MAT 3850 : Week 4

Fall 2023

App State

Section 1

Outline for the week

By the end of the week: Basic Regression

- Data Modeling
- Exploratory data analysis
- Linear regression

Section 2

Basic Regression

Basic Regression

- Now that we are equipped with
  - an understanding of how to import data
  - data visualization and
  - data wrangling skill
- Let's now proceed with **data modeling**.
- The fundamental premise of data modeling is to make explicit the relationship between:
  - an **outcome variable** y, also called a **dependent variable** or **response variable**, and
  - an e**xplanatory/predictor** variable x, also called an **independent variable** or **covariate**.

Data Modeling

Data modeling serves one of two purposes:

1. Modeling for explanation:
   - Describe and quantify the relationship between the outcome variable y and a set of explanatory variables x.
   - Determine the significance of any relationships.
   - Have measures summarizing these relationships.
   - Possibly identify any causal relationships between the variables.
2. Modeling for prediction:
   - Predict an outcome variable y based on the information contained in a set of predictor variables x.
   - Here, you don't care so much about understanding how all the variables relate and interact with one another.

Data Modeling

- For example, say you are interested in
  - an outcome variable y of whether patients develop lung cancer and
  - information x on their risk factors, such as smoking habits, age, and socioeconomic status.
- If we are modeling for explanation,
  - we would be interested in both describing and quantifying the effects of the different risk factors.
  - One reason could be that you want to design an intervention to reduce lung cancer incidence in a population, such as targeting smokers of a specific age group with advertising for smoking cessation programs.
- If we are modeling for prediction,
  - we wouldn't care so much about understanding how all the individual risk factors contribute to lung cancer,
  - but rather only whether we can make good predictions of which people will contract lung cancer.

Linear regression

- There are many techniques for modeling, such as
  - tree-based models and
  - neural networks,
- But in this class, we'll focus on one particular technique: **linear regression**.
- Linear regression involves a numerical outcome variable y and explanatory variables x that are either numerical or categorical.
  - the relationship between y and x is assumed to be linear, or in other words, a line.
  - However, we'll see that what constitutes a "line" will vary depending on the nature of your explanatory variables x.
  - Linear regression is one of the most commonly-used and easy-to-understand approaches to modeling.

Needed packages

Let's now load all the packages needed

```
library(ggplot2)  #  for data visualization
library(dplyr)    #  for data wrangling
library(readr)    # for importing spreadsheet data into R
library(moderndive) # package of datasets and regression functions
library(skimr)      # provides a simple-to-use functions
                    # for summary statistics
```

One numerical explanatory variable

- Researchers at the University of Texas in Austin, Texas (UT Austin) tried to answer the following research question:
    - what factors explain differences in instructor teaching evaluation scores?
- To this end, they collected instructor and course information on 463 courses.
- A full description of the study can be found at openintro.org.
- The data on the 463 courses at UT Austin can be found in the `evals` data frame included in the `moderndive` package.

One numerical explanatory variable

let's fully describe only the 4 variables we will focus on:

1. `ID`: An identification variable used to distinguish between the 1 through 463 courses in the dataset.
2. `score`: A numerical variable of the course instructor's average teaching score, where the average is computed from the evaluation scores from all students in that course. Teaching scores of 1 are lowest and 5 are highest. This is the outcome variable y of interest.
3. `bty_avg`: A numerical variable of the course instructor's average "beauty" score, where the average is computed from a separate panel of six students. "Beauty" scores of 1 are lowest and 10 are highest. This is the explanatory variable x of interest.
4. `age`: A numerical variable of the course instructor's age. This will be another explanatory variable x that we'll use later.

One numerical explanatory variable

We'll answer these questions by modeling the relationship between teaching scores and "beauty" scores using simple linear regression where we have:

1. A numerical outcome variable y (the instructor's teaching score) and
2. A single numerical explanatory variable x (the instructor's "beauty" score).

Exploratory data analysis

- A crucial step before doing any kind of analysis or modeling is performing an exploratory data analysis, or EDA for short.
  - Get distributions of the individual variables in your data,
  - Find out any potential relationships exist between variables,
  - Find out any outliers and/or missing values, and
  - (most importantly) helps you to decide how to build your model.
- Here are three common steps in an EDA:
  1. Most crucially, looking at the raw data values.
  2. Computing summary statistics, such as means, medians, and interquartile ranges.
  3. Creating data visualizations.

Step 1: looking at the raw data values

```
evals_ch5 <- evals %>%
  select(ID, score, bty_avg, age)   # take subset
glimpse(evals_ch5)
```

```
## Rows: 463
## Columns: 4
## $ ID      <int> 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,~
## $ score   <dbl> 4.7, 4.1, 3.9, 4.8, 4.6, 4.3, 2.8, 4.1, 3.4, 4.5, 3.8, 4.5, 4.~
## $ bty_avg <dbl> 5.000, 5.000, 5.000, 5.000, 3.000, 3.000, 3.000, 3.333, 3.333,~
## $ age     <int> 36, 36, 36, 36, 59, 59, 59, 51, 51, 40, 40, 40, 40, 40, 40, 40~
```

An alternative way to look at the raw data values is by choosing a random sample of the rows.

```
evals_ch5 %>%
  sample_n(size = 5)
```

```
## # A tibble: 5 x 4
##      ID score bty_avg   age
##   <int> <dbl>   <dbl> <int>
## 1   356   5      3.33    50
## 2    34   4.2    4.17    33
## 3   241   3.7    4.67    58
## 4   327   3.8    2.33    64
## 5   253   4.1    3.17    52
```

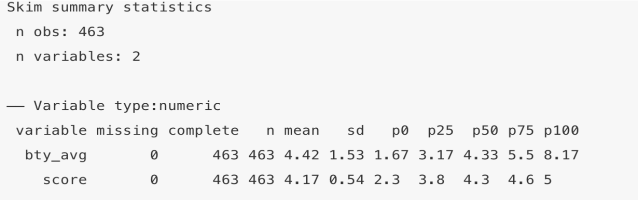
Step 2: summary statistics

```
evals_ch5 %>%
  summarize(mean_bty_avg = mean(bty_avg), mean_score = mean(score),
            median_bty_avg = median(bty_avg), median_score = median(score))
```

```
## # A tibble: 1 x 4
##   mean_bty_avg mean_score median_bty_avg median_score
##          <dbl>      <dbl>          <dbl>        <dbl>
## 1         4.42       4.17           4.33          4.3
```

`skim()` function from the `skimr` package, “skims” the date, and returns commonly used summary statistics

```
library(knitr)
evals_ch5 %>% select(score, bty_avg) %>% skim()
```



```
Skim summary statistics
 n obs: 463
 n variables: 2

── Variable type:numeric
 variable missing complete   n mean   sd   p0  p25  p50 p75 p100
  bty_avg       0      463 463 4.42 1.53 1.67 3.17 4.33 5.5 8.17
    score       0      463 463 4.17 0.54 2.3  3.8  4.3  4.6 5
```

Correlation coefficient r

When the two variables are numerical, we can compute the **correlation coefficient**.

- The correlation coefficient, denoted by r , measures the direction and strength of the linear relationship between two numerical variables. Is is given by the equation

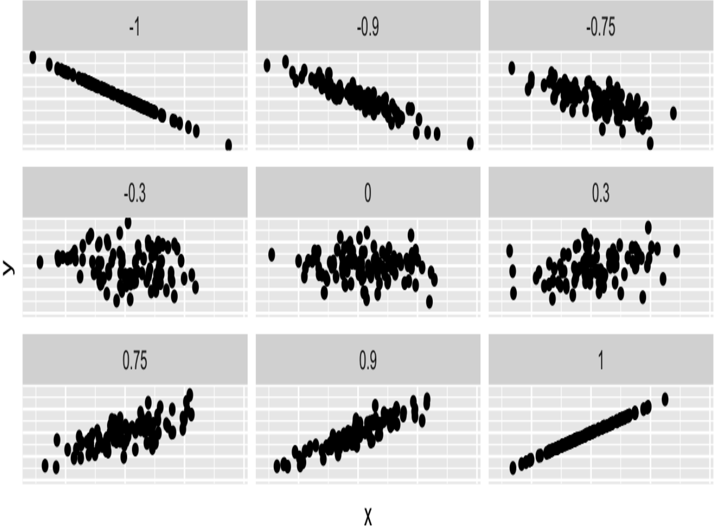
$$r = \frac{1}{(n-1)} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{s_x} \right) \left(\frac{y_i - \bar{y}}{s_y} \right) = \frac{\sum z_x z_y}{n-1}$$

where $\bar{x}$ and $\bar{y}$ represents the mean of the x and y variables. Also, s_x and s_y denotes the standard deviation. Then z_x and z_y are the z-scores.

Properties of r

- sign of r gives direction of association
- $-1 \leq r \leq 1$
  - -1 indicates a perfect negative relationship: As one variable increases, the value of the other variable tends to go down, following a straight line.
  - 0 indicates no relationship: The values of both variables go up/down independently of each other.
  - $+1$ indicates a perfect positive relationship: As the value of one variable goes up, the value of the other variable tends to go up as well in a linear fashion.
- $r_{x,y} = r_{y,x}$
- Correlation has no units.
- Correlation is not affected by multiplying or shifting data
- Correlation measures LINEAR association only
- Outliers affect correlation greatly

Correlation coefficient and scatterplot



Correlation coefficien: GPA Example

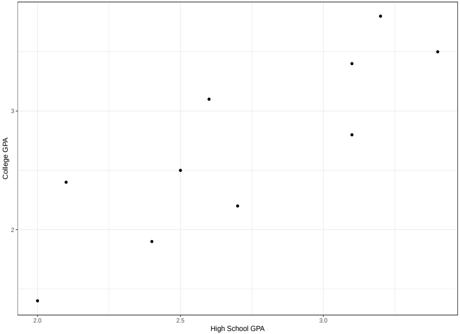
Following are the high school GPAs and the college GPAs at the end of the freshman year for ten different students from the `Gpa` data set of the `BSDA` package.

```
library(BSDA)  # INSTALL
head(Gpa)
```

```
## # A tibble: 6 x 2
##   hsgpa collgpa
##   <dbl>   <dbl>
## 1   2.7     2.2
## 2   3.1     2.8
## 3   2.1     2.4
## 4   3.2     3.8
## 5   2.4     1.9
## 6   3.4     3.5
```

Correlation coefficien: GPA Example

```
ggplot(data = Gpa, aes(x = hsgpa, y = collgpa)) +
  labs(x = "High School GPA", y = "College GPA") +
  geom_point() +
  theme_bw()
```



The scatterplot shows that the college GPA increases as the high school GPA increases

Correlation coefficien: GPA Example

```
values <- Gpa %>%
  mutate(y_ybar = collgpa - mean(collgpa), x_xbar = hsgpa - mean(hsgpa),
         zx = x_xbar/sd(hsgpa), zy = y_ybar/sd(collgpa))
values
```

```
## # A tibble: 10 x 6
##    hsgpa collgpa y_ybar  x_xbar      zx     zy
##    <dbl>   <dbl>  <dbl>   <dbl>   <dbl>  <dbl>
##  1   2.7     2.2 -0.5   -0.0100 -0.0210 -0.657
##  2   3.1     2.8  0.100  0.39    0.817   0.131
##  3   2.1     2.4 -0.300 -0.61   -1.28   -0.394
##  4   3.2     3.8  1.1    0.49    1.03    1.44
##  5   2.4     1.9 -0.8   -0.31   -0.650  -1.05
##  6   3.4     3.5  0.8    0.69    1.45    1.05
##  7   2.6     3.1  0.4   -0.110  -0.231   0.525
##  8   2       1.4 -1.3   -0.71   -1.49   -1.71
##  9   3.1     3.4  0.7    0.39    0.817   0.919
## 10   2.5     2.5 -0.200 -0.21   -0.440  -0.263
```

$$r = \frac{1}{(n-1)} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{s_x} \right) \left(\frac{y_i - \bar{y}}{s_y} \right) = \frac{\sum z_x z_y}{n-1}$$

Correlation coefficien: GPA Example

```
values %>%
  summarize(r = (1/9)*sum(zx*zy))
```

```
## # A tibble: 1 x 1
##       r
##   <dbl>
## 1 0.844
```

Using the build in `cor()` function:

```
Gpa %>%
  summarize(r = cor(collgpa, hsgpa))
```

```
## # A tibble: 1 x 1
##       r
##   <dbl>
## 1 0.844
```

Using `get_correlation()` function in the moderndive package.

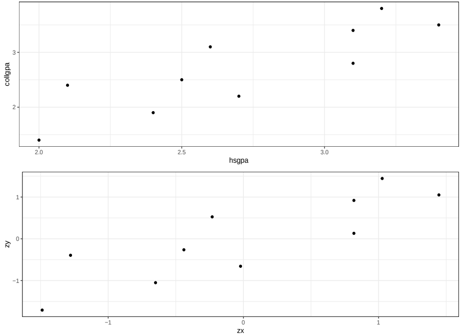
```
Gpa %>%
  get_correlation(formula =collgpa~ hsgpa)
```

```
## # A tibble: 1 x 1
##     cor
##   <dbl>
## 1 0.844
```

Correlation coefficien: GPA Example

Note:

```r
p1 <- ggplot(data = Gpa, aes(x = hsgpa, y = collgpa)) +
  geom_point() +
  theme_bw()
p2 <- ggplot(data = values, aes(x = zx, y = zy)) +
  geom_point() +
  theme_bw()
library(gridExtra)
grid.arrange(p1, p2, ncol = 1, nrow = 2)
```



Correlation coefficient: Teaching Evaluations Example

```
evals_ch5 %>%
  get_correlation(formula = score ~ bty_avg)
```

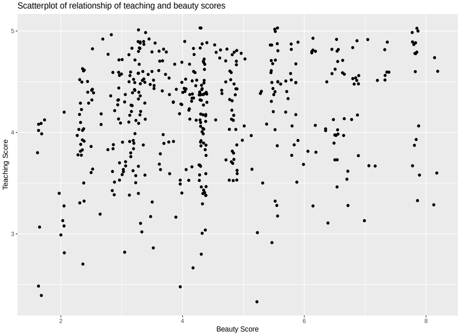
```
## # A tibble: 1 x 1
##     cor
##   <dbl>
## 1 0.187
```

```
evals_ch5 %>%
  summarize(correlation = cor(score, bty_avg))
```

```
## # A tibble: 1 x 1
##   correlation
##         <dbl>
## 1       0.187
```

Step 3: creating data visualizations

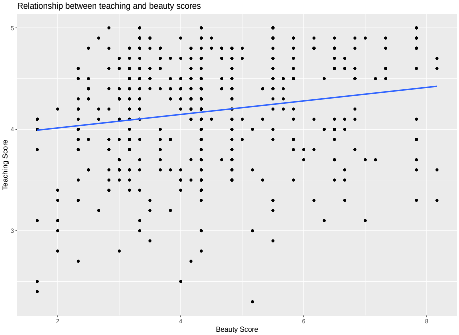
```
ggplot(evals_ch5, aes(x = bty_avg, y = score)) +
  geom_jitter() +
  labs(x = "Beauty Score", y = "Teaching Score",
       title = "Scatterplot of relationship of teaching and beauty scores")
```



Step 3: creating data visualizations

Add "best-fitting" line (regression line).

```
ggplot(evals_ch5, aes(x = bty_avg, y = score)) +
  geom_point() +
  labs(x = "Beauty Score", y = "Teaching Score",
       title = "Relationship between teaching and beauty scores") +
  geom_smooth(method = "lm", se = FALSE)
```



Section 3

Simple linear regression

Simple linear regression

You may recall from secondary/high school algebra that the equation of a line is:

$$y = a + b \cdot x$$

- The intercept coefficient is a is the value of y when $x = 0$.
- The slope coefficient b for x is the increase in y for every increase in x.

However, when defining regression equation line, we use slightly different notation.

Simple linear regression

The regression equation is given by:

$$y = \beta_0 + \beta_1 x + \epsilon$$

- where β_0 is the intercept
- β_1 is the slope.
- For the ith trial, we have:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Simple linear regression

The line that best fits the data is given by,

$$\hat{y} = b_0 + b_1 x$$

where b_0 and b_1 are estimates for the population parameters β_0 and β_1.

- Form the best fit line, we can compute the:
  - predicted $\hat{y}$ for each x and
  - measure the error of prediction.
- The error of prediction, e_i (also called residual) is the difference in the actual y_i and the predicted $\hat{y}_i$.

$$e_i = y_i - \hat{y}_i.$$

The least squares regression line

The least squares regression line is:

$$\hat{y} = b_0 + b_1 x$$

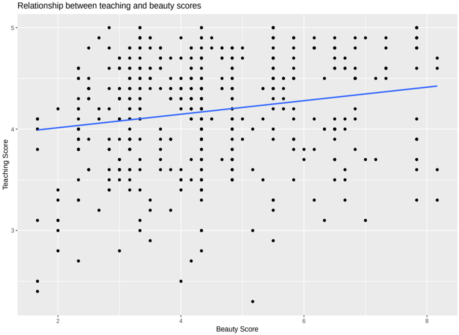
where

$$b_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = r \frac{s_y}{s_x}$$

and

$$b_0 = \bar{y} - b_1 \bar{x}.$$

Regression: Teaching Evaluations Example



- We know that the regression line has a positive slope b_1 corresponding to our explanatory x variable `bty_avg`.
- However, what is the numerical value of the slope b_1? What about the intercept b_0?

Regression: Teaching Evaluations Example

We obtain the regression line parameters in two steps:

1. We “fit” the linear regression model using the `lm()` function and save it, lets call it `score_model`.
2. We get the regression table by applying the `get_regression_table()` function from the `moderndive` package to `score_model`.

Regression: Teaching Evaluations Example

```r
# Fit regression model:
score_model <- lm(score ~ bty_avg, data = evals_ch5)
# Get regression table:
get_regression_table(score_model)
```

```
## # A tibble: 2 x 7
##   term      estimate std_error statistic p_value lower_ci upper_ci
##   <chr>        <dbl>     <dbl>     <dbl>   <dbl>    <dbl>    <dbl>
## 1 intercept    3.88      0.076     51.0        0    3.73     4.03
## 2 bty_avg      0.067     0.016      4.09       0    0.035    0.099
```

```r
# Use formula

evals_ch5 %>%
  summarize(b1 = cor(bty_avg, score)*sd(score)/sd(bty_avg),
            b0 = mean(score) - b1*mean(bty_avg))
```

```
## # A tibble: 1 x 2
##       b1    b0
##    <dbl> <dbl>
## 1 0.0666  3.88
```

Regression: Teaching Evaluations Example

Lets interpret the regression table. The equation of the regression line:

$$\begin{aligned}\hat{y} &= b_0 + b_1 \cdot x \\ \widehat{\text{score}} &= b_0 + b_{\text{bty\_avg}} \cdot \text{bty\_avg} \\ &= 3.88 + 0.067 \cdot \text{bty\_avg}\end{aligned}$$

- The intercept $b_0 = 3.88$
  - is the average teaching score $\hat{y} = \widehat{score}$ for those courses where the instructor had a “beauty” score `bty_avg` of 0.
  - Note however that `bty_avg` of 0 is impossible since the beauty scores ranges from 1 to 10.
- The slope b_1 of `bty_avg` is 0.067.
  - The sign is positive, suggesting a positive relationship between these two variables, meaning teachers with higher “beauty” scores also tend to have higher teaching scores.
  - For every increase of 1 unit in `bty_avg`, there is an associated increase of, on average, 0.067 units of `score`.

Observed/fitted values and residuals

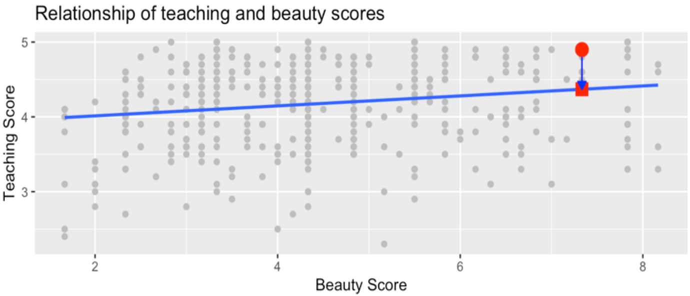
Now we are interested in information on individual observations. For example, let's focus on the 21st of the 463 courses in the evals_ch5 dataframe

```
# Fit regression model:
evals_ch5[21,]
```

```
## # A tibble: 1 x 4
##      ID score bty_avg   age
##   <int> <dbl>   <dbl> <int>
## 1    21   4.9    7.33    31
```

- We want to know what is the value $\hat{y}$ on the regression line corresponding to instructor's bty_avg "beauty" score of 7.333.

Observed/fitted values and residuals



- Square: The fitted value $\hat{y}_i$ is given by

$$\hat{y}_i = b_0 + b_1 \cdot x = 3.88 + 0.067 \cdot 7.333 = 4.369$$

- Circle: The observed value $y_i = 4.9$.
- Arrow: The length of the arrow is the residual or error and is given by

$$e_i = y_i - \hat{y}_i = 4.9 - 4.369 = 0.531.$$

Observed/fitted values and residuals

To compute both the fitted value and residual for all observations in the data we use the `get_regression_points()` function.

```r
regression_points <- get_regression_points(score_model)
regression_points
```

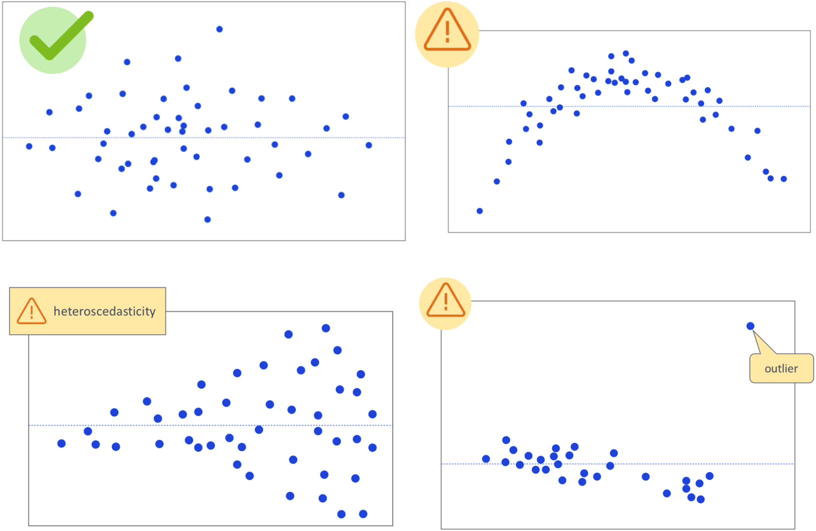
```
## # A tibble: 463 x 5
##       ID score bty_avg score_hat residual
##    <int> <dbl>   <dbl>     <dbl>    <dbl>
##  1     1   4.7    5         4.21    0.486
##  2     2   4.1    5         4.21   -0.114
##  3     3   3.9    5         4.21   -0.314
##  4     4   4.8    5         4.21    0.586
##  5     5   4.6    3         4.08    0.52
##  6     6   4.3    3         4.08    0.22
##  7     7   2.8    3         4.08   -1.28
##  8     8   4.1    3.33      4.10   -0.002
##  9     9   3.4    3.33      4.10   -0.702
## 10    10   4.5    3.17      4.09    0.409
## # i 453 more rows
```

Let's repeat the calculations for the instructor of the 1st course.

Assessing the fit

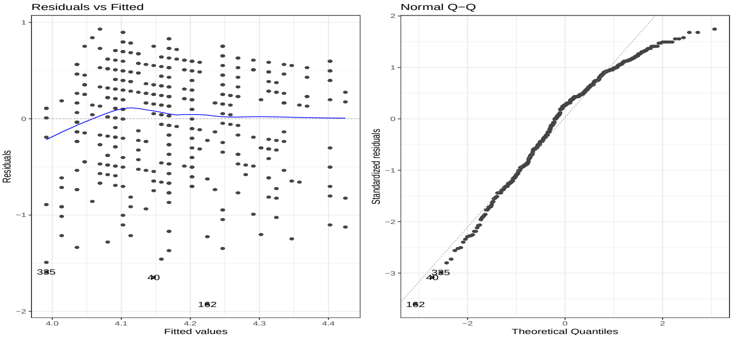
- The regression model is a good model if the scatterplot of residual vs. x-values or scatterplot of residual vs. $\hat{y}$-values has no interesting features.
  - No direction
  - No shape
  - No bends
  - No outliers
  - No identifiable pattern
  - Equal or constant variance (homoscedasticity)
- In addition, for a good model, the residuals are approximately normally distributed. Check histogram of the residuals.

Assessing the fit



Assessing the fit: Teaching Evaluations Example

```
library(ggfortify)
autoplot(score_model, ncol = 2, nrow = 1, which = 1:2) + theme_bw()
```



Section 4

Confidence Intervals and Tests

Estimating σ^2

The regression equations give the mean of the group. What is the standard deviation of this group? That is, we have to estimate σ.

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad Var(\epsilon_i) = \sigma^2.$$

A natural estimator for σ^2 is:

$$\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - E(\epsilon))^2 = \frac{1}{n}\sum_{i=1}^{n}(\epsilon_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2.$$

Since, β_0 and β_1 are unknown, we use the estimates:

$$\frac{1}{n}\sum_{i=1}^{n} e_i^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - b_0 - b_1 x_i)^2 = \frac{SSE}{n}$$

Estimating σ^2

Since b_0, b_1 are estimated, $\hat{e}_i^2$ are no longer independent, we use the following estimator:

$$s^2 = \frac{SSE}{n-2} = MSE,$$

where $n-2$ is the degree of freedom (df) (why?), and MSE stands for error mean square or residual mean square. Generally, $df =$ number of cases - number of parameters.

- The residual standard error, $s = \sqrt{MSE}$, gives the average error the model predicts.

Properties of OLS

If we assume that $\epsilon_i \sim N(0, \sigma^2)$, then the OLS estimates are also maximum likelihood estimates (MLE). Under the normal assumption,

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2}{\sum(x_i - \bar{x})^2}\right),$$

$$b_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum(x_i - \bar{x})^2}\right)\right),$$

These quantities will be used to construct confidence intervals, testing, and other statistical inferences.

Confidence Intervals

Linear model assumptions:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma^2)$$

Under this model:

$$\frac{b_0 - \beta_0}{S(b_0)} \sim t_{n-2}, \qquad \frac{b_1 - \beta_1}{S(b_1)} \sim t_{n-2}$$

- Hence, $100(1-\alpha)\%$ confidence interval for β_0 is

$$b_0 \pm t_{n-2,\alpha/2} S(b_0);$$

- $100(1-\alpha)\%$ confidence interval for β_1 is

$$b_1 \pm t_{n-2,\alpha/2} S(b_1);$$

Hypothesis Tests

- A hypothesis test of: $H_0 : \beta_0 = 0 \quad vs \quad H_a : \beta_0 \neq 0$,

is obtained by computing

$$t = \frac{b_0 - 0}{S(b_0)} = \frac{b_0}{S(b_0)} \sim t_{n-2}, \quad \text{under } H_0$$

Then, reject H_0 if $|t| > t_{n-2,\alpha/2}$. P-values can be computed as:

$$\text{P-value} = 2\,\mathrm{Pr}(T > t)$$

reject H_0 if P-value $< \alpha$.

Hypothesis Tests

Similarly, a hypothesis test of $H_0 : \beta_1 = 0 \quad vs \quad H_a : \beta_1 \neq 0$,

is obtained by computing

$$t = \frac{b_1 - 0}{S(b_1)} = \frac{b_1}{S(b_1)} \sim t_{n-2}, \quad \text{under } H_0$$

Then, reject H_0 if $|t| > t_{n-2,\alpha/2}$. P-values can be computed as:

$$\text{P-value} = 2\,\Pr(T > t)$$

reject H_0 if P-value $< \alpha$.

Regression: Teaching Evaluations Example

```
# Fit regression model:
score_model <- lm(score ~ bty_avg, data = evals_ch5)
# Get regression table:
get_regression_table(score_model,conf.level = 0.95) # conf.level = 0.95 is default
```

```
## # A tibble: 2 x 7
##   term      estimate std_error statistic p_value lower_ci upper_ci
##   <chr>        <dbl>     <dbl>     <dbl>   <dbl>    <dbl>    <dbl>
## 1 intercept    3.88      0.076     51.0        0    3.73     4.03
## 2 bty_avg      0.067     0.016      4.09       0    0.035    0.099
```

Regression: Teaching Evaluations Example

```
summary(score_model)
```

```
##
## Call:
## lm(formula = score ~ bty_avg, data = evals_ch5)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.9246 -0.3690  0.1420  0.3977  0.9309
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  3.88034    0.07614   50.96  < 2e-16 ***
## bty_avg      0.06664    0.01629    4.09 5.08e-05 ***
## ---
## Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
##
## Residual standard error: 0.5348 on 461 degrees of freedom
## Multiple R-squared:  0.03502,    Adjusted R-squared:  0.03293
## F-statistic: 16.73 on 1 and 461 DF,  p-value: 5.083e-05
```

Regression: Teaching Evaluations Example

To obtain the `residuals` for `score_model` use the function `resid` on a linear model object.

```r
eis <- resid(score_model)
RSS <- sum(eis^2)
RSS
```

```
## [1] 131.8684
```

```r
RSE <- sqrt(RSS/(dim(evals_ch5)[1]-2))
RSE
```

```
## [1] 0.5348351
```

```r
# Or
summary(score_model)$sigma
```

```
## [1] 0.5348351
```

Regression: Teaching Evaluations Example

```
b0<-coef(score_model)[1]
b1<-coef(score_model)[2]
c(b0,b1)
```

```
## (Intercept)     bty_avg
##  3.88033795  0.06663704
```

```
XTXI <- summary(score_model)$cov.unscaled
MSE <- summary(score_model)$sigma^2
var.cov.b <- MSE*XTXI
var.cov.b
```

```
##              (Intercept)       bty_avg
## (Intercept)  0.005797752 -0.0011725030
## bty_avg     -0.001172503  0.0002654016
```

```
seb0 <- sqrt(var.cov.b[1, 1])
seb1 <- sqrt(var.cov.b[2, 2])
c(seb0,seb1)
```

```
## [1] 0.07614297 0.01629115
```

Regression: Teaching Evaluations Example

```
# confidence interval
df<-dim(evals_ch5)[1]-2
df
```

```
## [1] 461
```

```
##b0
t_critical<-qt(0.975, df)
c(b0-t_critical*seb0,b0+t_critical*seb0)
```

```
## (Intercept) (Intercept)
##    3.730708    4.029968
```

```
##b1
c(b1-t_critical*seb1,b1+t_critical*seb1)
```

```
##    bty_avg    bty_avg
## 0.03462292 0.09865116
```

```
# Or
confint(score_model, level = 0.95)
```

```
##                  2.5 %     97.5 %
## (Intercept) 3.73070764 4.02996827
## bty_avg     0.03462292 0.09865116
```

Regression: Teaching Evaluations Example

```
tb0 <- b0/seb0
tb1 <- b1/seb1
c(tb0, tb1)
```

```
## (Intercept)     bty_avg
##   50.961212    4.090382
```

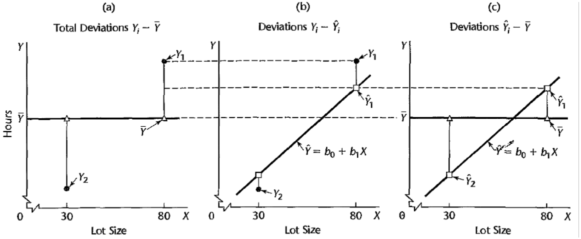
```
pvalues <- c(pt(tb0, df, lower = FALSE)*2, pt(tb1, df, lower = FALSE)*2)
pvalues
```

```
##   (Intercept)       bty_avg
## 1.561043e-191  5.082731e-05
```

Partition of Total sum of squares

- For the linear regression model: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$,
- We have fitted the line: $\hat{y}_i = \beta_0 + \beta_1 x_i$.

Partition of Total sum of squares:



- Total sum of squares (SST): $SST = \sum(y_i - \bar{y})^2$.
- Error sum of squares (SSE): $SSE = \sum(y_i - \hat{y}_i)^2$.
- Regression sum of squares (SSR): $SSR = \sum(\hat{y}_i - \bar{y})^2$

Partition of Total sum of squares

Then we have the following relation:

$$\sum(y_i - \bar{y})^2 = \sum(\hat{y}_i - \bar{y})^2 + \sum(y_i - \hat{y}_i)^2$$

That is: $SST = SSR + SSE$

Coefficient of Determination

A natural measure of the effect of x in reducing in variation in y is to express the reduction in variation as a proportion of the total variation:

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}.$$

We can also write:

$$R^2 = \frac{var(\hat{y})}{var(y)}$$

Thus, we have:

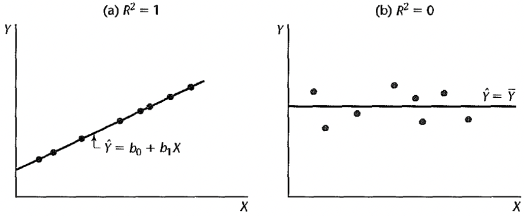
$$0 \leq R^2 \leq 1$$

Coefficient of Determination

Some misunderstanding of R^2:

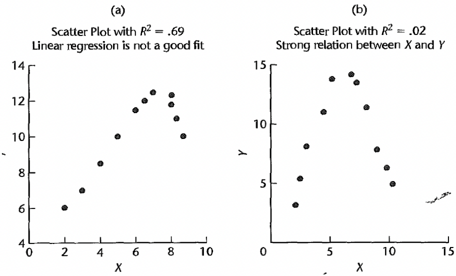
- A high coefficient of determination indicates the useful predictions can be made.
- A high coefficient of determination indicates the estimated regression line is a good fit.
- A coefficient of determination near zero indicates that x and y are not related.

Coefficient of Determination



(a) $R^2 = 1$

(b) $R^2 = 0$



(a)

Scatter Plot with $R^2 = .69$
Linear regression is not a good fit

(b)

Scatter Plot with $R^2 = .02$
Strong relation between X and Y

Regression: Teaching Evaluations Example

```
TSS <- sum((evals_ch5$score - mean(evals_ch5$score))^2)
c(RSS, TSS)
```

```
## [1] 131.8684 136.6543
```

```
R2 <- (TSS - RSS)/TSS
R2
```

```
## [1] 0.03502226
```

```
# Or
summary(score_model)$r.squared
```

```
## [1] 0.03502226
```

Regression: Teaching Evaluations Example

```r
get_regression_points(score_model)
```

```
## # A tibble: 463 x 5
##       ID score bty_avg score_hat residual
##    <int> <dbl>   <dbl>     <dbl>    <dbl>
##  1     1   4.7    5         4.21    0.486
##  2     2   4.1    5         4.21   -0.114
##  3     3   3.9    5         4.21   -0.314
##  4     4   4.8    5         4.21    0.586
##  5     5   4.6    3         4.08    0.52
##  6     6   4.3    3         4.08    0.22
##  7     7   2.8    3         4.08   -1.28
##  8     8   4.1    3.33      4.10   -0.002
##  9     9   3.4    3.33      4.10   -0.702
## 10    10   4.5    3.17      4.09    0.409
## # i 453 more rows
```

```r
Summarytable<-get_regression_points(score_model) %>%
  summarize(var_y = var(score),
                      var_y_hat = var(score_hat),
                      var_residual = var(residual))
Summarytable
```

```
## # A tibble: 1 x 3
##   var_y var_y_hat var_residual
##   <dbl>     <dbl>        <dbl>
## 1 0.296    0.0104        0.285
```

Regression: Teaching Evaluations Example

```r
## Calculate R2
Summarytable[2]/Summarytable[1]
```

```
##    var_y_hat
## 1 0.03502954
```

Regression: Teaching Evaluations Example

```
get_regression_table(score_model)
```

```
## # A tibble: 2 x 7
##   term      estimate std_error statistic p_value lower_ci upper_ci
##   <chr>        <dbl>     <dbl>     <dbl>   <dbl>    <dbl>    <dbl>
## 1 intercept    3.88      0.076     51.0        0    3.73     4.03
## 2 bty_avg      0.067     0.016      4.09       0    0.035    0.099
```

```
summary(score_model)
```

```
##
## Call:
## lm(formula = score ~ bty_avg, data = evals_ch5)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.9246 -0.3690  0.1420  0.3977  0.9309
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  3.88034    0.07614   50.96  < 2e-16 ***
## bty_avg      0.06664    0.01629    4.09 5.08e-05 ***
## ---
## Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
##
## Residual standard error: 0.5348 on 461 degrees of freedom
## Multiple R-squared:  0.03502,    Adjusted R-squared:  0.03293
## F-statistic: 16.73 on 1 and 461 DF,  p-value: 5.083e-05
```

Section 5

Prediction intervals

Interval Estimation for Mean Response and a Single Response

1. Estimating the mean of y at a given value of x, that is, $E(y|x) = \mu_{y|x}$

Example: A power company may want to estimate the mean daily power consumption for a given temperature. They need this estimate for a report.

2. Predicting a single value of y for a given value of x.

Example: The power company want to predict power consumption on a single day for a given temperature.

- They might know a hot day is coming up the next day and have good idea of what the high temperature will be, so what to predict the power consumption.
- We want to be 99.99% confident that they have enough access to power to cover demand.

Interval Estimation of Mean Response

Let x_h denote the level of x for which we wish to estimate the mean response. Then, by the regression equation we have:

$$E(y_h|x_h) = \hat{y}_h = b_0 + b_1 x_h$$

Variance:

$$Var(\hat{y}_h) = \sigma^2 \left[\frac{1}{n} + \frac{(x_h - \bar{x})^2}{\Sigma(x_i - \bar{x})^2} \right].$$

Estimated Variance:

$$S^2(\hat{y}_h) = MSE \left[\frac{1}{n} + \frac{(x_h - \bar{x})^2}{\Sigma(x_i - \bar{x})^2} \right].$$

Interval Estimation of Mean Response

- t-distribution

$$\frac{y_h - \hat{y}_h}{S(\hat{y}_h)} \sim t_{n-2}.$$

So, the $100(1-\alpha)\%$ confidence interval is:

$$\hat{y}_h \pm t_{\alpha/2,n-2} S(\hat{y}_h).$$

Prediction Intervals for a Single Response

In prediction of a single response, we can use the estimated mean function to predict it. Let x_* denote the level of x, then we have:

$$y_* = \beta_0 + \beta_1 x_* + \epsilon_*, \qquad Var(\epsilon_*) = \sigma^2.$$

A natural estimation is:

$$\hat{y}_* = b_0 + b_1 x_*.$$

- The variance of prediction error:

$$Var(pred) = \sigma^2 + \sigma^2 \left(\frac{1}{n} + \frac{(x_* - \bar{x})^2}{\sum (x_* - \bar{x})^2} \right)$$

- The estimated standard error of prediction at x_*:

$$S(pred) = \sqrt{MSE} \left(1 + \frac{1}{n} + \frac{(x_* - \bar{x})^2}{\sum (x_* - \bar{x})^2} \right)$$

Prediction Intervals for a Single Response

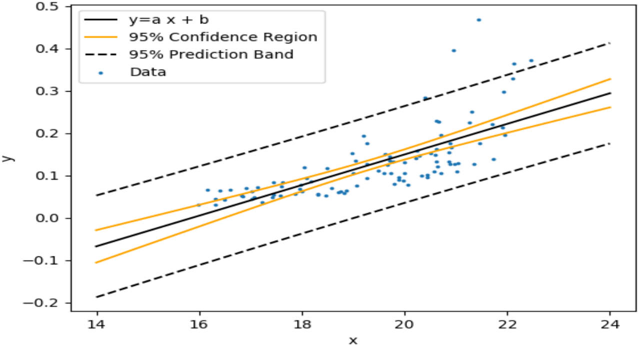
Hence,

$$\frac{y_* - \hat{y}_*}{S(pred)} \sim t_{n-2}.$$

So, the prediction interval for y_* is:

$$\hat{y}_* \pm t_{\alpha/2,n-2} S(pred).$$

Interval Estimation for Mean Response and a Single Response



Regression: Teaching Evaluations Example

```
# Using the build in function
predict(score_model, newdata = data.frame(bty_avg = 7.333))
```

```
##        1
## 4.368987
```

```
predict(score_model, newdata = data.frame(bty_avg = 7.333), interval = "conf", level = 0.90)
```

```
##        fit      lwr      upr
## 1 4.368987 4.280641 4.457333
```

```
predict(score_model, newdata = data.frame(bty_avg = 7.333), interval = "pred", level = 0.90)
```

```
##        fit      lwr    upr
## 1 4.368987 3.483074 5.2549
```

Section 6

Related topics

Correlation is not necessarily causation

- Throughout this chapter we've been cautious when interpreting regression slope coefficients.
  - We always discussed the "associated" effect of an explanatory variable x on an outcome variable y.
  - We include the term "associated" to be extra careful not to suggest we are making a **causal** statement.
- For example when we looked at the teaching score and "beauty" example:
  - For every increase of 1 unit in `bty_avg` there is an associated increase of on average 0.067 units of `score`.
  - while "beauty" score of `bty_avg` is positively correlated with teaching score, we can't necessarily make any statements about "beauty" scores' direct causal effect on teaching score without more information on how this study was conducted.

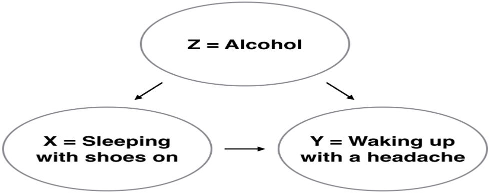
Correlation is not necessarily causation

Here is another example:

- A not-so-great medical doctor goes through medical records and finds that patients who slept with their shoes on tended to wake up more with headaches.
    - So this doctor declares, “Sleeping with shoes on causes headaches!”

Correlation is not necessarily causation

- However, there is a good chance that if someone is sleeping with their shoes on, it's potentially because they are intoxicated from alcohol.
  - Higher levels of drinking leads to more hangovers, and hence more headaches.
  - The amount of alcohol consumption here is what's known as a confounding/lurking variable.
  - It "lurks" behind the scenes, confounding the causal relationship (if any) of "sleeping with shoes on" with "waking up with a headache".



Z is a confounding variable.

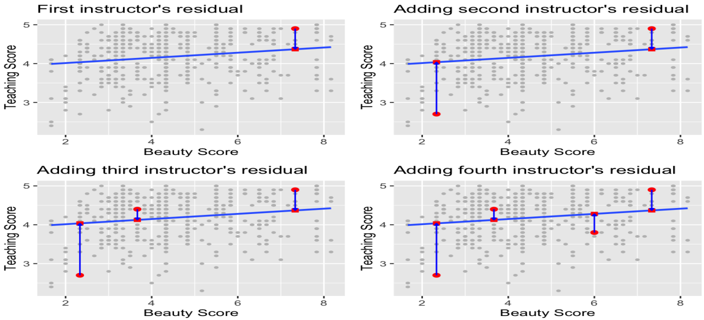
Correlation is not necessarily causation

- Establishing **causation** is a tricky problem
    - frequently takes either carefully designed experiments or methods to control for the effects of confounding variables.
    - Both these approaches attempt, as best they can, either to take all possible confounding variables into account or negate their impact.
    - This allows researchers to focus only on the relationship of interest: the relationship between the outcome variable Y and the treatment variable X.
- As you read news stories, be careful not to fall into the trap of thinking that correlation necessarily implies causation.

Best-fitting line

Regression lines are also known as “best-fitting” lines. But what do we mean by “best”?

- Lets use the Teaching Evaluations Example.



we mark the observed value y with a circle, the fitted value $\hat{y}$ with a square and the residual $y - \hat{y}$.

Best-fitting line

- Now say we repeated this process of computing residuals for all 463 courses' instructors,
  - then we squared all the residuals, and
  - then we summed them.
  - We call this quantity the sum of squared residuals;
- The **sum of squared residuals** is a measure of the lack of fit of a model.
  - Larger values of the sum of squared residuals indicate a bigger lack of fit. This corresponds to a worse fitting model.
  - If the regression line fits all the points perfectly, then the sum of squared residuals is 0
- The regression line minimizes the sum of the squared residuals:

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

Best-fitting line

```r
# Fit regression model:
score_model <- lm(score ~ bty_avg,
                  data = evals_ch5)

# Get regression points:
regression_points <- get_regression_points(score_model)
regression_points
```

```
## # A tibble: 463 x 5
##       ID score bty_avg score_hat residual
##    <int> <dbl>   <dbl>     <dbl>    <dbl>
##  1     1   4.7    5         4.21    0.486
##  2     2   4.1    5         4.21   -0.114
##  3     3   3.9    5         4.21   -0.314
##  4     4   4.8    5         4.21    0.586
##  5     5   4.6    3         4.08    0.52
##  6     6   4.3    3         4.08    0.22
##  7     7   2.8    3         4.08   -1.28
##  8     8   4.1    3.33      4.10   -0.002
##  9     9   3.4    3.33      4.10   -0.702
## 10    10   4.5    3.17      4.09    0.409
## # i 453 more rows
```

Best-fitting line

Any other straight line drawn in the figure would yield a sum of squared residuals greater than 132.

```r
# Compute sum of squared residuals
regression_points %>%
  mutate(squared_residuals = residual^2) %>%
  summarize(sum_of_squared_residuals = sum(squared_residuals))
```

```
## # A tibble: 1 x 1
##   sum_of_squared_residuals
##                      <dbl>
## 1                     132.
```

You can also get the residuals using the function `resid` on a linear model object.

```r
# Compute sum of squared residuals
eis <- resid(score_model)
RSS <- sum(eis^2)
RSS
```

```
## [1] 131.8684
```