# A Non-Rigorous Approach to the value of Euler's number $e$ (2.718281828459045 . . .)

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We first demonstrate the following-

**Theorem 1.** *If $f : \mathbb{R} \mapsto \mathbb{R}$, $f(x) = a^x, a \in \mathbb{R}^+$ be a real valued exponential function, then its derivative is proportional to itself, i.e.,*

$$f'(x) = k \cdot f(x), \quad k \in \mathbb{R}$$

*Proof.*

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} \\ &= \lim_{h \to 0} a^x \cdot \frac{a^h - 1}{h} \end{aligned}$$

It can be seen that $a^x$ is just a constant here as the limit concerns $h$. Thus, $a^x$ can be moved out of the limit sign. Therefore,

$$f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

Now, the term inside the limit is independent of $x$. Thus it will evaluate to a constant $k$, $k \in \mathbb{R}$. Rewriting the above limit as $k$, we get

$$\begin{aligned} f'(x) &= k \cdot a^x \\ &= k \cdot f(x) \end{aligned}$$

□

Now, it can be seen that the value of $k$ depends only on $a$, i.e., the base of the exponential function used. It is certainly natural to use a base such that $k = 1$. In that case, our differential equation reduces to-

$$f'(x) = f(x) \tag{1}$$

Setting $k = 1$, we get

$$\begin{aligned} k &= 1 \\ \implies \lim_{h \to 0} \frac{a^h - 1}{h} &= 1 \end{aligned}$$

Now, making $a$ the subject[1], we get-

$$\begin{aligned} \frac{a^h - 1}{h} &= 1 \\ \implies a^h - 1 &= h \\ \implies a^h \quad &= 1 + h \\ \implies a \quad &= (1 + h)^{\frac{1}{h}} \end{aligned}$$

Making the substitution $h = \frac{1}{n}$ and placing the limit back, we get

$$\begin{aligned} a &= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \\ &= \lim_{n \to \infty} \left(\binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n}\right) \\ &= \lim_{n \to \infty} \left(1 + \frac{n}{n} + \frac{n(n-1)}{2! \cdot n^2} + \cdots + \frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{n! \cdot n^n}\right) \\ &= \lim_{n \to \infty} \left(1 + 1 + \frac{\left(1 - \frac{1}{n}\right)}{2!} + \cdots + \left(\frac{\left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-2}{n}\right) \cdot \left(1 - \frac{n-1}{n}\right)}{n!}\right)\right) \\ &= \lim_{n \to \infty} \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) \\ &= \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} \\ &\approx 2.718281828459045 \ldots \end{aligned}$$

We call this number $e$, which is *exponential naturallis* in Greek.

[1]This is the non-rigorous part

Thus,

$$\frac{de^x}{dx} \stackrel{\text{def}}{=} e^x$$