Limit of a function

Concept of Limit: The most basic use of limit is to describe how a function behave as the independent variable approaches a given value.

Definition of Limit: If the values of *f(x)* can be made as close as we like to *l* by taking values of *x* sufficiently close to *a* (but not equal to *a*), then we write

$$\lim_{x \to a} f(x) = l$$

which is read "the limit of *f(x)* as *x* approaches *a* is *l* " or "*f(x)* approaches *l* as *x* approaches *a*."

One-sided limits: If the values of *f(x)* can be made as close as we like to *l* by taking values of *x* sufficiently close to *a* (but greater than *a*), then we write

$$\lim_{x \to a^+} f(x) = l \qquad (1)$$

and if the values of *f(x)* can be made as close as we like to *l* by taking values of *x* sufficiently close to *a* (but less than *a*), then we write

$$\lim_{x \to a^-} f(x) = l \qquad (2)$$

Expression (1) is read "the limit of *f(x)* as *x* approaches *a* from the right is *l*".
Similarly, expression (2) is read "the limit of *f(x)* as *x* approaches *a* from the left is *l*".

Problem: Show that, $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, where n is a rational number.

Solution: Let, $x = a + h$, when $x \to a$, then $h \to 0$

Then, $\lim_{x \to a} \frac{x^n - a^n}{x - a}$

$$= \lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a}$$

$$= \lim_{h \to 0} \frac{a^n(1+\frac{h}{a})^n - a^n}{h}$$

$$= \lim_{h \to 0} \frac{a^n}{h}[\left(1+\frac{h}{a}\right)^n]$$

$$= \lim_{h \to 0} \frac{a^n}{h}[1 + \frac{nh}{a} + \frac{n(n-1)}{2!}\frac{h^2}{a^2} + \frac{n(n-1)(n-2)}{3!}\frac{h}{a^3}^3 + \cdots \ldots \ldots \ldots \ldots . . -1]$$

$$= \lim_{h \to 0} \frac{a^n}{h}[\frac{nh}{a} + \frac{n(n-1)}{2!}\frac{h^2}{a^2} + \frac{n(n-1)(n-2)}{3!}\frac{h}{a^3}^3 + \cdots \ldots \ldots \ldots \ldots . .]$$

$$= \lim_{h \to 0} \frac{a^n}{h} \times \frac{nh}{a}.[1 + \frac{(n-1)}{2!}\frac{h}{a} + \frac{(n-1)(n-2)}{3!}\frac{h}{a^2}^2 + \cdots \ldots \ldots \ldots \ldots ..]$$

$$= \lim_{h \to 0} \frac{a^n}{h}\frac{nh}{a}. \lim_{h \to 0}[1 + \frac{(n-1)}{2!}\frac{h}{a} + \frac{(n-1)(n-2)}{3!}\frac{h}{a^2}^2 + \cdots \ldots \ldots \ldots \ldots . .]$$

$= \frac{a^n}{1} . \frac{n}{a} . 1 = na^{n-1}$ (Showed).

Problem: Find the value of the following Limit, $\lim_{x \to 2} \frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

Solution: $\lim_{x \to 2} \frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

$$= \lim_{x \to 2} \frac{(x^2-4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{\left(\sqrt{x+2}-\sqrt{3x-2}\right)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}$$

$$= \lim_{x \to 2} \frac{(x^2-4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{\left(\sqrt{x+2}\right)^2-\left(\sqrt{3x-2}\right)^2}$$

$$= \lim_{x \to 2} \frac{(x^2-4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{x+2-3x+2}$$

$$= \lim_{x \to 2} \frac{(x^2-4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{4-2x}$$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2(x-2)}$$

$$= \lim_{x \to 2} \frac{(x+2)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2}$$

$$= \lim_{x \to 2} \frac{(2+2)\left(\sqrt{2+2}+\sqrt{3\times 2-2}\right)}{-2}$$

$$= \frac{4(\sqrt{4}+\sqrt{4})}{-2}$$

$$= \frac{4(2+2)}{-2}$$

$$= -2(2+2)$$

$$= -2 \times 4 = -8 \text{ (Ans.)}$$

Problem: Find the value of the following Limit, $\lim_{x \to 1} \frac{\mathrm{x}-\sqrt{(2-\mathrm{x}^2)}}{2x-\sqrt{(2+2\mathrm{x}^2)}} = 2$

Solution:

$$= \lim_{x \to 1} \frac{\mathrm{x}-\sqrt{(2-\mathrm{x}^2)}}{2x-\sqrt{(2+2\mathrm{x}^2)}}$$

$$= \lim_{x \to 1} \frac{\{\mathrm{x}-\sqrt{(2-\mathrm{x}^2)}\}\{2x+\sqrt{(2+2\mathrm{x}^2}\}\{\mathrm{x}+\sqrt{(2-\mathrm{x}^2}\}}{\{2x-\sqrt{(2+2\mathrm{x}^2)}\}\{\mathrm{x}+\sqrt{(2-\mathrm{x}^2}\}\{2x+\sqrt{(2+2\mathrm{x}^2}\}}.$$

Prove that, the following limit, i) $\lim_{x \to 0} \frac{\mathrm{e}^x-1}{x} = 1$; ii) $\lim_{x \to 0} \frac{1}{x} \log(1+x) = 1$

Solution (i):

Let, $\lim_{x \to 0} \frac{\mathrm{e}^x-1}{x}$

$$= \lim_{x \to 0} \frac{1}{x}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \ldots \ldots - 1\right]$$

$= \lim_{x\to 0} \frac{1}{x} [\, x + \frac{x^2}{2!} \; + \frac{x^3}{3!} + \frac{x^4}{4!} \; + \cdots \ldots \ldots \,]$

$= \lim_{x\to 0} \frac{1}{x} \times x \,[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} \; + \cdots \ldots \ldots]$

$= \lim_{x\to 0} [1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} \; + \cdots \ldots \ldots]$

$= \; 1 \; + \frac{0}{2!} \; + \frac{0}{3!} + \frac{0}{4!} \; + \cdots \ldots \ldots$

= 1 (Proved)

Solution (ii):

Let, $\lim_{x\to 0} \frac{1}{x} \log(1+x)$

$= \lim_{x\to 0} \frac{1}{x} [\, x - \frac{x^2}{2!} \; + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots \,]$

$= \lim_{x\to 0} \frac{1}{x} \times x \,[1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} \; + \cdots]$

$= \lim_{x\to 0} [1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} \; + \cdots]$

$= \; 1 - \frac{0}{2!} \; + \frac{0}{3!} - \frac{0}{4!} \; + \cdots \ldots \ldots$

= 1 (proved).

Find the value of the following Limit

1. $\lim_{x\to 2} \left(\frac{\sqrt{(x+7)}-3}{x-2}\right)$; 2. $\lim_{x\to 1} \left(\frac{\sqrt{x+3}-2}{x-1}\right)$; 3. $\lim_{x\to -4} \left(\frac{2x+8}{x^2+x-12}\right)$; 4. $\lim_{x\to 0} \left(\frac{\sqrt{1+x}-1}{x}\right)$;

5. $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$; 6. $\lim_{x\to 0} \frac{\sqrt{4-x}-2}{x}$; 7. $\lim_{x\to 0} \frac{3x}{\sqrt{9-x}-3}$; 8. $\lim_{x\to \alpha} \frac{x^5}{e^x}$

9. $\lim_{x\to \infty} \left(\frac{2^x-2^{-x}}{2^x+2^{-x}}\right)$; 10. $\lim_{x\to \infty} \left(\frac{5^x-5^{-x}}{5^x+5^{-x}}\right)$; 11. $\lim_{x\to \infty} \frac{x^2}{(x-1)(x-2)}$; 12 $\lim_{x\to \infty} \frac{x^4}{e^x}$; 13. $\lim_{x\to 0} (1+x)^{1/x}$=e

<u>Solution</u>: (1) $\lim_{x\to 2} \left(\frac{\sqrt{(x+7)}-3}{x-2}\right)$

$=\lim_{x\to 2} \frac{\{\sqrt{(x+7)}-3\}\{\sqrt{(x+7)}+3\}}{(x-2)\{\sqrt{(x+7)}+3\}}$

$=\lim_{x\to 2} \frac{\{\sqrt{(x+7)}\}^2-(3)^2}{(x-2)\{\sqrt{(x+7)}+3\}}$

$=\lim_{x\to 2} \frac{x+7-9}{(x-2)\{\sqrt{(x+7)}+3\}}=\lim_{x\to 2} \frac{x-2}{(x-2)\{\sqrt{(x+7)}+3\}}=\lim_{x\to 2} \frac{1}{\{\sqrt{(x+7)}+3\}}$

$=\frac{1}{\sqrt{2+7}+3}=\frac{1}{6}$ Ans.....

2) $\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\right)$

$=\lim_{x\to 1}\frac{(\sqrt{x+3}+2)(\sqrt{x+3}-2)}{(x-1)(\sqrt{x+3}+2)}$

$=\lim_{x\to 1}\frac{(\sqrt{x+3})^2-(2)^2}{(x-1)(\sqrt{x+3}+2)}$

$=\lim_{x\to 1}\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}$

$=\lim_{x\to 1}\frac{(x-1)}{(x-1)(\sqrt{x+3}+2)}$

$=\lim_{x\to 1}\frac{1}{(\sqrt{x+3}+2)}$

$=\lim_{x\to 1}\frac{1}{\sqrt{1+3}+2}$

$=\frac{1}{2+2}$

$=\frac{1}{4}$ Ans.......

3) $\lim_{x\to -4}\left(\frac{2x+8}{x^2+x-12}\right)$

$=\lim_{x\to -4}\frac{2(x+4)}{x^2+4x-3x-12}$

$=\lim_{x\to -4}\frac{2(x+4)}{x(x+4)-3(x+4)}$

$=\lim_{x\to -4}\frac{2(x+4)}{(x+4)(x-3)}$

$=\lim_{x\to -4}\frac{2}{(x-3)}=\frac{2}{(-4-3)}=-\frac{2}{7}$ Ans.........

4) $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)$

$=\lim_{x\to 0}\frac{(\sqrt{1+x}+1)(\sqrt{1+x}-1)}{x(\sqrt{1+x}+1)}$

$=\lim_{x\to 0}\frac{(\sqrt{1+x})^2+1)}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1+x-1}{x(\sqrt{1+x}+1)}$

$=\lim_{x\to 0}\frac{x}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1}{(\sqrt{1+x})+1}$

$=\frac{1}{(\sqrt{1+0})+1}=\frac{1}{2}$ Ans……..

5) $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}$

$$= \lim_{x\to 0}\frac{\left(\sqrt{4+x}-2\right)\left(\sqrt{4+x}+2\right)}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0}\frac{(\sqrt{4+x})^2-(2)^2}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0}\frac{4+x-4}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0}\frac{1}{\left(\sqrt{4+x}+2\right)}$$

$$= \frac{1}{\sqrt{4+0}+2}$$

$= \frac{1}{2+2}= \frac{1}{4}$ (Ans:)

6) $\lim_{x\to 0}\frac{\sqrt{4-x}-2}{x}$

$$= \lim_{x\to 0}\frac{\left(\sqrt{4-x}-2\right)(\left(\sqrt{4-x}+2\right)}{x(\left(\sqrt{4-x}+2\right)}$$

$$= \lim_{x\to 0}\frac{(\sqrt{4-x})^2-(2)^2}{x\left(\sqrt{4-x}+2\right)}$$

$$= \lim_{x\to 0}\frac{4-x-4}{x\left(\sqrt{4-x}+2\right)}$$

$$= \lim_{x \to 0} \frac{-1}{\left(\sqrt{4-x}+2\right)}$$

$$= \frac{-1}{\sqrt{4-0}+2}$$

$$= \frac{-1}{2+2} \quad = -\frac{1}{4} \text{ (Ans:)}$$

7) $\lim_{x \to 0} \frac{3x}{\sqrt{9-x}-3}$

$$= \lim_{x \to 0} \frac{3x\left(\sqrt{9-x}+3\right)}{\left(\sqrt{9-x}-3\right)\left(\sqrt{9-x}+3\right)}$$

$$= \lim_{x \to 0} \frac{3x\left(\sqrt{9-x}+3\right)}{(\sqrt{9-x})^2-(3)^2}$$

$$= \lim_{x \to 0} \frac{3x\left(\sqrt{9-x}+3\right)}{9-x-9}$$

$$= \lim_{x \to 0} -3\left(\sqrt{9-x}+3\right)$$

$$= -3\left(\sqrt{9-0}+3\right)$$

$$= -3(3+3) \quad = -18 \text{ (Ans:)}$$

8) $\lim_{x \to \infty} \frac{x^5}{e^x}$

$$= \lim_{x \to \infty} \frac{x^5}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}}$$

$$= \lim_{x \to \infty} \frac{x^5}{x^5\left(\frac{1}{x^5}+\frac{1}{x^4}+\frac{1}{x^3 2!}+\frac{1}{x^2 3!}+\frac{1}{x4!}+\frac{1}{5!}+\frac{x}{6!}+\cdots\ldots..\right)}$$

$$= \lim_{x \to \infty} \frac{1}{\left(\frac{1}{x^5}+\frac{1}{x^4}+\frac{1}{x^3 2!}+\frac{1}{x^2 3!}+\frac{1}{x4!}+\frac{1}{5!}+\frac{x}{6!}+\cdots\ldots.\right)}$$

$$= \frac{1}{\frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{\infty}+\frac{1}{5!}+\frac{\infty}{6!}+\cdots}$$

$$= \frac{1}{\frac{1}{5!}+\infty} \quad = \frac{1}{\infty} = 0 \text{ (Ans:)}$$

9) $\lim_{x\to\infty} \frac{2^x-2^{-x}}{2^x+2^{-x}}$

$$= \lim_{x\to\infty} \frac{2^x(1-\frac{1}{2^{2x}})}{2^x(1+\frac{1}{2^{2x}})}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{(2^x)^2}}{1+\frac{1}{(2^x)^2}}$$

$$= \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}$$

$= 1$ Ans.

Exercises: Calculate the following limits

1.

$$\lim_{x\to 2} \frac{x^2-1}{x-1}$$

2.

$$\lim_{x\to -\infty} \frac{3-x}{\sqrt{x^2+3x}}$$

3.

$$\lim_{x\to 2^+} \frac{x-2}{|x-2|}$$

4.

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$$

5.

$$\lim_{x \to 0} \frac{\tan x - x}{\sin x}$$

6.

$$\lim_{x \to 0} \frac{\sin 4x}{\sin x}$$

Solutions to Above Exercises: 1) 3 ,2) 1 ,3) 1 ,4) 1/4 ,5) 0 ,6) 4