Essence of Complex Analysis

Adarsh Kishore

Contents

1 What is Complex Analysis?

Let us start this topic with a question: what the heck is this thing?

We've been studying calculus for quite a while. From the simple one variable version to the pumped up multivariable version to the mind-bending-in-3D vector calculus. But they all have one thing in common: in all of them, the domain is the real numbers only, it is $\mathbb{R}$.

This domain is the difference in complex analysis. Complex analysis is calculus with complex numbers, those things which have an $i = \sqrt{-1}$ somewhere in them. The domain of functions is now $\mathbb{C}$. You may be thinking that there cannot be much difference from what we've studied, and it is correct, but with a caveat. Since we will be studying complex analysis with a function of one complex variable only, the natural supposition is that it is analogous to normal single-variable calculus. No, in fact it is like **vector calculus** in two dimensions!

The reason for this wil become clear in due course, but right now I can give a hint as to why it is certainly not analogous to single-variable calculus. A complex number $z = a + i\,b$ has two parts, real and imaginary. Thus to represent a single complex number, we require two real numbers. So it is certainly not the one-variable-only calculus.

So in case you are planning to leave now saying that "I don't remember complex numbers at all", then no need to leave! Cause I plan to give a referesher over complex numbers before we can start our discussion.

2 Preliminaries

2.1 Imaginary Numbers

(a) Carl Friedrich Gauss

(b) Jean-Robert Argand

Figure 1: Gauss and Argand

So, what is an imaginary number? It is like a real number only, but with an imaginary constant multiplied to it. This imaginary constant is $i = \sqrt{-1}$, so a number of the form ai, $a \in \mathbb{R}$ is an imaginary number. Their special feature is the fact that their square is negative, $(ai)^2 = -a^2$, something which is not possible in $\mathbb{R}$.

The term "imaginary" given to them is unfortunate, because they have nothing imaginary to them. Consider that at one time, even negative numbers were regarded with suspicion, because for counting purposes, negative numbers make no sense. Similarly, these are just numbers with negative squares. What is new and groundbreaking in mathematics in one time becomes routine and usual afterwards. Consider that all the calculus we have studied is quite routine for us now, but it was certainly groundbreaking work and thought for the 17th century mathematicians like Newton, Leibniz, Fermat, etc.

So anyway, coming back to point, these numbers can be said to live in a different world than real numbers, because they have a property which none of the real numbers have. As such, things like "imaginary + real" have no meaning, because these are from different worlds. It is like saying "what is the value of 2 oranges + 3 bananas?"

So this term is undefined, yet. One of the things in mathematics is that we can define something to be anything we want, as long as it doesn't contradict any previously established definitions or theorems. So formally, this term "imaginary + real" is called **complex** (yet another unfortunate name because often complex numbers make things much simpler than using only real numbers. However, it was one of the greatest achievements of the mathematicians of the 17-18th century like Argand, Gauss to give a geometric interpretation to complex numbers.

2.2 Complex Numbers

So, a complex number, usually denoted by z is defined to be-

$$z := a + i\,b, \quad a, b \in \mathbb{R},\, i = \sqrt{-1} \tag{2.2.1}$$

Also, as we'll see it is helpful to define something akin to $-z$, but only for the imaginary part. This is called the **conjugate** of z, denoted by $\bar{z}$

$$\begin{aligned} & z = a + i\,b \\ \implies & \bar{z} := a + i\,(-b) \end{aligned} \tag{2.2.2}$$

In mathematics, everything at its root is a set. The set of real numbers $\mathbb{R}$ is a "set" as the name implies, so is $\mathbb{N}, \mathbb{Z}$ etc. A function is a mapping between two **sets**. Thus this z, this complex number, would also be in a set, the set of complex numbers we call it, and denote it by $\mathbb{C}$. Thus we can define $\mathbb{C}$ as -

$$\mathbb{C} := \{\, a + i\,b \mid \forall\, a, b \in \mathbb{R}, i = \sqrt{-1}\,\} \tag{2.2.3}$$

The a is a real number and is without any i, for this reason it is called the real part of z. Likewise, b is called the imaginary part of z. Thus

$$\begin{aligned} a &:= \Re(z) \\ b &:= \Im(z) \end{aligned}$$

(It looks like a fancy R and a fancy I).

Ok, now let us see how complex numbers behave under different algebra rules.

If $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$, then

$$z_1 \pm z_2 = (a_1 \pm a_2) + i\,(b_1 \pm b_2) \tag{2.2.4}$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1) \tag{2.2.5}$$

From (2.2.5), we can find our first important use of the conjugate. Specifically, what is $z\bar{z}$?

$$z\bar{z} = (aa - b(-b)) + i\,(ab - ab) = a^2 + b^2 \tag{2.2.6}$$

Equation (2.2.6) is important, because it gives us a way to get a real number out of a complex number. For reasons that will become clear when we study the geometry of complex numbers, we define the modulus of a complex number as $|z| = \sqrt{a^2 + b^2}$. In light of this, we can rewrite equation (2.2.6) as $z\bar{z} = |z|^2$.

This enables us to define properly what we mean by division of complex numbers. Specifically,

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} \tag{2.2.7}$$

The denomintor is just a real number, and the numerator is the product of two complex numbers, which we've defined already in (2.2.5).

2.3 Geometric Interpretation of Complex Numbers

2.3.1 Cartesian Form

The complex number $z = x + i\,y$ is reminiscent of the ordered pair (x, y). In fact all the definitions we gave earlier would still hold, albeit with a change of notation, if we replace $x + i\,y$ with (x, y). To show some examples-

$$(x_1, y_1) \pm (x_2, y_2) = (x_1 \pm y_1, x_2 \pm y_2) \tag{2.3.1}$$
$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \tag{2.3.2}$$
$$|(x, y)| = \sqrt{x^2 + y^2} \tag{2.3.3}$$

(x, y) is like a point which can be plotted on graph. However, this is not the XY-plane. The X-axis represents x, or $\Re(z)$ so it is called the real axis, while Y-axis is called the imaginary axis. Thus it is referred to as the Argand or Gauss plane.

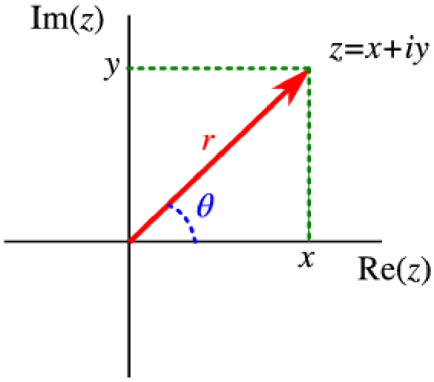


Figure 2: The Argand Plane

From Figure (2), we at once get a geometric meaning of the modulus. It is the distance of z from the origin. Also, we get another quantity θ, which is called the amplitude or argument of z. From the figure we can see that

$$\theta = \arg z = \arctan\left(\frac{y}{x}\right) \tag{2.3.4}$$

If $\theta \in [0, 2\pi)$, then it is called the **principal argument** of z.

2.3.2 Polar Form

We've got two quantities from z: one is $|z|$ and the other is $\arg z$. In other words, we have the distance of the point from the origin and the angle it makes with the real axis. This is exactly what we need for polar coordinates! So, in other words we can represent a complex number using $|z|$ and $\arg z$ using the ordered pair $z = (r, \theta)$, where $r = |z|$, $\theta = \arg z$.

As you know this is not a unique representation because we can add any multiple of 2π to θ and still land on the same spot. However, we can make it unique by demanding that only the principle value of argument be used. So now on, when I say argument I mean only principle argument.

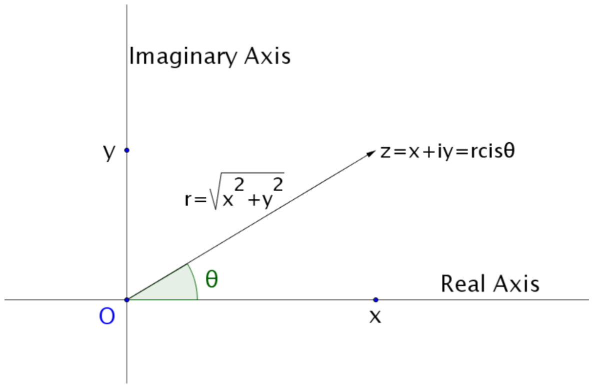


Figure 3: Relation between Cartesian and Polar

2.3.3 Relation between the two forms

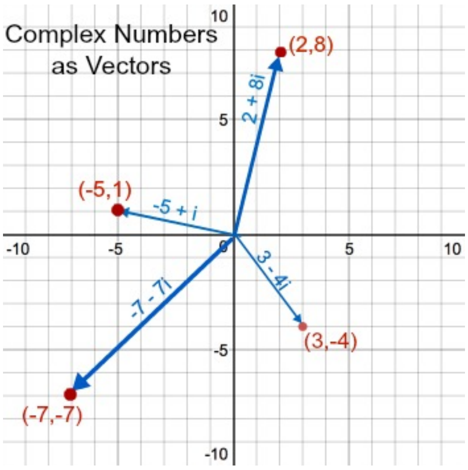
So what is the relation between $z = (r, \theta)$ and $z = (x, y)$? It is exactly the same thing as we've studied for polar coordinates in the real plane. In the Figure (3), you can see that

$$\begin{aligned} & x = r\cos(\theta) \\ & y = r\sin(\theta) \\ \Longrightarrow\ & z = r\cos(\theta) + i\,r\sin(\theta) := r\,\mathrm{cis}(\theta) \end{aligned} \tag{2.3.5}$$

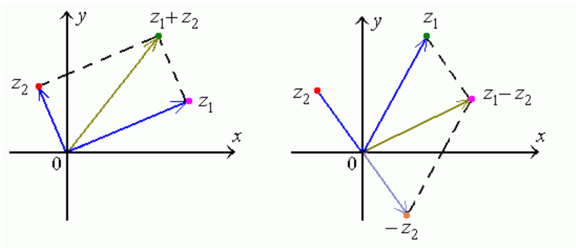
cis is a short-hand for writing the full thing. So up until now we've seen two representations. But I kinda saved the best for the last!

2.3.4 Complex Numbers as Vectors

If you see $z = x + i\,y$, the geometry reminds you of a very familiar thing in vectors. Real and imaginary parts are just perpendicular components in the plane and the basis is $\vec{i} = 1$, and $\vec{j} = i$. Thus we can write $z = x\vec{i} + y\vec{j}$.



(a) Representation on Argand Plane



(b) Addition of Complex Vectors

Figure 4: Vector Interpretation of Complex Numbers

These are vectors which can be **mutiplied** also, by equation (2.2.5). They can be divided also. That gives them much more power and versatility when compared to vectors in the real plane. To make things simple, we'll also define the **dot** and **cross** product of complex vectors analogous to their real counterparts.

$$z_1 \cdot z_2 = |z_1|\,|z_2|\cos(\theta) = (x_1 + i\,y_1)\cdot(x_2 + i\,y_2) = x_1x_2 + y_1y_2 \tag{2.3.6}$$
$$z_1 \times z_2 = |z_1|\,|z_2|\sin(\theta) = (x_1 + i\,y_1)\times(x_2 + i\,y_2) = x_1y_2 - x_2y_1 \tag{2.3.7}$$

The reason why I defined them is two-fold. First one, its really fun to see the analogies between complex numbers and vectors, and two, these serve the same purpose as before. The dot product (2.3.6) gives the angle between two complex vectors as $z_1 \cdot z_2 = |z_1|\,|z_2|\cos(\theta)$ and the cross product (2.3.7) gives the signed area of the parallelogram spanned by the two complex vectors.

Lastly, what does multiplication of vectors do. Equation (2.2.5) does not give a geometric picture at all to visualize what's happening under the hood. So after some time, you may think of using the polar representation, and indeed that shows how it works!

$$\begin{aligned}
z_1 z_2 &= r_1 \operatorname{cis}(\theta_1)\, r_2 \operatorname{cis}(\theta_2) \\
&= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\
&= r_1 r_2((\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i\,(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)) \\
&= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)
\end{aligned}$$

Multiplying two complex vectors multiplies their magnitude and adds their angles! That is a very beautiful result indeed, and if you look closer you might notice something interesting. **Multiply** complex numbers = **Add** angles. Kinda reminds you of $a^m \times a^n = a^{m+n}$? Multiplication is being converted to addition, that's like exponentials being multiplied. As we'll see later, this fact is no coincidence.

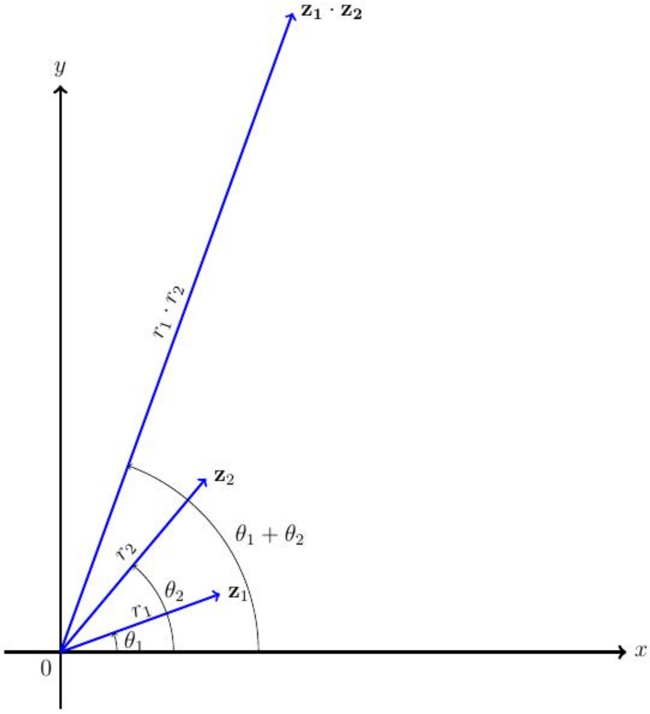


Figure 5: Multiplication of two complex numbers

I think that covers all the basics we need to understand the essence (read intuition) of complex analysis. Now we can get on to the new stuff!

3 Complex Functions

3.1 Introduction

What is a complex function? To understand this let us go back to the question of what is a real function. A function is a mapping between two sets, it takes elements from one set to produce an element in the other set. A real valued function is between $\mathbb{R}$ only, $f : \mathbb{R} \to \mathbb{R}$. A complex function f maps in $\mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$. For example, the function $f(z) = z^2$.

For analysis purposes, it is convenient to separate the real and imaginary parts of a complex function and study them. For example $z = x + i\,y$, therefore $f(z) = z^2 = (x^2 - y^2) + i\,(2xy)$.

So a complex function is of the form

$$w = f(z) = u(x, y) + i\,v(x, y) \tag{3.1.1}$$

To list some more examples-

$$\begin{aligned}
z = I(z) &= x + i\,y, \quad u(x, y) = x,\ v(x, y) = y \\
\bar{z} = f(z) &= x + i\,(-y), \quad u(x, y) = x,\ v(x, y) = -y \\
|z| = g(z) &= \sqrt{x^2 + y^2} + i\,(0), \quad u(x, y) = \sqrt{x^2 + y^2},\ v(x, y) = 0 \\
\frac{1}{z} = \frac{\bar{z}}{|z|^2} &= \left(\frac{x}{x^2 + y^2}\right) + i\left(\frac{-y}{x^2 + y^2}\right), \quad u(x, y) = \frac{x}{x^2 + y^2},\ v(x, y) = \frac{-y}{x^2 + y^2}
\end{aligned}$$

The function $f(z)$ itself is complex, but the functions $u(x, y)$ and $v(x, y)$ are real valued, they are $\mathbb{R}^2 \to \mathbb{R}$. They can be subjected to whatever we know in multivariable calculus, and because they are the components of a complex vector, complex analysis is similar to vector calculus in two dimensions, but not entirely because there is a notion of multiplication of complex numbers, which is not there for real vectors.

3.2 Visualize Complex Functions

Complex functions require an input z which is two dimensional as it requires a plane (the Argand plane) to represent it. Its output w would be another complex number again two dimensional. Two dimensions for both input and output means that if we wanted to draw the graph of a complex function, we would require a four-dimensional space, which is pretty much impossible in our 3D world.

So the old way of imaagining a function's graph wouldn't work here. We've got to think of a different way. A nice way to visualize it is as a **transformation**. A transformation is a mapping from a plane to a plane. If you've seen my Jacobian article, that is basically the essence behind it: transformation of one plane to another.

So for example consider the function $f(z) = z^2 = (x^2 - y^2) + i\,(2xy)$ If we take the outputs as the coordinates of a complex number in the uv-plane, then

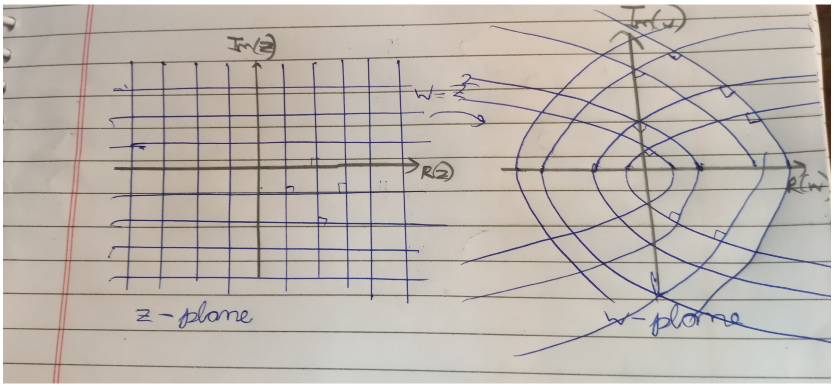
$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \\ \implies y &= \frac{v}{2x} \end{aligned}$$

Substituting this in the expression for u, we get $u = x^2 - \dfrac{v^2}{4x^2}$. Now suppose that you are travelling along the line $\Re(z) = k$ in the original domain plane. That means $x = k$, constant and

$$\begin{aligned} u &= k^2 - \frac{v^2}{4k^2} \\ \implies v^2 &= -4k^2(u - k^2) \end{aligned}$$

which is the equation of a left-opening parabola. We can say that under the transformation $z \to z^2$, all straight lines $x = k$ become parabolas in the uv-plane. And a little work will show that lines $y = k$ will also become parabolas, but right-opening parabolas.

Now suppose that we are travelling along the hyperbola $xy = c^2$ in the z-plane. Then $v = 2xy = 2c^2$ is a constant. So in the w-plane, the image of z would be tracing out the line $v = 2c^2$. And travelling along a different hyperbola, $x^2 - y^2 = a^2$ would give the line $u = a^2$ in w-plane. Really we can take any arbitrary shape in the z-plane and we would get its image figure in the w-plane.



(a) Straight lines $y = k$ and $x = k$ get mapped to parabolas

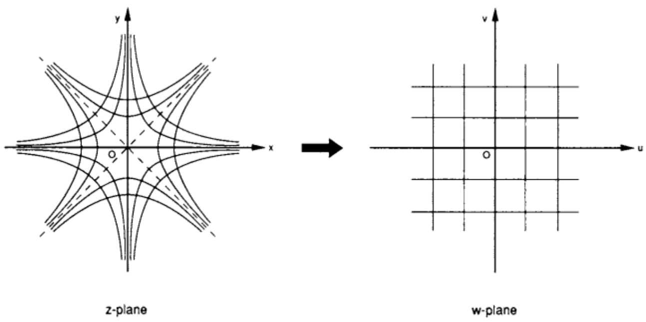


FIG. 71. Mapping by the complex function $w = z^2$.

(b) Hyperbolas get mapped to straight lines

Figure 6: Examples of the tranformation $w = f(z) = z^2$

It is worth noting that this transformation has preserved angles. If you see the z-plane, the hyperbolas $x^2 - y^2 = a^2$ and $xy = c^2$ intersect each other at right angles. And in the w-plane the lines $u = a^2$ and $v = 2c^2$ are obviously at right angles. In fact, even for the straight lines case, the parabolas in the w-plane intersect at right angles. This property of preserving angles is called **conformal** mapping, and complex functions with some conditions are always conformal. It need not be a right angle, any angle will be preserved.

So with this visualization in mind, let's delve deeper into this!

3.3 Limit of a Complex Function

What exactly do we mean by $z \to z_0$? It means both the real and imaginary parts of the z are simultaneously tending to that of z_0. But there are so many ways on a plane to approach a point. Line, parabola, circle, any arbitrary shape can be used. If the word limit has to make any sense, all these paths must lead to the same value. You cannot say that "go left and then down and you'll reach the garden" and then say "go down and then left and you'll reach the library".

Here comes the $\epsilon - \delta$ thing again. It says that a function $f(z)$ has a limit L at z_0 if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon$. In words, what it is saying is that if a point z is arbitrarily close to z_0, then the function value $f(z)$ should also get arbitrarily close to L.

As a self-exploration of sorts, you can try this definition with $w = z^2$ at the origin. Basically we've done this sort of stuff in multivariable calculus.

3.4 Continuity

The definition of continuity is exactly what you'd expect. The function value should change smoothly, there should be no sudden jumps anywhere. To say this mathematically,

1. $\lim_{z \to z_0} f(z)$ must exist.
2. $f(z_0)$ must exist
3. $\lim_{z \to z_0} f(z) = f(z_0)$

4 Derivative of a Complex Function

4.1 The Definition of Derivative

Here starts the interesting things. First of all, we can no longer interpret the derivative of $f(z)$ at z_0 as the slope of a tangent line at that point, because there is no tangent line in the first place! We can't visualize the graph of a complex function, and thus tangents make no sense.

Again referencing my article on Jacobian, I discussed the idea of a derivate as a measure of rate-changer there. Basically $\dfrac{\mathrm{d}w}{\mathrm{d}z}$ is a rate-measurer now, for a very small change in the value of z, it measures the corresponding change in the value of $w = f(z)$. The definition of the derivative formally remains the same, however it leads to very interesting results, primarily because the interpretation of the limit in the definition has changed compared to real number.

$$\frac{\mathrm{d}(f(z))}{\mathrm{d}z} := f'(z) := \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{4.1.1}$$

The part $\Delta z \to 0$ changes it all, no exaggeration. The definition of having a limit itself has changed compared to real numbers, and that will have ripple effects here. In fact, let's go and expand out this term.

$$\begin{aligned} f(z + \Delta z) - f(z) &= (u(x + \Delta x, y + \Delta y) + i\, v(x + \Delta x, y + \Delta y)) - (u(x, y) + i\, v(x, y)) \\ &= u(x + \Delta x, y + \Delta y) - u(x, y) + i\, (v(x + \Delta x, y + \Delta y) - v(x, y)) \end{aligned}$$

At this point, it is helpful to assume that u and v are differentiable functions of x and y. Reason? We can then write-

$$\begin{aligned} u(x + \Delta x, y + \Delta y) - u(x, y) &= u_x(x, y)\Delta x + u_y(x, y)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \\ \implies f(z + \Delta z) - f(z) &= u_x(x, y)\Delta x + u_y(x, y)\Delta y + i\, (v_x(x, y)\Delta x + v_y(x, y)\Delta y) + \psi \Delta x + \phi \Delta y \\ &= (u_x(x, y) + i\, v_x(x, y))\Delta x + (u_y(x, y) + i\, v_y(x, y))\Delta y + \psi \Delta x + \phi \Delta y \end{aligned}$$

ψ and $\phi \to 0$ as $\Delta x, \Delta y \to 0$. Finally, putting all this together,

$$f'(z) = \lim_{\Delta z \to 0} \frac{(u_x(x, y) + i\, v_x(x, y))\Delta x + (u_y(x, y) + i\, v_y(x, y))\Delta y + \psi \Delta x + \phi \Delta y}{\Delta x + i\, \Delta y} \tag{4.1.2}$$

This may look large and complicated, but it is not that difficult to comprehend when we see how we get here. Till now, we've been riding on one assumption: u, v are differentiable.

I've stressed enough I believe that for the limit to have any meaning it must be same however we make Δz approach 0. Certainly it should be same whether we approach by the real axis or the imaginary axis. If we approach via the real axis, then we are on the line $y = 0$ so $\Delta y = 0$. Then putting this in equation (4.1.2), we get

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{(u_x(x, y) + i\, v_x(x, y) + \psi)\Delta x}{\Delta x} \\ &= u_x(x, y) + i\, v_x(x, y) \end{aligned} \tag{4.1.3}$$

That cancellation was, to say the least, satisfyingly unexpected! Things fell together in place beautifully, and that is the theme in mathematics throughout. So, stopping my philosophical remarks xD, next we'll make Δz approach 0 from the imagniary axis. This means that we are on the line $x = 0$. Then putting this in equation (4.1.2), we get

$$
\begin{aligned}
f'(z) &= \lim_{(i\,\Delta y)\to 0} \frac{(u_y(x,y) + i\,v_y(x,y) + \phi)\Delta y}{i\,\Delta y} \\
&= v_y(x,y) + i\,(-u_y(x,y))
\end{aligned} \tag{4.1.4}
$$

Yet another satisfying result. Now, for the limit to have any meaning (see, I've said it again), (4.1.3) and (4.1.4) should be necessarily same. I'm not saying that they are same means there is a derivative, as there are an infinite number of paths and we've taken only two of them, but I'm saying that if the derivative exists they must be the same. So comparing the real and imaginary parts,

$$
u_x = v_y, u_y = -v_x \tag{4.1.5}
$$

We know that equations (4.1.5) are a **necessary** condition for the derivative to exist. Interestingly enough, it turns out that they are also **sufficient**, i.e, if this is satisfied the limit exists for sure and is the same for all paths. Is that amazing or what! By checking two paths we are able to conclude that all paths give the same thing! Let us see how this works. Going back to (4.1.2), and I will ignore the ϕ and ψ now, since they're anyway going to zero and also I'll drop the function arguments because they are understood from context,

$$
\begin{aligned}
f'(z) &= \lim_{\Delta z\to 0} \frac{(u_x + i\,v_x)\Delta x + (u_y + i\,v_y)\Delta y}{\Delta x + i\,\Delta y} \\
&= \lim_{\Delta z\to 0} \frac{(u_x + i\,v_x)\Delta x + ((-v_x) + i\,u_x)\Delta y}{\Delta x + i\,\Delta y} \\
&= \lim_{\Delta z\to 0} \frac{u_x(\Delta x + i\,\Delta y) + i\,v_x(\Delta x + i\,\Delta y)}{\Delta x + i\,\Delta y} \\
&= u_x + i\,v_x
\end{aligned} \tag{4.1.6}
$$

and this clearly does not depend on Δz at all, and thus is path independent. Not only have we proved that (4.1.5) are both necssary and sufficient, we've found an expression for the derivative also. And in light of (4.1.5), we can write it in several different ways.

$$
\begin{aligned}
&f(z) = u + i\,v \\
\implies &f'(z) = u_x + i\,v_x = v_y + i\,v_x = u_x + i\,(-u_y) = v_y + i\,(-u_y)
\end{aligned}
$$

If that is not beautiful, I don't know what is.
If (4.1.5) are so useful, it cannot be nameless right? They are called the **Cauchy-Riemann differential equations** in honour of the mathematicians Cauchy and Riemann, who made great contributions in the field of complex analysis. The Cauchy-Riemann equations are one of the most important set of differential equations in complex analysis and they give much of the power and beauty which complex analysis has.

The CR equations impose a very strong condition on a function. And in addition, the component functions need to be differentiable, as everything started from there. Since mere differentiability of

(a) Augustin-Louis Cauchy

(b) Bernhard Riemann

Figure 7: Cauchy and Riemann are the forerunners of complex analysis

a function has quite a different meaning as you'll agree with me, we call functions which satisfy the CR equations as **complex differentiable** or even better, **analytic**.

To get an idea of how strong a condition it is, consider $f(z) = \bar{z} = x + i\,(-y)$. Admittedly, this function looks quite nice and certainly polynomial-like functions are differentiable in the domain of real numbers. So it comes as a surprise to know that this function does **not** have a derivative at all, because it does not satisfy the CR equations. $u_x = 1 \neq -1 = v_y$, so it is not analytic.

4.2 Derivative Rules

The CR equations make the results of complex differentiation of complex functions completely analogous to their real counterparts. For example, take our favourite example $f(z) = z^2$. Then $u = x^2 - y^2$, $v = 2xy$. $u_x = 2x = v_y$ and $u_y = -2y = -v_x$, so this function is analytic. And $f'(z) = u_x + i\,v_x = 2x + i\,(2y) = 2z$, a result completely in agreement with the power rule for real numbers.

Some others are, $(z^n)' = nz^{n-1}, (\sin(z))' = \cos(z), (e^z)' = e^z$, basically almost all the standard functions in calculus of real numbers are analytic in complex domain. It will be good self-exploration if you try out all these functions and check that they are indeed analytic.

Another important thing is the derivative rules. Again, analytic functions have rules in complete analogy with what we'd know-

$$(z_1 \pm z_2)' = (z_1)' \pm (z_2)' \tag{4.2.1}$$

$$(z_1 z_2)' = z_1' z_2 + z_1 z_2' \tag{4.2.2}$$

$$\left(\frac{z_1}{z_2}\right)' = \frac{z_1' z_2 - z_1 z_2'}{z_2^2} \tag{4.2.3}$$

$$(f(g(z))' = f'(g(z))g'(z) \tag{4.2.4}$$

In addition with their analogy with vectors, we can differentiate the complex dot and cross product.

$$(z_1 \cdot z_2)' = (x_1x_2 + y_1y_2)' = x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2' = z_1' \cdot z_2 + z_1 \cdot z_2' \tag{4.2.5}$$

$$(z_1 \times z_2)' = (x_1y_2 - x_2y_1)' = x_1'y_2 + x_1y_2' - x_2'y_1 - x_2y_1' = z_1' \times z_2 + z_1 \times z_2' \tag{4.2.6}$$

Finally, remember ∇? The vector differental operator. In complex domain it can be defined as

$$\nabla = \frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} \tag{4.2.7}$$

$$\implies \nabla \cdot f(z) = u_x + v_y = u_x + u_x = 2u_x = 2v_y \tag{4.2.8}$$

$$\implies \nabla \times f(z) = v_x - u_y = 2v_x = -2u_y \tag{4.2.9}$$

Clearly, the CR equations allow a much simpler view of ∇, in fact it can be viewed as a multiple of partial derivative of the component functions. The following is nothing but a play of symbols only, yet it looks aesthetically pleasing,

$$f(z) = u + i\,v$$

$$\implies f'(z) = \frac{1}{2}(\nabla \cdot f(z) + i\,(\nabla \times f(z))) \tag{4.2.10}$$

4.3 Conformal Mapping

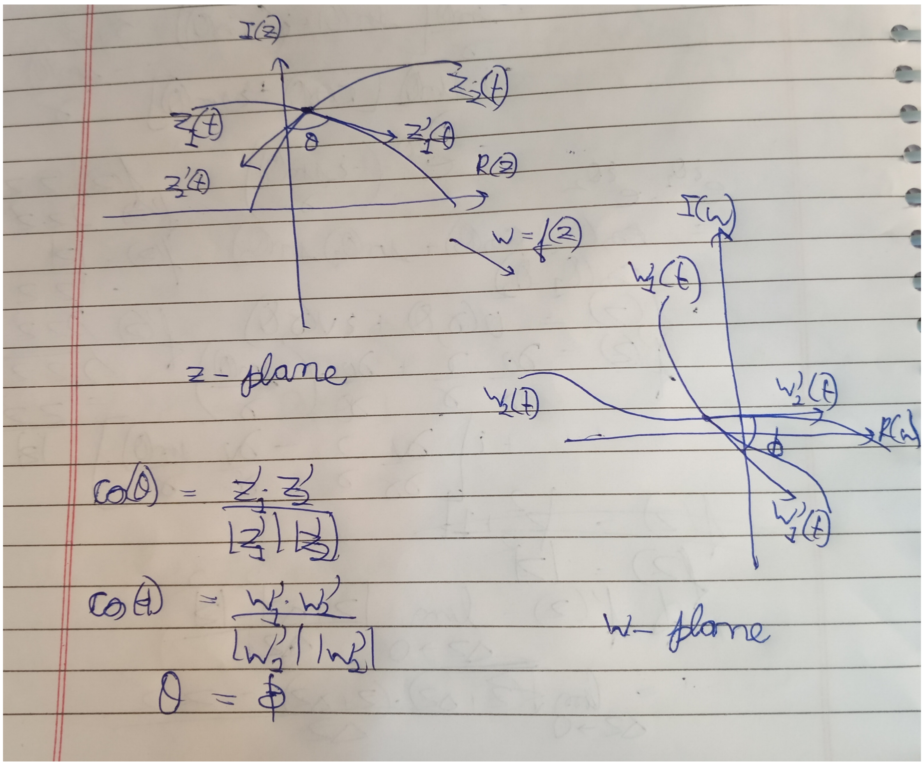


Figure 8: Analytic functions correspond to conformal transformations

As one of the first examples of the power of CR equations, let us investigate something which we had seen earlier, namely the property that analytic functions are equivalent to transformations which preserve angles. We'll take help from our old friend the dot product here (2.3.6), since the dot product is convenient when it comes to measuring angles.

Let us say there are two complex numbers z_1 and z_2, in general both of them will be some figure on the z-plane. For example if $|z_1| = 1$ then z_1 traces a unit circle around origin and $z_2 = x + i\,k$ is the line $y = k$. There can be other shapes also. Now the analogy between complex numbers and vectors would come to the forefront. Let us choose a variable of parameterization t and convert z_1 and z_2 to real vectors. For example, taking $|z_1| = 1$, the vector equation of this would be $\vec{z_1}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$. So in general we can write $\vec{z_1} = f_1(t)\vec{i} + g_1(t)\vec{j}$ and $\vec{z_2} = f_2(t)\vec{i} + g_2(t)\vec{j}$. Then their tangent vectors are $\vec{z_1'} = f_1'(t)\vec{i} + g_1'(t)\vec{j}$ and $\vec{z_2'} = f_2'(t)\vec{i} + g_2'(t)\vec{j}$.

The angle between the two curves at a point would then be given by $\cos(\theta) = \dfrac{z_1' \cdot z_2'}{|z_1'|\,|z_2'|}$.

Now, under some analytic transformation $z \to f(z)$, their images become w_1 and w_2 respectively. Then

$$\begin{aligned}
w_1 &= f(z_1) = u(f_1(t), g_1(t)) + i\,v(f_1(t), g_1(t)) \\
w_2 &= f(z_2) = u(f_2(t), g_2(t)) + i\,v(f_2(t), g_2(t)) \\
\implies \vec{w_1} &= u(f_1(t), g_1(t))\vec{i} + v(f_1(t), g_1(t))\vec{j} \\
\implies \vec{w_2} &= u(f_2(t), g_2(t))\vec{i} + v(f_2(t), g_2(t))\vec{j}
\end{aligned} \tag{4.3.1}$$

So now $\vec{w_1'}$ and $\vec{w_2'}$ would represent tangent vectors to the image of the curve in the w-plane, right? Since the components are just multivariable functions of real numbers only, we can apply the chain rule for multivariable functions.

$$\begin{aligned}
\vec{w_1'} &= (u_x f_1'(t) + u_y g_1'(t))\vec{i} + (v_x f_1'(t) + v_y g_1'(t))\vec{j} \\
&= (\nabla u \cdot \vec{z_1'})\vec{i} + (\nabla v \cdot \vec{z_1'})\vec{j} \\
\vec{w_2'} &= (\nabla u \cdot \vec{z_2'})\vec{i} + (\nabla v \cdot \vec{z_2'})\vec{j}
\end{aligned}$$

Then the angle between the two curves would be given by $\cos(\phi) = \dfrac{w_1' \cdot w_2'}{|w_1'|\,|w_2'|}$. Now this is where the CR equations kick in. If we calculate $|w_1|$, then

$$\begin{aligned}
|w_1'| &= \sqrt{(u_x f_1' + u_y g_1')^2 + (v_x f_1' + v_y g_1')^2} \\
&= \sqrt{(v_y f_1' - v_x g_1')^2 + (v_x f_1' + v_y g_1')^2} \\
&= \sqrt{v_y^2(f_1'^2 + g_1'^2) + v_x^2(f_1'^2 + g_1'^2)} \\
&= \sqrt{f_1'^2 + g_1'^2}\sqrt{v_x^2 + v_y^2}
\end{aligned}$$

$$|w_1'| = \sqrt{u_x^2 + v_x^2}\,|z_1'| \tag{4.3.2}$$

$$|w_2'| = \sqrt{u_x^2 + v_x^2}\,|z_2'| \tag{4.3.3}$$

Also,

$$\begin{aligned}
w_1' \cdot w_2' &= (u_x f_1' + u_y g_1')(u_x f_2' + u_y g_2') + (v_x f_1' + v_y g_1')(v_x f_2' + v_y g_2') \\
&= (u_x f_1' - v_x g_1')(u_x f_2' - v_x g_2') + (v_x f_1' + u_x g_1')(v_x f_2' + u_x g_2') \\
&= (u_x^2 + v_x^2)(f_1' f_2' + g_1' g_2') \\
&= (u_x^2 + v_x^2)(z_1' \cdot z_2')
\end{aligned} \tag{4.3.4}$$

Viola, now $\cos(\phi) = \dfrac{w_1' \cdot w_2'}{|w_1'|\,|w_2'|} = \dfrac{z_1' \cdot z_2'}{|z_1'|\,|z_2'|} = \cos(\theta)$ using all that we found above! Which means $\phi = \theta$ meaning that angles are preserved in an analytic transformation. And all of this is possible only because of the CR equations. Without them, we could not have factored out everything neatly leading to the cancellation of various terms.

4.4 Harmonic Functions

There are a special class of functions called **harmonic** functions. These are functions $f(x, y)$ which satisfy

$$f_{xx} + f_{yy} = 0 \tag{4.4.1}$$

The left hand side is usually called the **Laplacian** of the function. For harmonic functions, the Laplacian is uniformly 0.

The geometric interpretation can be seen as follows. In the single variable case, $f''(x)$ is the rate of change of the rate of change, but it can be thought of in another way also. Given a point on the function, you can draw an interval $(x - \epsilon, x + \epsilon)$ and see the function values in the neighbourhood of x. The second derivative gives the difference between the function value and the average value of its neighbours! For example, when x is a minimum point, all the points around x would have a value higher than x itself. So the average value of the neighbouring points is higher, making the difference between this average value and the function value positive. And indeed at points of minimum $f''(x) > 0$.

Similarly at points of maximum, the function value will be higher than the average of values of its neighbours. So at those points $f''(x) < 0$.

In a straight line, at any point, there are an equal number of points with values less than and more than the function value. So on average, the value is same as function value. And indeed for the straight line $y = f(x) = mx + c$, $f''(x) = 0$.

If you think of $y = e^x$, then on average the points after x have a much higher value compared to points before x (exponential growth) so the average value is higher. Which is why $f''(x) > 0$ always.

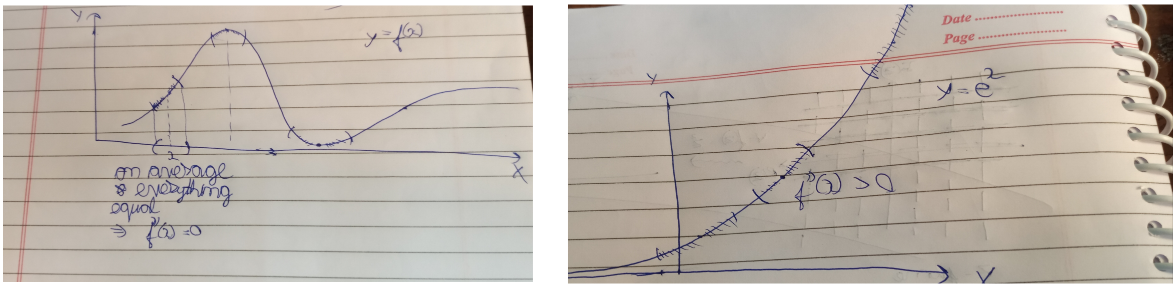


Figure 9: The Laplacian in one dimension is the double derivative

So, for multivariable functions, this analogy still holds. But this time, when we say neighbourhood, it is not an interval, but the disc of all points with center as the point we're measuring the Laplacian at.

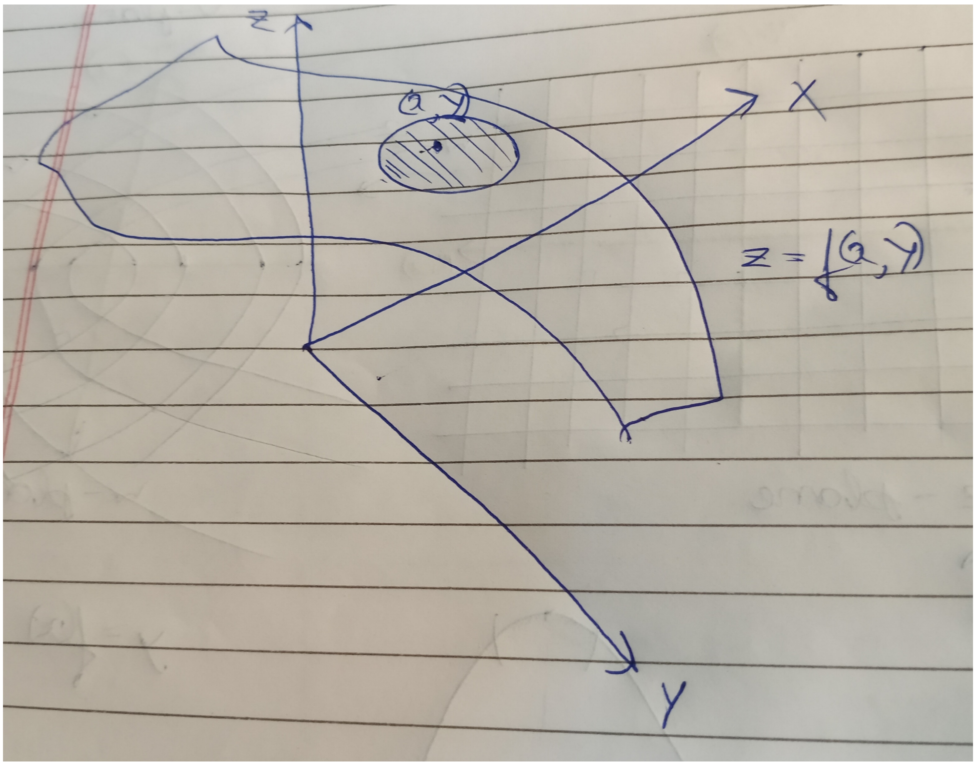


Figure 10: Laplacian in two dimensions

Harmonic functions have the property of the Laplacian being uniformly 0. What this means is that, no matter what point you choose, the average value of all its neighbour points on a disc will be same as the value at the centre of the disc! That sounds so complicated, measuring values of points on an arbitrary disc, then taking the average of it and then comparing it with the value at the centre. But harmonic functions say that it will always be equal, no matter what. Isn't that amazing?

For this reason, harmonic functions are everywhere in physics. They're useful in describing steady state problems, because in steady state, everything on average should be the same. For example, a room with a heater in it would tend towards a steady state in which the heat is uniformly distributed throughout the room. Meaning if $T(x, y, z)$ is the temperature at any point, their goal is to reach a state where $T_{xx} + T_{yy} + T_{zz} = 0$.

So, why did I bring up this topic at all? Sounds pretty unrelated to complex functions, right? It is not, in fact next I'll show that complex functions which are analytic have real and imaginary parts both harmonic!

$$\begin{aligned} f(z) &= u + i\,v \\ \implies f'(z) &= u_x + i\,v_x \end{aligned} \tag{4.4.2}$$

We'll have to assume one more thing in addition to the assumption that u and v are differentiable:

we assume that they satisfy Clairaut's theorem, i.e., $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. Then

$$\begin{aligned} & u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy} \\ \implies & u_{xx} + u_{yy} = 0 \end{aligned} \tag{4.4.3}$$

$$\begin{aligned} & v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = (-u_x)_y = (-v_y)_y = -v_{yy} \\ \implies & v_{xx} + v_{yy} = 0 \end{aligned} \tag{4.4.4}$$

Another spectacular juggling by the CR equations. And now you can pretty much imagine the use of complex analysis in physics. Any steady state problem can be modelled using complex numbers.

5 Integration of Complex Functions

5.1 Introduction

Now that we've seen derivatives, it is natural to ask for integrals. What is $\int f(z)\,\mathrm{d}z$? More importantly, what is the meaning of this thing now? As I've repeated several times before, the domain has changed, limit definition has changed, so the definition of integrals in terms of Riemann sums also has changed.

So what we can do is to break this in components into real and imaginary parts. Then those components are just real functions for whom integration is well defined. Specifically,

$$\begin{aligned} \int f(z)\,\mathrm{d}z &= \int (u + i\,v)(\,\mathrm{d}x + i\ \mathrm{d}y) \\ &= \int u\,\mathrm{d}x - v\,\mathrm{d}y + i\,(u\,\mathrm{d}y + v\,\mathrm{d}x) \\ &= \int u\,\mathrm{d}x - v\,\mathrm{d}y + i\int u\,\mathrm{d}y + v\,\mathrm{d}x \end{aligned} \tag{5.1.1}$$

Does equation (5.1.1) ring a bell? Doesn't it look like the line integrals we studied in vector calculus? In fact, it is exactly that. Complex integrals are **line integrals**. In hindsight this makes complete sense. The domain is two dimensional, so we can take any curve in the z-plane and integrate on the curve. Isn't this exactly what line integrals do?

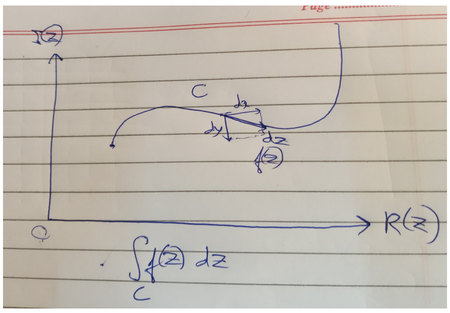


Figure 11: The complex line integral $\int_C f(z)\,\mathrm{d}z$

5.2 Complex Potential

In fact, let's go one step further and calculate the value of this line integral for a closed curve. In this case, we would be able to use Green's theorem for line integrals in a closed region.

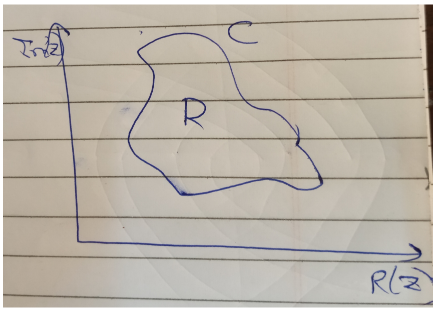


Figure 12: Complex line integrals in a closed curve

$$
\begin{aligned}
\oint_C f(z)\,\mathrm{d}z &= \oint_C u\,\mathrm{d}x - v\,\mathrm{d}y + i\oint_C u\,\mathrm{d}y + v\,\mathrm{d}x \\
&= \iint\limits_R (-v_x - u_y)\,\mathrm{d}A + i\iint\limits_R (u_x - v_y)\,\mathrm{d}A \\
&= \iint\limits_R (u_y - u_y)\,\mathrm{d}A + i\iint\limits_R (v_y - v_y)\,\mathrm{d}A \\
&= 0 \qquad (!)
\end{aligned}
$$

This result is amazing. It says that the value of the complex line integral of an analytic function around a closed curve is 0 always. Which also means that an analytic complex function is **conservative**, path independent. It depends only on the numbers at the end points. We can write

$$\int_{z_1}^{z_2} f'(z)\,\mathrm{d}z = f(z_2) - f(z_1) \tag{5.2.1}$$

This is the Fundamental Theorem of Calculus for complex numbers. The value of a complex line integral (of an analytic function, the CR equations have it all) depends on the end-points only. It makes sense to define a **potential** function for a complex function, called the complex potential. And it is really easy to calculate, it is just an antiderivative of the function! Consider that with vector potential functions, for which we had to repeatedly differentiate then integrate our vector function several times to get the potential function. Here, in one step we have it all!

This is also useful in various physical situations like fluid flow, gravitation, electricity and magnetism. The whole of quantum mechanics relies heavily on complex numbers. The CR equations give this power and beauty to complex analysis, which is why it is used so extensively in mathematical modelling and physics.

6 Conclusion

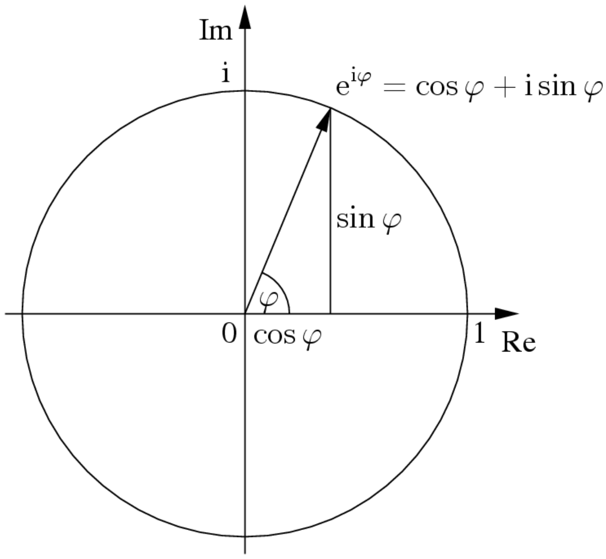
6.1 Multiplication Revisited

Remember how we noticed something in multiplication? Angles got added like exponential functions. With our new tools let us get to the bottom of this. Consider $z(t) = e^{it}$. Then $z'(t) = ie^{it}$ and $z''(t) = -e^{it} = -z(t)$. The geometric interpretation of multiplication by i is rotation by 90 degrees. So if $z(t)$ represents the position, then $z'(t)$ is the velocity. and $z''(t)$ is the acceleration. Acceleration is always in a direction perpendicular to velocity and is always equal and opposite to the position. This is possible only if the particle is in uniform circular motion. So the particle whose equation is $z(t) = e^{it}$ is going in a circle around the origin.

Circle of what radius? $z(0) = e^{(0)} = 1$ and in a circle radius doesn't change with time, so the radius is always 1. So e^{it} is a uniform circular motion in a circle around the origin of radius 1. But such a circle can also be parameterized by $z(t) = \text{cis}(t)$. Thus it must be that $\text{cis}(t) = e^{it}$. Or,

$$e^{it} = \cos(t) + i\sin(t) \tag{6.1.1}$$

(a) Leonhard Euler



(b) Euler form of Complex Number

This beautiful formula was first discovered by the mathematician Leonhard Euler, the Mozart of mathematics. In light of this, a general complex number can be written as $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$. Due to this representation, the multiplication of complex numbers becomes almost obvious now,

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{(\theta_1+\theta_2)} \tag{6.1.2}$$

6.2 The end?

This is the "The end" for my writing, but we've barely scratched the surface of complex analysis. My aim was to give an overview, an essence of complex analysis. And I hope I succeeded and that you enjoyed reading this!