Rank of Matrix

Echelon Matrix: A Matrix A is called an echelon Matrix or is said to be in echelon form. If the following conditions hold (Where a leading non-zero element of a row of A is the first non-zero element in the row).

i) All Zero rows if any are at the bottom of the matrix.

ii) The leading non-zero element (pivot) in any row is farther to the right than the leading non-zero element in the just above it.

iii) In each column containing a leading non-zero elements (pivot) the entries below that leading non-zero element are 0.

Example: The echelon form are following the Matrix whose pivot have been circled.

i) $\begin{bmatrix} 2 & 1 & 0 & 5 & 7 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -11 \\ 0 & 0 & 0 & 17 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row Canonical form: A matrix A is said to be in row canonical form if it is an echelon matrix and if it satisfies the following conditions:

i) Each pivot (leading non-zero entry) is 1.

ii) Each pivot is the only non-zero entry in its column.

Example:

i) $\begin{bmatrix} 2 & 3 & 5 & 0 & 7 & 8 \\ 0 & 0 & 1 & -3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

The 3rd Matrix is an example of a Matrix in row canonical form. The second Matrix is not row canonical form. Since, there is a non-zero element above the 2nd pivot in the 3rd column. The First Matrix is not in row canonical form. Since some pivot are not equal to 1 and there are non-zero elements above the pivots.

Note: i) The major difference between an echelon matrix and a matrix in row canonical form is that in an echelon matrix there must be zeros below the pivot, but in a matrix in row canonical form. Each pivot is equal to 1 and there must also be zeros above the pivots.

ii) The identify matrix I of any size is important special example of the matrix in row canonical form.

Rank of Matrix : The rank of a matrix A is the maximum number of linearly independent rows or columns in the matrix.

or, Let A be an matrix of order $m \times n$. The rank of the matrix A is the largest value of r for which there exists an $r \times r$ sub matrix of A with non-zero determinant.

or, Let A be an $m \times n$ matrix and let A_R be the row echelon form of A. Then the rank of A is the number non-zero rows of A_C.

Note: i) The rank of a matrix A is denoted by rank (A) or p (A).

ii) The rank of null matrix (zero matrix) is zero and the rank of a matrix of order $m \times n$

cannot be greater than m or n.

iii) An n-rowed square matrix A has a rank $r < n$ if $|A| = 0$. In this a is called a singular matrix, The matrix A has rank $r = n$ if $|A| \neq 0$ and is then called a non-singular matrix.

Example: The matrix $A = \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has two Linearly independent rows. So it rank is 2.

Row Rank of Matrix:

The row rank of a matrix A is the maximum number of linearly independent rows of A.

OR

The number of non-zero rows in the row echelon form of a matrix A is called the rank of the matrix A.

Example:

i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -3 \\ 1 & 1 & -3 \end{bmatrix}$ Row rank =3

ii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ Row rank =4

iii) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Row rank =2

Column Rank of Matrix: The column rank of a matrix A is the maximum number of linearly independent column of A.

or, The number of non-zero column in column reduced form (echelon form) of a matrix A is called the column rank of A.

Example:

i) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & -2 & 0 \end{bmatrix}$ Column rank =3

ii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ -1 & 3 & 5 & 0 & 0 \\ -2 & 5 & 3 & 0 & 0 \\ -1 & 1 & 2 & 1 & 0 \end{bmatrix}$ Column rank =4

iii) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ Column rank =2

Canonical Matrix: A matrix in which all the terms of the principal diagonal are one and zero but not all zero and all one rows or columns are precedes all zeros row or column is called canonical matrix.

Example:

i) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Problem: Find the rank of each of the following matrixes.

i) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Solution-(i):

Let, $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Here, $A = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

So, The rank is less than 2.
But not every element of A is zero.
i.e $|2| \neq 0$
So, the matrix rank is 1.

Solution-(ii):

Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$

$= 1(21 - 20) - 2(14 - 12) + 3(10 - 9)$

$= 1 - 4 + 3$

$= 4 - 4$

$= 0$

So, The rank is less than 3.
Now, let us take two- rowed minor (sub matrix) of A. Say

$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 3 - 4 = -1 \neq 0$

Here $|A| = 0$ but $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \neq 0$

So, the rank is 2.

Solution-(iii):

Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix}$

$$= 1(36-36) - 2(18-18) + 3(12-12)$$
$$= 0 - 0 + 0$$
$$= 0$$

So, the rank of A is less than 3.

Let us consider the two-rowed sub matrix (minor) of A. Say

$\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36 - 36 = 0;$ $\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 18 - 18 = 0;$

$\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 12 - 12 = 0;$ $\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18 - 18 = 0$

$\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 9 - 9 = 0;$ $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$

$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0;$ $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0;$

So, All determinant of the sub matrix of A are zero.

Thus the rank of A is less than 2.

But $|4| \neq 0$ or $|6| \neq 0$.

Hence rank of A is 1.

Problem: Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \\ 0 & 7 & 10 \end{bmatrix}$

Solution:

Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \\ 0 & 7 & 10 \end{bmatrix}$

First we will reduce the matrix A to row echelon form by the elementary row operation.

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & 10 \\ 0 & 7 & 10 \end{bmatrix} \quad \acute{R}_2 = R_2 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & 10 \\ 0 & 0 & 0 \end{bmatrix} \quad \acute{R}_3 = R_3 - R_2$$

The above matrix is in echelon (row) form and it has two non-zero row.

So, rank of the given matrix is 2.

Problem: Find the echelon form and canonical form (row reduced echelon form) of the matrix
$$A=\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 2 & 4 & 1 & -2 & 3\\ 3 & 6 & 2 & -6 & 5\end{bmatrix}.$$

Solution: First we will reduce the matrix A to echelon form by the elementary row operations.

$$\text{Given, } A=\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 2 & 4 & 1 & -2 & 3\\ 3 & 6 & 2 & -6 & 5\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & -6 & 1\\ 0 & 0 & 5 & -12 & 2\end{bmatrix}\quad \begin{matrix}\acute{R}_2 = R_2 - 2R_1\\ \acute{R}_3 = R_3 - 3R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & -6 & 1\\ 0 & 0 & 0 & -6 & 1\end{bmatrix}\quad \acute{R}_3 = 3R_3 - 5R_2$$

The Matrix is in row echelon form and the rank of A is 3.

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}\quad \acute{R}_2 = \frac{1}{3}R_2$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 2 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}\quad \acute{R}_1 = R_1 + R_2$$

$$\sim\begin{bmatrix}1 & 2 & 0 & 0 & \frac{4}{3}\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -\frac{1}{6}\end{bmatrix}\quad \acute{R}_1 = R_1 + 2R_3$$

This matrix is an canonical form.

Problem: Find the rank of the following matrix.

i) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 0 \\ -1 & 0 & -2 \\ 3 & -1 & 7 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 4 & -1 & 5 \\ 6 & 3 & 1 & 2 \\ -5 & -1 & 6 & 4 \\ 8 & 7 & 0 & 7 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 2 \\ 3 & 3 & 3 & 3 \\ 5 & 3 & 5 & 3 \end{bmatrix}$

Problem: Reduce $A = \begin{bmatrix} 3 & -10 & 5 \\ -1 & 12 & -2 \\ 1 & -5 & 2 \end{bmatrix}$ echelon form. Hence find the rank of the matrix.

Solution: Let, $A = \begin{bmatrix} 3 & -10 & 5 \\ -1 & 12 & -2 \\ 1 & -5 & 2 \end{bmatrix}$

Step- I: $\sim \begin{bmatrix} 1 & -5 & 2 \\ -1 & 12 & -2 \\ 3 & -10 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

Step- II: $\sim \begin{bmatrix} 1 & -5 & 2 \\ 0 & 7 & 0 \\ 0 & 5 & -1 \end{bmatrix}$ $\acute{R_2} = R_2 + R_1$, $\acute{R_3} = R_3 - 3R_1$

Step- III: $\sim \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$ $\acute{R_2} = \frac{1}{7} R_2$

Step- IV: $\sim \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\acute{R_3} = R_3 - 5R_2$

Which is a matrix in echelon form.
The above echelon matrix has thus non-zero rows. Hence the rank of the matrix is 3.

Problem: Reduce the following matrix to the normal or canonical form and hence obtain its rank. $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

Solution: We will apply both elementary column and row operation to the matrix A for reducing it to the normal form.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 3 & -2 & 1 & 5\\ -2 & 7 & 2 & 3\end{bmatrix} \quad \begin{matrix}\acute{C_2} = C_2 - 2C_1\\ \acute{C_4} = C_4 + C_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 3 & 0 & 1 & 0\\ -2 & 11 & 2 & -7\end{bmatrix} \quad \begin{matrix}\acute{C_2} = C_2 + 2C_3\\ \acute{C_4} = C_4 - 5C_3\end{matrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 4 & 0 & 1 & 0\\ 0 & 11 & 2 & 0\end{bmatrix} \quad \begin{matrix}\acute{C_1} = C_1 + C_3\\ \acute{C_4} = C_4 + \dfrac{7}{11}C_2\end{matrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 11 & 2 & 0\end{bmatrix} \quad \acute{R_2} = R_2 - 4R_1$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 11 & 0 & 0\end{bmatrix} \quad \acute{R_3} = R_3 - 2R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 11 & 0\end{bmatrix} \quad C_2 \leftrightarrow C_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} \quad \acute{C_3} = \frac{1}{11}C_3$$

$$\sim\begin{bmatrix}I_3 & 0\end{bmatrix} \qquad \text{where } I_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ and } 0 = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Hence the rank of A is 3.

Problem: Reduce the following matrix to the normal or canonical form and hence obtain its rank. $A = \begin{bmatrix}0 & 2 & 3 & 4\\ 2 & 3 & 5 & 4\\ 4 & 8 & 13 & 12\end{bmatrix}$

Solution: Given, $A = \begin{bmatrix}0 & 2 & 3 & 4\\ 2 & 3 & 5 & 4\\ 4 & 8 & 13 & 12\end{bmatrix}$

We will apply both elementary column and row operation to the matrix A for reducing it to the normal form.

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 2 & 8 & 13 & 12 \end{bmatrix} \quad \acute{C_1} = \frac{1}{2}C_1$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \quad \acute{R_3} = R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \acute{R_3} = R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \acute{C_2} = C_2 - 3C_1 \\ \acute{C_3} = C_3 - 5C_1 \\ \acute{C_4} = C_4 - 4C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \acute{C_2} = \frac{1}{2}C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \acute{C_3} = C_3 - 3C_2 \\ \acute{C_4} = C_4 - 4C_2 \end{matrix}$$

$$\sim \begin{bmatrix} I_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is normal form of the given matrix and it has two non-zero rows.
Therefore, rank of the matrix is 2.

Problem: Find the rank of the following matrix:

i) $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & -3 & 0 & -2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 3 & 5 & 6 \\ 4 & 1 & -2 & 4 \\ -2 & 0 & 3 & 1 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 3 & -2 & -1 \\ 2 & 6 & -4 & -2 \\ 1 & 3 & -2 & 1 \\ 2 & 6 & 1 & -1 \end{bmatrix}$