Solution to the problems on Ural Online Judge

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Problem 1 (1058 Chocolate)**.** Key point is to calculate the minimal length of pq for four points A, B, C, D satisfying that $S_{\triangle AXB} = S_{\triangle CXD}$ where $X = AD \cap BC$, $p \in AB, q \in CD$ and $S_{\triangle AYp} = S_{\triangle DYq}$. $AC//BD$ since $S_{\triangle AXB} = S_{\triangle CXD}$. Let $Z = AB \cap CD$, if $(\cos ZAD - \cos ZDA) * (\cos ZBC - \cos ZCB) < 0$, then the length of pq is given by

$$(\frac{pq}{2})^2 = S_{\triangle ZAD} \tan\frac{Z}{2} = \frac{\lambda}{1-\lambda} S_{\triangle BAD} \frac{\sin Z}{1+\cos Z},$$

$$\lambda = \frac{AC}{BD}, \quad \sin Z = \frac{AB \times CD}{|AB||CD|}, \quad \cos Z = \frac{AB \cdot CD}{|AB||CD|},$$

$$AB \times CD = AB \times D'A = BD' \times BA = (1-\lambda) BD \times BA, \quad S_{\triangle BAD} = \frac{AB \times AD}{2},$$

$$pq = \sqrt{\frac{2\lambda (AB \times AD)^2}{|AB||CD|(1+\cos Z)}}$$

Problem 2 (1199 Mouse)**.** Single source shortest path using Dijkstra. Key point is to generate the path of the mouse. Use $i = (i + N - 1)\%N$ instead of $i = (i - 1)\%N$ since taking modulo on negative integers will produce negative answers.

Problem 3 (1239 Ghost Busters)**.** Project the ghosts onto the unit sphere, they become spherical circles. Preprocess the ghosts so that their center lie on the unit sphere, we may assume ghost i has center $po_i = (x_i, y_i, z_i)$ and radius r_i. An spherical circle has a plane it lies on and radius in radian:

$$xx_i + yy_i + zz_i = c_i = \sqrt{1 - r_i^2}, \quad rad = \arcsin r_i,$$

For circle i and j, their intersections are determined as follows:

$$A = \begin{pmatrix} 1 & po_i \cdot po_j \\ po_j \cdot po_i & 1 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} c_i \\ c_j \end{pmatrix}, \quad \begin{pmatrix} x_{mid} \\ y_{mid} \\ z_{mid} \end{pmatrix} = \begin{pmatrix} x_i & x_j \\ y_i & y_j \\ z_i & z_j \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Here p_{mid} is the projection of the origin onto the line intersection of planes i and j. The above calculation find the least square solution to the equation below

$$xx_i + yy_i + zz_i = c_i, \quad xx_j + yy_j + zz_j = c_j,$$

The direction of the line is $dir = \frac{po_i \times po_j}{|po_i \times po_j|}$, with half the segment length $halfseg = \sqrt{1 - |p_{mid}|^2}$. So the intersections are

$$I_1 = p_{mid} + dir * halfseg, \quad I_2 = p_{mid} - dir * halfseg,$$

Scan the $\frac{1}{8}$ unit sphere from the north pole to equator. There are only three cases that would change the order of intersection segments of circles: inserting north endpoint, deleting south endpoint, and intersection between circles. So my strategy is to enumerate all such critical latitudes, for each latitude, scan from longitude 0 to $\frac{\pi}{2}$ and record the location that meets the most segments.

Problem 4 (1384 Goat in the Garden 4)**.** Non-convex optimization. $dirnum = 40, stepsizenum = 18, stepsize = 16./(1 << i), 0 \leq i \leq 17$. Initial seeds are mid points of edges and polygon vertices. Actually I implemented a gradient descent algorithm adopted from Boyd's book "Convex Optimization". Initial directions are randomly selected before each step.

Problem 5 (1420 Integer-Valued Complex Division)**.** Implemented struct GaussianQT in this problem. Since the norms of numerator and denominator of $\frac{a}{b}$ exceed the range of long long, I wrote BigInteger struct and got accepted for the first time. Question: why do I get WA13 using my BigInteger in problem 1661? Update: bug fixed.

Problem 6 (1460 Wires)**.** Claim: 1) auxiliary points has degree 3, and their 3 adjacent edges has pairwise angle $\frac{2\pi}{3}$. This can be proved by calculus of variations.

2) It is impossible to have 3 auxiliary points. Otherwise the total degree of vertices is at least 13, contradiction.

3) The trilinear coordinate of the Fermat point (actually the first isogonic center) of a given triangle ABC is $\sec(A - \frac{\pi}{6}) : \sec(B - \frac{\pi}{6}) : \sec(C - \frac{\pi}{6})$, its barycentric coordinate is $\frac{a}{\cos(A - \frac{\pi}{6})} : \frac{b}{\cos(B - \frac{\pi}{6})} : \frac{c}{\cos(C - \frac{\pi}{6})}$, and I used this formula in computer program calculation.

3) We may enumerate every possible configurations: when there are no auxiliary points, we calculate its minimal spanning tree using Prim's algorithm. When there is only one auxiliary point, this point is uniquely determined by the three points it connects to. When there are two auxiliary points, the configuration is uniquely determined by the permutation of $ABCD$. $auxpt1$ is the Fermat point of $A, B, auxpt2$, $auxpt2$ is the Fermat point of $C, D, auxpt1$.

Problem 7 (1464 Light)**.** Sort vertices according to their polar angles.

Problem 8 (1566 Triangular Postcards)**.** If $\triangle PQR$ can be included inside $\triangle ABC$, then there exist a position such that two vertices of $\triangle PQR$ lie on the sides of $\triangle ABC$. Assume that they are P, Q.

1) P, Q lie on the same side of $\triangle ABC$.

2) P, Q lie on different sides. Assume that $P \in CB, Q \in CA, CP = \lambda, CQ = \mu$, then

$$\lambda^2 + \mu^2 - 2\lambda\mu\cos C = r^2 = \sin^2\frac{C}{2}(\lambda + \mu)^2 + \cos^2\frac{C}{2}(\lambda - \mu)^2,$$

$$R = P + \frac{q}{r}\begin{pmatrix}\cos P & -\sin P \\ \sin P & \cos P\end{pmatrix}(Q - P) = \begin{pmatrix}\lambda + \frac{q}{r}(\mu\cos C\cos P - \lambda\cos P - \mu\sin C\sin P) \\ \frac{q}{r}(\mu\cos C\sin P - \lambda\sin P + \mu\sin C\cos P)\end{pmatrix}$$

Constraints are $0 \le \lambda \le a, 0 \le \mu \le b$, and three constraints depicting $R \in \triangle ABC$:

$$y_R \ge 0, \quad x_R \sin C - y_R \cos C \ge 0, \quad BA \times BR = (x_A - x_B)y_R - y_A x_R + y_A x_B \ge 0,$$

This problem can be solved by checking the sign of $f(\lambda, \mu) = \lambda^2 + \mu^2 - 2\lambda\mu \cos C$ on the boundary of the polygon determined by the above 7 constraints.

Method 2: actually we only need to check case 1 in the discussion above. Avoid using trigonometric functions helps improve numerical accuracy.

Problem 9 (1594 Aztec Treasure)**.** Calculate the number of domino tilings on a $m \times n$ rectangle grid. Let $m_1 = \lceil \frac{m}{2} \rceil, n_1 = \lceil \frac{n}{2} \rceil$, the formula is given by

$$Z_{m,n}(1,1) = \prod_{j,k=1}^{m_1,n_1} (4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1}),$$

where we define $g(h, v)$ to be the number of tilings with h horizontal and v vertical dominoes.

$$Z_{m,n}(x, y) = \sum_{h,v} g(h, v)x^h y^v, \quad h, v \ge 0, \quad 2(h + v) = mn,$$

Swap m, n if necessary to make sure that m is even.

$$Z_{m,n}(1,1) = \prod_{j,k=1}^{m_1,n_1} (4 + 2\cos \frac{2\pi j}{m+1} + 2\cos \frac{2\pi k}{n+1}),$$

Denote $P_{n_1}(x) = \prod_{k=1}^{n_1}(x + 2\cos \frac{2\pi k}{n+1})$, let $x_j = 4 + 2\cos \frac{2\pi j}{m+1}, 1 \le j \le m_1$, then the result $Z_{m,n}(1,1) = \prod_{j=1}^{m_1} P_{n_1}(x_j)$. We may calculate P_{n_1} by induction. 1) n is even, now

$$P_{n_1}(y + \frac{1}{y}) = \prod_{k=1}^{n_1}(y + \frac{1}{y} + 2\cos \frac{2\pi k}{n+1}) = y^{n_1} - y^{n_1-1} ... - y^{1-n_1} + y^{-n_1},$$

$$P_1 = x - 1, \quad P_2 = x^2 - x - 1, \quad P_{n_1} = xP_{n_1-1} - P_{n_1-2},$$

and I let $P_0 = P_{-1} = 1$ in my implementation. 2) n is odd, now

$$P_{n_1}(y + \frac{1}{y}) = \prod_{k=1}^{n_1}(y + \frac{1}{y} + 2\cos \frac{2\pi k}{n+1}) = \frac{(y^{n+1} - 1)(y - 1)}{y + 1},$$

$$P_1 = x - 2, \quad P_2 = x^2 - 2x, \quad P_{n_1} = xP_{n_1-1} - P_{n_1-2},$$

and I let $P_0 = 0$ in my implementation.

Key point is to implement a struct representing algebraic integers of the form

$$x_0 + \sum_{1 \le j \le m1} x_j 2\cos \frac{2\pi j}{m+1}, \quad m = 2m_1,$$

Its nontrivial arithmetic is essentially inside two methods named reduce() and totalreduce(). Notice that taking modulo is admissible since all the terms above are integral.

Problem 10 (1599 Winding Number)**.** Method 1: calculate $\sum_{i=1}^{n} \angle P_i X P_{i+1}$. Resulted in TLE-12.

Method 2: calculate intersection number of the polygon with ray $y = y_X, x \ge x_X$.

Problem 11 (1621 Definite Integral)**.** Roots finding algorithms. Given an integer coefficient degree 4 polynomial with $|a_i| \le 10^6, a_4 \neq 0$. Notice that the precision requirement is high, relative error or absolute error is no more than 10^{-9}.

$$\int_{|x|=\epsilon} \frac{1}{x} dx = \int_0^{2\pi} \frac{e^{-i\theta}}{\epsilon} d\epsilon e^{i\theta} = \int_0^{2\pi} i d\theta = 2\pi i,$$

So residue theorem says that if a meromorphic function $f = \frac{g}{x-x_0}$ where g is holomorphic near x_0, then $\int_{|x-x_0|=\epsilon} f dx = 2\pi i g(x_0)$. Given $P(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, we want to eliminate a_3 by translation $x' = x + \frac{a_3}{4}$.

$$\begin{aligned}
P'(x') =& P(x) = (x' - \frac{a_3}{4})^4 + a_3(x' - \frac{a_3}{4})^3 + a_2(x' - \frac{a_3}{4})^2 + a_1(x' - \frac{a_3}{4}) + a_0 \\
=& x'^4 + 6x'^2(\frac{a_3}{4})^2 - 4x'(\frac{a_3}{4})^3 + (\frac{a_3}{4})^4 - 3x'^2 a_3 \frac{a_3}{4} + 3x' a_3(\frac{a_3}{4})^2 - a_3(\frac{a_3}{4})^3 \\
&+ a_2 x'^2 - 2x' a_2 \frac{a_3}{4} + a_2(\frac{a_3}{4})^2 + a_1 x' - a_1 \frac{a_3}{4} + a_0 \\
=& x'^4 + x'^2(6(\frac{a_3}{4})^2 - 3a_3\frac{a_3}{4} + a_2) + x'(-4(\frac{a_3}{4})^3 + 3a_3(\frac{a_3}{4})^2 - 2a_2\frac{a_3}{4} + a_1) \\
&+ (\frac{a_3}{4})^4 - a_3(\frac{a_3}{4})^3 + a_2(\frac{a_3}{4})^2 - a_1\frac{a_3}{4} + a_0 \\
=& x'^4 + x'^2(-\frac{3a_3^2}{8} + a_2) + x'(\frac{a_3^3}{8} - \frac{a_2a_3}{2} + a_1) - \frac{3a_3^4}{256} + \frac{a_2a_3^2}{16} - \frac{a_1a_3}{4} + a_0,
\end{aligned}$$

So we may define

$$a_2' = -\frac{3a_3^2}{8} + a_2, \quad a_1' = \frac{a_3^3}{8} - \frac{a_2a_3}{2} + a_1, \quad a_0' = -\frac{3a_3^4}{256} + \frac{a_2a_3^2}{16} - \frac{a_1a_3}{4} + a_0,$$

Now we substitute x', P', a' by x, P, a, it becomes

$$P(x) = x^4 + a_2x^2 + a_1x + a_0 = (x^2 - 2ax + b)(x^2 + 2ax + c), \quad a > 0, \quad a^2 < b, c,$$

Assume that it has two roots in the upper half plane $x_1 = -a + ui, x_2 = a + vi, u, v > 0$, then the integral is

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{P(x)} dx = 2\pi i(\frac{1}{(x_1 - \overline{x_1})(x_1 - x_2)(x_1 - \overline{x_2})} + \frac{1}{(x_2 - x_1)(x_2 - \overline{x_1})(x_2 - \overline{x_2})}) \\
=& \frac{\pi}{u(4a^2 - u^2 + v^2 - 4aui)} + \frac{\pi}{v(4a^2 - v^2 + u^2 + 4avi)} = \frac{\pi((4a^2 - u^2 + v^2)/u + (4a^2 - v^2 + u^2)/v)}{16a^4 + 8a^2(u^2 + v^2) + (v^2 - u^2)^2},
\end{aligned}$$

where we used

$$(x_1 - x_2)(x_1 - \overline{x_2}) = (-2a + (u - v)i)(-2a + (u + v)i) = 4a^2 - (u^2 - v^2) - 4aui,$$

$$(x_2 - x_1)(x_2 - \overline{x_1}) = (2a + (v - u)i)(2a + (v + u)i) = 4a^2 - (v^2 - u^2) + 4avi,$$

and the imaginary part is

$$\frac{4au/u}{(4a^2 - u^2 + v^2)^2 + (4au)^2} + \frac{-4av/v}{(4a^2 - v^2 + u^2)^2 + (4av)^2} = 0,$$

$$(4a^2 - u^2 + v^2)^2 + (4au)^2 = (4a^2 - v^2 + u^2)^2 + (4av)^2 = 16a^4 + 8a^2(u^2 + v^2) + (v^2 - u^2)^2,$$

1) When $a_1 = 0$, either one of the following two cases occur: i)$a = 0$, the integral becomes

$$\int_{\mathbb{R}} \frac{1}{(x^2+u^2)(x^2+v^2)} dx = \frac{1}{v^2-u^2} \int_{\mathbb{R}} (\frac{1}{x^2+u^2} - \frac{1}{x^2+v^2}) dx = \frac{1}{v^2-u^2}(\frac{\pi}{u} - \frac{\pi}{v}),$$

where we used that

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+a^2} dx = \int_{-\infty}^{+\infty} \frac{1/a}{(\frac{x}{a})^2+1} d\frac{x}{a} = \frac{1}{a} \arctan \frac{x}{a} \Big|_{-\infty}^{+\infty} = \frac{\pi}{a},$$

This result agrees with our previous calculation since

$$\frac{\pi((-u^2+v^2)/u + (-v^2+u^2)/v)}{(v^2-u^2)^2} = \frac{1}{v^2-u^2}(\frac{\pi}{u} - \frac{\pi}{v}),$$

$$P(x) = (x^2+u^2)(x^2+v^2) = x^4 + (u^2+v^2)x^2 + u^2v^2,$$

and thus we can solve a quadratic equation to get values of u, v.

ii)$a > 0, u = v$, the integral becomes

$$\frac{8a^2\pi(1/u)}{16a^4+16a^2u^2} = \frac{\pi}{2u(a^2+u^2)},$$

$$P(x) = (x^2+2ax+a^2+u^2)(x^2-2ax+a^2+u^2) = x^4 + 2(u^2-a^2)x^2 + (a^2+u^2)^2,$$

and thus we can solve a linear equation to get values of a, u.

2) When $a_1 \neq 0$, notice that a^2 is a algebraic number with degree 3.

$$a = \frac{x_2+\overline{x_2}-x_1-\overline{x_1}}{4}, \quad \tilde{a}_1 = \frac{x_1+x_2-\overline{x_1}-\overline{x_2}}{4} = \frac{(u+v)i}{2}, \quad \tilde{a}_2 = \frac{x_1+\overline{x_2}-\overline{x_1}-x_2}{4} = \frac{(u-v)i}{2},$$

$$\tilde{a}_1^2 = -\frac{(u+v)^2}{4} \tilde{a}_2^2 = -\frac{(u-v)^2}{4}$$

Coefficients of P satisfy

$$P(x) = (x^2+2ax+a^2+u^2)(x^2-2ax+a^2+v^2) = x^4+(u^2+v^2-2a^2)x^2+2a(v^2-u^2)x+(a^2+u^2)(a^2+v^2),$$

$$a_2 = u^2+v^2-2a^2, \quad a_1 = 2a(v^2-u^2), \quad a_0 = (a^2+u^2)(a^2+v^2),$$

$$a^2 + \tilde{a}_1^2 + \tilde{a}_2^2 = a^2 - \frac{u^2+v^2}{2} = -\frac{a_2}{2},$$

$$a^2(\tilde{a}_1^2+\tilde{a}_2^2)+\tilde{a}_1^2\tilde{a}_2^2 = -\frac{a^2(u^2+v^2)}{2} + \frac{(u^2-v^2)^2}{16} = -\frac{a^2(u^2+v^2)}{2} + \frac{(u^2+v^2)^2}{16} - \frac{u^2v^2}{4} = \frac{1}{4}((\frac{a_2}{2})^2 - a_0),$$

$$a^2\tilde{a}_1^2\tilde{a}_2^2 = \frac{a^2(u^2-v^2)^2}{16} = \frac{1}{16}(\frac{a_1}{2})^2,$$

$$R(x) = x^3 + b_2x^2 + b_1x + b_0, \quad x' = x + \frac{b_2}{3},$$

$$b_2 = \frac{a_2}{2}, \quad b_1 = \frac{\frac{a_2^2}{4} - a_0}{4}, \quad b_0 = -\frac{a_1^2}{64},$$

$$R(x) = x'^3 + x'(-\frac{b_2^2}{3} + b_1) + \frac{2b_2^3}{27} - \frac{b_1b_2}{3} + b_0 = S(x'),$$

$$S(x) = x^3 + c_1 x + c_0, \quad c_1 = -\frac{b_2^2}{3} + b_1, \quad c_0 = \frac{2b_2^3}{27} - \frac{b_1 b_2}{3} + b_0, \quad x = y - \frac{c_1}{3y}$$

$$T(y) = S(x) = y^3 + c_0 - \frac{c_1^3}{27y^3}, \quad z = y^3, \quad \omega = e^{\frac{2\pi i}{3}},$$

$$x_1 = y_1 + y_2, \quad x_2 = y_1\omega + y_2\omega^2, \quad x_3 = y_1\omega^2 + y_2\omega,$$

Use long double and one step Newton method to improve result's accuracy.

Method 2: Find a square matrix such that $P(x)$ is its characteristic polynomial. According to rational canonical form, we may construct

$$A = \begin{pmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & 0 & -a_2 \\ & & 1 & -a_3 \end{pmatrix}, \quad \det(\lambda I - A) = P(\lambda),$$

Then we may apply iterative algorithms that finds unsymmetric eigenvalues of matrix A. But I haven't implemented this idea successfully.

Problem 12 (1626 Interfering Segment)**.** Reference: Computational geometry - algorithms and applications, chapter 2, line segment intersection. Introduction to algorithms 3rd edition, chapter 33, determining whether any pair of segments intersects.

Assume that the polygon is $P_1P_2...P_n$. $\triangle ABC$ is part of a legal triangulation $\iff$ each of AB, BC, AC divides the polygon into two parts. Only if part is trivial. If part: assume that $\Omega_A, \Omega_B, \Omega_C$ are the parts of the polygon that don't contain A, B, C after divided by BC, CA, AB respectively, then each of them is a polygon without self intersection. So $\Omega_A, \Omega_B, \Omega_C$ can be triangulated, adding $\triangle ABC$ to obtain a triangulation of the original polygon.

$$X \in \triangle ABC \iff X \notin \Omega_A, \Omega_B, \Omega_C,$$

Preprocessing: for any pair i, j not next to each other, judge if P_iP_j intersect with any edge besides $P_{i-1}P_i, P_iP_{i+1}, P_{j-1}P_j, P_jP_{j+1}$, store these boolean variables in $flag[i][j]$. If $flag[i][j] == true$, judge if X is on the $P_{i+1}...P_{j-1}$ side or $P_{j+1}...P_{i-1}$ side of P_iP_j, or is exactly on P_iP_j, and store it in $sideX[i][j]$. $flag[i][i+1]$ is always true. Denote by l_X the ray $y = y_X, x \geq x_X$, fix i and let j iterate from $i+1$ to $i-1$, we may maintain the intersection number of l_X with $P_iP_{i+1}...P_j$, and thus know $sideX[i][j]$. Similarly we get $sideY[i][j]$.

Problem 13 (1661 Dodecahedron)**.** The symmetry group of dodecahedron in $SO(3)$ is A_5. Assume that its edges are $e_1, ..., e_{30}$. Given $c_1, ..., c_{30} \in [30] = \{1, 2, ..., 30\}$, find the number of different dodecahedra. A coloring is to give each edge e_i a color $c[s_i]$, where $s \in S_{30}$ is a permutation. Two coloring s^1, s^2 are identical means there exists $\sigma \in A_5 \subset S_{30}$, such that for any $1 \leq i \leq 30$, $c[s^2(\sigma(i))] = c[s^1(\sigma(i))]$. Ignoring c and A_5, all the possible color assignments can be regarded as the permutation group S_{30}. There is a subgroup G of S_{30} determined by c, such that

$$c \circ s^2 = c \circ s^1 : [30] \to [30] \iff \text{exists } g \in G, g \circ s^2 = s^1, \quad s^1, s^2 \in S_{30}, \quad G = \prod_{x \in [30]} S_{c^{-1}(x)},$$

G is the product of permutation groups on each fiber of c. So S_{30} is given a double coset structure $G \curvearrowright S_{30} \curvearrowleft A_5$, and we are asked to calculate its cardinality.

The orbits of $S_{30} \curvearrowleft A_5$ are S_{30}/A_5, the set of right cosets of A_5 in S_{30}. Similarly, the orbits of $G \curvearrowright S_{30}$ are $G\backslash S_{30}$, the set of left cosets of G in S_{30}. We may assume that the image of c is $1, 2, ..., k$, and $|c^{-1}(i)| = n_i, \sum_{1 \leq i \leq k} n_i = 30$. Our task is to calculate $|G\backslash S_{30}/A_5|$.

$X = G\backslash S_{30}$ can be described as all the sequences $d_1, ..., d_{30}$ in which i appear n_i times, $1 \le i \le k$. Burnside's lemma says that

$$|X/A_5| = \frac{1}{|A_5|}\sum_{\sigma\in A_5}|X^\sigma|, \quad X^\sigma = \{x \in X, \sigma(x) = x\},$$

Elements of A_5 can be divided into 4 classes: identity, 15 order 2 elements, 20 order 3 elements, 24 order 5 elements. Let $\sigma_l \in A_5$ be an element with order l. i) $l = 1, 3, 5$, then $|X^{\sigma_l}| = \frac{(\frac{30}{l})!}{\prod(\frac{n_i}{l})!}$ when l divides each n_i, otherwise $|X^{\sigma_l}| = 0$. ii) $l = 2$, if the number of odds in n_i is larger than 2, $|X^{\sigma_l}| = 0$; if there are odds in n_i then $|X^{\sigma_l}| = \frac{(\frac{30}{l})!}{\prod(\frac{n_i}{l})!}$; otherwise there are 2 odds in n_i, $|X^{\sigma_l}| = \frac{2*(14)!}{\prod(\lfloor\frac{n_i}{l}\rfloor)!}$. Used Qifeng Chen's BigInteger struct for large number calculation, and made some modifications. One precious experience is that, never ever inherit from any std:: type, except for the standard types you're supposed to inherit from. Accepted using both implementations of BigInteger of mine and Qifeng Chen's.

Proof. Assume $G \curvearrowright X$ is a left group action.

$$X^g = \{x \in X, g(x) = x\}, \quad G^x = \{g \in G, g(x) = x\}, \quad |\text{orbit}(x)| = \frac{|G|}{|G^x|},$$

$$\frac{1}{|G|}\sum_{g\in G}|X^g| = \frac{1}{|G|}\sum_{x\in X}|G^x| = \sum_{x\in X}\frac{1}{|\text{orbit}(x)|} = |X/G|,$$

$\square$

Problem 14 (1662 Goat in the Garden 6)**.** Note that the bed is a convex polygon $P_1P_2...P_n$. Let $D_r = \bigcup_{p\in polygon}\odot(p, r)$, we need to determine whether $\bigcap_{p\in polygon}\odot(p, R) \cap \overline{D_r}$ is nonempty. $\bigcap_{p\in polygon}\odot(p, R) = \bigcap_{1\le i\le n}\odot(P_i, R)$ is an convex set. For any $p \notin D_r$, let q be its projection onto ∂D_r. If $p \in \bigcap_{1\le i\le n}\odot(P_i, R)$, then $q \in \bigcap_{1\le i\le n}\odot(P_i, R)$, so it sufficies to check if $\partial D_r \cap \bigcap_{1\le i\le n}\odot(P_i, R)$ is nonempty. ∂D_r can be divided into n arcs and n segments. 1) Segment circles intersection.

2) For an arc $\overline{AB} \in \odot(O, r)$ and $\odot(X, R)$, i) Both $AX > R, BX > R$ hold, then $\overline{AB} \cap \odot(X, R)$ is empty since O, X lie on the same side of AB. ii) $OX \le R - r$, check the next circle. iii) $OX > R - r$, get intersection C, D and judge that if $\overline{AB}$ split into two parts. Assume μ is the signed length of OH the direction of XH, $\nu = HC$, then

$$R^2 - r^2 = XH^2 - OH^2 = 2\mu XO + XO^2, \quad \nu = \sqrt{r^2 - \mu^2},$$

Problem 15 (1668 Death Star 2)**.** $A_{N\times M}, b_N$, find x_M such that $\|Ax - b\|_2^2$ reaches minimum. If the solution is ambiguous, output the one that $\|x\|_2^2$ is the minimum. Let $A = UsV^t$ be the singular value decomposition of A,

$$s^{inv}[i] = \begin{cases} \frac{1}{s[i]}, s[i] > 0, \\ 0, s[i] = 0. \end{cases}, \quad A^{inv} = Vs^{inv}U^t$$

Then we claim that $x = A^{inv}b$. While calculating SVD, we use Golub-Kahan bidiagonalization in phase 1. Householder reflection is given by

$$x = A_{k:m,k}, \quad v_k = \text{sign}(x_1)\|x\|_2 e_1 + x, \quad v_k = \frac{v_k}{\|v_k\|_2}, \quad A_{k:m,k:n} - = 2v_k(v_k^* A_{k:m,k:n})$$

Householder reflector: $x \mapsto Fx = \pm\|x\|e_1$. Givens rotation acting on the i,j-th rows or columns: $G(i,j,\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. How to use Givens rotations to eliminate the off-diagonal elements? Actually the remaining steps after bidiagonalization only use Givens rotations.

Problem 16 (1697 Sniper Shot)**.** Method 1: projection onto plane $z = 0$.

Method 2: projection onto the plane spanned by AB and e_z.

Problem 17 (1810 Antiequations)**.**

$$A : \mathbb{F}_3^n \to \mathbb{F}_3^k, \quad P_i = \{y_i = b_i\} \subset \mathbb{F}_3^k, \quad (y_1, ..., y_k) \in \mathbb{F}_3^k,$$

Assume that $l = im(A), \dim(l) = p$. We consider the case when all P_i cross intersects l first, let $l \cap P_i = Q_i$, then $\dim(Q_i) = p - 1$. It suffices to calculate the size of $l \backslash \bigcup_{1 \le i \le k} Q_i$. For each pair of $i, j \in \{1, ..., k\}$, there are three possible relations between Q_i and Q_j: 1) $Q_i = Q_j$ identical, 2) $Q_i \cap Q_j = \emptyset$ parallel, 3) $\dim(Q_i \cap Q_j) = p - 2$ cross intersect.

Method 2: counting points on the affine variety $X = Q_1 \cup Q_2 ... \cup Q_k$. Generally speaking, let

$$N_m = |X(\mathbb{F}_{q^m})|, \quad Z(X, t) = \exp(\sum_{m \ge 1} \frac{N_m}{m} t^m),$$

For example, when $p = k$, $Q_i : y_i = b_i$, we have

$$N_m = q^{mk} - (q^m - 1)^k = \sum_{i=0}^{k-1} q^{mi} \binom{k}{i} (-1)^{k+i+1},$$

$$\sum_{m \ge 1} \frac{N_m}{m} t^m = \sum_{i=0}^{k-1} \binom{k}{i} (-1)^{k+i+1} \sum_{m \ge 1} \frac{q^{mi}}{m} t^m = \sum_{i=0}^{k-1} \binom{k}{i} (-1)^{k+i} \log(1 - q^i t),$$

$$Z(X, t) = \exp(\sum_{i=0}^{k-1} \binom{k}{i} (-1)^{k+i} \log(1 - q^i t)) = \frac{(1 - q^{k-2} t)^{\binom{k}{2}} ...}{(1 - q^{k-1} t)^{\binom{k}{1}} (1 - q^{k-3} t)^{\binom{k}{3}} ...},$$

$$N_1 = \frac{dZ}{dt} \Big|_{t=0} = \binom{k}{1} q^{k-1} - \binom{k}{2} q^{k-2} + \binom{k}{3} q^{k-3} ... = q^k - (q-1)^k,$$

Another special case is when $b_i = 0$, each P_i is a codimension 1 subspace of $\mathbb{F}_3^k$. Claim: we may select a basis $s_1, ..., s_p$ of l such that their supports are pairwise disjoint.

Problem 18 (1814 Continued Fraction)**.** We need to implement quadratic extension of rational number field as a struct QuadraticRT. An element's inverse is given by

$$(\frac{x + y\sqrt{N}}{z})^{-1} = \frac{z(y\sqrt{N} - x)}{Ny^2 - x^2},$$

$nums[i]$ stores quadrtic rational $a_i + r_i$, where $a_i \in \mathbb{Z}_+, 0 < r_i < 1$, and $\sqrt{N} = nums[0]$. Formula of continued fraction is given by $nums[i+1] = \frac{1}{r_i}$. By the following theorem, we may check that if the block length is m, then

$$nums[m+1] = nums[1], \quad nums[m] = a_0 + nums[0] = a_0 + \sqrt{N},$$

Method 1: assume that $R_n = \frac{P_n}{Q_n} = [a_0; a_1, a_2, ..., a_n]$ and $1 \le k_1 \le m, k_1 \equiv k \mod m$, then

$$P_n = P_{n-1} a_n + P_{n-2}, \quad Q_n = Q_{n-1} a_n + Q_{n-2},$$

We start by calculating $[a_{k_1+1}, ...a_k]$. Let $P^l = P_{lm}, Q^l = Q_{lm}, l = (k-k_1)/m$ in this scenario, then

$$\begin{pmatrix} P^l \\ Q^l \end{pmatrix} = \begin{pmatrix} P_m & P_{m-1} \\ Q_m & Q_{m-1} \end{pmatrix} \begin{pmatrix} P^{l-1} \\ Q^{l-1} \end{pmatrix} = M^l \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

Turning back to the original problem, we have

$$P_k = P^l P_{k_1} + Q^l P_{k_1-1}, \quad Q_k = P^l Q_{k_1} + Q^l Q_{k_1-1},$$

Method 2: define $P_{-1} = 1, Q_{-1} = 0, P_{-2} = 0, Q_{-2} = 1$,

$$A_i = \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}, \quad M = A_m A_{m-1} ... A_1, \quad prodm = A_{k_1} ... A_1 M^l A_0,$$

Theorem 1. If $r \in \mathbb{Q}, r > 1$ is not a perfect square, then

$$\sqrt{r} = [a_0; \overline{a_1, a_2, ..., a_2, a_1, 2a_0}],$$

It has a repeating block of length m, in which the first $m-1$ partiail denominators form a palindromic string. In the continued fraction expansion of $\frac{P+\sqrt{D}}{Q}$, the largest partial denominator a_i in the expansion of $\sqrt{D}$ is less than $2\sqrt{D}$, and the block length $m = L(D)$ is less than $2D$. A sharper bound is

$$L(D) = O(\sqrt{D} \log D),$$

Problem 19 (1815 Farm in San Andreas)**.** Given coordinates of A, B, C, costs c_A, c_B, c_C, find the minimum of $c_A PA + c_B PB + c_C PC$. If P is different from A, B, C, assume that $\alpha = \angle BPC, \beta = \angle CPA, \gamma = \angle APB$, $\alpha_1 = \pi - \alpha, \beta_1 = \pi - \beta, \gamma_1 = \pi - \gamma$, then

$$c_B = c_A \cos \gamma_1 + c_C \cos \alpha_1, \quad c_A = c_C \cos \beta_1 + c_B \cos \gamma_1, \quad c_C = c_B \cos \alpha_1 + c_A \cos \beta_1,$$

$\alpha_1, \beta_1, \gamma_1$ are interior angles of the triangle *coststri* with edge lengths c_A, c_B, c_C. Geometrically we may construct point R (temppt in program) such that $BC : CR : BR = c_A : c_B : c_C$. Intersection of AR with the circumcircle of $\triangle CBR$ is the point P required.

Problem 20 (1845 Integer-valued Complex Determinant)**.** Calculate the determinant of a Gaussian integer valued matrix. Attention: my implementation of struct BigInteger is written in this program. It is modified from Qifeng Chen's implementation. Const qualifiers are used as improvements, and BigNumber uses std::vector instead of array to store x.

Method 1: Integer coefficient Gaussian elimination. GaussianZT is implemented as Gaussian integer struct. Use extended Euclidean algorithm to calculate GCD of Gaussian integers after pivoting, so that all the elementary row transformations have Gaussian integer coefficient. A prototype of it was implemented in IntegerElimination.cpp, in which I only tested on integer coefficient matrices but not Gaussian integer. There are three kinds of elementary row transformations:

$$a_j \to a_j - ca_i, \quad a_j \to -a_j, \quad a_i \leftrightarrow a_j,$$

and I try to turn a_{kk} into $\gcd(a_{kk}, a_{ik})$ for $i > k, a_{ik} \neq 0$ after pivoting. Let $u = a_{kk}, v = a_{ik}$, the output of extendedGCD function satisfies

$$xu + yv = d, \quad u = du_q, v = d_v 1, \quad xu_1 + yv_1 = 1, \quad u_1 v - v_1 u = 0, \quad |x| < |v_1|, |y| < |u_1|,$$

So we can take elementary row transform on the k, i-th rows resulting in

$$\det \begin{pmatrix} x & y \\ -v_1 & u_1 \end{pmatrix} = 1, \quad \begin{pmatrix} x & y \\ -v_1 & u_1 \end{pmatrix} \begin{pmatrix} a_{kk} \\ a_{ik} \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix},$$

and it doesn't change the determinant of A. Recursion of extendedeGCD satisfies

$$u\%v = u - \lambda v, \quad d = yv + x(u\%v) = xu + (y - \lambda x)v,$$

Note that generally speaking, we don't need auxilliary matrices P, L that appears in $PA = LU$ while calculating determinant.

Method 2: GaussianQT is implemented as Gaussian rational struct. Use raw Gaussian elimination after pivoting, but the current result is WA12 while using BigInteger struct, WA15 while using long long.

1 Unsymmetric eigenvaluve problems

Theorem 2 (Gershgorin Circle Theorem)**.** 1) If $X^{-1}AX = D + F$, where $D = \text{diag}(d_1, ...d_n)$ and F has zero diagonal entries, then

$$\sigma(A) \subset \bigcup_{i=1}^{n} D_i, \quad D_i = \{z \in \mathbb{C}, |z - d_i| \leq \sum_j = 1^n |f_{ij}|\},$$

2) If the Gershgorin disk D_i is isolated from other disks, then it contains precisely one eigenvalue of A.

Proof. 1) Suppose that $\lambda \in \sigma(A), \lambda \neq d_i, 1 \leq i \leq n$, $D - \lambda I + F = X^{-1}AX - \lambda I$ is singular.

$$(D - \lambda I)^{-1} = \text{diag}(\frac{1}{d_i - \lambda}), \quad G = (D - \lambda I)^{-1}F = (\frac{f_{ij}}{d_i - \lambda})_{1 \leq i,j \leq n},$$

We are interested in the l_∞ norm of operator G, which is defined as $\|G\|_\infty = \max_{x \neq 0} \frac{\|Gx\|_\infty}{\|x\|_\infty}$. Suppose $\|x\|_\infty = 1$,

$$Gx = (\sum_j g_{ij}x_j)_{1 \leq i \leq n}, \quad \|Gx\|_\infty \leq \max_i \sum_j |g_{ij}| = \max_i \sum_j \frac{|f_{ij}|}{|d_i - \lambda|},$$

and the equality holds. On the other hand, if $A+B$ is singular, A is non-singular, suppose $(A+B)u = 0, u \neq 0$, then

$$Au = -Bu, \quad u = -A^{-1}Bu, \quad \|A^{-1}Bu\|_p = \|u\|_p, \quad \|A^{-1}B\|_p \geq 1,$$

Let $A = D - \lambda I, B = F, p = \infty$, we have $1 \leq \sum_j \frac{|f_{kj}|}{|d_k - \lambda|}$ for some $1 \leq k \leq n$,

$$|d_k - \lambda| \leq \sum_j |f_{kj}|, \quad \lambda \in D_k,$$

Another proof: assume $\lambda \in \sigma(A)$ with eigenvector $u \in \mathbb{C}^n$, $k = \arg\max_j |u_j|$, then

$$|(d_k - \lambda)u_k| = |-\sum_j f_{kj}u_j| \leq |u_k| \sum_j |f_{kj}|, \quad |d_k - \lambda| \leq \sum_j |f_{kj}| = r_k,$$

2) If $D_k \cap D_i = \emptyset, i \neq k$, we show that there exist precisely one $\lambda \in \sigma(A), \lambda \in D_k$ and λ has multiplicity 1. i) Uniqueness: if $\lambda \in D_k$ is an eigenvalue of $D + F$, $u \in \mathbb{C}^n$ is an eigenvector of $D - \lambda I + F$. If $i = \arg\max_j |u_j|, i \neq k$ then

$$(d_i - \lambda)u_i = -\sum_j f_{ij}u_j, \quad |(d_i - \lambda)u_i| > r_i|u_i| \geq \sum_j |f_{ij}||u_j|,$$

contradiction! So we have $k = \arg\max_j |u_j|, \|u\| = |u_k|$. Without loss of generality, assume that $k = n$, $D'_{n-1}, F'_{n-1} \in \mathrm{Aut}(\mathbb{C}^{n-1})$ are the first $n-1$ rows and columns of D, F. Then for $x \in \mathbb{C}^{n-1}$,

$$((D' - \lambda I')^{-1}F'x)_i = \sum_{j=1}^{n-1} \frac{f_{ij}x_j}{d_i - \lambda}, \quad |((D' - \lambda I')^{-1}F'x)_i| \leq \sum_{j=1}^{n-1} \frac{|f_{ij}||x_j|}{|d_i - \lambda|} < \|x\|_\infty,$$

So $\|(D' - \lambda I')^{-1}F'\|_\infty < 1$, and we have the following expansion

$$(D' - \lambda I' + F')^{-1} = (D' - \lambda I')^{-1}(I' + (D' - \lambda I')^{-1}F')^{-1}$$

$$(I' + (D' - \lambda I')^{-1}F')^{-1} = \sum_{m\geq 0}((\lambda I' - D')^{-1}F')^m,$$

The right hand side above is absolutely convergent, so $D' - \lambda I' + F'$ is invertible. It follows that if $\lambda \in D_k$ is an eigenvalue of $D + F$, then it has multiplicity 1.

ii) Resolvent method:

$$R(\lambda) = (D - \lambda I + F)^{-1} : \mathbb{C}^n \to \mathbb{C}^n,$$

As an operator valued function, $R : \mathbb{C} \to \mathrm{Aut}(\mathbb{C}^n)$ is meromorphic in the following sense: for any $\phi \in (\mathbb{C}^n)^*, v \in \mathbb{C}^n$, $f(\phi, v, \lambda) = \langle \phi, R(\lambda)v\rangle$ is a meromorphic function of λ.

$$\langle \phi, R(\lambda_1)v\rangle - \langle \phi, R(\lambda_0)v\rangle = \langle \phi, (R(\lambda_1) - R(\lambda_0))v\rangle = \langle \phi, (\lambda_1 - \lambda_0)R(\lambda_1)R(\lambda_0)v\rangle,$$

$R(\lambda_1), R(\lambda_0)$ are commutative: $R(\lambda_1)R(\lambda_0) = R(\lambda_0)R(\lambda_1)$.

$$\frac{\langle \phi, R(\lambda_1)v\rangle - \langle \phi, R(\lambda_0)v\rangle}{\lambda_1 - \lambda_0} = \langle \phi, R(\lambda_1)R(\lambda_0)v\rangle$$

hence $f(\phi, v, \lambda)$ is meromorphic with $\frac{df(\phi,v,\lambda)}{d\lambda} = \langle \phi, R(\lambda)^2 v\rangle$.

$$\lambda \text{ singular} \iff \|R(\lambda)\| = \infty, \quad \|R(\lambda)\| = \max_{\|\phi\|=\|v\|=1} |\langle \phi, R(\lambda)v\rangle|,$$

In finite dimensional case, $D - \lambda I + F : \mathbb{C}^n \to \mathbb{C}^n$ induces an automorphism on $\bigwedge^n(\mathbb{C}^n)$:

$$e_1 \wedge e_2 ... \wedge e_n \mapsto \bigwedge_{i=1}^n (D - \lambda I + F)e_i = \det(D - \lambda I + F)e_1 \wedge e_2 ... \wedge e_n,$$

iii) Consider the eigenvalues of $D + \epsilon F, 0 \leq \epsilon \leq 1$, by 1), all of its eigenvalues lie in $\bigcup_{i=1}^n \epsilon D_i$. These eigenvalues vary continuously with respect to ϵ, and we may denote them as $\lambda_i(\epsilon), 1 \leq i \leq n$. When $\epsilon = 0$, we have $\lambda_i(0) = d_i$, so by continuity argument and the fact that $D_k \cap D_i = \emptyset, i \neq k$, we know that there is precisely one eigenvalue $\lambda_k(\epsilon) \in \epsilon D_k$. Take $\epsilon = 1$ finishes the proof. $\square$

Theorem 3 (Bauer-Fike)**.** If μ is an eigenvalue of $A+E \in \mathbb{C}^{n*n}$ and $X^{-1}AX = D = \mathrm{diag}(\lambda_q, ..., \lambda_n)$, then

$$\min_{\lambda\in\sigma(A)} |\lambda - \mu| \leq \kappa_p(X)\|E\|_p,$$

Proof. It suffices to consider the case $\mu \notin \sigma(A)$. If the matrix $X^{-1}(A+E-\mu I)X$ is singular, then so is $I+(D-\mu I)^{-1}X^{-1}EX$. Then we have

$$1 \leq \|(D-\mu I)^{-1}X^{-1}EX\|_p \leq \|(D-\mu I)^{-1}\|_p\|X\|_p\|E\|_p\|X^{-1}\|_p,$$

Since$\|(D-\mu I)^{-1}\|_p = \max_{\lambda\in\sigma(A)} \frac{1}{|\lambda-\mu|}$, we have finished our proof. □

Definition 1. For square matrix A define the condition number $\kappa(A) = \|A\|\|A^{-1}\|$, with the convention that $\kappa(A) = \infty$ for singular A. $\kappa(\cdot)$ depends on the underlying norm and subscripts are used accordingly.

$$\kappa_2(A) = \|A\|_2\|A^{-1}\|_2 = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}, \quad \frac{1}{\kappa_p(A)} = \min_{A+\Delta A \text{ singular}} \frac{\|\Delta A\|_p}{\|A\|_p},$$

$$\kappa(A) = \lim_{\epsilon\to 0} \sup_{\|\Delta A\|\leq\epsilon\|A\|} \frac{\|(A+\Delta A)^{-1}-A^{-1}\|}{\epsilon\|A^{-1}\|}$$

Theorem 4. Let $Q^HAQ = D+N$ be a Schur decomposition of $A \in \mathbb{C}^{n*n}$, i.e., $Q \in \mathbb{C}^{n*n}$ is unitary, $D = \text{diag}(\lambda_1,...,\lambda_n)$ and $N \in \mathbb{C}^{n*n}$ is strictly upper diagonal. Q can be chosen so that the eigenvalues λ_i appear in any order along the diagonal. If $\mu \in \sigma(A+E)$ and p is the smallest positive integer such that $N^p = 0$, then

$$\min_{\lambda\in\sigma(A)} |\lambda-\mu| \leq \max\{\theta, \theta^{\frac{1}{p}}\}, \quad \theta = \|E\|_2\sum_{k=0}^{p-1}\|N\|_2^k,$$

Extreme eigenvalue sensitivity for a matrix A cannot occur if A is normal. But A nonnormal matrix can have a mixture of well-conditioned and ill-conditioned eigenvalues. Suppose that λ is a simple eigenvalue of $A \in \mathbb{C}^{n*n}$ and that x and y satisfy $Ax = \lambda x, y^HA = \lambda y^H, \|x\|_2 = \|y\|_2 = 1$. If $Y^HAX = J$ is the Jordan decomposition with $Y^H = X^{-1}$, then y and x are nonzero multiples of $X(:,i), Y(:,i)$ for some i, so $y^Hx \neq 0$.

$$(A+\epsilon F)x(\epsilon) = \lambda(\epsilon)x(\epsilon), \quad \|F\|_2 = 1,$$

We refer to the reciprocal of $s(\lambda) = |y^Hx|$ as the condition of the eigenvalue λ. A small $s(\lambda)$ implies that A is near a matrix having a multiple eigenvalue. In particular, if λ is distinct and $s(\lambda) < 1$, then there exists an E such that λ is a repeated eigenvalue of $A+E$ and

$$\frac{\|E\|_2}{\|A\|_2} \leq \frac{s(\lambda)}{\sqrt{1-s(\lambda)^2}},$$

In general, if λ is a defective eigenvalue of A, then $O(\epsilon)$ perturbations in A can result in $O(\epsilon^{\frac{1}{p}})$ perturbations in λ if λ is associated with a p-dimensional Jordan block.

2 Symmetric eigenvalue problems

Theorem 5 (Gershgorin)**.** A is real symmetric, Q is orthogonal, if $Q^tAQ = D+F, D = \text{diag}(d_1,d_2,...,d_n)$ and F has zero diagonal entries, then

$$\sigma(A) \subset \bigcup_{i=1}^{n}[d_i-r_i, d_i+r_i], \quad r_i = \sum_j |f_{ij}|,$$

Proof. Exactly the same as the unsymmetric case, with an additional property that $\sigma(A) \subset \mathbb{R}$. $\square$

Theorem 6 (Wielandt-Hoffman)**.** If A and $A+E$ are $n*n$ symmetric matrices, then

$$\sum_{i=1}^{n}(\lambda_i(A+E)-\lambda_i(A))^2 \leq \|E\|_F^2 = \sum_{i,j=1}^{n}|e_{ij}|^2$$

Theorem 7. If A and $A+E$ are $n*n$ symmetric matrices, then

$$\lambda_k(A)+\lambda_n(E) \leq \lambda_k(A+E) \leq \lambda_k(A)+\lambda_1(E), \quad 1 \leq k \leq n,$$

$$|\lambda_k(A+E)-\lambda_k(A)| \leq \|E\|_2 = \max\{|\lambda_n(E)|,|\lambda_1(E)|\}, \quad 1 \leq k \leq n,$$

Theorem 8 (Interlacing property)**.** If $A \in \mathbb{R}^{n*n}$ is symmetric and $A_r = A(1:r,1:r)$, then

$$\lambda_{r+1}(A_{r+1}) \leq \lambda_r(A_r) \leq \lambda_r(A_{r+1}) \leq ... \leq \lambda_2(A_{r+1}) \leq \lambda_1(A_r) \leq \lambda_1(A_{r+1}), \quad 1 \leq r \leq n-1,$$

Theorem 9. Suppose $B = A+\tau cc^t, A \in \mathbb{R}^{n*n}, A = A^t, c \in \mathbb{R}^n, \|c\|_2 = 1, \tau \in \mathbb{R}.$ we have

$$\lambda_i(B) \in [\lambda_i(A), \lambda_{i-1}(A)], 2 \leq i \leq n, \text{ when } \tau \geq 0,$$

$$\lambda_i(B) \in [\lambda_{i+1}(A), \lambda_i(A)], 1 \leq i \leq n-1, \text{ when } \tau < 0,$$

In either case, there exist $m_1, m_2, ..., m_n \geq 0, m_1+m_2+...+m_n = 1$ such that

$$\lambda_i(B) = \lambda_i(A)+m_i\tau, \quad 1 \leq i \leq n,$$

Proposition 1. 1) If $T = QR$ is the QR factorization of a symmetric tridiagonal matrix $T \in \mathbb{R}^{n*n}$, then Q has lower bandwidth 1 and R has upper bandwidth 2 and it follows that $T_+ = RQ = Q^tTQ$ is also symmetric and tridiagonal.

2) If $s \in \mathbb{R}$ and $T-sI = QR$ is the QR factorization, then $T_+ = RQ+sI = Q^tTQ$ is also tridiagonal. This is called a shifted QR step.

3) If T is unreduced, then the first $n-1$ columns of $T-sI$ are independent regardless of s.

4) If $T \in \mathbb{R}^{n*n}$ is tridiagonal, then its QR factorization can be computed by applying a sequence of $n-1$ Givens rotations.

3 Solve univariate polynomial equations using $SL_2(\mathbb{R})$

Assume that a degree 4 real coefficient polynomial $P(x) = x^4+a_3x^3+a_2x^2+a_1x+a_0$ has no real roots. Assume that its four roots are

$$x_1, \overline{x_1}, x_2, \overline{x_2}, \quad x_1, x_2 \in \mathbb{H}, \quad \Im(x_1) > 0, \Im(x_2) > 0,$$

Assume that $x_1 = u_1+v_1i, x_2 = u_2+v_2i$. Half circle arc on the upper half plane which passes x_1, x_2 and meets real axis orthogonally is uniquely determined. Assume its center is $(u,0)$, then

$$(u-u_1)^2+v_1^2 = (u-u_2)^2+v_2^2, \quad u = \frac{u_2^2+v_2^2-u_1^2-v_1^2}{2(u_2-u_1)},$$

Radius of the half circle arc is

$$\begin{aligned} r_0^2 &= (u-u_1)+v_1^2 = \frac{(u_2^2-2u_1u_2+u_1^2+v_2^2-v_1^2)^2}{4(u_2-u_1)^2}+v_1^2 \\ &= \frac{((u_2-u_1)^2+v_2^2-v_1^2)^2+4(u_2-u_1)^2v_1^2}{4(u_2-u_1)^2} = \frac{(u_2-u_1)^4+2(u_2-u_1)^2(v_2^2+v_1^2)+(v_2^2-v_1^2)^2}{4(u_2-u_1)^2}, \end{aligned}$$

The action $SL_2(\mathbb{R}) \curvearrowright \overline{\mathbb{C}}$ is given by

$$x \mapsto g(x) = \frac{px+q}{rx+s}, \quad g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{R}),$$

Its kernel is $\pm id$. We want to determine g such that

$$\frac{px_1+q}{rx_1+s} = i, \quad \frac{px_2+q}{rx_2+s} \in \mathbb{R}_+ i,$$

4 Sendov's Conjecture

Conjecture 1 (Sendov's Conjecture)**.** For a polynomial $f(z) = (z-r_1)(z-r_2)...(z-r_n), n \geq 2$ with all roots $r_1, r_2, ..., r_n$ inside the closed unit disk $\{|z| \leq 1\}$, each of the n roots is at a distance no more than 1 from at least one root of $f'(z)$.

It suffices to show that for a fixed r_1, the following distance function has maximum no more than 1.

$$d(r_2, ..., r_n) = \min |r_1 - \xi_i|, \quad f'(z) = (z-\xi_1)(z-\xi_2)...(z-\xi_{n-1}),$$

Two near counter-examples are

$$f_1(z) = z^n - 1, \quad r_1 = e^{\frac{2\pi i}{n}}, \quad f_2(z) = z^n - z, \quad r_1 = 0,$$

$n^{-\frac{1}{n-1}} = 1 - O(\frac{\log n}{n})$,