Due: 2013/12/13

Homework 10

Note: When you want to apply the L.L.L.. Specify clearly what is the probability space; what are the events; how many events do you have; what is the dependency graph you want to use and why it satisfies the definition of a dependency graph; and show your calculations about the probabilities and degrees.

Problem 1. *Given a graph $G = (V, E)$, we uniformly randomly pick σ, a permutation of V. For each vertex $v \in V$, call v a seed if $v \prec_\sigma u$ for any $uv \in E$; and define X_v to be the indicator random variable for the event that v is a seed. And define $X = \sum_{v \in V} X_v$ to be the number of seeds in an outcome.*
(a) If G is the graph on 6 vertices that consists of two vertex-disjoint triangles, what is $\mathsf{E}(X)$, and what is the probability of $X = \mathsf{E}(X)$?
(b) If G is the cycle C_6, what is $\mathsf{E}(X)$, and what is the probability of $X = \mathsf{E}(X)$?

Solution. (a) Each triangle has exactly one seed. So $X = 2$ in every outcome. $\mathsf{E}(X) = 2$ and $\mathsf{Pr}(X = \mathsf{E}X) = 1$.
(b) For each v, the probability that v comes before its two neighbors is $1/3$. So $\mathsf{E}(X_v) = 1/3$ and $\mathsf{E}(X) = \sum_v \mathsf{E}(X_v) = 2$.
Because there can be at most one seed between any pair of adjacent vertices, so $X \in \{1, 2, 3\}$. It is a little bit easier to calculate $\mathsf{Pr}(X = 1)$ and $\mathsf{Pr}(X = 3)$. For $X = 1$, we randomly select the first point, then any subsequent point must be adjacent to one of the selected points.

$$\mathsf{Pr}(X = 1) = \frac{2}{5}\frac{2}{4}\frac{2}{3} = \frac{2}{15}.$$

For $X = 3$. Let $\sigma(1) = a$ and c, e be the two vertices having distance 2 to a. In order for $X = 3$, one of c and e must be $\sigma(2)$, then the other must come first among itself and its neighbors. So

$$\mathsf{Pr}(X = 3) = \frac{2}{5}\frac{1}{3} = \frac{2}{15}.$$

Therefore, $\mathsf{Pr}(X = 2) = 11/15$. □

Problem 2. *For any $n > 2$, show an example of a probability space and n events where any $n-1$ of them are mutually independent, but the n events are not mutually independent.*

Solution. Consider the cycle C_n. Independently randomly uniformly colour each vertex Y or B. For each edge e, let A_e be the event that the end points of e are of the same colour. So $\mathsf{Pr}(A_e) = 1/2$.
For any $k < n$ edges, they form a forest with $n-k$ components and the intersection of the corresponding events are exactly the event that every component is monochromatic. So

$$\mathsf{Pr}(\bigcap_{1\le i\le k} A_{e_i}) = 2^{n-k}/2^n = 2^{-k} = \prod_{1\le i\le k} \mathsf{Pr}(A_{e_i}).$$

This implies that any $t < n$ events are mutually independent.
However, the intersection of all the n events is the same as "all the points are of the same colour", it has probability $2^{1-n} \neq \prod_e \mathsf{Pr}(A_e) = 2^{-n}$. □

Problem 3. *This is the coldest winter in my hometown, i.e., the best of the winters to some of us. We are going to hold icecream parties all over the city in the following 3 days.*
There are many pubs in the city. Each person submits a wish list in the form

$$(P_1, d_1), (P_2, d_2), ..., (P_{40}, d_{40})$$

where the P_i's are distinct pubs, and $d_i \in [3]$. The person is satisfied *if at least one P_i on her wish list holds a party on the d_i-th day.*
The wish lists are so nice that each pub appears on at most 2013 *of them. Prove that we can arrange the parties so that each pub holds a party on only one day, and all the people are satisfied.*

Proof. For any pub, we randomly and independently pick one day to hold the party. For each person p, let A_p be the event that p is not satisfied, i.e., we picked a bad day for p for each pub on her list. Clearly $\mathsf{Pr}(A_p) = (1-1/3)^{40}$. We define a dependency graph on the people such that $p \to p'$ iff there is at least one common pub on their lists. It is clear to see this is a dependency graph for the events. Our problem assures that the max degree of the dependency graph is bounded above by $40 \cdot 2013$. So we have

$$4(1-1/3)^{40} 40 \cdot 2013 = 160 \cdot 2013 \cdot (16/81)^{10} < 160 \cdot 2013/5^{10} < 1.$$

By L.L.L., $\mathsf{Pr}(\cap_p \overline{A_p}) > 0$. There is one outcome where every person is satisfied.

$\square$

Problem 4. *Let $k \geq 10$ and $n = \lfloor 2^k/(10k) \rfloor$. A subset $X \subseteq [n]$ is called a* Za *if the elements of X form an arithmetic progression of length k.*
(a). *Show that for any Za, there are less than $1.25kn$ other Za's share some common points with it.*
(b). *Prove that we can colour each element of $[n]$ with yellow and blue such that no Za is monochromatic.*

Proof. (a). We prove any $x \in [n]$ is contained in less than $1.25n$ Za's. This is because a Za passing through x is decided by a pair (d, t), where d is the common difference of the arithmetic progression, and t is the rank of x in that progression. Clearly $1 \leq t \leq k$. The length of the interval $[1, n]$ is $n-1$, and the length of the Za is $d(k-1)$, so $d \leq (n-1)/(k-1)$. Therefore, the number of such Za's is at most $(n-1)k/(k-1) < 1.25n$ when $k \geq 10$.
(b). Randomly uniformly colour each vertex with Y or B. For each Za z, define E_z to be the event that z is monochromatic. So $\mathsf{Pr}(E_z) = 2^{1-k}$. Define a graph D on the events so that $E_z \to E_{z'}$ iff z and z' share some common points. It is easy to check D is a dependency graph for the events. The maximum (out-)degree of D is bounded by (a) as $1.25n$. And

$$4 \cdot 2^{1-k} \cdot 1.25 \cdot kn = 10 \cdot nk \cdot 2^{-k} \leq 1$$

by the assumption of our problem. So $\mathsf{Pr}(\bigcap_z \overline{E_z}) > 0$, there is one way to color $[n]$ so that no Za is monochromatic. $\square$