Chen-Ning Yang's conjecture of exterior product

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1 Introduction

Given a $\mathbb{R}$-vector space $V = \mathbb{R}^{2n}, n \in \mathbb{N}$, equipped with an inner product $\langle\ ,\ \rangle$, $\bigwedge^r V$ is the $\mathbb{R}$-vector space of degree r exterior forms. If $\{e_i\}_{i=1}^{2n}$ is an orthonormal basis of V, then $\{e_{i_1} \wedge e_{i_2} ... \wedge e_{i_r}, 1 \leq i_1 < i_2 ... < i_r \leq 2n\}$ is an orthonormal basis of $\bigwedge^r V$, $\dim \bigwedge^r V = \binom{2n}{r}$. For any $1 \leq i_1 < i_2 ... < i_r \leq 2n, 1 \leq j_1 < j_2 ... < j_r \leq 2n$,

$$\langle e_{i_1} \wedge e_{i_2} ... \wedge e_{i_r}, e_{j_1} \wedge e_{j_2} ... \wedge e_{j_r} \rangle = \delta_{i_1,j_1} \delta_{i_2,j_2} ... \delta_{i_r,j_r},$$

For any $1 \leq k, l \leq n, k + l \leq n$, $\xi \in \bigwedge^{2k} V, \eta \in \bigwedge^{2l} V$, find $\max_{\|\xi\|=\|\eta\|=1} \|\xi \wedge \eta\|$.

Conjecture 1 (Chen-Ning Yang)**.** The maximum is attained when

$$\xi_{max} = \frac{\omega^k}{\|\omega^k\|}, \quad \eta_{max} = \frac{\omega^l}{\|\omega^l\|}, \quad \omega = \sum_{i=1}^{n} e_{2i-1} \wedge e_{2i},$$

And we have

$$\omega^k = k! \sum_{1 \leq i_1 < i_2 ... < i_k \leq n} (e_{2i_1-1} \wedge e_{2i_1}) \wedge ... \wedge (e_{2i_k-1} \wedge e_{2i_k})$$

Notice that $(e_1 \wedge e_2) \wedge (e_3 \wedge e_4) = (e_3 \wedge e_4) \wedge (e_1 \wedge e_2)$ has an even permutation on indices.

$$\|\omega^k\|_2^2 = k!^2 \binom{n}{k}, \quad \|\omega^l\|_2^2 = l!^2 \binom{n}{l}, \quad \|\omega^{k+l}\|_2^2 = (k+l)!^2 \binom{n}{k+l},$$

$$\|\xi_{max} \wedge \eta_{max}\|_2^2 = \frac{\|\omega^{k+l}\|_2^2}{\|\omega^k\|_2^2 \|\omega^l\|_2^2} = \frac{(k+l)!(n-k)!(n-l)!}{k!l!n!(n-k-l)!} \triangleq C(n,k,l),$$

$$C(n,k,l) = \frac{(k+l)!(l+m)!(k+m)!}{k!l!m!(k+l+m)!}, \quad m = n-k-l,$$

Let's start by analysing some special cases.

Example 1. $k + l = n$, now $\bigwedge^{2k+2l} V = \bigwedge^{2n} V = \mathbb{R} e_1 \wedge e_2 \wedge ... e_{2n-1} \wedge e_{2n}$. Assume that $u = \sum_I u_I e_I \in \bigwedge^{2k} V$, $v = \sum_J v_J e_J \in \bigwedge^{2l} V$, where

$$I = (i_1, i_2, ..., i_{2k}), \quad 1 \leq i_1 < i_2 ... < i_{2k} \leq 2n, \quad e_I = e_{i_1} \wedge e_{i_2} ... \wedge e_{i_{2k}},$$

$$J = (j_1, j_2, ..., j_{2l}), \quad 1 \leq j_1 < j_2 ... < j_{2l} \leq 2n, \quad e_J = e_{j_1} \wedge e_{j_2} ... \wedge e_{j_{2l}},$$

then we have

$$u \wedge v = \sum_{I \cap J = \emptyset} u_I v_J e_I \wedge e_J = \sum_{I \cap J = \emptyset} u_I v_J \mathrm{sgn}(\sigma_{I,J}) e_{[2n]},$$

$$\sigma_{I,J}:(I,J)\to[2n]=(1,2,...,2n),\quad e_{[2n]}=e_1\wedge e_2\wedge ...e_{2n-1}\wedge e_{2n},$$

$\mathrm{sgn}(\sigma_{I,J})$ is the signature of permutation $\sigma_{I,J}$: $\mathrm{sgn}(\sigma_{I,J})=1$ if $\sigma_{I,J}$ is an even permutation, $\mathrm{sgn}(\sigma_{I,J})=-1$ if $\sigma_{I,J}$ is an odd permutation.

$$\|u\wedge v\|_2=|\sum_{I\cap J=\emptyset}u_Iv_J\mathrm{sgn}(\sigma_{I,J})|\leq\|u\|_2\|v\|_2,$$

holds by Cauchy's inequality, since for any given I, $J=[2n]\backslash I$ is uniquely determined. Since $C(n,k,l)=1$ in this case, the conjecture holds. Notice that in this case, for any given $u\in\bigwedge^{2k}V$ with $\|u\|=1$, there exists $v\in\bigwedge^{2l}V$ with $\|v\|=1$ such that the equality $\|u\wedge v\|=\|u\|\|v\|$ holds.

Property 1 (Construction of exterior product by tensor product and quotient map)**.** $\wedge:\bigwedge^{2k}V\times\bigwedge^{2l}V\to\bigwedge^{2k+2l}V$ is a bilinear map:

$$(\xi_1+\xi_2)\wedge\eta=\xi_1\wedge\eta+\xi_2\wedge\eta,\quad \xi\wedge(\eta_1+\eta_2)=\xi\wedge\eta_1+\xi\wedge\eta_2,$$

$$(a\xi)\wedge\eta=a(\xi\wedge\eta)=\xi\wedge(a\eta),\quad a\in\mathbb{R},$$

So we have the universal property of tensor product:
$$\begin{tikzcd}\bigwedge^{2k}V\times\bigwedge^{2l}V \arrow[r, "\varphi"] \arrow[dr, "\wedge"'] & \bigwedge^{2k}V\otimes\bigwedge^{2l}V \arrow[d, "\sim"] \\ & \bigwedge^{2k+2l}V\end{tikzcd}$$

where $\varphi:\bigwedge^{2k}V\times\bigwedge^{2l}V\to\bigwedge^{2k}V\otimes\bigwedge^{2l}V,\quad (u,v)\mapsto u\otimes v$ is the canonical bilinear map, and $\sim$: $\bigwedge^{2k}V\otimes\bigwedge^{2l}V\to\bigwedge^{2k+2l}V$ is the quotient map defined as follows: for $I\subset[2n],|I|=2k,J\subset[2n],|J|=2l$, if $I\cap J=\emptyset$, then let

$$\sim(e_I\otimes e_J)=\mathrm{sgn}(\sigma_{I,J})e_{I\cup J},\quad \sigma_{I,J}:(I,J)\to I\cup J,$$

where $I,J,I\cup J$ are sorted in ascending order, $\mathrm{sgn}(\sigma_{I,J})$ is the signature of permutation $\sigma_{I,J}$. If $I\cap J\neq\emptyset$, then let $\sim(e_I\otimes e_J)=0$.

Property 2 (Universal property of exterior product)**.** Assume $V=\mathbb{R}^n$, the space of degree m exterior forms on V is $\bigwedge^m V$. It has the following universal property: if $\varphi:V^{\times m}\to W$ is a m-linear antisymmetric (alternating) map where $V^{\times m}$ is the product of m copies of V and W is a $\mathbb{R}$-vector space, then there exists a unique m-linear antisymmetric map $\tilde{\varphi}$ such that the following diagram commutes:
$$\begin{tikzcd}V^{\times m} \arrow[r] \arrow[dr, "\varphi"'] & \bigwedge^m V \arrow[d, "\tilde{\varphi}"] \\ & W\end{tikzcd}$$

Especially, when $m=1$, $\bigwedge^1 V$ is the dual vector space V^* of V.

Property 3. Monotonicity of $C(n,k,l)$: for fixed $k,l\in\mathbb{N}_+$, $n\geq k+l$, we have

$$C(n+1,k,l)>C(n,k,l)...>C(k+l,k,l)=1,$$

Proof. Let $m=n+1-k-l$, since $\frac{(m+l)(m+k)}{m(m+k+l)}>1$, we have $C(n+1,k,l)>C(n,k,l)$. □

Property 4. If there exists a subspace $V_1\subset V$ with $\dim V_1=2k$, such that $\xi\in\bigwedge^{2k}V$ is the unit volume form of V_1, and there exists a subspace $V_2\subset V$ with $\dim V_2=2l$, such that $\eta\in\bigwedge^{2l}V$ is the unit volume form of V_2. Then $\|\xi\|=\|\eta\|=1$, and V_1 has an orthonormal basis $\{e_i\}_{i=1}^{2k}$, such that $\xi = e_1 \wedge e_2 ... \wedge e_{2k}$, V_2 has an orthonormal basis $\{e'_j\}_{j=1}^{2l}$, such that $\eta = e'_1 \wedge e'_2 ... \wedge e'_{2l}$. Then by Hadamard's inequality, we have

$$\|\xi \wedge \eta\| = \|e_1 \wedge e_2 ... \wedge e_{2k} \wedge e'_1 \wedge e'_2 ... \wedge e'_{2l}\| \leq \prod_{i=1}^{2k} \|e_i\| \prod_{j=1}^{2l} \|e'_j\| = 1,$$

If U, V are Hilbert spaces on $\mathbb{R}$ with inner products $\langle\ ,\ \rangle_U$ and $\langle\ ,\ \rangle_V$, we may construct an inner product on $U \otimes V$ as follows:

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{U \otimes V} = \langle u_1, u_2 \rangle_U \langle v_1, v_2 \rangle_V,$$

Notice that under this inner product on $U \otimes V$, the canonical bilinear map $\varphi : U \times V \to U \otimes V$ is an isometry onto $\mathrm{im}\varphi$ in the following sense: $\|u \otimes v\|_{U \otimes V} = \|u\|_U \|v\|_V$. φ is a Segre embedding, $\mathrm{im}\varphi$ is a Segre variety. If $\{e_i\}_{i=1}^m$ is an orthonormal basis of U, $\{e'_j\}_{j=1}^n$ is an orthonormal basis of V, then $\{e_i \otimes e'_j\} 1 \leq i \leq m, 1 \leq j \leq n$ is an orthonormal basis of $U \otimes V$. The exact form of φ is as follows:

$$u = \sum_{i=1}^m u_i e_i, \quad v = \sum_{j=1}^n v_j e'_j, \quad \varphi(u, v) = u \otimes v = \sum_{1 \leq i \leq m, 1 \leq j \leq n} u_i v_j e_i \otimes e'_j,$$

and we have $\dim \mathrm{im}\varphi = \dim U + \dim V - 1$. In the case of Yang's conjecture, we have

$$\dim \bigwedge^{2k} V = \tbinom{2n}{2k}, \quad \dim \bigwedge^{2l} V = \tbinom{2n}{2l}, \quad \dim \bigwedge^{2k+2l} V = \tbinom{2n}{2k+2l},$$

$$\dim \bigwedge^{2k} V \otimes \bigwedge^{2l} V = \tbinom{2n}{2k}\tbinom{2n}{2l}, \quad \dim \mathrm{im}\varphi = \tbinom{2n}{2k} + \tbinom{2n}{2l} - 1,$$

Example 2. $V = \mathbb{R}^2$, $u, v \in \bigwedge^1 V$, $u = u_1 e_1 + u_2 e_2$, $v = v_1 e_1 + v_2 e_2$, $u \wedge v = (u_1 v_2 - u_2 v_1) e_{12}$,

$$\frac{\|u \wedge v\|^2}{\|u\|^2 \|v\|^2} = \frac{(u_1 v_2 - u_2 v_1)^2}{(u_1^2 + u_2^2)(v_1^2 + v_2^2)} \leq 1,$$

If $u \otimes v = a_{11} e_1 \otimes e_1 + a_{12} e_1 \otimes e_2 + a_{21} e_2 \otimes e_1 + a_{22} e_2 \otimes e_2$, then $a_{11} a_{22} = a_{12} a_{21}$. If we remove the condition $a_{11} a_{22} = a_{12} a_{21}$ and consider arbitrary $w \in \bigwedge^1 V \otimes \bigwedge^1 V$, $w = a e_1 \otimes e_1 + b e_1 \otimes e_2 + c e_2 \otimes e_1 + d e_2 \otimes e_2$, then we have

$$\frac{\| \sim (w)\|^2}{\|w\|^2} = \frac{(b-c)^2}{a^2 + b^2 + c^2 + d^2} \leq 2,$$

Notice that the upper bound becomes loose. This example shows that we cannot only consider the quotient map $\sim: \bigwedge^1 V \otimes \bigwedge^1 V \to \bigwedge^2 V$ on the whole space $\bigwedge^1 V \otimes \bigwedge^1 V$, we must consider its restriction on the Segre variety $\mathrm{im}\varphi$. In the above case, if we add the constraint $ad = bc$, then we have

$$\frac{(b-c)^2}{a^2 + b^2 + c^2 + d^2} \leq \frac{(b-c)^2 + (a+d)^2}{a^2 + b^2 + c^2 + d^2} = 1,$$

Assume that $u = \sum_I u_I e_I \in \bigwedge^{2k} V$, $v = \sum_J v_J e_J \in \bigwedge^{2l} V$, then we have

$$u \wedge v = \sum_{I \cap J = \emptyset} u_I v_J e_I \wedge e_J = \sum_{|P| = 2k+2l} e_P \sum_{I \cup J = P} u_I v_J \mathrm{sgn}(\sigma_{I,J}),$$

$$\|u \wedge v\|^2 = \sum_{|P| = 2k+2l} (\sum_{I \cup J = P} u_I v_J \mathrm{sgn}(\sigma_{I,J}))^2 = \sum_{|P| = 2k+2l,\ I_1 \cup J_1 = I_2 \cup J_2 = P} u_{I_1} v_{J_1} u_{I_2} v_{J_2} \mathrm{sgn}(\sigma_{I_1,J_1}) \mathrm{sgn}(\sigma_{I_2,J_2}),$$

Example 3 (Volume form of a linear subspace)**.** Assume $V = \mathbb{R}^4$, $\xi \in \bigwedge^2 V$, and there exists a subspace $V_1 \subset V$, $\dim V_1 = 2$, such that $\xi = e_1' \wedge e_2'$ where $\{e_1', e_2'\}$ is an orthonormal basis of V_1. Assume that $e_j' = \sum_{i=1}^4 a_{ij} e_i$, then we have

$$(e_1'\ e_2') = (e_1\ e_2\ e_3\ e_4)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = (e_1\ e_2\ e_3\ e_4)A, \quad A^t A = I_2,$$

$$e_1' \wedge e_2' = \sum_{i<j} \det \begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix} e_i \wedge e_j,$$

More generally, assume $V = \mathbb{R}^n$, $\xi \in \bigwedge^m V$, and there exists a set of orthonormal vectors $\{e_j'\}_{j=1}^m$ such that $\xi = e_1' \wedge e_2' ... \wedge e_m'$. Assume that $e_j' = \sum_{i=1}^n a_{ij} e_i$, then we have

$$(e_1'\ e_2' ...\ e_m') = (e_1\ e_2 ...\ e_n)A, \quad A \in M(n, m, \mathbb{R}), \quad A^t A = I_m,$$

$$e_1' \wedge e_2' ... \wedge e_m' = \sum_{|I|=m} \det A_{I,[m]} e_I, \quad I = (i_1, i_2, ... i_m), \quad 1 \le i_1 < i_2 ... < i_m \le n,$$

Example 4. Assume $V = \mathbb{C}^G = L^2(G)$, $G = \mathbb{Z}/2n\mathbb{Z}$, $f, g \in V$. I tried to adopt the theory of Fourier analysis on finite abelian groups but failed.

Proposition 1 (Decomposition of 2-forms)**.** Assume $V = \mathbb{R}^{2n}$, $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, then there exists a basis $\{e_i'\}_{i=1}^{2n}$ of V such that

$$\xi = \sum_{i=1}^n a_i e_{2i-1}' \wedge e_{2i}', \quad \sum_{i=1}^n a_i^2 = 1, \quad a_1 \ge a_2 ... \ge a_n \ge 0,$$

Especially, when $n = 2$, we can write $\xi = \cos\theta e_1' \wedge e_2' + \sin\theta e_3' \wedge e_4'$; when $n = 3$, we can write $\xi = \cos\theta e_1' \wedge e_2' + \sin\theta\cos\phi e_3' \wedge e_4' + \sin\theta\sin\phi e_5' \wedge e_6'$.

Proof. This is equivalent to the fact that a real skew-symmetric matrix $A \in M(2n, 2n, \mathbb{R})$ can be written in the form $A = Q^t \Sigma Q$, where $Q \in SO(2n)$ and Σ is the real canonical form.

$$a_1 = \max_{e_1', e_2'} \langle \xi, e_1' \wedge e_2' \rangle,$$

$a_2, ... a_n$ can be determined recursively. □

Example 5. $n = 3$, $V = \mathbb{R}^6$, $k = l = 1$, $C(n,k,l) = \frac{2!2!2!}{1!1!1!3!} = \frac{4}{3}$. For each $P \subset [2n]$, $|P| = 2k + 2l = 4$, the bilinear form $\sum_{I \cup J = P} u_I v_J \mathrm{sgn}(\sigma_{I,J})$ contains $\binom{2k+2l}{2k} = 6$ terms. But if we use the decomposition of 2-forms, for a given $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = a_1 e_1 \wedge e_2 + a_2 e_3 \wedge e_4 + a_3 e_5 \wedge e_6, \quad a_1^2 + a_2^2 + a_3^2 = 1,$$

i) If $\eta = \sum_{i=1}^n b_i e_{2i-1} \wedge e_{2i}$, $\sum_{i=1}^n b_i^2 = 1$, then

$$\xi \wedge \eta = \sum_{1 \le i < j \le n} (a_i b_j + a_j b_i) e_{2i-1} \wedge e_{2i} \wedge e_{2j-1} \wedge e_{2j},$$

$$\|\xi \wedge \eta\|^2 = \sum_{1 \le i < j \le n} (a_i b_j + a_j b_i)^2 = (a_1 b_2 + a_2 b_1)^2 + (a_1 b_3 + a_3 b_1)^2 + (a_2 b_3 + a_3 b_2)^2,$$

$$\frac{4}{3}\|\xi\|^2\|\eta\|^2 - \|\xi\wedge\eta\|^2 = \frac{4}{3}(a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2) + \frac{1}{3}\sum_{i<j}(a_i^2b_j^2 + a_j^2b_i^2) - 2\sum_{i<j}a_ib_ja_jb_i$$
$$= \frac{1}{3}\sum_{i<j}(a_ib_j - a_jb_i)^2 + \frac{2}{3}\sum_{i<j}(a_ib_i - a_jb_j) \geq 0,$$

ii) If $\eta = \sum_{i<j} b_{ij}e_i\wedge e_j$, with $b_{2i-1,2i} = 0, 1 \leq i \leq n$, then η lies in the orthogonal complement of $\bigoplus_{i=1}^n \mathbb{R}e_{2i-1}\wedge e_{2i}$ in $\bigwedge^2 V$. Then we have

$$\|\xi\wedge\eta\|^2 = a_3^2(b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{24}^2) + a_2^2(b_{15}^2 + b_{16}^2 + b_{25}^2 + b_{26}^2) + a_1^2(b_{35}^2 + b_{36}^2 + b_{45}^2 + b_{46}^2),$$

So $\|\xi\wedge\eta\|^2 \leq \|\xi\|^2\|\eta\|^2 \leq \frac{4}{3}\|\xi\|^2\|\eta\|^2$ holds.

iii) Consider arbitrary $\eta = \sum_{i<j} b_{ij}e_i\wedge e_j$, $\|\xi\wedge\eta\|^2 \leq \frac{4}{3}\|\xi\|^2\|\eta\|^2$ holds by combining the two cases above.

Example 6. $n \geq 3$, $V = \mathbb{R}^{2n}$, $k = l = 1$, $C(n,k,l) = \frac{(n-1)!(n-1)!2!}{1!1!(n-2)!n!} = \frac{2(n-1)}{n}$. Using the decomposition of 2-forms, for a given $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = \sum_{i=1}^n a_ie_{2i-1}\wedge e_{2i}, \quad \sum_{i=1}^n a_i^2 = 1, \quad a_1 \geq a_2 ... \geq a_n \geq 0,$$

i) If $\eta = \sum_{i=1}^n b_ie_{2i-1}\wedge e_{2i}$, $\sum_{i=1}^n b_i^2 = 1$, then

$$\xi\wedge\eta = \sum_{1\leq i<j\leq n}(a_ib_j + a_jb_i)e_{2i-1}\wedge e_{2i}\wedge e_{2j-1}\wedge e_{2j},$$

$$\frac{2(n-1)}{n}\|\xi\|^2\|\eta\|^2 - \|\xi\wedge\eta\|^2 = \frac{2(n-1)}{n}(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) - \sum_{1\leq i<j\leq n}(a_ib_j + a_jb_i)^2$$
$$= \frac{2(n-1)}{n}(\sum_{i=1}^n a_i^2b_i^2) + \frac{n-2}{n}\sum_{i<j}(a_i^2b_j^2 + a_j^2b_i^2) - 2\sum_{i<j}a_ib_ja_jb_i$$
$$= \frac{n-2}{n}\sum_{i<j}(a_ib_j - a_jb_i)^2 + \frac{2}{n}\sum_{i<j}(a_ib_i - a_jb_j)^2 \geq 0,$$

ii) If $\eta = \sum_{i<j} b_{ij}e_i\wedge e_j$, with $b_{2i-1,2i} = 0, 1 \leq i \leq n$, then η lies in the orthogonal complement of $\bigoplus_{i=1}^n \mathbb{R}e_{2i-1}\wedge e_{2i}$ in $\bigwedge^2 V$. Then $\|\xi\wedge\eta\|^2 \leq \|\xi\|^2\|\eta\|^2 \leq \frac{2(n-1)}{n}\|\xi\|^2\|\eta\|^2$ holds.

iii) For arbitrary $\eta = \sum_{i<j} b_{ij}e_i\wedge e_j$, $\|\xi\wedge\eta\|^2 \leq \frac{2(n-1)}{n}\|\xi\|^2\|\eta\|^2$ holds by combining the two cases above. So we finished proving the case when $k = l = 1$.

Example 7. $k = 1, l \geq 2, n \geq l+1$, $C(n,k,l) = \frac{(l+1)!(n-1)!(n-l)!}{1!l!n!(n-1-l)!} = \frac{(l+1)(n-l)}{n}$. Using the decomposition of 2-forms, for a given $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = \sum_{i=1}^n a_ie_{2i-1}\wedge e_{2i}, \quad \sum_{i=1}^n a_i^2 = 1, \quad a_1 \geq a_2 ... \geq a_n \geq 0,$$

1) We first consider the case when $l = 2$, now $C(n,k,l) = \frac{3(n-2)}{n}$.

i) If $\eta = \sum_{1\leq i<j\leq n} b_{ij}e_{2i-1}\wedge e_{2i}\wedge e_{2j-1}\wedge e_{2j}$, $\sum_{1\leq i<j\leq n} b_{ij}^2 = 1$, then

$$\xi\wedge\eta = \sum_{p<i<j}(a_pb_{ij} + a_ib_{pj} + a_jb_{pi})e_{\{2p-1,2p,2i-1,2i,2j-1,2j\}},$$

$$\frac{3(n-2)}{n}\|\xi\|^2\|\eta\|^2-\|\xi\wedge\eta\|^2=\frac{3(n-2)}{n}(\sum_p a_p^2)(\sum_{i<j}b_{ij}^2)-\sum_{p<i<j}(a_pb_{ij}+a_ib_{pj}+a_jb_{pi})^2$$

$$=\frac{3(n-2)}{n}\sum_{i<j}(a_i^2+a_j^2)b_{ij}^2+\frac{2(n-3)}{n}\sum_{p<i<j}(a_p^2b_{ij}^2+a_i^2b_{pj}^2+a_j^2b_{pi}^2)$$

$$-2\sum_{p<i<j}(a_pb_{ij}a_ib_{pj}+a_pb_{ij}a_jb_{pi}+a_ib_{pj}a_jb_{pi})$$

$$=\frac{2(n-3)}{n}\sum_{p<i<j}(a_p^2b_{ij}^2+a_i^2b_{pj}^2+a_j^2b_{pi}^2-a_pb_{ij}a_ib_{pj}-a_pb_{ij}a_jb_{pi}-a_ib_{pj}a_jb_{pi})$$

$$+\frac{3}{n}\sum_{p<i<j}((a_p^2+a_i^2)b_{pi}^2+(a_p^2+a_j^2)b_{pj}^2+(a_i^2+a_j^2)b_{ij}^2-2(a_pb_{ij}a_ib_{pj}+a_pb_{ij}a_jb_{pi}+a_ib_{pj}a_jb_{pi}))\geq 0,$$

2) For arbitrary $l\geq 2$, now $C(n,k,l)=\frac{(l+1)(n-l)}{n}$.

i) If $\eta=\sum_{|I|=l,\ I\subset[n]}b_Ie_{\{2i-1,2i,\ i\in I\}}$, $\sum_{|I|=l,\ I\subset[n]}b_I^2=1$, then

$$\xi\wedge\eta=\sum_{|J|=l+1,\ J\subset[n]}(\sum_{i\in J}a_ib_{J\setminus\{i\}})e_{\{2j-1,2j,\ j\in J\}},$$

$$\frac{(l+1)(n-l)}{n}\|\xi\|^2\|\eta\|^2-\|\xi\wedge\eta\|^2=\frac{(l+1)(n-l)}{n}(\sum_p a_p^2)(\sum_{|I|=l,\ I\subset[n]}b_I^2)-\sum_{|J|=l+1,\ J\subset[n]}(\sum_{i\in J}a_ib_{J\setminus\{i\}})^2$$

$$=\frac{(l+1)(n-l)}{n}\sum_{|I|=l,\ I\subset[n]}(\sum_{i\in I}a_i^2)b_I^2+\frac{l(n-l-1)}{n}\sum_{|J|=l+1,\ J\subset[n]}(\sum_{i\in J}a_i^2b_{J\setminus\{i\}}^2)$$

$$-2\sum_{|J|=l+1,\ J\subset[n]}(\sum_{i<j\in J}a_ib_{J\setminus\{i\}}a_jb_{J\setminus\{j\}})$$

$$=\frac{n-1-l}{n}\sum_{|J|=l+1,\ J\subset[n]}(l\sum_{i\in J}a_i^2b_{J\setminus\{i\}}^2-2\sum_{i<j\in J}a_ib_{J\setminus\{i\}}a_jb_{J\setminus\{j\}})$$

$$+\frac{l+1}{n}\sum_{|J|=l+1,\ J\subset[n]}(\sum_{i\in J}(\sum_{j\in J\setminus\{i\}}a_j^2)b_{J\setminus\{i\}}^2-2\sum_{i<j\in J}a_ib_{J\setminus\{i\}}a_jb_{J\setminus\{j\}})\geq 0,$$

Conjecture 2 (Birch)**.** The only remaining case is when $n=5$, $z_1,z_2,...z_n\in\mathbb{C}$ satisfy the constraint $\sum_{i=1}^n|z_i|^2=n$, then

$$\Delta=\prod_{1\leq i<j\leq n}|z_i-z_j|^2\leq n^n,$$

The equality holds when $z_1,z_2,...z_n$ are the vertices of a pentagon on the unit circle.

$$f(z)=\prod_{i=1}^n(z-z_i),\quad \Delta=\prod_{i=1}^n|f'(z_i)|,\quad f_{max}(z)=z^n-c,\quad |f'_{max}(z_i)|=|nz_i^{n-1}|=n,$$

References

[1] Kailiang Lin, tba.