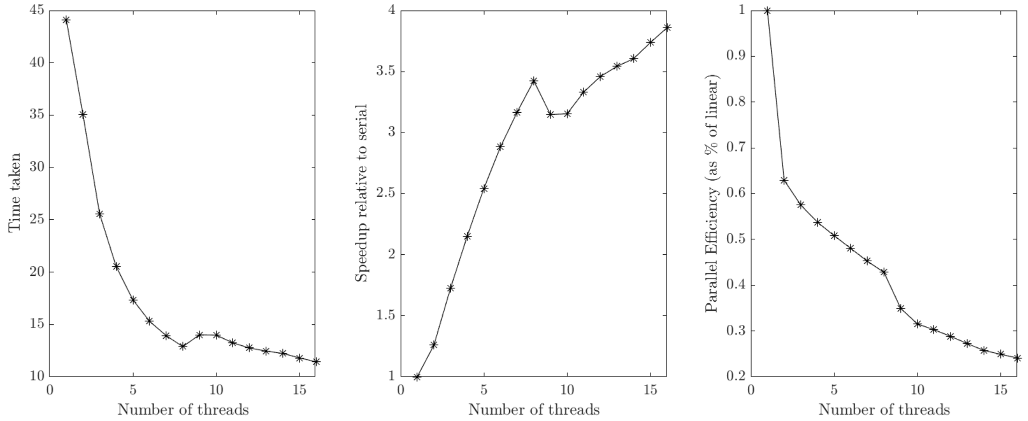


Figure 2

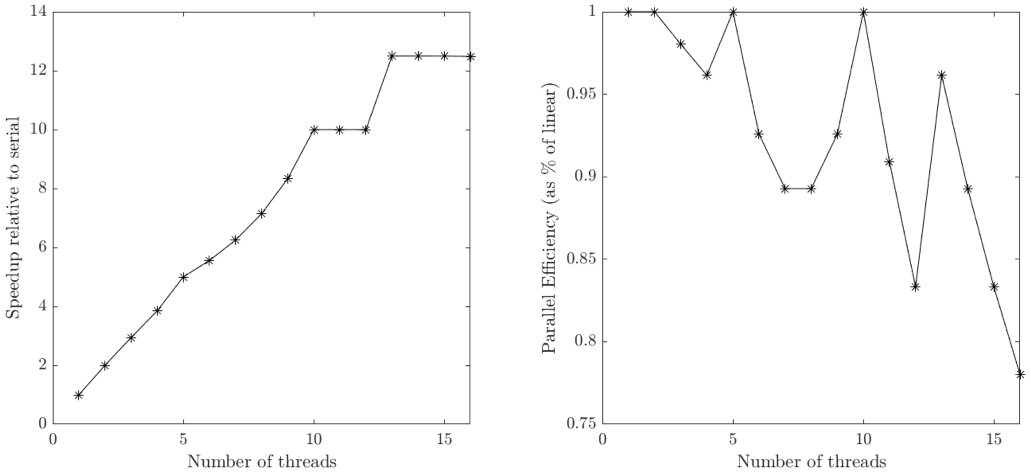
Large regression scaling study with replacement



Large regression scaling study w/o replacement



'Heavy Gradient' Efficiency Study



4 Efficiency Analysis

5 Applications: Judging Wine Quality

6 Conclusions and Future Work

References

[BCN16] Léon Bottou, Frank E. Curtis, and Jorge Nocedal. Optimization methods for large-scale machine learning, 2016.

[JP04] Jean Jacod and Philip Protter. *Probability Essentials*. Springer Berlin Heidelberg, 2004.

[NRRW11] Feng Niu, Benjamin Recht, Christopher Re, and Stephen J. Wright. Hogwild!: A lock-free approach to parallelizing stochastic gradient descent, 2011.

[Rec14] Benjamin Recht. HOGWILD! for machine learning on multicore. https://youtu.be/l5JqUvTdZts, June 2014.

[SZOR15] Christopher De Sa, Ce Zhang, Kunle Olukotun, and Christopher Ré. Taming the wild: A unified analysis of hogwild!-style algorithms, 2015.