

## Homework 11

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## Question 5

- a. Use mathematical induction to prove that for any positive integer  $n$ , 3 divides  $n^3 + 2n$  (leaving no remainder).

Hint: you may want to use the formula:  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ .

*Solution.* By induction on  $n$ .

**Base case:**  $n = 1$ .

$$1^3 + 2 \cdot 1 = 3$$

Therefore, for  $n = 1$ , 3 divides  $n^3 + 2n$ .

**Inductive step:** Suppose for any positive integer  $k \geq 1$ , 3 divide  $n^3 + 2n$ . Then we will show that 3 divides  $(n + 1)^3 + 2(n + 1)$ . First We have

$$(n + 1)^3 + 2(n + 1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + (3n^2 + 3n + 3)$$

By the inductive hypothesis we know that 3 divides  $(n^3 + 2n)$ , and it's also clear that 3 divides  $(3n^2 + 3n + 3)$ . Thus we have 3 divides  $(n + 1)^3 + 2(n + 1)$ . ■

- b. Use strong induction to prove that any positive integer  $n$  ( $n \geq 2$ ) can be written as a product of primes.

*Solution.* By strong induction on  $n$ .

**Base case:**  $n = 2$ . Since 2 is a prime number, it's already a product of one prime number: 2.

**Inductive step:** Suppose that for  $k \geq 2$ , any integer  $j$  in the range from 2 through  $k$  can be expressed as a product of prime numbers. We will show that  $k + 1$  can be expressed as a product of prime numbers. If  $k + 1$  is prime, then it is a product of one prime number:  $k + 1$ . If  $k + 1$  is not prime, then it is composite and thus can be expressed as the product of two integers,  $a$  and  $b$ , both of which are at least 2. We will show that both  $a$  and  $b$  are at most  $k$  to apply the inductive hypothesis. Since  $k + 1 = ab$ ,  $a = \frac{k+1}{b}$ . Furthermore, since  $b \geq 2$ ,  $a = \frac{k+1}{b} < k + 1$ , and thus  $a \leq k$ . By the symmetric argument we can show that  $b \leq k$ . Since  $a$  and  $b$  both fall in the range from 2 through  $k$ , the inductive hypothesis can be applied and they can each be expressed as a product of primes:

$$a = p_1 \cdot p_2 \cdots p_m$$

$$b = q_1 \cdot q_2 \cdots q_n$$

Now  $k + 1$  can be expressed as a product of primes:

$$k + 1 = ab = (p_1 \cdot p_2 \cdots p_m)(q_1 \cdot q_2 \cdots q_n)$$

■

## Question 6

Solve the following questions from the Discrete Math zyBook:

a) **Exercise 7.4.1**

Define  $P(n)$  to be the assertion that:

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that  $P(3)$  is true.

*Solution.*

$$\sum_{j=1}^3 j^2 = 1^2 + 2^2 + 3^2 = 14 = \frac{3(3+1)(2 \cdot 3 + 1)}{6}$$

■

(b) Express  $P(k)$ .

*Solution.*

$$\sum_{j=1}^k j^2 = \frac{k(k+1)(2k+1)}{6}$$

■

(c) Express  $P(k+1)$ .

*Solution.*

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

■

(d) In an inductive proof that for every positive integer  $n$ ,

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the base case?

*Solution.*  $P(1)$  is true.

■

- (e) In an inductive proof that for every positive integer  $n$ ,

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the inductive step?

*Solution.* For all positive integers  $k$ ,  $P(k)$  implies  $P(k+1)$ . ■

- (f) What would be the inductive hypothesis in the inductive step from your previous answer?

*Solution.*  $P(k)$ . ■

- (g) Prove by induction that for any positive integer  $n$ ,

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$$

*Solution.* By induction on  $n$ .

**Base case:**  $n = 1$ . We have

$$\sum_{j=1}^1 j^2 = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

**Inductive step:** Suppose that for any positive integer  $k \geq 1$ ,  $P(k)$  is true. Then we will show that  $P(k+1)$  is also true. We have

$$\begin{aligned} \sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^k j^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad (\text{by the inductive hypothesis}) \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

Therefore,  $P(k+1)$  is true. ■

b) **Exercise 7.4.3**

Prove each of the following statements using mathematical induction.

- (c) Prove that for  $n \geq 1$ ,  $\sum_{j=1}^n \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Hint: you may want to use the following fact:  $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$

*Solution.* By induction on  $n$ .

**Base case:**  $n = 1$ . We have  $\sum_{j=1}^1 \frac{1}{j^2} = 1 \leq 2 - \frac{1}{1}$ .

**Inductive step:** Suppose that for any positive integer  $k \geq 1$ ,  $\sum_{j=1}^k \frac{1}{j^2} \leq 2 - \frac{1}{k}$ . Then we

will show that  $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$ . We have

$$\begin{aligned} \sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^k \frac{1}{j^2} + \frac{1}{(k+1)^2} \\ &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \quad (\text{by the inductive hypothesis}) \\ &\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \quad (\text{by the hint}) \\ &= 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)} \\ &= 2 - \frac{1}{k+1} \end{aligned}$$

■

c) **Exercise 7.5.1**

Prove each of the following statements using mathematical induction.

- (a) Prove that for any positive integer  $n$ , 4 evenly divides  $3^{2n} - 1$ .

*Solution.* By induction on  $n$ .

**Base case:**  $n = 1$ . We have  $3^{2 \cdot 1} - 1 = 9 - 1 = 8$ , and it's clear that 4 divides 8.

**Inductive step:** Suppose that for any positive integer  $k \geq 1$ , 4 divides  $3^{2k} - 1$ . Then we will show that 4 divides  $3^{2(k+1)} - 1$ . First We have

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 9 \cdot 3^{2k} - 1 \\ &= 8 \cdot 3^{2k} + (3^{2k} - 1) \end{aligned}$$

By the inductive hypothesis we know that 4 divides  $3^{2k} - 1$ , and it's also clear that 4 divides  $8 \cdot 3^{2k}$ . Thus we have 4 divides  $3^{2k} - 1$ . ■