Alibaba math competition 2022 - Preliminaries

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Question 1. Consider all the possible shapes of the toy, what is the maximal distance between any two points on it?

Solution. Observing the topological structure of the toy, we know that both A and C_1 split into three points, B, C, D, A_1, B_1, D_1 don't split. The edges $PB, PC, PD, PA_1, PB_1, PD_1$ each splits into 3 edges, PA, PC_1 each splits into 6 edges. P splits into 12 points. A construction of local maxima with maximal distance $C_1^C A^{A_1} = 1 + \sqrt{6}$ is given below:

$$B_1(\frac{1}{\sqrt{2}}, \frac{1}{2}, 0), \quad B(\frac{1}{\sqrt{2}}, -\frac{1}{2}, 0), \quad A_1(0, \frac{1}{2} + \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}), \quad A^{A_1}(0, \frac{1}{2} + \frac{\sqrt{6}}{2}, 0),$$
$$D_1(-\frac{1}{\sqrt{2}}, \frac{1}{2}, 0), \quad D(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, 0), \quad C(0, -\frac{1}{2} - \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{3}}), \quad C_1^C(0, -\frac{1}{2} - \frac{\sqrt{6}}{2}, 0),$$
$$A^B(\frac{1}{\sqrt{2}} + \sqrt{\frac{2}{5}}, -\frac{1}{2}, -\sqrt{\frac{3}{5}}), \quad A^D(-\frac{1}{\sqrt{2}} - \sqrt{\frac{2}{5}}, -\frac{1}{2}, -\sqrt{\frac{3}{5}}),$$
$$C_1^{B_1}(\frac{1}{\sqrt{2}} + \sqrt{\frac{2}{5}}, \frac{1}{2}, \sqrt{\frac{3}{5}}), \quad C_1^{D_1}(-\frac{1}{\sqrt{2}} - \sqrt{\frac{2}{5}}, \frac{1}{2}, \sqrt{\frac{3}{5}}).$$

Another construction of local maxima which turns out to be the global maxima with maximal distance $C_1^C A^{A_1} = 1 + 2\sqrt{2}$ is given below:

$$B_1 = D_1(0, \frac{1}{2}, 0), \quad B = D(0, -\frac{1}{2}, 0), \quad A_1(0, \frac{1}{2} + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad A^{A_1}(0, \frac{1}{2} + \sqrt{2}, 0),$$
$$C(0, -\frac{1}{2} - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \quad C_1^C(0, -\frac{1}{2} - \sqrt{2}, 0),$$
$$A^B(1, -\frac{1}{2}, 0), \quad A^D(-1, -\frac{1}{2}, 0), \quad C_1^{B_1}(1, \frac{1}{2}, 0), \quad C_1^{D_1}(-1, \frac{1}{2}, 0).$$

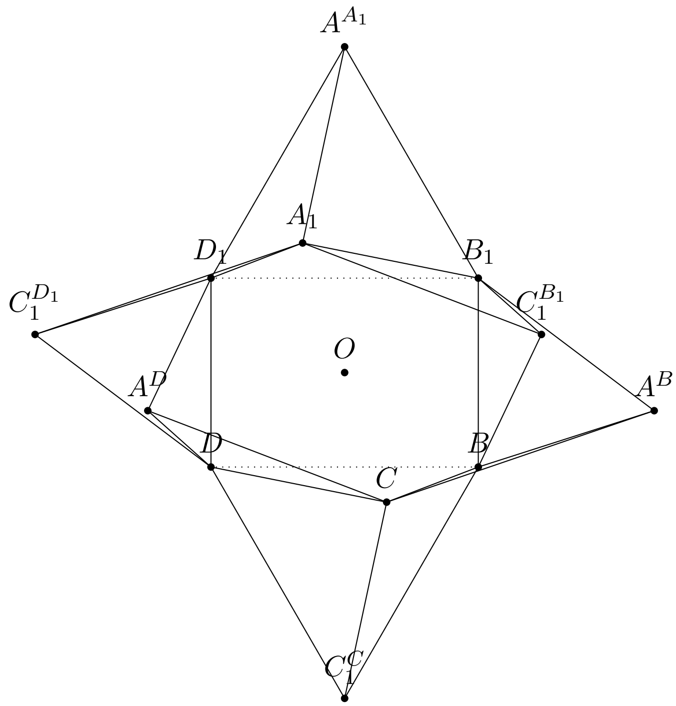
□

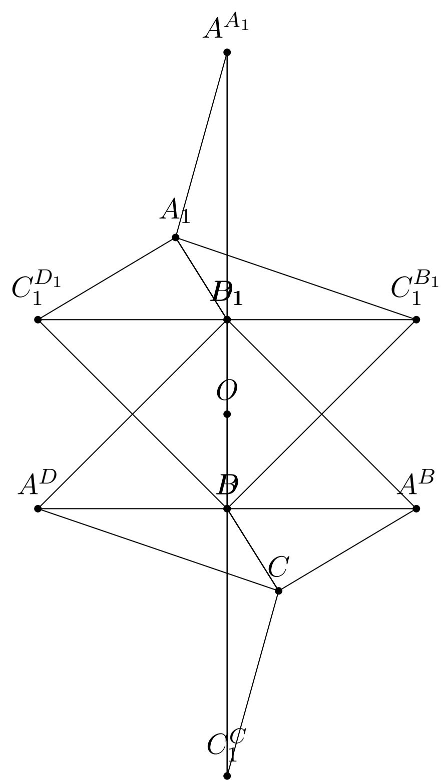
Question 2. Prove that there exists $c_1, c_2 > 0$, such that for any positive integer n, no matter how $A_1, A_2, ..., A_n$ choose their location, we always have $c_1\sqrt{n} \leq KP_n \leq c_2\sqrt{n}$.

Solution. □

Question 3. Prove that when there are 60 people, there exists such possibility that all the people can see the whole scene. But when there are 800 people, there must be someone that can't see the whole scene.

Solution. □





Question 4. What is the expectation of the milk packs needed to collect a whole collection of $HHSW$?

Solution.

$$p(stopping\ time = n) = p(not\ stops\ in\ n-1\ steps) - p(not\ stops\ in\ n\ steps),$$

$$Answer = \mathbb{E}(milk\ packs\ needed\ to\ collect\ HHSW) = \sum_{n\in\mathbb{N}} np(stopping\ time = n)$$

$$= \sum_{n\in\mathbb{N}} n(p(not\ stops\ in\ n-1\ steps) - p(not\ stops\ in\ n\ steps))$$

$$= \lim_{n\to+\infty}(\sum_{i=0}^{n-1} p(not\ stops\ in\ i\ steps) - np(not\ stops\ in\ n\ steps)).$$

$$p(not\ stops\ in\ n\ steps) = p(H\le 1) + p(S=0) + p(W=0)$$
$$-p(H\le 1, S=0) - p(H\le 1, W=0) - p(S=W=0) + p(H\le 1, S=W=0),$$
$$p(S=0) = p(W=0) = (\frac{2}{3})^n, \quad p(H\le 1) = p(H=0) + p(H=1) = (\frac{2}{3})^n + n\cdot\frac{1}{3}\cdot(\frac{2}{3})^{n-1}$$

$$p(H\le 1, S=0) = p(H=0,S=0) + p(H=1,S=0) = (\frac{1}{3})^n + n\cdot\frac{1}{3}\cdot(\frac{1}{3})^{n-1} = p(H\le 1, W=0),$$

$$p(S=W=0) = (\frac{1}{3})^n, \quad p(H\le 1, S=W=0) = \begin{cases} 1 & ,n=0,\\ \frac{1}{3} & ,n=1,\\ 0 & ,n\ge 2.\end{cases}$$

$$p(not\ stops\ in\ n\ steps) = ((\frac{2}{3})^n + n\cdot\frac{1}{3}\cdot(\frac{2}{3})^{n-1}) + (\frac{2}{3})^n + (\frac{2}{3})^n$$
$$-2\cdot((\frac{1}{3})^n + n\cdot\frac{1}{3}\cdot(\frac{1}{3})^{n-1}) - (\frac{1}{3})^n + \begin{cases} 1 & ,n=0,\\ \frac{1}{3} & ,n=1,\\ 0 & ,n\ge 2.\end{cases}$$

$$= 3\cdot(\frac{2}{3})^n + n\cdot\frac{1}{3}\cdot(\frac{2}{3})^{n-1} - 3\cdot(\frac{1}{3})^n - n\cdot\frac{2}{3}\cdot(\frac{1}{3})^{n-1} + \begin{cases} 1 & ,n=0,\\ \frac{1}{3} & ,n=1,\\ 0 & ,n\ge 2.\end{cases}$$

$$Answer = \sum_{n=0}^{+\infty} p(not\ stops\ in\ n\ steps) = 3\cdot\frac{1}{1-\frac{2}{3}} + \frac{1}{3}\cdot\frac{1}{(1-\frac{2}{3})^2} - 3\cdot\frac{1}{1-\frac{1}{3}}$$
$$-\frac{2}{3}\cdot\frac{1}{(1-\frac{1}{3})^2} + 1 + \frac{1}{3} = 9 + 3 - \frac{9}{2} - \frac{3}{2} + 1 + \frac{1}{3} = 7\frac{1}{3}.$$

□

Question 5. What are the values of p, q, r such that the expectation of the milk packs needed to collect a whole collection of $HHSW$ reaches minimum?

Solution. Assume $q = r$, so the constraint is $p + 2q = 1$.

$$p(S=0) = p(W=0) = (1-q)^n, \quad p(H\le 1) = p(H=0) + p(H=1) = (1-p)^n + n\cdot p(1-p)^{n-1},$$
$$p(H\le 1, S=0) = p(H=0,S=0) + p(H=1,S=0) = q^n + n\cdot pq^{n-1} = p(H\le 1, W=0),$$

$$p(S=W=0) = p^n, \quad p(H\le 1, S=W=0) = \begin{cases} 1 & ,n=0,\\ p & ,n=1,\\ 0 & ,n\ge 2.\end{cases}$$

$$p(not\ stops\ in\ n\ steps) = 2\cdot(1-q)^n + (1-p)^n + n\cdot p\cdot(1-p)^{n-1}$$

$$-p^n - 2\cdot(q^n + n\cdot pq^{n-1}) + \begin{cases} 1 & ,n=0, \\ \frac{1}{3} & ,n=1, \\ 0 & ,n\geq 2. \end{cases}$$

$$Answer = \sum_{n=0}^{+\infty} p(not\ stops\ in\ n\ steps) = 1\cdot\frac{1}{1-(1-q)} + \frac{1}{1-(1-p)} + p\cdot\frac{1}{(1-(1-p))^2}$$

$$-\frac{1}{1-p} - 2\cdot\frac{1}{1-q} - 2p\cdot\frac{1}{(1-q)^2} + 1 + p = \frac{2}{q} + \frac{2}{p} - \frac{1}{2q} - \frac{2}{1-q}(1+\frac{p}{1-q}) + 1 + p,$$

$$f(q) = Answer = \frac{3}{2q} + \frac{2}{1-2q} - \frac{2}{1-q} - \frac{2(1-2q)}{(1-q)^2} + 2 - 2q,$$

□

Question 6. Prove that left uniform is equivalent to right uniform.

Solution. □

Question 7. 1) Prove that $s_{[W]}$ doesn't depend on the orthonormal basis $\{v_1, ..., v_k\}$ that we choose.
2) Prove that $s_{[W]}^2 = id$.
3) Find the maximal number of elements of an interesting set in $Gr_k(V)$, and prove it.

Solution. □

Question 8. 1) Let $P_{1,n}$ be the probability that the distance between the salesman and the origin is larger than $\frac{n}{2}$ after $\lfloor n^{1.5}\rfloor$ steps. Prove that $\lim_{n\to+\infty} P_{1,n} = 1$.
2) Let $P_{2,n}$ be the probability that the salesman has returned to the origin in $\lfloor n^{1.5}\rfloor$ steps. Prove that $\lim_{n\to+\infty} P_{2,n} = 0$.
3) Let $P_{3,n}$ be the probability that the salesman has returned to the origin in 2^n steps. Prove that $\lim_{n\to+\infty} P_{3,n} = 1$.

Solution. 1) The Hoeffding's inequality is stated as follows: let $X_1, ..., X_n$ be independent random variables, $a_i \leq X_i \leq b_i$ almost surely. Consider the sum of these random variables $S_n = X_1 + ... + X_n$, then for any $t \geq 0$,

$$Pr(S_n - \mathbb{E}(S_n) \geq t) \leq \exp(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}).$$

In our case, we want to get a tail upper bound for binomial distribution. Let $p\in[0,1], n\geq 1$, X_i be Bernoulli random variables with expected value p for any $1\leq i\leq n$. $F(k;n,p) = Pr(S_n\leq k) = \sum_{i=0}^k C_n^i p^i (1-p)^{n-i}$, then by Hoeffding's inequality, for $k\leq np$, we have

$$F(k;n,p) = Pr(S_n\leq k) = Pr(np - S_n \geq np-k) \leq \exp(-\frac{2(np-k)^2}{\sum_{i=1}^n (1-0)^2}) = e^{-2n(p-\frac{k}{n})^2}$$

Suppose $x_k\in\mathbb{Z}^2$ is a random variable indicating the position of the salesman after k steps. So we have

$$Pr(|x_k - n| < \sigma) = (\frac{1}{4})^k \sum_l C_k^l \sum_{\frac{l-\sigma}{2} < x < \frac{l+\sigma}{2}} C_l^x 2^{k-l},$$

$$Pr(|x_k - n| \geq \sigma) = (\frac{1}{4})^k \sum_l C_k^l \sum_{|x-\frac{l}{2}|\geq\frac{\sigma}{2}} C_l^x 2^{k-l},$$

$$\sum_{|x-\frac{l}{2}|\geq\frac{\sigma}{2}} C_l^x = Pr(|S_l|\geq\sigma) \leq 2e^{-\frac{\sigma^2}{2l}},$$

Let $k=\lfloor n^{1.5}\rfloor, \sigma=\frac{n}{2}$, we get

$$Pr(|x_k-n|\geq\sigma) = (\frac{1}{2})^k \sum_{0\leq l\leq k} C_k^l \cdot 2e^{-\frac{\sigma^2}{2l}} = (\frac{1}{2})^k \sum_{0\leq l\leq k} C_k^l \cdot 2e^{-\frac{n^2}{8l}} \leq 2e^{-\frac{n^2}{8k}} \leq 2e^{-\frac{\sqrt{n}}{8}},$$

$$Pr(|x_k|>\frac{n}{2}) \geq 1-Pr(|x_k-n|\geq\frac{n}{2}) \geq 1-2e^{-\frac{\sqrt{n}}{8}}.$$

Letting $n\to+\infty$, the inequality above shows that $P_{1,n}=Pr(|x_k|>\frac{n}{2})\to 1$.
2) We may consider a simpler case of 1-dimensional random walk:

$$p(t+1,n)=\frac{1}{2}(p(t,n-1)+p(t,n+1)), \quad p(0,0)=1, \quad p(0,k)=0, k\neq 0$$

Goal is to estimate $p(t,n)=\begin{cases} C_t^{\frac{t+n}{2}}(\frac{1}{2})^t, 2\mid t+n, \\ 0, 2\nmid t+n \end{cases}$ for any $t\geq|n|$. De Moivre-Laplace theorem says that as n grows large, for k in the neighbourhood of np, we have

$$C_n^k p^k q^{n-k} \simeq \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(k-np)^2}{2npq}}, \quad p+q=1, \quad p,q>0.$$

In our case $p=q=\frac{1}{2}$, and it becomes $C_t^{\frac{t+n}{2}}(\frac{1}{2})^t \simeq \frac{2}{\sqrt{2\pi t}}e^{-\frac{n^2}{2t}}$. For 1-dimensional random walk, our question is: how to estimate the gap between LHS and RHS above for arbitrary t,n satisfying $2\mid t+n, t\geq|n|$? Information theoretical bounds for binomial coefficients:

$$\frac{1}{n+1}2^{nH(\frac{k}{n})} \leq \binom{n}{k} \leq 2^{nH(\frac{k}{n})}, \quad H(p)=-p\log_2 p-(1-p)\log_2(1-p), \quad 0\leq k\leq n.$$

$$\frac{1}{\sqrt{8npq}} \leq \binom{n}{k}2^{-nH(\frac{k}{n})} \leq \frac{1}{\sqrt{\pi npq}}, \quad p=\frac{k}{n}, q=1-p, \quad 1\leq k\leq n-1.$$

In 2-dimensional random walk of the salesman,

$$Pr(x_l=(0,0)) = (\frac{1}{4})^l \sum_p C_l^p C_p^{\frac{p-n}{2}} C_{l-p}^{\frac{l-p}{2}}$$

We may apply the 2-dimensional central limit theorem: X_i are independent random variables indicating a single step of the salesman, $\mu(X_i)=(0,0), \Sigma=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}, S_l=X_1+...+X_l$, then we have

$$\frac{S_l}{\sqrt{l}} \to \mathcal{N}((0,0),\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}), \quad S_l \to \mathcal{N}((0,0),\begin{pmatrix}\frac{l}{2} & 0\\ 0 & \frac{l}{2}\end{pmatrix}),$$

$$Pr(x_l=(0,0)) = Pr(S_l=(-n,0)) \simeq \frac{2}{2\pi\frac{l}{2}}\exp(-\frac{1}{2}(-n\ 0)\begin{pmatrix}\frac{2}{l} & 0\\ 0 & \frac{2}{l}\end{pmatrix}\begin{pmatrix}-n\\ 0\end{pmatrix}) = \frac{2}{\pi l}e^{-\frac{n^2}{l}}$$

For 2-dimensional random walk, our question is to estimate the gap between LHS and RHS above for arbitrary l,n, and show that $\sum_{1\leq l\leq k} Pr(x_l=(0,0))\to 0$ when $n\to+\infty$. The probability density function of multivariate normal distribution $\mathcal{N}(\mu,\Sigma)$ is given by

$$p(x)=\frac{\exp(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu))}{\sqrt{(2\pi)^k|\Sigma|}}$$

3) $k = 2^n$, The equation below is wrong!

$$1 - P_{3,n} = \prod_{n \le l \le k} (1 - Pr(x_l = (0,0))),$$

When $n \to +\infty$, we have

$$\log(1 - P_{3,n}) \sim -\sum_{n \le l \le k} Pr(x_l = (0,0)) \sim -\sum_{n \le l \le k} \frac{2}{\pi l} e^{-\frac{n^2}{l}}$$
$$\le -\sum_{n^2 \le l \le k} \frac{2}{\pi l} e^{-1} \sim -\frac{2e^{-1}}{\pi}(n \log 2 - 2\log n) \to -\infty.$$

So $P_{3,n} \to 1$. The right equation should be

$$1 - P_{3,n} = Pr(x_1, x_2, ..., x_k \ne (0,0))$$

We may define as follows and have

$$q_l = Pr(x_1, x_2, ...x_l \ne (0,0)), \quad r_l = Pr(x_1, x_2, ...x_{l-1} \ne (0,0), x_l = (0,0)),$$
$$q_{l-1} = q_l + r_l, \quad 1 = q_0 = q_k + \sum_{1 \le l \le k} r_l, \quad P_{3,n} = 1 - q_k = \sum_{1 \le l \le k} r_l,$$

Let us consider one dimensional random walk for heuristic. $x_0 = n > 0, x_l = x_{l-1} \pm 1$, $q_l(n) = Pr(x_0 = n, x_1, x_2, ...x_l \ne 0)$. Let $q_l(n,m) = Pr(x_0 = n, x_1, x_2, ...x_{l-1} \ne 0, x_l = m), m > 0$, $p_l(n,m) = Pr(x_0 = n, x_l = m) = \binom{l}{\frac{l+n-m}{2}}$. Let x' be a one dimensional random walk starting from $x'_0 = -n$, $p'_l(n,m) = Pr(x'_0 = -n, x'_l = m) = \binom{l}{\frac{l-n-m}{2}}$. Then we have $q_l(n,m) = p_l(n,m) - p'_l(n,m)$, so

$$q_l(n) = \sum_{m>0} q_l(n,m) = \sum_{m>0} p_l(n,m) - p'_l(n,m) = \sum_{m>0} \binom{l}{\frac{l+n-m}{2}} - \binom{l}{\frac{l-n-m}{2}},$$

Notice that when $n = m = 1, l = 2a$, it reduces to

$$q_l(n,m) = \binom{2a}{a} - \binom{2a}{a-1} = \binom{2a}{a}(1 - \frac{a}{a+1}) = \frac{1}{a+1}\binom{2a}{a},$$

which is the Catalan number. When $n = 1, m = 2, l = 2a+1$, it reduces to

$$q_l(n,m) = \binom{2a+1}{a} - \binom{2a+1}{a-1} = \binom{2a+1}{a}(1 - \frac{a}{a+2}) = \frac{2}{a+2}\binom{2a+1}{a},$$

More generally, when $n = 1, l = 2a, m = 2b-1$,

$$q_l(n,m) = \binom{2a}{a+1-b} - \binom{2a}{a-b} = \binom{2a}{a+1-b}(1 - \frac{a+1-b}{a+b}) = \frac{2b-1}{a+b}\binom{2a}{a+1-b},$$

When $n = 1, l = 2a+1, m = 2b$,

$$q_l(n,m) = \binom{2a+1}{a+1-b} - \binom{2a+1}{a-b} = \binom{2a+1}{a+1-b}(1 - \frac{a-b+1}{a+b+1}) = \frac{2b}{a+b+1}\binom{2a+1}{a+1-b},$$

The identities above can be summarized as, when $n = 1$,

$$q_l(n,m) = \frac{l!m}{(\frac{l+1-m}{2})!(\frac{l+1+m}{2})!},$$

When $n=2, l=2a, m=2b$,

$$
\begin{aligned}
q_l(n,m) &= \binom{2a}{a+1-b} - \binom{2a}{a-1-b} = \frac{(2a)!}{(a+1-b)!(a-1+b)!} - \frac{(2a)!}{(a-1-b)!(a+1+b)!} \\
&= \frac{(2a)!}{(a+1-b)!(a-1+b)!}\left(1-\frac{(a+1-b)(a-b)}{(a+1+b)(a+b)}\right) \\
&= \frac{(2a)!}{(a+1-b)!(a-1+b)!}\frac{2b(2a+1)}{(a+1+b)(a+b)} = \frac{(2a+1)!2b}{(a+1-b)!(a+1+b)!}
\end{aligned}
$$

Similarly, when $n=2, l=2a+1, m=2b-1$,

$$
\begin{aligned}
q_l(n,m) &= \binom{2a+1}{a+2-b} - \binom{2a+1}{a-b} = \frac{(2a+1)!}{(a+2-b)!(a-1+b)!} - \frac{(2a+1)!}{(a-b)!(a+1+b)!} \\
&= \frac{(2a+1)!}{(a+2-b)!(a-1+b)!}\left(1-\frac{(a+2-b)(a+1-b)}{(a+1+b)(a+b)}\right) \\
&= \frac{(2a+1)!}{(a+2-b)!(a-1+b)!}\frac{(2b-1)(2a+2)}{(a+1+b)(a+b)} = \frac{(2a+2)!(2b-1)}{(a+2-b)!(a+1+b)!}
\end{aligned}
$$

These identities can be summarized as

$$
q_l(n,m) = \frac{(l+1)!m}{(\frac{l+2-m}{2})!(\frac{l+2+m}{2})!},
$$

When $n=3, l=2a+1, m=2b$,

$$
\begin{aligned}
q_l(n,m) &= \binom{2a+1}{a+2-b} - \binom{2a+1}{a-1-b} = \frac{(2a+1)!}{(a+2-b)!(a-1+b)!} - \frac{(2a+1)!}{(a-1-b)!(a+2+b)!} \\
&= \frac{(2a+1)!}{(a+2-b)!(a-1+b)!}\left(1-\frac{(a+2-b)(a+1-b)(a-b)}{(a+2+b)(a+1+b)(a+b)}\right) \\
&= \frac{(2a+1)!}{(a+2-b)!(a-1+b)!}\frac{2b((a+2)(a+1)+(a+2)a+(a+1)a)+2b^3}{(a+2+b)(a+1+b)(a+b)} \\
&= \frac{(2a+1)!2b(3a^2+6a+2+b^2)}{(a+2-b)!(a+2+b)!},
\end{aligned}
$$

Clearly $q_l(n,m)=q_l(m,n)$, but I don't think there is a short expression for arbitrary $n\geq 3, m>0$.

$$
q_l(n,m) = \binom{l}{\frac{l+n-m}{2}} - \binom{l}{\frac{l-n-m}{2}} = \frac{l!}{(\frac{l+n-m}{2})!(\frac{l-n+m}{2})!} - \frac{l!}{(\frac{l-n-m}{2})!(\frac{l+n+m}{2})!}
$$

$$
q_l(n) = \sum_{m>0} q_l(n,m) = \sum_{m>0} \binom{l}{\frac{l+n-m}{2}} - \binom{l}{\frac{l-n-m}{2}} = \sum_{-n\leq t<n} \binom{l}{\frac{l+t}{2}}
$$

and we see that when $l>n^{1+\epsilon}, n\to+\infty$, the right hand side goes to zero. What about the two-dimensional case? Let $f(t,x)$ be the probability of salesman at position x after t steps without visiting $(0,0)$. $\widehat{f}(t,\xi)$ is the characteristic function of f with $\xi\in\mathbb{T}^2, \mathbb{T}=\mathbb{R}/\mathbb{Z}$.

$$
\begin{aligned}
&\widehat{f}(t,\xi) = \sum f(t,x)e^{-2\pi i x\cdot\xi} \\
=&\sum_{x\neq(0,0)} \frac{1}{4}(f(t-1,x+e_1)+f(t-1,x-e_1)+f(t-1,x+e_2)+f(t-1,x-e_2))e^{-2\pi i x\cdot\xi} \\
=&P_{\neq 0}\widehat{f}(t-1,\xi)\frac{e^{-2\pi i\xi_1}+e^{2\pi i\xi_1}+e^{-2\pi i\xi_2}+e^{2\pi i\xi_2}}{4} = P_{\neq 0}\frac{\cos(2\pi\xi_1)+\cos(2\pi\xi_2)}{2}\widehat{f}(t-1,\xi),
\end{aligned}
$$

We see that the right hand side above is the composition of a multiplier and a projection. Define

$$\mathcal{H} = \{f \in L^2(\mathbb{T}^2), \int_{\mathbb{T}^2} f = 0\} \subset L^2(\mathbb{T}^2), \quad \langle u, v\rangle = \int_{\mathbb{T}^2} u\overline{v},$$

$$M = \frac{\cos(2\pi\xi_1) + \cos(2\pi\xi_2)}{2}, \quad Tf = P_{\neq 0}Mf, \quad \langle u, Tv\rangle = \langle u, Mv\rangle = \langle Mu, v\rangle = \langle Tu, v\rangle,$$

$$\int uP_{\neq 0}\overline{v} = \int u(\overline{v} - \overline{v_0}) = \int u\overline{v} - u_0\overline{v_0} = \int (u - u_0)\overline{v},$$

so T is a self adjoint operator on $\mathcal{H}$, and $\|T\| \le 1$. □

Question 9. 1) Prove that there exists no periodic sequence $a_1, a_2, a_3, ...$, all the terms are ± 1, such that for any $\theta \in \mathbb{Q}$,

$$\sup_{N\in\mathbb{N}} |\sum_{n=1}^{N} a_n e^{2\pi in\theta}| < +\infty.$$

2) Prove that there exists no sequence $a_1, a_2, a_3, ...$, all the terms are ± 1, such that for any $\theta \in \mathbb{Q}$,

$$\sup_{N\in\mathbb{N}} |\sum_{n=1}^{N} a_n e^{2\pi in\theta}| < 2022.$$

3) Give an example of a sequence $a_1, a_2, a_3, ...$, all the terms are ± 1, such that there are infinitely many 1 and infinitely many -1, and for any $\theta \in \mathbb{Q}\backslash\mathbb{Z}$,

$$\sup_{N\in\mathbb{N}} |\sum_{n=1}^{N} a_n e^{2\pi in\theta}| < +\infty.$$

Solution. 1) Denote $f(N, \theta) = \sum_{n=1}^{N} a_n e^{2\pi in\theta}$, we may let θ to be any real number. Note that for any fixed $N \in \mathbb{N}$, $f(N, \theta)$ has a period 1 as a function of θ.
2) Suppose on the contrary that such sequence exists and it is $\{a_n\}_{n\ge 1}$. Let $N = 5000000 > 2022^2$, since $\mathbb{Q}\cap[0, 1]$ is dense in $[0, 1]$, and $f(N, \theta)$ is continuous as a function of θ, we know that $f(N, \theta) < 2022$ holds for any $\theta \in [0, 1]$. So

$$\int_0^1 |f(N, \theta)|^2 d\theta < \int_0^1 2022^2 d\theta = 2022^2.$$

But by Plancherel formula,

$$\int_0^1 |f(N, \theta)|^2 d\theta = \sum_{n=1}^{N} a_n^2 = N > 2022^2.$$

Contradiction! We conclude that no such sequence exists. □

Question 10.

$$\rho_t + ((u(t) - v)\rho)_v = \rho_{vv}, \quad u(t) = u_0 + u_1 N(t), \quad N(t) = \int_0^{+\infty} v\rho dv,$$

$$p_t + vp_x + ((u(t) - v)p)_v = p_{vv}, \quad p(t, 0, v) = p(t, 2\pi, v)$$

1) Prove that if $u_1 > 1$, then $N(t)$ is unbounded from above in the evolution process.
2) Prove that no matter what is the initial distribution, the UAV approaches a uniform distribution on the circle shortly.

Solution. 1) Let $N_1(t) = \int_{-\infty}^{+\infty} v\rho(t,v)dv$, we know that $\int_{-\infty}^{+\infty} \rho(t,v)dv = 1$ and $N(t) \geq N_1(t)$ holds for all $t \geq 0$.

$$\begin{aligned}\frac{dN_1(t)}{dt} &= \int_{-\infty}^{+\infty} v\rho_t dv = \int_{-\infty}^{+\infty} v(\rho_{vv} + ((v-u(t))\rho)_v)dv = \int_{-\infty}^{+\infty} vd\rho_v + \int_{-\infty}^{+\infty} vd((v-u(t))\rho) \\ &= -\int_{-\infty}^{+\infty} \rho_v dv - \int_{-\infty}^{+\infty} (v-u(t))\rho dv = 0 + \int_{-\infty}^{+\infty} \rho(u_0 + u_1 N(t) - v)dv \\ &= -N_1(t) + u_0 + u_1 N(t) \geq u_0 + (u_1 - 1)N(t) \geq u_0.\end{aligned}$$

So $N_1(t) \geq N_1(0) + u_0 t$ and we may pick $t_0 \geq \max(0, \frac{1-N_1(0)}{u_0})$, then $N_1(t_0) \geq N_1(0) + u_0 t_0 \geq 1$. The above inequality gives $\frac{dN_1(t)}{dt} \geq u_0 + (u_1-1)N(t) \geq u_0 + (u_1-1)N_1(t)$, so for $t > t_0$ we have

$$\frac{de^{(1-u_1)t}N_1(t)}{dt} = e^{(1-u_1)t}(\frac{dN_1(t)}{dt} + (1-u_1)N_1(t)) \geq e^{(1-u_1)t}u_0,$$

$$e^{(1-u_1)t}N_1(t) - e^{(1-u_1)t_0}N_1(t_0) \geq u_0 \int_{t_0}^{t} e^{(1-u_1)t'}dt' = \frac{u_0}{u_1-1}(e^{(1-u_1)t_0} - e^{(1-u_1)t}),$$

$$e^{(1-u_1)(t-t_0)}N_1(t) \geq N_1(t_0) + \frac{u_0}{u_1-1}(1 - e^{(1-u_1)(t-t_0)}) \geq 1,$$

$$N_1(t) \geq e^{(u_1-1)(t-t_0)}.$$

2) We may adjust the circle to have perimeter 1 using the transformation $\tilde{p}(t,\tilde{x},v) = 2\pi p(t,x,v), \tilde{x} = 2\pi x$. Now $p(t,x,v) = \frac{1}{2\pi}\tilde{p}(t,\tilde{x},v)$, $\partial_x p(t,x,v) = \frac{1}{(2\pi)^2}\partial_{\tilde{x}}\tilde{p}(t,\tilde{x},v)$, so the equation becomes

$$\tilde{p}_t + \frac{1}{2\pi}v\tilde{p}_{\tilde{x}} + ((u(t)-v)\tilde{p})_v = \tilde{p}_{vv}, \quad \tilde{p}(t,0,v) = \tilde{p}(t,1,v).$$

We use p, x instead of $\tilde{p}, \tilde{x}$ below. Let the scaled moments of p with respect to v be $q_n(t,x) = \int_{-\infty}^{+\infty} \frac{v^n}{n!}p(t,x,v)dv$, then we have:

$$\begin{aligned}n!\partial_t q_n &= \int_{-\infty}^{+\infty} v^n p_t dv = \int_{-\infty}^{+\infty} v^n(-\frac{1}{2\pi}vp_x - ((u(t)-v)p)_v + p_{vv})dv \\ &= -\frac{(n+1)!}{2\pi}\partial_x q_{n+1} + n\int_{-\infty}^{+\infty} v^{n-1}(u(t)-v)pdv - n\int_{-\infty}^{+\infty} v^{n-1}p_v dv \\ &= -\frac{(n+1)!}{2\pi}\partial_x q_{n+1} - n\cdot n!q_n + n!u(t)q_{n-1} + n!q_{n-2}.\end{aligned}$$

$$\partial_t q_n = -\frac{n+1}{2\pi}\partial_x q_{n+1} - nq_n + u(t)q_{n-1} + q_{n-2}$$

Let $\widehat{q_n}(t,k) = \int_0^1 q_n(t,x)e^{-2\pi ikx}dx$ be the Fourier transform of q_n with respect to x and define $\widehat{q_{-1}} = \widehat{q_{-2}} = 0$, then we have:

$$\partial_x\widehat{q_n}(t,k) = 2\pi ik\widehat{q_n}(t,k), \quad \partial_t\widehat{q_n} = -ik(n+1)\widehat{q_{n+1}} - n\widehat{q_n} + u(t)\widehat{q_{n-1}} + \widehat{q_{n-2}}$$

Our goal is to show that $q_0(t,x) \to 1$ holds for any $x \in [0,1]$ as t increases. Since $q_0(t,x) = \sum_{k\in\mathbb{Z}}\widehat{q_0}(t,k)e^{2\pi ikx}$, it suffices to show that

$$\widehat{q_n}(t,k) = \int_0^1 q_n(t,x)e^{-2\pi ikx}dx = \int_0^1\int_{-\infty}^{+\infty}\frac{v^n}{n!}p(t,x,v)e^{-2\pi ikx}dvdx$$

$$q_n(t,x) = \sum_{k\in\mathbb{Z}} \widehat{q_n}(t,k)e^{2\pi ikx} = \int_{-\infty}^{+\infty} \frac{v^n}{n!}p(t,x,v)dv$$

Denote $\widehat{Q}(t,k) = (\widehat{q_0}, \widehat{q_1}, ..., \widehat{q_n}, ...)(t,k)$. We use the ℓ_1 norm for the same mode of all moments. Let $\partial_t\widehat{Q}(t,k) = A(t,k)\widehat{Q}(t,k)$ where $A(t,k)$ is an unbounded linear operator from ℓ_1 to itself.

$$\begin{aligned}
&|\widehat{Q}(t,k)|_1 = \sum_{n\ge0} |\widehat{q_n}(t,k)| = \sum_{n\ge0} |\int_0^1 \int_{-\infty}^{+\infty} \frac{v^n}{n!}p(t,x,v)e^{-2\pi ikx}dvdx| \\
&\le \sum_{n\ge0} \int_0^1 \int_{-\infty}^{+\infty} |\frac{v^n}{n!}p(t,x,v)|dvdx = \int_0^1 \int_{-\infty}^{+\infty} \sum_{n\ge0} \frac{|v|^n}{n!}p(t,x,v)dvdx \\
&= \int_0^1 \int_{-\infty}^{+\infty} e^{|v|}p(t,x,v)dvdx < +\infty.
\end{aligned}$$

Then we may estimate the ℓ_1 norm of $A(t,k)\widehat{Q}(t,k)$ as follows:

$$\begin{aligned}
&|A(t,k)\widehat{Q}(t,k)|_1 = \sum_{n\ge0} |\partial_t\widehat{q_n}(t,k)| = \sum_{n\ge0} |-ik(n+1)\widehat{q_{n+1}} - n\widehat{q_n} + u(t)\widehat{q_{n-1}} + \widehat{q_{n-2}}| \\
&\le \sum_{n\ge0} |ik(n+1)\widehat{q_{n+1}}| + |n\widehat{q_n}| + |u(t)\widehat{q_{n-1}}| + |\widehat{q_{n-2}}| \\
&\le \int_0^1 \int_{-\infty}^{+\infty} \sum_{n\ge0} (k(n+1)\frac{|v|^{n+1}}{(n+1)!} + n\frac{|v|^n}{n!} + u(t)\frac{|v|^{n-1}}{(n-1)!} + \frac{|v|^{n-2}}{(n-2)!})p(t,x,v)dvdx \\
&= \int_0^1 \int_{-\infty}^{+\infty} ((k+1)|v| + u(t) + 1)e^{|v|}p(t,x,v)dvdx < +\infty.
\end{aligned}$$

We want to analyse the spectrum of $A(t,k)$. Our method is to estimate its resolvent. Let $\hat{p}(t,k,v) = \int_0^1 p(t,x,v)e^{-2\pi ikx}dx$,

$$((\lambda - A(t,k))Q(t,k))_n = ik(n+1)\widehat{q_{n+1}} + (\lambda+n)\widehat{q_n} - u(t)\widehat{q_{n-1}} - \widehat{q_{n-2}}$$

Question: How to divide the spectrum of $A(t,k)$ into discrete, essential, point, continuous and residual spectrums? What are the eigenspace corresponding to each eigenvalue? Is the collection of all its eigenvectors complete? I can compute the point spectrum of $A(t,k)$ here. We set $k=1$,

$$-i(n+1)\widehat{q_{n+1}} - n\widehat{q_n} + u(t)\widehat{q_{n-1}} + \widehat{q_{n-2}} = \lambda\widehat{q_n}$$

$$q_0 = 1, \quad q_1 = i\lambda, \quad q_2 = \frac{i}{2}((\lambda+1)q_1 - u) = -\frac{(\lambda+1)\lambda}{2} - \frac{iu}{2}$$

$$q_{n+1} = \frac{i}{n+1}((\lambda+n)q_n - u(t)q_{n-1} - q_{n-2})$$

We propose another question for heuristic: $q_0 = 1, q_{n+1} = \frac{\lambda+n}{n+1}q_n = \frac{(\lambda+n)...\lambda}{(n+1)!}$, find all the values of $\lambda \in \mathbb{C}$ such that the ℓ_1 norm $\sum_{n\ge0}|q_n|$ is finite.

$$\begin{aligned}
&\log|q_{n+1}|^2 = \sum_{0\le k\le n} \log|1 + \frac{\lambda-1}{k+1}|^2 = \sum_{0\le k\le n} \log\overline{(1 + \frac{\lambda-1}{k+1})}(1 + \frac{\lambda-1}{k+1}) \\
&= \sum_{0\le k\le n} \log(1 + \frac{2\Re(\lambda-1)}{k+1} + \frac{|\lambda-1|^2}{(k+1)^2}) \sim \sum_{1\le k\le n+1} (\frac{2\Re(\lambda-1)}{k} + \frac{|\lambda-1|^2}{k^2}) \\
&\sim 2\Re(\lambda-1)\log(n+1) + |\lambda-1|^2C, \qquad |q_n| \sim Cn^{\Re(\lambda-1)}
\end{aligned}$$

So the condition for $\lambda \in \mathbb{C}$ is that $\Re(\lambda) < 0$. Return to the original question, we expect a similar result that $\sigma_p(A(t,1)) \subset \{\lambda \in \mathbb{C}, \Re(\lambda) < 0\}$. For the sake of simplicity, we first consider the case $u(t) = 0$, and let $q_n = i^n \tilde{q}_n$, $\tilde{q}_0 = 1$.

$$q_{n+1} = \frac{i}{n+1}((\lambda+n)q_n - q_{n-2}), \quad \tilde{q}_{n+1} = \frac{1}{n+1}((\lambda+n)\tilde{q}_n + \tilde{q}_{n-2}),$$

Assume $\tilde{q}_n \sim Cn^\alpha$, we have

$$\begin{aligned}\frac{1}{n+1}((\lambda+n)\tilde{q}_n + \tilde{q}_{n-2}) &\sim \frac{\lambda+n}{n+1}Cn^\alpha + \frac{C}{n+1}(n-2)^\alpha \\ &\sim Cn^\alpha(1+\frac{\lambda-1}{n+1}+\frac{1}{n+1}) \sim Cn^\alpha(1+\frac{\lambda}{n}),\end{aligned}$$

$$C(n+1)^\alpha = Cn^\alpha(1+\frac{1}{n})^\alpha \sim Cn^\alpha(1+\frac{\alpha}{n}).$$

So heuristically we have $\alpha = \lambda$. We expect that if $\sum_{n\geq 0}|\tilde{q}_n|$ is finite, then $\Re(\lambda) < -1$.

Proof. Case 1: $\lambda \in \mathbb{R}, \lambda > 0$, we show that $\tilde{q}_n \geq (n+1)^\alpha > 0$ for $\alpha = \frac{\lambda}{2}$,

$$\tilde{q}_{n+1} = \frac{\lambda+n}{n+1}\tilde{q}_n + \frac{\tilde{q}_{n-2}}{n+1} \geq \frac{\lambda+n}{n+1}(n+1)^\alpha + \frac{(n-1)^\alpha}{n+1} = (n+1)^\alpha(1+\frac{\lambda-1}{n+1}+\frac{1}{n+1}(1-\frac{2}{n+1})^\alpha),$$

$$(1-\frac{2}{n+1})^\alpha \geq 1-\frac{3\alpha}{n+1}, \quad 1+\frac{\lambda-1}{n+1}+\frac{1}{n+1}(1-\frac{2}{n+1})^\alpha \geq 1+\frac{\lambda}{n+1}-\frac{3\alpha}{(n+1)^2},$$

$$(1+\frac{1}{n+1})^\alpha \leq 1+\frac{3\alpha}{2(n+1)}, \quad \alpha(\frac{3}{2(n+1)}+\frac{3}{(n+1)^2}) = \lambda(\frac{3}{4(n+1)}+\frac{3}{2(n+1)^2}) \leq \frac{\lambda}{n+1}$$

$$1+\frac{\lambda}{n+1}-\frac{3\alpha}{(n+1)^2} \geq 1+\frac{3\alpha}{2(n+1)}, \qquad \tilde{q}_{n+1} \geq (n+2)^\alpha.$$

Actually $\tilde{q}_n \geq C > 0$ when $n \geq 2$ is enough. Case 2: $\lambda \in \mathbb{R}, -\frac{1}{2} \leq \lambda \leq 0$, we show that $\tilde{q}_n \geq C(n+1)^\alpha$ for sufficiently large n and $\alpha = 2\lambda$.

$$\tilde{q}_{n+1} = \frac{\lambda+n}{n+1}\tilde{q}_n + \frac{\tilde{q}_{n-2}}{n+1} \geq C(n+1)^\alpha(1+\frac{\lambda-1}{n+1}+\frac{1}{n+1}(1-\frac{2}{n+1})^\alpha),$$

$$(1-\frac{2}{n+1})^\alpha \geq 1, \quad 1+\frac{\lambda-1}{n+1}+\frac{1}{n+1}(1-\frac{2}{n+1})^\alpha \geq 1+\frac{\lambda}{n+1},$$

$$(1+\frac{1}{n+1})^\alpha \leq 1+\frac{\alpha}{2(n+1)}, \quad \tilde{q}_{n+1} \geq C(n+2)^\alpha.$$

Case 3: $\lambda \in \mathbb{R}, \lambda < -2$, we show that $|\tilde{q}_n| \leq C(n+1)^\alpha$ for $n \geq 2, \alpha = 2\lambda$.

$$|\tilde{q}_{n+1}| = |\frac{\lambda+n}{n+1}\tilde{q}_n + \frac{\tilde{q}_{n-2}}{n+1}| \leq C(n+1)^\alpha(1+\frac{\lambda-1}{n+1}+\frac{1}{n+1}(1-\frac{2}{n+1})^\alpha),$$

$$(1-\frac{2}{n+1})^\alpha \leq 1-\frac{3\alpha}{n+1}, \quad 1+\frac{\lambda-1}{n+1}+\frac{1}{n+1}(1-\frac{2}{n+1})^\alpha \leq 1+\frac{\lambda}{n+1}-\frac{3\alpha}{(n+1)^2},$$

$$(1+\frac{1}{n+1})^\alpha \geq 1+\frac{\alpha}{n+1}, \quad 1+\frac{\lambda}{n+1}-\frac{3\alpha}{(n+1)^2} \leq 1+\frac{\alpha}{n+1},$$

$\square$

$$|(\lambda - A(t,k))\widehat{Q}(t,k)|_1 = \sum_{n\geq 0} |ik(n+1)\widehat{q_{n+1}} + (\lambda+n)\widehat{q_n} - u(t)\widehat{q_{n-1}} - \widehat{q_{n-2}}|$$

$$= \sum_{n\geq 0} |\int_0^1 \int_{-\infty}^{+\infty} (ik(n+1)\frac{v^{n+1}}{(n+1)!} + (\lambda+n)\frac{v^n}{n!} - u(t)\frac{v^{n-1}}{(n-1)!} - \frac{v^{n-2}}{(n-2)!})p(t,x,v)e^{-2\pi ikx}dvdx|$$

$$= \sum_{n\geq 0} |\int_{-\infty}^{+\infty} (ik(n+1)\frac{v^{n+1}}{(n+1)!} + (\lambda+n)\frac{v^n}{n!} - u(t)\frac{v^{n-1}}{(n-1)!} - \frac{v^{n-2}}{(n-2)!})\hat{p}(t,k,v)dv|.$$

By Hille-Yosida theorem, $A(t,k)$ is the generator of a contraction semigroup if and only if

$$(0,+\infty) \subset \rho(A(t,k)), \quad \|R_\lambda\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

So we need to find a norm such that $\|(\lambda - A(t,k))\widehat{Q}(t,k)\| \geq \lambda\|Q(t,k)\|$ holds for any $\lambda \geq 0$.

$$\langle(\lambda - A)Q, (\lambda - A)Q\rangle = \lambda^2\langle Q,Q\rangle - \lambda(\langle AQ,Q\rangle + \langle Q,AQ\rangle) + \langle AQ,AQ\rangle \geq \lambda^2\langle Q,Q\rangle,$$

$$\langle AQ,Q\rangle + \langle Q,AQ\rangle \leq 0,$$

When the inner product is trivial:

$$\langle Q,AQ\rangle = \sum_{n\geq 0} \overline{\widehat{q_n}}(-ik(n+1)\widehat{q_{n+1}} - n\widehat{q_n} + u(t)\widehat{q_{n-1}} + \widehat{q_{n-2}})$$

Assume $A_1 = A + A^* = A_{re} + iA_{im}$ When the inner product is not trivial, assume its form is $\langle Q,Q\rangle = Q^*C^*CQ$ where $*$ denotes conjugate transpose and C is a upper triangular matrix produced by Cholesky decomposition.

$$\langle Q,AQ\rangle = \sum_i \overline{(CQ)_i}(CAQ)_i = \sum_i \sum_j \overline{c_{i,j}q_j} \sum_l c_{i,l}(-ik(l+1)q_{l+1} - lq_l + uq_{l-1} + q_{l-2})$$

$$= \sum_{j,l} \overline{q_j}q_l \sum_i \overline{c_{i,j}}(-iklc_{i,l-1} - lc_{i,l} + uc_{i,l+1} + c_{i,l+2})$$

$$|\widehat{Q}(t,k)|_1 = \sum_{n\geq 0} |\widehat{q_n}(t,k)| = \sum_{n\geq 0} |\int_0^1 \int_{-\infty}^{+\infty} \frac{v^n}{n!}p(t,x,v)e^{-2\pi ikx}dvdx| = \sum_{n\geq 0} |\int_{-\infty}^{+\infty} \frac{v^n}{n!}\hat{p}(t,k,v)dv|.$$

$$\partial_t p = Lp = -\frac{1}{2\pi}vp_x - ((u(t)-v)p)_v + p_{vv}$$

$$\partial_t \int_0^1 \int_{-\infty}^{+\infty} \frac{p^2}{2}dvdx = \int_0^1 \int_{-\infty}^{+\infty} p(-\frac{1}{2\pi}vp_x - ((u(t)-v)p)_v + p_{vv})dvdx$$

$$= \int_0^1 \int_{-\infty}^{+\infty} -pd((u(t)-v)p)dx + \int_0^1 \int_{-\infty}^{+\infty} pdp_v dx$$

$$= \int_0^1 \int_{-\infty}^{+\infty} (u(t)-v)pp_v dvdx - \int_0^1 \int_{-\infty}^{+\infty} p_v^2 dvdx$$

$$= \int_0^1 \int_{-\infty}^{+\infty} \frac{p^2}{2}dvdx - \int_0^1 \int_{-\infty}^{+\infty} p_v^2 dvdx$$

$$\widehat{p_0}(t,v)=\hat{p}(t,0,v)=\int_0^1 p(t,x,v)dx,\quad \partial_t\widehat{p_0}=L'\widehat{p_0}=-((u(t)-v)\widehat{p_0})_v+\widehat{p_0}_{vv}$$

The operator $\partial_t - L'$ is uniformly parabolic under the definition of [1], Chapter 7.

$$\partial_t\int_{-\infty}^{+\infty}\frac{\widehat{p_0}^2}{2}dv=\int_{-\infty}^{+\infty}\widehat{p_0}(-((u(t)-v)\widehat{p_0})_v+\widehat{p_0}_{vv})dv$$
$$=\int_{-\infty}^{+\infty}-\widehat{p_0}d((u(t)-v)\widehat{p_0})+\int_{-\infty}^{+\infty}\widehat{p_0}d\widehat{p_0}_v$$
$$=\int_{-\infty}^{+\infty}(u(t)-v)\widehat{p_0}\widehat{p_0}_v dv-\int_{-\infty}^{+\infty}\widehat{p_0}_v^2dv$$
$$=\int_{-\infty}^{+\infty}\frac{\widehat{p_0}^2}{2}dv-\int_{-\infty}^{+\infty}\widehat{p_0}_v^2dv$$
$$\langle r,L'r\rangle=\frac{1}{2}\langle r,r\rangle-\langle r_v,r_v\rangle,\quad \langle r,(\frac{1}{2}-L')r\rangle=\langle r_v,r_v\rangle.$$

Evolutionary equations of momentum and kinetic energy:

$$\widehat{q_1}(t,0)=\int_0^1 q_1(t,x)dx,\quad \widehat{q_2}(t,0)=\int_0^1 q_2(t,x)dx,$$
$$\partial_t\widehat{q_1}(t,0)=-\widehat{q_1}(t,0)+u(t)\widehat{q_0}(t,0)=-\widehat{q_1}(t,0)+u(t)$$
$$\partial_t\widehat{q_2}(t,0)=-2\widehat{q_2}(t,0)+u(t)\widehat{q_1}(t,0)+\widehat{q_0}(t,0)=-\widehat{q_1}(t,0)+u(t)\widehat{q_1}(t,0)+1$$

Boltzmann's H-functional:

$$H(t)=-\int_0^1\int_{-\infty}^{+\infty}p\log p dvdx$$
$$\frac{dH(t)}{dt}=-\int_0^1\int_{-\infty}^{+\infty}p_t\log p+p\frac{p_t}{p}dvdx=-\int_0^1\int_{-\infty}^{+\infty}p_t(\log p+1)dvdx$$
$$=\int_0^1\int_{-\infty}^{+\infty}(\frac{1}{2\pi}vp_x+((u(t)-v)p)_v-p_{vv})(\log p+1)dvdx$$
$$=\int_0^1\int_{-\infty}^{+\infty}\frac{1}{2\pi}v(\log p+1)dvdp+\int_0^1\int_{-\infty}^{+\infty}(\log p+1)d((u(t)-v)p-p_v)dx$$
$$=-\int_0^1\int_{-\infty}^{+\infty}\frac{1}{2\pi}vpdvd(\log p+1)-\int_0^1\int_{-\infty}^{+\infty}((u(t)-v)p-p_v)d(\log p+1)dx$$
$$=-\int_0^1\int_{-\infty}^{+\infty}\frac{1}{2\pi}vp\frac{p_x}{p}dvdx-\int_0^1\int_{-\infty}^{+\infty}((u(t)-v)p-p_v)\frac{p_v}{p}dvdx$$
$$=-\int_0^1\int_{-\infty}^{+\infty}\frac{1}{2\pi}vp_xdvdx-\int_0^1\int_{-\infty}^{+\infty}((u(t)-v)p_v-\frac{p_v^2}{p})dvdx$$
$$=0+\int_0^1\int_{-\infty}^{+\infty}(p(u(t)-v)_v+\frac{p_v^2}{p})dvdx=\int_0^1\int_{-\infty}^{+\infty}(-p+\frac{p_v^2}{p})dvdx$$
$$H_0(t)=-\int_0^1 q_0\log q_0dx$$
$$\frac{dH_0(t)}{dt}=-\int_0^1\partial_tq_0(\log q_0+1)dx=\int_0^1\partial_xq_1(\log q_0+1)dx$$
$$=\int_0^1(\log q_0+1)dq_1=-\int_0^1 q_1d(\log q_0+1)=-\int_0^1 q_1\frac{\partial_xq_0}{q_0}dx$$

□

Appendix

Question 11. In the tail upper bound of binomial distribution given in question 8, let $k = 0$, we get the following inequality: $(1-p)^n \le e^{-2np^2}$. It is equivalent to $\log(1-p) \le -2p^2$.

Proof. Let $f(p) = -2p^2 - \log(1-p)$, $f'(p) = -4p + \frac{1}{1-p} = \frac{(1-2p)^2}{1-p} \ge 0$. Since $f(0) = 0$, this gives the above inequality. We may also prove that $(1-p)e^{2p^2} \le 1$. Let $g(p) = (1-p)e^{2p^2}$, $g'(p) = (4p(1-p)-1)e^{2p^2} = -(2p-1)^2 e^{2p^2} \le 0$, so $g(p) \le g(0) = 1$. $\square$

Question 12 (Maximum entropy distribution)**.** Suppose p is a probability density function on $\mathbb{R}$, such that

$$p(x) \ge 0, x \in \mathbb{R}, \quad \int_{\mathbb{R}} p(x)dx = 1, \quad \int_{\mathbb{R}} x^2 p(x)dx = r^2$$

Find the maximum value of $-\int_{\mathbb{R}} p(x)\log p(x)dx$, and the distribution p_0 to attain this maximum.

Solution.

$$p_t(x) = p_0(x) + tq(x), \quad \int_{\mathbb{R}} q(x)dx = 0, \quad \int_{\mathbb{R}} x^2 q(x)dx = 0.$$

$$H(t) = -\int_{\mathbb{R}} p_t(x)\log p_t(x)dx,$$

$$H'(0) = -\int_{\mathbb{R}} q(x)\log p_0(x) + p_0(x)\frac{q(x)}{p_0(x)}dx = -\int_{\mathbb{R}} q(x)(\log p_0(x)+1)dx = 0.$$

$$\log p_0(x) = ax^2 + b, \quad p_0(x) = \frac{1}{\sqrt{2\pi r^2}}e^{-\frac{x^2}{2r^2}}.$$

$\square$

$$\ge \left|\int_0^1 \int_{-\infty}^{+\infty} \sum_{n\ge 0}(ik(n+1)\frac{v^{n+1}}{(n+1)!} + (\lambda+n)\frac{v^n}{n!} - u(t)\frac{v^{n-1}}{(n-1)!} - \frac{v^{n-2}}{(n-2)!})p(t,x,v)e^{-2\pi ikx}dvdx\right|$$

$$= \left|\int_0^1 \int_{-\infty}^{+\infty} ((ik+1)v + \lambda - u(t) - 1)e^v p(t,x,v)e^{-2\pi ikx}dvdx\right|$$

$$|\tilde{q}_n| \le Cn^\alpha, \quad \alpha = \Re(\lambda),$$

$$\left|\frac{1}{n+1}((\lambda+n)\tilde{q}_n + \tilde{q}_{n-2})\right| \le \left|\frac{\lambda+n}{n+1}\right|Cn^\alpha + \frac{C}{n+1}(n-2)^\alpha,$$

$$\le Cn^\alpha\left(\left|\frac{\lambda+n}{n+1}\right| + \frac{1}{n+1}(1-\frac{2}{n})^\alpha\right)$$

$$\left|\frac{\lambda+n}{n+1}\right|^2 = (1+\Re(\frac{\lambda-1}{n+1}))^2 + (\frac{\Im(\lambda)}{n+1})^2, \quad \left|\frac{\lambda+n}{n+1}\right| \le (1+\Re(\frac{\lambda-1}{n+1})) + \frac{(\frac{\Im(\lambda)}{n+1})^2}{2(1+\Re(\frac{\lambda-1}{n+1}))}$$

References

[1] Evans, Partial differential equations.