APPENDIX B

FOURIER TRANSFORM

The Fourier Transform (FT) is widely used in audio signal analysis and synthesis. Understanding its properties is crucial for the design of audio processing systems.

Deriving the FTs fundamental properties is easier for continuous signals; we will thus focus on the continuous domain first and will then discuss the FT of windowed signals, the FT of sampled signals, and finally the Discrete Fourier Transform (DFT).

Periodic signals can be represented as a Fourier series as introduced in Eq. (2.3). The fundamental frequency ω_0 determines the "frequency resolution" of the series. For the analysis of non-periodic signals we let the period length grow $T_0 \rightarrow \infty$ (or equivalently $\omega_0 \rightarrow 0$). This has the effect that the previously discrete frequency resolution becomes continuous with $k\omega_0 \rightarrow \omega$. Due to the resulting infinite resolution of the frequency axis, the coefficients will decrease $a_k \rightarrow 0$. The formula for the Fourier series given in Eq. (2.3) thus changes into the FT:

$$X(\mathrm{j}\omega) = \mathfrak{F}\{x(t)\} = \int\limits_{-\infty}^{\infty} x(t)e^{-\mathrm{j}\omega t}\, dt \tag{B.1}$$

and $X(\mathrm{j}\omega)$ is the so-called *spectrum* of the signal $x(t)$.

The real and imaginary parts represent the cosine and sine functions, respectively. A common form of visualizing the results is to represent the spectrum as magnitude $|X(\mathrm{j}\omega)|$ and phase $\Phi_X(\mathrm{j}\omega)$ instead of real and imaginary parts. Frequently only the *magnitude spectrum* is being used for the visualization of the spectrum while the phase spectrum

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is ignored. Another common representation is the *power spectrum* which is the squared magnitude spectrum.

B.1 Properties of the Fourier Transformation

B.1.1 Inverse Fourier Transform

The FT is invertible. $X(\mathrm{j}\omega)$ can be converted back from the frequency domain into the time domain signal $x(t)$ by applying the *Inverse Fourier Transform (IFT)*:

$$x(t) = \mathfrak{F}^{-1}\left\{X(\mathrm{j}\omega)\right\} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\mathrm{j}\omega)e^{\mathrm{j}\omega t}\,d\omega. \tag{B.2}$$

That means that time and frequency representation are equivalent, i.e., no information is gained or lost by applying the FT to a signal; it just changes the representation of the signal.

It becomes also obvious from comparing Eqs. (B.1) and (B.2) that forward and inverse transform are very similar operations (see also Sect. B.1.6).

B.1.2 Superposition

If the signal $y(t)$ is the weighted addition of the signals $x_1(t)$ and $x_2(t)$

$$y(t) = c_1 \cdot x_1(t) + c_2 \cdot x_2(t), \tag{B.3}$$

then the same relationship is true for their frequency transformation:

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int\limits_{-\infty}^{\infty} \big(c_1 \cdot x_1(t) + c_2 \cdot x_2(t)\big) \cdot e^{-\mathrm{j}\omega t}\,dt \\ &= c_1 \cdot \int\limits_{-\infty}^{\infty} x_1(t)e^{-\mathrm{j}\omega t}\,dt + c_2 \cdot \int\limits_{-\infty}^{\infty} x_2(t)e^{-\mathrm{j}\omega t}\,dt \\ &= c_1 \cdot X_1(\mathrm{j}\omega) + c_2 \cdot X_2(\mathrm{j}\omega). \end{aligned} \tag{B.4}$$

B.1.3 Convolution and Multiplication

The convolution of signal $x(t)$ with the impulse response $h(t)$

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int\limits_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\,d\tau \end{aligned} \tag{B.5}$$

corresponds to a multiplication in the spectral domain

$$Y(\mathrm{j}\omega) = H(\mathrm{j}\omega) \cdot X(\mathrm{j}\omega). \tag{B.6}$$

The derivation involves clever grouping and expansion:

$$
\begin{aligned}
Y(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} y(t)e^{-\mathrm{j}\omega t}\,dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\,d\tau \right) e^{-\mathrm{j}\omega t}\,dt \\
&= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} x(t-\tau)e^{-\mathrm{j}\omega t}\,dt\,d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{-\mathrm{j}\omega\tau} \underbrace{\int_{-\infty}^{\infty} x(t-\tau)e^{-\mathrm{j}\omega(t-\tau)}\,d(t-\tau)}_{X(\mathrm{j}\omega)}\,d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{-\mathrm{j}\omega\tau}\,d\tau\cdot X(\mathrm{j}\omega) \\
&= H(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega). \qquad \text{(B.7)}
\end{aligned}
$$

This property allows the efficient computation of the convolution of a signal with an FIR filter with a long impulse response in the frequency domain [478].

The same relationship exists for convolution in the frequency domain. The convolution operation

$$Y(\mathrm{j}\omega) = H(\mathrm{j}\omega) * X(\mathrm{j}\omega) \qquad \text{(B.8)}$$

could be replaced by a multiplication in the time domain

$$y(t) = h(t)\cdot x(t). \qquad \text{(B.9)}$$

B.1.4 Parseval's Theorem

The energy of the signal can be calculated in both the time and the spectral domain:

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\,d\omega. \qquad \text{(B.10)}$$

This can be shown by using the equivalence between multiplication in the frequency domain and convolution in the time domain. Writing

$$\int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)e^{\mathrm{j}\omega t}d\omega \qquad \text{(B.11)}$$

and replacing $H(\mathrm{j}\omega)$ with the conjugate-complex spectrum $X^*(\mathrm{j}\omega)$ and $h(\tau)$ with $x(-\tau)$,[1] respectively, the result at $t = 0$ is

$$\begin{aligned}\int_{-\infty}^{\infty} x(-\tau)\cdot x(-\tau)\,d\tau &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\mathrm{j}\omega)\cdot X(\mathrm{j}\omega)\,d\omega,\\ \int_{-\infty}^{\infty} x^2(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\,d\omega. \end{aligned} \tag{B.12}$$

B.1.5 Time and Frequency Shift

The transform of a signal shifted by a constant in time $y(t) = x(t - t_0)$ is

$$Y(\mathrm{j}\omega) = X(\mathrm{j}\omega)e^{-\mathrm{j}\omega t_0}. \tag{B.13}$$

This means that the magnitude spectrum will be identical but the phase spectrum will have a linear offset $\Phi_Y(\omega) = \Phi_X(\omega) - \omega t_0$:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t-t_0)e^{-\mathrm{j}\omega t}\,dt &= \int_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\omega(\tau+t_0)}\,d\tau\\ &= e^{-\mathrm{j}\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-\mathrm{j}\omega\tau}\,d\tau\\ &= e^{-\mathrm{j}\omega t_0}\cdot X(\mathrm{j}\omega). \end{aligned} \tag{B.14}$$

Equivalently, the shifted spectrum[2] $Y(\mathrm{j}\omega) = X(\mathrm{j}(\omega-\omega_0))$ corresponds to the time domain signal $y(t) = x(t)\cdot e^{\mathrm{j}\omega_0 t}$ which is the original signal modulated by a sinusoidal signal:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathrm{j}\omega-\omega_0)e^{\mathrm{j}\omega t}\,d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathrm{j}\phi)e^{\mathrm{j}(\phi+\omega_0)t}\,d\phi\\ &= e^{\mathrm{j}\omega_0 t}\cdot x(t). \end{aligned} \tag{B.15}$$

B.1.6 Symmetry

If the time domain signal $x(t)$ is real-valued, then its frequency transform will be symmetric with $X(\mathrm{j}\omega) = X^*(-\mathrm{j}\omega)$. The magnitude is symmetric around the ordinate:

$$|X(\mathrm{j}\omega)| = |X(-\mathrm{j}\omega)| \tag{B.16}$$

while the phase is symmetric around the origin:

$$\Phi_X(\omega) = -\Phi_X(-\omega). \tag{B.17}$$

Vice versa, if the frequency transform is real-valued, then the time domain signal will be symmetric with $x(t) = x(-t)$ and if $X(\mathrm{j}\omega)$ is imaginary it means that $x(t) = -x(-t)$.

[1] Only real-valued time domain functions $x(t)$ are considered here.

[2] In real-valued time signals, this shift has to be applied symmetrically to the negative frequencies.

This can be shown by representing the time signal $x(t)$ as a sum of an even component x_e and an odd component $x_o(t)$:

$$x(t) = \underbrace{\frac{1}{2}\left(x(t) + x(-t)\right)}_{x_e(t)} + \underbrace{\frac{1}{2}\left(x(t) - x(-t)\right)}_{x_o(t)} . \tag{B.18}$$

The FT of the even and odd signal components is then

$$X_e(j\omega) = \int\limits_{-\infty}^{\infty} x_e(t)\cos(\omega t)\,dt - j\underbrace{\int\limits_{-\infty}^{\infty} x_e(t)\sin(\omega t)\,dt}_{=0}, \tag{B.19}$$

$$X_o(j\omega) = \underbrace{\int\limits_{-\infty}^{\infty} x_o(t)\cos(\omega t)\,dt}_{=0} -j\int\limits_{-\infty}^{\infty} x_o(t)\sin(\omega t)\,dt. \tag{B.20}$$

The transform of the even signal is thus purely real $X_e(j\omega) = \mathfrak{Re}[X(j\omega)]$, and the transform of the odd part is purely imaginary $X_o(j\omega) = \mathfrak{Im}[X(j\omega)]$. Furthermore, due to the property $\cos(\omega t) = \cos(-\omega t)$, it becomes clear that the real part is again an even function symmetric around $\omega = 0$. The imaginary part is odd due to $\sin(\omega t) = -\sin(-\omega t)$. It follows that the magnitude spectrum is an even function and the phase spectrum is an odd function.

We have seen that the FT is very similar to the IFT; thus, if $X(j\omega)$ is the FT of the signal $x(t)$, then it would also be true that $2\pi \cdot x(-j\omega)$ is the transform of $X(t)$. This can be shown by substituting t with $-\omega$ in the IFT:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega, \\ x(-t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}\,d\omega, \\ x(-j\omega) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(t)e^{-j\omega t}\,dt. \end{aligned} \tag{B.21}$$

B.1.7 Time and Frequency Scaling

The FT of a signal modified in the time domain by scaling the time axis $y(t) = x(c \cdot t)$ will be scaled inversely:

$$Y(j\omega) = \frac{1}{|c|}X\left(j\frac{\omega}{c}\right). \tag{B.22}$$

The derivation (for positive c) is

$$\begin{aligned} Y(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} x(c \cdot t) e^{-\mathrm{j}\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-\mathrm{j}\omega \frac{\tau}{c}}\, d\frac{\tau}{c} \\ &= \frac{1}{c} \int_{-\infty}^{\infty} x(\tau) e^{-\mathrm{j}\frac{\omega}{c}\tau}\, d\tau \\ &= \frac{1}{c} X\left(\mathrm{j}\frac{\omega}{c}\right). \end{aligned} \tag{B.23}$$

For negative c, the result is $Y(\mathrm{j}\omega) = -\frac{1}{c}X(\mathrm{j}\frac{\omega}{c})$. The spectrum of a stretched signal ($c > 1$) will thus be compressed and vice versa.

From the above equation it directly follows for $c = -1$ that

$$\mathfrak{F}\{x(-t)\} = X(-\mathrm{j}\omega) \tag{B.24}$$

and for a real-valued signal $x(t)$

$$\mathfrak{F}\{x(-t)\} = X^*(\mathrm{j}\omega). \tag{B.25}$$

B.1.8 Derivatives

The transform of the nth derivative of the signal has the following property (without derivation):

$$\mathfrak{F}\left\{\frac{d^n x(t)}{dt^n}\right\} = (\mathrm{j}\omega)^n X(\mathrm{j}\omega). \tag{B.26}$$

B.2 Spectrum of Example Time Domain Signals

B.2.1 Delta Function

The *delta function* $\delta(t)$, sometimes also named *dirac impulse* or *delta impulse*, equals zero for all points in time except $t = 0$. It represents an ideal impulse and is defined by

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1, \tag{B.27}$$

$$\delta(t) = 0 \text{ for all } t \neq 0. \tag{B.28}$$

This also means that the integration of the multiplication of signal $x(t)$ with this delta function results in

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t)\, dt = x(0). \tag{B.29}$$

Thus, the result of the FT is

$$\Delta(\mathrm{j}\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-\mathrm{j}\omega t}\, dt = e^{\mathrm{j}\omega \cdot 0} = 1. \tag{B.30}$$

The spectrum is therefore a real-valued constant; it follows that the delta function incorporates all frequencies with the same strength.

B.2.2 Constant

The symmetry of FT and IFT shown in Eq. (B.21), in combination with Eq. (B.30), tells us also that the spectrum of a constant valued time domain signal $x(t) = {}^1\!/_{2\pi}$ will be $X(\mathrm{j}\omega) = \delta(\omega)$.

B.2.3 Cosine

A sinusoidal time domain signal can be interpreted as a modulated constant value. Applying the frequency shift property from Eq. (B.15) thus shows that the spectrum of a cosine is the spectrum of a constant value shifted by the cosine's frequency ω_0, the delta function $\delta(\omega - \omega_0)$.

B.2.4 Rectangular Window

The rectangular window is defined by

$$w_\mathrm{R}(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}. \tag{B.31}$$

The spectrum of this window function is

$$\begin{aligned} W_\mathrm{R}(\mathrm{j}\omega) &= \int_{-\infty}^{\infty} w_\mathrm{R}(t) e^{-\mathrm{j}\omega t}\, dt \\ &= \int_{-1/2}^{1/2} e^{-\mathrm{j}\omega t}\, dt \\ &= \frac{1}{-\mathrm{j}\omega} \underbrace{\left(e^{-\mathrm{j}\omega/2} - e^{\mathrm{j}\omega/2}\right)}_{=-2j\sin(\omega/2)} \\ &= \frac{\sin\left(\omega/2\right)}{\omega/2} = \operatorname{sinc}\left(\frac{\omega}{2}\right). \end{aligned} \tag{B.32}$$

B.2.5 Delta Pulse

The *delta pulse* is a series of individual delta impulses, i.e., a superposition of delta functions. It is defined by

$$\delta_\mathrm{T}(t) = \sum_{i=-\infty}^{\infty} \delta(t - iT_0). \tag{B.33}$$

Each delta impulse has a distance T_0 from its neighbor. Using FT of a delta function given by Eq. (B.27) and the superposition property from Eq. (B.3) in combination with the time shift property from Eq. B.13, the FT of $\delta_{\mathrm{T}}(t)$ is

$$\Delta_{\mathrm{T}}(\mathrm{j}\omega) = \sum_{i=-\infty}^{\infty} e^{-\mathrm{j}\omega i T_0}. \tag{B.34}$$

With help from the geometric series it can be shown [7] that

$$\begin{aligned}\Delta_{\mathrm{T}}(\mathrm{j}\omega) &= \frac{2\pi}{T}\sum_{i=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi i}{T}\mathrm{j}\omega T\right)\\ &= \omega_T\delta_{\omega_{\mathrm{T}}}(\omega).\end{aligned} \tag{B.35}$$

B.3 Transformation of Sampled Time Signals

A sampled time signal can be represented by the multiplication of a continuous time signal $x(t)$ multiplied by a delta pulse $\delta_{\mathrm{T}}(t)$. Equation (B.8) states that a multiplication of two time signals corresponds to the convolution of their frequency transforms. This means that

$$\begin{aligned}\mathfrak{F}\{x(i)\} &= \mathfrak{F}\{x(t)\cdot\delta_{\mathrm{T}}(t)\}\\ &= \mathfrak{F}\{x(t)\} * \mathfrak{F}\{\delta_{\mathrm{T}}(t)\}\\ &= X(\mathrm{j}\omega) * \Delta_{\mathrm{T}}(\mathrm{j}\omega).\end{aligned} \tag{B.36}$$

Note that although the time domain signal is discrete, the resulting spectrum is still continuous. As can be seen from Eq. (B.36), the spectrum is repeated periodically with ω_{T}, the sample rate. This allows a very intuitive explanation of the sampling theorem stated in Eq. (2.9) since the periodically repeated spectra would overlap if signal $x(t)$ contains higher frequencies than $\omega_{\mathrm{T}}/2$ (see Fig. B.1). In that case, reconstruction of the original signal $x(t)$ is impossible, while otherwise perfect reconstruction is possible by applying an ideal low-pass filter with a cut-off frequency of $\omega_{\mathrm{T}}/2$ to the sampled signal $x(i)$. The effect of overlapping spectra is called aliasing and is visualized in Fig. B.1.

B.4 Short Time Fourier Transform of Continuous Signals

Up to this point, we have dealt mostly with signals unlimited in time. In the real world, signals will usually have a defined start and stop time. We might also be interested in transforming only segments of such signals. This can be seen as multiplying an infinite time signal with a window function that equals zero outside the time boundaries of interest. In signal analysis, typical segment lengths range — dependent on the task at hand — between 10 and 300 ms. Smith points out three reasons for choosing segments of this length [8]:

- "Perhaps most fundamentally, the ear similarly Fourier analyzes only a short segment of audio signals at a time (on the order of 10–20 ms worth). Therefore, to match our spectrum analysis to human hearing, we desire to limit the time window of the analysis."

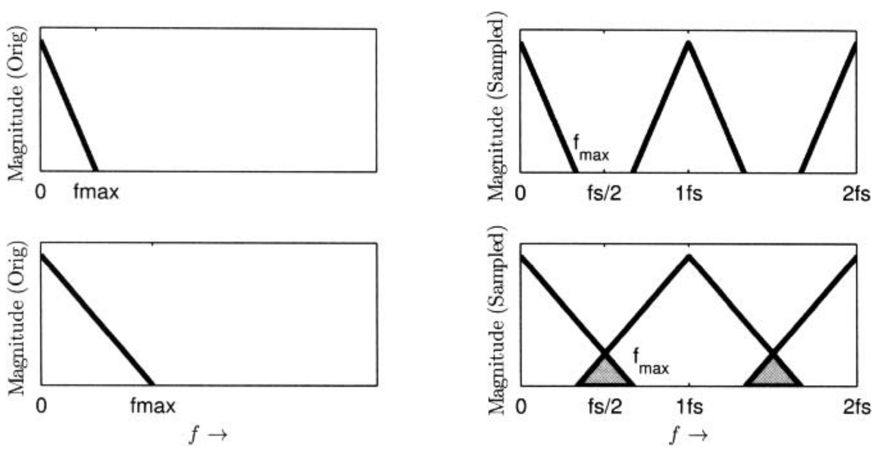

Figure B.1 Schematic visualization of the spectrum of a continuous time domain signal (left) and the sampled signal in accordance (top) and violation (bottom) of the sampling theorem

- "Audio signals typically have spectra which change over time. It is therefore usually most meaningful to restrict analysis to a time window over which the spectrum stays rather constant."
- "It can be extremely time consuming to compute the Fourier transform of an audio signal of typical length, and it will rarely fit in computer memory all at once."

B.4.1 Window Functions

Since every multiplication in the time domain corresponds to a convolution of the corresponding spectra, the spectrum of the signal is convolved with the spectrum of the window. The spectrum of the window thus has influence on the resulting spectrum. The most simple window in the time domain is a rectangular window introduced in Sect. B.2.4. The typical spectral shape of a window consists of a main lobe and many side lobes with more or less decreasing amplitude.

When the signal $x(t)$ of interest is a sinusoid, then the resulting FT of the windowed signal will therefore be a superposition of two window functions with their main lobes located at the signal's frequency $\omega_0, -\omega_0, \ldots$, so the delta functions are effectively "smeared" by windowing artifacts. This undesired side effect is referred to as spectral leakage. It is usually characterized by

- the width of the main lobe,
- the height of the first (closest) side lobe peak, and
- the rolloff or attenuation of the subsequent side lobe peaks.

In order to optimize these properties toward individual use cases, different window functions have been suggested in the past. Figure B.2 shows the presented window functions in time domain (left) and frequency domain (right).

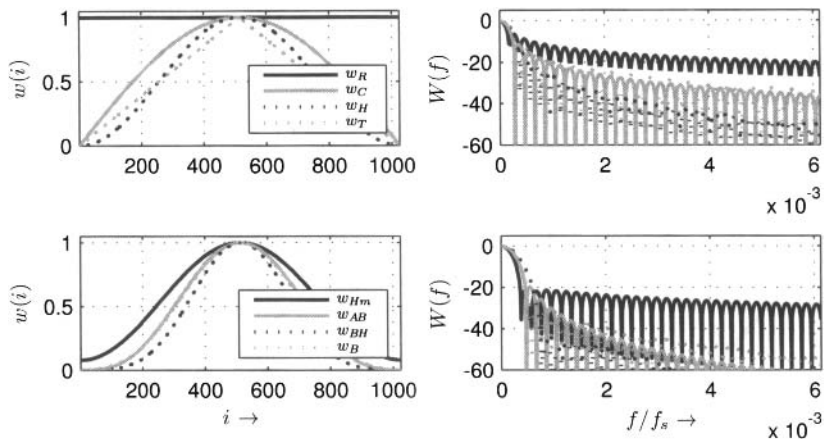

Figure B.2 Windows in time domain (left) and frequency domain (right)

B.4.1.1 Rectangular Window

The FT of the *rectangular window* has already been derived in Sect. B.2.4 to be a sinc function:

$$W_\mathrm{R}(\mathrm{j}\omega) = \mathrm{sinc}\left(\frac{\omega}{2}\right) \tag{B.37}$$

B.4.1.2 Bartlett Window

The *Bartlett window* has a triangular shape. It is defined by

$$\begin{aligned} w_\mathrm{T}(t) &= \begin{cases} t+1, & -1/2 \le t \le 0 \\ 1-t, & 0 \le t \le 1/2 \\ 0, & \text{otherwise} \end{cases} \\ &= w_\mathrm{R}(2t) * w_\mathrm{R}(2t). \end{aligned} \tag{B.38}$$

Using Eq. (B.7) it can be deduced that

$$\begin{aligned} W_\mathrm{T}(\mathrm{j}\omega) &= \mathfrak{F}\left\{w_\mathrm{R}(2t)\right\} \cdot \mathfrak{F}\left\{w_\mathrm{R}(2t)\right\} \\ &= \frac{1}{2} \cdot \mathrm{sinc}^2\left(\frac{\omega}{4}\right) \end{aligned} \tag{B.39}$$

B.4.1.3 Generalized Superposed Cosines

It is possible to generalize many window functions with

$$w_\mathrm{sup}(t) = w_\mathrm{R}(t) \sum_{j=0}^{\mathcal{O}-1} b_j \cos\left(\frac{\pi}{2} jt\right). \tag{B.40}$$

Different values for $\mathcal{O}$ result in different window families:

- $\mathcal{O} = 1$: *rectangular window* $w_\mathrm{R}(t)$

- $\mathcal{O} = 2$: Hamming family of windows:
  - *cosine window* $w_{\mathrm{C}}(t)$:
    $b_0 = 0,\ b_1 = 1$
  - *von-Hann window* $w_{\mathrm{H}}(t)$:
    $b_0 = {}^1\!/_2,\ b_1 = {}^1\!/_2$
  - *Hamming window* $w_{\mathrm{Hm}}(t)$:
    $b_0 = {}^{25}\!/_{46},\ b_1 = {}^{42}\!/_{46}$
- $\mathcal{O} = 3$: Blackman-Harris family of windows:
  - classic *Blackman window* $w_{\mathrm{B}}(t)$:
    $b_0 = {}^{7938}\!/_{18608},\ b_1 = {}^{9240}\!/_{18608},\ b_2 = {}^{1430}\!/_{18608}$
  - *Blackman-Harris window* $w_{\mathrm{BH}}(t)$:
    $b_0 = 0.4243801,\ b_1 = 0.4973406,\ b_2 = 0.0782793$

B.4.1.4 Generalized Power of Cosine

A different generalization of window functions is

$$w_{\mathrm{pow}}(t) = w_{\mathrm{R}}(t) \cos^{\beta}\left(\frac{\pi}{2} t\right). \tag{B.41}$$

Again, different windows can be derived for different β:

- *rectangular window* $w_{\mathrm{R}}(t)$:
  $\beta = 0$
- *cosine window* $w_{\mathrm{C}}(t)$:
  $\beta = 1$
- *von-Hann window* $w_{\mathrm{H}}(t)$:
  $\beta = 2$
- *alternative Blackman window* $w_{\mathrm{AB}}(t)$:
  $\beta = 0$

B.5 Discrete Fourier Transform

In computer applications, a discrete representation of the signal's spectrum is required; it can only be defined at discrete frequency bins. The frequency bins are evenly distributed over the interesting range of frequencies with the distance

$$\Delta\Omega = \frac{2\pi}{\mathcal{K}T_{\mathrm{S}}} = \frac{2\pi f_{\mathrm{S}}}{\mathcal{K}}. \tag{B.42}$$

The DFT of the nth block of the signal $x(i)$ will be referred to as STFT and is defined by

$$X(\mathrm{j}\Delta\Omega) = \sum_{i=0}^{\mathcal{K}-1} x(i) \exp\left(-\mathrm{j}ki\frac{2\pi}{\mathcal{K}}\right) \tag{B.43}$$

with $k = 0, 1, \ldots, \mathcal{K} - 1$.

Thus, the DFT of a block of samples of length $\mathcal{K}$ also consists of exactly $\mathcal{K}$ complex values; however, since the signal $x(i)$ is real, the result will be symmetric with

$$X(\mathcal{K} - k) = X^*(k) \tag{B.44}$$

and only $\mathcal{K}/2$ complex results need to be computed.

The spectrum $X(k, n)$ can be interpreted as the (continuous) FT of the block n of signal $x(i)$ sampled at equidistant bins at the positions $k \cdot \Delta\Omega$. It has to be periodic

$$X(k) = X(k + \mathcal{K}) \tag{B.45}$$

because the time domain signal is discrete.

The spectrum can only be discrete if the time domain signal is periodic (compare the Fourier series). Therefore, the DFT can be interpreted as the FT applied to the current block of samples periodically continued.

The *Inverse Discrete Fourier Transform (IDFT)* allows reconstruction of the time samples that had been transformed:

$$x(i) = \sum_{k=0}^{\mathcal{K}-1} X(k) e^{jki\Delta\Omega}. \tag{B.46}$$

The properties of the DFT correspond to the properties introduced for the continuous FT, but a few details have to be kept in mind: the multiplication of two DFTs corresponds to a circular convolution (similar to the CiCF) in the time domain. The same is true for time and frequency shift operations.

B.5.1 Window Functions

The discrete window functions are sampled (a potentially shifted) versions of the continuous window functions given above. The rectangular window is

$$w_{\mathrm{R}}(i) = \begin{cases} 1, & -\frac{\mathcal{K}-1}{2} \leq i \leq \frac{\mathcal{K}-1}{2} \\ 0, & \text{otherwise} \end{cases}, \tag{B.47}$$

and the superposed cosine window is

$$w_{\mathrm{sup}}(i) = w_{\mathrm{R}}(i) \sum_{j=0}^{\mathcal{O}-1} a_j \cos\left(\frac{j \cdot \pi}{\mathcal{K}} i\right). \tag{B.48}$$

The DFT of a rectangular window is

$$W_{\mathrm{R}}(k, n) = \exp\left(-j\frac{\mathcal{K} - 1}{2}\frac{2\pi k}{\mathcal{K}}\right) \cdot \frac{\sin\left(\frac{\mathcal{K}\frac{2\pi k}{\mathcal{K}}}{2}\right)}{\sin\left(\frac{\frac{2\pi k}{\mathcal{K}}}{2}\right)}. \tag{B.49}$$

Note that the phase shift term originates in moving the first window sample to sample 0. When transforming a windowed sine, the result will be shifted in the frequency domain

$$X(k, n) = \exp\left(-j\frac{\mathcal{K} - 1}{2}\frac{2\pi k}{\mathcal{K}} - \Omega_0\right) \cdot \frac{\sin\left(\frac{\mathcal{K}\frac{2\pi k}{\mathcal{K}}}{2} - \Omega_0\right)}{\sin\left(\frac{\frac{2\pi k}{\mathcal{K}}}{2} - \Omega_0\right)}. \tag{B.50}$$

Table B.1 Frequency domain properties of the most common windows (from [479])

Window	ΔB [Bins]	A_{SL} [dB]	S_{SL} [dB/Oct]	A_{WC} [dB]
w_{R}	0.89	−13	−6	3.92
w_{T}	1.28	−27	−12	3.07
w_{C}	1.20	−23	−12	3.01
w_{H}	1.44	−32	−18	3.18
w_{Hm}	1.30	−43	−6	3.10
w_{B}	1.68	−58	−18	3.47
w_{AB}	1.66	−39	−24	3.47
w_{BH}	1.66	−67	−6	3.45

If the sinusoidal frequency exactly fits the frequency of a bin with index k, then all bins will be zero for $k \neq k_0$. In this case, all the zero crossings of the window function fall at the spectral bin positions. However, if k_0 is between two frequency bins, then two artifacts appear: the main peak has lower level, the so-called *process loss*, and the frequency bins are now directly located at the main peaks of the side lobes. Two special cases are the *best case* and the *worst case* scenario with k_0 being directly on a bin or exactly between two bins. In the time domain, the best case means that one or more periods of the sinusoidal fit exactly into the window with length $\mathcal{K}$.

B.5.1.1 Discrete Window Properties

The following properties can be used to characterize the frequency domain representation of a window function:

- width of main lobe ΔB (3 dB bandwidth in bins),
- peak level of highest side lobe A_{SL} (dB),
- side lobe fall-off S_{SL} (dB/Oct),
- worst case process loss A_{WC} (dB).

A smaller main lobe width yields better frequency resolution and both a smaller side lobe peak level and higher side lobe fall-off results in less cross-talk between sinusoids of different frequencies. The smaller the worst case process loss, the higher the resulting amplitude accuracy.

Table B.1 summarizes these properties for common windows. For detailed introductions to spectral leakage see Harris [479] and Smith [8].

B.5.2 Fast Fourier Transform

An efficient way to compute the DFT is the *Fast Fourier Transform (FFT)*. The FFT is equivalent to the "normal" DFT; it just computes the result more efficiently. More specifically, the difference in the number of operations is approximately $O(\mathcal{K}^2)$ for the normal DFT compared to $O(\mathcal{K}\log\mathcal{K})$ for the FFT (compare [480, 481]). There are different algorithms to compute the FFT; most FFT implementations require an input block length which equals a power of 2.