Do not need to hand in

Homework 12

Problem 1. *Let A be the adjacency matrix of the complete bipartite graph $K_{3,3}$, compute the eigenvalues of A.*

Solution. One form of the adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It has rank 2, so the null space has dimension 4, therefore A has eigenvalue 0 with multiplicity 4. For the other two eigenvalues, notice that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

So A has eigenvalues $3, 0, 0, 0, 0, -3$. □

Problem 2. *Let A_1, A_2, ..., A_m be m distinct subsets of $[n]$ such that each $|A_i|$ is even and for any $i \neq j$, $|A_i \cap A_j|$ is odd. How big can m be? Prove your answer. (i.e., Prove the bound, and show examples where the bound is reached.)*

Lemma 1. *The matrix $J_n - I_n$ over the binary field has rank n if n is even, and $n-1$ if n is odd.*

Proof. In the binary field, plus and minus are the same, so the determinant (by its $n!$ terms expansion) equals the number of derangements. It is 1 iff n is even. When n is odd, the rank is less than n, but the first $n-1$ rows and columns has full rank, so its ranks is $n-1$. $\square$

Solution. For n odd, the set $\binom{[n]}{n-1}$ satisfies the requirement and has size n. For n even, the set $\binom{[n-1]}{n-2}$ satisfies the requirement and has size $n-1$. We prove this is best possible.
As in class, take the incidence matrix M where $M_{i,j} = 1$ if A_i contains j. As matrices over the binary field, $MM^T = J_m - I_m$.
If n is odd, there cannot be $n+1$ such sets. Otherwise, the $(n+1) \times (n+1)$ matrix $MM^T = J - I$ has full rank according to the lemma. But M has only n columns. A contradiction.
If n is even, there cannot be n such sets. Otherwise, the $n \times n$ matrix $J - I$ has full rank, but the columns vectors of M add up to the all 0 vector, so M has rank less than n. $\square$

Problem 3. *Let t be a fixed integer. Let A_1, A_2, ... A_m be distinct subsets of $[n]$ such that $|A_i \cap A_j| = t$ for any $i \neq j$. Prove that $m \leq n$.*

One should notice that the statement is not exactly true — when $t = 0$, we may have $m = n + 1$.
When $t > 0$, this is the Fisher's inequality. See 15.1 in Notes.