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Representing circles with five control points

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To the memory of Josef Hoschek

Abstract

We show that five is the minimal dimension of a space required to draw a complete circle with a unique control polygon. We identify all five-dimensional spaces invariant under translations and reflections where we can find shape preserving representations of a circle parameterized by its arc length.

Keywords: Trigonometric curves; Shape preserving; Totally positive basis; Critical length

1. Introduction

Many trigonometric curves play a key role in engineering. There are several alternatives for modeling trigonometric curves in Computer-Aided Geometric Design (CAGD). The most classical method is provided by the rational model. In (Piegl and Tiller, 1989; Chou, 1995; Bangert and Prautzsch, 1997; Mainar, 2001), we can find some advantages of the nonrational approach. The spaces of trigonometric polynomials

$$\mathcal{T}_n = \operatorname{span}\{1, \cos t, \sin t, \ldots, \cos(nt), \sin(nt)\}$$

have also been used in the literature. In (Peña, 1997) it was shown that these spaces are not adequate for design purposes on $[0, \pi]$ because they do not possess shape preserving representations. In contrast, for parameter domains of length less than π, these spaces admit normalized B-bases (i.e., bases with optimal shape preserving properties: see (Carnicer and Peña 1993, 1994; Peña, 1999)), as shown in (Sánchez-

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Reyes, 1997). Due to these limitations on the length of the parameter domain, we conclude that a complete circle with unit angular velocity cannot be represented in these spaces by means of a unique control polygon, although this task can be performed in spaces of trigonometric splines (see (Walz, 1997; Lyche, 1999; Han, 2002)).

In (Carnicer et al., 2003) it was shown that the space $\mathrm{span}\{1, t, t^2, t^3, \cos t, \sin t\}$ has a normalized B-basis on $[0, 2\pi]$. Hence it is possible to represent a complete circle with a unique control polygon of six control points. Then a natural question arises: is it possible to represent the complete circle in a space of dimension less than six? This would imply that the control polygon of a circle may have less than six control points. In Section 2, we provide an affirmative answer to the previous question. On one hand, we prove that five is the minimal number of points required for representing complete circles. On the other hand, we prove that the 5-dimensional spaces $T_w = \mathrm{span}\{1, \cos(wt), \sin(wt), \cos t, \sin t\}$ $(0 < w < 1/2)$ possess normalized B-basis on $[0, 2\pi]$. Moreover, we prove that they are the unique spaces of differentiable functions including $1, \cos t, \sin t$, invariant under translations and reflections with normalized B-bases on closed intervals of length 2π.

In (Carnicer et al., 2003), we showed, for any space of differentiable functions and invariant under translations, the existence of a critical length ℓ for design purposes in the sense that there exist normalized TP bases on and only on compact intervals of length less than ℓ. So, taking into account the results of (Peña, 1997; Sánchez-Reyes, 1997), the critical length for design purposes of the spaces $\mathcal{T}_n$ $(n \geqslant 1)$ is π. In Section 2 we prove that the critical length for design purposes of T_w, $0 < w < 1/2$, is greater than 2π and attains its maximum value 3π for $w = 1/3$. With greater critical lengths for design purposes, the intervals where we can define optimal bases are larger. On the other hand, in (Mainar, 2001) it is shown that, when the length of the interval of definition tends to the critical length for design purposes, then some basis functions degenerate to the zero function (see Figs. 10, 12, 14 of (Mainar, 2001)). In fact, the normalized B-basis on a given interval are more similar to the Bernstein basis for spaces with greater critical length.

The spaces T_w contain trigonometric functions with two different angular velocities, one of them being unit. A change of variable $t = w_1 s$ allows us to use the functions $\cos(w_1 s)$ and $\sin(w_1 s)$ instead of $\cos t$ and $\sin t$ and, defining $w_2 := w_1 w$, the functions $\cos(w_2 s)$ and $\sin(w_2 s)$, instead of $\cos(wt)$ and $\sin(wt)$. If $\ell(w)$ is the critical length for design purposes of T_w, then the corresponding length of $\langle 1, s, \cos(w_1 s), \sin(w_1 s), \cos(w_2 s), \sin(w_2 s)\rangle$ is $w_1^{-1}\ell(w_2/w_1)$. The normalized B-basis can be obtained by applying the change of variables to the optimal basis of T_w. Therefore, the discussion of the general case of spaces with two different angular velocities, is reduced to the case where one is unit. Note that, with a suitable reparameterization, $T_{1/3}$ corresponds to the space generated by $1, \cos t, \sin t, \cos(3t), \sin(3t)$, whose critical length for design purposes is π.

In Section 3 we construct the normalized B-bases for the spaces T_w, $0 < w < 1/2$, on $[0, 2\pi]$. We also provide the coefficients of some remarkable functions with respect to the normalized B-basis of $T_{1/3}$ on $[0, 2\pi]$. We also include some illustrative representations, including that of the complete circle with a unique control polygon.

2. Main results

Let U be a vector space of real-valued functions defined on $I \subseteq \mathbb{R}$ and $(u_0(t), \dots, u_n(t))$, $t \in I$, be a basis of U. If a sequence $P_0, \dots, P_n$ of points in $\mathbb{R}^k$ is given then we define a curve $\gamma(t) =$

$\sum_{i=0}^{n} P_i u_i(t)$, $t \in I$. The points $P_0, \dots, P_n$ are called *control points* and the polygon $P_0 \cdots P_n$ with vertices $P_0, \dots, P_n$ is called the *control polygon* of γ (see (Farin, 1997; Hoschek and Lasser, 1993)). In Computer Aided Geometric Design the functions $u_0, \dots, u_n$ are usually nonnegative and $\sum_{i=0}^{n} u_i(t) = 1$ for all $t \in [a, b]$ (i.e., the system $(u_0, \dots, u_n)$ is *normalized*) and in this case we say that $(u_0, \dots, u_n)$ is a *blending system*. The *convex hull property* means that the curve always lies in the convex hull of its control polygon. The convex hull property holds if and only if $(u_0, \dots, u_n)$ is a blending system.

These geometric properties correspond to some properties concerning the collocation matrices of the system of functions. The *collocation matrix* of $(u_0(t), \dots, u_n(t))$ at $t_0 < \dots < t_m$ in I is given by

$$M\begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_m \end{pmatrix} := \big(u_j(t_i)\big)_{i=0,\dots,m;\ j=0,\dots,n}. \tag{1}$$

Clearly, $(u_0, \dots, u_n)$ is blending if and only if all its collocation matrices are stochastic (that is, nonnegative and such that the sum of each row is one).

A matrix is *totally positive* if all its minors are nonnegative and a system of functions is totally positive when all its collocation matrices (1) are totally positive. In case of a normalized totally positive basis one knows that the curve imitates the shape of its control polygon, due to the variation diminishing properties of totally positive matrices (see (Goodman, 1989; Peña, 1999)). A space possessing normalized TP bases has a unique special basis, called the *normalized B-basis*, which presents optimal shape preserving properties (see (Carnicer and Peña, 1994) and Chapter 4 of (Peña, 1999)). Normalized B-bases have also optimal stability properties as shown in Chapter 5 of (Peña, 1999). Illustrative examples of normalized B-bases are the Bernstein basis in the corresponding polynomial space on a compact interval and the B-spline basis in the corresponding spline space. If the critical length for design purposes of a space is ℓ, it means that there exist normalized TP bases on and only on compact intervals of length less than ℓ (see (Carnicer et al., 2003)).

We say that a basis $(u_0, \dots, u_n)$ of an $(n+1)$-dimensional space U in $\mathcal{C}^n(I)$ is *canonical* at $t_0 \in I$ if the Wronskian matrix $W(u_0, \dots, u_n)(t_0)$ is lower triangular with nonzero diagonal entries. If $(u_0, \dots, u_n)$ is a basis such that $\det W(u_0, \dots, u_n)(t_0) \neq 0$, then a canonical basis at t_0 can be constructed by combining the functions of the basis $(u_0, \dots, u_n)$ in order to obtain a system of functions with zeros of increasing multiplicity at t_0. A *bicanonical basis* $(u_0, \dots, u_n)$ on the interval $[t_0, t_1]$ is a canonical basis at t_0 such that $(u_n, \dots, u_0)$ is canonical at t_1. As shown in (Carnicer et al., 2003), this concept is closely related to that of B-basis (see (Carnicer and Peña, 1994)).

In order to put together several pieces of curves, it is desirable for the designer to have a precise control over what happens at the ends of the curve. This leads to the *endpoint interpolation property*: the first control point always coincides with the start point of the curve and the last control point always coincides with the final point of the curve. A stronger requirement is that the first and last segments of its control polygon are tangent at the end points of the curve. Following (Aumann, 1997), when the curve generated by a blending system always satisfies this property, we say that the system satisfies the *boundary tangent property*.

Proposition 1. *Let U be a space of functions with a blending basis $(b_0, \dots, b_n)$ satisfying the boundary tangent property. If circles can be represented in the space U, then* $\dim U \geqslant 5$.

Proof. If $P_0 \cdots P_n$ is the control polygon of a complete circular arc with respect to the basis $(b_0, \dots, b_n)$, then by the end point interpolation property $P_0 = P_n$. Moreover, by the boundary tangent property the points P_0, P_1, P_{n-1}, P_n are collinear and by the convex hull property, the polygon $P_0 \cdots P_n$ cannot be contained in a line. Therefore $n \geqslant 4$ and that means that $\dim U \geqslant 5$. □

Observe that the previous argument can also be applied to the design of closed curves not contained in a line using a control polygon starting from a point of the curve where the tangent is defined and finishing at the same point.

A space of functions $U \subseteq \mathcal{C}(\mathbb{R})$ is *invariant under translations* if, for any $u \in U,\ \tau \in \mathbb{R}$, the function $v(t) := u(t-\tau),\ t \in \mathbb{R}$, belongs to U. Invariance under translations is useful for curve design, because it allows to represent curves whose parameter domains are larger than the interval of definition of the given basis, by simply joining several arcs.

A space of functions $U \subseteq \mathcal{C}(\mathbb{R})$ is *invariant under reflections* if for any $u \in U$ and $\tau \in \mathbb{R}$, the function $v(t) := u(\tau - t),\ t \in \mathbb{R}$ belongs to U. Invariance under reflections means that the curve with control polygon $P_n \cdots P_0$ is just the curve generated by $P_0 \cdots P_n$ reparameterized so as to reverse the orientation. Note that, if U is invariant under reflections, then U is invariant under translations.

We want to design complete circles in spaces of differentiable functions U invariant under translations and reflections of minimal dimension. By Proposition 1, $\dim U \geqslant 5$. Furthermore we want to represent complete circles with their arc-length parameterization and so, our space has to contain the functions $1, \cos t, \sin t$. Another requirement mentioned above is that the space has a normalized and totally positive basis on the interval of definition $[0,\alpha]$, whose length α should be not less than 2π.

Proposition 2. *If a* 5*-dimensional subspace of* $\mathcal{C}^1(\mathbb{R})$ *containing the functions* $1, \cos t, \sin t$ *is invariant under reflections, then it coincides with one of the following spaces*:

$$L := \langle 1, \cos t, \sin t, t\cos t, t\sin t\rangle, \tag{2}$$

$$T_w := \big\langle 1, \cos t, \sin t, \cos(wt), \sin(wt)\big\rangle, \quad w > 0,\ w \neq 1, \tag{3}$$

$$Q := \big\langle 1, \cos t, \sin t, t, t^2\big\rangle, \tag{4}$$

$$H_w := \big\langle 1, \cos t, \sin t, \cosh(wt), \sinh(wt)\big\rangle, \quad w > 0. \tag{5}$$

Proof. If U is a finite dimensional space of $\mathcal{C}^1(\mathbb{R})$ which is invariant under reflections, then $u(-h-t)$ and $u(t+h) = u(-(-h-t))$ belong to U for any $h \in \mathbb{R}$ and any $u \in U$. Furthermore $h^{-1}(u(t+h) - u(t)) \in U$ and, taking $h \to 0$, we deduce that $u' \in U \subset \mathcal{C}^1(\mathbb{R})$. So $U \subset \mathcal{C}^\infty(\mathbb{R})$. For a given basis $(u_0, \ldots, u_4)$, we have $(u_0', \ldots, u_4')^{\mathrm{T}} = A(u_0, \ldots, u_4)^{\mathrm{T}}$ for a given 5×5 matrix A. So $(u_0, \ldots, u_4)$ is a solution of a linear system of differential equations of first order with constant coefficients. Then, it can be easily shown that U is the set of solutions of a homogeneous linear differential equation of order 5 with constant coefficients

$$u^{(5)} + a_4 u^{(4)} + a_3 u^{(3)} + a_2 u'' + a_1 u' + a_0 u = 0.$$

The invariance under reflections implies that $a_4 = a_2 = a_0 = 0$ and so, the characteristic polynomial is of the form $p(t) = t^5 + a_3 t^3 + a_1 t$. Since it contains the functions $1, \cos t, \sin t$ we deduce that $p(t) = t(t^2+1)(t^2+b)$.

Let us discuss the different cases depending on the parameter b. If $b = 1$ then $U = L$. If $b > 0$, $b \neq 1$, we can write $b = w^2$, with $w > 0$, $w \neq 1$ and $U = T_w$. If $b = 0$, then $U = Q$. Finally, if $b < 0$, we can write $b = -w^2$ and $U = H_w$. □

The following auxiliary result is a consequence of (Carnicer et al., 2003).

Lemma 3. *Let U be a five-dimensional space of differentiable functions which is invariant under reflections. Let $(u_0, \dots, u_3)$ be a canonical basis at* 0 *of the space U' of its derivatives. Let $\phi(t) := u_3(t)$ and $\psi(t) := \det W(u_2, u_3)(t)$. Then the critical length for design purposes of U is $\min(z_\phi, z_\psi)$, where z_ϕ, z_ψ denote the first positive zero of ϕ and ψ, respectively.*

Proof. Since $d(u(a-t))/dt = -u'(a-t)$ we see that U' is invariant under reflections. By Proposition 3.2 of (Carnicer et al., 2003), U' is an extended Chebyshev space on and only on compact intervals of length less than $\min(z_\phi, z_\psi)$. By Theorem 4.1, U has a normalized B-basis on and only on such intervals. This implies that the critical length for design purposes of U is $\min(z_\phi, z_\psi)$. □

Theorem 4. *The spaces T_w defined in* (3) *with $0 < w < 1/2$ are the unique spaces satisfying the hypotheses of Proposition* 2 *with normalized B-bases on intervals $[0, \alpha]$ with $\alpha \geqslant 2\pi$.*

Proof. By Proposition 2, the only possibilities are the spaces provided by formulae (2)–(5). The space Q of (4) was considered in (Mainar, 2001) and it was proved there that it has normalized B-basis on $[0, \alpha]$ if and only if $0 < \alpha < 2\pi$.

Let us now consider the space L of (2). It can be checked that the system of functions $(\cos t, \sin t, t \sin t, \sin t - t \cos t)$ is a canonical basis at 0 of the space of its derivatives. Let $\chi(t) := \sin t - t \cos t$. Since $\chi(\pi) = \pi$ and $\chi(2\pi) = -2\pi$, the function $\chi(t)$ vanishes in $(\pi, 2\pi)$ and so, by Lemma 3, the critical length for design purposes of L is less than 2π.

Let us now consider the space H_w of (5). The space of its derivatives is $H_w' = \langle \cos t, \sin t, \cosh(wt), \sinh(wt) \rangle$. It can be easily checked that $(\cos t, \sin t, \cosh(wt) - \cos t, \sinh(wt) - w \sin t)$ is a canonical basis at 0 of H_w'. Since $\sinh(wt) - w \sin t > 0$ for all $t > 0$, by Lemma 3, the critical length for design purposes is given by the first positive zero of the Wronskian

$$\begin{aligned}\zeta(t) &:= \det W\bigl(\cosh(wt) - \cos t, \sinh(wt) - w \sin t\bigr)\\ &= 2w\bigl(1 - \cos t \cosh(wt)\bigr) + (w^2 - 1) \sin t \sinh(wt).\end{aligned}$$

It can be checked that $\zeta(t) = 4\zeta_1(t)\zeta_2(t)$ with

$$\begin{aligned}\zeta_1(t) &:= w \cos(t/2) \sinh(wt/2) + \sin(t/2) \cosh(wt/2),\\ \zeta_2(t) &:= w \sin(t/2) \cosh(wt/2) - \cos(t/2) \sinh(wt/2).\end{aligned}$$

Since $\zeta_1(\pi) = \cosh(w\pi/2) > 0$ and $\zeta_1(2\pi) = -w \sinh(w\pi) < 0$, $\zeta(t)$ vanishes in $(\pi, 2\pi)$ and therefore the critical length for design purposes of H_w is less than 2π.

Let us now consider the space T_w of (3). It can be checked that a canonical basis at 0 of the space of its derivatives is $(\cos t, \sin t, \cos(wt) - \cos t, \sin(wt) - w \sin t)$. Let

$$\phi(t) := \sin(wt) - w \sin t. \tag{6}$$

If $w > 1$, then $\phi(t)$ vanishes on $(0, \pi/(2w))$ and so the critical length for design purposes of T_w is less than $\pi/(2w) < 2\pi$.

For $w \in [1/2, 1)$, $\phi(\pi) = \sin(w\pi) \geqslant 0$ and $\phi(2\pi) = \sin(2w\pi) \leqslant 0$. Therefore $\phi(t)$ vanishes in $[\pi, 2\pi]$ and the critical length for design purposes of T_w is not greater than 2π.

Let us now see that, if $w \in (0, 1/2)$, then the critical length for design purposes is greater than 2π. Clearly,

$$\phi'(t) := w\big(\cos(wt) - \cos t\big) = 2w \sin\big((1+w)t/2\big) \sin\big((1-w)t/2\big).$$

Then $\phi'(t) > 0$ for all $0 < t < 2\pi/(1+w)$ and $\phi'(t) < 0$ for all $2\pi/(1+w) < t < 2\pi$. Since $\phi(0) = 0$ and $\phi(2\pi) = \sin(2w\pi) > 0$, then $\phi(t) > 0$, for all $t \in (0, 2\pi]$.

Let us now consider the Wronskian

$$\begin{aligned}\psi(t) &:= \det W\big(\cos(wt) - \cos t, \sin(wt) - w \sin t\big)\\ &= 2w\big(1 - \cos t \cos(wt)\big) - (1+w^2) \sin t \sin(wt).\end{aligned} \tag{7}$$

It can be checked that $\psi(t) = 4\psi_1(t)\psi_2(t)$ with

$$\psi_1(t) := \sin(t/2)\cos(wt/2) - w\cos(t/2)\sin(wt/2), \tag{8}$$

$$\psi_2(t) := w\sin(t/2)\cos(wt/2) - \cos(t/2)\sin(wt/2). \tag{9}$$

Clearly, $\psi_2'(t) = (1-w^2)\sin(t/2)\sin(wt/2)/2 > 0,\ t \in (0, 2\pi)$, and so $\psi_2(t)$ is increasing for all $t \in (0, 2\pi)$. Since $\psi_2(0) = 0$, we derive $\psi_2(t) > 0$ for all $t \in (0, 2\pi]$. On the other hand, $\psi_1(t) - w\psi_2(t) = (1-w^2)\sin(t/2)\cos(wt/2) \geqslant 0$ if $t \in (0, 2\pi]$ and so, $\psi_1(t) \geqslant w\psi_2(t)$ for all $t \in (0, 2\pi]$. In conclusion, for $w \in (0, 1/2)$, the functions ϕ and ψ are positive on $(0, 2\pi]$. So, by Lemma 3, the critical length for design purposes is greater than 2π. □

Theorem 5. *The critical length for design purposes of the spaces $T_w,\ 0 < w < 1/2$, attains its maximum value 3π at $w = 1/3$.*

Proof. Let w be such that $0 < w \leqslant 1/2$ and let $z_\phi(w)$ and $z_\psi(w)$ be the first zero of the functions $\phi(t)$ given by (6) and $\psi(t)$ given by (7), respectively. Let us define

$$f(t, w) := \frac{\sin(wt)}{w} - \sin t.$$

Since $f(t, w) = \phi(t)/w$, we have $f(t, w) \neq 0$ for all $t \in (0, z_\phi(w))$ and $w \in (0, 1/2)$. Taking into account that

$$\frac{\partial f}{\partial t}(t, w) = \cos(wt) - \cos t = 2\sin\left(\frac{1+w}{2}t\right)\sin\left(\frac{1-w}{2}t\right) > 0, \quad \forall t \in (0, 2\pi/w)$$

and that $f(0, w) = 0$, we have

$$f(t, w) > 0, \quad \forall t \in \big(0, z_\phi(w)\big),\ \forall w \in (0, 1/2). \tag{10}$$

Differentiating $f(t, w)$ with respect to w, we obtain

$$\frac{\partial f}{\partial w}(t, w) = \frac{wt\cos(wt) - \sin(wt)}{w^2} = -\frac{\chi(wt)}{w^2},$$

where $\chi(t) = \sin t - t\cos t$. Since $\chi'(t) = t\sin t > 0,\ t \in (0, \pi)$, we deduce that $\chi(t) > 0$ for all $t \in (0, \pi)$, we can conclude that $\frac{\partial f}{\partial w}(t, w) < 0$ for all $t \in (0, \pi/w)$. Let $0 < w_1 < w_2 \leqslant 1/2$. Then $f(t, w_1) > f(t, w_2)$ for all $t \in [0, 2\pi]$ and, evaluating at $t = z_\phi(w_1)$, we derive $f(z_\phi(w_1), w_2) < 0$ and, by (10), $z_\phi(w_1) > z_\phi(w_2)$. So we have shown that $z_\phi(w)$ is a strictly decreasing function of w.

The first positive zero of $\psi(t) = \psi_1(t)\psi_2(t)$ is $z_\psi(w)$. Let us observe that

$$w\psi_1(t) - \psi_2(t) = (1 - w^2)\cos\frac{t}{2}\sin\frac{wt}{2} < 0, \quad t \in (\pi, 3\pi),\ w \in (0, 1/2). \tag{11}$$

Let us now assume that $0 < w < \frac{1}{3}$. Since $\psi_2(\pi) = w\cos(w\pi/2) > 0$ and $\psi_2(3\pi) = -w\cos(3w\pi/2) \leqslant 0$, we have that ψ_2 has a zero in $(\pi, 3\pi)$. By (11), $\psi_1(t) < \frac{1}{w}\psi_2(t)$ and so $z_\psi(w)$ is the first positive zero of ψ_1. Taking into account the last part of Theorem 4, ψ_1 is positive on $(0, 2\pi)$. Thus, $z_\psi(w) \in [2\pi, 3\pi)$.

Let us define

$$g(t, w) := \tan t - w\tan(wt).$$

Since $g(t/2, w) = (\psi_1'(t)/(\cos(t/2)\cos(wt/2))$, we have $g(t, w) \neq 0$ for all $t \in (\pi, z_\psi(w)/2)$. Since $\lim_{t\to\frac{\pi}{2}^+} g(t, w) = -\infty$, we deduce that

$$g(t, w) < 0, \quad \forall t \in \big(\pi, z_\psi(w)/2\big). \tag{12}$$

By differentiating $g(t, w)$ with respect to w, we obtain

$$\frac{\partial g}{\partial w}(t, w) = -\tan(wt) - wt\big(1 + \tan^2(wt)\big) < 0, \quad \forall t \in (0, \pi/2w).$$

Let $0 < w_1 < w_2 \leqslant 1/3$. Then $g(t, w_1) > g(t, w_2)$ for all $t \in [0, 3\pi/2]$ and, evaluating at $t = z_\psi(w_2)/2 \in (\pi, 3\pi/2)$, we derive $g(z_\psi(w_2)/2, w_1) > 0$ and, by (12), $z_\psi(w_2) > z_\psi(w_1)$. So we have seen that $z_\psi(w)$ is a strictly increasing function of w on $[0, 1/3]$.

Let us now consider the case $w \in (1/3, 1/2)$. Since

$$\psi_1'(t) = (1 - w^2)\cos(wt/2)\cos(t/2)/2 < 0,$$

we have $\psi_1'(t) < 0$ for all $t \in (\pi, \pi/w)$ and $\psi_1'(t) > 0$ for all $t \in (\pi/w, 3\pi)$. So ψ_1 attains its minimum value on $[\pi, 3\pi]$ at $t = \pi/w$. Since $\psi_1(\pi/w) = -w\cos(\pi/2w) > 0$, we deduce that $\psi_1(t) > 0$ for all $t \in [\pi, 3\pi]$. Then, by (11), $\psi_2(t) > w\psi_1(t) > 0$ for all $t \in [\pi, 3\pi]$. So, neither ψ_1 nor ψ_2 vanish in $[\pi, 3\pi]$ and, taking into account that, by the last part of the proof of Theorem 4, they do not vanish in $[0, 2\pi]$, we conclude that $z_\psi(w) > 3\pi$.

Finally, let us assume that $w = 1/3$. Then we have

$$\phi(t) = \frac{4}{3}\sin^3(t/3), \qquad \psi(t) = \frac{16}{9}\sin^4(t/3)$$

and the first positive zero is 3π.

By Lemma 3, the critical length for design purposes of the spaces T_w is given by

$$\min\big\{z_\phi(w), z_\psi(w)\big\} = \begin{cases} z_\psi(w), & 0 < w < \frac{1}{3}, \\ 3\pi, & w = \frac{1}{3}, \\ z_\phi(w), & \frac{1}{3} < w < \frac{1}{2}. \end{cases}$$

Summarizing, the critical length for design purposes is an increasing function on $w \in (0, \frac{1}{3})$ attains its maximum value 3π at $w = \frac{1}{3}$ and decreases on $w \in (\frac{1}{3}, \frac{1}{2})$. $\square$

3. Normalized B-basis of T_w and illustrative examples

A canonical basis at 0 of the space T_w is the system

$$\begin{aligned}
&u_0(t) = 1, \quad u_1(t) = \sin t, \quad u_2(t) = 1 - \cos t,\\
&u_3(t) = \sin(wt) - w \sin t = \phi(t),\\
&u_4(t) = 1 - \cos(wt) - w^2(1 - \cos t) = 2\phi(t/2)\hat{\phi}(t/2),
\end{aligned}$$

where $\phi(t) = \sin(wt) - w \sin t$ is the function defined in (6) and $\hat{\phi}(t) = \sin(wt) + w \sin t$. We have proved in the previous section that all spaces T_w have a normalized B-basis on $[0, 2\pi]$ for any $w \in (0, \frac{1}{2})$. Our goal is to construct the normalized B-basis on that interval to represent the functions $\cos t$ and $\sin t$ on a compact interval whose length is the period 2π.

Following the procedure suggested in Theorem 2.4(iv) of (Carnicer et al., 2003) for constructing a B-basis, we first compute the Wronskian matrix of $(u_4, u_3, u_2, u_1, u_0)$ at $t = 2\pi$. In order to simplify the expressions as much as possible, we apply trigonometric formulae to reduce all trigonometric functions to the argument $w\pi$ and obtain

$$\begin{pmatrix}
2\sin^2(w\pi) & 2\sin(w\pi)\cos(w\pi) & 0 & 0 & 1\\
2w\sin(w\pi)\cos(w\pi) & -2w\sin^2(w\pi) & 0 & 1 & 0\\
-2w^2\sin^2(w\pi) & -2w^2\sin(w\pi)\cos(w\pi) & 1 & 0 & 0\\
-2w^3\sin(w\pi)\cos(w\pi) & w(2w^2\sin^2(w\pi) - (w^2-1)) & 0 & -1 & 0\\
w^2(2w^2\sin^2(w\pi) - (w^2-1)) & 2w^4\sin(w\pi)\cos(w\pi) & -1 & 0 & 0
\end{pmatrix}.$$

Then we compute the LU decomposition of the previous matrix and we use the matrix

$$U = \begin{pmatrix}
2\sin^2(w\pi) & 2\sin(w\pi)\cos(w\pi) & 0 & 0 & 1\\
0 & -2w & 0 & 1 & -w\cot(w\pi)\\
0 & 0 & 1 & 0 & w^2\\
0 & 0 & 0 & -(1-w^2)/2 & -w(1-w^2)\cot(w\pi)/2\\
0 & 0 & 0 & 0 & w^2(1-w^2)\csc^2(w\pi)/2
\end{pmatrix}$$

for obtaining a system $(\tilde{b}_0, \ldots, \tilde{b}_4)$ by solving the linear system

$$\big(\hat{b}_4, -\hat{b}_3, \hat{b}_2, -\hat{b}_1, \hat{b}_0\big)U = (u_4, u_3, u_2, u_1, u_0),$$

which is a B-basis. Using Remark 4.1 of (Carnicer et al., 2003) we may compute the coefficients in order to normalize the previous B-basis and obtain the normalized B-basis:

$$\begin{aligned}
&b_4(t) = \csc^2(w\pi)\phi(t/2)\hat{\phi}(t/2),\\
&b_3(t) = \cot(w\pi)\phi(t)/2 - \cos^2(w\pi)b_4(t),\\
&b_2(t) = 2w^2\sin^2(t/2) = w^2\big(1 - \cos(t)\big),\\
&b_1(t) = b_3(2\pi - t), \quad b_0(t) = b_4(2\pi - t).
\end{aligned} \tag{13}$$

Let us observe that the relations $b_1(t) = b_3(2\pi - t)$, $b_0(t) = b_4(2\pi - t)$ are due to the symmetry properties of the space. So, in order to obtain the B-basis we have just to compute $\hat{b}_4, \hat{b}_3, \hat{b}_2$ and normalize them, reducing considerably the amount of computations to be done.

In order to represent remarkable curves of the space, we need the coefficients of some particular functions of the space, shown in Table 1.

Table 1
Coefficients of some functions

Function	c_0	c_1	c_2	c_3	c_4
1	1	1	1	1	1
$\sin t$	0	$\tan(w\pi)/w$	0	$-\tan(w\pi)/w$	0
$1-\cos t$	0	0	$1/w^2$	0	0
$\sin(w(t-\pi))$	$-\sin(w\pi)$	0	0	0	$\sin(w\pi)$
$\cos(w(t-\pi))$	$\cos(w\pi)$	$1/\cos(w\pi)$	0	$1/\cos(w\pi)$	$\cos(w\pi)$
$\phi(t)/\sin(w\pi)$	0	0	0	$2/\cos(w\pi)$	$2\cos(w\pi)$

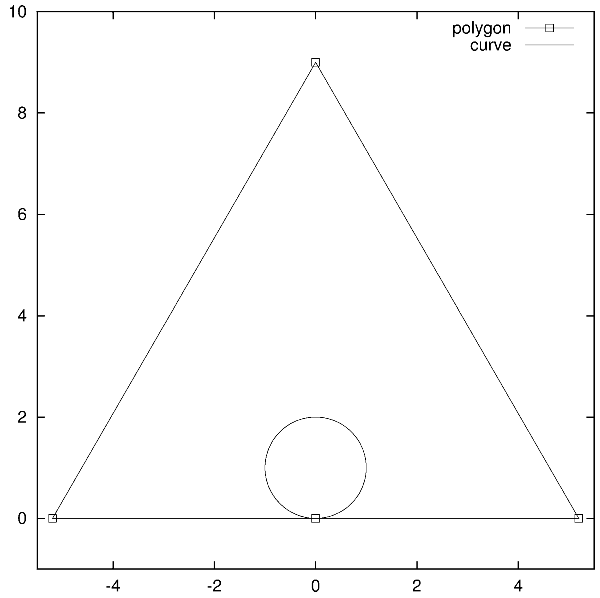

Fig. 1. A circle and its control polygon.

In this space we can represent any circle or ellipse. As an example, let us represent the circle $(\sin t, 1-\cos t)$, $t \in [0, 2\pi]$, whose control polygon is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tan(w\pi)/w \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/w^2 \end{pmatrix}, \begin{pmatrix} -\tan(w\pi)/w \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

shown in Fig. 1. Other curves which can be represented are arcs of epicycloids and hypocycloids.

Lissajous curves are often used for illustrating physical effects as well as in artistic design. These curves arise as the composition of two simple harmonic motions with different angular frequencies. We can choose a suitable reparameterization taking the largest frequency as unit and adjusting the phase. So we can represent any Lissajous curve in the form: $(A\sin t, B\sin(wt-\phi))$, $t \in \mathbb{R}$. Using the B-basis (13) we can represent arcs of Lissajous curves with $w \in (0, \frac{1}{2})$, corresponding to intervals of the parameter domain of length 2π. Joining several of these curves we can obtain large arcs of Lissajous curves. Let us remark that when w is a rational number then the Lissajous curves are closed.

In the particular case $w = 1/n$, the change of parameter $s = t/n$ transforms the space $T_{1/n}$ into $\operatorname{span}\{1, \cos s, \sin s, \cos(ns), \sin(ns)\}$. In this case, we are able to represent some additional polynomial

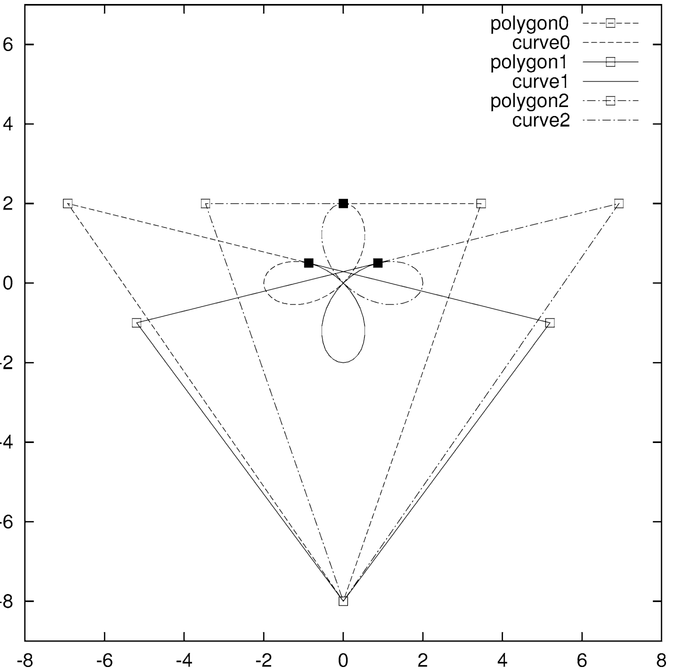

Fig. 2. A rose formed by 3 curves.

curves. The parameterization $(\cos s, \cos(ns))$, $s \in \mathbb{R}$, can be used to obtain the graph of the nth Chebyshev polynomial. In fact, this is a particular case of a closed Lissajous curve. In order to cover the whole abscissa range $[-1, 1]$ with the representation of the Chebyshev polynomial curve $(\cos(t/n), \cos t)$, we need a parameter interval of length $n\pi$. However our representation can only obtain curves on intervals of length 2π. So we can only represent a part of this graph with a single control polygon. If we need to represent larger arcs, we have to join several curve pieces.

The space $T_{1/3}$ can be used to represent even more curves. In addition to line segments we can also represent some cubic curves. More precisely, our representation is able to deal with arcs of graphs of polynomials $p(x)$ obtained as linear combination of $1, x, x^3$ as a consequence of the above comments on Chebyshev polynomial graph representation. The astroid and the four petals rose $(\cos^3(t/3), \sin^3(t/3))$, $(\cos(t) + \cos(t/3), \sin(t) - \sin(t/3))$, $t \in \mathbb{R}$, can also be represented in this space. The complete closed curves can be represented on the interval $[0, 6\pi]$. Since the length of our representation is 2π we need 3 pieces in order to represent the whole curve. The 3 control polygons corresponding to the rose are

$$\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -8 \end{pmatrix}, \begin{pmatrix} -4\sqrt{3} \\ -1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix},$$
$$\begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 3\sqrt{3} \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -8 \end{pmatrix}, \begin{pmatrix} -3\sqrt{3} \\ -1 \end{pmatrix}, \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix},$$
$$\begin{pmatrix} \sqrt{3}/2 \\ 3/2 \end{pmatrix}, \begin{pmatrix} 4\sqrt{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -8 \end{pmatrix}, \begin{pmatrix} -2\sqrt{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}.$$

Fig. 2 shows the graph of the rose and the corresponding control polygons.

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