Continuity of a Function

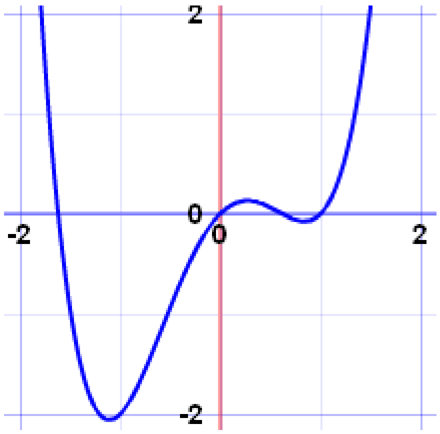
Continuous Functions

A function is continuous when its graph is a single unbroken curve

... that you could draw without lifting your pen from the paper.

That is not a formal definition, but it helps you understand the idea.

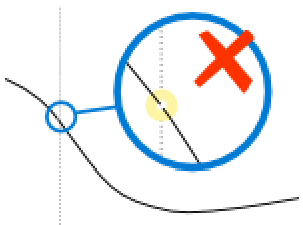
Here is a continuous function:

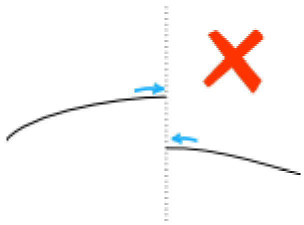


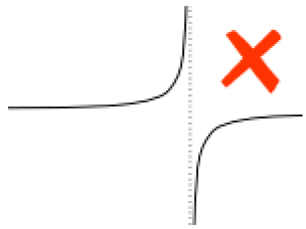
Examples

So what is **not continuous** (also called **discontinuous**) ?

Look out for holes, jumps or vertical asymptotes (where the function heads up/down towards infinity).







Not Continuous (hole) | **Not** Continuous (jump) | **Not** Continuous (vertical asymptote)

Continuity: A function f is said to be ***continuous*** at the point $x = a$, if the following conditions are satisfied:

i) If $f(a)$ is defined
ii) If left hand limit = Right hand limit
$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$
iii) If limiting values = Functional values.
$\lim_{x\to a} f(x) = f(a)$

OR

If $f(x)$ is defined $\lim_{x\to a} f(x)$exists and $\lim_{x\to a} f(x) = f(a)$, then $f(x)$ is said to be continuous at $x = 0$, Here $f(x)$ is the functional value of $f(x)$ at $x = 0$.

Problem: Show the function $\boldsymbol{f(x)}$ is continuous at $x = 0$ but is discontinuous at $x = \frac{3}{2}$,

where, $f(x) = \begin{cases} 3 + 2x & ; -\frac{3}{2} \le x < 0 \\ 3 - 2x & ; 0 \le x < \frac{3}{2} \\ -3 - 2x & ; x \ge \frac{3}{2} \end{cases}$

Tips: $x \to 0^- = x < 0;\ x \to 0^+ = x > 0;$

$x \to \frac{3}{2}^- = x < \frac{3}{2};\ x \to \frac{3}{2}^+ = x > \frac{3}{2}.$

Solve: For continuity at $\boldsymbol{x = 0}$

L.H.L$= \lim_{x\to 0-} f(x)$

$= \lim_{x\to 0-} (3 + 2x)$

$= 3 + 2 \times 0 = 3$

R.H.L$= \lim_{x\to 0+} f(x)$

$= \lim_{x\to 0+} (3 - 2x)$

$= 3 - 2 \times 0 = 3$

Here, $L.H.L = R.H.L$. So, $\lim_{x\to 0} f(x)$ exist.

Again at $x = 0$

$f(0) = 3 - 2 \times 0 = 3$

Here, $\lim_{x \to 0} f(x) = f(0) = 3$. So, given function is continuous at $x = 0$.

<u>For discontinuity at</u> $\boldsymbol{x = \frac{3}{2}}$

L.H.L= $\lim_{x \to \frac{3}{2}-} f(x)$

$= \lim_{x \to \frac{3}{2}-} (3 - 2x)$

$= 3 - 2 \times \frac{3}{2}$

$= 0$

R.H.L= $\lim_{x \to \frac{3}{2}+} f(x)$

$= \lim_{x \to \frac{3}{2}+} (-3 - 2x)$

$= -3 - 2 \times \frac{3}{2} = -6$

Here, L.H.L≠ R.H.L. So $\lim_{x \to \frac{3}{2}} f(x)$ dosen't exist. Hence given function is not continuous at $x = \frac{3}{2}$.

Problem: Show the function $\boldsymbol{f(x)}$ is continuous at $x = \frac{1}{2}$, where,

$$f(x) = \begin{cases} \frac{1}{2} - x; 0 \le x < \frac{1}{2} \\ \frac{1}{2}\ ; x = \frac{1}{2} \\ \frac{3}{2} - x\ ; \frac{1}{2} < x < 1 \end{cases}$$

<u>Tips:</u> $x \to \frac{1}{2}^{-} = x < \frac{1}{2}$; $x \to \frac{1}{2}^{+} = x > \frac{1}{2}$;

<u>Solve:</u> <u>For continuity at</u> $x = \frac{1}{2}$

L.H.L= $\lim_{x \to \frac{1}{2}-} f(x)$

$= \lim_{x \to \frac{1}{2}-} (\frac{1}{2} - x)$

$=\frac{1}{2}-\frac{1}{2}=0$

R.H.L= $\lim_{x\to\frac{1}{2}+} f(x)$

$= \lim_{x\to\frac{1}{2}+} (\frac{3}{2}-x)$

$=\frac{3}{2}-\frac{1}{2} \;=\frac{3-1}{2} \;=\frac{2}{2} \;=1$

Here, L.H.L≠ R.H.L. So $\lim_{x\to\frac{1}{2}} f(x)$ dosen't exist. Hence given function is not continuous at $x=\frac{1}{2}$.

Problem: Show that the function is discontinuous at x= 2

$$f(x)=\begin{cases} x^2+5\,;0\le x\le 2 \\ x^2-2\,;\;2\le x\ge 3 \end{cases}$$

Solution:

L.H.L = $\lim_{x\to 2-} f(x)$

= $\lim_{x\to 2-} (x^2+5)$

= $(2)^2$+ 5

= 4 + 5

= 9

R.H.L = $\lim_{x\to 2+} f(x)$

= $\lim_{x\to 2-} (x^2-2)$

= $(2)^2-2$

= 4 – 2

= 2

Here, L.H.L ≠ R.H.L, so $\lim_{x\to 2} f(x)$ dose not exist. Therefore, given function is discontinuous at x = 2 . **(Showed)**`

Problem: Show that the function is continuous at $x = \frac{1}{2}$

$$f(x) = \begin{cases} \mathrm{x} + \frac{5}{2}; -1 \leq x \leq \frac{1}{2} \\ 3;\ \frac{1}{2} \leq x \leq 1 \end{cases}$$

Solution: L.H.L = $\lim\limits_{x \to \frac{1}{2}-} f(x)$

$= \lim\limits_{x \to \frac{1}{2}-} \left(x - \frac{5}{2}\right)$

$= \left(\frac{1}{2} + \frac{5}{2} \right)$

$= \frac{1+5}{2}$

$= \frac{6}{2}$

$= 3$

R.H.L = $\lim\limits_{x \to \frac{1}{2}+} f(x)$

$= \lim\limits_{x \to \frac{1}{2}+} (3)$

$= 3$

Here, L.H.L = R.H.L. So, $\lim\limits_{x \to \frac{1}{2}} f(x)$ is exists.

Again, $\lim\limits_{x \to \frac{1}{2}} f(x) = 3$.

Here, $\lim\limits_{x \to \frac{1}{2}} f(x) = f(\frac{1}{2}) = 3$. So, given function is continuous at $x = \frac{1}{2}$.

Problem: Show the function $\boldsymbol{f(x)}$ is continuous at $x = \frac{1}{2}$, where,

$$f(x) = \begin{cases} \frac{3}{2} + x\,, 0 \leq x < \frac{1}{2} \\ x - \frac{3}{2}, \frac{1}{2} \leq x < 1 \end{cases}$$

Tips: $x \to \frac{1}{2}^{-} = x < \frac{1}{2}$; $x \to \frac{1}{2}^{+} = x > \frac{1}{2}$;

Solve:

L.H.L= $\lim_{x\to\frac{1}{2}-} f(x)$

$= \lim_{x\to\frac{1}{2}-} (\frac{3}{2} + x)$

$= \frac{3}{2} + \frac{1}{2} \qquad = \frac{3+1}{2} \qquad = 2$

R.H.L= $\lim_{x\to\frac{1}{2}+} f(x)$

$= \lim_{x\to\frac{1}{2}+} (x - \frac{3}{2})$

$= \frac{1}{2} - \frac{3}{2} \qquad = \frac{1-3}{2} \qquad = \frac{-2}{2} \qquad = -1$

Here, L.H.L≠ R.H.L. So $\lim_{x\to\frac{1}{2}} f(x)$ dosen't exist. Hence given function is not continuous at $x = \frac{1}{2}$.

Problem: Show the function $\boldsymbol{f(x)}$ is continuous at $x = \frac{1}{2}$, where,

$$f(x) = \begin{cases} 3 + 2x & ; -\frac{3}{2} \le x < 0 \\ 3 - 2x & ; 0 \le x < \frac{1}{2} \\ -3 - 2x & ; x \ge \frac{1}{2} \end{cases}$$

Tips: $x \to \frac{1}{2}^{-} = x < \frac{1}{2}$; $x \to \frac{1}{2}^{+} = x > \frac{1}{2}$;

Solution: L.H.L= $\lim_{x\to\frac{1}{2}-} f(x)$

$= \lim_{x\to\frac{1}{2}-} (3 - 2x)$

$= 3 - 2 \times \frac{1}{2} \qquad = 3 - 1 \qquad = 2$

R.H.L= $\lim_{x\to\frac{1}{2}+} f(x)$

$$= \lim_{x \to \frac{1}{2}+} (-3 - 2x) \quad = -3 - 2 \times \frac{1}{2} \quad = -3 - 1 \quad = -4$$

Here, L.H.L$\neq$ R.H.L. So $\lim_{x \to \frac{1}{2}} f(x)$ dosen't exist. Hence given function is not continuous at $x = \frac{1}{2}$.

Homework

PROBLEM1: Find the continuity / discontinuity at $x = 0$ of the following function

$$f(x) = \begin{cases} -x \ , \ \ where\ x < 0 \\ 0, where\ x = 0 \\ x, \quad where\ x > 0 \end{cases}$$

PROBLEM2: Find the continuity / discontinuity at $x = 0$ of the following function

$$f(x) = \begin{cases} 3 + 2x \ , \ \ where\ \dfrac{-3}{2} \leq x < 0 \\ 3 - 2x, where\ 0 \leq x \leq \dfrac{3}{2} \\ -3 - 2x, \quad where\ x \geq \dfrac{3}{2} \end{cases}$$

PROBLEM3: Check the continuity at $x = 0$ and $x = 1$ of the following functions

$$f(x) = \begin{cases} x^2 + 1 \ , \ \ where\ x < 0 \\ x, where\ 0 \leq x \ \leq 1 \\ \dfrac{1}{x}, \quad where\ x > 1 \end{cases}$$

PROBLEM4: Find the continuity / discontinuity at $x = 0$ and $x = \frac{1}{2}$ of the following function

$$f(x) = \begin{cases} 1 + 2x \ , \ \ where\ \dfrac{-1}{2} \leq x < 0 \\ 1 - 2x, where\ 0 \leq x \leq \dfrac{1}{2} \\ -1 + 2x, \quad where\ x > \dfrac{1}{2} \end{cases}$$

PROBLEM5: Check the continuity at $x = 0$ of the following functions

$$f(x) = \begin{cases} x^2 + 1\ , & where\ x > 0 \\ 1, & where\ x = 0 \\ 1 + x, & where\ x < 0 \end{cases}$$

PROBLEM6: Check the continuity at $x = 1$ of the following functions

$$f(x) = \begin{cases} x^2\ , & where\ x < 1 \\ 2.4\ , & where\ x = 1 \\ x^2 + 1, & where\ x > 1 \end{cases}$$

PROBLEM7: Check the continuity at $x = 2$ of the following functions

$$f(x) = \begin{cases} x^2\ , & where\ x < 2 \\ 3\ , & where\ x = 2 \\ x^2 - 1, & where\ x > 2 \end{cases}$$