wCorr Arguments

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This vignette explores two Boolean switches in the wCorr package. First, the `ML` switch allows for either a non-MLE (but consistent) esitimate of the nusiance parameters that define the binning process to be used (`ML=FALSE`) or for the nusiance parameters to be estimated using the MLE (`ML=TRUE`). Second the `fast` argument gives the option to use a pure R implementation (`fast=FALSE`) or an implementation that relies on the `Rcpp` and `RcppArmadillo` packages (`fast=TRUE`).

The *wCorr Formulas* vignette describes the statistical properties of the correlation estimators in the package and has a more complete derivation of the likelihood functions.

The `ML` switch

The correlation coefficients between two vectors of random variables that are jointly bivariate normal–call the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N \left[\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \boldsymbol{\Sigma} \right]$$

where $N(\mathbf{A}, \boldsymbol{\Sigma})$ is the bivariate normal distribution with mean $\boldsymbol{A}$ and covariance $\boldsymbol{\Sigma}$.

Polyserial computation

the likelihood function for an individual observation of the polyserial is[1]

$$\Pr\left(\rho = r, \boldsymbol{\theta}; Z = z_i, M = m_i\right) = \phi(z_i) \left[\Phi\left(\frac{\theta_{m_i+2} - r \cdot z_i}{\sqrt{1-r^2}} \right) - \Phi\left(\frac{\theta_{m_i+1} - r \cdot z_i}{\sqrt{1-r^2}} \right) \right]$$

where ρ is the correlation between $\boldsymbol{X}$ and $\boldsymbol{Y}$, $\boldsymbol{Z}$ is the normalized version of $\boldsymbol{X}$, and $\boldsymbol{M}$ is a discretized version of $\boldsymbol{Y}$, using $\boldsymbol{\theta}$ as cut points as described in the *wCorr Formulas* vignette.

The log-likelihood is then

$$\ell(\rho, \boldsymbol{\theta}; z, m) = \sum_i w_i \ln\left[\Pr\left(\rho = r, \boldsymbol{\theta}; Z = z_i, M = m_i\right)\right]$$

The derivatives of ℓ can be written down but are not readily computed and so when the `ML` argumet is set to `FALSE` (the default) a one dimensional optimization of ρ is calculated using `stats::optimize`. When the `ML` argument is set to `TRUE` a multi-dimensional optimization is done for ρ and $\boldsymbol{\theta}$ using `minqa::bobyqa`.

[1]See the *wCorr Formulas* vignette for a more complete description and motivation for the polyserial correlations's likelihood function.

Polychoric computation

For the polychoric the observed data is discreteized for both variables. Here the discretized version of $\boldsymbol{X}$ is $\boldsymbol{P}$ and the discretized version of $\boldsymbol{Y}$ remains $\boldsymbol{M}$.[2] The likelihood function for the polychoric is

$$\Pr\left(\rho = r, \boldsymbol{\theta}, \boldsymbol{\theta}'; P = p_i, M = m_i\right) = \int_{\theta'_{p_i+1}}^{\theta'_{p_i+2}} dx \int_{\theta_{m_i+1}}^{\theta_{m_i+2}} dy f(x, y|\rho = r)$$

where $f(x, y|r)$ is the noramlzied bivariate normal distribution with correlation ρ, and $\boldsymbol{\theta'}$ are the cut points used to discretize $\boldsymbol{X}$ into $\boldsymbol{P}$.

The log-likelihood is then

$$\ell(\rho, \boldsymbol{\theta}, \boldsymbol{\theta}'; \mathbf{p}, \mathbf{m}) = \sum_i w_i \ln\left[\Pr\left(\rho = r, \boldsymbol{\theta}, \boldsymbol{\theta}'; P = p_i, M = m_i\right)\right]$$

The derivatives of ℓ can be written down but are not readily computed and so when the `ML` argumet is set to `FALSE` (the default) a one dimensional optimization of ρ is calculated using `stats::optimize`. When the `ML` argument is set to `TRUE` a multi-dimensional optimization is done for ρ, $\boldsymbol{\theta}$, and $\boldsymbol{\theta}'$ using `minqa::bobyqa`.

General setup for the unweighted case

A simulation is run several times. For each itteration, the following procedure is used:

- select the number of observations (n)
- select a true correlation coefficient ρ
- generate $\boldsymbol{X}$ and $\boldsymbol{Y}$ to be bivariate normally distributed using a pseudo-Random Number Generator (RNG)
- using a pseudo-RNG, select the the number of bins for $\boldsymbol{M}$ and $\boldsymbol{P}$ (t and t') independantly from the set {2, 3, 4, 5}
- select the bin boundaries for $\boldsymbol{M}$ and $\boldsymbol{P}$ ($\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$) by sorting the results of $(t-1)$ and $(t'-1)$ draws, respectively, from a normal distribution using a pseudo-RNG
- confirm that at least 2 levels of each of $\boldsymbol{M}$ and $\boldsymbol{P}$ are occupied (if not, retrun to generating $\boldsymbol{X}$ and $\boldsymbol{Y}$)
- calculate and record relevant statistics

When the exact method of selecting a parameter (such as n) is not noted in the above description it is described as part of each simulation.

ML switch

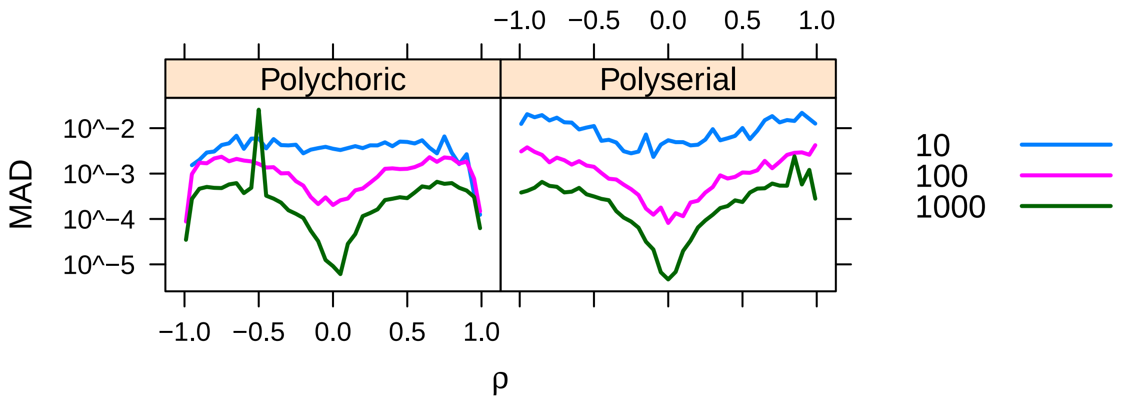
A simulation was done at each level of the cartesian product of $\texttt{ML} \in \{\texttt{TRUE}, \texttt{FALSE}\}$,
$\rho \in (-0.99, -0.95, -0.90, -0.85, ..., 0.95, 0.99)$, and $n \in \{10, 100, 1000\}$. For precision, each iteration is run three times. The compulation is run so that the same values of the variables are used for `ML=TRUE` as `ML=FALSE` and then the statistics are compared between the two sets of results. where MAD is the mean absolute difference and is given by

$$MAD = |r_{ML=TRUE} - r_{ML=FALSE}|$$

[2]See the “wCorr Formulas” vignette for a more complete description and motivation for the polychoric correlations’s likelihood function.

where $r_{ML=TRUE}$ is the estimated correlation when `ML=TRUE` and $r_{ML=FALSE}$ is the estimated correlation when `ML=FALSE`.

This is a plot of the MAD as a function of the true correlation coefficnet. It shows a decrease in MAD as n increases (change from line to line), a decrease when the correlations are in the neighborhood of zero that is more pronounced for larger n and a dip then the correlation is exactly 1 or -1.



This table shows the MAD by n and correlation type.

Correlation type	n	MAD
Polychoric	10	0.0037186
Polychoric	100	0.0012056
Polychoric	1000	0.0009111
Polyserial	10	0.0099031
Polyserial	100	0.0013888
Polyserial	1000	0.0003561

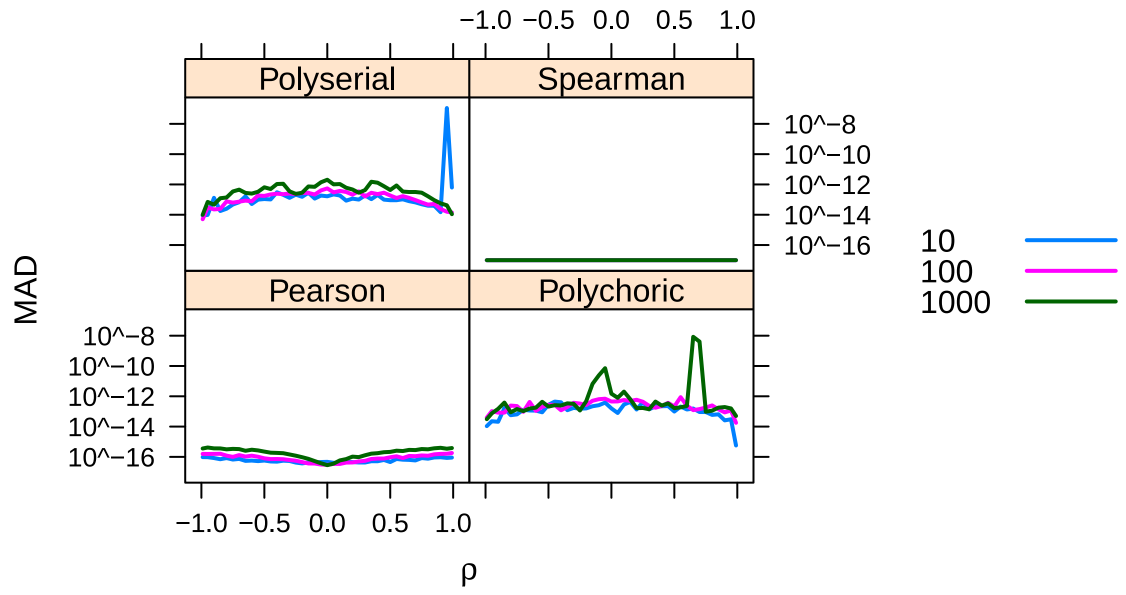
These resuls show that the side of the MAD decreases as n incrreases. When $n = 10$, the MAD is less than 0.02 for the polyserial and 0.002 for the polychoric.

fast switch

This section looks at the agreement between the pure R implementation of the optimizations and the `Rcpp` and `RcppArmadillo` impelemntation. The code can compute with either option by setting `fast=FALSE` (pure R) or `fast=TRUE` (Rcpp).

A simulation was done at each level of the cartesian product of `fast` $\in \{$`TRUE`, `FALSE`$\}$,
$\rho \in (-0.99, -0.95, -0.90, -0.85, ..., 0.95, 0.99)$, and $n \in \{10, 100, 1000\}$. Each iteration was run 100 times. The compulation is run so that the same values of the variables are used for `fast=TRUE` as `fast=FALSE` and then the statistics are compared between the two sets of results.

This is the summary of all differences between the `fast=TRUE` and `fast=FALSE runs` for the polyserial. Note that differences smaller than 10^{-16} are indistinguishable from 0 by the machine. However, a factor of 10^{-17} was added to the results so that they could all be shown on a log scale. Thus the Spearman never shows differences that are different from zero.

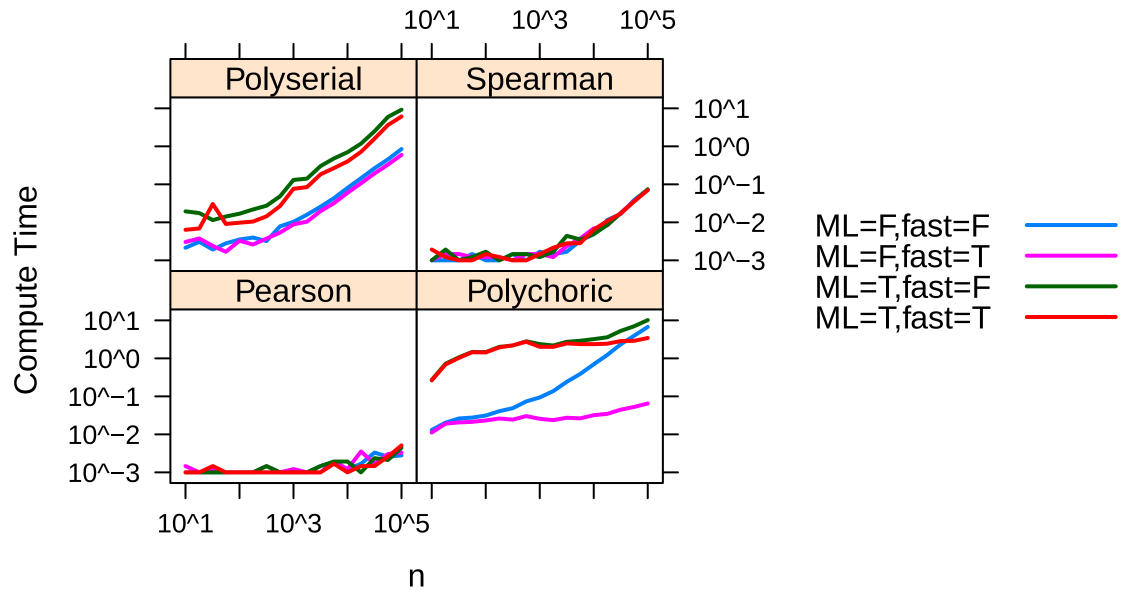


The above shows that differences as a result of the `fast` argument are never larger than 10^{-8} for any type.

Implications for speed

A simulation was done at each level of the cartesian product of $\texttt{ML} \in \{\texttt{TRUE}, \texttt{FALSE}\}$, $\texttt{fast} \in \{\texttt{TRUE}, \texttt{FALSE}\}$, $\rho \in (-0.99, -0.95, -0.90, -0.85, ..., 0.95, 0.99)$, and $n \in \{10^1, 10^{1.25}, 10^{1.5}, ..., 10^7\}$. For precision, each iteration is run 80 times when $n < 10^5$ and 20 times when $n \geq 10^5$. The compulation is run so that the same values of the variables are used all four levels of `ML` and `fast`. The variety of correlations is chosen so that the results represent an average of possible values of ρ.

The following plot shows the mean compute time versus n.



Conclusion

Using tables presented in this vignette, users who wish to use the more accurate `ML=TRUE` argument can compare the difference in compuation time and the difference in results.

the `fast` argument is provided primarily for comparison of the `Rcpp` and pure R code and shows agreement to within 10^{-8}.