Solutions to selected problems on Timus online judge

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Problem 1 (1058 Chocolate)**.** Key point is to calculate the minimal length of pq for four points A, B, C, D satisfying that $S_{\triangle AXB} = S_{\triangle CXD}$ where $X = AD \cap BC$, $p \in AB, q \in CD$ and $S_{\triangle AYp} = S_{\triangle DYq}$. $AC//BD$ since $S_{\triangle AXB} = S_{\triangle CXD}$. Let $Z = AB \cap CD$, if $(\cos ZAD - \cos ZDA) * (\cos ZBC - \cos ZCB) < 0$, then the length of pq is given by

$$(\frac{pq}{2})^2 = S_{\triangle ZAD} \tan\frac{Z}{2} = \frac{\lambda}{1-\lambda} S_{\triangle BAD} \frac{\sin Z}{1+\cos Z},$$

$$\lambda = \frac{AC}{BD}, \quad \sin Z = \frac{AB \times CD}{|AB||CD|}, \quad \cos Z = \frac{AB \cdot CD}{|AB||CD|},$$

$$AB \times CD = AB \times D'A = BD' \times BA = (1-\lambda) BD \times BA, \quad S_{\triangle BAD} = \frac{AB \times AD}{2},$$

$$pq = \sqrt{\frac{2\lambda (AB \times AD)^2}{|AB||CD|(1+\cos Z)}}$$

Problem 2 (1199 Mouse)**.** Single source shortest path using Dijkstra. Key point is to generate the path of the mouse. Use $i = (i + N - 1)\%N$ instead of $i = (i - 1)\%N$ since taking modulo on negative integers will produce negative answers.

Problem 3 (1239 Ghost Busters)**.** Project the ghosts onto the unit sphere, they become spherical circles. Preprocess the ghosts so that their center lie on the unit sphere, we may assume ghost i has center $po_i = (x_i, y_i, z_i)$ and radius r_i. An spherical circle has a plane it lies on and radius in radian:

$$xx_i + yy_i + zz_i = c_i = \sqrt{1 - r_i^2}, \quad rad = \arcsin r_i,$$

For circle i and j, their intersections are determined as follows:

$$A = \begin{pmatrix} 1 & po_i \cdot po_j \\ po_j \cdot po_i & 1 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} c_i \\ c_j \end{pmatrix}, \quad \begin{pmatrix} x_{mid} \\ y_{mid} \\ z_{mid} \end{pmatrix} = \begin{pmatrix} x_i & x_j \\ y_i & y_j \\ z_i & z_j \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Here p_{mid} is the projection of the origin onto the line intersection of planes i and j. The above calculation find the least square solution to the equation below

$$xx_i + yy_i + zz_i = c_i, \quad xx_j + yy_j + zz_j = c_j,$$

The direction of the line is $dir = \frac{po_i \times po_j}{|po_i \times po_j|}$, with half the segment length $halfseg = \sqrt{1 - |p_{mid}|^2}$. So the intersections are

$$I_1 = p_{mid} + dir * halfseg, \quad I_2 = p_{mid} - dir * halfseg,$$

Scan the $\frac{1}{8}$ unit sphere from the north pole to equator. There are only three cases that would change the order of intersection segments of circles: inserting north endpoint, deleting south endpoint, and intersection between circles. So my strategy is to enumerate all such critical latitudes, for each latitude, scan from longitude 0 to $\frac{\pi}{2}$ and record the location that meets the most segments. The original formula of inverse of a $2 * 2$ matrix has a bug, but now it is fixed.

Problem 4 (1368 Goat in the Garden 3)**.** When $K = n^2 + (n+1)^2$, $ans = 4n + 4$; when $K = 2n^2$, $ans = 4n + 2$; when $K = n(2n+1)$, $ans = 4n + 3$; when $K = (2n+1)(n+1)$, $ans = 4n + 5$. In the above four cases, we say that K is saturated. For non-saturated K value, construct the output of saturated K value first and modify it.

Problem 5 (1384 Goat in the Garden 4)**.** Non-convex optimization. $dirnum = 40, stepsizenum = 18, stepsize = 16./(1 << i), 0 \leq i \leq 17$. Initial seeds are mid points of edges and polygon vertices. Actually I implemented a gradient descent algorithm adopted from Boyd's book "Convex Optimization". Initial directions are randomly selected before each step.

Problem 6 (1420 Integer-Valued Complex Division)**.** Implemented struct GaussianQT in this problem. Since the norms of numerator and denominator of $\frac{a}{b}$ exceed the range of long long, I wrote BigInteger struct and got accepted for the first time. Question: why do I get WA13 using my BigInteger in problem 1661? Update: bug fixed.

Problem 7 (1460 Wires)**.** Claim: 1) auxiliary points has degree 3, and their 3 adjacent edges has pairwise angle $\frac{2\pi}{3}$. This can be proved by calculus of variations.

2) It is impossible to have 3 auxiliary points. Otherwise the total degree of vertices is at least 13, contradiction.

3) The trilinear coordinate of the Fermat point (actually the first isogonic center) of a given triangle ABC is $\sec(A - \frac{\pi}{6}) : \sec(B - \frac{\pi}{6}) : \sec(C - \frac{\pi}{6})$, its barycentric coordinate is $\frac{a}{\cos(A - \frac{\pi}{6})} : \frac{b}{\cos(B - \frac{\pi}{6})} : \frac{c}{\cos(C - \frac{\pi}{6})}$, and I used this formula in computer program calculation.

3) We may enumerate every possible configurations: when there are no auxiliary points, we calculate its minimal spanning tree using Prim's algorithm. When there is only one auxiliary point, this point is uniquely determined by the three points it connects to. When there are two auxiliary points, the configuration is uniquely determined by the permutation of $ABCD$. $auxpt1$ is the Fermat point of $A, B, auxpt2$, $auxpt2$ is the Fermat point of $C, D, auxpt1$.

Problem 8 (1464 Light)**.** Sort vertices according to their polar angles. A container is used to store segments in current region. Insertion or deletion are executed on boundary rays of regions. Comparator is dynamic during the process of sweeping, it is represented by the distance from lamp to the intersections of current region's bisector with segments. Supports deletion by key, query the nearest segment in current region, value is not needed since it's dynamic while sweeping, so I use std::set with delicately designed custom comparator. The comparator compares the distances from the lamp to the intersections of the region bisector and segments.

Problem 9 (1475 Ryaba Hen)**.** Let e_x be the direction parallel to the roof, e_y perpendicular to e_x and points upwards. Then in the direction of e_x, the egg's motion is uniformly accelerated with acceleration $g\sin\theta$, $\tan\theta = \frac{H}{l}$. In the direction of e_y, the egg's motion is periodic with period $T = \frac{2v_0}{g}$. In each peroid $[(n-1)T, nT]$, v_y is uniformly accerlerated with acceleration $-g\cos\theta$ and initial velocity $v_0\cos\theta$. Accepted using python and Qifeng Chen's BigNumber struct, but got TLE-3 using my implementation of BigNumber.

$$v_0 \sin\theta t + \frac{1}{2} g \sin\theta t^2 > \sqrt{H^2 + l^2}, \quad t = \frac{2nv_0}{g},$$

$$\sin\theta = \frac{H}{\sqrt{H^2 + l^2}}, \quad v_0 = \sqrt{2gh}, \quad n(n+1) > \frac{H^2 + l^2}{4Hh},$$

Problem 10 (1482 Triangle Game)**.** Notice that the original formula of inverse of $2*2$ matrix has a bug.

$$T_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = (I - T_\theta) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

New discovery is that operator= is automatically constructed in a custom struct.

Problem 11 (1566 Triangular Postcards)**.** If $\triangle PQR$ can be included inside $\triangle ABC$, then there exist a position such that two vertices of $\triangle PQR$ lie on the sides of $\triangle ABC$. Assume that they are P, Q.

1) P, Q lie on the same side of $\triangle ABC$.

2) P, Q lie on different sides. Assume that $P \in CB, Q \in CA, CP = \lambda, CQ = \mu$, then

$$\lambda^2 + \mu^2 - 2\lambda\mu\cos C = r^2 = \sin^2\frac{C}{2}(\lambda+\mu)^2 + \cos^2\frac{C}{2}(\lambda-\mu)^2,$$

$$R = P + \frac{q}{r}\begin{pmatrix} \cos P & -\sin P \\ \sin P & \cos P \end{pmatrix}(Q - P) = \begin{pmatrix} \lambda + \frac{q}{r}(\mu\cos C\cos P - \lambda\cos P - \mu\sin C\sin P) \\ \frac{q}{r}(\mu\cos C\sin P - \lambda\sin P + \mu\sin C\cos P) \end{pmatrix}$$

Constraints are $0 \le \lambda \le a, 0 \le \mu \le b$, and three constraints depicting $R \in \triangle ABC$:

$$y_R \ge 0, \quad x_R\sin C - y_R\cos C \ge 0, \quad BA \times BR = (x_A - x_B)y_R - y_A x_R + y_A x_B \ge 0,$$

This problem can be solved by checking the sign of $f(\lambda, \mu) = \lambda^2 + \mu^2 - 2\lambda\mu\cos C$ on the boundary of the polygon determined by the above 7 constraints.

Method 2: actually we only need to check case 1 in the discussion above. Avoid using trigonometric functions helps improve numerical accuracy.

Problem 12 (1594 Aztec Treasure)**.** Calculate the number of domino tilings on a $m \times n$ rectangle grid. Let $m_1 = \lceil \frac{m}{2} \rceil, n_1 = \lceil \frac{n}{2} \rceil$, the formula is given by

$$Z_{m,n}(1,1) = \prod_{j,k=1}^{m_1,n_1} (4\cos^2\frac{\pi j}{m+1} + 4\cos^2\frac{\pi k}{n+1}),$$

where we define $g(h, v)$ to be the number of tilings with h horizontal and v vertical dominoes.

$$Z_{m,n}(x,y) = \sum_{h,v} g(h,v)x^h y^v, \quad h, v \ge 0, \quad 2(h+v) = mn,$$

Swap m, n if necessary to make sure that m is even.

$$Z_{m,n}(1,1) = \prod_{j,k=1}^{m_1,n_1} (4 + 2\cos\frac{2\pi j}{m+1} + 2\cos\frac{2\pi k}{n+1}),$$

Denote $P_{n_1}(x) = \prod_{k=1}^{n_1}(x + 2\cos\frac{2\pi k}{n+1})$, let $x_j = 4 + 2\cos\frac{2\pi j}{m+1}, 1 \le j \le m_1$, then the result $Z_{m,n}(1,1) = \prod_{j=1}^{m_1} P_{n_1}(x_j)$. We may calculate P_{n_1} by induction. 1) n is even, now

$$P_{n_1}(y + \frac{1}{y}) = \prod_{k=1}^{n_1}(y + \frac{1}{y} + 2\cos\frac{2\pi k}{n+1}) = y^{n_1} - y^{n_1-1} ... - y^{1-n_1} + y^{-n_1},$$

$$P_1 = x - 1, \quad P_2 = x^2 - x - 1, \quad P_{n_1} = xP_{n_1-1} - P_{n_1-2},$$

and I let $P_0 = P_{-1} = 1$ in my implementation. 2) n is odd, now

$$P_{n_1}(y + \frac{1}{y}) = \prod_{k=1}^{n_1}(y + \frac{1}{y} + 2\cos\frac{2\pi k}{n+1}) = \frac{(y^{n+1} - 1)(y - 1)}{y + 1},$$

$$P_1 = x - 2, \quad P_2 = x^2 - 2x, \quad P_{n_1} = xP_{n_1-1} - P_{n_1-2},$$

and I let $P_0 = 0$ in my implementation.

Key point is to implement a struct representing algebraic integers of the form

$$x_0 + \sum_{1 \le j \le m1} x_j 2\cos\frac{2\pi j}{m+1}, \quad m = 2m_1,$$

Its nontrivial arithmetic is essentially inside two methods named reduce() and totalreduce(). Notice that taking modulo is admissible since all the terms above are integral.

Theorem 1 (Domino tilings)**.** 1) The number of ways to cover an $m \times n$ rectangle with $\frac{mn}{2}$ dominoes is given by

$$\prod_{j,k=1}^{\lceil\frac{m}{2}\rceil,\lceil\frac{n}{2}\rceil} (4\cos^2\frac{\pi j}{m+1} + 4\cos^2\frac{\pi k}{n+1}),$$

2) The number of tilings of an Aztec diamond of order n is $2^{\frac{n(n+1)}{2}}$.

3) The number of tilings of an augmented Aztec diamond of order n with 3 long rows in the middle rather than 2 is $D(n, n)$, a Delannoy number, defined as follows:

$$D(m,n) = \sum_{k=0}^{\min(m,n)} \binom{m+n-k}{m}\binom{m}{k} = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} 2^k,$$

$$\sum_{m,n=0}^{\infty} D(m,n)x^m y^n = (1 - x - y - xy)^{-1}, \quad \sum_{n=0}^{\infty} D(n,n)x^n = (1 - 6x + x^2)^{-\frac{1}{2}},$$

Problem 13 (1599 Winding Number)**.** Method 1: calculate $\sum_{i=1}^{n} \angle P_i X P_{i+1}$. Resulted in TLE-12.

Method 2: calculate intersection number of the polygon with ray $y = y_X, x \ge x_X$.

Problem 14 (1621 Definite Integral)**.** Roots finding algorithms. Given an integer coefficient degree 4 polynomial with $|a_i| \le 10^6, a_4 \neq 0$. Notice that the precision requirement is high, relative error or absolute error is no more than 10^{-9}.

$$\int_{|x|=\epsilon} \frac{1}{x} dx = \int_0^{2\pi} \frac{e^{-i\theta}}{\epsilon} d\epsilon e^{i\theta} = \int_0^{2\pi} i d\theta = 2\pi i,$$

So residue theorem says that if a meromorphic function $f = \frac{g}{x-x_0}$ where g is holomorphic near x_0, then $\int_{|x-x_0|=\epsilon} f dx = 2\pi i g(x_0)$. Given $P(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, we want to eliminate a_3 by translation $x' = x + \frac{a_3}{4}$.

$$\begin{aligned}
P'(x') =& P(x) = (x' - \frac{a_3}{4})^4 + a_3(x' - \frac{a_3}{4})^3 + a_2(x' - \frac{a_3}{4})^2 + a_1(x' - \frac{a_3}{4}) + a_0 \\
=& x'^4 + 6x'^2(\frac{a_3}{4})^2 - 4x'(\frac{a_3}{4})^3 + (\frac{a_3}{4})^4 - 3x'^2 a_3 \frac{a_3}{4} + 3x' a_3(\frac{a_3}{4})^2 - a_3(\frac{a_3}{4})^3 \\
&+ a_2 x'^2 - 2x' a_2 \frac{a_3}{4} + a_2(\frac{a_3}{4})^2 + a_1 x' - a_1 \frac{a_3}{4} + a_0 \\
=& x'^4 + x'^2(6(\frac{a_3}{4})^2 - 3a_3\frac{a_3}{4} + a_2) + x'(-4(\frac{a_3}{4})^3 + 3a_3(\frac{a_3}{4})^2 - 2a_2\frac{a_3}{4} + a_1) \\
&+ (\frac{a_3}{4})^4 - a_3(\frac{a_3}{4})^3 + a_2(\frac{a_3}{4})^2 - a_1\frac{a_3}{4} + a_0 \\
=& x'^4 + x'^2(-\frac{3a_3^2}{8} + a_2) + x'(\frac{a_3^3}{8} - \frac{a_2a_3}{2} + a_1) - \frac{3a_3^4}{256} + \frac{a_2a_3^2}{16} - \frac{a_1a_3}{4} + a_0,
\end{aligned}$$

So we may define

$$a_2' = -\frac{3a_3^2}{8} + a_2, \quad a_1' = \frac{a_3^3}{8} - \frac{a_2a_3}{2} + a_1, \quad a_0' = -\frac{3a_3^4}{256} + \frac{a_2a_3^2}{16} - \frac{a_1a_3}{4} + a_0,$$

Now we substitute x', P', a' by x, P, a, it becomes

$$P(x) = x^4 + a_2x^2 + a_1x + a_0 = (x^2 - 2ax + b)(x^2 + 2ax + c), \quad a > 0, \quad a^2 < b, c,$$

Assume that it has two roots in the upper half plane $x_1 = -a + ui, x_2 = a + vi, u, v > 0$, then the integral is

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{P(x)} dx = 2\pi i (\frac{1}{(x_1 - \overline{x_1})(x_1 - x_2)(x_1 - \overline{x_2})} + \frac{1}{(x_2 - x_1)(x_2 - \overline{x_1})(x_2 - \overline{x_2})}) \\
=& \frac{\pi}{u(4a^2 - u^2 + v^2 - 4aui)} + \frac{\pi}{v(4a^2 - v^2 + u^2 + 4avi)} = \frac{\pi((4a^2 - u^2 + v^2)/u + (4a^2 - v^2 + u^2)/v)}{16a^4 + 8a^2(u^2 + v^2) + (v^2 - u^2)^2},
\end{aligned}$$

where we used

$$(x_1 - x_2)(x_1 - \overline{x_2}) = (-2a + (u - v)i)(-2a + (u + v)i) = 4a^2 - (u^2 - v^2) - 4aui,$$

$$(x_2 - x_1)(x_2 - \overline{x_1}) = (2a + (v - u)i)(2a + (v + u)i) = 4a^2 - (v^2 - u^2) + 4avi,$$

and the imaginary part is

$$\frac{4au/u}{(4a^2 - u^2 + v^2)^2 + (4au)^2} + \frac{-4av/v}{(4a^2 - v^2 + u^2)^2 + (4av)^2} = 0,$$

$$(4a^2 - u^2 + v^2)^2 + (4au)^2 = (4a^2 - v^2 + u^2)^2 + (4av)^2 = 16a^4 + 8a^2(u^2 + v^2) + (v^2 - u^2)^2,$$

1) When $a_1 = 0$, either one of the following two cases occur: i)$a = 0$, the integral becomes

$$\int_{\mathbb{R}} \frac{1}{(x^2+u^2)(x^2+v^2)}dx = \frac{1}{v^2-u^2}\int_{\mathbb{R}}(\frac{1}{x^2+u^2} - \frac{1}{x^2+v^2})dx = \frac{1}{v^2-u^2}(\frac{\pi}{u} - \frac{\pi}{v}),$$

where we used that

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+a^2}dx = \int_{-\infty}^{+\infty} \frac{1/a}{(\frac{x}{a})^2+1}d\frac{x}{a} = \frac{1}{a}\arctan\frac{x}{a}\Big|_{-\infty}^{+\infty} = \frac{\pi}{a},$$

This result agrees with our previous calculation since

$$\frac{\pi((-u^2+v^2)/u + (-v^2+u^2)/v)}{(v^2-u^2)^2} = \frac{1}{v^2-u^2}(\frac{\pi}{u} - \frac{\pi}{v}),$$

$$P(x) = (x^2+u^2)(x^2+v^2) = x^4 + (u^2+v^2)x^2 + u^2v^2,$$

and thus we can solve a quadratic equation to get values of u, v.

ii)$a > 0, u = v$, the integral becomes

$$\frac{8a^2\pi(1/u)}{16a^4+16a^2u^2} = \frac{\pi}{2u(a^2+u^2)},$$

$$P(x) = (x^2+2ax+a^2+u^2)(x^2-2ax+a^2+u^2) = x^4 + 2(u^2-a^2)x^2 + (a^2+u^2)^2,$$

and thus we can solve a linear equation to get values of a, u.

2) When $a_1 \neq 0$, notice that a^2 is a algebraic number with degree 3.

$$a = \frac{x_2+\overline{x_2}-x_1-\overline{x_1}}{4}, \quad \tilde{a}_1 = \frac{x_1+x_2-\overline{x_1}-\overline{x_2}}{4} = \frac{(u+v)i}{2}, \quad \tilde{a}_2 = \frac{x_1+\overline{x_2}-\overline{x_1}-x_2}{4} = \frac{(u-v)i}{2},$$

$$\tilde{a}_1^2 = -\frac{(u+v)^2}{4}\tilde{a}_2^2 = -\frac{(u-v)^2}{4}$$

Coefficients of P satisfy

$$P(x) = (x^2+2ax+a^2+u^2)(x^2-2ax+a^2+v^2) = x^4+(u^2+v^2-2a^2)x^2+2a(v^2-u^2)x+(a^2+u^2)(a^2+v^2),$$

$$a_2 = u^2+v^2-2a^2, \quad a_1 = 2a(v^2-u^2), \quad a_0 = (a^2+u^2)(a^2+v^2),$$

$$a^2+\tilde{a}_1^2+\tilde{a}_2^2 = a^2 - \frac{u^2+v^2}{2} = -\frac{a_2}{2},$$

$$a^2(\tilde{a}_1^2+\tilde{a}_2^2)+\tilde{a}_1^2\tilde{a}_2^2 = -\frac{a^2(u^2+v^2)}{2} + \frac{(u^2-v^2)^2}{16} = -\frac{a^2(u^2+v^2)}{2} + \frac{(u^2+v^2)^2}{16} - \frac{u^2v^2}{4} = \frac{1}{4}((\frac{a_2}{2})^2 - a_0),$$

$$a^2\tilde{a}_1^2\tilde{a}_2^2 = \frac{a^2(u^2-v^2)^2}{16} = \frac{1}{16}(\frac{a_1}{2})^2,$$

$$R(x) = x^3 + b_2x^2 + b_1x + b_0, \quad x' = x + \frac{b_2}{3},$$

$$b_2 = \frac{a_2}{2}, \quad b_1 = \frac{\frac{a_2^2}{4} - a_0}{4}, \quad b_0 = -\frac{a_1^2}{64},$$

$$R(x) = x'^3 + x'(-\frac{b_2^2}{3} + b_1) + \frac{2b_2^3}{27} - \frac{b_1b_2}{3} + b_0 = S(x'),$$

$$S(x) = x^3 + c_1 x + c_0, \quad c_1 = -\frac{b_2^2}{3} + b_1, \quad c_0 = \frac{2b_2^3}{27} - \frac{b_1 b_2}{3} + b_0, \quad x = y - \frac{c_1}{3y}$$

$$T(y) = S(x) = y^3 + c_0 - \frac{c_1^3}{27y^3}, \quad z = y^3, \quad \omega = e^{\frac{2\pi i}{3}},$$

$$x_1 = y_1 + y_2, \quad x_2 = y_1\omega + y_2\omega^2, \quad x_3 = y_1\omega^2 + y_2\omega,$$

Use long double and one step Newton method to improve result's accuracy.

Method 2: Find a square matrix such that $P(x)$ is its characteristic polynomial. According to rational canonical form, we may construct

$$A = \begin{pmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & 0 & -a_2 \\ & & 1 & -a_3 \end{pmatrix}, \quad \det(\lambda I - A) = P(\lambda),$$

Then we may apply iterative algorithms that finds unsymmetric eigenvalues of matrix A. But I haven't implemented this idea successfully.

Problem 15 (1626 Interfering Segment)**.** Reference: Computational geometry - algorithms and applications, chapter 2, line segment intersection. Introduction to algorithms 3rd edition, chapter 33, determining whether any pair of segments intersects.

Assume that the polygon is $P_1 P_2 ... P_n$. $\triangle ABC$ is part of a legal triangulation $\iff$ each of AB, BC, AC divides the polygon into two parts. Only if part is trivial. If part: assume that $\Omega_A, \Omega_B, \Omega_C$ are the parts of the polygon that don't contain A, B, C after divided by BC, CA, AB respectively, then each of them is a polygon without self intersection. So $\Omega_A, \Omega_B, \Omega_C$ can be triangulated, adding $\triangle ABC$ to obtain a triangulation of the original polygon.

$$X \in \triangle ABC \iff X \notin \Omega_A, \Omega_B, \Omega_C,$$

Preprocessing: for any pair i, j not next to each other, judge if $P_i P_j$ intersect with any edge besides $P_{i-1}P_i, P_i P_{i+1}, P_{j-1}P_j, P_j P_{j+1}$, store these boolean variables in $flag[i][j]$. If $flag[i][j] == true$, judge if X is on the $P_{i+1}...P_{j-1}$ side or $P_{j+1}...P_{i-1}$ side of $P_i P_j$, or is exactly on $P_i P_j$, and store it in $sideX[i][j]$. $flag[i][i+1]$ is always true. Denote by l_X the ray $y = y_X, x \geq x_X$, fix i and let j iterate from $i+1$ to $i-1$, we may maintain the intersection number of l_X with $P_i P_{i+1} ... P_j$, and thus know $sideX[i][j]$. Similarly we get $sideY[i][j]$.

Problem 16 (1660 The Island of Bad Luck)**.** Method 1: calculating Apollonian circles, but I didn't use this method. 1) Assume that

$$l_0 : y = y_0, \quad l_1 : y = y_1, \quad y_0 > y_1, \quad l_2 : x^2 + (y - y_1 - r_2)^2 = r_2^2, \quad r_2 < \frac{y_0 - y_1}{2},$$

Find the equation of circle ω such that ω is tangent to l_0, l_1, l_2. $r_\omega = \frac{y_0 - y_1}{2}$. There are two solutions,

$$\omega_1 : (x - 2\sqrt{r_2 r_\omega})^2 + (y - \frac{y_0 + y_1}{2})^2 = r_\omega^2,$$

$$\omega_2 : (x + 2\sqrt{r_2 r_\omega})^2 + (y - \frac{y_0 + y_1}{2})^2 = r_\omega^2,$$

where ω_1 is on the left, ω_2 is on the right.

2) In the original problem, assume that the large circle and small circle are Γ_0, Γ_2, victim's circle is Γ_1. We've already known that Γ_0 has radius R, Γ_2 has radius r, distance of their centers is d.

Assume that Γ_0, Γ_1 are tangential at $P(0, R)$, and let it be the inversion center with radius $\sqrt{2}R$. More precisely, the inversion is

$$\varphi : (x, y) \mapsto (\frac{2R^2 x}{x^2 + (y-R)^2}, \frac{2R^2(y-R)}{x^2 + (y-R)^2} + R),$$

$$\Gamma_0 \mapsto l_0 : y = 0, \quad \Gamma_1 \mapsto l_1 : y = R - \frac{R}{r_1^2},$$

Assume that the center of the small circle is $(d \sin\theta, d\cos\theta)$, then the equation of Γ_2 is

$$\Gamma_2 : (x - d\sin\theta)^2 + (y - d\cos\theta)^2 = r^2,$$

$$l_2 = \varphi(\Gamma_2) : (\frac{2R^2 x}{x^2 + (y-R)^2} - d\sin\theta)^2 + (\frac{2R^2(y-R)}{x^2 + (y-R)^2} + R - d\cos\theta)^2 = r^2,$$

$$\frac{4R^4}{x^2 + (y-R)^2} + \frac{2R^2(y-R)(R - d\cos\theta) - 2R^2 x d\sin\theta}{x^2 + (y-R)^2} + R^2 - 2Rd\cos\theta + d^2 - r^2 = 0,$$

I didn't implement this method since Möbius transformation formulas turned out to be too complicated without using complex analysis.

Method 2: we only consider the case when $\theta = 0$.

$$\Gamma_2 : x^2 + (y-d)^2 = r^2, \quad \varphi(\Gamma_2) = \Gamma_4,$$

All the possible circle chain configurations can be regarded as the orbit of a particular configuration under the action of $SO(2)$. When $d = 0$, the action is exactly rotation around the common center of the two circles. When $d \neq 0$, assume that the center of the small circle is $(0, d)$, then there is a transformation that fixes Γ_0 and takes Γ_2 to a circle centered at the origin. More precisely, find the circle Γ_3 which is orthogonal to both Γ_1, Γ_2. Its center is on the radical axis l_{rad} of Γ_0, Γ_2, with radius equals to length of tangents, so its equation is

$$l_{rad} : 2dy = R^2 + d^2 - r^2, \quad \Gamma_3 : x^2 + (y - \frac{R^2 + d^2 - r^2}{2d})^2 = (\frac{R^2 + d^2 - r^2}{2d})^2 - R^2,$$

Using complex analysis, an orientation preserving auto morphism of the unit disk is

$$f : z \mapsto \frac{z - w}{1 - \overline{w}z}, \quad |w| < 1, \quad f'(z) = \frac{1 - w\overline{w}}{(1 - \overline{w}z)^2}, \quad f^{-1}(z) = \frac{z + w}{1 + \overline{w}z},$$

Let $w = x_0 + y_0 i, z = x + yi$, the formula of this transformation is

$$\begin{aligned} f(x + yi) &= \frac{(x - x_0) + (y - y_0)i}{1 - x_0 x - y_0 y + (y_0 x - x_0 y)i} = \frac{(z - w)(1 - w\overline{z})}{(1 - x_0 x - y_0 y)^2 + (y_0 x - x_0 y)^2} \\ &= \frac{x + yi + (x_0^2 - y_0^2 + 2x_0 y_0 i)(x - yi) - (x_0 + y_0 i)(1 + x^2 + y^2)}{1 - 2(x_0 x + y_0 y) + (x_0^2 + y_0^2)(x^2 + y^2)} \end{aligned}$$

It becomes too long, so I used its complex form in my implementation. In this scenario, we change the disk radius to R and let $\Re(w) = 0$. We have

$$f(z) = \frac{z - w}{1 - \frac{\overline{w}z}{R^2}}, \quad w = (\frac{R^2 + d^2 - r^2}{2d} - \sqrt{(\frac{R^2 + d^2 - r^2}{2d})^2 - R^2})i$$

Since $f(w) = 0, f'(w) \in \mathbb{R}$ we know that f maps Γ_3 to $\mathbb{R}$. We may regard it as a translation in the group of hyperbolic automorphisms of Γ_0. Denote the first and the last circles by Ω_A, Ω_B. Observe that the required minimum distance between Ω_A, Ω_B is obtained when they are symmetric along the y-axis. Γ_4 is the image of Γ_2 by translation, assume that it has radius r_4. The image of the first circle by translation is

$$|z - z_A| = \frac{R - r_4}{2}, \quad z_A = r_4 \sin \frac{\delta\theta}{2} + i \cos \frac{\delta\theta}{2},$$

where $\delta\theta$ is the angle between centers of $f(\Omega_A), f(\Omega_B)$. It is an invariant since Γ_0, Γ_4 are concentric. In my program, three points are selected on $f(\Omega_A)$, and Ω_A is obtained by finding the circumcircle of preimages of the three points.

Problem 17 (1661 Dodecahedron)**.** The symmetry group of dodecahedron in $SO(3)$ is A_5. Assume that its edges are $e_1, ..., e_{30}$. Given $c_1, ..., c_{30} \in [30] = \{1, 2, ..., 30\}$, find the number of different dodecahedra. A coloring is to give each edge e_i a color $c[s_i]$, where $s \in S_{30}$ is a permutation. Two coloring s^1, s^2 are identical means there exists $\sigma \in A_5 \subset S_{30}$, such that for any $1 \leq i \leq 30$, $c[s^2(\sigma(i))] = c[s^1(\sigma(i))]$. Ignoring c and A_5, all the possible color assignments can be regarded as the permutation group S_{30}. There is a subgroup G of S_{30} determined by c, such that

$$c \circ s^2 = c \circ s^1 : [30] \to [30] \iff \text{exists } g \in G, g \circ s^2 = s^1, \quad s^1, s^2 \in S_{30}, \quad G = \prod_{x \in [30]} S_{c^{-1}(x)},$$

G is the product of permutation groups on each fiber of c. So S_{30} is given a double coset structure $G \curvearrowright S_{30} \curvearrowleft A_5$, and we are asked to calculate its cardinality.

The orbits of $S_{30} \curvearrowleft A_5$ are S_{30}/A_5, the set of right cosets of A_5 in S_{30}. Similarly, the orbits of $G \curvearrowright S_{30}$ are $G \backslash S_{30}$, the set of left cosets of G in S_{30}. We may assume that the image of c is $1, 2, ..., k$, and $|c^{-1}(i)| = n_i, \sum_{1 \leq i \leq k} n_i = 30$. Our task is to calculate $|G \backslash S_{30}/A_5|$.

$X = G \backslash S_{30}$ can be described as all the sequences $d_1, ..., d_{30}$ in which i appear n_i times, $1 \leq i \leq k$. Burnside's lemma says that

$$|X/A_5| = \frac{1}{|A_5|} \sum_{\sigma \in A_5} |X^\sigma|, \quad X^\sigma = \{x \in X, \sigma(x) = x\},$$

Elements of A_5 can be divided into 4 classes: identity, 15 order 2 elements, 20 order 3 elements, 24 order 5 elements. Let $\sigma_l \in A_5$ be an element with order l. i) $l = 1, 3, 5$, then $|X^{\sigma_l}| = \frac{(\frac{30}{l})!}{\prod(\frac{n_i}{l})!}$ when l divides each n_i, otherwise $|X^{\sigma_l}| = 0$. ii) $l = 2$, if the number of odds in n_i is larger than 2, $|X^{\sigma_l}| = 0$; if there are odds in n_i then $|X^{\sigma_l}| = \frac{(\frac{30}{l})!}{\prod(\frac{n_i}{l})!}$; otherwise there are 2 odds in n_i, $|X^{\sigma_l}| = \frac{2*(14)!}{\prod(\lfloor\frac{n_i}{l}\rfloor)!}$. Used Qifeng Chen's BigInteger struct for large number calculation, and made some modifications. One precious experience is that, never ever inherit from any std:: type, except for the standard types you're supposed to inherit from. Accepted using both implementations of BigInteger of mine and Qifeng Chen's.

Proof. Assume $G \curvearrowright X$ is a left group action.

$$X^g = \{x \in X, g(x) = x\}, \quad G^x = \{g \in G, g(x) = x\}, \quad |\text{orbit}(x)| = \frac{|G|}{|G^x|},$$

$$\frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{1}{|G|} \sum_{x \in X} |G^x| = \sum_{x \in X} \frac{1}{|\text{orbit}(x)|} = |X/G|,$$

□

Problem 18 (1662 Goat in the Garden 6)**.** Note that the bed is a convex polygon $P_1P_2...P_n$. Let $D_r = \bigcup_{p\in polygon} \odot(p,r)$, we need to determine whether $\bigcap_{p\in polygon} \odot(p,R) \cap \overline{D_r}$ is nonempty. $\bigcap_{p\in polygon} \odot(p,R) = \bigcap_{1\le i\le n} \odot(P_i,R)$ is an convex set. For any $p \notin D_r$, let q be its projection onto ∂D_r. If $p \in \bigcap_{1\le i\le n} \odot(P_i,R)$, then $q \in \bigcap_{1\le i\le n} \odot(P_i,R)$, so it sufficies to check if $\partial D_r \cap \bigcap_{1\le i\le n} \odot(P_i,R)$ is nonempty. ∂D_r can be divided into n arcs and n segments. 1) Segment circles intersection.

2) For an arc $\overline{AB} \in \odot(O,r)$ and $\odot(X,R)$, i) Both $AX > R, BX > R$ hold, then $\overline{AB} \cap \odot(X,R)$ is empty since O, X lie on the same side of AB. ii) $OX \le R-r$, check the next circle. iii) $OX > R-r$, get intersection C, D and judge that if $\overline{AB}$ split into two parts. Assume μ is the signed length of OH the direction of XH, $\nu = HC$, then

$$R^2 - r^2 = XH^2 - OH^2 = 2\mu XO + XO^2, \quad \nu = \sqrt{r^2-\mu^2},$$

Problem 19 (1668 Death Star 2)**.** $A_{N\times M}, b_N$, find x_M such that $\|Ax-b\|_2^2$ reaches minimum. If the solution is ambiguous, output the one that $\|x\|_2^2$ is the minimum. Let $A = UsV^t$ be the singular value decomposition of A,

$$s^{inv}[i] = \begin{cases} \frac{1}{s[i]}, s[i] > 0, \\ 0, s[i] = 0. \end{cases}, \quad A^{inv} = Vs^{inv}U^t$$

Then we claim that $x = A^{inv}b$. While calculating SVD, we use Golub-Kahan bidiagonalization in phase 1. Householder reflection is given by

$$x = A_{k:m,k}, \quad v_k = \mathrm{sign}(x_1)\|x\|_2 e_1 + x, \quad v_k = \frac{v_k}{\|v_k\|_2}, \quad A_{k:m,k:n} -= 2v_k(v_k^* A_{k:m,k:n})$$

Householder reflector: $x \mapsto Fx = \pm\|x\|e_1$. Givens rotation acting on the i,j-th rows or columns: $G(i,j,\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. How to use Givens rotations to eliminate the off-diagonal elements? Actually the remaining steps after bidiagonalization only use Givens rotations.

Problem 20 (1691 Algorithm Complexity)**.** Given a graph G with n vertices and m edges, which doesn't contain multiple arcs but may contain loops. Let $F(N)$ be the number of walks of length N from vertex n_s to vertex n_t, find the growth order of $F(N)$. Assume that the adjacency matrix of G is A, then A is unsymmetric with 0-1 entries, $F(N) = A^N[n_s,n_t]$. Using the method of generating function, we get

$$\sum_{N\ge 0} A^N t^N = (I-At)^{-1}, \quad Answer = coeff < t^N, (I-At)^{-1}[n_s,n_t] >,$$

Consider the special case when G is a directed acyclic graph, and assume that vertices $v_1, v_2, ..., v_n$ are sorted topologically. Then A is a upper triangular 0-1 matrix. We may assume that $n_s = 1$ and $n_t = n$ by only considering the subgraph generated by vertices numbered from n_s to n_t. Now $F(N)$ reaches its maximum when $A[i,j] = 1$ for all $i \le j$. In this scenario, we may write

$$A = I + B + B^2 + ... + B^{n-1}, \quad B[i,i+1] = 1, \quad 1 \le i \le n-1, \quad B^n = 0, \quad A = (I-B)^{-1},$$

Goal is to calculate the coefficient of B^{n-1} in the expansion of $A^N = (I-B)^{-N}$.

$$(I-B)^{-N} = \sum_{0\le i\le n-1} \binom{i+N-1}{i} B^i, \quad \binom{n+N-2}{n-1} = O(N^{n-1}),$$

Assume that the order of $F(N)$ is f, we know that f is at most $n-1$. How to determine its exact value? There are two cases depending on whether v_n can be reached from v_{n-1}. Assume $F_0(N)$ is the number of length N paths that don't pass by v_{n-1}, $F_1(N)$ is the number of paths that pass by v_{n-1}.

1) $A[n-1,n]=0$, then no path from v_1 to v_n passes through v_{n-1}, we can remove this vertex and the number of vertices is reduced to $n-1$, $f=f_0$.

2) $A[n-1,n]=1$, then any path from v_1 to v_n can be classified by whether it passes by v_{n-1} or not. $F_0(N)$ can be calculated from the subgraph excluding v_{n-1}. $f=\max(f_0,f_1)$.

i) First we consider the case when $A[n,n]=1$. Assume that i is the last time that the path is at v_{n-1}, we have

$$F_1(N)=\sum_{i=1}^{N-1}F_1^0(i),\quad F_1^0(i)=\text{number of length } i \text{ paths from } v_1 \text{ to } v_{n-1},\quad f_1=f_1^0+1,$$

ii) $A[n,n]=0$. Now we have $F_1(N)=F_1^0(N-1), f_1=f_1^0$.

We can process the graph forwardly, and denote $f[i]$ the order of length i paths from v_1 to v_i. Then if $A[i,i]=1$, we let $f[i]=f[i]+1$; if $A[i,j]=1, i<j$, we let $f[j]=\max(f[j],f[i])$.

Next let us consider the general case when G is an arbitrary directed graph. I used Tarjan's algorithm to calculate its strongly connected components and they form a new graph which is acyclic. How to add edges on the new graph? There may be more than one paths between two components, and a component may not have self-loop. Depth first search is used to sort the components topologically. For each component, if it has more edges than vertices, then we think it has more than one self-loops; if its edge number equals to its vertex number, we think it has exactly one self-loop; otherwise we think it has no self-loop. The order of paths from v_s to v_t is the same as the order of paths between their components. Moreover, if $A[i,i]\geq 2$ in the components graph, we let $f[i]=+\infty$.

In a directed graph, accessibility from vertex A to vertex B is a partial order: 1) if $A\to B, B\to C$, then $A\to C$; 2) $A\to A$; 3) if $A\to B, B\to A$, then $A\simeq B$. The equality holds in the sense of mutually accessible, that is, A,B are in the same strongly connected component. Mutually accessible is an equivalence relation:

$$A\sim A,\quad A\sim B\iff B\sim A,\quad A\sim B, B\sim C, \Rightarrow A\sim C,$$

Problem 21 (1697 Sniper Shot)**.** Method 1: projection onto plane $z=0$.

Method 2: projection onto the plane spanned by AB and e_z.

Problem 22 (1810 Antiequations)**.**

$$A:\mathbb{F}_3^n\to\mathbb{F}_3^k,\quad P_i=\{y_i=b_i\}\subset\mathbb{F}_3^k,\quad (y_1,...,y_k)\in\mathbb{F}_3^k,$$

Assume that $l=im(A), \dim(l)=p$. We consider the case when all P_i cross intersects l first, let $l\cap P_i=Q_i$, then $\dim(Q_i)=p-1$. It suffices to calculate the size of $l\backslash\bigcup_{1\leq i\leq k}Q_i$. For each pair of $i,j\in\{1,...,k\}$, there are three possible relations between Q_i and Q_j: 1) $Q_i=Q_j$ identical, 2) $Q_i\cap Q_j=\emptyset$ parallel, 3) $\dim(Q_i\cap Q_j)=p-2$ cross intersect.

Method 2: counting points on the affine variety $X=Q_1\cup Q_2...\cup Q_k$. Generally speaking, let

$$N_m=|X(\mathbb{F}_{q^m})|,\quad Z(X,t)=\exp(\sum_{m\geq 1}\frac{N_m}{m}t^m),$$

For example, when $p=k$, $Q_i : y_i = b_i$, we have

$$N_m = q^{mk} - (q^m-1)^k = \sum_{i=0}^{k-1} q^{mi}\binom{k}{i}(-1)^{k+i+1},$$

$$\sum_{m\geq 1}\frac{N_m}{m}t^m = \sum_{i=0}^{k-1}\binom{k}{i}(-1)^{k+i+1}\sum_{m\geq 1}\frac{q^{mi}}{m}t^m = \sum_{i=0}^{k-1}\binom{k}{i}(-1)^{k+i}\log(1-q^i t),$$

$$Z(X,t) = \exp(\sum_{i=0}^{k-1}\binom{k}{i}(-1)^{k+i}\log(1-q^i t)) = \frac{(1-q^{k-2}t)^{\binom{k}{2}}...}{(1-q^{k-1}t)^{\binom{k}{1}}(1-q^{k-3}t)^{\binom{k}{3}}...},$$

$$N_1 = \frac{dZ}{dt}\Big|_{t=0} = \binom{k}{1}q^{k-1} - \binom{k}{2}q^{k-2} + \binom{k}{3}q^{k-3}... = q^k - (q-1)^k,$$

Another special case is when $b_i = 0$, each P_i is a codimension 1 subspace of $\mathbb{F}_3^k$. Claim: we may select a basis $s_1, ..., s_p$ of l such that their supports are pairwise disjoint.

Problem 23 (1814 Continued Fraction)**.** We need to implement quadratic extension of rational number field as a struct QuadraticRT. An element's inverse is given by

$$(\frac{x+y\sqrt{N}}{z})^{-1} = \frac{z(y\sqrt{N}-x)}{Ny^2-x^2},$$

$nums[i]$ stores quadrtic rational $a_i + r_i$, where $a_i \in \mathbb{Z}_+, 0 < r_i < 1$, and $\sqrt{N} = nums[0]$. Formula of continued fraction is given by $nums[i+1] = \frac{1}{r_i}$. By the following theorem, we may check that if the block length is m, then

$$nums[m+1] = nums[1], \quad nums[m] = a_0 + nums[0] = a_0 + \sqrt{N},$$

Method 1: assume that $R_n = \frac{P_n}{Q_n} = [a_0; a_1, a_2, ..., a_n]$ and $1 \leq k_1 \leq m, k_1 \equiv k \mod m$, then

$$P_n = P_{n-1}a_n + P_{n-2}, \quad Q_n = Q_{n-1}a_n + Q_{n-2},$$

We start by calculating $[a_{k_1+1}, ...a_k]$. Let $P^l = P_{lm}, Q^l = Q_{lm}, l = (k-k_1)/m$ in this scenario, then

$$\begin{pmatrix} P^l \\ Q^l \end{pmatrix} = \begin{pmatrix} P_m & P_{m-1} \\ Q_m & Q_{m-1} \end{pmatrix}\begin{pmatrix} P^{l-1} \\ Q^{l-1} \end{pmatrix} = M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

Turning back to the original problem, we have

$$P_k = P^l P_{k_1} + Q^l P_{k_1-1}, \quad Q_k = P^l Q_{k_1} + Q^l Q_{k_1-1},$$

Method 2: define $P_{-1} = 1, Q_{-1} = 0, P_{-2} = 0, Q_{-2} = 1$,

$$A_i = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}, \quad M = A_m A_{m-1} ... A_1, \quad prodm = A_{k_1} ... A_1 M^l A_0,$$

Theorem 2. If $r \in \mathbb{Q}, r > 1$ is not a perfect square, then

$$\sqrt{r} = [a_0; \overline{a_1, a_2, ..., a_2, a_1, 2a_0}],$$

It has a repeating block of length m, in which the first $m-1$ partiail denominators form a palindromic string. In the continued fraction expansion of $\frac{P+\sqrt{D}}{Q}$, the largest partial denominator a_i in the expansion of $\sqrt{D}$ is less than $2\sqrt{D}$, and the block length $m = L(D)$ is less than $2D$. A sharper bound is

$$L(D) = O(\sqrt{D}\log D),$$

Problem 24 (1815 Farm in San Andreas)**.** Given coordinates of A, B, C, costs c_A, c_B, c_C, find the minimum of $c_A PA + c_B PB + c_C PC$. If P is different from A, B, C, assume that $\alpha = \angle BPC, \beta = \angle CPA, \gamma = \angle APB$, $\alpha_1 = \pi - \alpha, \beta_1 = \pi - \beta, \gamma_1 = \pi - \gamma$, then

$$c_B = c_A \cos\gamma_1 + c_C \cos\alpha_1, \quad c_A = c_C \cos\beta_1 + c_B \cos\gamma_1, \quad c_C = c_B \cos\alpha_1 + c_A \cos\beta_1,$$

$\alpha_1, \beta_1, \gamma_1$ are interior angles of the triangle *coststri* with edge lengths c_A, c_B, c_C. Geometrically we may construct point R (temppt in program) such that $BC : CR : BR = c_A : c_B : c_C$. Intersection of AR with the circumcircle of $\triangle CBR$ is the point P required.

Problem 25 (1840 Victim of Advertising)**.** The trajectory of the skater is uniquely determined since only one segment can be extended if there is no arc connecting two consecutive directed segments without breaks. There are three constraints:

$$v \leq 10\text{m/s}, \quad a_{tan} \leq 1\text{m/s}^2, \quad a_n \leq 1\text{m/s}^2,$$

My approach is to draw a $\frac{v^2}{2} - s$ diagram, $\frac{v^2}{2}$ is the kinetic energy.

$$a_{tan} = \frac{dv^2}{2ds} \leq 1, \quad a_n = \frac{v^2}{R} \leq 1, \quad \frac{v^2}{2} \leq \frac{R}{2},$$

In this motion planning problem, the trajectory consists of N segments and $N-1$ arcs, separated by $2N$ endpoints. Assume that the kinetic energy and traveled length at the i-th endpoint are $kinetic[i]$ and $s[i]$, then constraints are

$$|kinetic[i] - kinetic[i+1]| \leq s[i+1] - s[i], \quad kinetic[2i+1], kinetic[2i+2] \leq \min(\frac{arcs[i].radius}{2}, 50),$$

Notice that if solution 1 and 2 are legal, then their pointwise maximum is also legal. So we may regard it as a linear programming problem that maximizes $\sum_{1 \leq i \leq 2N-1} kinetic[i]$. We may solve this problem in a way similar to Ford-Fulkerson algorithm by finding possible increments of $kinetic[i]$ in every iteration. Condition that $kinetic[i]$ can't increase is that at least one of the three constraints at endpoint i holds.

Thus we get $kinetic[i]$ exactly from the greedy algorithm above. trapzoidtime() is used to calculate for costed time in trapzoidal regions of the $k - s$ diagram, and elapsedtime() calculates costed time between two consecutive endpoints. I got memory limit exceeded sevaral times, possibly because call elapsedtime() recursively. Changing parameter types from T to constant reference doesn't cost much memory when T is small.

Problem 26 (1845 Integer-valued Complex Determinant)**.** Calculate the determinant of a Gaussian integer valued matrix. Attention: my implementation of struct BigInteger is written in this program. It is modified from Qifeng Chen's implementation. Const qualifiers are used as improvements, and BigNumber uses std::vector instead of array to store x.

Method 1: Integer coefficient Gaussian elimination. GaussianZT is implemented as Gaussian integer struct. Use extended Euclidean algorithm to calculate GCD of Gaussian integers after pivoting, so that all the elementary row transformations have Gaussian integer coefficients. A prototype of it was implemented in IntegerElimination.cpp, in which I only tested on integer coefficient matrices but not Gaussian integer. There are three kinds of elementary row transformations:

$$a_j \to a_j - c a_i, \quad a_j \to -a_j, \quad a_i \leftrightarrow a_j,$$

and I try to turn a_{kk} into $\gcd(a_{kk}, a_{ik})$ for $i > k, a_{ik} \neq 0$ after pivoting. Let $u = a_{kk}, v = a_{ik}$, the output of extendedGCD function satisfies

$$xu + yv = d, \quad u = du_q, v = d_v1, \quad xu_1 + yv_1 = 1, \quad u_1v - v_1u = 0, \quad |x| < |v_1|, |y| < |u_1|,$$

So we can take elementary row transform on the k, i-th rows resulting in

$$\det \begin{pmatrix} x & y \\ -v_1 & u_1 \end{pmatrix} = 1, \quad \begin{pmatrix} x & y \\ -v_1 & u_1 \end{pmatrix} \begin{pmatrix} a_{kk} \\ a_{ik} \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix},$$

and it doesn't change the determinant of A. Recursion of extendedeGCD satisfies

$$u\%v = u - \lambda v, \quad d = yv + x(u\%v) = xu + (y - \lambda x)v,$$

Note that generally speaking, we don't need auxilliary matrices P, L that appears in $PA = LU$ while calculating determinant.

Method 2: GaussianQT is implemented as Gaussian rational struct. Use raw Gaussian elimination after pivoting, but the current result is WA12 while using BigInteger struct, WA15 while using long long.

Problem 27 (2076 Vasiana)**.**

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}, \quad \begin{pmatrix} t^2 \\ t \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

1) $x, y, 1$ are linearly independent. The trajectory of (x, y) must be a parabola.

$$ax + by + c = (dx + ey + f)^2, \quad x' = \frac{dx + ey}{\sqrt{d^2 + e^2}}, \quad y' = \frac{-ex + dy}{\sqrt{d^2 + e^2}},$$

$$T = \frac{1}{\sqrt{d^2 + e^2}} \begin{pmatrix} d & e \\ -e & d \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = T^t \begin{pmatrix} x' \\ y' \end{pmatrix},$$

$$\begin{pmatrix} t^2 \\ t \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} T^t & \\ & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a' & b' & c \\ d' & 0 & f \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}, \quad d' = \sqrt{d^2 + e^2},$$

$$(d'x' + f)^2 = a'x' + b'y' + c, \quad y' = a''(x' - x'_{mid})^2 + c'',$$

$$x'_{mid} = \frac{-f + \frac{a'}{2d'}}{d'}, \quad a'' = \frac{d'^2}{b'}, \quad c'' = \frac{f^2 - c}{b'},$$

The equation for circle parabola intersection is

$$x^2 + (y - y_0)^2 = R^2, \quad y = a''x^2 + c'',$$

$$f(x^2) = a''^2x^4 + (2a''(c'' - y_0) + 1)x^2 + (c'' - y_0)^2 - R^2 = 0,$$

Criterion for existence of intersection is, f has non-negative real root.

2) $x, y, 1$ are linearly dependent. i) $a_0 = d_0 = 0$, a) $b_0 = e_0 = 0$, it means that the trajectory of (x, y) is a single point. b) $b_0 \neq 0$ or $e_0 \neq 0$, it means that the trajectory of (x, y) is a full straight line. ii)$a_0 \neq 0$ or $d_0 \neq 0$, now the trajectory of (x, y) is a ray.

$$P_{critical}(x_{critical}, y_{critical}), \quad x' = \frac{a_0x + d_0y}{\sqrt{a_0^2 + d_0^2}}, \quad y' = \frac{-d_0x + a_0y}{\sqrt{a_0^2 + d_0^2}},$$

1 Unsymmetric eigenvaluve problems

Theorem 3 (Gershgorin Circle Theorem)**.** 1) If $X^{-1}AX = D + F$, where $D = \mathrm{diag}(d_1, ...d_n)$ and F has zero diagonal entries, then

$$\sigma(A) \subset \bigcup_{i=1}^{n} D_i, \quad D_i = \{z \in \mathbb{C}, |z - d_i| \le \sum_j = 1^n |f_{ij}|\},$$

2) If the Gershgorin disk D_i is isolated from other disks, then it contains precisely one eigenvalue of A.

Proof. 1) Suppose that $\lambda \in \sigma(A), \lambda \ne d_i, 1 \le i \le n$, $D - \lambda I + F = X^{-1}AX - \lambda I$ is singular.

$$(D - \lambda I)^{-1} = \mathrm{diag}(\frac{1}{d_i - \lambda}), \quad G = (D - \lambda I)^{-1}F = (\frac{f_{ij}}{d_i - \lambda})_{1 \le i,j \le n},$$

We are interested in the l_∞ norm of operator G, which is defined as $\|G\|_\infty = \max_{x \ne 0} \frac{\|Gx\|_\infty}{\|x\|_\infty}$. Suppose $\|x\|_\infty = 1$,

$$Gx = (\sum_j g_{ij}x_j)_{1 \le i \le n}, \quad \|Gx\|_\infty \le \max_i \sum_j |g_{ij}| = \max_i \sum_j \frac{|f_{ij}|}{|d_i - \lambda|},$$

and the equality holds. On the other hand, if $A+B$ is singular, A is non-singular, suppose $(A+B)u = 0, u \ne 0$, then

$$Au = -Bu, \quad u = -A^{-1}Bu, \quad \|A^{-1}Bu\|_p = \|u\|_p, \quad \|A^{-1}B\|_p \ge 1,$$

Let $A = D - \lambda I, B = F, p = \infty$, we have $1 \le \sum_j \frac{|f_{kj}|}{|d_k - \lambda|}$ for some $1 \le k \le n$,

$$|d_k - \lambda| \le \sum_j |f_{kj}|, \quad \lambda \in D_k,$$

Another proof: assume $\lambda \in \sigma(A)$ with eigenvector $u \in \mathbb{C}^n$, $k = \arg\max_j |u_j|$, then

$$|(d_k - \lambda)u_k| = |-\sum_j f_{kj}u_j| \le |u_k| \sum_j |f_{kj}|, \quad |d_k - \lambda| \le \sum_j |f_{kj}| = r_k,$$

2) If $D_k \cap D_i = \emptyset, i \ne k$, we show that there exist precisely one $\lambda \in \sigma(A), \lambda \in D_k$ and λ has multiplicity 1. i) Uniqueness: if $\lambda \in D_k$ is an eigenvalue of $D + F$, $u \in \mathbb{C}^n$ is an eigenvector of $D - \lambda I + F$. If $i = \arg\max_j |u_j|, i \ne k$ then

$$(d_i - \lambda)u_i = -\sum_j f_{ij}u_j, \quad |(d_i - \lambda)u_i| > r_i|u_i| \ge \sum_j |f_{ij}||u_j|,$$

contradiction! So we have $k = \arg\max_j |u_j|, \|u\| = |u_k|$. Without loss of generality, assume that $k = n$, $D'_{n-1}, F'_{n-1} \in \mathrm{Aut}(\mathbb{C}^{n-1})$ are the first $n-1$ rows and columns of D, F. Then for $x \in \mathbb{C}^{n-1}$,

$$((D' - \lambda I')^{-1}F'x)_i = \sum_{j=1}^{n-1} \frac{f_{ij}x_j}{d_i - \lambda}, \quad |((D' - \lambda I')^{-1}F'x)_i| \le \sum_{j=1}^{n-1} \frac{|f_{ij}||x_j|}{|d_i - \lambda|} < \|x\|_\infty,$$

So $\|(D'-\lambda I')^{-1}F'\|_\infty < 1$, and we have the following expansion

$$(D'-\lambda I'+F')^{-1} = (D'-\lambda I')^{-1}(I'+(D'-\lambda I')^{-1}F')^{-1}$$

$$(I'+(D'-\lambda I')^{-1}F')^{-1} = \sum_{m\geq 0}((\lambda I'-D')^{-1}F')^m,$$

The right hand side above is absolutely convergent, so $D'-\lambda I'+F'$ is invertible. It follows that if $\lambda \in D_k$ is an eigenvalue of $D+F$, then it has multiplicity 1.

ii) Resolvent method:

$$R(\lambda) = (D-\lambda I+F)^{-1} : \mathbb{C}^n \to \mathbb{C}^n,$$

As an operator valued function, $R : \mathbb{C} \to \mathrm{Aut}(\mathbb{C}^n)$ is meromorphic in the following sense: for any $\phi \in (\mathbb{C}^n)^*, v \in \mathbb{C}^n$, $f(\phi, v, \lambda) = \langle \phi, R(\lambda)v\rangle$ is a meromorphic function of λ.

$$\langle \phi, R(\lambda_1)v\rangle - \langle \phi, R(\lambda_0)v\rangle = \langle \phi, (R(\lambda_1)-R(\lambda_0))v\rangle = \langle \phi, (\lambda_1-\lambda_0)R(\lambda_1)R(\lambda_0)v\rangle,$$

$R(\lambda_1), R(\lambda_0)$ are commutative: $R(\lambda_1)R(\lambda_0) = R(\lambda_0)R(\lambda_1)$.

$$\frac{\langle \phi, R(\lambda_1)v\rangle - \langle \phi, R(\lambda_0)v\rangle}{\lambda_1-\lambda_0} = \langle \phi, R(\lambda_1)R(\lambda_0)v\rangle$$

hence $f(\phi, v, \lambda)$ is meromorphic with $\frac{df(\phi,v,\lambda)}{d\lambda} = \langle \phi, R(\lambda)^2 v\rangle$.

$$\lambda \text{ singular} \iff \|R(\lambda)\| = \infty, \quad \|R(\lambda)\| = \max_{\|\phi\|=\|v\|=1} |\langle \phi, R(\lambda)v\rangle|,$$

In finite dimensional case, $D-\lambda I+F : \mathbb{C}^n \to \mathbb{C}^n$ induces an automorphism on $\bigwedge^n(\mathbb{C}^n)$:

$$e_1 \wedge e_2 ... \wedge e_n \mapsto \bigwedge_{i=1}^{n}(D-\lambda I+F)e_i = \det(D-\lambda I+F)e_1 \wedge e_2 ... \wedge e_n,$$

iii) Consider the eigenvalues of $D+\epsilon F, 0 \leq \epsilon \leq 1$, by 1), all of its eigenvalues lie in $\bigcup_{i=1}^n \epsilon D_i$. These eigenvalues vary continuously with respect to ϵ, and we may denote them as $\lambda_i(\epsilon), 1 \leq i \leq n$. When $\epsilon = 0$, we have $\lambda_i(0) = d_i$, so by continuity argument and the fact that $D_k \cap D_i = \emptyset, i \neq k$, we know that there is precisely one eigenvalue $\lambda_k(\epsilon) \in \epsilon D_k$. Take $\epsilon = 1$ finishes the proof. □

Theorem 4 (Bauer-Fike)**.** If μ is an eigenvalue of $A+E \in \mathbb{C}^{n*n}$ and $X^{-1}AX = D = \mathrm{diag}(\lambda_q, ..., \lambda_n)$, then

$$\min_{\lambda\in\sigma(A)} |\lambda-\mu| \leq \kappa_p(X)\|E\|_p,$$

Proof. It suffices to consider the case $\mu \notin \sigma(A)$. If the matrix $X^{-1}(A+E-\mu I)X$ is singular, then so is $I+(D-\mu I)^{-1}X^{-1}EX$. Then we have

$$1 \leq \|(D-\mu I)^{-1}X^{-1}EX\|_p \leq \|(D-\mu I)^{-1}\|_p\|X\|_p\|E\|_p\|X^{-1}\|_p,$$

Since$\|(D-\mu I)^{-1}\|_p = \max_{\lambda\in\sigma(A)} \frac{1}{|\lambda-\mu|}$, we have finished our proof. □

Definition 1. For square matrix A define the condition number $\kappa(A) = \|A\|\|A^{-1}\|$, with the convention that $\kappa(A) = \infty$ for singular A. $\kappa(\cdot)$ depends on the underlying norm and subscripts are used accordingly.

$$\kappa_2(A) = \|A\|_2\|A^{-1}\|_2 = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}, \quad \frac{1}{\kappa_p(A)} = \min_{A+\Delta A \text{ singular}} \frac{\|\Delta A\|_p}{\|A\|_p},$$

$$\kappa(A) = \lim_{\epsilon\to 0} \sup_{\|\Delta A\|\leq\epsilon\|A\|} \frac{\|(A+\Delta A)^{-1}-A^{-1}\|}{\epsilon\|A^{-1}\|}$$

Theorem 5. Let $Q^H AQ = D + N$ be a Schur decomposition of $A \in \mathbb{C}^{n*n}$, i.e., $Q \in \mathbb{C}^{n*n}$ is unitary, $D = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and $N \in \mathbb{C}^{n*n}$ is strictly upper diagonal. Q can be chosen so that the eigenvalues λ_i appear in any order along the diagonal. If $\mu \in \sigma(A + E)$ and p is the smallest positive integer such that $N^p = 0$, then

$$\min_{\lambda \in \sigma(A)} |\lambda - \mu| \leq \max\{\theta, \theta^{\frac{1}{p}}\}, \quad \theta = \|E\|_2 \sum_{k=0}^{p-1} \|N\|_2^k,$$

Extreme eigenvalue sensitivity for a matrix A cannot occur if A is normal. But A nonnormal matrix can have a mixture of well-conditioned and ill-conditioned eigenvalues. Suppose that λ is a simple eigenvalue of $A \in \mathbb{C}^{n*n}$ and that x and y satisfy $Ax = \lambda x, y^H A = \lambda y^H, \|x\|_2 = \|y\|_2 = 1$. If $Y^H AX = J$ is the Jordan decomposition with $Y^H = X^{-1}$, then y and x are nonzero multiples of $X(:, i), Y(:, i)$ for some i, so $y^H x \neq 0$.

$$(A + \epsilon F)x(\epsilon) = \lambda(\epsilon)x(\epsilon), \quad \|F\|_2 = 1,$$

We refer to the reciprocal of $s(\lambda) = |y^H x|$ as the condition of the eigenvalue λ. A small $s(\lambda)$ implies that A is near a matrix having a multiple eigenvalue. In particular, if λ is distinct and $s(\lambda) < 1$, then there exists an E such that λ is a repeated eigenvalue of $A + E$ and

$$\frac{\|E\|_2}{\|A\|_2} \leq \frac{s(\lambda)}{\sqrt{1 - s(\lambda)^2}},$$

In general, if λ is a defective eigenvalue of A, then $O(\epsilon)$ perturbations in A can result in $O(\epsilon^{\frac{1}{p}})$ perturbations in λ if λ is associated with a p-dimensional Jordan block.

2 Symmetric eigenvalue problems

Theorem 6 (Gershgorin)**.** A is real symmetric, Q is orthogonal, if $Q^t AQ = D+F, D = \mathrm{diag}(d_1, d_2, ..., d_n)$ and F has zero diagonal entries, then

$$\sigma(A) \subset \bigcup_{i=1}^{n} [d_i - r_i, d_i + r_i], \quad r_i = \sum_j |f_{ij}|,$$

Proof. Exactly the same as the unsymmetric case, with an additional property that $\sigma(A) \subset \mathbb{R}$. □

Theorem 7 (Wielandt-Hoffman)**.** If A and $A + E$ are $n * n$ symmetric matrices, then

$$\sum_{i=1}^{n} (\lambda_i(A + E) - \lambda_i(A))^2 \leq \|E\|_F^2 = \sum_{i,j=1}^{n} |e_{ij}|^2$$

Theorem 8. If A and $A + E$ are $n * n$ symmetric matrices, then

$$\lambda_k(A) + \lambda_n(E) \leq \lambda_k(A + E) \leq \lambda_k(A) + \lambda_1(E), \quad 1 \leq k \leq n,$$

$$|\lambda_k(A + E) - \lambda_k(A)| \leq \|E\|_2 = \max\{|\lambda_n(E)|, |\lambda_1(E)|\}, \quad 1 \leq k \leq n,$$

Theorem 9 (Interlacing property)**.** If $A \in \mathbb{R}^{n*n}$ is symmetric and $A_r = A(1 : r, 1 : r)$, then

$$\lambda_{r+1}(A_{r+1}) \leq \lambda_r(A_r) \leq \lambda_r(A_{r+1}) \leq ... \leq \lambda_2(A_{r+1}) \leq \lambda_1(A_r) \leq \lambda_1(A_{r+1}), \quad 1 \leq r \leq n - 1,$$

Theorem 10. Suppose $B = A + \tau cc^t, A \in \mathbb{R}^{n*n}, A = A^t, c \in \mathbb{R}^n, \|c\|_2 = 1, \tau \in \mathbb{R}$. we have

$$\lambda_i(B) \in [\lambda_i(A), \lambda_{i-1}(A)], 2 \leq i \leq n, \text{ when } \tau \geq 0,$$

$$\lambda_i(B) \in [\lambda_{i+1}(A), \lambda_i(A)], 1 \leq i \leq n-1, \text{ when } \tau < 0,$$

In either case, there exist $m_1, m_2, ..., m_n \geq 0, m_1 + m_2 + ... + m_n = 1$ such that

$$\lambda_i(B) = \lambda_i(A) + m_i\tau, \quad 1 \leq i \leq n,$$

Proposition 1. 1) If $T = QR$ is the QR factorization of a symmetric tridiagonal matrix $T \in \mathbb{R}^{n*n}$, then Q has lower bandwidth 1 and R has upper bandwidth 2 and it follows that $T_+ = RQ = Q^tTQ$ is also symmetric and tridiagonal.

2) If $s \in \mathbb{R}$ and $T - sI = QR$ is the QR factorization, then $T_+ = RQ + sI = Q^tTQ$ is also tridiagonal. This is called a shifted QR step.

3) If T is unreduced, then the first $n-1$ columns of $T - sI$ are independent regardless of s.

4) If $T \in \mathbb{R}^{n*n}$ is tridiagonal, then its QR factorization can be computed by applying a sequence of $n-1$ Givens rotations.

3 Solve univariate polynomial equations using $SL_2(\mathbb{R})$

Assume that a degree 4 real coefficient polynomial $P(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ has no real roots. Assume that its four roots are

$$x_1, \overline{x_1}, x_2, \overline{x_2}, \quad x_1, x_2 \in \mathbb{H}, \quad \Im(x_1) > 0, \Im(x_2) > 0,$$

Assume that $x_1 = u_1 + v_1i, x_2 = u_2 + v_2i$. Half circle arc on the upper half plane which passes x_1, x_2 and meets real axis orthogonally is uniquely determined. Assume its center is $(u, 0)$, then

$$(u - u_1)^2 + v_1^2 = (u - u_2)^2 + v_2^2, \quad u = \frac{u_2^2 + v_2^2 - u_1^2 - v_1^2}{2(u_2 - u_1)},$$

Radius of the half circle arc is

$$\begin{aligned} r_0^2 &= (u - u_1) + v_1^2 = \frac{(u_2^2 - 2u_1u_2 + u_1^2 + v_2^2 - v_1^2)^2}{4(u_2 - u_1)^2} + v_1^2 \\ &= \frac{((u_2 - u_1)^2 + v_2^2 - v_1^2)^2 + 4(u_2 - u_1)^2v_1^2}{4(u_2 - u_1)^2} = \frac{(u_2 - u_1)^4 + 2(u_2 - u_1)^2(v_2^2 + v_1^2) + (v_2^2 - v_1^2)^2}{4(u_2 - u_1)^2}, \end{aligned}$$

The action $SL_2(\mathbb{R}) \curvearrowright \overline{\mathbb{C}}$ is given by

$$x \mapsto g(x) = \frac{px + q}{rx + s}, \quad g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{R}),$$

Its kernel is $\pm id$. We want to determine g such that

$$\frac{px_1 + q}{rx_1 + s} = i, \quad \frac{px_2 + q}{rx_2 + s} \in \mathbb{R}_+i,$$

4 Sendov's Conjecture

Conjecture 1 (Sendov's Conjecture)**.** For a polynomial $f(z) = (z - r_1)(z - r_2)...(z - r_n), n \geq 2$ with all roots $r_1, r_2, ..., r_n$ inside the closed unit disk $\{|z| \leq 1\}$, each of the n roots is at a distance no more than 1 from at least one root of $f'(z)$.

It suffices to show that for a fixed r_1, the following distance function has maximum no more than 1.

$$d(r_2, ..., r_n) = \min |r_1 - \xi_i|, \quad f'(z) = (z - \xi_1)(z - \xi_2)...(z - \xi_{n-1}),$$

Two near counter-examples are

$$f_1(z) = z^n - 1, \quad r_1 = e^{\frac{2\pi i}{n}}, \quad f_2(z) = z^n - z, \quad r_1 = 0,$$

In the latter case the distance from 0 to any root of $f_2'(z)$ is $n^{-\frac{1}{n-1}} = 1 - O(\frac{\log n}{n})$,