Thiele's differential equation

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We prove Thiele's differential equation in continuous time from scratch (ignoring some technical details). We are sometimes referring to M. Koller's lecture notes on "Selected Topics in Life Insurance" which are not publicly available, but we hope the text should be readable without this reference at hand.

The setting

We start with the setting in which we will prove Thiele's differential equation: The *general insurance model*. It consists of a regular Markov chain, cash flow functions, and an interest rate function.

A *regular Markov chain* is a family of random variables $(X_t)_{t\geq 0}$ on a common probability space (with probability measure P) taking values in a finite set S of states. The states can for example be "alive", "dead", "disabled", etc., and the random variable X_t represents the state the insured is in at time t.

In a Markov chain, the transition probabilities $p_{ij}(s,t) = P(X_t = j|X_s = i)$ only depend on the state at time s, and not on the history of states before time s. In a Markov chain the Chapman-Kolmogorov equation

$$P(X_u = k|X_s = i) = \sum_{j\in S} P(X_u = k|X_t = j) \cdot P(X_t = j|X_s = i)$$

for times $s < t < u$ and states i, j and k holds which we will use later on. Also note that the transition probabilities only depend on the time passed $t - s$, and not on s.

When we say the Markov chain is *regular*, we mean that the transition probabilities are differentiable as follows:

$$\frac{d}{dt} p_{ij}(s,t) = \mu_{ij}(t) \text{ (for } i \neq j)$$
$$\frac{d}{dt} p_{ii}(s,t) = 1 - \mu_i(t).$$

The μ_{ij} and μ_i are called *transition intensities*. The distinction between the cases $i = j$ and $i \neq j$ is in a sense artificial (we could redefine $\mu_i(t)$ as $1 - \mu_i(t)$) but it allows the interpretation that for t close to s, the probability of leaving a given state $p_{ij}(s,t)$ should be small, and the probability of remaining in the given state $p_{ii}(s,t)$ should be close to 1. In both cases, for the definition chosen here, the transition intensities are close to 0.

The *cash flow functions* are given as infinitesimal monetary amounts at time t, $da_i(t)$ and $da_{ij}(t)$. The $da_i(t)$ are the regular payments at time t by or to the insured in state i (e.g., premiums or pension). The $da_{ij}(t)$ are the lump sums payed in case the insured changes the state at time t. We assume these functions are sufficiently nice: We want to write $da_i(t) = a_i(t)dt$ and $da_{ij}(t) = a_{ij}(t)dN_{ij}(t)$ (note the abuse of notation) (the N_{ij} counts the total number of state transitions from state i to state j up to time t). We also want the integrals $\int_t^\infty a_i(\tau)d\tau$ and $\int_t^\infty a_{ij}(\tau)d\tau$ to exist.

The *total cash flow* up to time t is denoted as $A(t)$, and the infinitesimal change of total cash flow at time t is denoted as $dA(t)$. This infinitesimal cash flow at time t depends on which state the insured is in (for regular payments) and on whether a state transition has happened (for lump sums). Using indicator functions $I_i(t)$ for the state i of the insured at time t we can summarize this in closed form as

$$dA(t) = \sum_{i \in S} I_i(t) a_i(t) dt + \sum_{i,j \in S;\ i \neq j} a_{ij}(t) dN_{ij}(t).$$

The final part of the regular insurance model is the *interest rate*. This is a function $\delta(t)$ representing the risk-free interest rate. We are using continuous compounding, i.e., the interest obtained by investing one unit of currency for time t is $\exp(\int_0^t \delta(\tau) d\tau)$. We assume δ is such that this integral exists. We define the *discount factors* as $v(t) = \exp(-\int_0^t \delta(\tau) d\tau)$. This means that one unit of currency at time t is worth $v(t)$ today (the present value). We will also look at the "present" value of money at a future time: one unit of currency at time t is worth $v(t)/v(s)$ at time s.

Reserves

We can now write down the present value at time t of the future cash flow as

$$V^+(t, A) = \frac{1}{v(t)} \int_t^\infty v(\tau) dA(\tau).$$

The *reserve* is the defined as the expected value thereof, given the insured is in state i at time t:

$$V_i^+(t) = E\left[\frac{1}{v(t)} \int_t^\infty v(\tau) dA(\tau) \middle| X_t = i \right].$$

This value represents today's ($t = 0$) view of the expected value at time t of the cash flows after time t. In other words, on average, the insurance company needs to be prepared to put aside this much money at time t to cover future claims. The reserve can be large and positive (e.g., if the insured has become disabled and is not paying in further premiums) or small and positive (at the start of a life insurance: the premium is adjusted such that the expected value of the life insurance is zero at the beginning, or at the end of a life insurance: e.g., if the insured is still healthy the probability of becoming disabled or of dying over a small time period is small). The reserve is negative if the insurer expects to profit from the insurance contract (e.g., if premiums were agreed upon in favor of the insurance company).

Preparations

We can use the linearity of the expected value and the break-down of the infinitesimal change in cash flow in lump sums and regular payments to write the reserve as a linear combination of reserves for lump sums and regular payments

$$\begin{aligned} V_i^+(t) &= E\left[\frac{1}{v(t)} \int_t^\infty v(\tau) dA(\tau) \middle| X_t = i \right] \\ &= \sum_{j \in S} \frac{1}{v(t)} E\left[\int_t^\infty v(\tau) I_j(\tau) a_j(\tau) d\tau \middle| X_t = i \right] + \sum_{j,k \in S;\ j \neq k} \frac{1}{v(t)} E\left[\int_t^\infty v(\tau) a_{jk}(\tau) dN_{jk}(\tau) \middle| X_t = i \right]. \end{aligned}$$

We are therefore interested in obtaining nicer expressions for the reserves for lump sums and regular payments. This is the content of the following two equations (Theorem 4.6.3 in the lecture notes):

$$
\begin{aligned}
E\left[\int_t^\infty v(\tau)I_j(\tau)a_j(\tau)d\tau \middle| X_t=i\right] &= \int_t^\infty v(\tau)p_{ij}(t,\tau)a_j(\tau)d\tau,\\
E\left[\int_t^\infty v(\tau)a_{jk}(\tau)dN_{jk}(\tau) \middle| X_t=i\right] &= \int_t^\infty v(\tau)p_{ij}(t,\tau)\mu_{jk}(\tau)a_{jk}(\tau)d\tau.
\end{aligned}
$$

These two equations can be proved by changing the order of integration (pulling the outer expected value inside).

Using these two expressions for the reserves of lump sums and regular payments leads to the following integral formula for the reserve (Theorem 4.6.10):

$$
\begin{aligned}
V_i^+(t) &= \sum_{j\in S}\frac{1}{v(t)}E\left[\int_t^\infty v(\tau)I_j(\tau)a_j(\tau)d\tau \middle| X_t=i\right] + \sum_{j,k\in S;\ j\neq k}\frac{1}{v(t)}E\left[\int_t^\infty v(\tau)a_{jk}(\tau)dN_{jk}(\tau) \middle| X_t=i\right]\\
&= \sum_{j\in S}\frac{1}{v(t)}\int_t^\infty v(\tau)p_{ij}(t,\tau)a_j(\tau)d\tau + \sum_{j,k\in S;\ j\neq k}\frac{1}{v(t)}\int_t^\infty v(\tau)p_{ij}(t,\tau)\mu_{jk}(\tau)a_{jk}(\tau)d\tau\\
&= \sum_{j\in S}\frac{1}{v(t)}\int_t^\infty v(\tau)p_{ij}(t,\tau)\left(a_j(\tau)+\sum_{k\in S;k\neq j}\mu_{jk}(\tau)a_{jk}(\tau)\right)d\tau\\
&= \frac{1}{v(t)}\int_t^\infty v(\tau)\sum_{j\in S}p_{ij}(t,\tau)\left(a_j(\tau)+\sum_{k\in S;k\neq j}\mu_{jk}(\tau)a_{jk}(\tau)\right)d\tau.
\end{aligned}
$$

The last line is how this equation is presented in the lecture notes (using the assumption that the cash flow functions are “nice” in the sense that $da_i(t)=a_i(t)dt$ and $da_{ij}(t)=a_{ij}(t)dt$).

We now prove another equation (Lemma 4.7.2 in the lecture notes) in which we split up the integral from t to ∞ in a part from t to an intermediary time u and the remaining time from u to infinity. Ignoring the part from t to u for now, we are looking at the integral

$$
\frac{1}{v(t)}\int_u^\infty v(\tau)\sum_{j\in S}p_{ij}(t,\tau)\left(a_j(\tau)+\sum_{k\in S;k\neq j}\mu_{jk}(\tau)a_{jk}(\tau)\right)d\tau
$$

for $t<u$. This is almost the same as the reserve at time u in state i, except that the transition probabilities start earlier at t instead of u. This is where we use the Chapman-Kolmogorov equation to replace the $p_{ij}(t,\tau)$ by $\sum_l p_{il}(t,u)p_{lj}(u,\tau)$ (note that $t\le u\le\tau$). The sum $\sum_j p_{ij}(t,\tau)$ over j therefore becomes a double sum over j and l and we pull the sum over l and the transition probabilities $p_{il}(t,u)$ outside the integral (they don't depend on τ). What one ends up with is

$$
\sum_{l\in S}p_{il}(t,u)\frac{1}{v(t)}\int_u^\infty v(\tau)\sum_{j\in S}p_{lj}(u,\tau)\left(a_j(\tau)+\sum_{k\in S;k\neq j}\mu_{jk}(\tau)a_{jk}(\tau)\right)d\tau.
$$

The integral part is now the reserve $V_l^+(u)$ in state l at time u.

To summarize (remember that we did not do anything about the integral from t to u), we showed that

$$
\begin{aligned}
V_i^+(t) =& \frac{1}{v(t)}\int_t^\infty v(\tau)\sum_{j\in S}p_{ij}(t,\tau)\left(a_j(\tau)+\sum_{k\in S;k\neq j}\mu_{jk}(\tau)a_{jk}(\tau)\right)d\tau\\
=& \frac{1}{v(t)}\int_t^u v(\tau)\sum_{j\in S}p_{ij}(t,\tau)\left(a_j(\tau)+\sum_{k\in S;k\neq j}\mu_{jk}(\tau)a_{jk}(\tau)\right)d\tau + \frac{v(u)}{v(t)}\sum_{l\in S}p_{il}(t,u)V_l^+(u).
\end{aligned}
$$

To simplify notation a little bit, we define

$$W_j^+(t) = v(t)V_j^+(t).$$

This is the reserve at time t expressed in units of currency at time t. Using this definition, the previous equation becomes easier to read because all the discount factors outside of the integrals disappear:

$$W_i^+(t) = \int_t^u v(\tau) \sum_{j \in S} p_{ij}(t,\tau) \left(a_j(\tau) + \sum_{k \in S; k \neq j} \mu_{jk}(\tau) a_{jk}(\tau) \right) d\tau + \sum_{j \in S} p_{ij}(t,u) W_j^+(u).$$

I know of two ways how to motivate the idea to split up the integral in the reserve from t to infinity into two parts from t to u and from u to infinity.

The first is related to the recursive formula for the reserve in the discrete setting (called Thiele's difference equation, Theorem 4.7.3 in the lecture notes). This difference equation gives a recursive formula for the reserve at time t consisting of the cash flows during the time interval $[t, t+1)$ and the reserve at time $t+1$.

The second is that having in mind that in order to prove Thiele's differential equation one might be interested in a quantity like $(W_i^+(t+\Delta t) - W_i^+(t))/\Delta t$ for small Δt. Hence it would perhaps be interesting to find a relation between the reserves at time t and $t + \Delta t$.

Statement and Proof

We can now formulate and prove Thiele's differential equation (Theorem 5.2.1 in the lecture notes):

Theorem 1 (Thiele's differential equation in continuous time)**.**

$$\frac{\partial W_i^+(t)}{\partial t} = \mu_i(t) W_i^+(t) - \sum_{j \in S; j \neq i} \mu_{ij}(t) W_j^+(t) - v(t) \left(a_i(t) + \sum_{j \in S; j \neq i} \mu_{ij}(t) a_{ij}(t) \right).$$

To prove this equation, we use the recursive formula for the reserve in continuous time with $u = t + \Delta t$, approximate the integral from t to $t + \Delta t$ by Δt times the integrand at time t, and use that the transition probabilities are differentiable and that their derivatives are given by the transition intensities ($p_{ij}(t, t+\Delta t) \approx \mu_{ij}(t)\Delta t$ and $p_{ii}(t, t+\Delta t) \approx 1 - \mu_{ii}(t)\Delta t$). In doing so, we obtain

$$\begin{aligned} W_i^+(t) &= \int_t^{t+\Delta t} v(\tau) \sum_{j \in S} p_{ij}(t,\tau) \left(a_j(\tau) + \sum_{k \in S; k \neq j} \mu_{jk}(\tau) a_{jk}(\tau) \right) d\tau + \sum_{j \in S} p_{ij}(t, t+\Delta t) W_j^+(t+\Delta t) \\ &= v(t) \left(a_i(t) + \sum_{k \in S; k \neq i} \mu_{ik}(t) a_{ik}(t) \right) \Delta t + (1 - \mu_i(t)\Delta t) W_i^+(t+\Delta t) + \sum_{j \in S; j \neq i} \mu_{ij}(t) \Delta t W_j^+(t+\Delta t). \end{aligned}$$

Using this expression in $(W_i^+(t+\Delta t) - W_i^+(t))/\Delta t$ and letting Δt go to zero gives Thiele's equation.