Due: 2013/12/06

Homework 9

Problem 1. *For a fixed $0 < p < 1$, consider the probability space of random graphs $\mathcal{G}_{n,p}$. Let E be the event that the graph contains the Petersen graph as an induced subgraph. Prove that*

$$\lim_{n\to\infty} Pr(E) \to 1.$$

Definition 1. *An* independent set *of a hypergraph $\mathcal{H} = (V, E)$ is a set $S \subseteq V$ such that S does not contain any hyperedge, i.e. $E \cap 2^S = \emptyset$. Define $\alpha(\mathcal{H})$ to be the size of the largest independent set in $\mathcal{H}$.*

Problem 2. *What is $\alpha(\mathcal{H})$ when $\mathcal{H}$ is the Fano configuration?*

Problem 3. *Prove that, for any r and n (for simplicity, $r|n$), there is a r-uniform hypergraph $\mathcal{H}$ on $[n]$ with at least $\binom{n}{r}/e^r$ hyperedges, and*

$$\alpha(\mathcal{H}) \geq n(1 - \frac{1}{r}).$$

Problem 4. *Prove that, for any constant $c_1 > 0$, there is another constant $c_2 > 0$, such that for any 3-uniform hypergraph $\mathcal{H}$ with n vertices and m hyperedges where $m \geq c_1 n$,*

$$\alpha(\mathcal{H}) \geq \frac{c_2 n\sqrt{n}}{\sqrt{m}}.$$

Problem 5. *Let A be a set of $2r + 1$ points*

$$A = \{a_1, ..., a_r, b_1, ..., b_r, c\}.$$

Uniformly pick a random permutation σ of A. Define the random variables

$$x := |\{a_i | a_i \prec_\sigma c\}|,$$

$$y := |\{b_i | b_i \prec_\sigma c\}|.$$

Let $0 \leq p \leq 1$ be fixed.
(a) When $r = 1$ and $r = 2$, compute $\mathsf{E}_\sigma[(1+p)^x(1-p)^y]$.
(b) Prove that $\mathsf{E}_\sigma[(1+p)^x(1-p)^y] \leq 1$. (Hint: Let c_i be the number of elements in $\{a_i, b_i\}$ that are before c, $(c_1, ..., c_r)$ is a sequence of 0-1-2 of length r. Partition all the outcomes by conditioning on such sequences.)