Assignment (I)

1. (Simultaneous Recursion) Let

$$\begin{aligned} h_1(x,0) &= f_1(x), \\ h_2(x,0) &= f_2(x), \\ h_1(x,t+1) &= g_1(x,h_1(x,t),h_2(x,t)), \\ h_2(x,t+1) &= g_2(x,h_1(x,t),h_2(x,t)). \end{aligned}$$

   Prove that if f_1, f_2, g_1, g_2 are recurisve, then h_1, h_2 are also.

2. Show that the *Euler's function* $\phi(x)$ is primitive recursive, where $\phi(x)$ is defined by "$\phi(x) =$ the number of positive integers less than x which are relatively prime to x".

3. Construct a lambda term M s.t. $\forall m,n \in \mathbb{N}.M\lceil m\rceil\lceil n\rceil = \lceil gcd(m,n)\rceil$, where "$gcd(m,n) =$ the greatest common divisor of m,n".($\lceil n\rceil$ represents the Barendregt numeral of n.)

4. The class $\mathcal{E}$ of *elementary functions* is the smallest class such that

   (a) $x+1$, $U_i^n(x_1,\ldots,x_n)$, $x+y$, $x\dot{-}y$, xy are all in $\mathcal{E}$,

   (b) $\mathcal{E}$ is closed under composition,

   (c) $\mathcal{E}$ is closed under bounded sum and bounded products (i.e. if $f(x,z)$ is in $\mathcal{E}$ the so are the functions $\sum_{z<y} f(\mathbf{x},z)$ and $\Pi_{z<y} f(\mathbf{x},z)$).

   Let the binary function $exp_y(x)$ defined by

$$\begin{aligned} \exp_0(x) &= x \\ \exp_{y+1}(x) &= 2^{\exp_y(x)}. \end{aligned}$$

   (a) Show that for every elementary function $f(x_1,\ldots,x_n)$ there is a constant k such that $f(x_1,\ldots,x_n) \leq \exp_k(\max\{x_1,\ldots,x_n\})$.[Hinit: Show that for every n there is an $m \geq n$ such that $x \cdot \exp_n(x) \leq \exp_m(x)$ for all x.]

   (b) Prove that $\mathcal{E} \subsetneq \mathcal{PR}$, where $\mathcal{PR}$ is the class of primitive functions.

5. Describe an algorithm that transforms a closed λ-term to a Turing Machine. The machine should meet the specification: Whenever it starts with a closed λ-term written on its input tape, it carries out the leftmost reduction, and stops if the reduction terminates.

6. Describe an algorithm that transforms a TM to a URM.