Vector Calculus Theorems: An Intuitive Approach

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Contents

1 Vector Functions

So I'm sure everyone would have come across vector equations like $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ etc. Now in this, the coefficients of the basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are constants, but let us make them variable. Then we get a **vector function** in which the coefficients are not simple constants but in general are functions of some independent variable(s).

For example, let's say we're measuring the velocity of a projectile with time. Then its X-component v_x remains constant and is some v_{x_0} say. However in the vertical direction gravity is acting. If originally v_y was say v_{y_0}, and the acceleration due to gravity is g downwards then at any time t, $v_y = v_{y_0} - gt$. Then v_y is a function of time, denoted by $v_y(t)$. Then $\mathbf{v} = v_{x_0}\mathbf{i} + v_y(t)\mathbf{j}$ is a vector function.

In a more general case, say near an electric charge where the force is radial as given by Coulomb's law, all the components can vary with time. This is the vector equation in one variable, denoted usually by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. This is like three functions packaged together into a single entity, with well-defined operations like the dot and cross product. So we can use these properties to create a richer calculus than is given by single or multivariable functions alone.

2 The Concept of Circulation

I'm sure you know what **work** is (the physics work). If a force is acting in some direction, and it causes a body to displace in some direction, then it is the product of the magnitude of component of force in the direction of displacement. I know it sounds like a mouthfull, but what it's saying is take the force and measure its projection along displacement, cause that's what is pushing the object in that direction. Mathematically,

$$W = FS\cos(\theta) = \mathbf{F} \cdot \mathbf{S} \tag{1}$$

where θ is the angle between force and displacement. Now this holds if the force is constant along the path, but what if it changes in path? That's the job for integration. We can divide the path into several small strips of $\mathrm{d}\mathbf{S}$, which is a vector and then in that interval $\mathbf{F}$ is like almost constant. We can calculate the small work on each strip using the formula (1), and add them all together by integration, i.e

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{S} \tag{2}$$

Where C is the path of integration. So this is the work. How is it related to circulation?

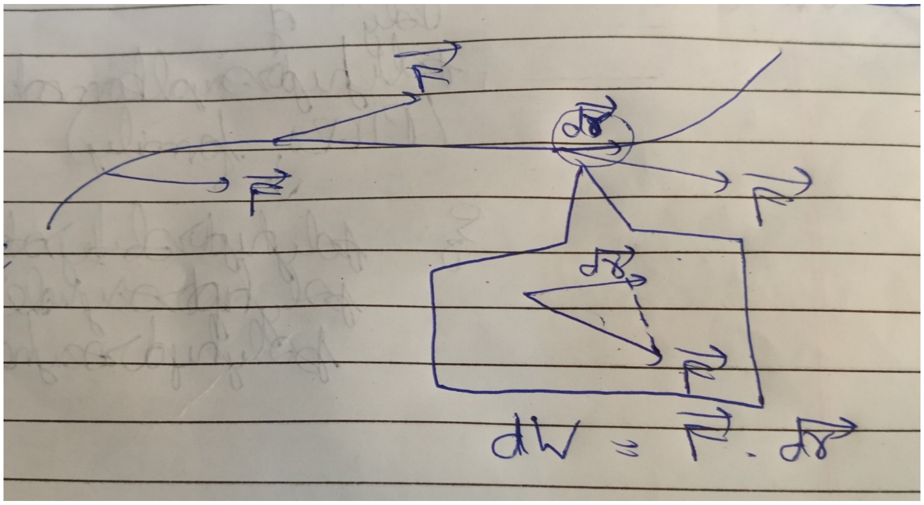

Figure 1: The concept of circultion

Work *is* circulation! Work is in physics, but equation (2) is pretty general, it works for any vector function $\mathbf{F}$, not necessarily force only, and any path C.

For convenience, I'll do a little change of notation. It is customary to call the tiny displacement by d$\mathbf{r}$ rather than d$\mathbf{S}$. So the equation of the circulation of $\mathbf{F}$ becomes,

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{3}$$

In practice, we can parameterize this curve by a variable, and calculate d$\mathbf{r}$. For example, if the curve is the circle $x^2 + y^2 = a^2$, then a parameterization is $x = a\cos(\theta), y = a\sin(\theta)$ where θ can lie between 0 and 2π. Then the vector equation of it becomes $\mathbf{r}(\theta) = a\cos(\theta)\mathbf{i} + a\sin(\theta)\mathbf{j}$ from which it is easy to calculate d$\mathbf{r}$.

So in short equation 3 is called the **circulation** of $\mathbf{F}$ around the curve C, which may or may not be a closed curve. Now it can be seen how this relates to the tangent vector at each point. Let the vector equation be a function of a single variable, say t so that $\mathbf{r} = \mathbf{r}(t)$. Then what does $\mathbf{r}'(t)$ mean geometrically. If we expand it out, it is saying that,

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

If you ignore the denominator h for a second, as it's a scalar and won't affect the direction of the vector, this is actually a vector joining two points at t and

$t+h$, as $h \to 0$. In other words, it is a vector joining two very nearby points on the curve. And that vector is actually the vector **tangent** to the curve at t. And dividing by h simply scales it, changes its length. Thus the unit tangent vector $\mathbf{T}$ is given by,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{4}$$

So our equation (3) can also be written like this,

$$W = \int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, \mathrm{d}t = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{T}(t) |\mathbf{r}'(t)|\, \mathrm{d}t$$

Now, $|\mathbf{r}'(t)|\mathrm{d}t$ represents the length of the tangent vector for a small change in t, and if you see figure (1) carefully, you will see that the length of the tangent vector is exactly the length of the curve at that point! This length is generally denoted by $\mathrm{d}s$, called **arc length**. In light of this, we can write this circulation in multiple ways.

$$W = \int\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, \mathrm{d}t = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{T}(t) |\mathbf{r}'(t)|\, \mathrm{d}t = \int_0^l \mathbf{F}(s) \cdot \mathbf{T}(s) \mathrm{d}s \tag{5}$$

where l is the length of the entire curve.

3 The Concept of Flux

In the last section, we considered the component of $\mathbf{F}$ in the direction of the curve C. A natural question would be to see if it is meaningful to take the component of **F normal** to the curve, i.e., in the direction of a vector normal to the curve at each point. Mathematically, we are investigating if the expression $\int\limits_C \mathbf{F}(s) \cdot \mathbf{n}(s)\, \mathrm{d}s$ makes any sense, where $\mathbf{n}$ is the normal vector at each point now.

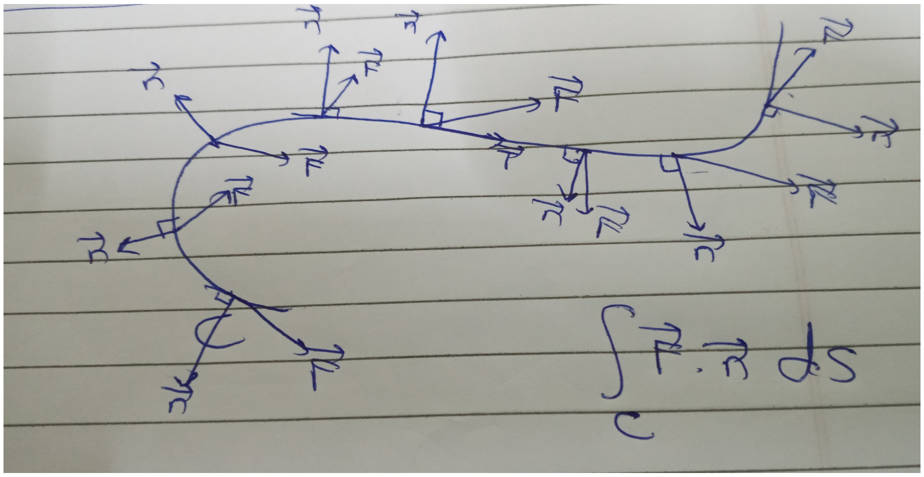

Figure 2: The geometric significance of $\int_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s$

So there's one ambiguity. There are two normals at any point, and we can take anyone of them. Usually its specified which to take, or if it is a closed curve then by convention normal means **outward pointing normal**, unless specified otherwise.

This integral indeed seems to make sense mathematically, but what is its physical significance? It is visible only for closed curves actually. What this is saying is how much the field is towards the normal. Since this is the outward normal for closed curves, it's saying how much the field is getting out of the curve. This is a pretty useful concept and is called the **flux** of field **F** on the curve C. It has applications in fluid flow and more and in electricity and magnetism. It is often denoted by the letter ϕ.

$$\phi = \int_C \mathbf{F}(s) \cdot \mathbf{n}(s) \, \mathrm{d}s \tag{6}$$

It is easy to find an expression for the normal vector in 2D. Say our curve C is $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$. Then the tangent vector is given by
$\mathbf{T}(s) = \mathbf{r}'(s) = x'(s)\mathbf{i} + y'(s)\mathbf{j}$ as $|\mathbf{r}'(s)|$ being the length of the unit tangent vector is 1. The normal is specified by the fact that $\mathbf{n} \cdot \mathbf{T} = 0$. It is not too hard to see that such a normal vector is given by $\mathbf{n}(s) = \pm(y'(s)\mathbf{i} - x'(s))\mathbf{j}$. Putting this in equation (6) we can get a regular integral which we can evaluate easily.

4 Divergence and Curl

Now, we've already seen how to calculate the circulation and flux for a given vector field and curve. Let's see what else we can do with vector fields. Vectors have two basic operations, the dot and cross product, and functions have the operation of differentiation. Vector functions are their combination, and interesting things result from combining these two kinds of operations.

To make things simpler, we've an operator ∇ (called "del"), which you can think of as a vector of partial derivatives.

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \qquad \text{(For 2D)}$$
$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \qquad \text{(For 3D)}$$

Divergence is kind of like the force coming out or going in, in a very small area or volume. Since this is measured by flux, so divergence is called the **flux density**.

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\text{Flux}}{\text{Volume}} \tag{7}$$

Similarly, the curl is kind of like the amount of rotational effect produced by the vector field, in a very small area or volume. Since this is measured by circulation, the curl is also called the **circulation density**.

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \frac{\text{Circulation}}{\text{Volume}} \tag{8}$$

Now we're equipped fully to see the actual meaning behind the theorems like Green's theorem, Stoke's theorem and Divergence theorem.

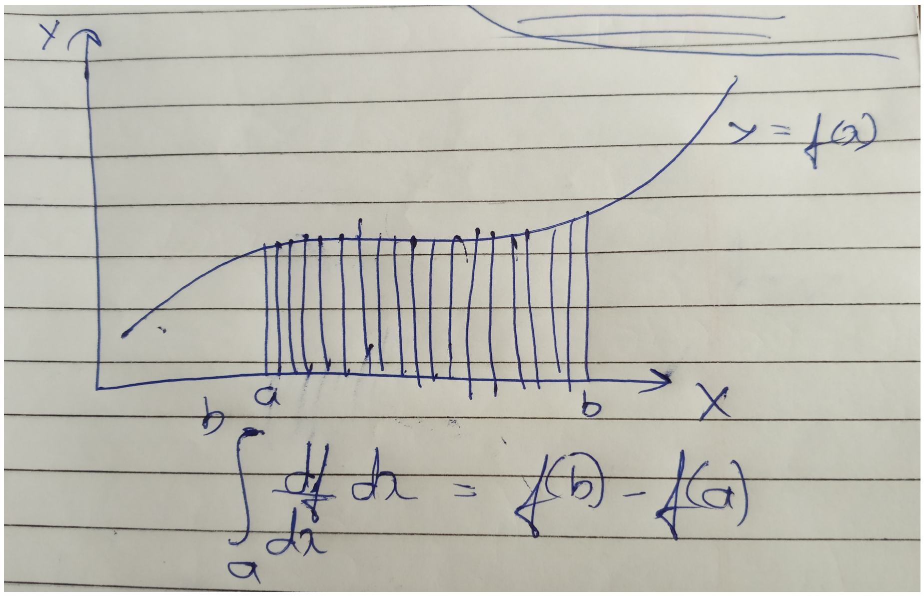

Figure 3: The Fundamental Theorem says that "the total change on outside = sum of little changes on inside"

5 The Fundamental Theorem of Calculus

As most calculus students know, the FTC is the most important theorem in calculus, relating the operations of differentiation and integration. Just that, it does not actually, it is relating derivatives and boundaries actually.

If you see figure (3), this is the standard example given when FTC is first taught. I'd like to introduce a bit of notation to simplify things. If D is the domain, then the boundary of D is denoted by ∂D. For example if the domain D is a disc, like say $x^2 + y^2 \leq a^2$ then its boundary ∂D is the circle $x^2 + y^2 = a^2$. Really it is like we denote the region by R and its bounding curve by C, then $C = \partial R$. This is just a notation, the ∂ is not an operator, just a symbol here.

So for a single dimensional domain $[a, b]$ what is $\partial[a, b]$? It is just the end points a and b. Also, when we say boundary we mean **oriented** boundary. So in this case since a comes before b its orientation can be said to be negative while b is positive. So when we calculate $f(a)$ in FTC, we should make it negative, and positive for $f(b)$.

Also, we can see the interval $[a, b]$ as a sort of 1D closed curve. It is a curve in which the paths have been sort of squished on top of each other to fit in a single dimension. (I'm taking the analogy with 2D to the extreme here, but bear with me). So we can denote it by a domain C, with the oriented boundary being just two points $\{a, b\} = \partial C$.

So, for all this, the equation becomes

$$\int_C \mathrm{d}f = \int_{\partial C} f \tag{9}$$

This is the ultimate **generalized Stoke's theorem**. It is the ultimate equation in calculus, and all of single variable, multivariable and vector calculus is a consequence of (9). If you dissect its meaning, on the left hand side we are summing up $\mathrm{d}f$ over a region C, like the sum of all small parts of f over C, and on the right hand side we are summing up f on ∂C, like finding total change in f across the boundary ∂C, which is the outside of the region C.

Thus the FTC actually says that "the total change on the outside = sum of all the little changes on the inside". This general principle holds in all dimensions and we'll leverage it to see the theorems pop out naturally, as all of them basically are a generalization of FTC to higher dimensions. So we'll begin with circulation.

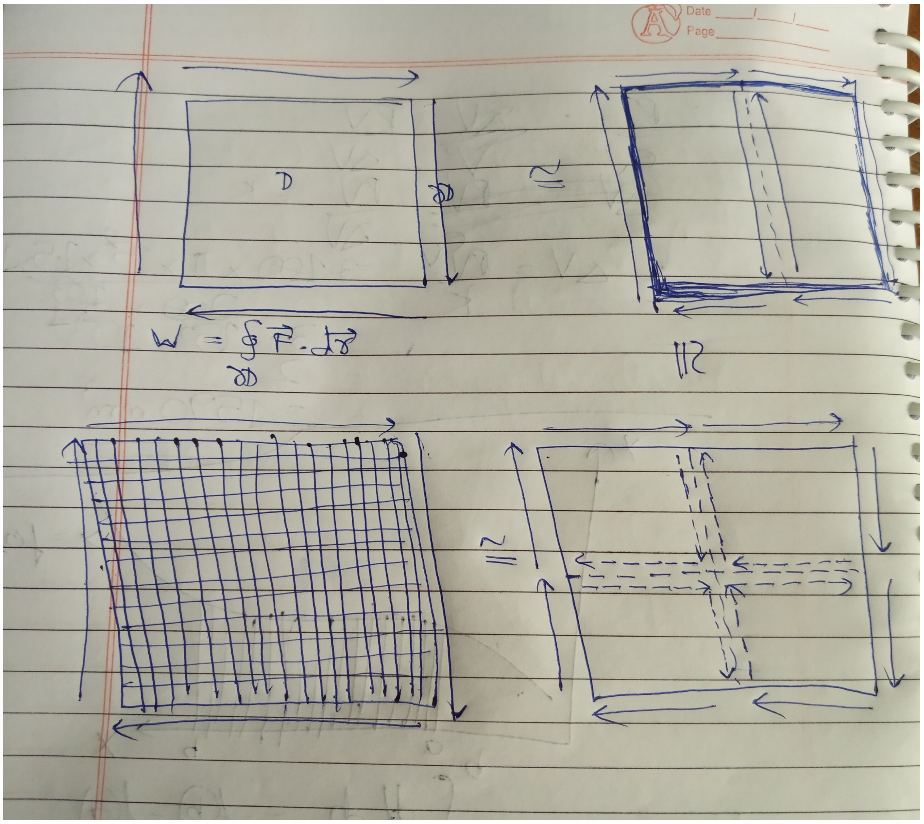

Figure 4: Visualization of Stoke's theorem using circulation

6 Circulation and Stoke's Theorem

Circulation is in some sense the measure of how much a vector field "circulates" around a curve. Thus it is helpful to visualize circulation as some sort of whirlpool arrows around a curve (This is not rigorous, but then my aim is intuition first, rigor later :).

Follow the figures in Figure (4) clockwise, starting from top left. In that figure I've basically considered a region D with a boundary ∂D. Also, there is a vector field $\mathbf{F}$ in the region, and I've drawn only the component of $\mathbf{F}$ which is tangential as only that part contributes to circulation. The field itself is not tangential to the boundary at all points.

If you follow the next figure, what I've done is basically I've dissected the region into two halves by a vertical line in between Also, I drew two equal and opposite arrows, which are the tangential component of $\mathbf{F}$ on that line and its negative. They cancel out each other, leaving the total circulation unchanged. But on the other hand, they serve as the circulation around each of the smaller compartments. The arrows before and the arrows I've added now are like the circulation for these two subsets of D.

Similarly, if I now draw a horizontal line and add equal and opposite arrows, they all cancel out each other, giving the total circulation. On the other hand, these are ciruclations themselves for the small regions now. Clearly, I can continue this process indefinitely, and still get the same total circulation.

So in the last figure, I've divided the region by a number of vertical and horizontal areas, leaving squares of infintesimal area $\mathrm{d}A$. In such a small region, the flux is approximately given by the flux density times the area of the region. And totally we're getting the circulation around the entire region D.

Putting this in equation (9) now reads "total circulation on the outside = sum of little circulations inside". The total circulation on the outside is along $C = \partial D$ which is $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. The circle denotes that we are now integrating over a closed curve. The little circulation = circulation density times area. Since circulation density is $\nabla \times \mathbf{F}$, which is a vector, and we should take the component along the normal to the curve $\mathbf{n}$ at each point in order to get the scalar circulation density.

$$\oint\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint\limits_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, \mathrm{dA} \tag{10}$$

which is precisely Stoke's theorem. And although I drew the region flat on paper, it need not be so. It can be a curved surface itself, because to find the normal at each point we just have to make a tangent plane approximation and find its normal vector.

However, in the special case that it is indeed flat, the normal vector is a constant. If D is confined to the XY-plane, the normal is given by $\mathbf{k}$. Then,

$$\begin{gathered}
\mathbf{F} = P \, \mathbf{i} + Q \, \mathbf{j} \\
\implies (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\
\implies \oint\limits_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \\
\oint\limits_C P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{dA}
\end{gathered} \tag{11}$$

which is the tangential form of Green's theorem.

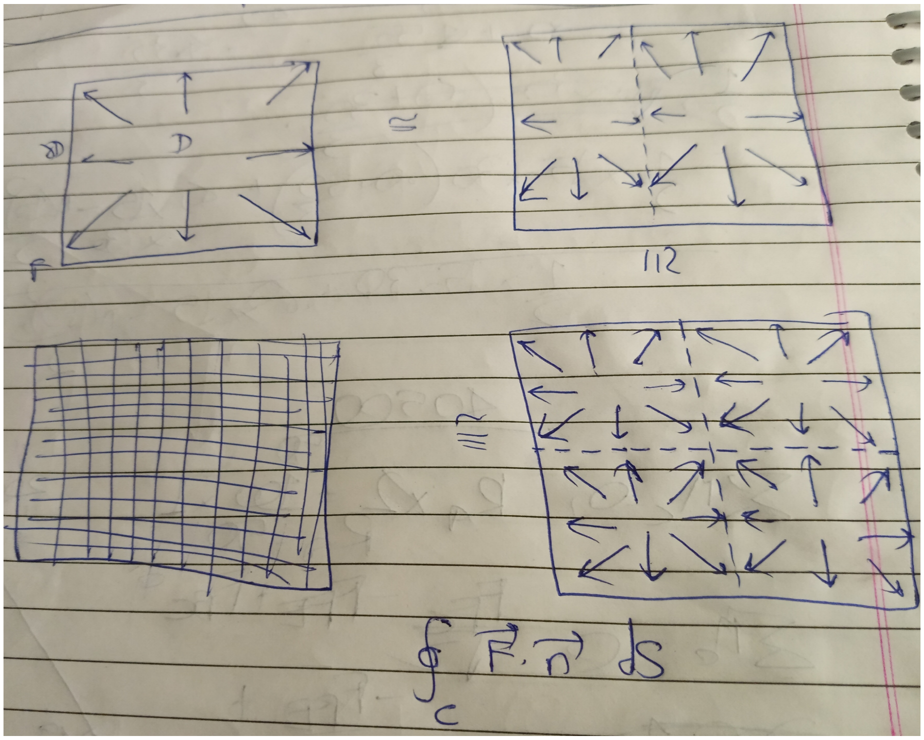

Figure 5: Visualization of Divergence theorem using circulation

7 Flux and Divergence Theorem

We use a similar logic as we discussed in Stoke's theorem. Again I've drawn a region D in Figure (5), and a vector field $\mathbf{F}$, but this time I've shown only its normal component as that contributes in flux calculation. If you follow clockwise from top left, you will see that as I partition the domain into subdomains and add equal and opposite arrows, they cancel out their normal components. Since the tangential part left does not contribute to flux anyway, in effect I can divide the domain as I like and still get the same total flux.

So I divide the domain into many rectangles of infinitesimal area dA. In such a small region, the flux is approximately the flux density times area of that region. In terms of equation (9), "the total **flux** on the outside = sum of little **flux**es on the inside".

Now, the total flux is $\oint_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s$, while each little flux is flux density into area. The flux density is nothing but the **divergence** of $\mathbf{F}$, so every flux $= \nabla \cdot \mathbf{F} \, \mathrm{dA}$. Thus finally,

$$\oint\limits_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \iint\limits_D \nabla \cdot \mathbf{F} \, \mathrm{dA} \tag{12}$$

which is the Divergence theorem in two dimensions, or if you like, normal form of Green's theorem. Because in two dimensions,

$$\begin{aligned}
\mathbf{F} &= P \, \mathbf{i} + Q \, \mathbf{j} \\
C : \mathbf{r}(s) &= x(s) \, \mathbf{i} + y(s) \, \mathbf{j} \\
\implies \mathbf{n} &= y'(s) \, \mathbf{i} - x'(s) \, \mathbf{j} \\
\implies \mathbf{F} \cdot \mathbf{n} &= P \, y'(s) - Q \, x'(s) \\
\implies \oint\limits_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s &= P \, \mathrm{d}y - Q \, \mathrm{d}x \\
\nabla \cdot \mathbf{F} &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}
\end{aligned}$$

$$\therefore \oint\limits_C P \, \mathrm{d}y - Q \, \mathrm{d}x = \iint\limits_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, \mathrm{dA} \tag{13}$$

Similarly, instead of a flat plane, think of a surface in three dimensions, like a cube or sphere or other surfaces. For simplicity imagine a cube, then by the same logic as above, we can divide it into a number of small cubes of infinitesimal volume $\mathrm{d}V$, and sum up the little fluxes to get the total flux. Since to calculate the flux we will have to integrate over a surface now, the left hand side of equation (12) will becomes a surface integral. And since its volume now not area, the right side will become a triple integral because we have to integrate over the 3D domain now.

$$\therefore \iint\limits_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}\sigma = \iiint\limits_D \nabla \cdot \mathbf{F} \, \mathrm{d}V \tag{14}$$

which is the Divergence theorem in three dimensions. With this, I hope that I've derived all the theorems in an intuitive sense.

8 Conclusion

With this, we bring our discussion to an end. Hope you enjoyed!