NYU Computer Science Bridge to Tandon Course Winter 2021

Homework 11

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Question 5

a. Use mathematical induction to prove that for any positive integer n, 3 divides $n^3 + 2n$ (leaving no remainder).
Hint: you may want to use the formula: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Solution. By induction on n.

Base case: $n = 1$.

$$n^3 + 2n = 1^3 + 2 \cdot 1 = 3$$

Therefore, for $n = 1$, 3 divides $n^3 + 2n$.

Inductive step: We will show that for any positive integer $k \geq 1$, if 3 divide $k^3 + 2k$, then 3 also divides $(k + 1)^3 + 2(k + 1)$. First We have

$$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + (3k^2 + 3k + 3)$$

By the inductive hypothesis we know that 3 divides $(k^3 + 2k)$, and it's also clear that 3 divides $(3k^2 + 3k + 3)$. Thus we have 3 divides $(k + 1)^3 + 2(k + 1)$. ∎

b. Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.

Solution. By strong induction on n.

Base case: $n = 2$. Since 2 is a prime number, it's already a product of one prime number: 2.

Inductive step: Suppose that for $k \geq 2$, any integer j in the range from 2 through k can be expressed as a product of prime numbers. We will show that $k+1$ can be expressed as a product of prime numbers. If $k + 1$ is prime, then it is a product of one prime number: $k + 1$. If $k + 1$ is not prime, then it is composite and thus can be expressed as the product of two integers, a and b, both of which are at least 2. We will show that both a and b are at most k to apply the inductive hypothesis. Since $k + 1 = ab$, $a = \frac{k+1}{b}$. Furthermore, since $b \geq 2$, $a = \frac{k+1}{b} < k + 1$, and thus $a \leq k$. By the symmetric argument we can show that $b \leq k$. Since a and b both fall in the range from 2 through k, the inductive hypothesis can be applied and they can each be expressed as a product of primes:

$$a = p_1 \cdot p_2 \cdots p_m$$
$$b = q_1 \cdot q_2 \cdots q_n$$

Now $k + 1$ can be expressed as a product of primes:

$$k + 1 = ab = (p_1 \cdot p_2 \cdots p_m)(q_1 \cdot q_2 \cdots q_n)$$

∎

Question 6

Solve the following questions from the Discrete Math zyBook:

a) **Exercise 7.4.1**

Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that $P(3)$ is true.

Solution.

$$\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = 14 = \frac{3(3+1)(2 \cdot 3+1)}{6}$$

■

(b) Express $P(k)$.

Solution.

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

■

(c) Express $P(k+1)$.

Solution.

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

■

(d) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the base case?

Solution. $P(1)$ is true. ■

(e) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the inductive step?

Solution. For all positive integers k, $P(k)$ implies $P(k+1)$. ■

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

Solution. $P(k)$. ■

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution. By induction on n.
Base case: $n = 1$. We have

$$\sum_{j=1}^{1} j^2 = 1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$$

Inductive step: We will show that for any positive integer $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is also true. We have

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Therefore, $P(k+1)$ is true. ■

b) **Exercise 7.4.3**

Prove each of the following statements using mathematical induction.

(c) Prove that for $n \geq 1$, $\displaystyle\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Hint: you may want to use the following fact: $\displaystyle\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$

Solution. By induction on n.

Base case: $n = 1$. We have $\displaystyle\sum_{j=1}^{1} \frac{1}{j^2} = 1 \leq 2 - \frac{1}{1}$.

Inductive step: We will shot that for any positive integer $k \geq 1$, if $\displaystyle\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$, then $\displaystyle\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$. We have

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \quad \text{(by the inductive hypothesis)} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \quad \text{(by the hint)} \\
&= 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)} \\
&= 2 - \frac{1}{k+1}
\end{aligned}$$

■

c) **Exercise 7.5.1**

Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Solution. By induction on n.
Base case: $n = 1$. We have $3^{2 \cdot 1} - 1 = 9 - 1 = 8$, and it's clear that 4 divides 8.
Inductive step: We will show that for any positive integer $k \geq 1$, if 4 divides $3^{2k} - 1$, then 4 also divides $3^{2(k+1)} - 1$. First We have

$$\begin{aligned}
3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\
&= 9 \cdot 3^{2k} - 1 \\
&= 8 \cdot 3^{2k} + (3^{2k} - 1)
\end{aligned}$$

By the inductive hypothesis we know that 4 divides $3^{2k} - 1$, and it's also clear that 4 divides $8 \cdot 3^{2k}$. Thus 4 divides $3^{2k} - 1$. ■