Due: 2013/12/13

Homework 10

Note: When you want to apply the L.L.L.. Specify clearly what is the probability space; what are the events; how many events do you have; what is the dependency graph you want to use and why it satisfies the definition of a dependency graph; and show your calculations about the probabilities and degrees.

Problem 1. *Given a graph $G = (V, E)$, we uniformly randomly pick σ, a permutation of V. For each vertex $v \in V$, call v a seed if $v \prec_\sigma u$ for any $uv \in E$; and define X_v to be the indicator random variable for the event that v is a seed. And define $X = \sum_{v \in V} X_v$ to be the number of seeds in an outcome.*
(a) If G is the graph on 6 vertices that consists of two vertex-disjoint triangles, what is $\mathsf{E}(X)$, and what is the probability of $X = \mathsf{E}(X)$?
(b) If G is the cycle C_6, what is $\mathsf{E}(X)$, and what is the probability of $X = \mathsf{E}(X)$?

Problem 2. *For any $n > 2$, show an example of a probability space and n events where any $n-1$ of them are mutually independent, but the n events are not mutually independent.*

Problem 3. *This is the coldest winter in my hometown, i.e., the best of the winters to some of us. We are going to hold icecream parties all over the city in the following 3 days.*
There are many pubs in the city. Each person submits a wish list in the form

$$(P_1, d_1), (P_2, d_2), ..., (P_{40}, d_{40})$$

where the P_i's are distinct pubs, and $d_i \in [3]$. The person is satisfied *if at least one P_i on her wish list holds a party on the d_i-th day.*
The wish lists are so nice that each pub appears on at most 2013 *of them. Prove that we can arrange the parties so that each pub holds a party on only one day, and all the people are satisfied.*

Problem 4. *Let $k \geq 10$ and $n = \lfloor 2^k/(10k) \rfloor$. A subset $X \subseteq [n]$ is called a* Za *if the elements of X form an arithmetic progression of length k.*

(a). *Show that for any Za, there are less than* $1.25kn$ *other Za's share some common points with it.*
(b). *Prove that we can color each element of* $[n]$ *with yellow and blue such that no Za is monochromatic.*