

# Linear stability of RMHD equations on 2D finite channel

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## Abstract

This is a note about linear stability of RMHD equations on 2D finite channel.

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## 1 Introduction

$\Omega = [0, 1] \times \mathbb{T}$ ,  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ . The equilibrium is  $V_s = 0$ ,  $B_s = (0, b(x))$ ,  $P_s = 0$ , total magnetic field is  $\tilde{B} = B_s + B$ , where  $B$  is the perturbation. We assume that  $b$  is monotone positive. Total velocity field is  $V_s + v = v$ , total pressure is  $P_s + p = p$ , and they are the same as the perturbations  $v, p$ , then the original RMHD system has the following form:

$$\partial_t v + v \cdot \nabla v + \nabla p = \tilde{B} \cdot \nabla \tilde{B}, \quad (1)$$

$$\partial_t \tilde{B} - \eta \Delta \tilde{B} + v \cdot \nabla \tilde{B} = \tilde{B} \cdot \nabla v, \quad (2)$$

$$\nabla \cdot v = \nabla \cdot \tilde{B} = 0, \quad (3)$$

Nonlinear equations for the perturbations are

$$\begin{aligned} \partial_t v_x &= -\partial_x p - (v_x \partial_x + v_y \partial_y) v_x + (B_x \partial_x + (b + B_y) \partial_y) B_x, \\ \partial_t v_y &= -\partial_y p - (v_x \partial_x + v_y \partial_y) v_y + B_x \partial_x (b + B_y) + (b + B_y) \partial_y B_y, \\ \partial_t B_x &= (B_x \partial_x + (b + B_y) \partial_y) v_x - (v_x \partial_x + v_y \partial_y) B_x + \eta \Delta B_x, \\ \partial_t B_y &= (B_x \partial_x + (b + B_y) \partial_y) v_y - v_x \partial_x (b + B_y) - v_y \partial_y B_y + \eta \Delta B_y, \\ \nabla \cdot v &= \nabla \cdot B = 0, \end{aligned}$$

Linearized equations for the perturbations are:

$$\partial_t v_x = -\partial_x p + b \partial_y B_x, \quad (4)$$

$$\partial_t v_y = -\partial_y p + b \partial_y B_y + b' B_x, \quad (5)$$

$$\partial_t B_x = b \partial_y v_x + \eta \Delta B_x, \quad (6)$$

$$\partial_t B_y = b \partial_y v_y - b' v_x + \eta \Delta B_y, \quad (7)$$

$$\nabla \cdot v = \nabla \cdot B = 0, \quad (8)$$

with Navier slip boundary conditions

$$v_x|_{x=0,1} = B_x|_{x=0,1} = 0,$$

Taking Fourier transform in  $y$ , we get for  $\alpha \neq 0$ ,

$$\partial_t \widehat{v}_x = -\partial_x \widehat{p} + i\alpha b \widehat{B}_x, \quad (9)$$

$$\partial_t \widehat{v}_y = -i\alpha \widehat{p} + i\alpha b \widehat{B}_y + b' \widehat{B}_x, \quad (10)$$

$$\partial_t \widehat{B}_x = i\alpha b \widehat{v}_x + \eta(\partial_x^2 - \alpha^2) \widehat{B}_x, \quad (11)$$

$$\partial_t \widehat{B}_y = i\alpha b \widehat{v}_y - b' \widehat{v}_x + \eta(\partial_x^2 - \alpha^2) \widehat{B}_y, \quad (12)$$

$$\partial_x \widehat{v}_x + i\alpha \widehat{v}_y = 0, \quad \partial_x \widehat{B}_x + i\alpha \widehat{B}_y = 0, \quad (13)$$

Eliminating  $\widehat{p}, \widehat{v}_y, \widehat{B}_y$  from [equation 10](#) gives

$$\partial_t \partial_x \widehat{v}_x = \partial_t (-i\alpha \widehat{v}_y) = -\alpha^2 \widehat{p} + i\alpha b \partial_x \widehat{B}_x - i\alpha b' \widehat{B}_x,$$

$$\partial_t \partial_x^2 \widehat{v}_x = -\alpha^2 \partial_x \widehat{p} + i\alpha \partial_x (b \partial_x \widehat{B}_x) - i\alpha \partial_x (b' \widehat{B}_x), \quad \partial_x \widehat{p} = -\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x,$$

$$\begin{aligned} \partial_t \partial_x^2 \widehat{v}_x &= -\alpha^2 (-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha \partial_x (b \partial_x \widehat{B}_x) - i\alpha \partial_x (b' \widehat{B}_x) \\ &= -\alpha^2 (-\partial_t \widehat{v}_x + i\alpha b \widehat{B}_x) + i\alpha (b \partial_x^2 \widehat{B}_x - b'' \widehat{B}_x), \end{aligned}$$

$$\partial_t (\partial_x^2 - \alpha^2) \widehat{v}_x = \alpha b (\partial_x^2 - \alpha^2) i \widehat{B}_x - \alpha b'' i \widehat{B}_x, \quad \partial_t i \widehat{B}_x = -\alpha b \widehat{v}_x + \eta (\partial_x^2 - \alpha^2) i \widehat{B}_x,$$

Let  $\xi = \widehat{v}_x, \psi = i \widehat{B}_x$ , we have the following system of evolution:

$$\partial_t \xi = \alpha (\partial_x^2 - \alpha^2)^{-1} (b (\partial_x^2 - \alpha^2) \psi - b'' \psi), \quad (14)$$

$$\partial_t \psi = -\alpha b \xi + \eta (\partial_x^2 - \alpha^2) \psi, \quad (15)$$

Denote  $\mathcal{L}_\alpha$  the linear operator of the above equations, and consider the eigenvalue problem of operator  $\mathcal{L}_\alpha$ . If  $c \in \sigma_p(\mathcal{L}_\alpha)$  with associated eigenfunctions  $\xi, \psi$ , then

$$c\xi = \alpha (\partial_x^2 - \alpha^2)^{-1} (b (\partial_x^2 - \alpha^2) \psi - b'' \psi), \quad c\psi = -\alpha b \xi + \eta (\partial_x^2 - \alpha^2) \psi,$$

$$\begin{aligned} \alpha b \xi &= -(c - \eta (\partial_x^2 - \alpha^2)) \psi, \quad c (\partial_x^2 - \alpha^2) \xi = \alpha (b (\partial_x^2 - \alpha^2) - b'') \psi, \\ -c (\partial_x^2 - \alpha^2) b^{-1} (c - \eta (\partial_x^2 - \alpha^2)) \psi &= \alpha^2 (b (\partial_x^2 - \alpha^2) - b'') \psi, \end{aligned}$$

Let  $\psi = bg$ , then

$$\begin{aligned} b^{-1} (c - \eta (\partial_x^2 - \alpha^2)) bg &= cg - c\eta b^{-1} (\partial_x^2 - \alpha^2) bg, \\ \alpha^2 (b (\partial_x^2 - \alpha^2) - b'') bg &= \alpha^2 (b^2 (\partial_x^2 - \alpha^2) + 2bb' \partial_x) g, \end{aligned}$$

Summing up the above two equations, we get the Orr-Sommerfeld type equation for linearized RMHD system on a 2-dimensional finite channel:

$$(c^2 + \alpha^2 b^2) (\partial_x^2 - \alpha^2) g + 2\alpha^2 b b' \partial_x g - c\eta (\partial_x^2 - \alpha^2) b^{-1} (\partial_x^2 - \alpha^2) bg = 0,$$

Boundary conditions are

$$\psi|_{x=0,1} = 0, \quad \xi|_{x=0,1} = 0, \quad (c - \eta (\partial_x^2 - \alpha^2)) \psi|_{x=0,1} = 0,$$

$$g|_{x=0,1} = 0, \quad \partial_x^2 (bg)|_{x=0,1} = b \partial_x^2 g + 2b' \partial_x g|_{x=0,1} = 0,$$

Let us denote by  $OS_\alpha$  the Orr-Sommerfeld type fourth-order operator

$$OS_\alpha(g) \triangleq (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g + 2\alpha^2 b b' \partial_x g - c\eta(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)bg,$$

Following [11], we study the resolvent estimates of the linearized operator under the Navier-slip boundary conditions. More precisely, we consider the equation

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g + 2\alpha^2 b b' \partial_x g - c\eta(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)bg = F,$$

$$g|_{x=0,1} = 0, \quad b\partial_x^2 g + 2b'\partial_x g|_{x=0,1} = 0,$$

Substitute  $h = b^{-1}(\partial_x^2 - \alpha^2)bg = b^{-1}(\partial_x^2 - \alpha^2)\psi$ , we have

$$h|_{x=0,1} = 0, \quad g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh,$$

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh + 2\alpha^2 b b' \partial_x b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh - c\eta(\partial_x^2 - \alpha^2)h,$$

Using Green's function, the operator  $(\partial_x^2 - \alpha^2)^{-1}$  with Dirichlet's boundary condition can be represented as an integral operator:

$$G(x, x') = \begin{cases} -\frac{\sinh \alpha(1-x') \sinh \alpha x}{\alpha \sinh \alpha}, & x \leq x' \\ -\frac{\sinh \alpha x' \sinh \alpha(1-x)}{\alpha \sinh \alpha}, & x > x' \end{cases}, \quad (\partial_x^2 - \alpha^2)^{-1}u(x) = \int_0^1 G(x, x')u(x')dx',$$

$$g(x) = b^{-1}(x)[(\partial_x^2 - \alpha^2)^{-1}bh](x) = b^{-1}(x) \int_0^1 G(x, x')b(x')h(x')dx',$$

As an integral operator,  $\partial_x(\partial_x^2 - \alpha^2)^{-1}$  is represented as follows:

$$\partial_x G(x, x') = \begin{cases} -\frac{\alpha \sinh \alpha(1-x') \cosh \alpha x}{\alpha \sinh \alpha}, & x \leq x' \\ \frac{\alpha \sinh \alpha x' \cosh \alpha(1-x)}{\alpha \sinh \alpha}, & x > x' \end{cases}, \quad \partial_x(\partial_x^2 - \alpha^2)^{-1}u(x) = \int_0^1 \partial_x G(x, x')u(x')dx',$$

We investigate the case of exponential background magnetic profile  $b(x) = e^{\lambda x}$  for convenience. The Sobolev space we concern is

$$H_0^1([0, 1]) = \{u : [0, 1] \rightarrow \mathbb{C}, \|u\|_{H^1} < +\infty, u(0) = u(1) = 0\}, \quad \|u\|_{H^1}^2 = \int_0^1 \partial_x u \overline{\partial_x u} dx,$$

A set of orthonormal basis of  $H_0^1([0, 1])$  is  $\{e_k = \frac{\sqrt{2} \sin k\pi x}{k\pi}, k \in \mathbb{Z}_+\}$ , and they are all the eigenfunctions of operator  $\partial_x^2$ , with eigenvalues  $\partial_x^2 e_k = -k^2 \pi^2 e_k$ . Under the exponential background magnetic profile, we have

$$(\partial_x^2 - \alpha^2)g = (\partial_x^2 - \alpha^2)b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh = h + 2(b^{-1})'\partial_x(\partial_x^2 - \alpha^2)^{-1}bh + (b^{-1})''(\partial_x^2 - \alpha^2)^{-1}bh,$$

$$\partial_x g = \partial_x b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh = (b^{-1})'(\partial_x^2 - \alpha^2)^{-1}bh + b^{-1}\partial_x(\partial_x^2 - \alpha^2)^{-1}bh,$$

The integral form of the above two equations are:

$$\begin{aligned} (\partial_x^2 - \alpha^2)g(x) &= h(x) + \lambda b^{-1}(x)[(\lambda - 2\partial_x)(\partial_x^2 - \alpha^2)^{-1}bh](x) \\ &= h(x) + \lambda b^{-1}(x) \int_0^1 (\lambda - 2\partial_x)G(x, x')b(x')h(x')dx', \end{aligned}$$

$$\partial_x g(x) = b^{-1}(x)[(\partial_x - \lambda)(\partial_x^2 - \alpha^2)^{-1}bh](x) = b^{-1}(x) \int_0^1 (\partial_x - \lambda)G(x, x')b(x')h(x')dx',$$

The Orr-Sommerfeld type equation in terms of  $h$  becomes

$$\begin{aligned}
F &\triangleq OS_\alpha(g) = (c^2 + \alpha^2 b^2)(h + 2(b^{-1})' \partial_x (\partial_x^2 - \alpha^2)^{-1} b h + (b^{-1})'' (\partial_x^2 - \alpha^2)^{-1} b h) \\
&\quad + 2\alpha^2 b b' ((b^{-1})' (\partial_x^2 - \alpha^2)^{-1} b h + b^{-1} \partial_x (\partial_x^2 - \alpha^2)^{-1} b h) - c \eta (\partial_x^2 - \alpha^2) h, \\
F &= (c^2 + \alpha^2 b^2)(h - 2\lambda b^{-1} \partial_x (\partial_x^2 - \alpha^2)^{-1} b h + \lambda^2 b^{-1} (\partial_x^2 - \alpha^2)^{-1} b h) + 2\alpha^2 \lambda b (\partial_x - \lambda) (\partial_x^2 - \alpha^2)^{-1} b h - c \eta (\partial_x^2 - \alpha^2) h, \\
F &= c^2 (h - 2\lambda b^{-1} \partial_x (\partial_x^2 - \alpha^2)^{-1} b h + \lambda^2 b^{-1} (\partial_x^2 - \alpha^2)^{-1} b h) + \alpha^2 b^2 h - \lambda^2 \alpha^2 b (\partial_x^2 - \alpha^2)^{-1} b h - c \eta (\partial_x^2 - \alpha^2) h \\
&= c^2 (h + \lambda b^{-1} (\lambda - 2\partial_x) (\partial_x^2 - \alpha^2)^{-1} b h) + \alpha^2 b^2 h - \lambda^2 \alpha^2 b (\partial_x^2 - \alpha^2)^{-1} b h - c \eta (\partial_x^2 - \alpha^2) h,
\end{aligned}$$

While taking inner product with  $h$ , the first and the third term in the right hand side above is nontrivial. We give a closer look at the first term below:

$$\begin{aligned}
(\partial_x^2 - \alpha^2)g &= h + \lambda b^{-1} (\lambda - 2\partial_x) (\partial_x^2 - \alpha^2)^{-1} b h, \\
\langle (\partial_x^2 - \alpha^2)g, h \rangle &= \int_0^1 b^{-1} (\partial_x^2 - \alpha^2)g (\partial_x^2 - \alpha^2) b \bar{g} = \int_0^1 b^{-1} (\partial_x^2 - \alpha^2)g (b (\partial_x^2 - \alpha^2) \bar{g} + \lambda^2 b \bar{g} + 2\lambda b \partial_x \bar{g}) \\
&= \|(\partial_x^2 - \alpha^2)g\|_2^2 + \lambda^2 \int_0^1 (\partial_x^2 - \alpha^2)g \bar{g} + 2\lambda \int_0^1 (\partial_x^2 - \alpha^2)g \cdot \partial_x \bar{g} \\
&= \|(\partial_x^2 - \alpha^2)g\|_2^2 - \lambda^2 \|g'\|_2^2 - \alpha^2 \lambda^2 \|g\|_2^2 + 2\lambda (\langle g'', g' \rangle - \alpha^2 \langle g, g' \rangle), \\
\|(\partial_x^2 - \alpha^2)g\|_2^2 &= \|g''\|_2^2 + \alpha^4 \|g\|_2^2 + 2\alpha^2 \|g'\|_2^2,
\end{aligned}$$

The third term is dealt with as follows:

$$\begin{aligned}
b(\partial_x^2 - \alpha^2)^{-1} b h &= b \psi, \quad h = b^{-1} (\partial_x^2 - \alpha^2) \psi, \\
\langle b \psi, h \rangle &= \int_0^1 \psi (\partial_x^2 - \alpha^2) \bar{\psi} dx = -\|\psi'\|_2^2 - \alpha^2 \|\psi\|_2^2,
\end{aligned}$$

Combining the equations above together, we have

$$\begin{aligned}
\langle F, ch \rangle &= \langle c^2 (\partial_x^2 - \alpha^2)g, ch \rangle + \bar{c} \alpha^2 \|b h\|_2^2 - \bar{c} \lambda^2 \alpha^2 \langle b \psi, h \rangle - |c|^2 \eta \langle (\partial_x^2 - \alpha^2)h, h \rangle \\
&= |c|^2 c (\|(\partial_x^2 - \alpha^2)g\|_2^2 - \lambda^2 \|g'\|_2^2 - \alpha^2 \lambda^2 \|g\|_2^2 + 2\lambda (\langle g'', g' \rangle - \alpha^2 \langle g, g' \rangle)) \\
&\quad + \bar{c} (\alpha^2 \|b h\|_2^2 + \lambda^2 \alpha^2 \|\psi'\|_2^2 + \lambda^2 \alpha^4 \|\psi\|_2^2) + |c|^2 \eta (\|h'\|_2^2 + \alpha^2 \|h\|_2^2) \\
&= |c|^2 c (\|g''\|_2^2 + (2\alpha^2 - \lambda^2) \|g'\|_2^2 + \alpha^2 (\alpha^2 - \lambda^2) \|g\|_2^2 + 2\lambda (\langle g'', g' \rangle - \alpha^2 \langle g, g' \rangle)) \\
&\quad + \bar{c} (\alpha^2 \|b h\|_2^2 + \lambda^2 \alpha^2 \|\psi'\|_2^2 + \lambda^2 \alpha^4 \|\psi\|_2^2) + |c|^2 \eta (\|h'\|_2^2 + \alpha^2 \|h\|_2^2)
\end{aligned}$$

We see that when  $\Re(c) > 0$ ,  $-1 \leq \lambda \leq 1$ , the real part of the right hand side of the above equation is strictly positive for non-zero  $h$ .

$$\|b h\|_2^2 = \|(\partial_x^2 - \alpha^2) \psi\|_2^2 = \|\psi''\|_2^2 + 2\alpha^2 \|\psi'\|_2^2 + \alpha^4 \|\psi\|_2^2,$$

**Question 1.** 1) Prove that when  $\eta = 0$ , if  $c \in \sigma(\mathcal{L}_\alpha)$  then there exist  $x_c \in [0, 1]$  such that  $c = \pm i \alpha b(x_c)$ . It means that  $\mathcal{L}_\alpha$  can only have embedding eigenvalues. An equivalent form of this proposition appears in [10].

2) Does the Rayleigh equation for Euler's equation only admits embedding eigenvalues?

*Proof.* Method 1: When  $c^2 + \alpha^2 b^2 \neq 0$ , rewrite the equation as follows

$$(\partial_x^2 - \alpha^2)g + \frac{2\alpha^2 b b' \partial_x g}{c^2 + \alpha^2 b^2} = 0, \quad g(0) = g(1) = 0,$$

It is an second order elliptic ordinary differential equation on  $[0, 1]$  with Dirichlet's boundary condition.

□

$$\mathcal{H} = \{(\psi, \xi), \psi, \xi \in H_0^1([0, 1])\}, \quad \|(\psi, \xi)\|_{\mathcal{H}}^2 = \int_0^1 \partial_x \psi \overline{\partial_x \psi} + \partial_x \xi \overline{\partial_x \xi},$$

$$\partial_t \xi = \alpha(\partial_x^2 - \alpha^2)^{-1}(b(\partial_x^2 - \alpha^2)\psi - b''\psi) = \alpha(b\psi + K_1\psi),$$

where  $K_1$  is a compact operator defined by

$$K_1\psi = (\partial_x^2 - \alpha^2)^{-1}(-2b'\partial_x\psi - 2b''\psi) = -2(\partial_x^2 - \alpha^2)^{-1}\partial_x(b'\psi),$$

and we have  $\overline{K_1\psi} = K_1\overline{\psi}$ . Notice that  $v_x, B_x \in \mathbb{R}, \xi = \widehat{v}_x, \psi = i\widehat{B}_x$  only implies that  $\widehat{v}_x(\alpha) = \overline{\widehat{v}_x(-\alpha)}, \widehat{B}_x(\alpha) = \overline{\widehat{B}_x(-\alpha)}$ ,  $\xi, \psi$  are complex-valued functions.

$$\int_0^1 \partial_t \xi \overline{\xi} = \int_0^1 \alpha(b\psi + K_1\psi) \overline{\xi}, \quad \int_0^1 \partial_t \psi \overline{\psi} = \int_0^1 (-\alpha b \xi + \eta(\partial_x^2 - \alpha^2)\psi) \overline{\psi},$$

$$\partial_t(\|\psi\|_2^2 + \|\xi\|_2^2) = \int_0^1 \alpha(K_1\psi \overline{\xi} + \xi K_1\overline{\psi}) + \eta(\partial_x^2 - \alpha^2)\psi \overline{\psi} + \psi \eta(\partial_x^2 - \alpha^2)\overline{\psi},$$

When  $b(x) = e^{\lambda x}$ , we have  $K_1\psi = (\partial_x^2 - \alpha^2)^{-1}(-2\lambda b\partial_x\psi - 2\lambda^2 b\psi) = -2\lambda(\partial_x^2 - \alpha^2)^{-1}\partial_x(b\psi)$ .

$$\partial_t \psi = -\alpha b \xi + \eta(\partial_x^2 - \alpha^2)\psi = -\alpha b(\xi(0) + \int_0^t \alpha(b\psi(t') + K\psi(t'))dt') + \eta(\partial_x^2 - \alpha^2)\psi,$$

So the evolutionary equation of  $\psi$  takes the following form, which is similar to the wave equation:

$$\partial_t^2 \psi = -\alpha^2 b(b\psi + K_1\psi) + \eta(\partial_x^2 - \alpha^2)\partial_t \psi,$$

Substitute  $\psi = bg$  and let  $\eta = 0, B = b^2$ , we have

$$\partial_t^2(bg) = -\alpha^2 b(b^2g + Kbg), \quad K_1bg = -2(\partial_x^2 - \alpha^2)^{-1}\partial_x(b'bg) = -(\partial_x^2 - \alpha^2)^{-1}\partial_x(B'g),$$

$$\partial_t^2 g + \alpha^2(b^2g + K_1bg) = \partial_t^2 g + \alpha^2(Bg - (\partial_x^2 - \alpha^2)^{-1}\partial_x(B'g)) = 0,$$

the above calculation is the same as equation (2.5) in [10]. We define

$$Kg = K_1bg = -(\partial_x^2 - \alpha^2)^{-1}\partial_x(B'g),$$

**Proposition 1** (Energy conservation on each frequency).

## 2 Spectral method

**Theorem 1.** Reference: Lecture notes on functional analysis II by Gongqing Zhang, p53 problem 5.5.10.  $\mathcal{H}$  is a Hilbert space,  $N$  is a normal operator on  $\mathcal{H}$  and its spectrum  $\sigma(N)$  is countable, then  $\mathcal{H}$  has a orthonormal basis  $B = \{y\}$  where  $y$  are eigenfunctions of  $N$ , and the Fourier expansion holds:

$$x = \sum_{y \in B} (x, y)y, \quad x \in \mathcal{H},$$

the Fourier coefficients  $(x, y)$  only have countably many nonzero elements.

*Proof.* 1) Eigenspaces of different eigenvalues are orthogonal. If  $f_1, f_2$  are two eigenfunctions of  $N$  with different eigenvalues  $\lambda_1, \lambda_2$ . When  $N$  is self-adjoint, we have  $\lambda_1, \lambda_2 \in \mathbb{R}$ ,

$$\lambda_1 \langle f_1, f_2 \rangle = \langle Nf_1, f_2 \rangle = \langle f_1, Nf_2 \rangle = \lambda_2 \langle f_1, f_2 \rangle,$$

Since  $\lambda_1 \neq \lambda_2$ ,  $\langle f_1, f_2 \rangle = 0$ . □

**Problem 1.** Eigenfunctions of  $\partial_x^2$  on  $H_0^1([0, 1])$ .

$$\partial_x^2 f = \lambda f, \quad -\xi^2 \hat{f} = \lambda \hat{f}, \quad \lambda = -\xi^2, \quad \text{supp } \hat{f} = \{\pm \xi\}, \quad f(x) = \hat{f}(\xi)e^{i\xi x} + \hat{f}(-\xi)e^{-i\xi x},$$

Boundary conditions are

$$f(0) = \hat{f}(\xi) + \hat{f}(-\xi) = 0, \quad f(1) = \hat{f}(\xi)e^{i\xi} + \hat{f}(-\xi)e^{-i\xi} = 0,$$

So we have  $e^{i\xi} = e^{-i\xi}, \xi = k\pi, k \in \mathbb{Z} \setminus \{0\}, \lambda = -k^2\pi^2, f = C \sin k\pi x$ .

Consider the case when  $b(x)$  is a positive constant. We have

$$OS_\alpha(g) = (c^2 + \alpha^2 b^2)(\partial_x^2 - \alpha^2)g - c\eta(\partial_x^2 - \alpha^2)^2 g, \quad g|_{x=0,1} = g'|_{x=0,1} = 0,$$

Let  $h = (\partial_x^2 - \alpha^2)g$ , we claim that when  $\Re(c) > 0, f \in H_0^1([0, 1])$ , the solution to  $OS_\alpha(g) = f$  uniquely exists. We have

$$\mathcal{L}_\alpha h \triangleq OS_\alpha(g) = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = f,$$

Assume that  $f = \sum_k (f, e_k) e_k$ , then  $(\partial_x^2 - \alpha^2)e_k = -(k^2\pi^2 + \alpha^2)e_k$ ,

$$(f, e_k) = (h, e_k)(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)), \quad (h, e_k) = \frac{(f, e_k)}{c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2)},$$

$$(g, e_k) = -\frac{(h, e_k)}{(k^2\pi^2 + \alpha^2)} = -\frac{(f, e_k)}{(c^2 + \alpha^2 b^2 + c\eta(k^2\pi^2 + \alpha^2))(k^2\pi^2 + \alpha^2)},$$

**Problem 2.** Eigenvalue problem: when  $b$  is constant, for what values of  $c$  is  $\mathcal{L}_\alpha$  not injective?

$$\mathcal{L}_\alpha h = (c^2 + \alpha^2 b^2)h - c\eta(\partial_x^2 - \alpha^2)h = 0,$$

When  $\eta = 0$ , eigenvalues are  $c = \pm i\alpha b$ , and their invariant space is the whole  $H_0^1([0, 1])$ . When  $\eta > 0$ , 1)  $c^2 + \alpha^2 b^2 = 0$ , then  $c \neq 0$ , since  $\partial_x^2 - \alpha^2$  is injective, we have  $h = 0$ .

2)  $c^2 + \alpha^2 b^2 \neq 0$ , take Fourier transform  $h = \sum_k \hat{h}_k e_k$ , we have

$$(c^2 + \alpha^2 b^2 + c\eta\alpha^2 + c\eta k^2\pi^2)\hat{h}_k = 0, \quad k \neq 0,$$

For small  $k$ ,  $\eta\alpha^2 + \eta k^2\pi^2 \leq 2|\alpha|b$ , the roots are two conjugate complex numbers with negative real parts and norm  $\alpha b$ . For large  $k$ ,  $\eta\alpha^2 + \eta k^2\pi^2 > 2|\alpha|b$ , the two real roots are

$$c_{1,2} = -\frac{\eta\alpha^2 + \eta k^2\pi^2 \pm \sqrt{(\eta\alpha^2 + \eta k^2\pi^2)^2 - 4\alpha^2 b^2}}{2},$$

$$A = \int_0^1 e^{\lambda x} \sin k\pi x dx = -\int_0^1 \frac{e^{\lambda x}}{\lambda} k\pi \cos k\pi x dx = \frac{k\pi}{\lambda^2} - (-1)^k e^{\lambda} \frac{k\pi}{\lambda^2} - \int_0^1 \frac{e^{\lambda x}}{\lambda^2} k^2\pi^2 \sin k\pi x dx,$$

$$\int_0^1 \frac{e^{\lambda x}}{\lambda^2} k^2\pi^2 \sin k\pi x dx = \frac{k^2\pi^2}{\lambda^2} A, \quad A = \frac{(1 - (-1)^k e^{\lambda})k\pi}{\lambda^2 + k^2\pi^2},$$

$$B = \int_0^1 e^{\lambda x} \cos m\pi x dx = (-1)^m \frac{e^{\lambda}}{\lambda} - \frac{1}{\lambda} + \int_0^1 \frac{e^{\lambda x}}{\lambda} m\pi \sin m\pi x dx,$$

$$\int_0^1 \frac{e^{\lambda x}}{\lambda} m\pi \sin m\pi x dx = - \int_0^1 \frac{e^{\lambda x}}{\lambda^2} m^2 \pi^2 \sin m\pi x dx = - \frac{m^2 \pi^2}{\lambda^2} B, \quad B = \frac{((-1)^m e^\lambda - 1)\lambda}{\lambda^2 + m^2 \pi^2},$$

When  $\lambda < 0$ , we may also use complex analysis techniques to calculate the above integrals:

$$\int_0^{+\infty} e^{\lambda x} \sin k\pi x dx = \Im \int_0^{+\infty} e^{(\lambda + ik\pi)x} dx = \Im \frac{1}{-\lambda - ik\pi} = \frac{k\pi}{\lambda^2 + k^2 \pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \sin k\pi x dx = A(1 + (-1)^k e^\lambda + (-1)^{2k} e^{2\lambda} + \dots) = \frac{A}{1 - (-1)^k e^\lambda}, \quad A = \frac{(1 - (-1)^k e^\lambda)k\pi}{\lambda^2 + k^2 \pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \cos m\pi x dx = \Re \int_0^{+\infty} e^{(\lambda + im\pi)x} dx = \Re \frac{1}{-\lambda - im\pi} = \frac{-\lambda}{\lambda^2 + m^2 \pi^2},$$

$$\int_0^{+\infty} e^{\lambda x} \cos m\pi x dx = B(1 + (-1)^m e^\lambda + (-1)^{2m} e^{2\lambda} + \dots) = \frac{B}{1 - (-1)^m e^\lambda}, \quad B = \frac{((-1)^m e^\lambda - 1)\lambda}{\lambda^2 + m^2 \pi^2},$$

I try to express the relationship of  $h$  and  $g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh$  on the frequency domain. Recall that  $e_k = \frac{\sqrt{2} \sin k\pi x}{k\pi}$ ,  $k \in \mathbb{Z}_+$ ,

$$g = \sum_k \widehat{g}_k e_k, \quad \widehat{g}_k = \langle g, e_k \rangle = \int_0^1 \partial_x g \overline{\partial_x e_k} dx,$$

$$h = \sum_k \widehat{h}_k e_k, \quad \widehat{h}_k = \langle h, e_k \rangle = \int_0^1 \partial_x h \overline{\partial_x e_k} dx,$$

$$\begin{aligned} (\widehat{bg})_k &= \langle bg, e_k \rangle = \int_0^1 \partial_x (bg) \overline{\partial_x e_k} = - \int_0^1 bg \overline{\partial_x^2 e_k} = \int_0^1 bg k^2 \pi^2 \overline{e_k} \\ &= \int_0^1 b \left( \sum_l \widehat{g}_l e_l \right) k^2 \pi^2 \overline{e_k} = \sum_l \widehat{g}_l \int_0^1 b k^2 \pi^2 e_l \overline{e_k}, \end{aligned}$$

$$e_l \overline{e_k} = \frac{2 \sin l\pi x \sin k\pi x}{lk\pi^2} = \frac{\cos(l-k)\pi x - \cos(l+k)\pi x}{lk\pi^2},$$

$$\int_0^1 b k^2 \pi^2 e_l \overline{e_k} = \int_0^1 b \frac{k}{l} (\cos(l-k)\pi x - \cos(l+k)\pi x),$$

$$\int_0^1 b \frac{k}{l} \cos(l-k)\pi x = \frac{k}{l} \int_0^1 e^{\lambda x} \cos(l-k)\pi x dx = \frac{k}{l} \frac{((-1)^{l-k} e^\lambda - 1)\lambda}{\lambda^2 + (l-k)^2 \pi^2} = B_1,$$

$$\int_0^1 b \frac{k}{l} \cos(l+k)\pi x = \frac{k}{l} \frac{((-1)^{l+k} e^\lambda - 1)\lambda}{\lambda^2 + (l+k)^2 \pi^2} = B_2,$$

$$B_1 - B_2 = \frac{k}{l} ((-1)^{l-k} e^\lambda - 1)\lambda \left( \frac{1}{\lambda^2 + (l-k)^2 \pi^2} - \frac{1}{\lambda^2 + (l+k)^2 \pi^2} \right) = \frac{4k^2 \pi^2 ((-1)^{l-k} e^\lambda - 1)\lambda}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2 k^2 \pi^4},$$

$$(\widehat{bg})_k = \sum_l \widehat{g}_l \frac{4k^2 \pi^2 ((-1)^{l-k} e^\lambda - 1)\lambda}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2 k^2 \pi^4},$$

If  $u, v \in H_0^1([0, 1])$ ,  $u = \sum_k \widehat{u}_k e_k$ ,  $v = \sum_k \widehat{v}_k e_k$  satisfy  $v = (\partial_x^2 - \alpha^2)u$ , then their Fourier coefficients satisfy

$$\widehat{v}_k = -(k^2 \pi^2 + \alpha^2) \widehat{u}_k, \quad \widehat{u}_k = -\frac{\widehat{v}_k}{k^2 \pi^2 + \alpha^2},$$

Since  $b^{-1}(x) = e^{-\lambda x}$ , the action of multiplier  $b^{-1}$  on the Fourier side is given by

$$(\widehat{b^{-1}g})_k = \sum_l \widehat{g}_l \frac{4k^2\pi^2((-1)^{l-k}e^{-\lambda} - 1)(-\lambda)}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

We may denote

$$K_\lambda(k, l) = \frac{4k^2\pi^2}{(\lambda^2 + (l^2 + k^2)\pi^2)^2 - 4l^2k^2\pi^4},$$

Consider representing the Fourier coefficients of  $h$  in terms of Fourier coefficients of  $g$ , where  $g = b^{-1}(\partial_x^2 - \alpha^2)^{-1}bh$ , we have

$$v = bh, \quad \psi = (\partial_x^2 - \alpha^2)^{-1}v, \quad g = b^{-1}\psi,$$

$$\widehat{v}_k = \sum_l \widehat{h}_l((-1)^{l-k}e^\lambda - 1)\lambda K_\lambda(k, l), \quad \widehat{\psi}_k = -\frac{\widehat{v}_k}{k^2\pi^2 + \alpha^2},$$

$$\begin{aligned} \widehat{g}_m &= \sum_k \widehat{\psi}_k((-1)^{l-k}e^{-\lambda} - 1)(-\lambda)K_\lambda(m, k) = \sum_k \widehat{v}_k \frac{((-1)^{l-k}e^{-\lambda} - 1)\lambda}{k^2\pi^2 + \alpha^2} K_\lambda(m, k) \\ &= \sum_{k,l} \frac{((-1)^{k-m}e^{-\lambda} - 1)\lambda}{k^2\pi^2 + \alpha^2} K_\lambda(m, k)((-1)^{l-k}e^\lambda - 1)\lambda K_\lambda(k, l)\widehat{h}_l \\ &= \sum_l \widehat{h}_l \sum_k \frac{\lambda^2((-1)^{k-m}e^{-\lambda} - 1)((-1)^{l-k}e^\lambda - 1)}{k^2\pi^2 + \alpha^2} K_\lambda(m, k)K_\lambda(k, l) \end{aligned}$$

### 3 Energy conservation of nonlinear RVMHD

Suppose  $u$  and  $b$  are the velocity field and magnetic field,  $p$  is the total pressure, we consider the original nonlinear RVMHD equations' energy conservation:

$$\partial_t(\|u\|_2^2 + \|b\|_2^2) = \partial_t \int_\Omega u \cdot u + b \cdot b = 2 \int_\Omega u \cdot (-\nabla p - u \cdot \nabla u + b \cdot \nabla b + \mu \Delta u) + b \cdot (b \cdot \nabla u - u \cdot \nabla b + \eta \Delta b),$$

$$\int_\Omega u \cdot \nabla p = \int_\Omega u \cdot \nabla p + p \nabla \cdot u = \int_\Omega \nabla \cdot (pu) = 0,$$

$$u \cdot (u \cdot \nabla u) = u \cdot \nabla \frac{|u|^2}{2}, \quad \nabla \cdot (|u|^2 u) = u \cdot \nabla |u|^2 + |u|^2 \nabla \cdot u,$$

$$\nabla \cdot (|b|^2 u) = u \cdot \nabla |b|^2 + |b|^2 \nabla \cdot u, \quad b \cdot (u \cdot \nabla b) = u \cdot \nabla \frac{|b|^2}{2},$$

$$\int_\Omega u \cdot (u \cdot \nabla u) = \frac{1}{2} \int_\Omega \nabla \cdot (|u|^2 u) = 0, \quad \int_\Omega b \cdot (u \cdot \nabla b) = \frac{1}{2} \int_\Omega \nabla \cdot (|b|^2 u) = 0,$$

$$\nabla \cdot ((u \cdot b)b) = (b \cdot \nabla)(u \cdot b) + (u \cdot b)(\nabla \cdot b), \quad (b \cdot \nabla)(u \cdot b) = u \cdot (b \cdot \nabla)b + b \cdot (b \cdot \nabla)u,$$

$$\int_\Omega u \cdot (b \cdot \nabla)b + b \cdot (b \cdot \nabla)u = \int_\Omega \nabla \cdot ((u \cdot b)b) = 0,$$

$$\int_\Omega u \cdot \Delta u = - \int_\Omega \nabla u_x \cdot \nabla u_x + \nabla u_y \cdot \nabla u_y + \nabla u_z \cdot \nabla u_z = - \int_\Omega |\nabla u|^2, \quad \int_\Omega b \cdot \Delta b = - \int_\Omega |\nabla b|^2,$$



Energy conservation for perturbations are

Representing  $(\partial_x^2)^{-1}$  with Dirichlet's boundary condition on  $[0, 1]$  as an integral operator:

$$\partial_x^2 v = u, \quad u|_{x=0,1} = 0, \quad v|_{x=0,1} = 0, \quad v(x) = \int_0^1 G(x, x') u(x') dx',$$

$$G(0, x') = G(1, x') = 0, \quad G(x, x') = \begin{cases} (x' - 1)x, & x \leq x' \\ (x - 1)x', & x > x' \end{cases},$$

In this case,  $\partial_x^2 G(x, x') = 0$  on  $[0, x')$  and  $(x', 1]$ , so it is linear on the above two intervals. Let

$$\partial_x G(x, x')|_{x=x'_-} = a, \quad \partial_x G(x, x')|_{x=x'_+} = b, \quad a = b - 1, \quad G(x', x') = ax' = -b(1 - x'),$$

so its solution is  $a = x' - 1, b = x'$ .

Representing  $(\partial_x^2 - \alpha^2)^{-1}$  on  $[0, 1]$  as an integral operator when  $\alpha \neq 0$ :

$$(\partial_x^2 - \alpha^2)v = u, \quad u|_{x=0,1} = 0, \quad v|_{x=0,1} = 0, \quad v(x) = \int_0^1 G(x, x') u(x') dx',$$

$$G(0, x') = G(1, x') = 0, \quad (\partial_x^2 - \alpha^2)v(x) = \int_0^1 (\partial_x^2 - \alpha^2)G(x, x') u(x') dx' = u(x),$$

$$g(x) = G(x, x'), \quad (\partial_x^2 - \alpha^2)g(x) = \delta_{x'}(x) = \delta(x - x'), \quad g(x) = \sum_{k \geq 1} \hat{g}(k) \sin k\pi x,$$

$$\hat{g}(k) = 2 \int_0^1 g(x) \sin k\pi x dx = 2 \int_0^1 \hat{g}(k) \sin^2 k\pi x dx = \hat{g}(k),$$

$$\hat{\delta}_{x'}(k) = 2 \int_0^1 \delta_{x'}(x) \sin k\pi x dx = 2 \sin k\pi x', \quad \hat{g}(k) = \frac{\hat{\delta}_{x'}(k)}{-k^2\pi^2 - \alpha^2} = \frac{2 \sin k\pi x'}{-k^2\pi^2 - \alpha^2},$$

In this case  $(\partial_x^2 - \alpha^2)g(x) = 0$  on  $[0, x')$  and  $(x', 1]$ , so it has the following forms on the above two intervals.

$$g(x) = A \sinh \alpha x, \quad x \in [0, x'), \quad g(x) = B \sinh \alpha(1 - x), \quad x \in (x', 1],$$

$$A \sinh \alpha x' = B \sinh \alpha(1 - x'), \quad \alpha A \cosh \alpha x' = -\alpha B \cosh \alpha(1 - x') - 1,$$

There exists  $\lambda < 0$  such that  $A = \lambda \sinh \alpha(1 - x'), B = \lambda \sinh \alpha x'$ ,

$$-1 = \lambda \alpha (\cosh \alpha x' \sinh \alpha(1 - x') + \cosh \alpha(1 - x') \sinh \alpha x') = \lambda \alpha \sinh \alpha, \quad \lambda = -\frac{1}{\alpha \sinh \alpha},$$

$$A = -\frac{\sinh \alpha(1 - x')}{\alpha \sinh \alpha}, \quad B = -\frac{\sinh \alpha x'}{\alpha \sinh \alpha}, \quad G(x, x') = \begin{cases} -\frac{\sinh \alpha(1-x') \sinh \alpha x}{\alpha \sinh \alpha}, & x \leq x' \\ -\frac{\sinh \alpha x' \sinh \alpha(1-x)}{\alpha \sinh \alpha}, & x > x' \end{cases},$$

Notice that if we let  $\alpha \rightarrow 0_+$ , the above Green's function tends to the case  $\alpha = 0$ .

## References

- [1] Evans, Partial differential equations, 2010.
- [2] Young-Ping Pao, Kerner, Analytic theory of stable resistive magnetohydrodynamic modes, 1984.
- [3] Kurt Riedel, The spectrum of resistive viscous magnetohydrodynamics, 1986.
- [4] Kurt Riedel, The pseudospectrum of the resistive magnetohydrodynamics operator, 1994.
- [5] Helffer, Spectral theory and its applications, 2013.
- [6] Drazin, Hydrodynamic stability, second edition, 2004.
- [7] Wolfgang Kerner, Lerbinger, Kurt Riedel, Resistive Alfvén spectrum of tokamaklike configurations in straight cylindrical geometry, 1986.
- [8] Emmanuel Grenier, Yan Guo, Toan T. Nguyen, Spectral instability of general symmetric shear flows in a two-dimensional channel, 2016.
- [9] Dongyi Wei, Zhifei Zhang, Weiren Zhao, Linear inviscid damping for a class of monotone shear flow in Sobolev spaces, 2015.
- [10] Siqi Ren, Weiren Zhao, Linear damping of alfvén waves by phase mixing, 2017.
- [11] Qi Chen, Te Li, Dongyi Wei, Zhifei Zhang, Transition threshold for the 2-d Couette flow in a finite channel, 2018.