Linear stability of RMHD equations on 2D finite channel

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Abstract

This is a note about linear stability of RMHD equations on 2D finite channel.

Contents

1 Introduction

$\Omega = [0,1] \times \mathbb{T}, \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The equilibrium is $V_s = 0, B_s = (0, b(x)), P_s = 0$, total magnetic field is $\tilde{B} = B_s + B$, where B is the pertubation. We assume that b is monotone positive. Total velocity field is $V_s + v = v$, total pressure is $P_s + p = p$, and they are the same as the pertubations v, p, then the original RMHD system has the following form:

$$\partial_t v + v \cdot \nabla v + \nabla p = \tilde{B} \cdot \nabla \tilde{B}, \tag{1}$$

$$\partial_t \tilde{B} - \eta \Delta \tilde{B} + v \cdot \nabla \tilde{B} = \tilde{B} \cdot \nabla v, \tag{2}$$

$$\nabla \cdot v = \nabla \cdot \tilde{B} = 0, \tag{3}$$

Nonlinear equations for the perturbations are

$$\begin{aligned}
\partial_t v_x &= -\partial_x p - (v_x \partial_x + v_y \partial_y) v_x + (B_x \partial_x + (b + B_y)\partial_y) B_x, \\
\partial_t v_y &= -\partial_y p - (v_x \partial_x + v_y \partial_y) v_y + B_x \partial_x (b + B_y) + (b + B_y) \partial_y B_y, \\
\partial_t B_x &= (B_x \partial_x + (b + B_y)\partial_y) v_x - (v_x \partial_x + v_y \partial_y) B_x + \eta \Delta B_x, \\
\partial_t B_y &= (B_x \partial_x + (b + B_y)\partial_y) v_y - v_x \partial_x (b + B_y) - v_y \partial_y B_y + \eta \Delta B_y, \\
\nabla \cdot v &= \nabla \cdot B = 0,
\end{aligned}$$

Linearized equations for the pertubations are:

$$\partial_t v_x = -\partial_x p + b \partial_y B_x, \tag{4}$$

$$\partial_t v_y = -\partial_y p + b \partial_y B_y + b' B_x, \tag{5}$$

$$\partial_t B_x = b \partial_y v_x + \eta \Delta B_x, \tag{6}$$

$$\partial_t B_y = b \partial_y v_y - b' v_x + \eta \Delta B_y, \tag{7}$$

$$\nabla \cdot v = \nabla \cdot B = 0, \tag{8}$$

with Navier slip boundary conditions

$$v_x|_{x=0,1} = B_x|_{x=0,1} = 0,$$

Taking Fourier transform in y, we get for $\alpha \neq 0$,

$$\partial_t \widehat{v}_x = -\partial_x \widehat{p} + i\alpha b \widehat{B}_x, \tag{9}$$

$$\partial_t \widehat{v}_y = -i\alpha \widehat{p} + i\alpha b \widehat{B}_y + b' \widehat{B}_x, \tag{10}$$

$$\partial_t \widehat{B}_x = i\alpha b \widehat{v}_x + \eta(\partial_x^2 - \alpha^2)\widehat{B}_x, \tag{11}$$

$$\partial_t \widehat{B}_y = i\alpha b \widehat{v}_y - b'\widehat{v}_x + \eta(\partial_x^2 - \alpha^2)\widehat{B}_y, \tag{12}$$

$$\partial_x \widehat{v}_x + i\alpha \widehat{v}_y = 0, \quad \partial_x \widehat{B}_x + i\alpha \widehat{B}_y = 0, \tag{13}$$

Eliminating $\widehat{p}, \widehat{v}_y, \widehat{B}_y$ from equation 10 gives

$$\partial_t \partial_x \widehat{v}_x = \partial_t(-i\alpha \widehat{v}_y) = -\alpha^2 \widehat{p} + i\alpha b \partial_x \widehat{B}_x - i\alpha b' \widehat{B}_x,$$

$$\partial_t \partial_x^2 \widehat{v}_x = -\alpha^2 \partial_x \widehat{p} + i\alpha \partial_x (b \partial_x \widehat{B}_x) - i\alpha \partial_x (b' \widehat{B}_x), \quad \partial_x \widehat{p} = -\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x,$$

$$\begin{aligned} \partial_t \partial_x^2 \widehat{v}_x &= -\alpha^2(-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha \partial_x (b \partial_x \widehat{B}_x) - i\alpha \partial_x (b' \widehat{B}_x) \\ &= -\alpha^2(-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha (b \partial_x^2 \widehat{B}_x - b'' \widehat{B}_x), \end{aligned}$$

$$\partial_t(\partial_x^2 - \alpha^2)\widehat{v}_x = \alpha b(\partial_x^2 - \alpha^2) i\widehat{B}_x - \alpha b'' i\widehat{B}_x, \quad \partial_t i\widehat{B}_x = -\alpha b \widehat{v}_x + \eta(\partial_x^2 - \alpha^2) i\widehat{B}_x,$$

Let $\xi = \widehat{v}_x, \psi = i\widehat{B}_x$, we have

$$\partial_t \xi = \alpha(\partial_x^2 - \alpha^2)^{-1}(b(\partial_x^2 - \alpha^2)\psi - b''\psi), \quad \partial_t \psi = -\alpha b \xi + \eta(\partial_x^2 - \alpha^2)\psi,$$

Denote $\mathcal{L}_\alpha$ the linear operator of the above equations, and consider the eigenvalue problem of operator $\mathcal{L}_\alpha$. If $c \in \sigma_p(\mathcal{L}_\alpha)$ with associated eigenfunctions ξ, ψ, then

$$c\xi = \alpha(\partial_x^2 - \alpha^2)^{-1}(b(\partial_x^2 - \alpha^2)\psi - b''\psi), \quad c\psi = -\alpha b \xi + \eta(\partial_x^2 - \alpha^2)\psi,$$

$$\alpha b \xi = -(c - \eta(\partial_x^2 - \alpha^2))\psi, \quad c(\partial_x^2 - \alpha^2)\xi = \alpha(b(\partial_x^2 - \alpha^2) - b'')\psi,$$

$$-c(\partial_x^2 - \alpha^2) b^{-1}(c - \eta(\partial_x^2 - \alpha^2))\psi = \alpha^2(b(\partial_x^2 - \alpha^2) - b'')\psi,$$

Let $\psi = bg$, then

$$b^{-1}(c - \eta(\partial_x^2 - \alpha^2))bg = cg - c\eta b^{-1}(\partial_x^2 - \alpha^2)bg,$$

$$\alpha^2(b(\partial_x^2 - \alpha^2) - b'')bg = \alpha^2(b^2(\partial_x^2 - \alpha^2) + 2bb'\partial_x)g,$$

Summing up the above two equations, we get the Orr-Sommerfeld type equation for linearized RMHD system on a 2-dimensional finite channel:

$$(c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g + 2\alpha^2 bb'\partial_x g - c\eta(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)bg = 0,$$

Boundary conditions are

$$\psi|_{x=0,1} = 0, \quad \xi|_{x=0,1} = 0, \quad (c - \eta(\partial_x^2 - \alpha^2))\psi|_{x=0,1} = 0,$$

$$g|_{x=0,1} = 0, \quad \partial_x^2(bg)|_{x=0,1} = b\partial_x^2 g + 2b'\partial_x g|_{x=0,1} = 0,$$

Let us denote by OS_α the Orr-Sommerfeld type fourth-order operator

$$OS_\alpha(g) \triangleq (c^2+\alpha^2 b^2)(\partial_x^2-\alpha^2)g + 2\alpha^2 bb'\partial_x g - c\eta(\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)bg,$$

Following [11], we study the resolvent estimates of the linearized operator under the Navier-slip boundary conditions. More precisely, we consider the equation

$$OS_\alpha(g) = (c^2+\alpha^2 b^2)(\partial_x^2-\alpha^2)g + 2\alpha^2 bb'\partial_x g - c\eta(\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)bg = F,$$

$$g|_{x=0,1}=0, \quad b\partial_x^2 g + 2b'\partial_x g|_{x=0,1}=0,$$

Substitute $h = b^{-1}(\partial_x^2-\alpha^2)bg = b^{-1}(\partial_x^2-\alpha^2)\psi$, we have

$$h|_{x=0,1}=0, \quad g = b^{-1}(\partial_x^2-\alpha^2)^{-1}bh,$$

$$OS_\alpha(g) = (c^2+\alpha^2 b^2)(\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)^{-1}bh + 2\alpha^2 bb'\partial_x b^{-1}(\partial_x^2-\alpha^2)^{-1}bh - c\eta(\partial_x^2-\alpha^2)h,$$

We investigate the case of exponential background magnetic profile $b(x)=e^{\lambda x}$ for convenience. The Sobolev space we concern is

$$H_0^1([0,1]) = \{u:[0,1]\to\mathbb{C}, \|u\|_{H^1}<+\infty, u(0)=u(1)=0\}, \quad \|u\|_{H^1}^2 = \int_0^1 \partial_x u\overline{\partial_x u}dx,$$

A set of orthonormal basis of $H_0^1([0,1])$ is $\{e_k = \frac{\sqrt{2}\sin k\pi x}{k\pi}, k\in\mathbb{Z}_+\}$, and they are all the eigenfunctions of operator ∂_x^2, with eigenvalues $\partial_x^2 e_k = -k^2\pi^2 e_k$. Under the exponential background magnetic profile, we have

$$(\partial_x^2-\alpha^2)g = (\partial_x^2-\alpha^2)b^{-1}(\partial_x^2-\alpha^2)^{-1}bh = h + 2(b^{-1})'\partial_x(\partial_x^2-\alpha^2)^{-1}bh + (b^{-1})''(\partial_x^2-\alpha^2)^{-1}bh,$$

$$\partial_x g = \partial_x b^{-1}(\partial_x^2-\alpha^2)^{-1}bh = (b^{-1})'(\partial_x^2-\alpha^2)^{-1}bh + b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh,$$

The Orr-Sommerfeld type equation in terms of h becomes

$$\begin{aligned} F \triangleq OS_\alpha(g) &= (c^2+\alpha^2 b^2)(h + 2(b^{-1})'\partial_x(\partial_x^2-\alpha^2)^{-1}bh + (b^{-1})''(\partial_x^2-\alpha^2)^{-1}bh) \\ &+ 2\alpha^2 bb'((b^{-1})'(\partial_x^2-\alpha^2)^{-1}bh + b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh) - c\eta(\partial_x^2-\alpha^2)h, \end{aligned}$$

$$F = (c^2+\alpha^2b^2)(h-2\lambda b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh+\lambda^2 b^{-1}(\partial_x^2-\alpha^2)^{-1}bh)+2\alpha^2\lambda b(\partial_x-\lambda)(\partial_x^2-\alpha^2)^{-1}bh - c\eta(\partial_x^2-\alpha^2)h,$$

$$\begin{aligned} F &= c^2(h - 2\lambda b^{-1}\partial_x(\partial_x^2-\alpha^2)^{-1}bh + \lambda^2 b^{-1}(\partial_x^2-\alpha^2)^{-1}bh) + \alpha^2 b^2 h - \lambda^2\alpha^2 b(\partial_x^2-\alpha^2)^{-1}bh - c\eta(\partial_x^2-\alpha^2)h \\ &= c^2(h + \lambda b^{-1}(\lambda-2\partial_x)(\partial_x^2-\alpha^2)^{-1}bh) + \alpha^2 b^2 h - \lambda^2\alpha^2 b(\partial_x^2-\alpha^2)^{-1}bh - c\eta(\partial_x^2-\alpha^2)h, \end{aligned}$$

While taking inner product with h, the first and the third term in the right hand side above is nontrivial. We give a closer look at the first term below:

$$(\partial_x^2-\alpha^2)g = h + \lambda b^{-1}(\lambda-2\partial_x)(\partial_x^2-\alpha^2)^{-1}bh,$$

$$\begin{aligned} \langle(\partial_x^2-\alpha^2)g, h\rangle &= \int_0^1 b^{-1}(\partial_x^2-\alpha^2)g(\partial_x^2-\alpha^2)b\overline{g} = \int_0^1 b^{-1}(\partial_x^2-\alpha^2)g(b(\partial_x^2-\alpha^2)\overline{g} + \lambda^2 b\overline{g} + 2\lambda b\partial_x\overline{g}) \\ &= \|(\partial_x^2-\alpha^2)g\|_2^2 + \lambda^2\int_0^1(\partial_x^2-\alpha^2)g\overline{g} + 2\lambda\int_0^1(\partial_x^2-\alpha^2)g\cdot\partial_x\overline{g} \\ &= \|(\partial_x^2-\alpha^2)g\|_2^2 - \lambda^2\|g'\|_2^2 - \alpha^2\lambda^2\|g\|_2^2 + 2\lambda(\langle g'', g'\rangle - \alpha^2\langle g, g'\rangle), \end{aligned}$$

$$\|(\partial_x^2-\alpha^2)g\|_2^2 = \|g''\|_2^2 + \alpha^4\|g\|_2^2 + 2\alpha^2\|g'\|_2^2,$$

The third term is dealt with as follows:

$$b(\partial_x^2-\alpha^2)^{-1}bh=b\psi,\quad h=b^{-1}(\partial_x^2-\alpha^2)\psi,$$

$$\langle b\psi,h\rangle=\int_0^1\psi(\partial_x^2-\alpha^2)\overline{\psi}dx=-\|\psi'\|_2^2-\alpha^2\|\psi\|_2^2,$$

Combining the equations above together, we have

$$\begin{aligned}&\langle F,ch\rangle=\langle c^2(\partial_x^2-\alpha^2)g,ch\rangle+\bar{c}\alpha^2\|bh\|_2^2-\bar{c}\lambda^2\alpha^2\langle b\psi,h\rangle-|c|^2\eta\langle(\partial_x^2-\alpha^2)h,h\rangle\\=&|c|^2c(\|(\partial_x^2-\alpha^2)g\|_2^2-\lambda^2\|g'\|_2^2-\alpha^2\lambda^2\|g\|_2^2+2\lambda(\langle g'',g'\rangle-\alpha^2\langle g,g'\rangle))\\&+\bar{c}(\alpha^2\|bh\|_2^2+\lambda^2\alpha^2\|\psi'\|_2^2+\lambda^2\alpha^4\|\psi\|_2^2)+|c|^2(\eta\|h'\|_2^2+\eta\alpha^2\|h\|_2^2)\\=&|c|^2c(\|g''\|_2^2+(2\alpha^2-\lambda^2)\|g'\|_2^2+\alpha^2(\alpha^2-\lambda^2)\|g\|_2^2+2\lambda(\langle g'',g'\rangle-\alpha^2\langle g,g'\rangle))\\&+\bar{c}(\alpha^2\|bh\|_2^2+\lambda^2\alpha^2\|\psi'\|_2^2+\lambda^2\alpha^4\|\psi\|_2^2)+|c|^2(\eta\|h'\|_2^2+\eta\alpha^2\|h\|_2^2)\end{aligned}$$

We see that when $\Re(c)>0,-1\leq\lambda\leq1$, the real part of the right hand side of the above equation is strictly positive for non-zero h.

$$\|bh\|_2^2=\|(\partial_x^2-\alpha^2)\psi\|_2^2=\|\psi''\|+2\alpha^2\|\psi'\|_2^2+\alpha^4\|\psi\|_2^2,$$

$$\mathcal{H}=\{(\psi,\xi),\psi,\xi\in H_0^1([0,1])\},\quad\|(\psi,\xi)\|_{\mathcal{H}}^2=\int_0^1\partial_x\psi\overline{\partial_x\psi}+\partial_x\xi\overline{\partial_x\xi},$$

$$\partial_t\xi=\alpha(\partial_x^2-\alpha^2)^{-1}(b(\partial_x^2-\alpha^2)\psi-b''\psi)=\alpha(b\psi+K_1\psi),$$

where K_1 is a compact operator defined by

$$K_1\psi=(\partial_x^2-\alpha^2)^{-1}(-2b'\partial_x\psi-2b''\psi)=-2(\partial_x^2-\alpha^2)^{-1}\partial_x(b'\psi),$$

and we have $\overline{K_1\psi}=K_1\overline{\psi}$. Notice that $v_x,B_x\in\mathbb{R},\xi=\widehat{v}_x,\psi=i\widehat{B}_x$ only implies that $\widehat{v}_x(\alpha)=\overline{\widehat{v}_x(-\alpha)},\widehat{B}_x(\alpha)=\overline{\widehat{B}_x(-\alpha)}$, ξ,ψ are complex-valued functions.

$$\int_0^1\partial_t\xi\overline{\xi}=\int_0^1\alpha(b\psi+K_1\psi)\overline{\xi},\quad\int_0^1\partial_t\psi\overline{\psi}=\int_0^1(-\alpha b\xi+\eta(\partial_x^2-\alpha^2)\psi)\overline{\psi},$$

$$\partial_t(\|\psi\|_2^2+\|\xi\|_2^2)=\int_0^1\alpha(K_1\psi\overline{\xi}+\xi K_1\overline{\psi})+\eta(\partial_x^2-\alpha^2)\psi\overline{\psi}+\psi\eta(\partial_x^2-\alpha^2)\overline{\psi},$$

When $b(x)=e^{\lambda x}$, we have $K_1\psi=(\partial_x^2-\alpha^2)^{-1}(-2\lambda b\partial_x\psi-2\lambda^2b\psi)=-2\lambda(\partial_x^2-\alpha^2)^{-1}\partial_x(b\psi)$.

$$\partial_t\psi=-\alpha b\xi+\eta(\partial_x^2-\alpha^2)\psi=-\alpha b(\xi(0)+\int_0^t\alpha(b\psi(t')+K\psi(t'))dt')+\eta(\partial_x^2-\alpha^2)\psi,$$

$$\partial_t^2\psi=-\alpha^2b(b\psi+K_1\psi)+\eta(\partial_x^2-\alpha^2)\partial_t\psi,$$

Substitute $\psi=bg$ and let $\eta=0,B=b^2$, we have

$$\partial_t^2(bg)=-\alpha^2b(b^2g+Kbg),\quad K_1bg=-2(\partial_x^2-\alpha^2)^{-1}\partial_x(b'bg)=-(\partial_x^2-\alpha^2)^{-1}\partial_x(B'g),$$

$$\partial_t^2g+\alpha^2(b^2g+K_1bg)=\partial_t^2g+\alpha^2(Bg-(\partial_x^2-\alpha^2)^{-1}\partial_x(B'g))=0,$$

the above calculation is the same as equation (2.5) in [10]. We define

$$Kg=K_1bg=-(\partial_x^2-\alpha^2)^{-1}\partial_x(B'g),$$

Proposition 1 (Energy conservation on each frequency)**.**

2 Spectral method

Theorem 1. Reference: Lecture notes on functional analysis II by Gongqing Zhang, p53 problem 5.5.10. $\mathcal{H}$ is a Hilbert space, N is a normal operator on $\mathcal{H}$ and its spectrum $\sigma(N)$ is countable, then $\mathcal{H}$ has a orthonormal basis $B = \{y\}$ where y are eigenfunctions of N, and the Fourier expansion holds:

$$x = \sum_{y \in B} (x, y)y, \quad x \in \mathcal{H},$$

the Fourier coefficients (x, y) only have countably many nonzero elements.

Proof. 1) Eigenspaces of different eigenvalues are orthogonal. If f_1, f_2 are two eigenfunctions of N with different eigenvalues λ_1, λ_2. When N is self-adjoint, we have $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\lambda_1 \langle f_1, f_2 \rangle = \langle N f_1, f_2 \rangle = \langle f_1, N f_2 \rangle = \lambda_2 \langle f_1, f_2 \rangle,$$

Since $\lambda_1 \neq \lambda_2$, $\langle f_1, f_2 \rangle = 0$. $\square$

Problem 1. Eigenfunctions of ∂_x^2 on $H_0^1([0, 1])$.

$$\partial_x^2 f = \lambda f, \quad -\xi^2 \widehat{f} = \lambda \widehat{f}, \quad \lambda = -\xi^2, \quad \mathrm{supp}\widehat{f} = \{\pm\xi\}, \quad f(x) = \widehat{f}(\xi)e^{i\xi x} + \widehat{f}(-\xi)e^{-i\xi x},$$

Boundary conditions are

$$f(0) = \widehat{f}(\xi) + \widehat{f}(-\xi) = 0, \quad f(1) = \widehat{f}(\xi)e^{i\xi} + \widehat{f}(-\xi)e^{-i\xi} = 0,$$

So we have $e^{i\xi} = e^{-i\xi}, \xi = k\pi, k \in \mathbb{Z}\backslash\{0\}$, $\lambda = -k^2\pi^2, f = C \sin k\pi x$.

Consider the case when $b(x)$ is a positive constant. We have

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g - c\eta(\partial_x^2 - \alpha^2)^2 g, \quad g|_{x=0,1} = g''|_{x=0,1} = 0,$$

Let $h = (\partial_x^2 - \alpha^2)g$, we claim that when $\Re(c) > 0, f \in H_0^1([0, 1])$, the solution to $OS_\alpha(g) = f$ uniquely exists. We have

$$\mathcal{L}_\alpha h \triangleq OS_\alpha(g) = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = f,$$

Assume that $f = \sum_k (f, e_k)e_k$, then $(\partial_x^2 - \alpha^2)e_k = -(k^2\pi^2 + \alpha^2)e_k$,

$$(f, e_k) = (h, e_k)(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)), \quad (h, e_k) = \frac{(f, e_k)}{c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)},$$

$$(g, e_k) = -\frac{(h, e_k)}{(k^2\pi^2 + \alpha^2)} = -\frac{(f, e_k)}{(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2))(k^2\pi^2 + \alpha^2)},$$

Problem 2. Eigenvalue problem: when b is constant, for what values of c is $\mathcal{L}_\alpha$ not injective?

$$\mathcal{L}_\alpha h = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = 0,$$

When $\eta = 0$, eigenvalues are $c = \pm i\alpha b$, and their invariant space is the whole $H_0^1([0, 1])$. When $\eta > 0$, 1) $c^2 + \alpha^2 b^2 = 0$, then $c \neq 0$, since $\partial_x^2 - \alpha^2$ is injective, we have $h = 0$.

2) $c^2 + \alpha^2 b^2 \neq 0$, take Fourier transform $h = \sum_k \widehat{h}_k e_k$, we have

$$(c^2 + \alpha^2 b^2 + c\eta\alpha^2 + c\eta k^2\pi^2)\widehat{h}_k = 0, \quad k \neq 0,$$

For small k, $\eta\alpha^2 + \eta k^2\pi^2 \leq 2|\alpha|b$, the roots are two conjugate complex numbers with negative real parts and norm αb. For large k, $\eta\alpha^2 + \eta k^2\pi^2 > 2|\alpha|b$, the two real roots are

$$c_{1,2} = -\frac{\eta\alpha^2 + \eta k^2\pi^2 \pm \sqrt{(\eta\alpha^2 + \eta k^2\pi^2)^2 - 4\alpha^2 b^2}}{2},$$

$$A=\int_0^1 e^{\lambda x}\sin k\pi x dx=-\int_0^1 \frac{e^{\lambda x}}{\lambda}k\pi\cos k\pi x dx=\frac{k\pi}{\lambda^2}-(-1)^k e^{\lambda}\frac{k\pi}{\lambda^2}-\int_0^1\frac{e^{\lambda x}}{\lambda^2}k^2\pi^2\sin k\pi x dx,$$

$$\int_0^1\frac{e^{\lambda x}}{\lambda^2}k^2\pi^2\sin k\pi x dx=\frac{k^2\pi^2}{\lambda^2}A,\quad A=\frac{(1-(-1)^k e^{\lambda})k\pi}{\lambda^2+k^2\pi^2},$$

$$B=\int_0^1 e^{\lambda x}\cos m\pi x dx=(-1)^m\frac{e^{\lambda}}{\lambda}-\frac{1}{\lambda}+\int_0^1\frac{e^{\lambda x}}{\lambda}m\pi\sin m\pi x dx,$$

$$\int_0^1\frac{e^{\lambda x}}{\lambda}m\pi\sin m\pi x dx=-\int_0^1\frac{e^{\lambda x}}{\lambda^2}m^2\pi^2\sin m\pi x dx=-\frac{m^2\pi^2}{\lambda^2}B,\quad B=\frac{((-1)^m e^{\lambda}-1)\lambda}{\lambda^2+m^2\pi^2},$$

When $\lambda<0$, we may also use complex analysis techniques to calculate the above integrals:

$$\int_0^{+\infty}e^{\lambda x}\sin k\pi x dx=\Im\int_0^{+\infty}e^{(\lambda+ik\pi)x}dx=\Im\frac{1}{-\lambda-ik\pi}=\frac{k\pi}{\lambda^2+k^2\pi^2},$$

$$\int_0^{+\infty}e^{\lambda x}\sin k\pi x dx=A(1+(-1)^k e^{\lambda}+(-1)^{2k}e^{2\lambda}+...)=\frac{A}{1-(-1)^k e^{\lambda}},\quad A=\frac{(1-(-1)^k e^{\lambda})k\pi}{\lambda^2+k^2\pi^2},$$

$$\int_0^{+\infty}e^{\lambda x}\cos m\pi x dx=\Re\int_0^{+\infty}e^{(\lambda+im\pi)x}dx=\Re\frac{1}{-\lambda-im\pi}=\frac{-\lambda}{\lambda^2+m^2\pi^2},$$

$$\int_0^{+\infty}e^{\lambda x}\cos m\pi x dx=B(1+(-1)^m e^{\lambda}+(-1)^{2m}e^{2\lambda}+...)=\frac{B}{1-(-1)^m e^{\lambda}},\quad B=\frac{((-1)^m e^{\lambda}-1)\lambda}{\lambda^2+m^2\pi^2},$$

I try to express the relationship of h and $g=b^{-1}(\partial_x^2-\alpha^2)^{-1}bh$ on the frequency domain. Recall that $e_k=\frac{\sqrt{2}\sin k\pi x}{k\pi}, k\in\mathbb{Z}_+$,

$$g=\sum_k\widehat{g}_k e_k,\quad \widehat{g}_k=\langle g,e_k\rangle=\int_0^1\partial_x g\overline{\partial_x e_k}dx,$$

$$h=\sum_k\widehat{h}_k e_k,\quad \widehat{h}_k=\langle h,e_k\rangle=\int_0^1\partial_x h\overline{\partial_x e_k}dx,$$

$$\widehat{(bg)}_k=\langle bg,e_k\rangle=\int_0^1\partial_x(bg)\overline{\partial_x e_k}=-\int_0^1 bg\overline{\partial_x^2 e_k}=\int_0^1 bgk^2\pi^2\overline{e_k}$$

$$=\int_0^1 b(\sum_l\widehat{g}_l e_l)k^2\pi^2\overline{e_k}=\sum_l\widehat{g}_l\int_0^1 bk^2\pi^2 e_l\overline{e_k},$$

$$e_l\overline{e_k}=\frac{2\sin l\pi x\sin k\pi x}{lk\pi^2}=\frac{\cos(l-k)\pi x-\cos(l+k)\pi x}{lk\pi^2},$$

$$\int_0^1 bk^2\pi^2 e_l\overline{e_k}=\int_0^1 b\frac{k}{l}(\cos(l-k)\pi x-\cos(l+k)\pi x),$$

$$\int_0^1 b\frac{k}{l}\cos(l-k)\pi x=\frac{k}{l}\int_0^1 e^{\lambda x}\cos(l-k)\pi x dx=\frac{k}{l}\frac{((-1)^{l-k}e^{\lambda}-1)\lambda}{\lambda^2+(l-k)^2\pi^2}=B_1,$$

$$\int_0^1 b\frac{k}{l}\cos(l+k)\pi x=\frac{k}{l}\frac{((-1)^{l+k}e^{\lambda}-1)\lambda}{\lambda^2+(l+k)^2\pi^2}=B_2,$$

$$B_1 - B_2 = \frac{k}{l}((-1)^{l-k}e^{\lambda} - 1)\lambda(\frac{1}{\lambda^2 + (l-k)^2\pi^2} - \frac{1}{\lambda^2 + (l+k)^2\pi^2}) = \frac{4k^2\pi^2((-1)^{l-k}e^{\lambda} - 1)\lambda}{(\lambda^2 + (l^2+k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

$$\widehat{(bg)}_k = \sum_l \widehat{g}_l \frac{4k^2\pi^2((-1)^{l-k}e^{\lambda} - 1)\lambda}{(\lambda^2 + (l^2+k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

If $u, v \in H_0^1([0,1])$, $u = \sum_k \widehat{u}_k e_k$, $v = \sum_k \widehat{v}_k e_k$ satisfy $v = (\partial_x^2 - \alpha^2)u$, then their Fourier coefficients satisfy

$$\widehat{v}_k = -(k^2\pi^2 + \alpha^2)\widehat{u}_k, \quad \widehat{u}_k = -\frac{\widehat{v}_k}{k^2\pi^2 + \alpha^2},$$

Since $b^{-1}(x) = e^{-\lambda x}$, the action of multiplier b^{-1} on the Fourier side is given by

$$\widehat{(b^{-1}g)}_k = \sum_l \widehat{g}_l \frac{4k^2\pi^2((-1)^{l-k}e^{-\lambda} - 1)(-\lambda)}{(\lambda^2 + (l^2+k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

We may denote

$$K_\lambda(k,l) = \frac{4k^2\pi^2}{(\lambda^2 + (l^2+k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

Consider representing the Fourier coefficients of h in terms of Fourier coefficients of g, where $g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh$, we have

$$v = bh, \quad \psi = (\partial_x^2 - \alpha^2)^{-1}v, \quad g = b^{-1}\psi,$$

$$\widehat{v}_k = \sum_l \widehat{h}_l((-1)^{l-k}e^{\lambda} - 1)\lambda K_\lambda(k,l), \quad \widehat{\psi}_k = -\frac{\widehat{v}_k}{k^2\pi^2 + \alpha^2},$$

$$\begin{aligned}
\widehat{g}_m &= \sum_k \widehat{\psi}_k((-1)^{l-k}e^{-\lambda} - 1)(-\lambda)K_\lambda(m,k) = \sum_k \widehat{v}_k \frac{((-1)^{l-k}e^{-\lambda} - 1)\lambda}{k^2\pi^2 + \alpha^2} K_\lambda(m,k) \\
&= \sum_{k,l} \frac{((-1)^{k-m}e^{-\lambda} - 1)\lambda}{k^2\pi^2 + \alpha^2} K_\lambda(m,k)((-1)^{l-k}e^{\lambda} - 1)\lambda K_\lambda(k,l)\widehat{h}_l \\
&= \sum_l \widehat{h}_l \sum_k \frac{\lambda^2((-1)^{k-m}e^{-\lambda} - 1)((-1)^{l-k}e^{\lambda} - 1)}{k^2\pi^2 + \alpha^2} K_\lambda(m,k)K_\lambda(k,l)
\end{aligned}$$

3 Energy conservation of nonlinear RVMHD

Suppose u and b are the velocity field and magnetic field, p is the total pressure, we consider the original nonlinear RVMHD equations:

$$\partial_t(\|u\|_2^2 + \|b\|_2^2) = \partial_t \int_\Omega u \cdot u + b \cdot b = 2\int_\Omega u \cdot (-\nabla p - u \cdot \nabla u + b \cdot \nabla b + \mu\Delta u) + b \cdot (b \cdot \nabla u - u \cdot \nabla b + \eta\Delta b),$$

$$\int_\Omega u \cdot \nabla p = \int_\Omega u \cdot \nabla p + p\nabla \cdot u = \int_\Omega \nabla \cdot (pu) = 0,$$

$$u \cdot (u \cdot \nabla u) = u \cdot \nabla \frac{|u|^2}{2}, \quad \nabla \cdot (|u|^2 u) = u \cdot \nabla |u|^2 + |u|^2 \nabla \cdot u,$$

$$\nabla \cdot (|b|^2 u) = u \cdot \nabla |b|^2 + |b|^2 \nabla \cdot u, \quad b \cdot (u \cdot \nabla)b = u \cdot \nabla \frac{|b|^2}{2},$$

$$\int_\Omega u\cdot(u\cdot\nabla u)=\frac{1}{2}\int_\Omega \nabla\cdot(|u|^2u)=0,\quad \int_\Omega b\cdot(u\cdot\nabla b)=\frac{1}{2}\int_\Omega\nabla\cdot(|b|^2u)=0,$$

$$\nabla\cdot((u\cdot b)b)=(b\cdot\nabla)(u\cdot b)+(u\cdot b)(\nabla\cdot b),\quad (b\cdot\nabla)(u\cdot b)=u\cdot(b\cdot\nabla)b+b\cdot(b\cdot\nabla)u,$$

$$\int_\Omega u\cdot(b\cdot\nabla)b+b\cdot(b\cdot\nabla)u=\int_\Omega\nabla\cdot((u\cdot b)b)=0,$$

$$\int_\Omega u\cdot\Delta u=-\int_\Omega \nabla u_x\cdot\nabla u_x+\nabla u_y\cdot\nabla u_y+\nabla u_z\cdot\nabla u_z=-\int_\Omega|\nabla u|^2,\quad \int_\Omega b\cdot\Delta b=-\int_\Omega|\nabla b|^2,$$

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