

Chen-Ning Yang's conjecture of exterior product

Wang Xiang

October 9, 2022

1 Introduction

Given a \mathbb{R} -vector space $V = \mathbb{R}^{2n}$, $n \in \mathbb{N}$, equipped with an inner product $\langle \cdot, \cdot \rangle$, $\bigwedge^r V$ is the \mathbb{R} -vector space of degree r exterior forms. If $\{e_i\}_{i=1}^{2n}$ is an orthonormal basis of V , then $\{e_{i_1} \wedge e_{i_2} \dots \wedge e_{i_r}, 1 \leq i_1 < i_2 \dots < i_r \leq 2n\}$ is an orthonormal basis of $\bigwedge^r V$, $\dim \bigwedge^r V = \binom{2n}{r}$. For any $1 \leq i_1 < i_2 \dots < i_r \leq 2n, 1 \leq j_1 < j_2 \dots < j_r \leq 2n$,

$$\langle e_{i_1} \wedge e_{i_2} \dots \wedge e_{i_r}, e_{j_1} \wedge e_{j_2} \dots \wedge e_{j_r} \rangle = \delta_{i_1, j_1} \delta_{i_2, j_2} \dots \delta_{i_r, j_r},$$

For any $1 \leq k, l \leq n, k + l \leq n$, $\xi \in \bigwedge^{2k} V, \eta \in \bigwedge^{2l} V$, find $\max_{\|\xi\|=\|\eta\|=1} \|\xi \wedge \eta\|$.

Conjecture 1 (Chen-Ning Yang). The maximum is attained when

$$\xi_{max} = \frac{\omega^k}{\|\omega^k\|}, \quad \eta_{max} = \frac{\omega^l}{\|\omega^l\|}, \quad \omega = \sum_{i=1}^n e_{2i-1} \wedge e_{2i},$$

And we have

$$\omega^k = k! \sum_{1 \leq i_1 < i_2 \dots < i_k \leq n} (e_{2i_1-1} \wedge e_{2i_1}) \wedge \dots \wedge (e_{2i_k-1} \wedge e_{2i_k})$$

Notice that $(e_1 \wedge e_2) \wedge (e_3 \wedge e_4) = (e_3 \wedge e_4) \wedge (e_1 \wedge e_2)$ has an even permutation on indices.

$$\|\omega^k\|_2^2 = k!^2 \binom{n}{k}, \quad \|\omega^l\|_2^2 = l!^2 \binom{n}{l}, \quad \|\omega^{k+l}\|_2^2 = (k+l)!^2 \binom{n}{k+l},$$

$$\|\xi_{max} \wedge \eta_{max}\|_2^2 = \frac{\|\omega^{k+l}\|_2^2}{\|\omega^k\|_2^2 \|\omega^l\|_2^2} = \frac{(k+l)!(n-k)!(n-l)!}{k!l!n!(n-k-l)!} \triangleq C(n, k, l),$$

$$C(n, k, l) = \frac{(k+l)!(l+m)!(k+m)!}{k!l!m!(k+l+m)!}, \quad m = n - k - l,$$

Let's start by analysing some special cases.

Example 1. $k + l = n$, now $\bigwedge^{2k+2l} V = \bigwedge^{2n} V = \mathbb{R}e_1 \wedge e_2 \wedge \dots \wedge e_{2n-1} \wedge e_{2n}$. Assume that $u = \sum_I u_I e_I \in \bigwedge^{2k} V, v = \sum_J v_J e_J \in \bigwedge^{2l} V$, where

$$I = (i_1, i_2, \dots, i_{2k}), \quad 1 \leq i_1 < i_2 \dots < i_{2k} \leq 2n, \quad e_I = e_{i_1} \wedge e_{i_2} \dots \wedge e_{i_{2k}},$$

$$J = (j_1, j_2, \dots, j_{2l}), \quad 1 \leq j_1 < j_2 \dots < j_{2l} \leq 2n, \quad e_J = e_{j_1} \wedge e_{j_2} \dots \wedge e_{j_{2l}},$$

then we have

$$u \wedge v = \sum_{I \cap J = \emptyset} u_I v_J e_I \wedge e_J = \sum_{I \cap J = \emptyset} u_I v_J \text{sgn}(\sigma_{I,J}) e_{[2n]},$$

$$\sigma_{I,J} : (I, J) \rightarrow [2n] = (1, 2, \dots, 2n), \quad e_{[2n]} = e_1 \wedge e_2 \wedge \dots \wedge e_{2n-1} \wedge e_{2n},$$

$\text{sgn}(\sigma_{I,J})$ is the signature of permutation $\sigma_{I,J}$: $\text{sgn}(\sigma_{I,J}) = 1$ if $\sigma_{I,J}$ is an even permutation, $\text{sgn}(\sigma_{I,J}) = -1$ if $\sigma_{I,J}$ is an odd permutation.

$$\|u \wedge v\|_2 = \left| \sum_{I \cap J = \emptyset} u_I v_J \text{sgn}(\sigma_{I,J}) \right| \leq \|u\|_2 \|v\|_2,$$

holds by Cauchy's inequality, since for any given I , $J = [2n] \setminus I$ is uniquely determined. Since $C(n, k, l) = 1$ in this case, the conjecture holds. Notice that in this case, for any given $u \in \bigwedge^{2k} V$ with $\|u\| = 1$, there exists $v \in \bigwedge^{2l} V$ with $\|v\| = 1$ such that the equality $\|u \wedge v\| = \|u\| \|v\|$ holds.

Property 1 (Construction of exterior product by tensor product and quotient map). $\wedge : \bigwedge^{2k} V \times \bigwedge^{2l} V \rightarrow \bigwedge^{2k+2l} V$ is a bilinear map:

$$(\xi_1 + \xi_2) \wedge \eta = \xi_1 \wedge \eta + \xi_2 \wedge \eta, \quad \xi \wedge (\eta_1 + \eta_2) = \xi \wedge \eta_1 + \xi \wedge \eta_2,$$

$$(a\xi) \wedge \eta = a(\xi \wedge \eta) = \xi \wedge (a\eta), \quad a \in \mathbb{R},$$

So we have the universal property of tensor product:

$$\begin{array}{ccc} \bigwedge^{2k} V \times \bigwedge^{2l} V & \xrightarrow{\varphi} & \bigwedge^{2k} V \otimes \bigwedge^{2l} V \\ & \searrow \wedge & \downarrow \sim \\ & & \bigwedge^{2k+2l} V \end{array}$$

where $\varphi : \bigwedge^{2k} V \times \bigwedge^{2l} V \rightarrow \bigwedge^{2k} V \otimes \bigwedge^{2l} V$, $(u, v) \mapsto u \otimes v$ is the canonical bilinear map, and $\sim : \bigwedge^{2k} V \otimes \bigwedge^{2l} V \rightarrow \bigwedge^{2k+2l} V$ is the quotient map defined as follows: for $I \subset [2n]$, $|I| = 2k$, $J \subset [2n]$, $|J| = 2l$, if $I \cap J = \emptyset$, then let

$$\sim(e_I \otimes e_J) = \text{sgn}(\sigma_{I,J}) e_{I \cup J}, \quad \sigma_{I,J} : (I, J) \rightarrow I \cup J,$$

where $I, J, I \cup J$ are sorted in ascending order, $\text{sgn}(\sigma_{I,J})$ is the signature of permutation $\sigma_{I,J}$. If $I \cap J \neq \emptyset$, then let $\sim(e_I \otimes e_J) = 0$.

Property 2 (Universal property of exterior product). Assume $V = \mathbb{R}^n$, the space of degree m exterior forms on V is $\bigwedge^m V$. It has the following universal property: if $\varphi : V^{\times m} \rightarrow W$ is a m -linear antisymmetric (alternating) map where $V^{\times m}$ is the product of m copies of V and W is a \mathbb{R} -vector space, then there exists a unique m -linear antisymmetric map $\tilde{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} V^{\times m} & \longrightarrow & \bigwedge^m V \\ & \searrow \varphi & \downarrow \tilde{\varphi} \\ & & W \end{array}$$

Especially, when $m = 1$, $\bigwedge^1 V$ is the dual vector space V^* of V .

Property 3. Monotonicity of $C(n, k, l)$: for fixed $k, l \in \mathbb{N}_+$, $n \geq k + l$, we have

$$C(n+1, k, l) > C(n, k, l) \dots > C(k+l, k, l) = 1,$$

Proof. Let $m = n + 1 - k - l$, since $\frac{(m+l)(m+k)}{m(m+k+l)} > 1$, we have $C(n+1, k, l) > C(n, k, l)$. \square

Property 4. If there exists a subspace $V_1 \subset V$ with $\dim V_1 = 2k$, such that $\xi \in \bigwedge^{2k} V$ is the unit volume form of V_1 , and there exists a subspace $V_2 \subset V$ with $\dim V_2 = 2l$, such that $\eta \in \bigwedge^{2l} V$ is the unit volume form of V_2 . Then $\|\xi\| = \|\eta\| = 1$, and V_1 has an orthonormal basis $\{e_i\}_{i=1}^{2k}$, such

that $\xi = e_1 \wedge e_2 \dots \wedge e_{2k}$, V_2 has an orthonormal basis $\{e'_j\}_{j=1}^{2l}$, such that $\eta = e'_1 \wedge e'_2 \dots \wedge e'_{2l}$. Then by Hadamard's inequality, we have

$$\|\xi \wedge \eta\| = \|e_1 \wedge e_2 \dots \wedge e_{2k} \wedge e'_1 \wedge e'_2 \dots \wedge e'_{2l}\| \leq \prod_{i=1}^{2k} \|e_i\| \prod_{j=1}^{2l} \|e'_j\| = 1,$$

If U, V are Hilbert spaces on \mathbb{R} with inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$, we may construct an inner product on $U \otimes V$ as follows:

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{U \otimes V} = \langle u_1, u_2 \rangle_U \langle v_1, v_2 \rangle_V,$$

Notice that under this inner product on $U \otimes V$, the canonical bilinear map $\varphi : U \times V \rightarrow U \otimes V$ is an isometry onto $\text{im}\varphi$ in the following sense: $\|u \otimes v\|_{U \otimes V} = \|u\|_U \|v\|_V$. φ is a Segre embedding, $\text{im}\varphi$ is a Segre variety. If $\{e_i\}_{i=1}^m$ is an orthonormal basis of U , $\{e'_j\}_{j=1}^n$ is an orthonormal basis of V , then $\{e_i \otimes e'_j\}_{1 \leq i \leq m, 1 \leq j \leq n}$ is an orthonormal basis of $U \otimes V$. The exact form of φ is as follows:

$$u = \sum_{i=1}^m u_i e_i, \quad v = \sum_{j=1}^n v_j e'_j, \quad \varphi(u, v) = u \otimes v = \sum_{1 \leq i \leq m, 1 \leq j \leq n} u_i v_j e_i \otimes e'_j,$$

and we have $\dim \text{im}\varphi = \dim U + \dim V - 1$. In the case of Yang's conjecture, we have

$$\dim \bigwedge^{2k} V = \binom{2n}{2k}, \quad \dim \bigwedge^{2l} V = \binom{2n}{2l}, \quad \dim \bigwedge^{2k+2l} V = \binom{2n}{2k+2l},$$

$$\dim \bigwedge^{2k} V \otimes \bigwedge^{2l} V = \binom{2n}{2k} \binom{2n}{2l}, \quad \dim \text{im}\varphi = \binom{2n}{2k} + \binom{2n}{2l} - 1,$$

Example 2. $V = \mathbb{R}^2$, $u, v \in \bigwedge^1 V$, $u = u_1 e_1 + u_2 e_2$, $v = v_1 e_1 + v_2 e_2$, $u \wedge v = (u_1 v_2 - u_2 v_1) e_{12}$,

$$\frac{\|u \wedge v\|^2}{\|u\|^2 \|v\|^2} = \frac{(u_1 v_2 - u_2 v_1)^2}{(u_1^2 + u_2^2)(v_1^2 + v_2^2)} \leq 1,$$

If $u \otimes v = a_{11} e_1 \otimes e_1 + a_{12} e_1 \otimes e_2 + a_{21} e_2 \otimes e_1 + a_{22} e_2 \otimes e_2$, then $a_{11} a_{22} = a_{12} a_{21}$. If we remove the condition $a_{11} a_{22} = a_{12} a_{21}$ and consider arbitrary $w \in \bigwedge^1 V \otimes \bigwedge^1 V$, $w = a e_1 \otimes e_1 + b e_1 \otimes e_2 + c e_2 \otimes e_1 + d e_2 \otimes e_2$, then we have

$$\frac{\|\sim(w)\|^2}{\|w\|^2} = \frac{(b-c)^2}{a^2 + b^2 + c^2 + d^2} \leq 2,$$

Notice that the upper bound becomes loose. This example shows that we cannot only consider the quotient map $\sim: \bigwedge^1 V \otimes \bigwedge^1 V \rightarrow \bigwedge^2 V$ on the whole space $\bigwedge^1 V \otimes \bigwedge^1 V$, we must consider its restriction on the Segre variety $\text{im}\varphi$. In the above case, if we add the constraint $ad = bc$, then we have

$$\frac{(b-c)^2}{a^2 + b^2 + c^2 + d^2} \leq \frac{(b-c)^2 + (a+d)^2}{a^2 + b^2 + c^2 + d^2} = 1,$$

Assume that $u = \sum_I u_I e_I \in \bigwedge^{2k} V$, $v = \sum_J v_J e_J \in \bigwedge^{2l} V$, then we have

$$u \wedge v = \sum_{I \cap J = \emptyset} u_I v_J e_I \wedge e_J = \sum_{|P|=2k+2l} e_P \sum_{I \cup J = P} u_I v_J \text{sgn}(\sigma_{I,J}),$$

$$\|u \wedge v\|^2 = \sum_{|P|=2k+2l} \left(\sum_{I \cup J = P} u_I v_J \text{sgn}(\sigma_{I,J}) \right)^2 = \sum_{|P|=2k+2l, I_1 \cup J_1 = I_2 \cup J_2 = P} u_{I_1} v_{J_1} u_{I_2} v_{J_2} \text{sgn}(\sigma_{I_1, J_1}) \text{sgn}(\sigma_{I_2, J_2}),$$

Example 3 (Volume form of a linear subspace). Assume $V = \mathbb{R}^4$, $\xi \in \bigwedge^2 V$, and there exists a subspace $V_1 \subset V$, $\dim V_1 = 2$, such that $\xi = e'_1 \wedge e'_2$ where $\{e'_1, e'_2\}$ is an orthonormal basis of V_1 . Assume that $e'_j = \sum_{i=1}^4 a_{ij}e_i$, then we have

$$(e'_1 \ e'_2) = (e_1 \ e_2 \ e_3 \ e_4) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = (e_1 \ e_2 \ e_3 \ e_4)A, \quad A^t A = I_2,$$

$$e'_1 \wedge e'_2 = \sum_{i < j} \det \begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix} e_i \wedge e_j,$$

More generally, assume $V = \mathbb{R}^n$, $\xi \in \bigwedge^m V$, and there exists a set of orthonormal vectors $\{e'_j\}_{j=1}^m$ such that $\xi = e'_1 \wedge e'_2 \dots \wedge e'_m$. Assume that $e'_j = \sum_{i=1}^n a_{ij}e_i$, then we have

$$(e'_1 \ e'_2 \dots e'_m) = (e_1 \ e_2 \dots e_n)A, \quad A \in M(n, m, \mathbb{R}), \quad A^t A = I_m,$$

$$e'_1 \wedge e'_2 \dots \wedge e'_m = \sum_{|I|=m} \det A_{I, [m]} e_I, \quad I = (i_1, i_2, \dots, i_m), \quad 1 \leq i_1 < i_2 < \dots < i_m \leq n,$$

Example 4. Assume $V = \mathbb{C}^G = L^2(G)$, $G = \mathbb{Z}/2n\mathbb{Z}$, $f, g \in V$. I tried to adopt the theory of Fourier analysis on finite abelian groups but failed.

Proposition 1 (Decomposition of 2-forms). Assume $V = \mathbb{R}^{2n}$, $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, then there exists a basis $\{e'_i\}_{i=1}^{2n}$ of V such that

$$\xi = \sum_{i=1}^n a_i e'_{2i-1} \wedge e'_{2i}, \quad \sum_{i=1}^n a_i^2 = 1, \quad a_1 \geq a_2 \dots \geq a_n \geq 0,$$

Especially, when $n = 2$, we can write $\xi = \cos \theta e'_1 \wedge e'_2 + \sin \theta e'_3 \wedge e'_4$; when $n = 3$, we can write $\xi = \cos \theta e'_1 \wedge e'_2 + \sin \theta \cos \phi e'_3 \wedge e'_4 + \sin \theta \sin \phi e'_5 \wedge e'_6$.

Proof. This is equivalent to the fact that a real skew-symmetric matrix $A \in M(2n, 2n, \mathbb{R})$ can be written in the form $A = Q^t \Sigma Q$, where $Q \in SO(2n)$ and Σ is the real canonical form.

$$a_1 = \max_{e'_1, e'_2} \langle \xi, e'_1 \wedge e'_2 \rangle,$$

a_2, \dots, a_n can be determined recursively. □

Example 5. $n = 3$, $V = \mathbb{R}^6$, $k = l = 1$, $C(n, k, l) = \frac{2!2!2!}{1!1!1!3!} = \frac{4}{3}$. For each $P \subset [2n]$, $|P| = 2k + 2l = 4$, the bilinear form $\sum_{I \cup J = P} u_I v_J \text{sgn}(\sigma_{I, J})$ contains $\binom{2k+2l}{2k} = 6$ terms. But if we use the decomposition of 2-forms, for a given $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = a_1 e_1 \wedge e_2 + a_2 e_3 \wedge e_4 + a_3 e_5 \wedge e_6, \quad a_1^2 + a_2^2 + a_3^2 = 1,$$

i) If $\eta = \sum_{i=1}^n b_i e_{2i-1} \wedge e_{2i}$, $\sum_{i=1}^n b_i^2 = 1$, then

$$\xi \wedge \eta = \sum_{1 \leq i < j \leq n} (a_i b_j + a_j b_i) e_{2i-1} \wedge e_{2i} \wedge e_{2j-1} \wedge e_{2j},$$

$$\|\xi \wedge \eta\|^2 = \sum_{1 \leq i < j \leq n} (a_i b_j + a_j b_i)^2 = (a_1 b_2 + a_2 b_1)^2 + (a_1 b_3 + a_3 b_1)^2 + (a_2 b_3 + a_3 b_2)^2,$$

$$\begin{aligned} \frac{4}{3}\|\xi\|^2\|\eta\|^2 - \|\xi \wedge \eta\|^2 &= \frac{4}{3}(a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2) + \frac{1}{3}\sum_{i<j}(a_i^2b_j^2 + a_j^2b_i^2) - 2\sum_{i<j}a_ib_ja_jb_i \\ &= \frac{1}{3}\sum_{i<j}(a_ib_j - a_jb_i)^2 + \frac{2}{3}\sum_{i<j}(a_ib_i - a_jb_j)^2 \geq 0, \end{aligned}$$

ii) If $\eta = \sum_{i<j} b_{ij}e_i \wedge e_j$, with $b_{2i-1,2i} = 0, 1 \leq i \leq n$, then η lies in the orthogonal complement of $\bigoplus_{i=1}^n \mathbb{R}e_{2i-1} \wedge e_{2i}$ in $\bigwedge^2 V$. Then we have

$$\|\xi \wedge \eta\|^2 = a_3^2(b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{24}^2) + a_2^2(b_{15}^2 + b_{16}^2 + b_{25}^2 + b_{26}^2) + a_1^2(b_{35}^2 + b_{36}^2 + b_{45}^2 + b_{46}^2),$$

So $\|\xi \wedge \eta\|^2 \leq \|\xi\|^2\|\eta\|^2 \leq \frac{4}{3}\|\xi\|^2\|\eta\|^2$ holds.

iii) Consider arbitrary $\eta = \sum_{i<j} b_{ij}e_i \wedge e_j$, $\|\xi \wedge \eta\|^2 \leq \frac{4}{3}\|\xi\|^2\|\eta\|^2$ holds by combining the two cases above.

Example 6. $n \geq 3$, $V = \mathbb{R}^{2n}$, $k = l = 1$, $C(n, k, l) = \frac{(n-1)!(n-1)!2!}{1!1!(n-2)!n!} = \frac{2(n-1)}{n}$. Using the decomposition of 2-forms, for a given $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = \sum_{i=1}^n a_i e_{2i-1} \wedge e_{2i}, \quad \sum_{i=1}^n a_i^2 = 1, \quad a_1 \geq a_2 \dots \geq a_n \geq 0,$$

i) If $\eta = \sum_{i=1}^n b_i e_{2i-1} \wedge e_{2i}$, $\sum_{i=1}^n b_i^2 = 1$, then

$$\xi \wedge \eta = \sum_{1 \leq i < j \leq n} (a_i b_j + a_j b_i) e_{2i-1} \wedge e_{2i} \wedge e_{2j-1} \wedge e_{2j},$$

$$\begin{aligned} \frac{2(n-1)}{n}\|\xi\|^2\|\eta\|^2 - \|\xi \wedge \eta\|^2 &= \frac{2(n-1)}{n} \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) - \sum_{1 \leq i < j \leq n} (a_i b_j + a_j b_i)^2 \\ &= \frac{2(n-1)}{n} \left(\sum_{i=1}^n a_i^2 b_i^2 \right) + \frac{n-2}{n} \sum_{i<j} (a_i^2 b_j^2 + a_j^2 b_i^2) - 2 \sum_{i<j} a_i b_j a_j b_i \\ &= \frac{n-2}{n} \sum_{i<j} (a_i b_j - a_j b_i)^2 + \frac{2}{n} \sum_{i<j} (a_i b_i - a_j b_j)^2 \geq 0, \end{aligned}$$

ii) If $\eta = \sum_{i<j} b_{ij}e_i \wedge e_j$, with $b_{2i-1,2i} = 0, 1 \leq i \leq n$, then η lies in the orthogonal complement of $\bigoplus_{i=1}^n \mathbb{R}e_{2i-1} \wedge e_{2i}$ in $\bigwedge^2 V$. Then $\|\xi \wedge \eta\|^2 \leq \|\xi\|^2\|\eta\|^2 \leq \frac{2(n-1)}{n}\|\xi\|^2\|\eta\|^2$ holds.

iii) For arbitrary $\eta = \sum_{i<j} b_{ij}e_i \wedge e_j$, $\|\xi \wedge \eta\|^2 \leq \frac{2(n-1)}{n}\|\xi\|^2\|\eta\|^2$ holds by combining the two cases above. So we finished proving the case when $k = l = 1$.

Example 7. $k = 1, l \geq 2, n \geq l + 1$, $C(n, k, l) = \frac{(l+1)!(n-1)!(n-l)!}{1!l!n!(n-1-l)!} = \frac{(l+1)(n-l)}{n}$. Using the decomposition of 2-forms, for a given $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = \sum_{i=1}^n a_i e_{2i-1} \wedge e_{2i}, \quad \sum_{i=1}^n a_i^2 = 1, \quad a_1 \geq a_2 \dots \geq a_n \geq 0,$$

1) We first consider the case when $l = 2$, now $C(n, k, l) = \frac{3(n-2)}{n}$.

i) If $\eta = \sum_{1 \leq i < j \leq n} b_{ij}e_{2i-1} \wedge e_{2i} \wedge e_{2j-1} \wedge e_{2j}$, $\sum_{1 \leq i < j \leq n} b_{ij}^2 = 1$, then

$$\xi \wedge \eta = \sum_{p < i < j} (a_p b_{ij} + a_i b_{pj} + a_j b_{pi}) e_{\{2p-1, 2p, 2i-1, 2i, 2j-1, 2j\}},$$

$$\begin{aligned}
& \frac{3(n-2)}{n} \|\xi\|^2 \|\eta\|^2 - \|\xi \wedge \eta\|^2 = \frac{3(n-2)}{n} \left(\sum_p a_p^2 \right) \left(\sum_{i < j} b_{ij}^2 \right) - \sum_{p < i < j} (a_p b_{ij} + a_i b_{pj} + a_j b_{pi})^2 \\
&= \frac{3(n-2)}{n} \sum_{i < j} (a_i^2 + a_j^2) b_{ij}^2 + \frac{2(n-3)}{n} \sum_{p < i < j} (a_p^2 b_{ij}^2 + a_i^2 b_{pj}^2 + a_j^2 b_{pi}^2) \\
&\quad - 2 \sum_{p < i < j} (a_p b_{ij} a_i b_{pj} + a_p b_{ij} a_j b_{pi} + a_i b_{pj} a_j b_{pi}) \\
&= \frac{2(n-3)}{n} \sum_{p < i < j} (a_p^2 b_{ij}^2 + a_i^2 b_{pj}^2 + a_j^2 b_{pi}^2 - a_p b_{ij} a_i b_{pj} - a_p b_{ij} a_j b_{pi} - a_i b_{pj} a_j b_{pi}) \\
&\quad + \frac{3}{n} \sum_{p < i < j} ((a_p^2 + a_i^2) b_{pi}^2 + (a_p^2 + a_j^2) b_{pj}^2 + (a_i^2 + a_j^2) b_{ij}^2 - 2(a_p b_{ij} a_i b_{pj} + a_p b_{ij} a_j b_{pi} + a_i b_{pj} a_j b_{pi})) \geq 0,
\end{aligned}$$

2) For arbitrary $l \geq 2$, now $C(n, k, l) = \frac{(l+1)(n-l)}{n}$.

i) If $\eta = \sum_{|I|=l, I \subset [n]} b_I e_{\{2i-1, 2i, i \in I\}}$, $\sum_{|I|=l, I \subset [n]} b_I^2 = 1$, then

$$\begin{aligned}
& \xi \wedge \eta = \sum_{|J|=l+1, J \subset [n]} \left(\sum_{i \in J} a_i b_{J \setminus \{i\}} \right) e_{\{2j-1, 2j, j \in J\}}, \\
& \frac{(l+1)(n-l)}{n} \|\xi\|^2 \|\eta\|^2 - \|\xi \wedge \eta\|^2 = \frac{(l+1)(n-l)}{n} \left(\sum_p a_p^2 \right) \left(\sum_{|I|=l, I \subset [n]} b_I^2 \right) - \sum_{|J|=l+1, J \subset [n]} \left(\sum_{i \in J} a_i b_{J \setminus \{i\}} \right)^2 \\
&= \frac{(l+1)(n-l)}{n} \sum_{|I|=l, I \subset [n]} \left(\sum_{i \in I} a_i^2 \right) b_I^2 + \frac{l(n-l-1)}{n} \sum_{|J|=l+1, J \subset [n]} \left(\sum_{i \in J} a_i^2 b_{J \setminus \{i\}}^2 \right) \\
&\quad - 2 \sum_{|J|=l+1, J \subset [n]} \left(\sum_{i < j \in J} a_i b_{J \setminus \{i\}} a_j b_{J \setminus \{j\}} \right) \\
&= \frac{n-1-l}{n} \sum_{|J|=l+1, J \subset [n]} \left(l \sum_{i \in J} a_i^2 b_{J \setminus \{i\}}^2 - 2 \sum_{i < j \in J} a_i b_{J \setminus \{i\}} a_j b_{J \setminus \{j\}} \right) \\
&\quad + \frac{l+1}{n} \sum_{|J|=l+1, J \subset [n]} \left(\sum_{i \in J} \left(\sum_{j \in J \setminus \{i\}} a_j^2 \right) b_{J \setminus \{i\}}^2 - 2 \sum_{i < j \in J} a_i b_{J \setminus \{i\}} a_j b_{J \setminus \{j\}} \right) \geq 0,
\end{aligned}$$

Example 8 (Generalizing Hadamard's inequality). If $\xi \in \bigwedge^{2k} V$ satisfies that there exists $\xi_1, \xi_2, \dots, \xi_k \in \bigwedge^2 V$ such that $\xi = \xi_1 \wedge \xi_2 \dots \wedge \xi_k$, and $\eta \in \bigwedge^{2l} V$ satisfies that there exists $\eta_1, \eta_2, \dots, \eta_l \in \bigwedge^2 V$ such that $\eta = \eta_1 \wedge \eta_2 \dots \wedge \eta_l$, then

Conjecture 2 (Birch). The only remaining case is when $n = 5$, $z_1, z_2, \dots, z_n \in \mathbb{C}$ satisfy the constraint $\sum_{i=1}^n |z_i|^2 = n$, then

$$\Delta = \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 \leq n^n,$$

The equality holds when z_1, z_2, \dots, z_n are the vertices of a pentagon on the unit circle.

$$f(z) = \prod_{i=1}^n (z - z_i), \quad \Delta = \prod_{i=1}^n |f'(z_i)|, \quad f_{\max}(z) = z^n - c, \quad |f'_{\max}(z_i)| = |nz_i^{n-1}| = n,$$

References

[1] Kailiang Lin, tba.