NYU Computer Science Bridge to Tandon Course	Winter 2021

Homework 5

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Question 3

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.1.3

(b) $f(x) = 1/(x^2 - 4)$

Solution. The function is not well-defined for $x = 2$ and $x = -2$. ■

(c) $f(x) = \sqrt{x^2}$

Solution. The function is well-defined. The range is $\mathbb{R}^+ \cup \{0\}$. ■

b) Exercise 4.1.5

(b) *Solution.* $\{4, 9, 16, 25\}$. ■

(d) *Solution.* $\{0, 1, 2, 3, 4, 5\}$. ■

(h) *Solution.* $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$. ■

(i) *Solution.* $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$. ■

(l) *Solution.* $\{\varnothing, \{2\}, \{3\}, \{2, 3\}\}$. ■

Question 4

I. Solve the following questions from the Discrete Math zyBook:

a. Exercise 4.2.2

(c) $h : \mathbb{Z} \to \mathbb{Z}$. $h(x) = x^3$

Solution. One-to-one but not onto. There is no integer x such that $h(x) = 2$. ∎

(g) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 1, 2y)$

Solution. One-to-one but not onto. There is no integer pair (x, y) such that $f(x, y) = (0, 1)$. ∎

(k) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x, y) = 2^x + y$.

Solution. Not one-to-one. $f(2, 1) = f(1, 3) = 5$. Not onto. There is no positive integer pair (x, y) such that $f(x, y) = 1$. ∎

b. Exercise 4.2.4

(b) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.

Solution. Not one-to-one. $f(000) = f(100) = 100$. Not onto. There is no string $s \in \{0, 1\}^3$ such that $f(s) = 000$. ∎

(c) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.

Solution. One-to-one and onto. ∎

(d) $f : \{0, 1\}^3 \to \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$.

Solution. One-to-one but not onto. There is no string $s \in \{0, 1\}^3$ such that $f(s) = 1000$. ∎

(g) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B = \{1\}$. $f : P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - B$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

Solution. Not one-to-one. Let $X_1 = \{1, 2\}$ and $X_2 = \{2\}$, then $f(X_1) = f(X_2) = \{2\}$. Not onto, there is no $X \in P(A)$ such that $f(X) = \{1\}$. ∎

II. Give an example of a function from the set of integers to the set of positive integers that is:

a. one-to-one, but not onto.

Solution.

$$f(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x + 3 & \text{if } x \leq 0 \end{cases}$$

■

b. onto, but not one-to-one.

Solution. $f(x) = |x| + 1$. ■

c. one-to-one and onto.

Solution.

$$f(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x + 1 & \text{if } x \leq 0 \end{cases}$$

■

d. neither one-to-one nor onto

Solution. $f(x) = 1$. ■

Question 5

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.3.2

(c) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = 2x + 3$.

Solution. $f^{-1}(x) = \dfrac{x-3}{2}$. ∎

(d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. $f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For $X \subseteq A$, $f(X) = |X|$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

Solution. The function f is not one-to-one, so f^{-1} is not well-defined. ∎

(g) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$.

Solution. $f^{-1} = f$. That is, the output of f^{-1} is obtained by taking the input string and reversing the bits. ∎

(i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 5, y - 2)$

Solution. $f^{-1}(x, y) = (x - 5, y + 2)$. ∎

b) Exercise 4.4.8

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$
- $h(x) = x^2 + 1$

(c) $f \circ h$

Solution. $f \circ h(x) = 2x^2 + 5$. ∎

(d) $h \circ f$

Solution. $h \circ f(x) = 4x^2 + 12x + 10$. ∎

c) Exercise 4.4.2

- $f(x) = x^2$
- $g(x) = 2^x$
- $h(x) = \left\lceil \frac{x}{5} \right\rceil$

(b) $f \circ h(52)$

Solution. $f \circ h(52) = \left(\left\lceil \frac{52}{5} \right\rceil \right)^2 = 11^2 = 121.$ ■

(c) $g \circ h \circ f(4)$

Solution. $g \circ h \circ f(4) = g \circ h(16) = g(4) = 16.$ ■

(d) $h \circ f$

Solution. $h \circ f = \left\lceil \frac{x^2}{5} \right\rceil.$ ■

d) Exercise 4.4.6

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
- $g : \{0,1\}3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

(c) $h \circ f(010)$

Solution. $h \circ f(010) = h(110) = 111.$ ■

(d) The range of $h \circ f$

Solution. The range of $h \circ f$ is $\{101, 111\}$. ■

(e) The range of $g \circ f$

Solution. The range of $g \circ f$ is $\{001, 011, 101, 111\}$. ■

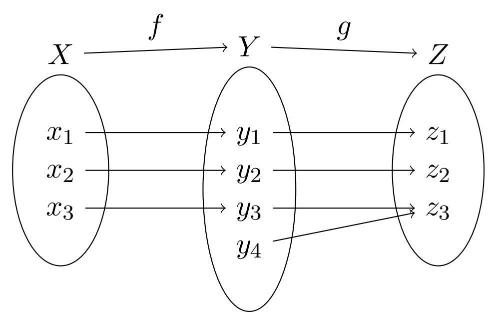
e) **Extra Credit:** Exercise 4.4.4

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Solution. No. We will show that if $g \circ f$ is one-to-one, then f must be one-to-one. Let $x_1 \in X$ and $x_2 \in X$ such that $x_1 \neq x_2$. Since $g \circ f$ is one-to-one, we have $g(f(x_1)) \neq g(f(x_2))$, so we must have $f(x_1) \neq f(x_2)$. Hence f is one-to-one. ■

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Solution. Yes. The diagram below illustrates an example:



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