XII. Elementary Function

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What is the class of arithmetic functions we use in mathematics?

Definition

The class ${\mathcal E}$ of elementary function is constructed from

- 1. zero, successor, projection, and
- 2. subtraction x y (for defining conditionals); and is
- 3. closed under composition and
- 4. bounded sum/product (bounded recursion).

Remark.

- 1. Addition and multiplication can be defined using bounded sum; (hyper) exponential can be defined using bounded product.
- 2. Lower elementary functions are constructed by dropping (4).
- 3. A predicate is elementary if its characterization function is elementary.

Bounded Minimalisation is Elementary

Fact. \mathcal{E} is closed under bounded minimalisation.

Proof.

Suppose f(x,z) is elementary. Then $\mu z < y.f(x,z) = 0$ is

$$\sum_{v < y} \prod_{u \le v} \operatorname{sg}(f(x, u)).$$

It is easy to see that sg is elementary.

Logical Operation

Fact. The set of elementary predicates is closed under negation, conjunction, disjunction and bounded quantifiers.

Basic Arithmetic Functions are Elementary

- 1. The exponential x^y is defined by $\prod_{z < y} U_1^2(x, z)$.
- 2. The function p_x is defined by

$$p_x = \mu y < 2^{2^x} \cdot (x = 0 \text{ or } y \text{ is the } x \text{th prime})$$

$$= \mu y < 2^{2^x} \cdot \left(x = \sum_{z \le y} Pr(z) \right)$$

$$= \mu y < 2^{2^x} \cdot \left(x - \sum_{z \le y} Pr(z) = 0 \right).$$

Bounded Recursion

Fact. Let f(x) and g(x, y, z) be elementary and h(x, y) be defined from f, g via primitive recursion. If $h(x, y) \le b(x, y)$ for some elementary function b(x, y), then h(x, y) is elementary.

Proof.

Observe that

$$2^{h(x,0)}3^{h(x,1)}\dots p_{y+1}^{h(x,y)} \leq \prod_{z\leq y} p_{z+1}^{b(x,z)}.$$

So we can define h(x, y) by

$$\mu e \leq \prod_{z \leq y} p_{z+1}^{b(x,z)}. ((e)_1 = f(x) \land \forall z < y. ((e)_{z+2} = g(x,z,(e)_{z+1}))).$$

We are done.

Gödel Encoding is Elementary

Fact. Gödel encoding functions are elementary.

Kleene's Predicate is Elementary

Fact. The Kleene's functions σ_n , c_n and j_n are elementary.

Elementary Time Functions

A computable function ϕ_e is in elementary time if $t_e(x) \leq b(x)$ almost everywhere for some elementary function b(x).

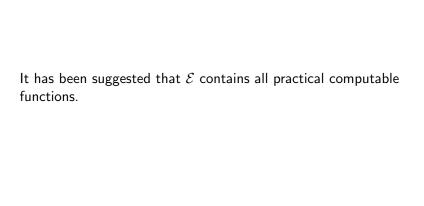
Fact. The elementary time functions are elementary.

Proof.

 $\phi_{\mathrm{e}}(\mathrm{x})$ is almost everywhere computable by the elementary function

$$(c_n(e,\widetilde{x},\mu t \leq b(\widetilde{x}).j_n(e,\widetilde{x},t)=0))_1,$$

which implies that $\phi_e(x)$ is elementary.



A computable function f(x) is practically computable if it can be computed in

$$exp_k(x) = \underbrace{2^{2\cdots 2^x}}_{k}$$

steps for some k. We let $2^{exp_k(x)}$ stand for $exp_{k+1}(x)$.

Upper Bound of Elementary Functions

Theorem. For each elementary function $f(\widetilde{x})$ there is some k such that $f(\widetilde{x}) \leq exp_k(\max{\{\widetilde{x}\}})$.

Proof.

The basic elementary functions satisfy the upper bound. The elementary operations preserves the upper bound.

Corollary. $exp_x(x)$ is primitive recursive but not elementary.

Proof.

The function $exp_x(x)$ is defined by g(x,x), where

$$g(x,0) = x,$$

 $g(x,y+1) = 2^{g(x,y)}.$

We are done.

Elementary Functions are Elementary Time

Lemma. Suppose $f(\widetilde{x})$ and $g(\widetilde{x}, y, z)$ are in elementary time and $h(\widetilde{x}, y)$ is defined from f, g via recursion. If $h(\widetilde{x}, y)$ is elementary, then $h(\widetilde{x}, y)$ is in elementary time.

Proof.

The standard program that calculates h does it in elementary time.

Elementary Functions are Elementary Time

Theorem. If $f(\widetilde{x})$ is elementary, then there is a program P for f such that $t_P^n(\widetilde{x})$ is elementary.

Proof.

Use the above lemma.

Complexity Theoretical Characterization

Theorem. A total function $f(\widetilde{x})$ is elementary iff it is computable in time $exp_k(\max{\{\widetilde{x}\}})$ for some k.

ELEMENTARY = $TIME(2^n) \cup TIME(2^{2^n}) \cup TIME(2^{2^{2^n}}) \cup \dots$