MacMahon Master Theorem

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Reference: [1], page 118, exercise 19.

Theorem 1. The Dixon's identity is: for $a, b, c \in \mathbb{N}$,

$$\sum_k (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!} = \binom{a+b+c}{a,b,c}.$$

Proof. $\square$

Question 1 (Ning Jiang)**.** 1) Find the constant coefficient in the expansion of $(1-\frac{x}{y})^m(1-\frac{y}{x})^n$.
2) Find the constant coefficient in the expansion of $\prod_{1\le i\ne j\le n}(1-\frac{x_i}{x_j})$.
3) Find the constant coefficient in the expansion of $\prod_{1\le i\ne j\le n}(1-\frac{x_i}{x_j})^{d_j}$, $d_i \in \mathbb{N}$.

Solution. 1) $\sum_k \binom{m}{k}\binom{n}{k} = \sum_k \binom{m}{k}\binom{n}{n-k} = \binom{m+n}{n}$.
2) $n!$.
3) In the Dixon's identity, when $a = b = c$, the equality becomes $\sum_k (-1)^k \binom{2a}{a+k}^3 = \frac{(3a)!}{(a!)^3}$. When $n = 3, d_1 = a, d_2 = b, d_3 = c$, $(-1)^k \sum_i \binom{b}{i}\binom{a}{i-k} = (-1)^k \sum_i \binom{b}{i}\binom{a}{a+k-i} = (-1)^k\binom{a+b}{a+k}$ is the coefficient of $(\frac{x_1}{x_2})^k$ in the expansion of $(1-\frac{x_1}{x_2})^b(1-\frac{x_2}{x_1})^a$. Similarly we may list 3 equalities below:

$$coeff < (\frac{x_1}{x_2})^k, (1-\frac{x_1}{x_2})^b(1-\frac{x_2}{x_1})^a >= (-1)^k\binom{a+b}{a+k}$$
$$coeff < (\frac{x_2}{x_3})^k, (1-\frac{x_2}{x_3})^c(1-\frac{x_3}{x_2})^b >= (-1)^k\binom{b+c}{b+k}$$
$$coeff < (\frac{x_3}{x_1})^k, (1-\frac{x_3}{x_1})^a(1-\frac{x_1}{x_3})^c >= (-1)^k\binom{c+a}{c+k}$$

Putting the above equalities together, we conclude that when $n = 3$,

$$coeff < 1, \prod_{1\le i\ne j\le 3}(1-\frac{x_i}{x_j})^{d_j} >= \frac{(d_1+d_2+d_3)!}{d_1!d_2!d_3!} = \binom{d_1+d_2+d_3}{d_1,d_2,d_3}.$$

We want to imitate the proof of the MacMahon Master Theorem. Define

$$G(d_1,...,d_n) = coeff < x_1^{(n-1)d_1}...x_n^{(n-1)d_n}, \prod_{1\le i\ne j\le n}(x_j - x_i)^{d_j} >$$
$$= coeff < 1, \prod_{1\le i\ne j\le n}(1-\frac{x_i}{x_j})^{d_j} > .$$

Let $t_1, ..., t_n$ be another set of formal variables, then

$$\sum_{d_1,...,d_n \geq 0} G(d_1,...,d_n) t_1^{(n-1)d_1} ... t_n^{(n-1)d_n} = \sum_{d_1,...,d_n \geq 0} coeff < 1, \prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^{d_j} > t_1^{(n-1)d_1} ... t_n^{(n-1)d_n}$$

$$= coeff < 1, \sum_{d_1,...,d_n \geq 0} \prod_{1 \leq i \neq j \leq n} ((1 - \frac{x_i}{x_j}) t_j)^{d_j} >= coeff < 1, \prod_{1 \leq j \leq n} (1 - \prod_{i \neq j} (1 - \frac{x_i}{x_j}) t_j)^{-1} >$$

$$g(x_1, ..., x_n) = \prod_{1 \leq j \leq n} (1 - \prod_{i \neq j} (1 - \frac{x_i}{x_j}) t_j), \quad (Lx)_j = (1 - \prod_{i \neq j} (1 - \frac{x_i}{x_j}) t_j) x_j.$$

$$\frac{\partial (Lx)_j}{x_i} = \prod_{p \neq i,j} (1 - \frac{x_p}{x_j}) t_j^{n-1} x_j, \quad \frac{\partial (Lx)_j}{x_j} = 1 - \prod_{i \neq j} (1 - \frac{x_i}{x_j}) t_j^{n-1} + ?.$$

$$\begin{aligned}
LHS &= \int_{\mathbb{T}^m} \frac{1}{g} d\theta_1 ... d\theta_m = \int_{\mathbb{T}^m} \prod_{1 \leq j \leq m} \frac{x_j}{(Lx)_j} d\theta_1 ... d\theta_m \\
&= \int_{\mathbb{T}^m} \prod_{1 \leq j \leq m} \frac{1}{2\pi J} \frac{dx_j}{(Lx)_j} = \frac{1}{(2\pi J)^m} \int_{L\mathbb{T}^m} \prod_{1 \leq j \leq m} \frac{d(L^{-1}y)_j}{y_j} \\
&= \frac{1}{(2\pi J)^m} \int_{L\mathbb{T}^m} \det(L^{-1}) \frac{dy_1 ... dy_m}{y_1 ... y_m} \\
&= \det(L^{-1}) \int_{\mathbb{T}^m} d\theta_1' ... d\theta_m' = \det(L^{-1}) = \frac{1}{\det(I_m - TA)}.
\end{aligned}$$

□

Theorem 2 (MacMahon Master Theorem)**.** Let $A = (a_{i,j})_{m*m}$ be a complex matrix, and let $x_1, ... x_m$ be formal variables. Consider a coefficient

$$G(k_1, ..., k_m) = coeff < x_1^{k_1} ... x_m^{k_m}, \prod_{1 \leq i \leq m} (a_{i,1} x_1 + ... + a_{i,m} x_m)^{k_i} > .$$

Let $t_1, ..., t_m$ be another set of formal variables, and let $T = (\delta_{i,j} t_i)_{m*m}$ be a diagonal matrix, then

$$\sum_{k_1,...,k_m \geq 0} G(k_1, ..., k_m) t_1^{k_1} ... t_m^{k_m} = \frac{1}{\det(I_m - TA)}.$$

Proof.

$$\begin{aligned}
& coeff < x_1^{k_1} ... x_m^{k_m}, \prod_{1 \leq i \leq m} (a_{i,1} x_1 + ... + a_{i,m} x_m)^{k_i} > \\
=& coeff < 1, \prod_{1 \leq i \leq m} (\frac{a_{i,1} x_1 + ... + a_{i,m} x_m}{x_i})^{k_i} >
\end{aligned}$$

Denote $LHS = \sum_{k_1,...,k_m \geq 0} G(k_1, ..., k_m) t_1^{k_1} ... t_m^{k_m}$, then

$$\begin{aligned}
LHS =& \sum_{k_1,...,k_m \geq 0} coeff < 1, \prod_{1 \leq i \leq m} (\frac{a_{i,1} x_1 + ... + a_{i,m} x_m}{x_i})^{k_i} > t_1^{k_1} ... t_m^{k_m} \\
=& coeff < 1, \sum_{k_1,...,k_m \geq 0} \prod_{1 \leq i \leq m} (\frac{a_{i,1} x_1 + ... + a_{i,m} x_m}{x_i})^{k_i} t_i^{k_i} > \\
=& coeff < 1, \prod_{1 \leq i \leq m} (1 - (\frac{a_{i,1} x_1 + ... + a_{i,m} x_m}{x_i}) t_i)^{-1} >
\end{aligned}$$

Fix $t_1, ..., t_m$ with small norm, let

$$f(x_1, ..., x_m) = \prod_{1\le i\le m}(1 - (\frac{a_{i,1}x_1 + ... + a_{i,m}x_m}{x_i})t_i)^{-1},$$

$$g(x_1, ..., x_m) = \prod_{1\le i\le m}(1 - (\frac{a_{i,1}x_1 + ... + a_{i,m}x_m}{x_i})t_i).$$

Let $x = (x_1, ..., x_m)^{\mathrm{tr}}$ be a column vector. Then we may express g as

$$g(x_1, ..., x_m) = \prod_{1\le i\le m}(1 - \frac{(TAx)_i}{x_i}) = \prod_{1\le i\le m}\frac{(Lx)_i}{x_i}.$$

Here $L = I_m - TA$ is an invertible matrix since $|T|$ is small. Take $x_i = e^{2\pi J\theta_i}$, $J = \sqrt{-1}$, we have $dx_i = 2\pi Jx_i d\theta_i$. Moreover, let $Lx = y$ and take $y_i = r_i e^{2\pi J\theta'_i}, r_i > 0$, $dy_i = 2\pi Jy_i d\theta'_i$,

$$\begin{aligned}
LHS &= \int_{\mathbb{T}^m}\frac{1}{g}d\theta_1...d\theta_m = \int_{\mathbb{T}^m}\prod_{1\le i\le m}\frac{x_i}{(Lx)_i}d\theta_1...d\theta_m \\
&= \int_{\mathbb{T}^m}\prod_{1\le i\le m}\frac{1}{2\pi J}\frac{dx_i}{(Lx)_i} = \frac{1}{(2\pi J)^m}\int_{L\mathbb{T}^m}\prod_{1\le i\le m}\frac{d(L^{-1}y)_i}{y_i} \\
&= \frac{1}{(2\pi J)^m}\int_{L\mathbb{T}^m}\det(L^{-1})\frac{dy_1...dy_m}{y_1...y_m} \\
&= \det(L^{-1})\int_{\mathbb{T}^m}d\theta'_1...d\theta'_m = \det(L^{-1}) = \frac{1}{\det(I_m - TA)}.
\end{aligned}$$

□

We may also calculate other coefficients. Let $l_1, ...l_m \in \mathbb{Z}, l_1 + ... + l_m = 0$, then

$$\begin{aligned}
LHS &= coeff < x_1^{l_1}...x_m^{l_m}, \prod_{1\le i\le m}(1 - (\frac{a_{i,1}x_1 + ... + a_{i,m}x_m}{x_i})t_i)^{-1} > \\
&= \int_{\mathbb{T}^m}\frac{1}{g}x_1^{-l_1}...x_m^{-l_m}d\theta_1...d\theta_m = \int_{\mathbb{T}^m}\prod_{1\le i\le m}\frac{x_i^{1-l_i}}{(Lx)_i}d\theta_1...d\theta_m \\
&= \int_{\mathbb{T}^m}\prod_{1\le i\le m}\frac{1}{2\pi J}\frac{x_i^{-l_i}dx_i}{(Lx)_i} = \frac{1}{(2\pi J)^m}\int_{L\mathbb{T}^m}\prod_{1\le i\le m}\frac{(L^{-1}y)_i^{-l_i}d(L^{-1}y)_i}{y_i} \\
&= \frac{\det(L^{-1})}{(2\pi J)^m}\int_{L\mathbb{T}^m}\prod_{1\le i\le m}\frac{(L^{-1}y)_i^{-l_i}dy_i}{y_i} \\
&= \det(L^{-1})\int_{\mathbb{T}^m}\prod_{1\le i\le m}(L^{-1}y)_i^{-l_i}d\theta'_1...d\theta'_m \\
&= \det(L^{-1})\int_{\mathbb{T}^m}\prod_{1\le i\le m}((I_m - TA)^{-1}y)_i^{-l_i}d\theta'_1...d\theta'_m
\end{aligned}$$

Theorem 3 (Feuerbach's theorem)**.** The nine-point circle of a triangle ABC is tangent to its inner inscribed circle.

Proof. It suffices to show that $O_1I = \frac{R}{2} - r$.

$$IO_1 = IO + \frac{1}{2}OH, \quad OH = OA + OB + OC, \quad OI = \frac{a}{a+b+c}OA + \frac{b}{a+b+c}OB + \frac{c}{a+b+c}OC,$$

$$IO_1 = \frac{1}{a+b+c}(xOA + yOB + zOC), \quad x = p-a, y = p-b, z = p-c, p = \frac{a+b+c}{2},$$

$$|IO_1|^2 = \frac{R^2}{(a+b+c)^2}(x^2+y^2+z^2+2xy(1-2\sin^2 C)+2yz(1-2\sin^2 A)+2xz(1-2\sin^2 B))$$

$$xy = (p-a)(p-b) = \frac{c^2-(a-b)^2}{4} = R^2(\sin^2 C - (\sin A - \sin B)^2)$$

$$\begin{aligned} 1 &= p^2 - 4xy\sin^2 C - 4yz\sin^2 A - 4xz\sin^2 B = p^2 - 4R^2(\sin^2 C - (\sin A - \sin B)^2)\sin^2 C \\ &\quad - 4R^2(\sin^2 A - (\sin B - \sin C)^2)\sin^2 A - 4R^2(\sin^2 B - (\sin C - \sin A)^2)\sin^2 B, \end{aligned}$$

$$\begin{aligned} 1 - p^2 &= -4R^2(\sin^4 A + \sin^4 B + \sin^4 C) + 8R^2(\sin^2 A\sin^2 B + \sin^2 B\sin^2 C + \sin^2 C\sin^2 A) \\ &\quad - 8R^2\sin A\sin B\sin C(\sin A + \sin B + \sin C), \end{aligned}$$

$$r = \frac{4R^2\sin A\sin B\sin C}{a+b+c}, \quad (\frac{R}{2} - r)^2 = \frac{R^2}{(a+b+c)^2}(p - 4R\sin A\sin B\sin C)^2$$

$$2 = p^2 - p\cdot 8R\sin A\sin B\sin C + 16R^2\sin^2 A\sin^2 B\sin^2 C, \quad p = R(\sin A + \sin B + \sin C),$$

$$2 - p^2 = -8R^2(\sin A + \sin B + \sin C)\sin A\sin B\sin C + 16R^2\sin^2 A\sin^2 B\sin^2 C,$$

$$\begin{aligned} &2(\sin^2 A\sin^2 B + \sin^2 B\sin^2 C + \sin^2 C\sin^2 A) - \sin^4 A - \sin^4 B - \sin^4 C - 4\sin^2 A\sin^2 B\sin^2 C \\ &= \sin^2 A(1-2\sin^2 B)(1-2\sin^2 C) + \sin^2 A + 2\sin^2 B\sin^2 C - \sin^4 A - \sin^4 B - \sin^4 C \\ &= -\sin^2 A\cos 2B\cos 2C + \sin^2 A\cos^2 A - (\sin^2 B - \sin^2 C)^2, \quad \sin^2 B - \sin^2 C = \sin A\sin(B-C), \end{aligned}$$

$$-\cos 2B\cos 2C + \cos^2 A - \sin^2(B-C) = \frac{1}{2}\cos 2A - \frac{1}{2}\cos 2(B-C) + \frac{1+\cos 2A}{2} - \frac{1-\cos 2(B-C)}{2} = 0.$$

□

Theorem 4 (Casey's theorem)**.** The circumcenter of $\triangle ABC$ is O, there's another circle centered at O_1. t_A, t_B, t_C are lengths of tangent segments from A, B, C to circle O_1. Then circle O_1 and circle O are inscribed to each other is equivalent to $at_A + bt_B = ct_C$, where $a = BC, b = AC, c = AB$.

Proof. 1) If circle O_1 and circle O are inscribed to each other, then

2) If $at_A + bt_B = ct_C$, then

$$c^2t_C^2 - a^2t_A^2 - b^2t_B^2 = 2abt_At_B,$$

$$LHS = c^2(t_{C0}^2 + r_0^2 - r^2) - a^2(t_{A0}^2 + r_0^2 - r^2) - b^2(t_{B0}^2 + r_0^2 - r^2) = LHS_0 + (r_0^2 - r^2)(c^2 - a^2 - b^2),$$

$$RHS^2 = 4a^2b^2t_A^2t_B^2 = 4a^2b^2(t_{A0}^2 + r_0^2 - r^2)(t_{B0}^2 + r_0^2 - r^2) = 4a^2b^2(t_{A0}^2t_{B0}^2 + (r_0^2 - r^2)(t_{A0}^2 + t_{B0}^2) + (r_0^2 - r^2)^2),$$

$$LHS^2 = LHS_0^2 + 2LHS_0(r_0^2 - r^2)(c^2 - a^2 - b^2) + (r_0^2 - r^2)^2(c^2 - a^2 - b^2)^2, \quad c^2 - a^2 - b^2 = -2ab\cos C,$$

$$RHS^2 - LHS^2 = (r_0^2 - r^2)4a^2b^2(t_{A0}^2 + t_{B0}^2 + 2t_{A0}t_{B0}\cos C) + (r_0^2 - r^2)^2 4a^2b^2\sin^2 C,$$

$$t_{A0} = \sqrt{\frac{l}{R}}AD, \quad t_{B0} = \sqrt{\frac{l}{R}}BD, \quad \cos C = -\cos D, \quad t_{A0}^2 + t_{B0}^2 + 2t_{A0}t_{B0}\cos C = \frac{l}{R}c^2,$$

$$RHS^2 - LHS^2 = (r_0^2 - r^2)\frac{4a^2b^2c^2l}{R} + (r_0^2 - r^2)^2\frac{a^2b^2c^2}{R^2} = (r_0^2 - r^2)\frac{a^2b^2c^2}{R^2}(4Rl + (r_0^2 - r^2)),$$

$$r^2 = r_0^2, \text{ or } r^2 = r_0^2 + 4Rl = (R-l)^2 + 4Rl = (R+l)^2$$

In both of the above two cases, circle O_1 and circle O are inscribed to each other. But actually if $r = R + l$, then there are no tangent segments from A, B, C to circle O_1. $\square$

Question 2 (2013 China TST p14)**.** Suppose $\angle API = \alpha$, since $\angle AEF = \angle APE$, we have

$$\tan\alpha = \tan\angle AEF = \frac{EA \times FA}{AE \cdot FE},$$

$$AE \cdot FE = AP\sin\alpha DF - (DP - AP\cos\alpha - DQ)DQ, \quad DF = AI \cdot \frac{DP}{AP},$$

$$EA \times FA = (DP - DQ)\sin\alpha FA, \quad FA = \frac{DP}{\cos\alpha} - AP,$$

$$\begin{aligned}\frac{EA \times FA}{\tan\alpha} &= (DP - DQ)\sin\alpha(\frac{DP}{\cos\alpha} - AP)\frac{\cos\alpha}{\sin\alpha} = (DP - DQ)(DP - AP\cos\alpha)\\ &= AI \cdot DP\sin\alpha - (DP - AP\cos\alpha - DQ)DQ,\end{aligned}$$

$$DP(DP - AP\cos\alpha) = AI \cdot DP\sin\alpha + DQ^2, \quad DP^2 - DQ^2 = AI \cdot DP\sin\alpha + AP \cdot DP\cos\alpha = DP \cdot PI,$$

$$LHS = BP^2 - BQ^2 = 4R^2(\cos^2\frac{A}{2} - \sin^2\frac{A}{2}) = 4R^2\cos A, \quad \frac{DP}{IP} = \frac{PM}{PM - r} = \frac{2R\cos^2\frac{A}{2}}{2R\cos^2\frac{A}{2} - r},$$

$$\begin{aligned}PI^2 = AI^2 + AP^2 &= 4R^2((\cos\frac{C-B}{2} - \sin\frac{A}{2})^2 + \sin^2\frac{C-B}{2})\\ &= 4R^2(1 + \sin^2\frac{A}{2} - \cos B - \cos C) = 4R^2\cos A\frac{1 + \cos A - \frac{r}{R}}{1 + \cos A}\end{aligned}$$

$$\frac{r}{R} = \frac{r}{AI}\frac{AI}{R} = \sin\frac{A}{2} \cdot 2(\cos\frac{C-B}{2} - \sin\frac{A}{2}) = \cos B + \cos C - 1 + \cos A$$

$$(1 + \cos A - \frac{r}{R})\cos A = (1 + \sin^2\frac{A}{2} - \cos B - \cos C)(1 + \cos A),$$

$$(2 - \cos B - \cos C)\cos A = (1 + \cos A)(1 + \sin^2\frac{A}{2} - \cos A - \cos B - \cos C),$$

$$2\cos A = (1 + \cos A)(1 + \sin^2\frac{A}{2}) - \cos B - \cos C = 1 + \cos A + \sin^2\frac{A}{2}(1 + \cos A) - \cos B - \cos C,$$

$$\frac{r}{R} = \sin^2\frac{A}{2}(1 + \cos A) = 2\sin^2\frac{A}{2}\cos^2\frac{A}{2} = \frac{\sin^2 A}{2}, \quad \sin^2 A = \frac{2r}{R},$$

Question 3 (2021 China TST p20)**.** Let $Q = MN \cap FI$, $RFIA, MQIA, RFQM$ are cocyclic, $LI \perp NM$, it suffices to show that $\angle MLI = \angle KNM$.

$$\angle LIA = C + \frac{A}{2} = \angle NRD, \quad \angle LAI = \angle NDR, \quad \triangle LIA \sim \triangle NRD, \quad LI = AI \cdot \frac{RN}{RD},$$

Let $P = AE \cap \odot O$, then since $\angle NRI = \angle FAI = \angle EAI$, RIP are colinear. Coincidence: AQE are colinear. Let I' be the incenter of $\triangle DBC$, $Q' = II' \cap AE$, it suffices to show that $Q = Q'$, i.e.,

$$I'Q' = \frac{1}{2}I'I = \frac{b-c}{2}, \quad \frac{I'Q'}{r} = \frac{p - c - c\cos B}{c\sin B} = \frac{b-c}{2r}, \quad r = \frac{ac\sin B}{a+b+c}$$

$$\frac{2a(p - c - c\cos B)}{a+b+c} = \frac{(a+b-c)a - (a^2 + c^2 - b^2)}{a+b+c} = b - c, \quad 2r(p - c - c\cos B) = (b-c)c\sin B,$$

$$\frac{RN}{RD} = \frac{\sin \angle NPI}{\sin \angle DPI} = \frac{\sin \angle INP \cdot IN}{\sin \angle IDP \cdot ID}, \quad IK = AI \cdot \frac{\sin \angle FAI}{\sin \angle AKI},$$

$$\angle AKI = \angle AID - \angle EAI = B - C + \angle I'AN - \angle EAI = B - C + \angle I'AE,$$

$$\tan \angle KNM = \frac{IQ + IK \cos\alpha}{NQ - IK \sin\alpha}, \quad \alpha = \angle IDA, \quad \tan \angle MLI = \frac{MQ}{QI + LI},$$

$$IQ = \frac{b-c}{2}, \quad NQ = 2R \sin\frac{A}{2} \cos\frac{B-C}{2}, \quad MQ = 2R - NQ,$$

$$\beta = \angle FAD = \angle PDA, \quad AP = 2R \sin\beta, \quad \angle FAI = \beta - C - \frac{A}{2}, \quad \angle AKI = \pi - \alpha - \beta,$$

Assume $Y = AF \cap BC$, then

$$IN \sin \angle INP = AP\frac{IN}{2R} = AP \sin\frac{A}{2}, \quad ID \sin \angle IDP = d(I', AF) = EY \sin\beta + r\cos\beta,$$

It suffices to show that

$$(2R - NQ)(NQ - IK\sin\alpha) = (IQ + LI)(IQ + IK\cos\alpha),$$

$$IK \sin\alpha = AI\frac{\sin(\beta - C - \frac{A}{2})}{\sin(\alpha+\beta)} \sin\alpha, \quad IK\cos\alpha = AI\frac{\sin(\beta - C - \frac{A}{2})}{\sin(\alpha+\beta)}\cos\alpha,$$

It suffices to show that

$$(2R-NQ)(NQ\sin(\alpha+\beta) - AI\sin(\beta - C - \frac{A}{2})\sin\alpha) = (IQ+LI)(IQ\sin(\alpha+\beta) + AI\sin(\beta-C-\frac{A}{2})\cos\alpha),$$

$$\begin{aligned}
&\frac{2}{R}(IQ\sin(\alpha+\beta) + AI\sin(\beta - C - \frac{A}{2})\cos\alpha) \\
=&2(\sin B - \sin C)\sin(\alpha+\beta) + 4(\cos\frac{B-C}{2} - \sin\frac{A}{2})\sin(\beta - C - \frac{A}{2})\cos\alpha \\
=&\cos(\alpha+\beta-B) - \cos(\alpha+\beta+B) - \cos(\alpha+\beta-C) + \cos(\alpha+\beta+C) \\
&+ 2(\cos\frac{B-C}{2} - \sin\frac{A}{2})(\sin(\alpha+\beta-C-\frac{A}{2}) + \sin(-\alpha+\beta-C-\frac{A}{2})) \\
=&-\cos(\alpha+\beta+B-C) - \cos(\alpha+\beta) - \cos(-\alpha+\beta+B-C) - \cos(-\alpha+\beta) \\
&+ \cos(\alpha+\beta-B) + \cos(\alpha+\beta+C) + \cos(-\alpha+\beta+B) + \cos(-\alpha+\beta-C),
\end{aligned}$$

$$\begin{aligned}
&\frac{2}{R}(NQ\sin(\alpha+\beta) - AI\sin(\beta - C - \frac{A}{2})\sin\alpha) \\
=&4\sin\frac{A}{2}\cos\frac{B-C}{2}\sin(\alpha+\beta) - 4(\cos\frac{B-C}{2} - \sin\frac{A}{2})\sin(\beta - C - \frac{A}{2})\sin\alpha \\
=&2(\cos B + \cos C)\sin(\alpha+\beta) + 2(\sin\frac{A}{2} - \cos\frac{B-C}{2}) \\
&(\cos(-\alpha+\beta-C-\frac{A}{2}) - \cos(\alpha+\beta-C-\frac{A}{2})) \\
=&\sin(\alpha+\beta+B) + \sin(\alpha+\beta-B) + \sin(\alpha+\beta+C) + \sin(\alpha+\beta-C) \\
&+ \sin(-\alpha+\beta-C) + \sin(-\alpha+\beta+B) - \sin(-\alpha+\beta+B-C) - \sin(-\alpha+\beta) \\
&- \sin(\alpha+\beta-C) - \sin(\alpha+\beta+B) + \sin(\alpha+\beta+B-C) + \sin(\alpha+\beta) \\
=&\sin(\alpha+\beta-B) + \sin(\alpha+\beta+C) + \sin(-\alpha+\beta-C) + \sin(-\alpha+\beta+B) \\
&- \sin(-\alpha+\beta+B-C) - \sin(-\alpha+\beta) + \sin(\alpha+\beta+B-C) + \sin(\alpha+\beta),
\end{aligned}$$

$$\cot\alpha = \frac{p-c-c\cos B}{c\sin B - r}, \quad \frac{BY}{YC} = \frac{c\sin\angle EAC}{b\sin\angle BAE} = \frac{c^2(p-b)}{b^2(p-c)},$$

$$BY = c\sin B\cot\beta + c\cos B = \frac{ac^2(p-b)}{c^2(p-b)+b^2(p-c)},$$

$$EY\sin\beta + r\cos\beta = (p-c-c\cos B)\sin\beta + (r-c\sin B)\cos\beta = \frac{c\sin B - r}{\sin\alpha}\sin(\beta-\alpha),$$

$$c\sin B - r = AI\cos\frac{B-C}{2}, \quad LI = \frac{2R\sin\beta\sin\frac{A}{2}\sin\alpha}{\cos\frac{B-C}{2}\sin(\beta-\alpha)},$$

Theorem 5 (Hardy's inequality)**.** 1) If $a_1, a_2, a_3, ...$ is a sequence of non-negative real numbers, then for any real number $p > 1$, we have

$$\sum_{n\geq 1}(\frac{a_1+a_2+...+a_n}{n})^p \leq (\frac{p}{p-1})^p \sum_{n\geq 1} a_n^p.$$

2) Integral version: if f is a measuable function with non-negative values, then

$$\int_0^{+\infty}(\frac{1}{x}\int_0^x f(t)dt)^p \leq \int_0^{+\infty} f(x)^p dx$$

Question: what if $p \leq 1$ in the statement of Hardy's inequality? The case $p = -1$ is a classical mathematical olympiad problem.

Theorem 6 (Carleman's inequality)**.** 1) Let $a_1, a_2, a_3, ...$ be a sequence of non-negative real numbers, then we have

$$\sum_{n\geq 1}(a_1a_2...a_n)^{1/n} \leq e\sum_{n\geq 1} a_n.$$

2) Integral version: if f is a measuable function with non-negative values, then

$$\int_0^{+\infty}\exp(\frac{1}{x}\int_0^x \log f(t)dt)dx \leq e\int_0^{+\infty} f(x)dx.$$

Theorem 7 (Carleson's inequality)**.** For any convex function g with $g(0) = 0$, and for any $-1 < p < +\infty$, we have

$$\int_0^{+\infty} x^p e^{-\frac{g(x)}{x}}dx \leq e^{p+1}\int_0^{+\infty} x^p e^{-g'(x)}dx.$$

Carleman's inequality follows from the case $p = 0$.

Theorem 8 (Sobolev's inequality)**.**

Theorem 9 (Hilbert's inequality)**.** 1) Show that for every pair of sequences of real numbers $\{a_n\}$ and $\{b_n\}$ we have

$$\sum_{m,n=1}^{\infty}\frac{a_mb_n}{m+n} \leq \pi\sqrt{(\sum_{m=1}^{\infty}a_m^2)(\sum_{n=1}^{\infty}b_n^2)},$$

and the constant π is optimal.

2) For any nonnegative sequences $\{a_n\}$ and $\{b_n\}$ we have

$$\sum_{m,n=1}^{\infty}\frac{a_mb_n}{m+n} \leq \frac{\pi}{\sin(\frac{\pi}{p})}(\sum_{m=1}^{\infty}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}b_n^q)^{\frac{1}{q}},$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is optimal.

3) Generalization with order of denominator modified

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m+n)^\tau} < (\frac{\pi}{\sin(\frac{\pi(q-1)}{\tau q})})^\tau \|a\|_p \|b\|_q,$$

valid for $p, q > 1, \tau > 0, \frac{1}{p} + \frac{1}{q} \geq 1$ and $\tau + \frac{1}{p} + \frac{1}{q} = 2$.

4) Harder Hilbert inequality

$$|\sum_{m \neq n} \frac{a_m b_n}{m-n}| \leq \pi \sqrt{(\sum_{m=1}^{\infty} |a_m|^2)(\sum_{n=1}^{\infty} |b_n|^2)}$$

Proof. 1)

$$(\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n})^2 \leq (\sum_{m,n=1}^{\infty} \frac{a_m^2}{m+n} (\frac{m}{n})^{2\lambda})(\sum_{m,n=1}^{\infty} \frac{b_n^2}{m+n} (\frac{n}{m})^{2\lambda}),$$

$$\sum_{m,n=1}^{\infty} \frac{a_m^2}{m+n} (\frac{m}{n})^{2\lambda} = \sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} \frac{1}{m+n} (\frac{m}{n})^{2\lambda},$$

$$\frac{1}{m+n} (\frac{m}{n})^{2\lambda} \leq \int_0^\infty \frac{1}{m+x} \frac{m^{2\lambda}}{x^{2\lambda}} dx = \int_0^\infty \frac{1}{(1+y) y^{2\lambda}} dy = \frac{\pi}{\sin 2\pi\lambda},$$

Choose $\lambda = \frac{1}{4}$ finishes the proof of the original L^2 Hilbert's inequality.

2) Using Holder's inequality, we have

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq (\sum_{m,n=1}^{\infty} \frac{a_m^p}{m+n} (\frac{m}{n})^{p\lambda})^{\frac{1}{p}} (\sum_{m,n=1}^{\infty} \frac{b_n^q}{m+n} (\frac{n}{m})^{q\lambda})^{\frac{1}{q}}$$

$$\sum_{m,n=1}^{\infty} \frac{a_m^p}{m+n} (\frac{m}{n})^{p\lambda} = \sum_{m=1}^{\infty} a_m^p \sum_{n=1}^{\infty} \frac{1}{m+n} (\frac{m}{n})^{p\lambda}, \quad \sum_{n=1}^{\infty} \frac{1}{m+n} (\frac{m}{n})^{p\lambda} \leq \frac{\pi}{\sin p\pi\lambda},$$

$$\sum_{m,n=1}^{\infty} \frac{b_n^q}{m+n} (\frac{n}{m})^{q\lambda} = \sum_{n=1}^{\infty} b_n^q \sum_{m=1}^{\infty} \frac{1}{m+n} (\frac{n}{m})^{q\lambda}, \quad \sum_{m=1}^{\infty} \frac{1}{m+n} (\frac{n}{m})^{q\lambda} \leq \frac{\pi}{\sin q\pi\lambda},$$

$$\text{minimize } (\frac{\pi}{\sin p\pi\lambda})^{\frac{1}{p}} (\frac{\pi}{\sin q\pi\lambda})^{\frac{1}{q}}, \quad F(\lambda) = \frac{1}{p} \log \sin p\pi\lambda + \frac{1}{q} \log \sin q\pi\lambda,$$

$$F'(\lambda) = \cot p\pi\lambda + \cot q\pi\lambda = 0, \quad \lambda = \frac{1}{p+q}, \quad \frac{p}{p+q} = \frac{1}{q}, \quad \frac{q}{p+q} = \frac{1}{p},$$

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\frac{\pi}{p})} (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} (\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}},$$

Observation: let $f(z) = \sum_{n>0} a_n e^{-nz}$, then

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n} = \int_0^{+\infty} |f(t)|^2 dt, \quad 2\pi \sum_{n=1}^{\infty} a_n^2 = \int_0^{2\pi i} |f(t)|^2 dt,$$

How to convert the inequality to terms inside $\Omega = \{z \in \mathbb{C}, \Re(z) \geq 0, 0 \leq \Im(z) \leq 2\pi\}$? □

Question 4. 1) Guaranted positivity: show that for any real numbers $a_1, a_2, ..., a_n$ and positive $\lambda_1, \lambda_2, ..., \lambda_n$ one has

$$\sum_{i,j=1}^{n} \frac{a_i a_j}{i+j} \geq 0, \quad \sum_{i,j=1}^{n} \frac{a_i a_j}{\lambda_i + \lambda_j} \geq 0,$$

2) Show that if the complex array $\{a_{jk}\}$ satisfies the bound

$$|\sum_{j,k} a_{jk} x_j y_k| \leq M \|x\|_2 \|y\|_2,$$

then one also has the bound

$$|\sum_{j,k} a_{jk} h_{jk} x_j y_k| \leq \alpha\beta M \|x\|_2 \|y\|_2,$$

provided that the factors h_{jk} have an integral representation of the form

$$h_{jk} = \int_D f_j(x) g_k(x) dx$$

and for all j, k one has the bounds

$$\int_D |f_j(x)|^2 dx \leq \alpha^2, \quad \int_D |g_k(x)|^2 dx \leq \beta^2,$$

3) Show that for every pair of sequences of real numbers $\{a_n\}$ and $\{b_n\}$ one has

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{\max(m,n)} \leq 4 \sqrt{(\sum_{m=1}^{\infty} a_m^2)(\sum_{n=1}^{\infty} b_n^2)},$$

and the constant 4 is optimal.

4) Carlson's inequality:

$$(\sum_{k=1}^{n} a_k)^4 \leq \pi^2 (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} k^2 a_k^2)$$

5) Hilbert's inequality via the Toeplitz method: the Fourier coefficients of $t - \pi, 0 \leq t \leq 2\pi$ are

$$\int_0^{2\pi} (t-\pi) e^{int} dt = \frac{(t-\pi)e^{int}}{in} \Big|_0^{2\pi} - \frac{1}{in} \int_0^{2\pi} e^{int} dt = \frac{2\pi}{in},$$

so for real $a_k, b_k, k \geq 1$ one has

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} = \frac{i}{2\pi} \int_0^{2\pi} (t-\pi)(\sum_{m\geq 1} a_m e^{imt})(\sum_{n \geq 1} b_n e^{int}) dt,$$

$$RHS \leq \frac{\|t-\pi\|_\infty}{2\pi} |\int_0^{2\pi} (\sum_{m\geq 1} a_m e^{imt})(\sum_{n\geq 1} b_n e^{int}) dt| \leq \pi \|a\|_2 \|b\|_2,$$

the last step used the following fact: $\tilde{a}(t) = \sum_{m \geq 1} a_m e^{imt}$, $\tilde{b}(t) = \sum_{n \geq 1} b_n e^{int}$,

$$|\int_0^{2\pi} \tilde{a}(t) \tilde{b}(t) dt| \leq \|\tilde{a}\|_2 \|\tilde{b}\|_2 = 2\pi \|a\|_2 \|b\|_2,$$

Theorem 10 (Pólya's random walk theorem)**.** A random walk is said to be recurrent if it returns to its initial position with probability one. A random walk which is not recurrent is called transient. Pólya's classical result says: the simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$.

Proof. □

Question 5 (Spectrum of one dimensional quantum harmonic oscillator)**.** Find the values of λ such that

$$-u'' + x^2 u = \lambda u, \quad x \in \mathbb{R}, u \neq 0,$$

we may assume that u is a Schwartz function.

Proof. All the eigenvalues are $\lambda_n = 2n+1, n \in \mathbb{N}$, with eigenfunctions $u_n = e^{-\frac{x^2}{2}} H_n(x)$ where H_n is the degree n Hermite polynomial. Substitute $u = e^{-\frac{x^2}{2}}\tilde{u}$, we have

$$-u'' + x^2 u = e^{-\frac{x^2}{2}}(-\tilde{u}'' + 2x\tilde{u}' - (x^2-1)\tilde{u} + x^2\tilde{u}),$$

$$-\tilde{u}'' + 2x\tilde{u}' = (\lambda - 1)\tilde{u},$$

First, we show that all eigenvalues are nonnegetive. It follows by

$$\langle -u'' + x^2 u, u\rangle = \|u'\|_2^2 + \|xu\|_2^2 \geq 0, \quad \lambda \geq 0,$$

1) $\tilde{u}$ is a nonzero constant, then it is a valid solution with eigenvalue $\lambda = 1$.
2) $\tilde{u}$ is not a constant. Let $v = \partial_x \tilde{u}$, then it satisfies

$$\partial_x(-\tilde{u}'' + 2x\tilde{u}') = -v'' + 2xv' + 2v = (\lambda-1)v, \quad -v'' + 2xv' = (\lambda - 3)v,$$

so v is an eigenfunction with eigenvalue $\lambda - 2$.

We use an induction on $\lfloor \lambda \rfloor$ to find all eigenvalues and eigenfunctions. i) If $\lfloor \lambda \rfloor \leq 1$ and $\tilde{u}$ is not a constant, then $v = \partial_x \tilde{u}$ is an eigenfunction with eigenvalue $\lambda - 2 < 0$, contradiction! So if $\lfloor \lambda \rfloor \leq 1$, $\tilde{u}$ must be a constant and $\lambda = 1$. ii) If $\lfloor \lambda \rfloor \geq 2$, then $\tilde{u}$ mustn't be a constant. $v = \partial_x \tilde{u}$ is an eigenfunction with eigenvalue $\lambda - 2$. Since $\lfloor \lambda - 2 \rfloor = \lfloor \lambda \rfloor - 2$, v is determined by induction. Then $\tilde{u}$ is determined by equation

$$(\lambda - 1)\tilde{u} = -v' + 2xv,$$

Variation of parameters: assume $u = e^{Q(x)}\tilde{u}$, then

$$u'' = e^{Q(x)}(\tilde{u}'' + 2Q'(x)\tilde{u}' + (Q''(x) + Q'(x)^2)\tilde{u})$$

to the leading order let $Q'(x)^2 = x^2$, we get $Q = -\frac{x^2}{2} + C$. □

Question 6 (Spectrum of higher dimensional quantum harmonic oscillator)**.** Find the values of λ such that

$$\mathcal{L}_0 u = -\Delta u + |x|^2 u = \lambda u, \quad x \in \mathbb{R}^n, u \neq 0,$$

we may assume that u is a Schwartz function.

Proof. Variation of parameters: $u = f\tilde{u}, f = e^{-\frac{|x|^2}{2}}$, we have

$$\Delta u = f\Delta\tilde{u} + 2\nabla f \cdot \nabla\tilde{u} + \tilde{u}\Delta f, \quad \nabla f = -\overline{x}f, \quad \Delta f = (|x|^2 - n)f,$$

$$\mathcal{L}\tilde{u} = -\Delta\tilde{u} + 2\overline{x} \cdot \nabla\tilde{u} = (\lambda - n)\tilde{u},$$

Energy estimate shows that all eigenvalues are nonnegative:

$$\langle -\Delta u + |x|^2 u, u\rangle = \|\nabla u\|_2^2 + \|\overline{x}u\|_2^2 \geq 0, \quad \lambda \geq 0,$$

1) $\tilde{u}$ is a nonzero constant, then it is a valid solution with eigenvalue $\lambda = n$.
2) $\tilde{u}$ is not a constant. Let $v = \partial_x\tilde{u}$, then it satisfies

$$\partial_x(-\Delta\tilde{u} + 2\overline{x} \cdot \nabla\tilde{u}) = -\Delta v + 2\overline{x} \cdot \nabla v + 2v = (\lambda - n)v, \quad \mathcal{L}v = (\lambda - n - 2)v,$$

so v is an eigenfunction with eigenvalue $\lambda - 2$.

□

References

[1] Stanley, Enumerative combinatorics.