NYU Computer Science Bridge to Tandon Course Winter 2021

Homework 3

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Question 7

Solve the following questions from the Discrete Math zyBook:

a) Exercise 3.1.1

$$
\begin{aligned}
A &= \{x \in \mathbb{Z} : x \text{ is an integer multiple of } 3\} \\
B &= \{x \in \mathbb{Z} : x \text{ is a perfect square}\} \\
C &= \{4, 5, 9, 10\} \\
D &= \{2, 4, 11, 14\} \\
E &= \{3, 6, 9\} \\
F &= \{4, 6, 16\}
\end{aligned}
$$

(a) $27 \in A$

Solution. True. Clearly $27 = 9 * 3$ is an integer multiple of 3. ■

(b) $27 \in B$

Solution. False. There is no integer y such that $27 = y^2$. ■

(c) $100 \in B$

Solution. True. Clearly $100 = 10^2$ is a perfect square. ■

(d) $E \subseteq C$ or $C \subseteq E$

Solution. False. $E \nsubseteq C$ since $3 \in E$ but $3 \notin C$. $C \nsubseteq E$ since $4 \in C$ but $4 \notin E$. ■

(e) $E \subseteq A$

Solution. True. Clearly each element of A is an integer multiple of 3. ■

(f) $A \subset E$

Solution. False. Counterexample: 12. ■

(g) $E \in A$

Solution. False. E is a set while the elements of A are integers. ■

b) Exercise 3.1.2

$$\begin{aligned}
A &= \{x \in \mathbb{Z} : x \text{ is an integer multiple of } 3\} \\
B &= \{x \in \mathbb{Z} : x \text{ is a perfect square}\} \\
C &= \{4, 5, 9, 10\} \\
D &= \{2, 4, 11, 14\} \\
E &= \{3, 6, 9\} \\
F &= \{4, 6, 16\}
\end{aligned}$$

(a) $15 \subset A$

Solution. False. 15 is not a set. ■

(b) $\{15\} \subset A$

Solution. True. Clearly 15 is an integer multiple of 3. Also, $3 \in A$ but $3 \notin \{15\}$. ■

(c) $\varnothing \subset A$

Solution. The statement is vacuously true. ■

(d) $A \subseteq A$

Solution. The statement is clearly true. ■

(e) $\varnothing \in B$

Solution. False. The empty set is a set while the elements of B are integers. ■

c) Exercise 3.1.5

(b) $\{3, 6, 9, 12, \ldots\}$

Solution. $\{x \in \mathbb{Z}^+ : x \text{ is an integer multiple of } 3\}$. The set is infinite. ■

(d) $\{0, 10, 20, 30, \ldots, 1000\}$

Solution. $\{x \in \mathbb{N} : x \text{ is an integer multiple of } 10 \text{ and } x \leq 1000\}$; the cardinality is 101. ■

d) Exercise 3.2.1

$$X = \{1, \{1\}, \{1, 2\}, 2, \{3\}, 4\}$$

(a) $2 \in X$

Solution. True. ■

(b) $\{2\} \subseteq X$

Solution. True since $2 \in X$. ■

(c) $\{2\} \in X$

Solution. False. ■

(d) $3 \in X$

Solution. False. ■

(e) $\{1, 2\} \in X$

Solution. True. ■

(f) $\{1, 2\} \subseteq X$

Solution. True since both 1 and 2 are elements of X. ■

(g) $\{2, 4\} \subseteq X$

Solution. True since both 2 and 4 are elements of X. ■

(h) $\{2, 4\} \in X$

Solution. False. ■

(i) $\{2, 3\} \subseteq X$

Solution. False since $3 \notin X$. ■

(j) $\{2, 3\} \in X$

Solution. False. ■

(k) $|X| = 7$

Solution. False. $|X| = 6$. ■

Question 8

Solve Exercise 3.2.4, section b from the Discrete Math zyBook.

(c) Let $A = \{1, 2, 3\}$. What is $\{X \in P(A) : 2 \in X\}$?

Solution. Namely X is an element of the power set of A such that 2 is an element of X. Therefore we have

$$X = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

■

Question 9

Solve the following questions from the Discrete Math zyBook:

a) Exercise 3.3.1

$$\begin{aligned} A &= \{-3, 0, 1, 4, 17\} \\ B &= \{-12, -5, 1, 4, 6\} \\ C &= \{x \in \mathbb{Z} : x \text{ is odd}\} \\ D &= \{x \in \mathbb{Z} : x \text{ is positive}\} \end{aligned}$$

(c) $A \cap C$

Solution. $\{-3, 1, 17\}$. $\blacksquare$

(d) $A \cup (B \cap C)$

Solution. $\{-5, -3, 0, 1, 4, 17\}$. $\blacksquare$

(e) $A \cap B \cap C$

Solution. $\{1\}$. $\blacksquare$

b) Exercise 3.3.3

- $A_i = \{i^0, i^1, i^2\}$
- $B_i = \{x \in \mathbb{R} : -i \leq x \leq 1/i\}$
- $C_i = \{x \in \mathbb{R} : -1/i \leq x \leq 1/i\}$

(a) $\bigcap_{i=2}^{5} A_i$

Solution.

$$\begin{aligned} \bigcap_{i=2}^{5} A_i &= \{1, 2, 4\} \cap \{1, 3, 9\} \cap \{1, 4, 16\} \cap \{1, 5, 25\} \\ &= \{1\}. \end{aligned}$$

$\blacksquare$

(b) $\bigcup_{i=2}^{5} A_i$

Solution.

$$\begin{aligned} \bigcap_{i=2}^{5} A_i &= \{1, 2, 4\} \cup \{1, 3, 9\} \cup \{1, 4, 16\} \cup \{1, 5, 25\} \\ &= \{1, 2, 3, 4, 5, 9, 16, 25\}. \end{aligned}$$

$\blacksquare$

(e) $\bigcap_{i=1}^{100} C_i$

Solution. Note that both $\dfrac{-1}{i}$ and $\dfrac{1}{i}$ approach 0 as i increases. Thus we have

$$C_i \supseteq C_j \text{ for } i \leq j.$$

It follows that

$$\begin{aligned}\bigcap_{i=1}^{100} C_i &= C_{100} \\ &= \left\{x \in \mathbb{R} : \frac{-1}{100} \leq x \leq \frac{1}{100}\right\}.\end{aligned}$$

■

(f) $\bigcup_{i=1}^{100} C_i$

Solution. Similar to (e) we have

$$C_i \supseteq C_j \text{ for } i \leq j.$$

It follows that

$$\begin{aligned}\bigcup_{i=1}^{100} C_i &= C_1 \\ &= \{x \in \mathbb{R} : -1 \leq x \leq 1\}.\end{aligned}$$

■

c) Exercise 3.3.4

Use the set definitions $A = \{a, b\}$ and $B = \{b, c\}$ to express each set below. Use roster notation in your solutions.

(b) $P(A \cup B)$

Solution. First we have

$$A \cup B = \{a, b, c\}.$$

Therefore

$$P(A \cup B) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

■

(d) $P(A) \cup P(B)$

Solution. First we have

$$\begin{aligned}P(A) &= \{\varnothing, \{a\}, \{b\}, \{a, b\}\} \\ P(B) &= \{\varnothing, \{b\}, \{c\}, \{b, c\}\}.\end{aligned}$$

Therefore

$$P(A) \cup P(B) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}.$$

■

Question 10

Solve the following questions from the Discrete Math zyBook:

a) Exercise 3.5.1

The sets A, B, and C are defined as follows:

$$A = \{\text{tall}, \text{grande}, \text{venti}\}$$
$$B = \{\text{foam}, \text{no-foam}\}$$
$$C = \{\text{non-fat}, \text{whole}\}$$

Use the definitions for A, B, and C to answer the questions. Express the elements using n-tuple notation, not string notation.

(b) Write an element from the set $B \times A \times C$.

Solution. One example: (foam, tall, non-fat). ■

(c) Write the set $B \times C$ using roster notation.

Solution.

$$B \times C = \{(\text{foam}, \text{non-fat}), (\text{foam}, \text{whole}), (\text{no-foam}, \text{non-fat}), (\text{no-foam}, \text{whole})\}.$$

■

b) Exercise 3.5.3

Indicate which of the following statements are true.

(b) $\mathbb{Z}^2 \subseteq \mathbb{R}^2$

Solution. True. If $(x, y) \in \mathbb{Z}^2$, then x and y are both elements of $\mathbb{Z}$. Since $\mathbb{Z} \subseteq \mathbb{R}$, then x and y are also elements of $\mathbb{R}$. Therefore $(x, y) \in \mathbb{R}^2$. ■

(c) $\mathbb{Z}^2 \cap \mathbb{Z}^3 = \varnothing$

Solution. The elements in $\mathbb{Z}^2$ are pairs. The elements in $\mathbb{Z}^3$ are triples. Therefore the two sets have no elements in common. ■

(e) For any three sets, A, B, and C, if $A \subseteq B$, then $A \times C \subseteq B \times C$.

Solution. True. If $(x, y) \in A \times C$, then $x \in A$ and $y \in C$. Since $A \subseteq B$, then x is also an element of B. Therefore $(x, y) \in B \times C$. ■

c) Exercise 3.5.6

Express the following sets using the roster method. Express the elements as strings, not n-tuples.

(d) $\{xy : \text{where } x \in \{0\} \cup \{0\}^2 \text{ and } y \in \{1\} \cup \{1\}^2\}$

Solution. $\{01, 011, 001, 0011\}$. ■

(e) $\{xy : x \in \{\text{aa}, \text{ab}\} \text{ and } y \in \{\text{a}\} \cup \{\text{a}\}^2\}$

Solution. $\{\text{aaa}, \text{aaaa}, \text{aba}, \text{abaa}\}$. ■

d) Exercise 3.5.7

Use the following set definitions to specify each set in roster notation. Except where noted, express elements of Cartesian products as strings.

- $A = \{\text{a}\}$
- $B = \{\text{b}, \text{c}\}$
- $C = \{\text{a}, \text{b}, \text{d}\}$

(c) $(A \times B) \cup (A \times C)$

Solution. $\{\text{aa}, \text{ab}, \text{ac}, \text{ad}\}$. ■

(f) $P(A \times B)$

Solution. $\{\varnothing, \{\text{ab}\}, \{\text{ac}\}, \{\text{ab}, \text{ac}\}\}$. ■

(g) $P(A) \times P(B)$. Use ordered pair notation for elements of the Cartesian product.

Solution. $\{(\varnothing, \varnothing), (\varnothing, \{\text{b}\}), (\varnothing, \{\text{c}\}), (\varnothing, \{\text{b}, \text{c}\}), (\{\text{a}\}, \varnothing), (\{\text{a}\}, \{\text{b}\}), (\{\text{a}\}, \{\text{c}\}), (\{\text{a}\}, \{\text{b}, \text{c}\})\}$. ■

Question 11

Solve the following questions from the Discrete Math zyBook:

a) Exercise 3.6.2

Use the set identities given in the table to prove the following new identities. Label each step in your proof with the set identity used to establish that step.

(b) $(B \cup A) \cap (\overline{B} \cup A) = A$

Solution.

$(B \cup A) \cap (\overline{B} \cup A)$	
$(B \cap \overline{B}) \cup A$	Distrubitive law
$\varnothing \cup A$	Complement law
A	Identity law

■

(c) $\overline{A \cap \overline{B}} = \overline{A} \cup B$

Solution.

$\overline{A \cap \overline{B}}$	
$\overline{A} \cup \overline{\overline{B}}$	De Morgan's law
$\overline{A} \cup B$	Double complement law

■

b) Exercise 3.6.3

Show that each set equation given below is not a set identity.

(b) $A - (B \cap A) = A$

Solution. Let $A = \{1, 2\}$ and $B = \{2, 3\}$, then $A - (B \cap A) = \{1\}$. ■

(d) $(B - A) \cup A = A$

Solution. Let $A = \{1, 2\}$ and $B = \{2, 3\}$, then $(B - A) \cup A = \{1, 2, 3\}$. ■

c) Exercise 3.6.4

(b) $A \cap (B - A) = \varnothing$

Solution.

$A \cap (B - A)$	
$A \cap (B \cap \overline{A})$	Set subtraction law
$A \cap (\overline{A} \cap B)$	Commutative law
$(A \cap \overline{A}) \cap B$	Associative law
$\varnothing \cap B$	Complement law
$\varnothing$	Domination law

■

(c) $A \cup (B - A) = A \cup B$

Solution.

$A \cup (B - A)$	
$A \cup (B \cap \overline{A})$	Set subtraction law
$(A \cup B) \cap (A \cup \overline{A})$	De Morgan's law
$(A \cup B) \cap U$	Complement law
$A \cup B$	Identity law

■