APPENDIX A

CONVOLUTION PROPERTIES

Convolution is one of the most regularly applied operation in audio signal processing. It applies to all linear and quasi-linear systems such as filters and rooms. In this chapter the most fundamental properties of this operation will be derived.

A.1 Identity

The result of a convolution with a delta function is the signal itself:

$$x(i) = \delta(i) * x(i). \tag{A.1}$$

The result for each individual sample can be computed by the sum of the sample itself (weighted by 1) and all other samples weighted by 0, i.e., the sample value itself [compare Eq. (B.29)].

A.2 Commutativity

Changing the order of operands does change the result of the convolution operation. That means that the distinction between impulse response and signal is of no mathematical consequence in the context of convolution:

$$h(i) * x(i) = x(i) * h(i). \tag{A.2}$$

An Introduction to Audio Content Analysis: Applications in Signal Processing and Music Informatics, First Edition. Alexander Lerch.

This can be shown by substituting $j' = i - j$:

$$\begin{aligned} x(i) * h(i) &= \sum_{j=-\infty}^{\infty} h(j) \cdot x(i-j) \\ &= \sum_{j'=-\infty}^{\infty} h(i-j') \cdot x(j') \\ &= \sum_{j'=-\infty}^{\infty} x(j') \cdot h(i-j'). \end{aligned} \tag{A.3}$$

A.3 Associativity

The associative property of convolution means that changing the order of subsequent convolution operations does not change the overall result. When applying two or more filters to a signal, the output will be identical for every order of filters.[1] This means that

$$\big(g(i) * h(i)\big) * x(i) = g(i) * \big(h(i) * x(i)\big). \tag{A.4}$$

This can be derived by changing the order of sums and shifting the operands as shown below:

$$\begin{aligned} \big(g(i) * h(i)\big) * x(i) &= \sum_{j=-\infty}^{\infty} \big(g(j) * h(j)\big) \cdot x(i-j) \\ &= \sum_{j=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} g(l) \cdot h(j-l) \right) \cdot x(i-j) \\ &= \sum_{j=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} g(l) \cdot h(j-l) \cdot x(i-j) \right) \\ &= \sum_{l=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g(l) \cdot h(j-l) \cdot x(i-j) \\ &= \sum_{l=-\infty}^{\infty} g(l) \cdot \sum_{j=-\infty}^{\infty} h(j-l) \cdot x(i-j) \\ &= \sum_{l=-\infty}^{\infty} g(l) \cdot \sum_{j=-\infty}^{\infty} h(j) \cdot x(i-l-j) \\ &= \sum_{l=-\infty}^{\infty} g(l) \cdot \big(h(i-l) * x(i-l)\big) \\ &= g(i) * \big(h(i) * x(i)\big). \end{aligned} \tag{A.5}$$

[1] Strictly speaking this is only true for unlimited word length. The lower the word length the more the output signal differs from the expected signal.

A.4 Distributivity

The order of different linear operations is irrelevant due to the distributive property of the convolution, for example:

$$g(i) * \big(h(i) + x(i)\big) = g(i) * h(i) + g(i) * x(i). \tag{A.6}$$

This means that two signals, one computed by applying a filter to two different signals and summing them together afterwards, the other computed by applying the filter to the sum of the signals, are identical:

$$\begin{aligned}
g(i) * \big(h(i) + x(i)\big) &= \sum_{j=-\infty}^{\infty} g(j) \cdot \big(h(i-j) + x(i-j)\big) \\
&= \sum_{j=-\infty}^{\infty} g(j) \cdot h(i-j) + g(j) \cdot x(i-j) \\
&= \sum_{j=-\infty}^{\infty} g(j) \cdot h(i-j) + \sum_{j=-\infty}^{\infty} g(j) \cdot x(i-j) \\
&= g(i) * h(i) + g(i) * x(i).
\end{aligned} \tag{A.7}$$

A.5 Circularity

The convolution with a periodic signal will result in a periodic output signal. The periodic signal $x(i)$ is the sum of the shifted (fundamental) periods with length N:

$$x(i) = \sum_{n=-\infty}^{\infty} x_N(i + nN). \tag{A.8}$$

With $x_N(i) = 0$ for $i < 0 \vee i \geq N$. We can show that

$$\begin{aligned}
x(i) * h(i) &= \sum_{j=-\infty}^{\infty} h(i-j) \sum_{n=-\infty}^{\infty} x_N(j + nN) \\
&= \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h(i-j) \cdot x_N(j + nN) \\
&= \sum_{n=-\infty}^{\infty} x_N(i + nN) * h(i).
\end{aligned} \tag{A.9}$$

The multiplication of two spectra computed with the DFT will result in a circular convolution; the result will be the convolution of the two periodically continued sample blocks.