Due: 2013/12/20

Homework 11

Problem 1. *Find seven points in the plane such that when we connect pairs of points with distance 1, the resulting graph has chromatic number 4.*

Solution. Pick any two points b and c where $bc = 1$. The two unit circles centered at b and c intersect at two points a and d. a, b, c, and d form a diamond where $ab = bc = ca = bd = cd = 1$. We make a copy of this diamond $(ab'c'd')$, and let it rotate clockwise around a, until the point d' is unit distance away from d. Now consider the unit distance graph on these 7 points. We prove by contradiction that it is not 3-colourable.
Suppose there is a proper colouring with YRB. WLOG, a is coloured Y, since abc is a triangle, b and c must be coloured by RB or RB. And since bcd is a triangle, d must be coloured Y. By the same reason, d' is also coloured Y. But then d and d' has the same colour, and they are adjacent in the unit distance graph.

□

Problem 2. *Given* $2n$ *points in the plane,* n *red and* n *blue. Prove that one can always 1-1 pair the red and blue points such that these* n *segments do not intersect.*

Note: The statement is not correct if there are collinear points. In the following we assume no three points are collinear.

Proof. There are $n!$ possible parings. Assume σ is one with the minimum total segment length. We claim that all the segments in σ do not intersect. Otherwise, suppose r_i is paired with b_j, r_k paired with b_l and the segments r_ib_j and r_kb_l intersect at p. Let σ' be the pairing almost identical to σ, except that r_i is paired with b_l and r_k with b_j. In these two pairings, all the other segments are identical, and by triangle inequality,

$$r_ib_l + r_kb_j \le r_ip + pb_l + r_kp + pb_j = r_ib_j + r_kb_l,$$

contradicts the assumption that σ is one with the minimum total segment length. □

Problem 3. *Given a set V of n points in the plane, call a line* magic *if it contains exactly 3 points in V. Prove that the number of magic lines is at most $n^2/6$.*

Proof. Let m_k be the number of lines with exactly k points from V on it. Count the pairs

$$(\{a,b\},l) : a \in V \cap l, b \in V \cap l$$

One one hand, any two points has exactly one such line. On the other hand, each line with k points from V on it appears $\binom{k}{2}$ times. So

$$\binom{n}{2} = \binom{2}{2}m_2 + \binom{3}{2}m_3 + ... + \binom{k}{2}m_k + ...$$

So $3m_3 \leq \binom{n}{2}$. The number of magic lines is at most $\binom{n}{2}/3 \leq n^2/6$. □

Problem 4. *Given a point set V in the plane, call a line* magic *if it contains exactly 3 points in V. Let M_n be the maximum number of magic lines over all configuration of n points. Prove a lower bound on the order of M_n as better as you can.*

This problem, known as the *Orchard problem*, is probably the oldest problem in discrete geomrtry. About 150 years ago, J. J. Sylvester showed that $n^2/6$ can be "almost" achieved. The result got improved several times. Last year Green and Tao claimed a complete solution to this problem.
For this homework assignment, I think a lower bound in the order of $n \log n$ is good enough.