Eigen value and Eigen vectors

Let T: $v -> v$ be a linear operator on a vector space v over a field k. A scalar t ∈ k is called an $\boldsymbol{Eigen\ value}$ of T if there exists a non zero vector $v \in V$ for which $T(v) = tv$.
Every vector satisfying this relation is called $\boldsymbol{Eigen\ vector}$ belonging to the eigen value t.

Problem – 01: Find the Eigen value and associated non zero Eigen vector for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix};$$

Solution:

We seek a scalar t and a non zero vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$, such that

$$AX = tX$$

$$=> \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} x \\ y \end{pmatrix}$$

The above matrix equation is equivalent to the homogeneous

$$\Rightarrow \begin{cases} x+2y \\ 3x+2y \end{cases} = \begin{matrix} tx \\ ty \end{matrix}$$

$$\text{Or, } \begin{cases} x+2y = tx \\ 3x+2y = ty \end{cases}$$

$$\text{Or, } \begin{cases} x+2y-tx = 0 \\ 3x+2y-ty = 0 \end{cases}$$

$$\text{Or, } \begin{cases} x(1-t)+2y = 0 \\ 3x+(2-t)y = 0 \end{cases} \qquad \text{..........................(1)}$$

The homogeneous system has a non zero solution if and only if the determinant of the matrix of coefficient is 0 :

$$\begin{vmatrix} 1-t & 2 \\ 3 & 2-t \end{vmatrix} = 0$$

$$\Rightarrow (1-t)(2-t) - 6 = 0$$
$$\Rightarrow 2 - t - 2t + t^2 - 6 = 0$$
$$\Rightarrow t^2 - 3t - 4 = 0$$
$$\Rightarrow (t-4)(t+1) = 0$$

Thus, t is an *Eigen value* of A if only if $t = 4$ and $t = 1$

Setting $t = 4\ in\ (1)$ then,

$$\begin{cases} x(1-4)+2y &= 0 \\ 3x+(2-4)y &= 0 \end{cases}$$

Or, $\begin{cases} -3x+2y &= 0 \\ 3x-2y &= 0 \end{cases}$ simply $3x - 2y = 0$

Thus $v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a non zero Eigen vector belongin to the Eigen value $t = 4$

$setting\ t = 1\ in\ (1)\ then$

$$\begin{cases} (1+1)x+2y &= 0 \\ 3x+(2+1)y &= 0 \end{cases}$$

Or, $\begin{cases} 2x+2y &= 0 \\ 3x+2y &= 0 \end{cases}$ simply $x + y = 0$

Thus $w = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a non zero Eigen vector belongin to the $Eigen\ value$ $t = -1$

$\boldsymbol{Problem - 02}$ (HW):

Find the $eigen\ value$ and associated non-zero eigen vector for matrix:

i) A = $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ ii) B = $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$\boldsymbol{Solution}$ $(\boldsymbol{i})$:

Let,

$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ We seek a scalar to and a non-zero vector $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$

Such that $\boldsymbol{AX} = \boldsymbol{tX}$

$$=> \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} x \\ y \end{pmatrix}$$

The above matrix equivalent to the homogenous,

$$=> \begin{cases} x + 4y = tx \\ 2x + 3y = ty \end{cases}$$

$$or, \begin{cases} x + 4y - ty = 0 \\ 2x + 3y - tx = 0 \end{cases}$$

$$or \begin{cases} (1-t)x + 4y = 0 \\ 2x + (3-t)y = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The homogenous system has a nonzero solution if and only if the determinant of the matrix of coefficient is 0

$$\begin{vmatrix} 1-t & 4 \\ 2 & 3-t \end{vmatrix} = 0$$

$(1-t)(3-t)-8=0$
$=> 3-t-3t+t^2-8=0$
$=> t^2-4t-5=0$
$=> t^2-5t+t-5=0$
$=> t(t-5)+1(t-5)=0$
$=> (t-5)(t+1)=0$
$\therefore t=5, \quad t=-1$

Thus, t is an Eigen value of A if only if $t=5, \; and \; t=-1$

$Setting\ t=5\ in\ (i)\ then,\ \begin{cases}(1-5)x+4y=0 \\ 2x+(3-t)y=0\end{cases}$

$or \begin{cases}-4x+4y=0 \\ 2x-2y=0\end{cases} \qquad Simply\ 2x-2y=0$

Thus, $v=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$ is a nonzero Eigen vector belonging to the Eigen Value $t=5$
Setting value $t=-1$ in (i) then,

or, $\begin{cases}(1+1)x+4y=0 \\ 2x+(3+1)y=0\end{cases}$

or, $\begin{cases}2x+4y=0 \\ 2x+4y=0\end{cases} \qquad Simply\ \ x+2y=0$

Thus $w=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}$ is a nonzero Eigen vector belonging to the Eigen $Value=-1$

$\boldsymbol{Solution\ (ii)}$:

We seek a scalar t and non zero vector, $X=\begin{pmatrix}x\\y\end{pmatrix}$
Such that $AX = tX$

⇨ $\begin{pmatrix}3 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ = t $\begin{pmatrix}x\\y\end{pmatrix}$

The above matrix equation to the homogeneous,

⇨ $\begin{cases}3x-y \\ x+y\end{cases}$ = $\begin{matrix}tx \\ ty\end{matrix}$

Or, $\begin{cases}3x-y = tx \\ x+y = ty\end{cases}$

Or, $\begin{cases}3x-y-tx=0 \\ x+y-ty=0\end{cases}$

Or, $\begin{cases} x(3-t) - y = 0 \\ x + y\,(1 - t) = 0 \end{cases}$ -------------------------------- (i)

The homogenous system has a non zero solution if and only if the determinant of the matrix of coefficient is 0

$$\begin{vmatrix} 3-t & -1 \\ 1 & 1-t \end{vmatrix} = 0$$

⇨ $(3-t)(1-t) + 1 = 0$

⇨ $3 - 3t - t + t2 + 1 = 0$

⇨ $t2 - 4t + 4 = 0$

⇨ $t2 - 2.2.t + 22 = 0$

⇨ $(t - 2)2 = 0$

⇨ $t = 2$

Thus it is an Eigen value of 0 if only if $t = 2$

Setting $t = 2\, in\ (i)$ then $\begin{cases} x(3-2) - y = 0 \\ x + y\,(1-2) = 0 \end{cases}$

Or, $\begin{cases} x - y = 0 \\ x - y = 0 \end{cases}$ simply $\mathrm{x} - \mathrm{y} = 0$

Thus $v = \begin{pmatrix} x \\ y \end{pmatrix}$ = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a non zero Eigen vector belonging to the Eigen value t $= 2$.

Home work : Find the $eigen\ value$ and associated non-zero eigen vector for matrix:

i) A = $\begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}$ ii) B = $\begin{pmatrix} -6 & -1 \\ 1 & 2 \end{pmatrix}$