# Discussion & Problem Solving on Integral Calculus

## Definite Integral

Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into $n$ subintervals of equal width, $\Delta x$, and from each interval choose a point, $x_i^*$. Then the **definite integral of** $f(x)$ **from** a **to** b is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

There is also a little bit of terminology that we should get out of the way here. The number "a" that is at the bottom of the integral sign is called the **lower limit** of the integral and the number "b" at the top of the integral sign is called the **upper limit** of the integral. Also, despite the fact that a and b were given as an interval the lower limit does not necessarily need to be smaller than the upper limit. Collectively we'll often call a and b the **interval of integration**.

### Fundamental Theorem of Integral calculus:

If $f$ is continuous on the closed interval $[a, b]$then, $\int_a^b f(x)\, dx = F(b) - F(a)$, where $F$ is any antiderivative of $f$.

### Properties of Integration:

1. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.
2. $\int_a^b f(x)\, dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.
3. $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.
4. $\int_a^b f(x) \pm g(x) dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$, We can break up definite integrals across a sum or difference.

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a, c]$ and $[c, b]$. Note however that c doesn't need to be between a and b.
6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$, The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

**Questions:** Given that, $\int_6^{-10} f(x)\,dx = 23$ and $\int_{-10}^{6} g(x)\,dx = -9$, determine the value of $\int_{-10}^{6} 2f(x) - 10g(x)\,dx$.

**Solution:** We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\int_{-10}^{6} 2f(x) - 10g(x)\,dx = \int_{-10}^{6} 2f(x)\,dx - \int_{-10}^{6} 10g(x)\,dx$$

$$= 2\int_{-10}^{6} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above (and adding a minus sign of course). Once this is done we can plug in the known values of the integrals.

$$\int_{-10}^{6} 2f(x) - 10g(x)\,dx = -2\int_{6}^{-10} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$

$$= -2(23) - 10(-9) = 44 \text{ Ans.}$$

**Questions:** Given that, $\int_{12}^{-10} f(x)\,dx = 6$, $\int_{100}^{-10} f(x)\,dx = -2$ and $\int_{100}^{-5} f(x)\,dx = 4$ , determine the value of $\int_{-5}^{12} f(x)\,dx$.

**Solution:** Here,

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{-10} f(x)\,dx + \int_{-10}^{12} f(x)\,dx$$

$$\int_{-5}^{12} f(x)\,dx = -\int_{100}^{-5} f(x)\,dx + \int_{100}^{-10} f(x)\,dx - \int_{12}^{-10} f(x)\,dx$$
$$= -4 - 2 - 6 = -12 \text{ Ans.}$$

**Home work:**

1. Given that, $\int_{6}^{-100} f(x)\,dx = 20$ and $\int_{16}^{-100} g(x)\,dx = -5$, determine the value of $\int_{16}^{6} 20f(x) - 10g(x)\,dx$.
2. Given that, $\int_{12}^{-10} f(x)\,dx = 6$, $\int_{100}^{-10} f(x)\,dx = -2$ and $\int_{100}^{-5} f(x)\,dx = 4$ , determine the value of $\int_{-5}^{12} f(x)\,dx$.

**Find the value of the following integration:**

$i)\ \int_0^3 \frac{dx}{9+x^2}$ $\quad ii)\ \int_0^2 \frac{dx}{16-x^2}$ $\quad iii)\ \int_0^{\frac{\pi}{4}} cos^2\theta\, d\theta$ $\quad iv)\ \int_0^1 \frac{tan^{-1}x}{1+x^2}\,dx$

***Solved – (i):***

$Let,\ \ I = \int_0^3 \frac{dx}{9+x^2}$

$$= \int_0^3 \frac{dx}{3^2+x^2}$$

$$= \left[\frac{1}{3} tan^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{1}{3} tan^{-1}\frac{3}{3} - \frac{1}{3} tan^{-1}\frac{0}{3}$$

$$= \frac{1}{3} tan^{-1}.1 - \frac{1}{3} tan^{-1}.0$$

$$= \frac{1}{3}.\frac{\pi}{4} - 0 = \frac{\pi}{12}$$

***Solved – (ii):***

$Let,\ \ I = \int_0^2 \frac{dx}{16-x^2}$

$$= \int_0^2 \frac{dx}{4^2-x^2}$$

$$= \left[\frac{1}{2.4} ln\left|\frac{4+x}{4-x}\right|\right]_0^2$$

$$=\frac{1}{8}\left[ln\left|\frac{4+2}{4-2}\right|-ln\left|\frac{4+0}{4-o}\right|\right]$$

$$=\frac{1}{8}[ln3-ln1]$$

$$=\frac{1}{8}ln3-0=\frac{1}{8}ln3$$

***Solved – (iii):***

$$Let,\ \ I=\int_0^{\frac{\pi}{4}}cos^2\theta\ d\theta$$

$$=\tfrac{1}{2}\int_0^{\frac{\pi}{4}}2cos^2\theta\ d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1+cos2\theta)\ d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1+cos2\theta)\ d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{sin2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left[\frac{\pi}{4}+\frac{sin2\times\frac{\pi}{4}}{2}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}+\frac{sin\frac{\pi}{2}}{2}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}+\frac{1}{2}\right]=\frac{\pi+2}{8}.$$

***Solved – (iv):***

$$Let,\ \ I=\int_0^1\frac{tan^{-1}x}{1+x^2}\ dx$$

$$=\int_0^{\frac{\pi}{4}}z\ dz$$

$$=\left[\frac{z^2}{2}\right]_0^{\frac{\pi}{4}}$$

$$=\tfrac{1}{2}\left\{\left(\tfrac{\pi}{4}\right)^2-0\right\}=\tfrac{\pi^2}{32}$$

Let, $tan^{-1}x=z$

$\Rightarrow\frac{1}{1+x^2}\ dx=dz$

| | | |
|---|---|---|
| $x$ | 0 | 1 |
| $z$ | 0 | $\pi/4$ |

***Solved – (iv):***

$$Let,\ \ I = \int xlnxdx$$

$$= lnx \int x\,dx - \int \left\{\frac{d}{dx}(\ln x) \int x\,dx\right\} dx$$

$$= lnx.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2} dx$$

$$= \frac{x^2}{2} lnx - \frac{1}{2}.\frac{x^2}{2}$$

$$= \frac{x^2}{2} lnx - \frac{x^2}{4}$$

Now, $\int_0^e x\,lnx\ dx = \left[\frac{x^2}{2} lnx - \frac{x^2}{4}\right]_0^e$

$$= \left[\frac{e^2}{2} lne - \frac{e^2}{4}\right] - \left[\frac{0^2}{2} ln0 - \frac{0^2}{4}\right]$$

$$= \left[\frac{e^2}{2} - \frac{e^2}{4}\right] - 0$$

$$= \left(\frac{2e^2 - e^2}{4}\right)$$

$$= \frac{e^2}{4}$$

**Find the value of the following integration:**

$i)\ \int_0^{\frac{\pi}{3}} \frac{cosx}{3+4sinx}$ $\quad ii)\ \int_0^1 x^3\sqrt{1+3x^4}\ dx$ $\quad iii)\ \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+2cos\theta}$ $\quad iv)\ \int_0^1 \frac{dx}{2+cosx}$

***Solved – (i):***

$$Let,\ \ I = \int_0^{\frac{\pi}{3}} \frac{cosx}{3+4sinx}$$

$$= \frac{1}{4} \int_3^{3+2\sqrt{3}} \frac{dz}{z}$$

$$= \frac{1}{4} [lnz]_3^{3+2\sqrt{3}}$$

$$= \frac{1}{4} \left[ln\left(3+2\sqrt{3}\right) - ln3\right]$$

$$= \frac{1}{4} \left(ln\frac{3+2\sqrt{3}}{3}\right)$$

Let, $3 + 4sinx = z$

$\Rightarrow 4cosx\ dx = dz$

$\Rightarrow cosx\ dx = \frac{1}{4} dz$

| $x$ | 0 | π/3 |
|---|---|---|
| $z$ | 3 | $3+2\sqrt{3}$ |

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**Solved – (ii):**

$$Let,\ \ I = \int_0^1 x^3\sqrt{1+3x^4}\ dx$$

$$= \frac{1}{12}\int_1^4 \sqrt{z}\, dz$$

$$= \frac{1}{12}\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4$$

$$= \frac{1}{12}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{18}\left[\left(\sqrt{z}\right)^3\right]_1^4$$

$$= \frac{1}{18}\left\{\left(\sqrt{4}\right)^3 - \left(\sqrt{1}\right)^3\right\}$$

$$= \frac{1}{18}(8-1)$$

$$= \frac{7}{18}$$

| Let, $1 + 3x^4 = z$ <br> $\Rightarrow x^3 dx = \frac{1}{12}\, dz$ | | |
|---|---|---|
| $x$ | 0 | 1 |
| $z$ | 1 | 4 |

**Solved – (iii):**

$$Let,\ \ I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+2cos\theta}$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+2\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\frac{1+tan^2\frac{x}{2}+2-2tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{sec^2\frac{\theta}{2}}{3-tan^2\frac{x}{2}}\ dx$$

$$= 2\int_0^1 \frac{dz}{3-z^2}$$

$$= 2\int_0^1 \frac{dz}{\sqrt{3}-z^2}$$

$$= 2\left[\frac{1}{2.\sqrt{3}}\ln\left|\frac{\sqrt{3}+z}{\sqrt{3}-z}\right|\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left[\ln\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| - \ln\left|\frac{\sqrt{3}+0}{\sqrt{3}-0}\right|\right]$$

| Let, $tan\frac{\theta}{2} = z$ <br> $\Rightarrow sec^2\frac{\theta}{2}d\theta = 2dz$ | | |
|---|---|---|
| $x$ | 0 | π/2 |
| $z$ | 0 | 1 |

$$= \frac{1}{\sqrt{3}}\left[\ln\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| - \ln\left|\frac{\sqrt{3}}{\sqrt{3}}\right|\right]$$

$$= \frac{1}{\sqrt{3}}\{ln(\sqrt{3}+1) - ln(\sqrt{3}-1) - \ln 1\}$$

$$= \frac{1}{\sqrt{3}}\{ln(\sqrt{3}+1) - ln(\sqrt{3}-1)\}$$

***Solved – (iv):***

$$Let,\ \ I = \int_0^1 \frac{dx}{2+cosx}$$

$$= \int_0^{\pi} \frac{dx}{2+\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int_0^{\pi} \frac{dx}{\frac{2+2tan^2\frac{x}{2}+1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}$$

$$= \int_0^{\pi} \frac{sec^2\frac{x}{2}}{3+tan^2\frac{x}{2}}\,dx$$

$$= 2\int_0^{\infty} \frac{dz}{3+z^2}$$

$$= 2\int_0^{\infty} \frac{dz}{(\sqrt{3})^2+z^2}$$

$$= 2\left[\frac{1}{\sqrt{3}}tan^{-1}\frac{z}{\sqrt{3}}\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}}\left[tan^{-1}\frac{\infty}{\sqrt{3}} - tan^{-1}\frac{0}{\sqrt{3}}\right]$$

$$= \frac{2}{\sqrt{3}}[tan^{-1}\infty - tan^{-1}0]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{\sqrt{3}}$$

Let, $tan\frac{x}{2} = z$

$\Rightarrow sec^2\frac{x}{2}dx = 2dz$

| $x$ | 0 | π |
|---|---|---|
| $z$ | 0 | ∞ |

**Show that:** $i)\ \int_0^{\frac{\pi}{2}} sin^2\theta\, cos^3\theta\, d\theta = \frac{2}{15}$ $\quad ii)\ \int_0^{\frac{\pi}{2}} \frac{cosx\, dx}{(1+sinx)(2+sinx)} = ln\frac{4}{3}$

***Solved – (i):***

$$L.H.S,\ I = \int_0^{\frac{\pi}{2}} sin^2\theta\, cos^3\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} sin^2\theta\, cos^2\theta\, cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} sin^2\theta\ (1 - sin^2\theta)\ cos\theta\ d\theta$$

$$= \int_0^1 z^2\ (1 - z^2)\ dz$$

$$= \int_0^1 (z^2 - z^4)\ dz$$

$$= \left[\frac{z^3}{3} - \frac{z^5}{5}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{5}$$

$$= \frac{2}{15} \quad = R.H.S$$

| Let, $sin\theta = z$ $\Rightarrow cos\theta\ d\theta = dz$ | | |
|---|---|---|
| $x$ | 0 | π/2 |
| $z$ | 0 | 1 |

$So, L.H.S = R.H.S\ is\ Equal.$

***Solved – (ii):***

$$L.H.S,\ I = \int_0^{\frac{\pi}{2}} \frac{cosx\ dx}{(1+sinx)(2+sinx)}$$

$$= \int_0^1 \frac{dz}{(1+z)(2+z)}$$

$$= \int_0^1 \left[\frac{1}{1+z} - \frac{1}{2+z}\right] dz$$

$$= [ln(1+z) - ln(2+z)]_0^1$$

$$= \{ln(1+1) - ln(2+1)\} - \{ln(1+0) - ln(2+0)\}$$

$$= (ln2 - ln3) - (ln1 - ln2)$$

$$= ln2 - ln3 - 0 + ln2$$

$$= 2ln2 - ln3$$

$$= ln4 - ln3$$

$$= ln\frac{4}{3} = R.H.S$$

| Let, $sinx = z$ $\Rightarrow cosxdx = dz$ | | |
|---|---|---|
| $x$ | 0 | π/2 |
| $z$ | 0 | 1 |

$So, L.H.S = R.H.S\ is\ Equal.$

# Discussion & Problem Solving on Integral Calculus

**Find the value of the following integration:**

*i)* $\int_1^{e^2} \frac{dx}{x(1+lnx)^2}$  *ii)* $\int_0^{\frac{\pi}{4}} \frac{cosx\ 2x-1}{cos2x+1} dx$  *iii)* $\int_1^{e^2} \frac{dx}{x(1+lnx)}$

***Solved – (i):*** $Let,\ \ I = \int_1^{e^2} \frac{dx}{x(1+lnx)^2}$

$$= \int_1^3 \frac{dz}{z^2}$$

$$= \int_1^3 z^{-2} dz$$

$$= \left[\frac{z^{-2+1}}{-2+1}\right]_1^3$$

$$= \left[\frac{z^{-1}}{-1}\right]$$

$$= -\left[\frac{1}{z}\right]$$

$$= -\left[\frac{1}{3} - 1\right] = \frac{2}{3}$$

Let, $1 + lnx = z$

$\Rightarrow \frac{1}{x} dx = dz$

| | | |
|---|---|---|
| $x$ | 1 | $e^2$ |
| $z$ | 1 | 3 |

***Solved – (ii):***

$$Let,\ \ I = \int_0^{\frac{\pi}{4}} \frac{cosx\ 2x-1}{cos2x+1} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{-(1-cos2x)}{1+cos2x} dx$$

$$= -\int_0^{\frac{\pi}{4}} \frac{1-cos2x}{1+cos2x} dx$$

$$= -\int_0^{\frac{\pi}{4}} \frac{2sin^2x}{2cos^2x} dx$$

$$= -\int_0^{\frac{\pi}{4}} tan^2x\ dx$$

$$= -\int_0^{\frac{\pi}{4}} (sec^2x - 1)\ dx$$

$$= \int_0^{\frac{\pi}{4}} (1 - sec^2x)\ dx$$

$$= [x - tanx]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} - tan\frac{\pi}{4}\right) - 0$$

$$= \left(\frac{\pi}{4} - 1\right)$$


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**Solved – (iii):**

$$Let,\ \ I = \int_1^{e^2} \frac{dx}{x(1+lnx)}$$

$$= \int_1^3 \frac{1}{z}\, dz$$

$$= [lnz]_1^3$$

$$= (ln3 - ln1)$$

$$= ln3$$

| Let, | $1 + lnx = z$<br>$\Rightarrow \frac{1}{x}\, dx = dz$ | |
|---|---|---|
| $x$ | 1 | $e^2$ |
| $z$ | 1 | 3 |

**Evaluate following integrals:**

$i)\ \int_0^{\frac{\pi}{4}} \frac{sin2x}{sin^4x+cos^4x}\, dx \quad ii)\ \int_0^{\frac{\pi}{2}} cos^3x(sinx)^{\frac{1}{4}}dx$

**Solved – (i):**

$$\textbf{Let,}\quad I = \int_0^{\frac{\pi}{4}} \frac{sin2x}{sin^4x+cos^4x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2sinx.cosx}{cos^4x\left(\frac{sin^4x}{cos^4x}+1\right)}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2sinx}{cos^3x\,(tan^4x+1)}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2.\frac{sinx}{cosx}.\frac{1}{cos^2x}}{tan^4x+1}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2tanx.sec^2x}{(tan^2x)^2+1}\, dx$$

$$= \int_0^1 \frac{dz}{z^2+1}$$

$$= [tan^{-1}z]_0^1$$

$$= tan^{-1}1 - tan^{-1}0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

| Let, | $tan^2x = z$<br>$\Rightarrow 2tanx.sec^2x\, dx = dz$ | |
|---|---|---|
| $x$ | 0 | π/2 |
| $z$ | 0 | 1 |

**Solved – (ii):**

$$I = \int_0^{\frac{\pi}{2}} cos^3x(sinx)^{\frac{1}{4}}dx$$

$$= \int_0^{\frac{\pi}{2}} cosx.cos^2x(sinx)^{\frac{1}{4}}dx$$

$$= \int_0^{\frac{\pi}{2}} cosx(1 - sin^2x)(sinx)^{\frac{1}{4}}dx$$

$$= \int_0^1 (1 - z^2)z^{\frac{1}{4}}dz$$

| Let, $sinx = z$ <br> $\Rightarrow cosx\ dx = dz$ | | |
|---|---|---|
| $x$ | 0 | π/2 |
| $z$ | 0 | 1 |

$$= \int_0^1 \left(z^{\frac{1}{4}} - z^{\frac{9}{4}}\right) dz$$

$$= \left[\frac{z^{\frac{5}{4}}}{\frac{5}{4}} - \frac{z^{\frac{13}{4}}}{\frac{13}{4}}\right]_0^1$$

$$= \left(\frac{1}{\frac{5}{4}} - \frac{1}{\frac{13}{4}}\right) - \left(\frac{0}{\frac{5}{4}} - \frac{0}{\frac{13}{4}}\right)$$

$$= \frac{4}{5} - \frac{4}{13}$$

$$= \frac{32}{65}$$