Matrix Algebra-01

Linear Algebra has Two Parts:

i) Matrix and ii) Vector

Matrix : A matrix is a rectangular array of numbers (real or complex) enclosed by a pair of brackets (or double vertical bars) and the numbers in the array are called the entries or the element of the matrix . i.e. a rectangular array of numbers of the form,

$$\begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ a_{m1} & a_{m2} \dots & a_{mn} \end{bmatrix}$$

Is called a matrix .the numbers $a_{11}a_{12}$ ... ... a_{mn} are called the entries or elements of the matrix . the above matrix has m rows and n columns and is called on $(m \times n)$ matrix (read "m by n" matrix) . the matrix of m rows and n column is said to be of order "m by n" or (m×n) Matrix are enerally denoted by capital letters A,B,X,Y……

Ex: (i) $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & 8 \end{bmatrix}$ is a matrix of order (2×3)over .the real field IR and also over the complex field C

(ii) $B = \begin{bmatrix} i & 0 & -2 \\ 1 & -i & 4 \\ 1+i & 0 & 7i \end{bmatrix}$is matrix of order (3×3) over the complex field of C

(iii) $C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ is a matrix order (2×2) over the real field

(iv) $D = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}$is a matrix over the real field of order (3×2) matrix

i) Rectangular Matrix: if the number of rows and the number of columns of a matrix are not equal then it is called Rectangular matrix.

ii) Square Matrix : A Matrix with same number of rows and columns is called a square matrix .

Ex: $A = \begin{bmatrix} 2 & 7 \\ -5 & 0 \end{bmatrix}$ $and\ B = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 7 & 0 \\ 1 & 2 & 7 \end{bmatrix}$

iii) Horizontal Matrix: if in a matrix the number of columns is more than the number of rows then it is called horizontal matrix.

iv) Vertical Matrix: if in a matrix the number of columns is less than the number of rows then it is called

Vertical Matrix.

v) Row Matrix: if in a matrix there is only one row then it is called row matrix.

Ex: $\begin{bmatrix} 1 & 2 \end{bmatrix}$

vi) Column Matrix: if in a matrix there is only one column then it is called columns matrix

Ex: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

vii) Zero Matrix: if in a matrix all the value is zero then it is called zero matrix.

Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

viii) Unit Matrix/Identity Matrix: A square matrix having unity for its element in the leading diagonal and all other elements as zero then it is called identity matrix/ unit matrix.

Ex: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} is\ the\ matrix\ of\ order\ 3 \times 3$

ix) Diagonal Matrix: A square matrix in which all elements except those in main diagonal are zero, it is called diagonal matrix.

Ex: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Order: $m \times n$, m by n [here m is row and n is column]

Matrix Multiplication: $A_{m\times n} \quad B_{p\times q}$, if $n = p$ then multiplication is applicable .

$A_{3\times 2} = B_{3\times 2}$; $3\times 2 \neq 3 \times 2$

Problem: Let A= $\begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$ then final $2A, A + B$, and $A - B$.

Solution:

Given A= $\begin{bmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{bmatrix}$

Now 2A=2$\begin{bmatrix}1 & -2 & 3\\5 & 1 & -4\end{bmatrix}$

$=\begin{bmatrix}2.1 & 2.(-2) & 2.3\\2.5 & 2.1 & 2.(-4)\end{bmatrix}$

$=\begin{bmatrix}2 & -4 & 6\\10 & 2 & -8\end{bmatrix}$

A+B= $\begin{bmatrix}1 & -2 & 3\\5 & 1 & -4\end{bmatrix}$+$\begin{bmatrix}2 & 3 & 5\\1 & 4 & -2\end{bmatrix}$

=$\begin{bmatrix}1+2 & -2+3 & 3+5\\5+1 & 1+4 & -4-2\end{bmatrix}$

=$\begin{bmatrix}3 & 1 & 8\\6 & 5 & -6\end{bmatrix}$

A−B= $\begin{bmatrix}1 & -2 & 3\\5 & 1 & -4\end{bmatrix}-\begin{bmatrix}2 & 3 & 5\\1 & 4 & -2\end{bmatrix}$

=$\begin{bmatrix}1-2 & -2-3 & 3-5\\5-1 & 1-4 & -4+2\end{bmatrix}$

=$\begin{bmatrix}-1 & -5 & -2\\4 & -3 & -2\end{bmatrix}$.(Ans).

Problem: Let $A=\begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$ and B=$\begin{bmatrix}5 & 0\\2 & 5\end{bmatrix}$ then find AB, and BA.

Solution:

Here, $A=\begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$ and B=$\begin{bmatrix}5 & 0\\2 & 5\end{bmatrix}$

Then AB=$\begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}\begin{bmatrix}5 & 0\\2 & 5\end{bmatrix}$

=$\begin{bmatrix}5+0 & 0+0\\0+10 & 0+5\end{bmatrix}$

=$\begin{bmatrix}5 & 0\\10 & 25\end{bmatrix}$

BA=$\begin{bmatrix}5 & 0\\2 & 5\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$

=$\begin{bmatrix}5+0 & 0+0\\2+0 & 0+5\end{bmatrix}$

$=\begin{bmatrix} 5 & 0 \\ 2 & 5 \end{bmatrix}$

Here we see that AB≠BA (Ans).

Problem: $If\ A = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 2 & 6 \end{bmatrix}\ and\ B = \begin{bmatrix} 10 & 9 & 8 \\ 1 & 2 & 3 \end{bmatrix}\ then\ find\ A + B.$

Solution:

$\therefore A + B = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 10 & 9 & 8 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+10 & 3+9 & 5+8 \\ 4+1 & 2+2 & 6+3 \end{bmatrix}$

$= \begin{bmatrix} 11 & 12 & 13 \\ 5 & 4 & 9 \end{bmatrix}$ **Answer:**

Problem: $\text{I}f\ A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & 2 \end{bmatrix}\ and\ B = \begin{bmatrix} -11 & 0 & -9 \\ 2 & 5 & -12 \end{bmatrix}\ then\ find\ A - B.$

Solution:

Given, $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & 2 \end{bmatrix}\ and\ B = \begin{bmatrix} -11 & 0 & -9 \\ 2 & 5 & -12 \end{bmatrix}$

$Now\ \ \therefore A - B = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & 2 \end{bmatrix} - \begin{bmatrix} -11 & 0 & -9 \\ 2 & 5 & -12 \end{bmatrix}$

$= \begin{bmatrix} 1+11 & -1-0 & 3+9 \\ 0-2 & 5-5 & 2+12 \end{bmatrix}$

$= \begin{bmatrix} 12 & -1 & 12 \\ -2 & 0 & 14 \end{bmatrix}$ **Answer:**

Problem: $if\ A \begin{bmatrix} 0 & 5 & 9 \\ -1 & 2 & 3 \\ -5 & -4 & 7 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 5 & 5 \\ 0 & -9 & -9 \end{bmatrix}$, find the addition & subtraction.

Solution:

Given, $A = \begin{bmatrix} 0 & 5 & 9 \\ -1 & 2 & 3 \\ -5 & -4 & 7 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 5 & 5 \\ 0 & -9 & -9 \end{bmatrix}$

$\therefore A + B = \begin{bmatrix} 0 & 5 & 9 \\ -1 & 2 & 3 \\ -5 & -4 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -3 \\ -3 & 5 & 5 \\ 0 & -9 & -9 \end{bmatrix}$

$$= \begin{bmatrix} 0+1 & 5+2 & 9-3 \\ -1-3 & 2+5 & 3+5 \\ -5+0 & -4-9 & 7-9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 & 6 \\ -4 & 7 & 8 \\ -5 & -13 & -2 \end{bmatrix}$$ **Answer:**

$$again, \therefore A - B = \begin{bmatrix} 0 & 5 & 9 \\ -1 & 2 & 3 \\ -5 & -4 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ -3 & 5 & 5 \\ 0 & -9 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0-1 & 5-2 & 9+3 \\ -1+3 & 2-5 & 3-5 \\ -5+3 & -4-5 & 7+9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 12 \\ 2 & -3 & -2 \\ -2 & -9 & 16 \end{bmatrix}$$ **Answer:**

$*$ ***Find the value of 3A − 4B , where A*** & B *value assuming from previous question.*

Now, $3A - 4B = 3\begin{bmatrix} 0 & 5 & 9 \\ -1 & 2 & 3 \\ -5 & -4 & 7 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -3 \\ -3 & 5 & 5 \\ 0 & -9 & -9 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 15 & 27 \\ -3 & 6 & 9 \\ -15 & -12 & 21 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -12 \\ -12 & 20 & 20 \\ 0 & -36 & -36 \end{bmatrix}$$

$$= \begin{bmatrix} 0-4 & 15-8 & 27+12 \\ -3+12 & 6-20 & 9-20 \\ -15-0 & -12+36 & 21+36 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 7 & 39 \\ 9 & -14 & -11 \\ -15 & 24 & 57 \end{bmatrix}$$ **Answer:**

Problem: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix}$, *so find* AB & BA.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+2\times3+3\times5 & 1\times2+2\times4+3\times0 \\ 4\times1+5\times3+6\times5 & 4\times2+5\times4+6\times0 \end{bmatrix}$$

$$= \begin{vmatrix} 1+6+15 & 2+8+0 \\ 4+15+30 & 8+20+0 \end{vmatrix}$$

$\therefore AB = \begin{bmatrix} 22 & 10 \\ 49 & 28 \end{bmatrix}$ **Answer:**

$$again, BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+2\times4 & 1\times2+2\times5 & 1\times3+2\times6 \\ 3\times1+4\times4 & 3\times2+4\times5 & 3\times3+4\times6 \\ 5\times1+0\times4 & 5\times2+0\times5 & 5\times3+0\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & 2+10 & 3+12 \\ 3+16 & 6+20 & 9+24 \\ 5+0 & 10+0 & 15+0 \end{bmatrix}$$

$\therefore BA = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 5 & 10 & 15 \end{bmatrix}$ **Answer:**

Problem: if $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$ then find AB where BA exists? Give reason.

<u>Solution</u>:

$$\boldsymbol{Given\ that}, AB = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times1+1\times2+2\times1 & 3\times4+1\times2+2\times0 \\ 0\times1+1\times2+1\times1 & 0\times4+1\times2+1\times0 \\ 1\times1+2\times2+0\times1 & 1\times4+2\times2+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2+2 & 12+2+0 \\ 0+2+1 & 0+2+0 \\ 1+4+0 & 4+4+0 \end{bmatrix}$$

$= \begin{bmatrix} 7 & 14 \\ 3 & 2 \\ 5 & 8 \end{bmatrix}$ Here B is a matrix of order 3×2 and A is a matrix of order 3×3.hence BA does not

Exits as number columns in B is not equal to the number of rows in A**.**

Problem: if $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ *and B* $\begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ *then prove that* $AB \neq BA$

Solution:

Here, AB = $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\times1+3\times-1+4\times0 & 2\times3+3\times2+4\times0 & 2\times0+3\times1+4\times2 \\ 1\times1+2\times-1+3\times0 & 1\times3+2\times2+3\times0 & 1\times0+2\times1+3\times2 \\ -1\times1+1\times-1+2\times0 & -1\times3+1\times2+2\times0 & -1\times0+1\times1+2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3+0 & 6+6+0 & 0+3+8 \\ 1-2+0 & 3+4+0 & 0+2+6 \\ -1-1+0 & -2+2+0 & 0+2+4 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$$

$$now, BA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+3\times1+0\times-1 & 1\times3+3\times2+0\times1 & 1\times4+3\times3+0\times2 \\ -1\times2+2\times1+1\times-1 & -1\times3+2\times2+1\times1 & -1\times4+2\times3+1\times2 \\ 0\times2+0\times1+2\times-1 & 0\times3+0\times2+2\times1 & 0\times4+0\times3+2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+0 & 3+6+0 & 4+9+0 \\ -2+2-1 & -3+4+1 & -4+6+2 \\ 0+0-2 & 0+0+2 & 0+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$$

$\therefore AB \neq BA$ **Proved:**

Problem: *Let,* $f(x) = x^2 - 4x - 5$. Find f(A) if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, *that* $f(A)$ *is the null matrix.*

Solution: ***Given that,***

$$f(x) = x^2 - 4x - 5.$$

$\therefore f(A) = A^2 - 4A - 5I,\ Where\ I\ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $A^2 = A.A$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times1 \\ 2\times1+1\times2+2\times2 & 2\times2+1\times1+2\times2 & 2\times2+1\times2+2\times1 \\ 2\times1+2\times2+1\times2 & 2\times2+2\times1+1\times2 & 2\times2+2\times2+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ **Showed:**

Problem: $Let, g(y) = y^2 - 5y + 6, find\ f(B), if\ B = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Solution: Given That, $g(y) = y^2 - 5y + 6$

$$so\ f(B) = B^2 - 5B + 6I\ ,\ \ Where\ I\ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $B^2 = B.B$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 2 + 0\times 2 + 1\times 1 & 2\times 0 + 0\times 1 + 1\times -1 & 2\times 1 + 0\times 3 + 1\times 0 \\ 2\times 2 + 1\times 2 + 3\times 1 & 2\times 0 + 1\times 1 + 3\times -1 & 2\times 1 + 1\times 3 + 3\times 0 \\ 1\times 2 + (-1)\times 2 + 0\times 1 & 1\times 0 + (-1)\times 1 + 0\times -1 & 1\times 1 + (-1)\times 3 + 0\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1-0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore B^2 - 5B + 6I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 2 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 2 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10-6 & -11-0+0 & 2-5+0 \\ 9-10+0 & -2-5-6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2+0-6 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -1 & -3 \\ -1 & -13 & -10 \\ -5 & 4 & -8 \end{bmatrix}$$

Answer:

Problem : If A = $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ $then\ show\ that\ A^3 + A^2 - 21A - 45I = 0$

Solution : Here , $A^2 = A.A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 4+4+3 & -4+2+6 & 6-12+0 \\ -4+2+6 & 4+1+12 & -6+6+0 \\ 2-4-0 & -2-2-0 & 3+12+0 \end{bmatrix} = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}$$

And $A^3 = A^2.A = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 43 & -114 \\ -19 & -38 & 30 \end{bmatrix}$$

Now $A^3 + A^2 - 21A - 45I$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 43 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} - 21\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \quad 45\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 38 & -57 \\ 38 & 43 & -114 \\ -19 & -38 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix} -$$

$$\begin{bmatrix} -42 & 42 & -63 \\ 42 & 21 & -126 \\ -21 & -42 & 0 \end{bmatrix} - \quad \begin{bmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{bmatrix}$$

$$= \begin{bmatrix} -8+11+42-45 & 38+4-42-0 & -57-6+63-0 \\ 38+4-42-0 & 43+17-21-45 & -114-12+126-0 \\ -13-2+21-0 & -38-4+42-0 & 30+15-0-45 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore A^3 + A^2 - 21A - 45I = 0$ (Proved)

Problem : If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ $and\ B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & -2 \end{bmatrix}$ $verify$

The result, $(A+B)^2 = A^2 + AB + BA + B^2$

Solution : We have $A + B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 1+3 & 2+(-1) & -1+1 \\ 2+0 & 0+0 & 3+2 \\ 0+4 & 1+(-3) & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix}$$

Then $(A+B)^2 = (A+B)(A+B)$

$$= \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times 2+1\times 2+0\times 4 & 4\times 1+1\times 0+0\times -2 & 4\times 0+1\times 5+0\times 4 \\ 2\times 4+0\times 2+5\times 4 & 2\times 1+0\times 0+5\times -2 & 2\times 0+0\times 5+5\times 4 \\ 4\times 4+-2\times 2+4\times 4 & 41+-2\times 0+4\times 2 & 4\times 0+-2\times 5+4\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}$$

Now $A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+0 & 2+0-1 & -1+6-2 \\ 2+0+0 & 4+0+3 & -2+0+6 \\ 0+2+0 & 0+0+2 & 0+3+4 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 7 & 4 \\ 2 & 2 & 7 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0-4 & -1+0+3 & 1+4+2 \\ 6+0+12 & -2+0-3 & 2+0+6 \\ 0+0+8 & 0+0-6 & 0+2+4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 18 & -11 & 8 \\ 8 & -6 & 6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2+0 & 6+0+1 & -3-3+2 \\ 0+0+0 & 0+0+2 & 0+0+4 \\ 4-6+0 & 8+0+2 & -4-9+4 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 4 \\ 0 & 2 & 4 \\ -2 & 10 & -9 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & -2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9+0+4 & -3+0-3 & 3-2+2 \\ 0+0+8 & 0+0-6 & 0+0+4 \\ 12-0+8 & -4+0-6 & 4-6+4 \end{bmatrix} = \begin{bmatrix} 13 & -6 & 3 \\ 8 & -6 & 4 \\ 20 & -10 & 2 \end{bmatrix}$$

$$\therefore A^2 + AB + BA + B^2 = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 7 & 4 \\ 2 & 2 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 3 \\ 18 & -11 & 8 \\ 8 & -6 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 7 & 4 \\ 0 & 2 & 4 \\ -2 & 10 & -9 \end{bmatrix} + \begin{bmatrix} 13 & -6 & 3 \\ 8 & -6 & 4 \\ 20 & -10 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-1+1+13 & 1+2+7-6 & 3+3-4+3 \\ 2+18+0+8 & 7-11+2-6 & 4+8+4+4 \\ 2+8-2+20 & 2-6+10-10 & 7+6-9+2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}$$

thus $(A+B)^2 = A^2 + AB + BA + B^2$(Proved)

Problem(Home work) : 1) if A= $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ And B= $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$=Then Prove that, $(A+B)^2 = A^2 + AB + BA + B^2$

2) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$ And $B = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 0 & 1 \\ 0 & 5 & 1 \end{bmatrix}$ Then Find $3A, AB, BA$. Show that $AB \neq BA$

3) If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Then Show that, $A^3 = A^2A = AA^2 = I$

4) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ Then Show that, $A^3 - 3A^2 - A + 9I = 0$

5) $Let, g(z) = z^3 - 6z + 2, find\ the\ f(B), if\ B = \begin{bmatrix} 2 & -5 & 1 \\ 2 & 1 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

Problem 01: If A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ show that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

Solution:

$\boldsymbol{Given\ that},\ \ A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$Now, A^2 = A.A$

$= \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 3 \times 3 + (-4) \times 1 & 3 \times -4 + (-4) \times -1 \\ 1 \times 3 + (-1) \times 1 & 1 \times -4 + (-1) \times (-1) \end{bmatrix}$

$= \begin{bmatrix} 9-4 & -12+4 \\ 3-1 & -4+1 \end{bmatrix}$

$= \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 1+2(2) & -4(2) \\ 2 & 1-2(2) \end{bmatrix}$

$= A^n\ where\ n = 2$

$\therefore A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}\ \ holds\ \ when\ n = 2$

Now $A^{n+1} = A^n.A$

$= \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} (1+2n) \times 3 + (-4n) \times 1 & (1+2n) \times (-4) + (-4n) \times (-1) \\ n \times 3 + (1-2n) \times 1 & n \times (-4) + (1-2n) \times (-1) \end{bmatrix}$

$= \begin{bmatrix} 3+6n-4n & -4-8n+4n \\ 3n+1-2n & -4n-1+2n \end{bmatrix}$

$= \begin{bmatrix} 3+2n & -4-4n \\ n+1 & -1-2n \end{bmatrix}$

$$= \begin{bmatrix} 1+2(n+1) & -4(n+1) \\ n+1 & 1-2(n+1) \end{bmatrix}$$

That is $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ $holds\ for$ 'n'=n+1

Also we have shown above that it holds for $n = 2$, hence by mathematical induction it is true for all positive integers.

Problem. 02 : Show that $\begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}^n = \begin{bmatrix} Cosn\theta & -Sinn\theta \\ Sinn\theta & Cosn\theta \end{bmatrix}$

Solution:

$$let\ A = \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix} \dots\dots\dots\dots\dots\dots\dots\ (i)$$

$$\therefore A^2 = A.A$$

$$= \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix} \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cos\theta Cos\theta + (-Sin\theta)Sin\theta) & Cos\theta(-Sin\theta) + (-Sin\theta)Cos\theta \\ Sin\theta Cos\theta + Cos\theta Sin\theta & -Sin\theta Sin\theta + Cos\theta Cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cos^2\theta - Sin^2\theta & -Cos\theta Sin\theta - Sin\theta Cos\theta \\ Sin\theta Cos\theta + Cos\theta Sin\theta & -Sin^2\theta + Cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cos2\theta & -Sin2\theta \\ Sin2\theta & Cos2\theta \end{bmatrix} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

$$\therefore A^3 = A^2.A$$

$$= \begin{bmatrix} Cos2\theta & -Sin2\theta \\ Sin2\theta & Cos2\theta \end{bmatrix} = \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cos2\theta Cos\theta + (-Sin2\theta)Sin\theta & Cos2\theta(-Sin\theta) + (-Sin\theta)Cos\theta \\ Sin2\theta Cos\theta + Cos2\theta Sin\theta & Sin2\theta(-sin2\theta) + Cos2\theta Cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cos(2\theta+\theta) & -Sin(2\theta+\theta) \\ Sin(2\theta+\theta) & -Cos(2\theta+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} Cos3\theta & -Sin3\theta \\ Sin3\theta & -Cos3\theta \end{bmatrix} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(iii)$$

$In\ the\ light\ of\ (i), (ii), (iii)\ let\ us\ assume\ that,$

$$A^n = \begin{bmatrix} Cosn\theta & -Sinn\theta \\ Sinn\theta & Cosn\theta \end{bmatrix} \dots\dots\dots\dots\dots\dots\dots\dots\dots.(iv)$$

$$Now\ A^{n+1} = A^n.A$$

$$= \begin{bmatrix} Cosn\theta & -Sinn\theta \\ Sinn\theta & Cosn\theta \end{bmatrix} \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cosn\theta Cos\theta + (-Sinn\theta)Sin\theta & Cosn\theta(-sin\theta) + (-sinn\theta)Cos\theta \\ Sinn\theta Cos\theta + Cosn\theta Sin\theta & Sinn\theta(-Sin\theta) + Cosn\theta Cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cos(n\theta + \theta) & -Sin(n\theta + \theta) \\ \mathrm{Sin}(n\theta + \theta) & -Cos(n\theta + \theta) \end{bmatrix}$$

$$= \begin{bmatrix} Cos(n+1)\theta & -Sin(n+1)\theta \\ Sin(n+1)\theta & -Cos(n+1)\theta \end{bmatrix}$$

$That\ is\ (iv) holds\ for\ n+1\ if\ it\ is\ true\ for\ n.$

$Hence\ , (iv)\ holds\ for\ all\ positive\ integral\ \ Value\ of\ n.$

$$\therefore A^n = \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}^n$$

$$= \begin{bmatrix} Cosn\theta & -Sinn\theta \\ Sinn\theta & Cosn\theta \end{bmatrix} \qquad [\boldsymbol{Hence\ Proved}]$$