Due: 2013/11/29

Homework 8

Problem 1. *Consider the cycle $C_4 = (V, E)$, where $V = \{a, b, c, d\}$, and $E = \{ab, bc, cd, da\}$. We randomly color each of the 4 edges as red with probability $1/3$, and blue with probability $2/3$. For each outcome, let $R = (V, \{e \in E : e \text{ is red}\})$ and $B = (V, \{e \in E : e \text{ is blue}\})$.*
(a) The probability space contains 16 possible outcomes, for each of the resulting graph, draw it, and compute it's probability mass. (As usual, I don't care if you draw red edge with a red pen or not.)
(b) Define the random variable X to be the number of connected components in R, what is $\mathsf{E}(X)$?
(c) Define the random variable Y to be the indicator random variable for the event "B is bipartite". What is $\mathsf{E}(Y)$?
(d) Define the event $E :=$ both R and B are disconnected. What is $\mathsf{Pr}(E)$?

Problem 2. *A rabbit is jumping on the $[n]$-world. In the beginning, she picks a uniformly random point in $[n]$ as the starting point, then in each step she picks, among all the points that has* not been visited yet*, a uniformly random point and jumps to it. She will stop when all the points are visited.*
The distance between the points i and j is $|i - j|$. What is the expected total length she will jump from the starting point to the end? Give the answer in closed form, and justify.

Problem 3. *Let $n \geq 4$ and $t \geq 3\sqrt{n}$. Prove that, for any $n \times n$ matrix A with distinct real entries, we can transform it to a new matrix B by permuting columns, such that in B none of the rows contain any monotone sub-sequence of length t.*

In the problems below, we use some standard notations. For a graph $G = (V, E)$, and a set of colors C, a *proper (vertex) coloring* is a function $s : V \to C$ such that $s(u) \neq s(v)$ whenever uv is an edge. The *chromatic number* $\chi(G)$ is the smallest number k such that there is a proper colouring of G by $C = [k]$.

Problem 4. *Let G be a bipartite graph on n vertices, and let C be a set of $t > \log_2 n$ colors. For each vertex v, suppose we have a list of candidate colors $S(v) \subseteq C$ such that $|S(v)| > \log_2 n$. Prove that G has a proper coloring s where $s(v) \in S(v)$ for each vertex v.*