$\underline{\textit{Inverse of a Matrix}}$

Definition: A square matrix, in which all elements above the main diagonal are zero, is called a **lower triangle** matrix.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix}$

Definition: A square matrix, in which all elements below the main diagonal are zero, is called a **upper triangle** matrix.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

Definition: A square matrix A, such that $|A| = 0$, is called a **singular** matrix.

Example: Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is singular matrix.

Definition: A square matrix A, such that $|A| \neq 0$, is called **nonsingular** matrix.

Example: Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ is nonsingular matrix.

Solution: $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix} = \mathbf{1(9-16) - 2(3-4) + 3(4-3)}$

$$\mathbf{= -7 + 2 + 3 = -2 \neq 0}$$

Let, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Formula for co-factor, $A_{11} = (-1)^{i+j} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$= a_{22}a_{33} - a_{32}a_{23}$$

$Q.\ Find\ the\ adjoint\ of\ A, where\ A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

Solution: $The\ cofactors\ of\ A\ are$

$A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 9 - 16 = -7$

$A_{12=} - \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -(3 - 4) = -(-1) = 1$

$A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 = 1$

$A_{21=} - \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -(6 - 12) = -(-6) = 6$

$A_{22=} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3 - 3 = 0$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -(4 - 2) = -2$

$A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1$

$A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -(4 - 3) = -1$

$A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$

$So\ the\ adjoint\ matrix\ of\ A = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$ ***Answer***:

Now the determinant of A is, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix}$

$$= \mathbf{1(9 - 16) - 2(3 - 4) + 3(4 - 3)}$$

$$= \mathbf{-7 + 2 + 3 = -2}$$

So the inverse of A is, $\boldsymbol{A^{-1}} = -\frac{\mathbf{1}}{\mathbf{2}} \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 - 3 & -2 + 2 \\ -6 + 6 & -3 + 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$$BA = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 2-2 \\ -6+6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

There fore $A\ and\ B$ are invertible and are inverse of each other that is $A^{-I} = B$and $B^{-I} = A$.

$\boldsymbol{Example}$: Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\ and\ B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+0-5 & -2+0+2 & 2+0-2 \\ 15-15+0 & -5+6+0 & 5-5+0 \\ 0-15+15 & 0+6-6 & 0-5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly $BA = I$(Identity Matrix)

Therefore A and B are invertible and are inverse of each other. That is $A^{-1} = B\ and\ B^{-1} = A$

$\boldsymbol{Problem}$: $If\ A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}\ then\ verify\ A.adjA = adjA.A = |A|.I$

$\boldsymbol{Given\ that}$, $A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

the Co-factors are,

$$A_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 9-2 = 7$$

$$A_{12=} -\begin{vmatrix} 5 & 1 \\ 3 & 3 \end{vmatrix} = -(15-3) = -12$$

$$A_{13} = \begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} = 10-9 = 1$$

$$A_{21=} -\begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} = -(3-6) = -(-9) = 9$$

$$A_{22=} \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = 6-9 = -3$$

$A_{23} = -\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = -(4+3) = -7$

$A_{31} = \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = -1 - 9 = -10$

$A_{32} = -\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = -(2-15) = -(-13) = 13$

$A_{33} = \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 6 + 5 = 11$

$$\therefore Adj\ of\ A = \begin{bmatrix} 7 & -12 & 1 \\ 9 & -3 & -7 \\ -10 & 13 & 11 \end{bmatrix}^t = \begin{bmatrix} 7 & 9 & -10 \\ -12 & -3 & 13 \\ 1 & -7 & 11 \end{bmatrix}$$

$$\therefore A\,.\,adj\ A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 7 & 9 & -10 \\ -12 & -3 & 13 \\ 1 & -7 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 7 + (-1)\times(-12) + 3\times 1 & 2\times 9 + (-1)\times(-3) + 3\times(-7) & 2\times(-10) + (-1)\times 13 + 3\times 11 \\ 5\times 7 + 3\times(-12) + 1\times 1 & 5\times 9 + 3\times(-3) + 1\times(-7) & 5\times(-10) + 3\times 13 + 1\times 11 \\ 3\times 7 + 2\times(-12) + 3\times 1 & 3\times 9 + 2\times(-3) + 3\times(-7) & 3\times(-10) + 2\times 13 + 3\times 11 \end{bmatrix}$$

$$= \begin{bmatrix} 14+12+3 & 18+3-21 & -20-13+33 \\ 35-36+1 & 45-9-7 & -50+39-11 \\ 21-24+3 & 27-6-21 & -30+26+33 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 29 \end{bmatrix}$$

$$adj\ A.A = \begin{bmatrix} 7 & 9 & -10 \\ -12 & -3 & 13 \\ 1 & -7 & 11 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7\times 2 + 9\times 5 + (-10)\times 3 & 7\times(-1) + 9\times 3 + (-10)\times 2 & 7\times 3 + 9\times 1 + (-10)\times 3 \\ -12\times 2 + (-3)\times 5 + 13\times 3 & -12\times(-1) + (-3)\times 3 + 13\times 2 & -12\times 3 + (-3)\times 1 + 13\times 3 \\ 1\times 2 + (-7)\times 5 + 11\times 3 & 1\times(-1) + (-7)\times 3 + 11\times 2 & 1\times 3 + (-7)\times 1 + 11\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14+45-30 & -7+27-20 & 21+9-30 \\ -24-15+39 & 12-9+26 & -36-3+39 \\ 2-35+33 & -1-21+22 & 3-7+33 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 29 \end{bmatrix}$$

$$\therefore\ |A| = \begin{vmatrix} 2 & 1 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 3 \end{vmatrix}$$

$$= 2(9-1) - \{-1(15-3)\} + 3(10-9)$$

$$= 2 \times 7 + 1 \times 12 + 3 \times 1$$

$$= 14 + 12 + 3 \quad = 29$$

$$\therefore\ |A|.I = 29 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 29 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 29 \end{bmatrix}$$

$\therefore A.adjA = adjA.A = |A|.I$ $\quad\quad\quad$ $[\boldsymbol{Verified}]$

Home work: Find the $adjoint$ and $inverse$ for the following matrices

1. $A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & -3 & 1 \\ 3 & 2 & 0 \end{bmatrix}$
2. $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -5 \\ 3 & 2 & 3 \end{bmatrix}$
3. $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 2 & 1 \\ 3 & -2 & 3 \end{bmatrix}$
4. $A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & 1 \\ -3 & 2 & 3 \end{bmatrix}$
5. $A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & -1 \\ 3 & 0 & 3 \end{bmatrix}$
6. $\boldsymbol{A = -\frac{1}{2}} \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$