Homework 3

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(It’ s hard for me to describe my solution for some problems in English, so I answer some problems in Chinese)

1. (a) Countably infinite

   (b) Countably infinite

   (c) Countably infinite

   (d) Not countably infinite

   (e) Not countably infinite

   (f) Not countably infinite

   (g) Not countably infinite

   (h) Countably infinite (If the character set is finite)

   (i) Countably infinite (If the character set is finite)

   (j) Not countably infinite

   (k) Countably infinite

   (l) Not countably infinite

   (m) Not countably infinite

   (n) Not countably infinite

   (o) Countably infinite

2. Let P(n) be the proposition that 1+2+3+…+n = n(n+1)/2.

   BASIS STEP: ∵ 1=1*(1+1)/2 ∴ P(1) is true.

INDUCTIVE STEP: Assume P(k) is true, $k \in N^*$. Under this assumption,

$\because$ 1+2+3+…+k = k(k+1)/2.

$\therefore$ 1+2+3+…+k+k+1 = k(k+1)/2+k+1

$\therefore$ 1+2+3+…+k+k+1 = (k+1)((k+1)+1)/2

$\therefore$ P(k+1) is true

So by mathematical induction, P(n) is true for all positive integers n.

3. (a) Let P(n) be the proposition that $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

BASIS STEP: $\because (x+y)^1 = \sum_{i=0}^{1} \binom{1}{i} x^i y^{1-i}$ $\therefore$P(1) is true.

INDUCTIVE STEP: Assume P(k) is true, $k \in N^*$. Under this assumption,

$\because\ (x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i}$

$\therefore\ (x+y)(x+y)^k = (x+y)\sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i}$

$$\therefore\ (x+y)^{k+1} = \sum_{i=0}^{k} \binom{k}{i} x^{i+1} y^{k-i} + \sum_{i=0}^{k} \binom{k}{i} x^i y^{k-i+1}$$

$$= \sum_{i=0}^{k} \binom{k}{i} x^{i+1} y^{k-i} + \sum_{i=-1}^{k} \binom{k}{i+1} x^{i+1} y^{k-i}$$

$$= y^{k+1} + \sum_{i=0}^{k} [\binom{k}{i} + \binom{k}{i+1}] x^{i+1} y^{k-i}$$

$$= \binom{k+1}{0} x^0 y^{k+1} + \sum_{i=0}^{k} \binom{k+1}{i+1} x^{i+1} y^{k-i}$$

$$= \binom{k+1}{0} x^0 y^{k+1} + \sum_{i=1}^{k+1} \binom{k+1}{i} x^i y^{k+1-i}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} x^i y^{k+1-i}$$

$\therefore$ P(k+1) is true

So by mathematical induction, P(n) is true for all positive integers n.

(b) Let P(n) be the proposition that $\sum_{i=1}^{n} (2\mathrm{i}-1) = \mathrm{n}^2$

BASIS STEP: $\because \sum_{i=1}^{1} (2\mathrm{i}-1) = 1^2$ $\therefore$P(1) is true.

INDUCTIVE STEP: Assume P(k) is true, $k \in N^*$. Under this assumption,

$\because\ \sum_{i=1}^{k}(2\mathrm{i}-1)=k^2$

$\therefore\ 2(\mathrm{k}+1)-1+\sum_{i=1}^{k}(2\mathrm{i}-1)=2(k+1)-1+k^2$

$\therefore\ \sum_{i=1}^{k+1}(2\mathrm{i}-1)=2(k+1)-1+k^2$

$=(k+1)^2$

$\therefore$ P(k+1) is true

So by mathematical induction, P(n) is true for all positive integers n.

(c) Let P(n) be the proposition that $\sum_{i=1}^{n}\mathrm{i}(\mathrm{i}+1)=\frac{1}{3}\mathrm{n}(\mathrm{n}+1)(\mathrm{n}+2)$

BASIS STEP: It is easy to check that P(1) is true.

INDUCTIVE STEP: Assume P(k) is true, $\mathrm{k}\in\mathrm{N}^*$. Under this assumption,

$\because\ \sum_{i=1}^{k}\mathrm{i}(\mathrm{i}+1)=\frac{1}{3}\mathrm{k}(\mathrm{k}+1)(\mathrm{k}+2)$

$\therefore\ (\mathrm{k}+1)(\mathrm{k}+2)+\sum_{i=1}^{k}\mathrm{i}(\mathrm{i}+1)=(k+1)(k+2)+\frac{1}{3}\mathrm{k}(\mathrm{k}+1)(\mathrm{k}+2)$

$\therefore\ \sum_{i=1}^{k+1}\mathrm{i}(\mathrm{i}+1)=(k+1)(k+2)+\frac{1}{3}\mathrm{k}(\mathrm{k}+1)(\mathrm{k}+2)$

$=\frac{1}{3}(\mathrm{k}+1)((\mathrm{k}+1)+1)((\mathrm{k}+1)+2)$

$\therefore$ P(k+1) is true

So by mathematical induction, P(n) is true for all positive integers n.

(d) (The problem seems to be wrong and I think we should replace $\frac{2}{3}$ by $\frac{1}{3}$)

Let P(n) be the proposition that $\sum_{i=1}^{n}(2\mathrm{i}-1)^2=\frac{1}{3}\mathrm{n}(2\mathrm{n}+1)(2\mathrm{n}-1)$

BASIS STEP: It is easy to check that P(1) is true.

INDUCTIVE STEP: Assume P(k) is true, $\mathrm{k}\in\mathrm{N}^*$. Under this assumption,

$\because\ \sum_{i=1}^{k}(2\mathrm{i}-1)^2=\frac{1}{3}\mathrm{k}(2\mathrm{k}+1)(2\mathrm{k}-1)$

$\therefore\ (2(\mathrm{k}+1)-1)^2+\sum_{i=1}^{k}(2\mathrm{i}-1)^2=(2(\mathrm{k}+1)-1)^2+\frac{1}{3}\mathrm{k}(2\mathrm{k}+1)(2\mathrm{k}-1)$

$\therefore\ \sum_{i=1}^{k+1}(2\mathrm{i}-1)^2=(2(\mathrm{k}+1)-1)^2+\frac{1}{3}\mathrm{k}(2\mathrm{k}+1)(2\mathrm{k}-1)$

$=\frac{1}{3}(\mathrm{k}+1)(2(\mathrm{k}+1)+1)(2(\mathrm{k}+1)-1)$

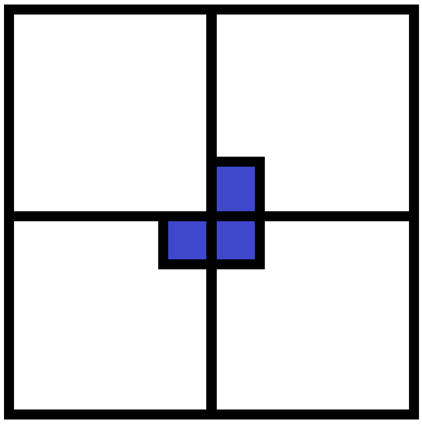
$\therefore$ P(k+1) is true

So by mathematical induction, P(n) is true for all positive integers n.

(e) It's easy to see that (i) is the special case of (ii), so I just prove (ii). Let P(n) be the proposition that any $2^n \times 2^n$ checker board can be covered with L-shaped tiles except for any designated square.

BASIS STEP: It is easy to check that P(1) is true.

INDUCTIVE STEP: Assume P(k) is true, $\mathrm{k} \in \mathrm{N}^*$. For a $2^{k+1} \times 2^{k+1}$ checker board, divide it to four $2^k \times 2^k$ checker boards. There's must be exactly one of them has the designated square. Without loss of generality, we can assume the left-top $2^k \times 2^k$ checker board has the designated square. Put an L-shaped tile(*) on the center of the board as the following picture. And because P(k) is true, the 4 parts can be covered with L-shaped tiles(the designated square which shouldn't be covered of the left-top board is the designated square which shouldn't be covered of the whole board and for the other three boards, the designated square which shouldn't be covered is the square which is covered by (*)). So P(k+1) is true.



So by mathematical induction, P(n) is true for all positive integers n.

(f)(1)首先若接受到的第一个字符是 1，则由图中描述直接接受字符串，因而下设接受到的第一个字符是 0，则进入图中左上的状态。

(2)若当前在图中左上的状态，则由图可知上一个接受的字符必为 0，则若当前接受的字符为 0，则接受字符串，否则进入图中左下状态。

(3)若当前在图中左下的状态，则由图可知上一个接受的字符必为 1，则若当前接受的字符为 1，则接受字符串，否则进入图中左上状态。

由(1)(2)(3)可见，只要字符串以 1 开头，或者字符串中含有 00 或 11，则字符串被接受。

(g)It is obvious that from ii we can come to i, so we just need to prove that from i we can come to ii.

Let $\{a_n\}$ be one of the path from x to y, a_0=x, a_m=y.

If m=1, then there is a direct edge from x to y. ∴ The two statements are equivalent.

If m=2, ∵There is an edge from x to a_1 and an edge from a_1 to y.

∴There is a direct edge from x to y. ∴ The two statements are equivalent.

If m=k>1 and the two statements are equivalent. When m=k+1,

∵ There is an edge from x to a_{m-1} and an edge from a_{m-1} to y

∴ There is an edge from x to y.

So by mathematical induction, the two statements are equivalent.

(h) (1)如果当前局面下，当前的 player 可以直接获得胜局（距离终局的步数为 1），则显然当前的 player 有必胜策略。

(2)设距离局终的步数小于等于 k 时，某一方 player 有必胜策略。则当前局面距离局终的步数为 k+1 时，不妨设当前的 player 为 player1，若 player1 行动后所得的任意局面均是 player2 有必胜策略，则 player2 有必胜策略，否则 player1 有必胜策略。

由（1）（2）可证命题成立。

4. 对于获得的 Hamilton 回路，每次在对方询问前对顶点编号进行重排。每次对方可以进行以下两种询问中的一种：（1）要求证明重排编号后的图与原图等价（但不提供 Hamilton 回路）。（2）提供重排编号后的图中的 Hamilton 回路（但不提供重排编号与原图编号的关系）。