Linear stability of RMHD equations on 2D finite channel

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Abstract

This is a note about linear stability of RMHD equations on 2D finite channel.

Contents

1 Introduction

$\Omega = [0,1] \times \mathbb{T}, \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The equilibrium is $V_s = 0, B_s = (0, b(x)), P_s = 0$, total magnetic field is $\tilde{B} = B_s + B$, where B is the perturbation. We assume that b is monotone positive. Total velocity field is $V_s + v = v$, total pressure is $P_s + p = p$, and they are the same as the perturbations v, p, then the original RMHD system has the following form:

$$\partial_t v + v \cdot \nabla v + \nabla p = \tilde{B} \cdot \nabla \tilde{B}, \tag{1}$$

$$\partial_t \tilde{B} - \eta \Delta \tilde{B} + v \cdot \nabla \tilde{B} = \tilde{B} \cdot \nabla v, \tag{2}$$

$$\nabla \cdot v = \nabla \cdot \tilde{B} = 0, \tag{3}$$

Nonlinear equations for the perturbations are

$$\begin{aligned}
&\partial_t v_x = -\partial_x p - (v_x\partial_x + v_y\partial_y)v_x + (B_x\partial_x + (b + B_y)\partial_y)B_x,\\
&\partial_t v_y = -\partial_y p - (v_x\partial_x + v_y\partial_y)v_y + B_x\partial_x(b + B_y) + (b + B_y)\partial_y B_y,\\
&\partial_t B_x = (B_x\partial_x + (b + B_y)\partial_y)v_x - (v_x\partial_x + v_y\partial_y)B_x + \eta\Delta B_x,\\
&\partial_t B_y = (B_x\partial_x + (b + B_y)\partial_y)v_y - v_x\partial_x(b + B_y) - v_y\partial_y B_y + \eta\Delta B_y,\\
&\nabla \cdot v = \nabla \cdot B = 0,
\end{aligned}$$

Linearized equations for the perturbations are:

$$\partial_t v_x = -\partial_x p + b\partial_y B_x, \tag{4}$$

$$\partial_t v_y = -\partial_y p + b\partial_y B_y + b' B_x, \tag{5}$$

$$\partial_t B_x = b\partial_y v_x + \eta\Delta B_x, \tag{6}$$

$$\partial_t B_y = b\partial_y v_y - b' v_x + \eta\Delta B_y, \tag{7}$$

$$\nabla \cdot v = \nabla \cdot B = 0, \tag{8}$$

with Navier slip boundary conditions

$$v_x|_{x=0,1} = B_x|_{x=0,1} = 0,$$

Taking Fourier transform in y, we get for $\alpha \neq 0$,

$$\partial_t \widehat{v}_x = -\partial_x \widehat{p} + i\alpha b \widehat{B}_x, \tag{9}$$
$$\partial_t \widehat{v}_y = -i\alpha \widehat{p} + i\alpha b \widehat{B}_y + b' \widehat{B}_x, \tag{10}$$
$$\partial_t \widehat{B}_x = i\alpha b \widehat{v}_x + \eta(\partial_x^2 - \alpha^2)\widehat{B}_x, \tag{11}$$
$$\partial_t \widehat{B}_y = i\alpha b \widehat{v}_y - b' \widehat{v}_x + \eta(\partial_x^2 - \alpha^2)\widehat{B}_y, \tag{12}$$
$$\partial_x \widehat{v}_x + i\alpha \widehat{v}_y = 0, \quad \partial_x \widehat{B}_x + i\alpha \widehat{B}_y = 0, \tag{13}$$

Eliminating $\widehat{p}, \widehat{v}_y, \widehat{B}_y$ from equation 10 gives

$$\partial_t \partial_x \widehat{v}_x = \partial_t(-i\alpha \widehat{v}_y) = -\alpha^2 \widehat{p} + i\alpha b \partial_x \widehat{B}_x - i\alpha b' \widehat{B}_x,$$

$$\partial_t \partial_x^2 \widehat{v}_x = -\alpha^2 \partial_x \widehat{p} + i\alpha \partial_x(b \partial_x \widehat{B}_x) - i\alpha \partial_x(b' \widehat{B}_x), \quad \partial_x \widehat{p} = -\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x,$$

$$\begin{aligned} \partial_t \partial_x^2 \widehat{v}_x &= -\alpha^2(-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha \partial_x(b \partial_x \widehat{B}_x) - i\alpha \partial_x(b' \widehat{B}_x) \\ &= -\alpha^2(-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha(b \partial_x^2 \widehat{B}_x - b'' \widehat{B}_x), \end{aligned}$$

$$\partial_t(\partial_x^2 - \alpha^2)\widehat{v}_x = \alpha b(\partial_x^2 - \alpha^2) i\widehat{B}_x - \alpha b'' i\widehat{B}_x, \quad \partial_t i\widehat{B}_x = -\alpha b \widehat{v}_x + \eta(\partial_x^2 - \alpha^2) i\widehat{B}_x,$$

Let $\xi = \widehat{v}_x, \psi = i\widehat{B}_x$, we have the following system of evolution:

$$\partial_t \xi = \alpha(\partial_x^2 - \alpha^2)^{-1}(b(\partial_x^2 - \alpha^2)\psi - b''\psi), \tag{14}$$
$$\partial_t \psi = -\alpha b \xi + \eta(\partial_x^2 - \alpha^2)\psi, \tag{15}$$

Denote $\mathcal{L}_\alpha$ the linear operator of the above equations, and consider the eigenvalue problem of operator $\mathcal{L}_\alpha$. If $c \in \sigma_p(\mathcal{L}_\alpha)$ with associated eigenfunctions ξ, ψ, then

$$c\xi = \alpha(\partial_x^2 - \alpha^2)^{-1}(b(\partial_x^2 - \alpha^2)\psi - b''\psi), \quad c\psi = -\alpha b \xi + \eta(\partial_x^2 - \alpha^2)\psi,$$

$$\alpha b \xi = -(c - \eta(\partial_x^2 - \alpha^2))\psi, \quad c(\partial_x^2 - \alpha^2)\xi = \alpha(b(\partial_x^2 - \alpha^2) - b'')\psi,$$
$$-c(\partial_x^2 - \alpha^2) b^{-1}(c - \eta(\partial_x^2 - \alpha^2))\psi = \alpha^2(b(\partial_x^2 - \alpha^2) - b'')\psi,$$

Let $\psi = bg$, then

$$b^{-1}(c - \eta(\partial_x^2 - \alpha^2)) bg = cg - c\eta b^{-1}(\partial_x^2 - \alpha^2) bg,$$
$$\alpha^2(b(\partial_x^2 - \alpha^2) - b'') bg = \alpha^2(b^2(\partial_x^2 - \alpha^2) + 2bb'\partial_x) g,$$

Summing up the above two equations, we get the Orr-Sommerfeld type equation for linearized RMHD system on a 2-dimensional finite channel:

$$(c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2) g + 2\alpha^2 bb' \partial_x g - c\eta(\partial_x^2 - \alpha^2) b^{-1}(\partial_x^2 - \alpha^2) bg = 0,$$

Boundary conditions are

$$\psi|_{x=0,1} = 0, \quad \xi|_{x=0,1} = 0, \quad (c - \eta(\partial_x^2 - \alpha^2))\psi|_{x=0,1} = 0,$$

$$g|_{x=0,1} = 0, \quad \partial_x^2(bg)|_{x=0,1} = b\partial_x^2 g + 2b'\partial_x g|_{x=0,1} = 0,$$

Let us denote by OS_α the Orr-Sommerfeld type fourth-order operator

$$OS_\alpha(g) \triangleq (c^2+\alpha^2 b^2)(\partial_x^2-\alpha^2)g + 2\alpha^2 bb'\partial_x g - c\eta(\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)bg,$$

Following [11], we study the resolvent estimates of the linearized operator under the Navier-slip boundary conditions. More precisely, we consider the equation

$$OS_\alpha(g) = (c^2+\alpha^2 b^2)(\partial_x^2-\alpha^2)g + 2\alpha^2 bb'\partial_x g - c\eta(\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)bg = F,$$

$$g|_{x=0,1}=0, \quad b\partial_x^2 g + 2b'\partial_x g|_{x=0,1}=0,$$

Substitute $h = b^{-1}(\partial_x^2-\alpha^2)bg = b^{-1}(\partial_x^2-\alpha^2)\psi$, we have

$$h|_{x=0,1}=0, \quad g = b^{-1}(\partial_x^2-\alpha^2)^{-1}bh,$$

$$OS_\alpha(g) = (c^2+\alpha^2 b^2)(\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)^{-1}bh + 2\alpha^2 bb'\partial_x b^{-1}(\partial_x^2-\alpha^2)^{-1}bh - c\eta(\partial_x^2-\alpha^2)h,$$

Using Green's function, the operator $(\partial_x^2-\alpha^2)^{-1}$ with Dirichlet's boundary condition can be represented as an integral operator:

$$G(x,x') = \begin{cases} -\frac{\sinh\alpha(1-x')\sinh\alpha x}{\alpha\sinh\alpha}, x\le x' \\ -\frac{\sinh\alpha x'\sinh\alpha(1-x)}{\alpha\sinh\alpha}, x > x' \end{cases}, \quad (\partial_x^2-\alpha^2)^{-1}u(x) = \int_0^1 G(x,x')u(x')dx',$$

$$g(x) = b^{-1}(x)[(\partial_x^2-\alpha^2)^{-1}bh](x) = b^{-1}(x)\int_0^1 G(x,x')b(x')h(x')dx',$$

As an integral operator, $\partial_x(\partial_x^2-\alpha^2)^{-1}$ is represented as follows:

$$\partial_x G(x,x') = \begin{cases} -\frac{\alpha\sinh\alpha(1-x')\cosh\alpha x}{\alpha\sinh\alpha}, x\le x' \\ \frac{\alpha\sinh\alpha x'\cosh\alpha(1-x)}{\alpha\sinh\alpha}, x > x' \end{cases}, \quad \partial_x(\partial_x^2-\alpha^2)^{-1}u(x) = \int_0^1 \partial_x G(x,x')u(x')dx',$$

We investigate the case of exponential background magnetic profile $b(x)=e^{\lambda x}$ for convenience. The Sobolev space we concern is

$$H_0^1([0,1]) = \{u:[0,1]\to\mathbb{C}, \|u\|_{H^1}<+\infty, u(0)=u(1)=0\}, \quad \|u\|_{H^1}^2 = \int_0^1 \partial_x u\overline{\partial_x u}dx,$$

A set of orthonormal basis of $H_0^1([0,1])$ is $\{e_k = \frac{\sqrt{2}\sin k\pi x}{k\pi}, k\in\mathbb{Z}_+\}$, and they are all the eigenfunctions of operator ∂_x^2, with eigenvalues $\partial_x^2 e_k = -k^2\pi^2 e_k$. Under the exponential background magnetic profile, we have

$$(\partial_x^2-\alpha^2)g = (\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)^{-1}bh = h + 2(b^{-1})'\partial_x(\partial_x^2-\alpha^2)^{-1}bh + (b^{-1})''(\partial_x^2-\alpha^2)^{-1}bh,$$

$$\partial_x g = \partial_x b^{-1}(\partial_x^2-\alpha^2)^{-1}bh = (b^{-1})'(\partial_x^2-\alpha^2)^{-1}bh + b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh,$$

The integral form of the above two equations are:

$$\begin{aligned}(\partial_x^2-\alpha^2)g(x) &= h(x) + \lambda b^{-1}(x)[(\lambda-2\partial_x)(\partial_x^2-\alpha^2)^{-1}bh](x) \\ &= h(x) + \lambda b^{-1}(x)\int_0^1 (\lambda-2\partial_x)G(x,x')b(x')h(x')dx',\end{aligned}$$

$$\partial_x g(x) = b^{-1}(x)[(\partial_x-\lambda)(\partial_x^2-\alpha^2)^{-1}bh](x) = b^{-1}(x)\int_0^1 (\partial_x-\lambda)G(x,x')b(x')h(x')dx',$$

The Orr-Sommerfeld type equation in terms of h becomes

$$F \triangleq OS_\alpha(g) = (c^2+\alpha^2 b^2)(h + 2(b^{-1})'\partial_x(\partial_x^2-\alpha^2)^{-1}bh + (b^{-1})''(\partial_x^2-\alpha^2)^{-1}bh) + 2\alpha^2 bb'((b^{-1})'(\partial_x^2-\alpha^2)^{-1}bh + b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh) - c\eta(\partial_x^2-\alpha^2)h,$$

$$F = (c^2+\alpha^2 b^2)(h-2\lambda b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh+\lambda^2 b^{-1}(\partial_x^2-\alpha^2)^{-1}bh)+2\alpha^2\lambda b(\partial_x-\lambda)(\partial_x^2-\alpha^2)^{-1}bh-c\eta(\partial_x^2-\alpha^2)h,$$

$$F = c^2(h-2\lambda b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh+\lambda^2 b^{-1}(\partial_x^2-\alpha^2)^{-1}bh)+\alpha^2 b^2 h-\lambda^2\alpha^2 b(\partial_x^2-\alpha^2)^{-1}bh-c\eta(\partial_x^2-\alpha^2)h$$
$$= c^2(h+\lambda b^{-1}(\lambda-2\partial_x)(\partial_x^2-\alpha^2)^{-1}bh)+\alpha^2 b^2 h-\lambda^2\alpha^2 b(\partial_x^2-\alpha^2)^{-1}bh-c\eta(\partial_x^2-\alpha^2)h,$$

While taking inner product with h, the first and the third term in the right hand side above is nontrivial. We give a closer look at the first term below:

$$(\partial_x^2-\alpha^2)g = h+\lambda b^{-1}(\lambda-2\partial_x)(\partial_x^2-\alpha^2)^{-1}bh,$$

$$\begin{aligned}\langle(\partial_x^2-\alpha^2)g,h\rangle &= \int_0^1 b^{-1}(\partial_x^2-\alpha^2)g(\partial_x^2-\alpha^2)b\overline{g} = \int_0^1 b^{-1}(\partial_x^2-\alpha^2)g(b(\partial_x^2-\alpha^2)\overline{g}+\lambda^2 b\overline{g}+2\lambda b\partial_x\overline{g})\\ &= \|(\partial_x^2-\alpha^2)g\|_2^2+\lambda^2\int_0^1(\partial_x^2-\alpha^2)g\overline{g}+2\lambda\int_0^1(\partial_x^2-\alpha^2)g\cdot\partial_x\overline{g}\\ &= \|(\partial_x^2-\alpha^2)g\|_2^2-\lambda^2\|g'\|_2^2-\alpha^2\lambda^2\|g\|_2^2+2\lambda(\langle g'',g'\rangle-\alpha^2\langle g,g'\rangle),\end{aligned}$$

$$\|(\partial_x^2-\alpha^2)g\|_2^2 = \|g''\|_2^2+\alpha^4\|g\|_2^2+2\alpha^2\|g'\|_2^2,$$

The third term is dealt with as follows:

$$b(\partial_x^2-\alpha^2)^{-1}bh = b\psi,\quad h = b^{-1}(\partial_x^2-\alpha^2)\psi,$$

$$\langle b\psi,h\rangle = \int_0^1\psi(\partial_x^2-\alpha^2)\overline{\psi}dx = -\|\psi'\|_2^2-\alpha^2\|\psi\|_2^2,$$

Combining the equations above together, we have

$$\begin{aligned}&\langle F,ch\rangle = \langle c^2(\partial_x^2-\alpha^2)g,ch\rangle+\overline{c}\alpha^2\|bh\|_2^2-\overline{c}\lambda^2\alpha^2\langle b\psi,h\rangle-|c|^2\eta\langle(\partial_x^2-\alpha^2)h,h\rangle\\ =&|c|^2c(\|(\partial_x^2-\alpha^2)g\|_2^2-\lambda^2\|g'\|_2^2-\alpha^2\lambda^2\|g\|_2^2+2\lambda(\langle g'',g'\rangle-\alpha^2\langle g,g'\rangle))\\ &+\overline{c}(\alpha^2\|bh\|_2^2+\lambda^2\alpha^2\|\psi'\|_2^2+\lambda^2\alpha^4\|\psi\|_2^2)+|c|^2\eta(\|h'\|_2^2+\alpha^2\|h\|_2^2)\\ =&|c|^2c(\|g''\|_2^2+(2\alpha^2-\lambda^2)\|g'\|_2^2+\alpha^2(\alpha^2-\lambda^2)\|g\|_2^2+2\lambda(\langle g'',g'\rangle-\alpha^2\langle g,g'\rangle))\\ &+\overline{c}(\alpha^2\|bh\|_2^2+\lambda^2\alpha^2\|\psi'\|_2^2+\lambda^2\alpha^4\|\psi\|_2^2)+|c|^2\eta(\|h'\|_2^2+\alpha^2\|h\|_2^2)\end{aligned}$$

We see that when $\Re(c)>0, -1\le\lambda\le 1$, the real part of the right hand side of the above equation is strictly positive for non-zero h.

$$\|bh\|_2^2 = \|(\partial_x^2-\alpha^2)\psi\|_2^2 = \|\psi''\|_2^2+2\alpha^2\|\psi'\|_2^2+\alpha^4\|\psi\|_2^2,$$

Question 1. 1) Prove that when $\eta=0$, if $c\in\sigma(\mathcal{L}_\alpha)$ then there exist $x_c\in[0,1]$ such that $c=\pm i\alpha b(x_c)$. It means that $\mathcal{L}_\alpha$ can only have embedding eigenvalues. An equivalent form of this proposition appears in [10].

2) Does the Rayleigh equation for Euler's equation only admits embedding eigenvalues?

Proof. Method 1: When $c^2+\alpha^2 b^2\neq 0$, rewrite the equation as follows

$$(\partial_x^2-\alpha^2)g+\frac{2\alpha^2 bb'\partial_x g}{c^2+\alpha^2 b^2} = 0,\quad g(0)=g(1)=0,$$

It is an second order elliptic ordinary differential equation on $[0,1]$ with Dirichlet's boundary condition.

□

$$\mathcal{H}=\{(\psi,\xi),\psi,\xi\in H_0^1([0,1])\},\quad \|(\psi,\xi)\|_{\mathcal{H}}^2=\int_0^1\partial_x\psi\overline{\partial_x\psi}+\partial_x\xi\overline{\partial_x\xi},$$

$$\partial_t\xi=\alpha(\partial_x^2-\alpha^2)^{-1}(b(\partial_x^2-\alpha^2)\psi-b''\psi)=\alpha(b\psi+K_1\psi),$$

where K_1 is a compact operator defined by

$$K_1\psi=(\partial_x^2-\alpha^2)^{-1}(-2b'\partial_x\psi-2b''\psi)=-2(\partial_x^2-\alpha^2)^{-1}\partial_x(b'\psi),$$

and we have $\overline{K_1\psi}=K_1\overline{\psi}$. Notice that $v_x,B_x\in\mathbb{R},\xi=\widehat{v}_x,\psi=i\widehat{B}_x$ only implies that $\widehat{v}_x(\alpha)=\overline{\widehat{v}_x(-\alpha)},\widehat{B}_x(\alpha)=\overline{\widehat{B}_x(-\alpha)}$, ξ,ψ are complex-valued functions.

$$\int_0^1\partial_t\xi\overline{\xi}=\int_0^1\alpha(b\psi+K_1\psi)\overline{\xi},\quad \int_0^1\partial_t\psi\overline{\psi}=\int_0^1(-\alpha b\xi+\eta(\partial_x^2-\alpha^2)\psi)\overline{\psi},$$

$$\partial_t(\|\psi\|_2^2+\|\xi\|_2^2)=\int_0^1\alpha(K_1\psi\overline{\xi}+\xi K_1\overline{\psi})+\eta(\partial_x^2-\alpha^2)\psi\overline{\psi}+\psi\eta(\partial_x^2-\alpha^2)\overline{\psi},$$

When $b(x)=e^{\lambda x}$, we have $K_1\psi=(\partial_x^2-\alpha^2)^{-1}(-2\lambda b\partial_x\psi-2\lambda^2b\psi)=-2\lambda(\partial_x^2-\alpha^2)^{-1}\partial_x(b\psi)$.

$$\partial_t\psi=-\alpha b\xi+\eta(\partial_x^2-\alpha^2)\psi=-\alpha b(\xi(0)+\int_0^t\alpha(b\psi(t')+K\psi(t'))dt')+\eta(\partial_x^2-\alpha^2)\psi,$$

So the evolutionary equation of ψ takes the following form, which is similar to the wave equation:

$$\partial_t^2\psi=-\alpha^2b(b\psi+K_1\psi)+\eta(\partial_x^2-\alpha^2)\partial_t\psi,$$

Substitute $\psi=bg$ and let $\eta=0,B=b^2$, we have

$$\partial_t^2(bg)=-\alpha^2b(b^2g+Kbg),\quad K_1bg=-2(\partial_x^2-\alpha^2)^{-1}\partial_x(b'bg)=-(\partial_x^2-\alpha^2)^{-1}\partial_x(B'g),$$

$$\partial_t^2g+\alpha^2(b^2g+K_1bg)=\partial_t^2g+\alpha^2(Bg-(\partial_x^2-\alpha^2)^{-1}\partial_x(B'g))=0,$$

the above calculation is the same as equation (2.5) in [10]. We define

$$Kg=K_1bg=-(\partial_x^2-\alpha^2)^{-1}\partial_x(B'g),$$

Proposition 1 (Energy conservation on each frequency)**.**

2 Spectral method

Theorem 1. Reference: Lecture notes on functional analysis II by Gongqing Zhang, p53 problem 5.5.10. $\mathcal{H}$ is a Hilbert space, N is a normal operator on $\mathcal{H}$ and its spectrum $\sigma(N)$ is countable, then $\mathcal{H}$ has a orthonormal basis $B=\{y\}$ where y are eigenfunctions of N, and the Fourier expansion holds:

$$x=\sum_{y\in B}(x,y)y,\quad x\in\mathcal{H},$$

the Fourier coefficients (x,y) only have countably many nonzero elements.

Proof. 1) Eigenspaces of different eigenvalues are orthogonal. If f_1,f_2 are two eigenfunctions of N with different eigenvalues λ_1,λ_2. When N is self-adjoint, we have $\lambda_1,\lambda_2\in\mathbb{R}$,

$$\lambda_1\langle f_1,f_2\rangle=\langle Nf_1,f_2\rangle=\langle f_1,Nf_2\rangle=\lambda_2\langle f_1,f_2\rangle,$$

Since $\lambda_1\neq\lambda_2$, $\langle f_1,f_2\rangle=0$. □

Problem 1. Eigenfunctions of ∂_x^2 on $H_0^1([0,1])$.

$$\partial_x^2 f = \lambda f, \quad -\xi^2 \widehat{f} = \lambda \widehat{f}, \quad \lambda = -\xi^2, \quad \mathrm{supp}\widehat{f} = \{\pm\xi\}, \quad f(x) = \widehat{f}(\xi)e^{i\xi x} + \widehat{f}(-\xi)e^{-i\xi x},$$

Boundary conditions are

$$f(0) = \widehat{f}(\xi) + \widehat{f}(-\xi) = 0, \quad f(1) = \widehat{f}(\xi)e^{i\xi} + \widehat{f}(-\xi)e^{-i\xi} = 0,$$

So we have $e^{i\xi} = e^{-i\xi}, \xi = k\pi, k \in \mathbb{Z}\backslash\{0\}$, $\lambda = -k^2\pi^2, f = C\sin k\pi x$.

Consider the case when $b(x)$ is a positive constant. We have

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g - c\eta(\partial_x^2 - \alpha^2)^2 g, \quad g|_{x=0,1} = g''|_{x=0,1} = 0,$$

Let $h = (\partial_x^2 - \alpha^2)g$, we claim that when $\Re(c) > 0, f \in H_0^1([0,1])$, the solution to $OS_\alpha(g) = f$ uniquely exists. We have

$$\mathcal{L}_\alpha h \triangleq OS_\alpha(g) = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = f,$$

Assume that $f = \sum_k (f, e_k)e_k$, then $(\partial_x^2 - \alpha^2)e_k = -(k^2\pi^2 + \alpha^2)e_k$,

$$(f, e_k) = (h, e_k)(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)), \quad (h, e_k) = \frac{(f, e_k)}{c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)},$$

$$(g, e_k) = -\frac{(h, e_k)}{(k^2\pi^2 + \alpha^2)} = -\frac{(f, e_k)}{(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2))(k^2\pi^2 + \alpha^2)},$$

Problem 2. Eigenvalue problem: when b is constant, for what values of c is $\mathcal{L}_\alpha$ not injective?

$$\mathcal{L}_\alpha h = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = 0,$$

When $\eta = 0$, eigenvalues are $c = \pm i\alpha b$, and their invariant space is the whole $H_0^1([0,1])$. When $\eta > 0$, 1) $c^2 + \alpha^2 b^2 = 0$, then $c \neq 0$, since $\partial_x^2 - \alpha^2$ is injective, we have $h = 0$.

2) $c^2 + \alpha^2 b^2 \neq 0$, take Fourier transform $h = \sum_k \widehat{h}_k e_k$, we have

$$(c^2 + \alpha^2 b^2 + c\eta\alpha^2 + c\eta k^2\pi^2)\widehat{h}_k = 0, \quad k \neq 0,$$

For small k, $\eta\alpha^2 + \eta k^2\pi^2 \le 2|\alpha|b$, the roots are two conjugate complex numbers with negative real parts and norm αb. For large k, $\eta\alpha^2 + \eta k^2\pi^2 > 2|\alpha|b$, the two real roots are

$$c_{1,2} = -\frac{\eta\alpha^2 + \eta k^2\pi^2 \pm \sqrt{(\eta\alpha^2 + \eta k^2\pi^2)^2 - 4\alpha^2 b^2}}{2},$$

$$A = \int_0^1 e^{\lambda x}\sin k\pi x dx = -\int_0^1 \frac{e^{\lambda x}}{\lambda} k\pi\cos k\pi x dx = \frac{k\pi}{\lambda^2} - (-1)^k e^\lambda \frac{k\pi}{\lambda^2} - \int_0^1 \frac{e^{\lambda x}}{\lambda^2} k^2\pi^2 \sin k\pi x dx,$$

$$\int_0^1 \frac{e^{\lambda x}}{\lambda^2} k^2\pi^2 \sin k\pi x dx = \frac{k^2\pi^2}{\lambda^2} A, \quad A = \frac{(1 - (-1)^k e^\lambda)k\pi}{\lambda^2 + k^2\pi^2},$$

$$B = \int_0^1 e^{\lambda x}\cos m\pi x dx = (-1)^m \frac{e^\lambda}{\lambda} - \frac{1}{\lambda} + \int_0^1 \frac{e^{\lambda x}}{\lambda} m\pi \sin m\pi x dx,$$

$$\int_0^1 \frac{e^{\lambda x}}{\lambda} m\pi \sin m\pi x dx = -\int_0^1 \frac{e^{\lambda x}}{\lambda^2} m^2\pi^2 \sin m\pi x dx = -\frac{m^2\pi^2}{\lambda^2}B, \quad B = \frac{((-1)^m e^\lambda - 1)\lambda}{\lambda^2 + m^2\pi^2},$$

When $\lambda < 0$, we may also use complex analysis techniques to calculate the above integrals:

$$\int_0^{+\infty} e^{\lambda x} \sin k\pi x dx = \Im \int_0^{+\infty} e^{(\lambda + ik\pi)x} dx = \Im \frac{1}{-\lambda - ik\pi} = \frac{k\pi}{\lambda^2 + k^2\pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \sin k\pi x dx = A(1 + (-1)^k e^\lambda + (-1)^{2k} e^{2\lambda} + ...) = \frac{A}{1 - (-1)^k e^\lambda}, \quad A = \frac{(1 - (-1)^k e^\lambda)k\pi}{\lambda^2 + k^2\pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \cos m\pi x dx = \Re \int_0^{+\infty} e^{(\lambda + im\pi)x} dx = \Re \frac{1}{-\lambda - im\pi} = \frac{-\lambda}{\lambda^2 + m^2\pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \cos m\pi x dx = B(1 + (-1)^m e^\lambda + (-1)^{2m} e^{2\lambda} + ...) = \frac{B}{1 - (-1)^m e^\lambda}, \quad B = \frac{((-1)^m e^\lambda - 1)\lambda}{\lambda^2 + m^2\pi^2},$$

I try to express the relationship of h and $g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh$ on the frequency domain. Recall that $e_k = \frac{\sqrt{2}\sin k\pi x}{k\pi}, k \in \mathbb{Z}_+$,

$$g = \sum_k \widehat{g}_k e_k, \quad \widehat{g}_k = \langle g, e_k \rangle = \int_0^1 \partial_x g \overline{\partial_x e_k} dx,$$

$$h = \sum_k \widehat{h}_k e_k, \quad \widehat{h}_k = \langle h, e_k \rangle = \int_0^1 \partial_x h \overline{\partial_x e_k} dx,$$

$$\widehat{(bg)}_k = \langle bg, e_k \rangle = \int_0^1 \partial_x(bg)\overline{\partial_x e_k} = -\int_0^1 bg\overline{\partial_x^2 e_k} = \int_0^1 bgk^2\pi^2\overline{e_k}$$

$$= \int_0^1 b(\sum_l \widehat{g}_l e_l)k^2\pi^2\overline{e_k} = \sum_l \widehat{g}_l \int_0^1 bk^2\pi^2 e_l\overline{e_k},$$

$$e_l\overline{e_k} = \frac{2\sin l\pi x \sin k\pi x}{lk\pi^2} = \frac{\cos(l-k)\pi x - \cos(l+k)\pi x}{lk\pi^2},$$

$$\int_0^1 bk^2\pi^2 e_l\overline{e_k} = \int_0^1 b\frac{k}{l}(\cos(l-k)\pi x - \cos(l+k)\pi x),$$

$$\int_0^1 b\frac{k}{l}\cos(l-k)\pi x = \frac{k}{l}\int_0^1 e^{\lambda x}\cos(l-k)\pi x dx = \frac{k}{l}\frac{((-1)^{l-k}e^\lambda - 1)\lambda}{\lambda^2 + (l-k)^2\pi^2} = B_1,$$

$$\int_0^1 b\frac{k}{l}\cos(l+k)\pi x = \frac{k}{l}\frac{((-1)^{l+k}e^\lambda - 1)\lambda}{\lambda^2 + (l+k)^2\pi^2} = B_2,$$

$$B_1 - B_2 = \frac{k}{l}((-1)^{l-k}e^\lambda - 1)\lambda(\frac{1}{\lambda^2 + (l-k)^2\pi^2} - \frac{1}{\lambda^2 + (l+k)^2\pi^2}) = \frac{4k^2\pi^2((-1)^{l-k}e^\lambda - 1)\lambda}{(\lambda^2 + (l^2+k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

$$\widehat{(bg)}_k = \sum_l \widehat{g}_l \frac{4k^2\pi^2((-1)^{l-k}e^\lambda - 1)\lambda}{(\lambda^2 + (l^2+k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

If $u, v \in H_0^1([0,1]), u = \sum_k \widehat{u}_k e_k, v = \sum_k \widehat{v}_k e_k$ satisfy $v = (\partial_x^2 - \alpha^2)u$, then their Fourier coefficients satisfy

$$\widehat{v}_k = -(k^2\pi^2 + \alpha^2)\widehat{u}_k, \quad \widehat{u}_k = -\frac{\widehat{v}_k}{k^2\pi^2 + \alpha^2},$$

Since $b^{-1}(x) = e^{-\lambda x}$, the action of multiplier b^{-1} on the Fourier side is given by

$$\widehat{(b^{-1}g)}_k = \sum_l \widehat{g}_l \frac{4k^2\pi^2((-1)^{l-k}e^{-\lambda}-1)(-\lambda)}{(\lambda^2+(l^2+k^2)\pi^2)^2-4l^2k^2\pi^4},$$

We may denote

$$K_\lambda(k,l) = \frac{4k^2\pi^2}{(\lambda^2+(l^2+k^2)\pi^2)^2-4l^2k^2\pi^4},$$

Consider representing the Fourier coefficients of h in terms of Fourier coefficients of g, where $g = b^{-1}(\partial_x^2-\alpha^2)^{-1}bh$, we have

$$v = bh, \quad \psi = (\partial_x^2-\alpha^2)^{-1}v, \quad g = b^{-1}\psi,$$

$$\widehat{v}_k = \sum_l \widehat{h}_l((-1)^{l-k}e^{\lambda}-1)\lambda K_\lambda(k,l), \quad \widehat{\psi}_k = -\frac{\widehat{v}_k}{k^2\pi^2+\alpha^2},$$

$$\begin{aligned}
\widehat{g}_m = \sum_k \widehat{\psi}_k((-1)^{l-k}e^{-\lambda}-1)(-\lambda)K_\lambda(m,k) &= \sum_k \widehat{v}_k \frac{((-1)^{l-k}e^{-\lambda}-1)\lambda}{k^2\pi^2+\alpha^2}K_\lambda(m,k) \\
= \sum_{k,l} \frac{((-1)^{k-m}e^{-\lambda}-1)\lambda}{k^2\pi^2+\alpha^2}K_\lambda(m,k)((-1)^{l-k}e^{\lambda}-1)\lambda K_\lambda(k,l)\widehat{h}_l \\
= \sum_l \widehat{h}_l \sum_k \frac{\lambda^2((-1)^{k-m}e^{-\lambda}-1)((-1)^{l-k}e^{\lambda}-1)}{k^2\pi^2+\alpha^2}K_\lambda(m,k)K_\lambda(k,l)
\end{aligned}$$

3 Energy conservation of nonlinear RVMHD

Suppose u and b are the velocity field and magnetic field, p is the total pressure, we consider the original nonlinear RVMHD equations' energy conservation:

$$\partial_t(\|u\|_2^2+\|b\|_2^2) = \partial_t\int_\Omega u\cdot u+b\cdot b = 2\int_\Omega u\cdot(-\nabla p-u\cdot\nabla u+b\cdot\nabla b+\mu\Delta u)+b\cdot(b\cdot\nabla u-u\cdot\nabla b+\eta\Delta b),$$

$$\int_\Omega u\cdot\nabla p = \int_\Omega u\cdot\nabla p+p\nabla\cdot u = \int_\Omega \nabla\cdot(pu) = 0,$$

$$u\cdot(u\cdot\nabla u) = u\cdot\nabla\frac{|u|^2}{2}, \quad \nabla\cdot(|u|^2u) = u\cdot\nabla|u|^2+|u|^2\nabla\cdot u,$$

$$\nabla\cdot(|b|^2u) = u\cdot\nabla|b|^2+|b|^2\nabla\cdot u, \quad b\cdot(u\cdot\nabla)b = u\cdot\nabla\frac{|b|^2}{2},$$

$$\int_\Omega u\cdot(u\cdot\nabla u) = \frac{1}{2}\int_\Omega\nabla\cdot(|u|^2u) = 0, \quad \int_\Omega b\cdot(u\cdot\nabla b) = \frac{1}{2}\int_\Omega\nabla\cdot(|b|^2u) = 0,$$

$$\nabla\cdot((u\cdot b)b) = (b\cdot\nabla)(u\cdot b)+(u\cdot b)(\nabla\cdot b), \quad (b\cdot\nabla)(u\cdot b) = u\cdot(b\cdot\nabla)b+b\cdot(b\cdot\nabla)u,$$

$$\int_\Omega u\cdot(b\cdot\nabla)b+b\cdot(b\cdot\nabla)u = \int_\Omega\nabla\cdot((u\cdot b)b) = 0,$$

$$\int_\Omega u\cdot\Delta u = -\int_\Omega\nabla u_x\cdot\nabla u_x+\nabla u_y\cdot\nabla u_y+\nabla u_z\cdot\nabla u_z = -\int_\Omega|\nabla u|^2, \quad \int_\Omega b\cdot\Delta b = -\int_\Omega|\nabla b|^2,$$

Energy conservation for perturbations are

Representing $(\partial_x^2)^{-1}$ with Dirichlet's boundary condition on $[0,1]$ as an integral operator:

$$\partial_x^2 v = u, \quad u|_{x=0,1} = 0, \quad v|_{x=0,1} = 0, \quad v(x) = \int_0^1 G(x,x')u(x')dx',$$

$$G(0,x') = G(1,x') = 0, \quad G(x,x') = \begin{cases} (x'-1)x, x \le x' \\ (x-1)x', x > x' \end{cases},$$

In this case, $\partial_x^2 G(x,x') = 0$ on $[0,x')$ and $(x',1]$, so it is linear on the above two intervals. Let

$$\partial_x G(x,x')|_{x=x'_-} = a, \quad \partial_x G(x,x')|_{x=x'_+} = b, \quad a = b-1, \quad G(x',x') = ax' = -b(1-x'),$$

so its solution is $a = x'-1, b = x'$.

Representing $(\partial_x^2 - \alpha^2)^{-1}$ on $[0,1]$ as an integral operator when $\alpha \neq 0$:

$$(\partial_x^2 - \alpha^2)v = u, \quad u|_{x=0,1} = 0, \quad v|_{x=0,1} = 0, \quad v(x) = \int_0^1 G(x,x')u(x')dx',$$

$$G(0,x') = G(1,x') = 0, \quad (\partial_x^2 - \alpha^2)v(x) = \int_0^1 (\partial_x^2 - \alpha^2)G(x,x')u(x')dx' = u(x),$$

$$g(x) = G(x,x'), \quad (\partial_x^2 - \alpha^2)g(x) = \delta_{x'}(x) = \delta(x-x'), \quad g(x) = \sum_{k\ge 1} \widehat{g}(k)\sin k\pi x,$$

$$\widehat{g}(k) = 2\int_0^1 g(x)\sin k\pi x dx = 2\int_0^1 \widehat{g}(k)\sin^2 k\pi x dx = \widehat{g}(k),$$

$$\widehat{\delta}_{x'}(k) = 2\int_0^1 \delta_{x'}(x)\sin k\pi x dx = 2\sin k\pi x', \quad \widehat{g}(k) = \frac{\widehat{\delta}_{x'}(k)}{-k^2\pi^2 - \alpha^2} = \frac{2\sin k\pi x'}{-k^2\pi^2 - \alpha^2},$$

In this case $(\partial_x^2 - \alpha^2)g(x) = 0$ on $[0,x')$ and $(x',1]$, so it has the following forms on the above two intervals.

$$g(x) = A\sinh\alpha x, \quad x \in [0,x'), \quad g(x) = B\sinh\alpha(1-x), \quad x \in (x',1],$$

$$A\sinh\alpha x' = B\sinh\alpha(1-x'), \quad \alpha A\cosh\alpha x' = -\alpha B\cosh\alpha(1-x') - 1,$$

There exists $\lambda < 0$ such that $A = \lambda\sinh\alpha(1-x'), B = \lambda\sinh\alpha x'$,

$$-1 = \lambda\alpha(\cosh\alpha x'\sinh\alpha(1-x') + \cosh\alpha(1-x')\sinh\alpha x') = \lambda\alpha\sinh\alpha, \quad \lambda = -\frac{1}{\alpha\sinh\alpha},$$

$$A = -\frac{\sinh\alpha(1-x')}{\alpha\sinh\alpha}, \quad B = -\frac{\sinh\alpha x'}{\alpha\sinh\alpha}, \quad G(x,x') = \begin{cases} -\frac{\sinh\alpha(1-x')\sinh\alpha x}{\alpha\sinh\alpha}, x \le x' \\ -\frac{\sinh\alpha x'\sinh\alpha(1-x)}{\alpha\sinh\alpha}, x > x' \end{cases},$$

Notice that if we let $\alpha \to 0_+$, the above Green's function tends to the case $\alpha = 0$.

Question 2. Δ^{-1} is the inverse of Laplacian with Dirichlet's boundary condition,

$$\Delta = \partial_r^2 - \frac{k^2}{r^2}, \quad H(r) = e^{ikB\frac{t}{r^2}}, \quad g = H^{-1}\Delta^{-1}H\Delta\psi, \quad r \in [1,2],$$

Show that for some positive C independent to k, B, t, we have

$$\|\psi'\|_2^2 + (k^2 - \frac{1}{4})\|\frac{\psi}{r}\|_2^2 \geq C\|g'\|_2^2,$$

Modification: 1) Under Dirichlet's boundary condition,

$$\Delta = \partial_x^2 - k^2, \quad H(x) = e^{\lambda x}, \quad g = H^{-1}\Delta^{-1}H\Delta\psi, \quad x \in [0,1],$$

Does the following inequality hold for some positive C independent to k, B, t?

$$\|\psi'\|_2^2 + k^2\|\psi\|_2^2 \geq C\|g'\|_2^2,$$

2) Let $H(x) = e^{i\lambda x}$ be purely oscillatory, $\lambda \in \mathbb{R}_+, x \in [0,1]$, $g(x) \in \mathbb{R}$, then we have

$$\|\psi'\|_2^2 + k^2\|\psi\|_2^2 \geq \|g'\|_2^2 + k^2\|g\|_2^2,$$

3) Remove the condition $g(x) \in \mathbb{R}$, what can we say about ψ and g?

Proof. 1) Let the inner product on $[0,1]$ be $\langle u, v\rangle = \int_0^1 u\overline{v}dx$,

$$-\langle \Delta g, \psi\rangle = -\langle g, H^{-1}\Delta Hg\rangle = -\langle H^{-1}g, \Delta Hg\rangle = \langle \nabla H^{-1}g, \nabla Hg\rangle,$$
$$\langle (H^{-1}g)', (Hg)'\rangle = \langle H^{-1}g' - \lambda H^{-1}g, Hg' + \lambda Hg\rangle = \|g'\|_2^2 - \lambda^2\|g\|_2^2,$$
$$\langle \nabla H^{-1}g, \nabla Hg\rangle = \|g'\|_2^2 + (k^2 - \lambda^2)\|g\|_2^2,$$

2) Notice that $\overline{H^{-1}} = H$, $|H| = 1$,

$$-\langle \Delta g, \psi\rangle = -\langle g, H^{-1}\Delta Hg\rangle = -\langle Hg, \Delta Hg\rangle = \langle \nabla Hg, \nabla Hg\rangle,$$

$$\langle (Hg)', (Hg)'\rangle = \langle Hg' + i\lambda Hg, Hg' + i\lambda Hg\rangle = \|g'\|_2^2 + \lambda^2\|g\|_2^2 + \int_0^1 i\lambda(g\overline{g'} - g'\overline{g}),$$

If $g(x) \in \mathbb{R}$, then the last term above is 0 because $g\overline{g'} = g'\overline{g}$. So we have

$$\|g'\|_2^2 + (k^2 + \lambda^2)\|g\|_2^2 = \langle \nabla Hg, \nabla Hg\rangle = -\langle \Delta g, \psi\rangle = \langle \nabla g, \nabla\psi\rangle \leq \|\nabla g\|_2\|\nabla\psi\|_2,$$

and we get a slightly stronger inequality

$$\|\nabla\psi\|_2 \geq \frac{\|\nabla g\|_2^2 + \lambda^2\|g\|_2^2}{\|\nabla g\|_2} \geq \|\nabla g\|_2,$$

3) Assume $g(x) = u(x) + iv(x)$, the region that it encloses is Ω, then

$$\int_0^1 \frac{g\overline{g'} - g'\overline{g}}{2i} = \int_0^1 \Im(g\overline{g'}) = \int_0^1 \Im(u + iv)(u' - iv') = \int_0^1 u'v - uv' = -2Area(\Omega),$$

where $Area(\Omega)$ denotes signed area with positive sign if $g(x)$ travels counter-clockwisely.

$$\int_0^1 i\lambda(g\overline{g'} - g'\overline{g}) = -2\lambda\int_0^1 \Im(g\overline{g'}) = 4\lambda Area(\Omega),$$

i) By isoperimetric inequality on $\mathbb{R}^2$, let $L(\partial\Omega)=\int_0^1|g'|$ be the perimeter of $\partial\Omega$, we have

$$4\lambda Area(\Omega)\le\frac{\lambda}{\pi}L(\partial\Omega)^2=\frac{\lambda}{\pi}\|g'\|_1^2\le\frac{\lambda}{\pi}\|g'\|_2^2,$$

$$\langle\nabla Hg,\nabla Hg\rangle=\|\nabla g\|_2^2+\lambda^2\|g\|_2^2+4\lambda Area(\Omega)\ge\|\nabla g\|_2^2+\lambda^2\|g\|_2^2-\frac{\lambda}{\pi}\|g'\|_2^2,$$

So when $0<\lambda<\pi$, we have

$$\|\nabla g\|_2\|\nabla\psi\|_2\ge\langle\nabla g,\nabla\psi\rangle=\langle\nabla Hg,\nabla Hg\rangle\ge(1-\frac{\lambda}{\pi})\|\nabla g\|_2^2+(\frac{k^2\lambda}{\pi}+\lambda^2)\|g\|_2^2,$$

But the constant factor $C=1-\frac{\lambda}{\pi}$ on the right hand side depends on λ.

ii)

$$\int_0^1\Im(g\overline{g'})\le\int_0^1|g||g'|\le\|g\|_2\|g'\|_2,\quad\langle\nabla Hg,\nabla Hg\rangle\ge\|\nabla g\|_2^2+\lambda^2\|g\|_2^2-2\lambda\|g\|_2\|g'\|_2,$$

Directly use $\lambda^2\|g\|_2^2-2\lambda\|g\|_2\|g'\|_2+\|g'\|_2^2\ge0$ exploits all first order term in $\|\nabla g\|_2^2$, but it is independent of λ. So we need a more delicate estimate.

iii) We want to show that for some $\epsilon>0$,

$$\langle\nabla Hg,\nabla Hg\rangle=\|\nabla g\|_2^2+\lambda^2\|g\|_2^2-2\lambda\int_0^1\Im(g\overline{g'})\ge\epsilon\|\nabla g\|_2^2,$$

$$\int_0^1\Re(g\overline{g'})=\int_0^1\frac{g\overline{g'}+\overline{g}g'}{2}=\int_0^1(\frac{|g|^2}{2})'=0,$$

So $\int_0^1 g\overline{g'}$ is purely imaginary. Assume that $g=\sum_{j>0}\widehat{g}_je_j$, $e_j=\frac{\sqrt{2}\sin j\pi x}{j\pi}$, then

$$\int_0^1 g\overline{g'}=\int_0^1(\sum_{j>0}\widehat{g}_j\frac{\sqrt{2}\sin j\pi x}{j\pi})(\sum_{l>0}\overline{\widehat{g}_l}\sqrt{2}\cos l\pi x)=\int_0^1\sum_{j,l>0}\widehat{g}_j\overline{\widehat{g}_l}\frac{\sin(j+l)\pi x+\sin(j-l)\pi x}{j\pi}$$

$$=\sum_{j,l>0,\ 2\nmid j+l}\widehat{g}_j\overline{\widehat{g}_l}\frac{1}{j\pi}(\frac{2}{(j+l)\pi}+\frac{2}{(j-l)\pi})=\sum_{j,l>0,\ 2\nmid j+l}\widehat{g}_j\overline{\widehat{g}_l}\frac{4}{(j^2-l^2)\pi^2},$$

$$\sum_{j,l>0,\ 2\nmid j+l}\widehat{g}_j\overline{\widehat{g}_l}\frac{4}{(j^2-l^2)\pi^2}=\sum_{j,l>0,\ 2\nmid j,\ 2|l}\widehat{g}_j\overline{\widehat{g}_l}\frac{4}{(j^2-l^2)\pi^2}-\sum_{j,l>0,\ 2\nmid j,\ 2|l}\overline{\widehat{g}_j}\widehat{g}_l\frac{4}{(j^2-l^2)\pi^2},$$

$$(1-\epsilon)\|\nabla g\|_2^2+\lambda\|g\|_2^2=\sum_{j>0}|\widehat{g}_j|^2((1-\epsilon)(1+\frac{k^2}{j^2\pi^2})+\frac{\lambda^2}{j^2\pi^2}),$$

However, numerical simulation doesn't support that there exists $\epsilon>0$ such that

$$(1-\epsilon)\|\nabla g\|_2^2+\lambda\|g\|_2^2\ge2\lambda|\int_0^1 g\overline{g'}|=2\lambda|\sum_{j,l>0,\ 2\nmid j+l}\widehat{g}_j\overline{\widehat{g}_l}\frac{4}{(j^2-l^2)\pi^2}|,$$

holds uniformly for all $\lambda>0$. $\square$

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