Walles formula

Statement: If n is a positive integer then

$$I \quad = \int_0^{\frac{\pi}{2}} sin^n x\, dx = \int_0^{\frac{\pi}{2}} cos^n x\, dx$$

$$= \frac{(n-1)(n-3)(n-5)\ldots\ldots\ldots\ldots 5.3.1}{n(n-2)(n-4)\ldots\ldots\ldots\ldots\ldots\ldots 6.4.2} \times \frac{\pi}{2} \quad when\ n\ is\ even$$

$$= \frac{(n-1)(n-3)(n-5)\ldots\ldots\ldots\ldots 6.4.2}{n(n-2)(n-4)\ldots\ldots\ldots\ldots\ldots\ldots 5.3.1} \quad when\ n\ is\ odd$$

Problem: Find the value by using walles formula:

$i)\ \int_0^{\frac{\pi}{2}} sin^7 x\, dx$ $\quad ii)\ \int_0^{\pi} sin^9 x\, dx$ $\quad iii)\ \int_0^{\pi} sin^8 x\, dx$ $\quad iv)\ \int_0^{\frac{\pi}{2}} cos^6 x\, dx$

$v)\ \int_0^{\frac{\pi}{2}} cos^7 x\, dx.$

Solved – (i): **Let,**

$$I \ = \int_0^{\frac{\pi}{2}} sin^7 x\, dx \qquad Here, n = 7\ which\ is\ odd$$

$$so,\ \ I \ = \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)}$$

$$= \frac{48}{105}$$

$$= \frac{16}{35}$$

Solved – (ii): $Let, I = \int_0^{\pi} sin^9 x\, dx$

$$= 2\int_0^{\frac{\pi}{2}} sin^9 x\, dx$$

$$Here, n = 9\ which\ is\ odd$$

$$so, I = 2\ \frac{(9-1)(9-3)(9-5)(9-7)}{9(9-2)(9-4)(9-6)(9-8)}$$

$$= 2\ \frac{348}{945} = \frac{256}{215}$$

Solved – (iii): $Let, I = \int_0^{\pi} sin^8 x\, dx$

$$= 2\int_0^{\frac{\pi}{2}} sin^8 x\, dx$$

$$Here, n = 8\ which\ is\ even\ number$$

$$so, I = 2 \times \frac{(8-1)(8-3)(8-5)(8-7)}{8(8-2)(8-4)(8-6)} \times \frac{\pi}{2}$$

$$= 2 \times \frac{7.5.3.1}{8.6.4.2} \times \frac{\pi}{2}$$

$$= \frac{35\pi}{128}$$

Solved – (iv): $Let, I = \int_0^{\frac{\pi}{2}} cos^7 x\, dx$

$$Here, n = 7 \text{ which is odd}$$

$$so, I = \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)} = \frac{48}{105} = \frac{16}{35}.$$

Gamma and Beta function:

Beta Function: An integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, where $m > 0, n > 0$ is called *beta function* and is denoted by $\beta(m, n)$.

Gamma Function: An integral $\int_0^\infty e^{-x}x^{n-1}dx$, where $n > 0$ is called *gamma function* and is denoted by Γn.

Problem-1: *Show that,*

$$i)\ \int_0^\infty e^{-y^{\frac{1}{n}}}dy = \Gamma(n+1); \quad ii)\ \beta(m,n) = \beta(n,m)$$

Solved – (i):

$$L.H.S, \quad I = \int_0^\infty e^{-y^{\frac{1}{n}}}dy$$

$$= \int_0^\infty e^{-x}nx^{n-1}dx$$

$$= n\int_0^\infty e^{-x}x^{n-1}dx$$

$$= n\Gamma n$$

Let, $y^{\frac{1}{n}} = x$

$\Rightarrow y = x^n$

$\Rightarrow dy = nx^{n-1}dx$

x	0	∞
y	0	∞

$$\therefore \int_0^\infty e^{-y^{\frac{1}{n}}}dy = \Gamma(n+1) \qquad (Showed)$$

Solved – (ii):

We know,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= -\int_1^0 (1-y)^{m-1}y^{n-1}dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

Let, $1-x=y$
$\Rightarrow 0-dx=dy$
$\Rightarrow dx=-dy$

x	0	1
y	1	0

$\therefore\ \beta(m,n) = \beta(n,m)$ $\quad$ *(Showed)*

Problem -2: *Show that,*

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

Solved: *We know,*

$$\Gamma m = \int_0^\infty e^{-x}x^{m-1}dx$$

$$= \int_1^\infty e^{-\lambda y}.(\lambda y)^{m-1}\lambda dy$$

Let, $x=\lambda y$
$\Rightarrow dx=\lambda dy$

x	0	∞
y	1	∞

$$= \int_1^\infty e^{-\lambda y}\lambda^{m-1}y^{m-1}\lambda dy$$

$$= \int_1^\infty e^{-\lambda y}y^{m-1}\lambda^{m}\lambda dy \quad \ldots\ldots\ldots(i)$$

again, $\Gamma n = \int_0^\infty e^{-\lambda}\lambda^{n-1}d\lambda \quad \ldots\ldots\ldots(ii)$

Multiplying (i) *and* (ii) *we get,*

$$\Gamma m\Gamma n = \int_0^\infty\int_0^\infty e^{-\lambda y}e^{-\lambda}\, y^{m-1}\lambda^{n-1}\lambda^m d\lambda dy$$

$$= \int_0^\infty\left[\int_0^\infty e^{-\lambda(1+y)}\,\lambda^{m+n-1}d\lambda\right]y^{m-1}dy$$

$$= \int_0^\infty \frac{\Gamma(m+n)}{(y+1)^{m+n}}y^{m-1}dy$$

$$= \Gamma(m+n)\int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}dy$$

$$\Gamma m\Gamma n = \Gamma(m+n).\beta(m,n)$$

$$\therefore\ \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\quad (Showed).$$

Problem-3: *Show that,*

$$\int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}.$$

Solved:

We know,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \int_0^{\frac{\pi}{2}}(sin^2\theta)^{m-1}(1-sin^2\theta)^{n-1}2sin\theta cos\theta d\theta =$$

$$2\int_0^{\frac{\pi}{2}} sin^{2m-2}\theta cos^{2n-2}\theta sin\theta cos\theta d\theta$$

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$

Let, $x = sin^2\theta$ $\Rightarrow dx = 2sin\theta cos\theta d\theta$		
x	0	1
y	0	$\pi/2$

Again, $2m-1 = p$ $\qquad 2n-1 = q$

$$m = \frac{p+1}{2} \qquad n = \frac{q+1}{2}$$

Therefore,

$$\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

$$\therefore \int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}.\ \text{(Showed).}$$

Problem-4: Show that, $\int_0^{\frac{\pi}{2}} sin^p\theta d\theta = \int_0^{\frac{\pi}{2}} cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{1}{2}\right)$

$$= \frac{1}{2}\beta\left(\frac{1}{2},\frac{p+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}, where\ p > -1.$$

Solve: We know,

$$\int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) \ldots\ldots\ldots\ldots\ldots(1)$$

Putting $q = 0$, then we get

$$\int_0^{\frac{\pi}{2}} sin^p\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{1}{2}\right) \ldots\ldots\ldots\ldots(2)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} sin^p\left(\frac{\pi}{2}-\theta\right)d\theta = \frac{1}{2}\beta\left(\frac{1}{2},\frac{p+1}{2}\right)$$

[Since, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and $\beta(m,n) = \beta(n,m)$]

$$\Rightarrow \int_0^{\frac{\pi}{2}} cos^p\theta d\theta = \frac{1}{2}\beta\left(\frac{1}{2},\frac{p+1}{2}\right) \ldots\ldots\ldots\ldots(3)$$

Again, $\frac{1}{2}\beta\left(\frac{1}{2},\frac{p+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

From (2), (3) & (4) we get,

$$\int_0^{\frac{\pi}{2}} sin^p\theta d\theta = \int_0^{\frac{\pi}{2}} cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{2},\frac{p+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}, where\ p > -1.$$ (showed)

Problem-5: Show that, $\Gamma\frac{1}{2} = \sqrt{\pi}$

Solved:

We know, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$= \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx$$

$$= \int_0^1 \frac{1}{\sqrt{x(1-x)}}dx$$

Let, $1 - x = y$

$\Rightarrow dx = -dy$

x	0	1
y	1	0

$$= \int_0^1 \frac{1}{\sqrt{(x-x^2)}}dx$$

$$= \int_0^1 \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}dx$$

$$= \left[sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= sin^{-1}\left(\frac{1-\frac{1}{2}}{\frac{1}{2}}\right) - sin^{-1}\left(\frac{0-\frac{1}{2}}{\frac{1}{2}}\right)$$

$$= sin^{-1}\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) - sin^{-1}\left(\frac{-\frac{1}{2}}{\frac{1}{2}}\right)$$

$$= sin^{-1}(1) - sin^{-1}(-1)$$

$$= sin^{-1}(1) + sin^{-1}(1)$$

$$= 2sin^{-1}(1)$$

$$= 2 \times \frac{\pi}{2}$$

$$\Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \pi$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1} = \pi;$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi \ [\because \Gamma 1 = 1]$$

$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, (Hence proved).

Problem-6: Find the value,

$$\text{(i)}\Gamma\left(\frac{3}{2}\right) \ \text{(ii)}\Gamma\left(\frac{5}{2}\right) \ \text{(iii)}\Gamma\left(\frac{7}{2}\right) \ \text{(iv)}\Gamma\left(-\frac{1}{2}\right) \ \text{(v)}\Gamma\left(-\frac{3}{2}\right)$$

Solution: We know, $\Gamma(n+1) = n\Gamma n$

(i)$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

(ii)$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$.

(ii) $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$.

Again, $\Gamma n = \frac{\Gamma(n+1)}{n}$

(iv) $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \left(-\frac{2}{1}\right)\sqrt{\pi} = -2\sqrt{\pi}$.

(v) $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \left(-\frac{2}{3}\right)\Gamma\left(-\frac{1}{2}\right) = \left(-\frac{2}{3}\right)\left(-2\sqrt{\pi}\right) = \frac{4\sqrt{\pi}}{3}$.