Basic Discussion on Integral calculus

In mathematics, an **integral** assigns numbers to functions in a way that can describe displacement, area, volume, and other concepts that arise by combining infinitesimal data. Integration is one of the two main operations of calculus; its inverse operation, differentiation, is the other. Given a function f of a real variable x and an interval $[a, b]$ of the real line, the **definite integral**

$$\int_a^b f(x)\,dx$$

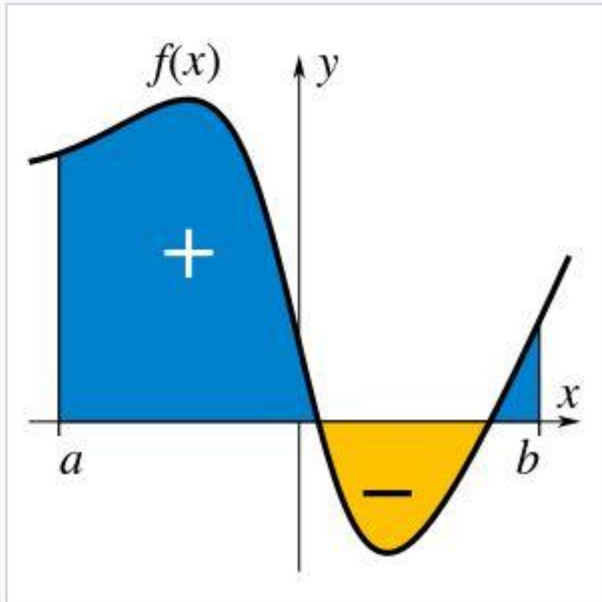
can be interpreted informally as the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. The area above the x-axis

adds to the total and that below the x-axis subtracts from the total. The operation of integration, up to an additive constant, is the inverse of the operation of differentiation. For this reason, the term *integral* may also refer to the related notion of the antiderivative, a function F whose derivative is the given function f. In this case, it is called an **indefinite integral** and is written:

$$F(x) = \int f(x)\,dx.$$

The integrals discussed in this article are those termed *definite integrals*. It is the fundamental theorem of calculus that connects differentiation with the definite integral: if f is a continuous real-valued function defined on a closed interval $[a, b]$, then, once an antiderivative F of f is known, the definite integral of f over that interval is given by,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$



A definite integral of a function can be represented as the signed area of the region bounded by its graph.

The principles of integration were formulated independently by Isaac Newton and Gottfried Wilhelm Leibniz in the late 17th century, who thought of the integral as an infinite sum of rectangles of infinitesimal width. Bernhard Riemann gave a rigorous mathematical definition of integrals. It is based on a limiting procedure that approximates the area of a curvilinear region by breaking the region into thin vertical slabs. Beginning in the 19th century, more sophisticated notions of integrals began to appear, where the type of the function as well as the domain over which the integration is performed has been generalised. A line integral is defined for functions of two or more variables, and the interval of integration $[a, b]$ is replaced by a curve connecting the two endpoints. In a surface integral, the curve is replaced by a piece of a surface in three-dimensional space.

Leibniz and Newton

The major advance in integration came in the 17th century with the independent discovery of the fundamental theorem of calculus by Leibniz and Newton. Leibniz published his work on calculus before Newton. The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Leibniz and Newton developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern calculus, whose notation for integrals is drawn directly from the work of Leibniz.

Historical notation

The notation for the indefinite integral was introduced by Gottfried Wilhelm Leibniz in 1675 (Burton 1988, p. 359; Leibniz 1899, p. 154). He adapted the integral symbol, ∫, from the letter ſ (long s), standing for *summa* (written as *ſumma*; Latin for "sum" or "total"). The modern notation for the definite integral, with limits above and below the integral sign, was first used by Joseph Fourier in *Mémoires* of the French Academy around 1819–20, reprinted in his book of 1822 (Cajori 1929, pp. 249–250; Fourier 1822, §231).

Isaac Newton used a small vertical bar above a variable to indicate integration, or placed the variable inside a box. The vertical bar was easily confused with $\dot{x}$ or x', which are used to indicate differentiation, and the box notation was difficult for printers to reproduce, so these notations were not widely adopted.

Process of Integration

As the name should hint itself, the process of Integration is actually the reverse/inverse of the process of Differentiation. It is represented by the symbol $\int$, for example,

$$\int \frac{1}{x}\,dx = lnx + c$$

where,

- $\frac{1}{x}$ – the integrand
- dx – denotes that x is the variable with respect to which the integrand has to be integrated
- lnx – the resultant function
- c – the constant of integration

You may note that,

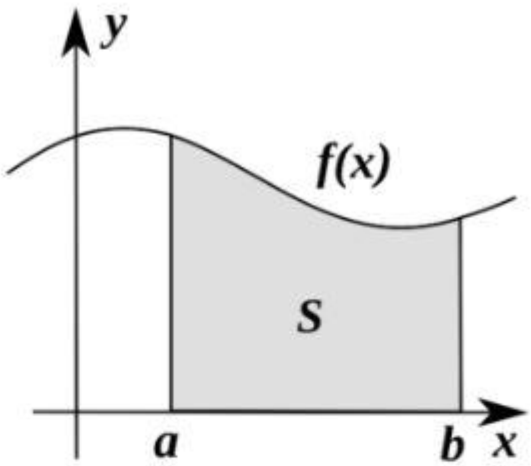
$$\frac{d}{dx}(lnx) = \frac{1}{x}$$

Comparing with the example for integration, you can see it for yourself how the integration and differentiation of a function are interconnected. Besides, from the formulae of differentiation, we also have –

$$\frac{d}{dx}(lnx + c) = \frac{1}{x}$$

Thus, you always need to attach a constant of integration with the result (unless provided with some initial or boundary conditions), because the differentiation of such a function also gives the same result.

Since you must already be aware of the formulae for derivatives of various known functions, we can use those results in the opposite way here now. Even if you don't remember the exact formulae, you may verify the result yourself by differentiating the integrand and looking for yourself. Thus, building upon this simple connection with differentiation, we can understand the following basic formulae for integration.



The Method of Substitution (Change of Variable)

This method is used to reduce a seemingly complex integrand to a known simple form, for which the integration formula is known already. With enough practice and a good understanding of the integration formulae, you'll understand yourself what substitutions to make.

For a given complex integral, $\int F(h(x))dx$, you may make the substitution $h(x) = z$ (your new variable of integration. Followed by this step, you'll also have to change the variable of integration. Thus, dx will have to be changed to dz. From $h(x) = z$; you may arrive at $h'(x)dx = dz$.

Substituting this in the place of dz, and proceeding with the new variable, you may now be able to successfully integrate the resulting function. Let us look at a couple of important applications below for a better idea of this process –

$$\Rightarrow \int f(ax)dx = ?$$

Substitute $z = ax$. Then on differentiating, we have $dz = adx$. Use this in the integral to get –

$$\int f(ax)dx = \int f(z)\frac{dz}{a} = \frac{1}{a}\int f(z)dz$$

Thus if you actually know how to solve the integral, $\int f(x)dx$, the integral $\int f(ax)dx$ with a = a real number, can be easily evaluated using the change in the variable of integration. Obviously in the last step here, $\int f(z)dz = \int f(x)dx$.

$$\Rightarrow \int \frac{f\prime(x)}{f(x)}dx = ?$$

Substitute $f(x) = z$ in this case. On differentiating, we have $f'(x)dx = dz$. Put this back in the integral to get –

$$\int \frac{f'(x)}{f(x)}dx = \int \frac{dz}{z}$$

Using the formulae for integration,

$$= lnz + c$$

Re-substituting for z in the result,

$$= lnf(x) + c$$

These two results arrived at, above are very useful in evaluating simple integrals and find great application in problems.

Solved Examples on Integral Calculus using Method of Substitution

Find the value of the following integration:

$i)\ \int \frac{sinx}{\sqrt{1+cosx}}\,dx \quad ii)\ \int e^{tan^{-1}x}\frac{1}{1+x^2}\,dx \quad iii)\ \int \frac{1}{\sqrt{x}}cos\sqrt{x}\,dx$

*** c is an integrating constant***

Solved – (i):

Let, $I = \int \frac{sinx}{\sqrt{1+cosx}}\,dx$

$= -\int \frac{dz}{\sqrt{z}}$

$= -\int z^{-\frac{1}{2}}\,dz$

$= -\left[\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + c$

$= -2\sqrt{z} + c$

$= -2\sqrt{1+cosx} + c.$

Let, $1 + cosx = z$

$\Rightarrow \frac{d}{dx}(1+cosx) = \frac{dz}{dx}$

$\Rightarrow -sinx = \frac{dz}{dx}$

$\Rightarrow sinx\,dx = -dz$

Solved – (ii):

$Let,\ I = \int e^{tan^{-1}x}\frac{1}{1+x^2}\,dx$

$= \int e^z\,dz$

$= e^z + c$

$= e^{tan^{-1}x} + c$

Let, $tan^{-1}x = z$

$\Rightarrow \frac{d}{dx}(tan^{-1}x) = \frac{dz}{dx}$

$\Rightarrow \frac{1}{1+x^2}\,dx = dz$

Solved – (iii):

$Let,\ I = \int \frac{1}{\sqrt{x}}cos\sqrt{x}\,dx$

$= 2\int cosz\,dz$

$= 2sinz + c$

$= 2\sin(\sqrt{x}) + c$

Let, $\sqrt{x} = z \Rightarrow \frac{d}{dx}(\sqrt{x}) = \frac{dz}{dx}$

$\Rightarrow \frac{1}{2\sqrt{x}} = \frac{dz}{dx} \Rightarrow \frac{1}{\sqrt{x}}dx = 2dz$

Find the value of the following integration:

$i)\ \int \frac{\sqrt{1+lnx}}{x}\,dx$ $\qquad ii)\ \int sin^4x\, cos^3x\, dx$ $\qquad iii)\ \int cos^3x\, dx$

Solved – (i):

$$Let,\ \ I = \int \frac{\sqrt{1+lnx}}{x}\,dx$$
$$= \int \sqrt{z}\,dz$$
$$= \left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + c$$
$$= \frac{z^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \frac{2}{3}\sqrt{z^3} + c$$
$$= \frac{2}{3}\sqrt{(1+lnx)^3} + c$$

Let, $1 + lnx = z$

$$\Rightarrow \frac{d}{dx}(1+lnx) = \frac{dz}{dx}$$
$$\Rightarrow \frac{1}{x}\,dx = dz$$

Solved – (ii):

$$Let,\ \ I = \int sin^4x\, cos^3x\, dx$$
$$= \int sin^4x\, cos^2x.cosx\, dx$$
$$= \int sin^4x\,(1 - sin^2x)\, cosx\, dx$$
$$= \int z^4\,(1 - z^2)\,dz$$
$$= \int\ (z^4 - z^6)\,dz$$
$$= \left[\frac{z^{4+1}}{4+1} - \frac{z^{6+1}}{6+1}\right] + c$$
$$= \left[\frac{z^5}{5} - \frac{z^7}{7}\right] + c$$
$$= \frac{1}{5}sin^5x - \frac{1}{7}sin^7x + c$$

Let, $sinx = z$

$$\Rightarrow \frac{d}{dx}(sinx) = \frac{dz}{dx}$$
$$\Rightarrow cosx\, dx = dz$$

Solved – (iii):

$$Let,\ \ I = \int cos^3x\, dx$$
$$= \int cos^2x. cosx\, dx$$
$$= \int (1 - sin^2x). cosx\, dx$$
$$= \int (1 - z^2)\, dz$$
$$= z - \frac{z^3}{3} + c$$
$$= sinx - \frac{1}{3}sin^3x + c.$$

Let, $sinx = z$

$$\Rightarrow \frac{d}{dx}(sinx) = \frac{dz}{dx}$$
$$\Rightarrow cosx\, dx = dz$$

Find the value of the following integration:

$i)\ \int \frac{1}{a^2+x^2}\, dx$ $\quad ii)\ \int \frac{x^4}{4+x^{10}}\, dx$ $\quad iii)\ \int \frac{2dx}{x\{1+(lnx)^2\}}\, dx$

$iv)\ \int \frac{sinx\, cosx}{sin^4x+cos^4x}dx$ $\quad v)\ \int \frac{dx}{\sqrt{a^2-x^2}}$

Solved – (i):

$$Let,\ \ I = \int \frac{1}{a^2+x^2}\, dx$$
$$= \int \frac{asec^2\theta\, d\theta}{a^2+a^2tan^2\theta}$$
$$= \int \frac{asec^2\theta\, d\theta}{a^2(1+tan^2\theta)}$$
$$= \frac{1}{a}\int \frac{sec^2\theta}{sec^2\theta}\, d\theta$$
$$= \frac{1}{a}\int d\theta$$
$$= \frac{1}{a}\, \theta + c$$
$$= \frac{1}{a}\, tan^{-1}\frac{x}{a} + c$$

Let, $x = atan\theta$

$$\Rightarrow \frac{dx}{d\theta} = asec^2\theta$$
$$\Rightarrow dx = asec^2\theta\, d\theta$$

Again, $tan\theta = \frac{x}{a}$

$$\Rightarrow \theta = tan^{-1}\frac{x}{a}$$

Solved – (ii):

$$Let,\ \ I = \int \frac{x^4}{4+x^{10}}\ dx$$
$$= \int \frac{x^4}{4+(x^5)^2}\ dx$$
$$= \frac{1}{5}\int \frac{dz}{2^2+z^2}$$
$$= \frac{1}{5}\left[\frac{1}{2}\ tan^{-1}\frac{z}{2}\right] +\ c$$
$$= \frac{1}{10}\ tan^{-1}\frac{x^5}{2} +\ c$$

Let, $x^5 = z$

$$\Rightarrow 5x^4 dx = dz$$
$$\Rightarrow x^4 dx = \frac{1}{5}dz$$

Solved – (iii):

$$Let,\ \ I = \int \frac{2}{x\{1+(lnx)^2\}}\ dx$$
$$= 2\int \frac{dz}{1+z^2}$$
$$= 2tan^{-1}(z) + c$$
$$= 2tan^{-1}(lnx) + c$$

Let, $lnx = z$

$$\Rightarrow \frac{d}{dx}(lnx) = \frac{dz}{dx}$$
$$\Rightarrow \frac{1}{x}dx = dz$$

Solved – (iv):

$$Let,\ \ I = \int \frac{sinx\ cosx}{sin^4x+cos^4x}\ dx$$
$$= \int \frac{sinx\ cosx}{\left(\frac{sin^4x}{cox^4x}+1\right)cos^4x}\ dx$$
$$= \int \frac{sinx}{cosx.cos^2x(tan^4x+1)}\ dx$$
$$= \int \frac{tanx.sec^2x}{1+tan^4x}\ dx$$
$$= \int \frac{tanx.sec^2x}{1+\ (tan^2x)^2}\ dx$$
$$= \frac{1}{2}\int \frac{dz}{1+z^2}\ \ = \frac{1}{2}\ tan^{-1}z + c\ \ = \frac{1}{2}\ tan^{-1}(tan^2x) + c.$$

Let, $tan^2x = z$

$$\Rightarrow (tanx)^2 = z$$
$$\Rightarrow 2tanx.sec^2x\ dx = dz$$
$$\Rightarrow tanx.sec^2x\ dx = \frac{1}{2}dz$$

Solved – (v):

$$Let,\ \ I = \int \frac{dx}{\sqrt{a^2-x^2}}$$
$$= \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\, d\theta$$
$$= \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\, d\theta$$
$$= \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\, d\theta$$
$$= \int \frac{\cos\theta}{\cos\theta}\, d\theta$$
$$= \int d\theta$$
$$= \theta + c = \sin^{-1}\frac{x}{a} + c$$

Let, $x = a\sin\theta$

$$\Rightarrow \frac{dx}{d\theta} = a\cos\theta$$
$$\Rightarrow dx = a\cos\theta\, d\theta$$

$Again,\ a\sin\theta = x$

$$\Rightarrow \theta = \sin^{-1}\frac{x}{a}$$

Find the value of the following integration:

$i)\ \int \frac{x^2+\sin^2x}{1+x^2}\sec^2x\, dx$ $\quad ii)\ \int \frac{\cos x}{5+4\cos^2x}\, dx$ $\quad iii)\ \int \frac{dx}{(1+x^2)\sqrt{1-(\tan^{-1}x)^2}}$

$iv)\ \int \frac{x^4}{\sqrt{x^{10}+1}}\, dx$

Solved – (i):

$$Let,\ \ I = \int \frac{x^2+\sin^2x}{1+x^2}\sec^2x\, dx$$
$$= \int \frac{x^2+1-\cos^2x}{1+x^2}\sec^2x\, dx$$
$$= \int \frac{(1+x^2)-\cos^2x}{1+x^2}\sec^2x\, dx$$
$$= \int \frac{(1+x^2)\sec^2x}{1+x^2}\, dx - \int \frac{\cos^2x.\sec^2x}{1+x^2}\, dx$$
$$= \int \sec^2x\, dx - \int \frac{1}{1+x^2}\, dx$$
$$= \tan x - \tan^{-1}x + c$$

Solved – (ii):

$$Let,\ \ I = \int \frac{cosx}{5+4cos^2x}\,dx$$
$$= \int \frac{cosx}{5+4(1-sin^2x)}\,dx$$
$$= \int \frac{cosx}{5+4-4sin^2x}\,dx$$
$$= \int \frac{cosx}{9-\ 4sin^2x}\,dx$$
$$= \int \frac{cosx}{4(\frac{9}{4}-\ sin^2x)}\,dx$$
$$= \frac{1}{4}\int \frac{cosx}{(\frac{3}{2})^2-\ sin^2x}\,dx$$

Let, $sinx = z$

$$\Rightarrow \frac{d}{dx}(sinx) = \frac{dz}{dx}$$
$$\Rightarrow cosx\,dx = dz$$

$$= \frac{1}{4}\int \frac{dz}{(\frac{3}{2})^2-\ z^2};\qquad \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$
$$= \frac{1}{4}\left[\frac{1}{2\times\frac{3}{2}}\ ln\left|\frac{\frac{3}{2}+z}{\frac{3}{2}-z}\right|\right] + c$$
$$= \frac{1}{12}\ ln\left|\frac{3+2z}{3-2z}\right| + c$$
$$= \frac{1}{12}\ ln\left|\frac{3+2sinx}{3-2sinx}\right| + c. \text{ Ans.}$$

Solved – (iii):

$$Let,\ \ I = \int \frac{dx}{(1+x^2)\sqrt{1-(tan^{-1}x)^2}}$$

Let, $tan^{-1}x = z$

$$\Rightarrow \frac{1}{1+x^2}\,dx = dz$$

$$= \int \frac{dz}{\sqrt{1-z^2}}$$
$$= sin^{-1}z + c$$
$$= sin^{-1}(tan^{-1}x) + c. \text{ Ans.}$$

Solved – (iv):

$$Let,\ \ I = \int \frac{x^4}{\sqrt{x^{10}+1}}\,dx$$

$$= \int \frac{x^4}{\sqrt{(x^5)^2+1}}\, dx$$

$$= \frac{1}{5}\int \frac{dz}{\sqrt{z^2+1}}$$

$$= \frac{1}{5}\, ln\left|z+\sqrt{z^2+1}\right| + c$$

$$= \frac{1}{5}\, ln\left|x^5+\sqrt{(x^5)^2+1}\right| + c$$

$$= \frac{1}{5}\, ln\left|x^5+\sqrt{x^{10}+1}\right| + c \text{ . Ans.}$$

Let, $x^5 = z$

$$\Rightarrow 5x^4dx = dz$$

$$\Rightarrow x^4dx = \frac{1}{5}dz$$

Find the value of the following integration:

$i)\ \int \frac{tanx}{\ln(cosx)}\, dx \quad ii)\ \int \frac{\tan(lnx)}{x}\, dx \quad iii)\ \int \frac{(sin^{-1}x)}{\sqrt{1-x^2}}\, dx$

<u>Solved – (i):</u>

$$Let,\ I = \int \frac{tanx}{\ln(cosx)}\, dx$$

$$= \int \frac{-dz}{z}$$

$$= -\int \frac{1}{z}\, dz$$

$$= -lnz + c$$

$$= -\ln(\ln(cosx)) + c. \text{ Ans.}$$

Let, $lncosx = z$

$$\Rightarrow \frac{1}{cosx}.(-sinx) = \frac{dz}{dx}$$

$$\Rightarrow tanx\ dx = -dz$$

<u>Solved – (ii):</u>

$$Let,\ I = \int \frac{\tan(lnx)}{x}\, dx$$

$$= \int tanz\ dz$$

$$= \ln|cosz| + c$$

$$= \ln|\cos(lnx)| + c.$$

Let, $lnx = z$

$$\Rightarrow \frac{d}{dx}(lnx) = \frac{dz}{dx}$$

$$\Rightarrow \frac{1}{x}dx = dz$$

<u>Solved – (iii):</u>

$$Let,\ I = \int \frac{(sin^{-1}x)^2}{\sqrt{1-x^2}}\, dx$$

$$= \int z^2\, dz$$

$$= \frac{z^{2+1}}{2+1} + c$$

$$= \frac{1}{3}\, z^3 + c$$

$$= \frac{1}{3}\,(sin^{-1}x)^3 + c.$$

Let, $sin^{-1}x = z$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dz$$

Find the value of the following integration:

$i)\ \int \frac{cosx}{cosx+sinx}\,dx \quad ii)\ \int \frac{dx}{2x^2+x+1} \quad iii)\ \int \frac{dx}{4+5sinx}$

<u>Solved – (i):</u>

$$Let,\ \ I = \int \frac{cosx}{cosx+sinx}\,dx$$

$$= \frac{1}{2}\int \frac{cosx+sinx+cosx-sinx}{cosx+sinx}$$

$$= \frac{1}{2}\int \frac{cosx+sinx}{cosx+sinx}\,dx + \frac{1}{2}\int \frac{cosx-sinx}{sinx+cosx}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\,ln|sinx+cosx|$$

$$= \frac{1}{2}x + \frac{1}{2}\,ln|sinx+cosx| + c.\ \text{Ans.}$$

Let, $sinx + cosx = z$

$$\Rightarrow (cosx - sinx)dx = dz$$

<u>Solved – (ii):</u>

$$Let,\ \ I = \int \frac{dx}{2x^2+x+1}$$

$$= \int \frac{dx}{2\left(x^2+\frac{x}{2}+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\int \frac{dx}{x^2+2.\frac{1}{4}.x+\left(\frac{1}{4}\right)^2+\frac{1}{2}-\left(\frac{1}{4}\right)^2}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{\frac{\sqrt{7}}{4}} tan^{-1} \frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}}\right] + c$$

$$= \frac{2\times 4}{\sqrt{7}} tan^{-1} \frac{x+\frac{1}{4}}{\sqrt{7}} + c$$

$$= \frac{8}{\sqrt{7}} tan^{-1} \frac{x+\frac{1}{4}}{\sqrt{7}} + c. \text{ Ans.}$$

<u>Solved – (iii):</u>

Let, $I = \int \frac{dx}{4+5sinx}$ Formula: $\boldsymbol{sin2x = \frac{2tanx}{1+tan^2x}}$

$$= \int \frac{dx}{4+5\frac{2tan\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int \frac{dx}{\frac{4+4tan^2\frac{x}{2}+10tan\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int \frac{sec^2\frac{x}{2}dx}{4tan^2\frac{x}{2}+10tan\frac{x}{2}+4}$$

$$= \int \frac{2dz}{4z^2+10z+4}$$

Let, $tan\frac{x}{2} = z$

$$\Rightarrow \frac{1}{2}sec^2\frac{x}{2} = \frac{dz}{dx}$$

$$\Rightarrow sec^2\frac{x}{2}dx = 2dz$$

$$= \int \frac{2dz}{4\left(z^2+\frac{10}{4}z+1\right)}$$

$$= \frac{1}{2}\int \frac{dz}{z^2+\frac{5}{2}z+1}$$

$$= \frac{1}{2}\int \frac{dz}{z^2+2\frac{5}{4}z+\left(\frac{5}{4}\right)^2+1-\left(\frac{5}{4}\right)^2}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z+\frac{5}{4}\right)^2+1-\frac{25}{16}}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

Assistant Professor & Coordinator,
Dept. of CSE,
Uttara University

$$= \frac{1}{2}\left[\frac{1}{2\times\frac{3}{4}}\ ln\left|\frac{z+\frac{5}{4}-\frac{3}{4}}{z+\frac{5}{4}+\frac{3}{4}}\right|\right] + c$$

$$= \frac{1}{2}\left[\frac{2}{3}\ ln\left|\frac{4z+5-3}{4z+5+3}\right|\right] + c$$

$$= \frac{1}{3}\ ln\left|\frac{4tan\frac{x}{2}+2}{4tan\frac{x}{2}+8}\right| + c, \text{ Ans.}$$

Find the value of the following integration:

$i)\ \int \frac{dx}{5+4cos2x}$ $\quad ii)\ \int \frac{dx}{5+3cosx}$ $\quad iii)\ \int \frac{dx}{3+2cosx}$

<u>Solved – (i):</u>

$$Let,\ \ I = \int \frac{dx}{5+4cos2x}$$

Formula: $\boldsymbol{cos2x = \frac{1-tan^2x}{1+tan^2x}}$

$$= \int \frac{dx}{5+4\frac{1-tan^2x}{1+tan^2x}}$$

$$= \int \frac{dx}{5+\frac{4-4tan^2x}{1+tan^2x}}$$

$$= \int \frac{dx}{\frac{5+5tan^2x+4-4tan^2x}{1+tan^2x}}$$

$$= \int \frac{1+tan^2x}{9+tan^2x}\ dx$$

$$= \int \frac{sec^2x}{9+tan^2x}\ dx$$

$$= \int \frac{dz}{(3)^2+z^2}$$

$$= \frac{1}{3}tan^{-1}\frac{z}{3} + c$$

$$= \frac{1}{3}tan^{-1}\left(\frac{tanx}{3}\right) + c$$

Let, $tanx = z$

$$\Rightarrow sec^2x = \frac{dz}{dx}$$

$$\Rightarrow sec^2xdx = dz$$

Solved – (ii):

$$Let,\ \ I = \int \frac{dx}{5+3cosx}$$

$$= \int \frac{dx}{5+3\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int \frac{dx}{5+\frac{3-3tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int \frac{dx}{\frac{5+5tan^2\frac{x}{2}+3-3tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int \frac{sec^2\frac{x}{2}}{8+2tan^2\frac{x}{2}}\ dx$$

$$= \frac{1}{2}\int \frac{sec^2\frac{x}{2}}{4+tan^2\frac{x}{2}}\ dx$$

$$= \frac{1}{2}\int \frac{2dz}{2^2+z^2}$$

Let, $tan\frac{x}{2} = z$

$$\Rightarrow sec^2\frac{x}{2}dx = 2dz$$

$$= \int \frac{dz}{2^2+z^2}$$

$$= \left[\frac{1}{2}tan^{-1}\frac{z}{2}\right]$$

$= \frac{1}{2}tan^{-1}\left(\frac{tan\frac{x}{2}}{2}\right) + c$, Ans.

Solved – (iii):

$$Let,\ \ I = \int \frac{dx}{3+2cosx}$$

$$= \int \frac{dx}{3+2\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int \frac{dx}{3+\frac{2-2tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int \frac{\boldsymbol{dx}}{\frac{\boldsymbol{3+3tan^2\frac{x}{2}+2-2tan^2\frac{x}{2}}}{\boldsymbol{1+tan^2\frac{x}{2}}}}$$

$$= \int \frac{\boldsymbol{sec^2\frac{x}{2}}}{5+\boldsymbol{tan^2\frac{x}{2}}}\ dx$$

$$= \int \frac{2dz}{\sqrt{5}+\boldsymbol{z^2}}$$

$$= \frac{2}{\sqrt{5}} tan^{-1} \frac{z}{\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}} tan^{-1} \left(\frac{tan\frac{x}{2}}{\sqrt{5}}\right) + c$$

Let, $tan^{-1}x\,\frac{x}{2} = z$

$$\Rightarrow sec^2\frac{x}{2}dx = 2dz$$