Integrating by Parts

In calculus, and more generally in mathematical analysis, **integration by parts** or **partial integration** is a process that finds the integral of a product of functions in terms of the integral of the product of their derivative and antiderivative. It is frequently used to transform the antiderivative of a product of functions into an antiderivative for which a solution can be more easily found. The rule can be thought of as an integral version of the product rule of differentiation.

Formula: $\int uvdx = u\int vdx - \int\left\{\frac{du}{dx}\int vdx\right\}dx,$ where u is differentiable function and v is integrable function of x.

Find the value of the following integration:

$i)\ \int x\,cosx\,dx$ $\quad ii)\ \int lnx\,dx$ $\quad iii)\ \int \frac{xe^x}{(1+x)^2}dx$ $\quad iv)\ \int (sin^{-1}x)^2 dx$

$v)\ \int x^2 sin^2 x\,dx$ $\quad vi)\ \int \frac{ln(1+x)}{\sqrt{x+1}}dx$

Solved – (i):

$$Let,\ \ I = \int x\,cosx\,dx$$

$$= x\int cosx\,dx - \int\left\{\frac{dx}{dx}\int cosx\,dx\right\}dx$$

$$= x\,sinx - \int 1.sinx\,dx$$

$$= x\,sinx - \int sinx\,dx$$

$$= x\,sinx - (-cosx) + c$$

$$= x\,sinx + cosx + c$$

Solved – (ii):

Let, $\quad I = \int lnx\,dx$

$$= \int 1.lnx\,dx$$

$$= lnx \int dx - \int \left\{\frac{d}{dx} lnx \int dx\right\} dx$$

$$= lnx \times x - \int \frac{1}{x} \times x\, dx$$

$$= x\, lnx - \int dx$$

$$= x\, lnx - x + c. \quad \text{Ans.}$$

Solved – (iii):

$$Let,\; I = \int \frac{xe^x}{(1+x)^2}\, dx$$

$$= \int e^x \left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\} dx$$

$$= \int e^x \{f(x) + f'(x)\}\, dx$$

$$= e^x f(x) + c$$

$$= e^x \frac{1}{1+x} + c \quad \text{Ans.}$$

Let, $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$$\Rightarrow f'(x) = -\frac{1}{(1+x)^2}$$

Solved – (iv):

Let, $\quad I = \int (sin^{-1}x)^2 dx$

$$= \int 1.(sin^{-1}x)^2 dx$$

$$= (sin^{-1}x)^2 \int dx - \int \left\{\frac{d}{dx}(sin^{-1}x)^2 \int dx\right\} dx$$

$$= (sin^{-1}x)^2.x - \int \left\{2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}.x\right\} dx$$

$$= x.(sin^{-1}x)^2 - 2\int \frac{xsin^{-1}x}{\sqrt{1-x^2}} dx$$

$$= x.(sin^{-1}x)^2 - 2\int sinz.zdz$$

$$= x.(sin^{-1}x)^2 - 2\int z\, sinz\, dz$$

$$= x.(sin^{-1}x)^2 - 2\left[z\int sinz\, dz - \int \left\{\frac{dz}{dz}\int sinz\, dz\right\} dz\right]$$

Let, $sin^{-1}x = z$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dz$$

Again, $sin^{-1}x = z$

$$\Rightarrow x = sinz$$

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$$= x.(sin^{-1}x)^2 - 2\left[z(-cosz) + \int cosz\ dz\right]$$

$$= x.(sin^{-1}x)^2 - 2\left[-zcosz + \int cosz\ dz\right]$$

$$= x.(sin^{-1}x)^2 + 2z\ cosz - 2sinz + c$$

$$= x.(sin^{-1}x)^2 + 2sin^{-1}x\ cos(sin^{-1}x) - 2sin(sin^{-1}x) + c$$

<u>Solved – (v):</u>

$$Let,\ \ I = \int x^2 sin^2 x\ dx$$

$$= \tfrac{1}{2}\textstyle\int x^2 sin^2 x\ dx$$

$$= \frac{1}{2}\int x^2(1 - cos2x)dx$$

$$= \frac{1}{2}\int x^2 dx - \frac{1}{2}\int x^2 cos2x\ dx$$

$$= \frac{1}{2}\frac{x^3}{3} - \frac{1}{2}\left[x^2\int cos2x - \int\left\{\frac{d}{dx}x^2\int cos2x\ dx\right\}dx\right]$$

$$= \frac{1}{2}\frac{x^3}{3} - \frac{1}{2}\left[x^2\frac{sin2x}{2} - \int\left\{2x - \frac{sin2x}{2}\right\}dx\right]$$

$$= \frac{1}{2}\frac{x^3}{3} - \frac{1}{4}x^2 sin2x + \frac{1}{2}\int x\ sin2x\ dx$$

$$= \frac{1}{2}\frac{x^3}{3} - \frac{1}{4}x^2 sin2x + \frac{1}{2}\left[x\int \sin 2x\ dx - \int\left\{\frac{dx}{dx}\int \sin 2x\ dx\right\}dx\right]$$

$$= \frac{1}{2}\frac{x^3}{3} - \frac{1}{4}x^2 sin2x + \frac{1}{2}\left[x.\frac{-cos2x}{2} - \int\frac{-cos2x}{2}dx\right]$$

$$= \frac{1}{2}\frac{x^3}{3} - \frac{1}{4}x^2 sin2x + \frac{1}{4}(-xcos2x) + \frac{1}{4}\int cos2x\ dx$$

$$= \tfrac{1}{2}\tfrac{x^3}{3} - \tfrac{1}{4}x^2 sin2x - \tfrac{1}{4}xcos2x + \tfrac{1}{4}\tfrac{sin2x}{2} + c$$

Solved – (vi):

$$Let,\ \ I = \int \frac{ln(1+x)}{\sqrt{x+1}}\ dx$$

$$= ln(1+x)\int \frac{1}{\sqrt{1+x}}dx - \int\left\{\frac{d}{dx}ln(1+x)\int\frac{1}{\sqrt{1+x}}dx\right\}dx$$

$$= ln(1+x)\ 2\sqrt{1+x} - \int\left\{\frac{1}{1+x}.\,2\sqrt{1+x}\right\}dx$$

$$= 2\sqrt{1+x}.\,ln(1+x)\ - \int\frac{2}{\sqrt{1+x}}dx$$

$$= 2\sqrt{1+x}.\,ln(1+x)\ - 2\int\frac{1}{\sqrt{1+x}}dx$$

$$= 2\sqrt{1+x}.\,ln(1+x)\ - 2.2\sqrt{1+x} + c$$

$$= 2\sqrt{1+x}\{ln(1+x)\ - 2\} + c$$